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Consistency Measures for Pairwise Comparison Matrices

JONATHAN BARZILAI
School of Computer Science, Technical University of Nova Scotia, Halifax, NS B3J 2X4, Canada

ABSTRACT

We propose new measures of consistency of additive and multiplicative pairwise comparison matrices. These measures, the *relative consistency and relative error*, are easy to compute and have clear and simple algebraic and geometric meaning, interpretation and properties. The correspondence between these measures in the additive and multiplicative cases reflects the same correspondence which underpins the algebraic structure of the problem and relates naturally to the corresponding optimization models and axiom systems. The *relative consistency and relative error* are related to one another by the theorem of Pythagoras through the decomposition of comparison matrices into their consistent and error components. One of the conclusions of our analysis is that inconsistency is not a sufficient reason for revision of judgements. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: decision analysis; AHP; consistency; error measurement

1. INTRODUCTION

The analytic hierarchy process (AHP) is a widely used multicriteria decision analysis methodology (Saaty, 1980). In previous work on the mathematical foundations of the AHP (Barzilai *et al*., 1987; Barzilai *et al*., 1992; Barzilai and Golany, 1990, 1994; Barzilai, 1996, 1997) we studied the problem of deriving weight vectors from AHP pairwise comparison matrices. In this paper we study the related problem of measuring consistency of additive and multiplicative pairwise comparison matrices.

Typically, measures of consistency of pairwise comparison matrices are heuristics which are unrelated to the underlying mathematical structure of the problem, are defined in non-standard and non-intuitive ways, possess properties which are not fully understood and have no clear algebraic and geometric interpretation.

Saaty's (1980, p. 21) consistency index is defined in terms of the principal eigenvalue $\lambda_{\max}$ of A. Unfortunately, there is no sense in which $\lambda_{\max}(A_1) < \lambda_{\max}(A_2)$ corresponds to A_1 being more consistent than A_2 when both matrices are inconsistent, although this is exactly the relation such measures are supposed to convey. Furthermore, there is no explanation in the AHP literature of the meaning of comparisons involving the consistency index when comparing consistency of matrices of different dimensions, i.e. an explanation of the meaning of the relation $(\lambda_1 - n_1)/(n_1 - 1) < (\lambda_2 - n_2)/(n_2 - 1)$. Similar questions hold for the role of randomization in the construction of Saaty's consistency ratio and the meaning of the AHP 10% cut-off rule.

Golden and Wang (1989) consider Saaty's consistency ratio and the 10% cut-off rule 'somewhat arbitrary' and propose a measure based on the geometric mean. Their paper gives no reason for the use of the geometric mean and it is difficult to see what the properties of their measure are or how it relates to the structure of the problem. The papers by Islei and Lockett (1988) and Liang and Sheng (1990) are also relevant to our subject.

This paper proposes new measures of consistency, the *relative consistency and relative error* of pairwise comparison matrices. Section 2 provides the framework for our study by outlining the mathematical structure of the problem. Section 3 deals with optimization models and derived consistency measures. The *relative error* is defined and its basic properties are explored in Section 4. Section 5 deals with the decomposition of pairwise comparison matrices and the definition of the *relative consistency* measure. The results are extended to the multiplicative case in Section 6 and to hierarchies in Section 9. Numerical examples are given in Section 7 and Section 8 deals with the AHP measures. Conclusions from the analysis on the issue of revising judgements are drawn in Section 10 and the results are summarized in Section 11.

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2. STRUCTURE AND NOTATION

2.1. Conventions

Throughout this paper, all matrices are $n \times n$, vectors are n-dimensional and vector and matrix operations apply componentwise.

2.2. Definitions

1. $A = (a_{ij})$ is a pairwise multiplicative matrix if $0 < a_{ij} = 1/a_{ji}$.
2. $w = (w_k)$ is a *multiplicative weight vector* if $w_k > 0$ and $\Pi_{k=1}^n w_k = 1$.
3. $A = (a_{ij})$ is a *multiplicative consistent matrix* if $a_{ij} = w_i/w_j$ for some multiplicative weight vector w.
4. $A^\times$, $w^\times$ and $C^\times$ are the sets of all pairwise multiplicative matrices, multiplicative weight vectors and multiplicative consistent matrices respectively.
5. $f^\times$ is the set of all mappings from $A^\times$ to $w^\times$.
6. $A = (a_{ij})$ is a *pairwise additive matrix* if $a_{ij} = -a_{ji}$.
7. $w = (w_k)$ is an *additive weight vector* if $\Sigma_{k=1}^n w_k = 0$.
8. $A = (a_{ij})$ is an *additive consistent matrix* if $a_{ij} = w_i - w_j$ for some additive weight vector w.
9. A^+, w^+ and C^+ are the sets of all pairwise additive matrices, additive weight vectors and additive consistent matrices respectively.
10. f^+ is the set of all mappings from A^+ to w^+.

2.3. Structure

Proofs of the following observations are elementary and are omitted.

1. $A^\times$, $w^\times$ and $C^\times$ are all groups under componentwise multiplication; $C^\times$ is isomorphic to $w^\times$ and is a subgroup of $A^\times$.
2. A^+, w^+ and C^+ are all groups under componentwise addition; C^+ is isomorphic to w^+ and is a subgroup of A^+.
3. $A^\times$, $w^\times$ and $C^\times$ are isomorphic to A^+, w^+ and C^+ respectively. (The logarithmic function with any fixed basis applied componentwise is an isomorphism and the corresponding exponential function is its inverse.)

The normalization of weight vectors is necessary to ensure uniqueness. The normalization in the additive case is natural as the data are given in terms of differences, the fundamental nature of the problem is additive and the zero matrix and vector are the units of the groups A^+, C^+ and w^+. (See the discussion in Barzilai *et al.* (1987)). If w is a weight vector satisfying $\Sigma_{k=1}^n w_k = 0$, then $w' = w + \alpha/n$ satisfies $\Sigma_{k=1}^n w'_k = \alpha$. In particular, $w' = w + 1/n$ satisfies the common normalization $\Sigma_{k=1}^n w'_k = 1$.

3. OPTIMIZATION MODELS AND ERROR MEASUREMENT

We have established (Barzilai *et al.*, 1987; Barzilai and Golany, 1990, 1994; Barzilai, 1996, 1997) that the only acceptable solution for the additive problem of deriving weights from additive pairwise comparison matrices is the arithmetic mean mapping.

$$w_i = \frac{1}{n} \sum_{j=1}^n a_{ij}$$

The corresponding solution of the multiplicative problem is the geometric mean

$$w_i = \left(\prod_{j=1}^n a_{ij} \right)^{1/n}$$

and the logarithmic/exponential isomorphisms of Section 2.3 link the two means:

$$\ln\left[\left(\prod_{j=1}^n e^{a_{ij}} \right)^{1/n} \right] = \frac{1}{n} \sum_{j=1}^n a_{ij}$$

We have also shown (Barzilai et al., 1992) that the arithmetic mean is the solution of the optimization problem

$$\min_w \quad \sum_{i=1}^n \sum_{j=1}^n [a_{ij} - (w_i - w_j)]^2$$
$$\text{s.t.} \quad \sum_{i=1}^n w_i = 0$$

naturally associated with the additive problem of specifying $f \in f^+$ and that the geometric mean is the solution of the optimization problem

$$\min_w \quad \sum_{i=1}^n \sum_{j=1}^n [\log a_{ij} - \log(w_i/w_j)]^2$$
$$\text{s.t.} \quad \prod_{i=1}^n w_i = 1, \quad w_i > 0, \quad i = 1, \ldots, n$$

The group structures provided within the framework of Section 2 justify the association between the two optimization problems by means of the logarithmic isomorphism relating $A^\times$ and A^+.

The extremal values of the objective functions of the optimization problems provide measures of consistency in a natural way. In the additive case this measure is given by $\Sigma_{i,j=1}^{n} e_{ij}^2$, where $e_{ij} = a_{ij} - (w_i - w_j)$ are the error terms. (See Section 7 for numerical examples.) For the arithmetic mean solution the error terms satisfy

$$e_{ij} = a_{ij} - \frac{1}{n}\sum_{k=1}^{n}(a_{ik} - a_{jk}) = \frac{1}{n}\sum_{k=1}^{n}(a_{ij} + a_{jk} + a_{ki})$$

Since consistency means $a_{ij} + a_{jk} = a_{ik}$ or $a_{ij} + a_{jk} + a_{ki} = 0$, e_{ij} is the average inconsistency over all triplets with fixed i and j. Similarly, in the multiplicative case we have

$$e_{ij} = \left(\prod_{k=1}^{n} a_{ij}a_{jk}a_{ki}\right)^{1/n}.$$

The eigenvector solution is not applicable to the additive problem. Elegant as it is, the spectral analysis of positive matrices is not relevant to our decision analysis problem. While Saaty (1980, pp. 53, 223 and 234; 1983, pp. 248–250) does refer to difference and interval scales and to the arithmetic mean, the eigenvector solution limits the applicability of the AHP to positive matrices and hence to the multiplicative case. For example, Saaty (1983) proposes

$$\frac{1}{n^2}\sum_{i,j=1}^{n} a_{ij}$$

as a measure of inconsistency for additive matrices. There is no natural relationship between this measure and the eigenvalue-based multiplicative (AHP) consistency measure and this measure is meaningless since its value is zero for every matrix in A^+.

4. RELATIVE ERROR MEASURE

We first consider the additive case. Let $A = (a_{ij}) \in A^+$ and

$$w_i = \frac{1}{n}\sum_{j=1}^{n} a_{ij}$$

$$C = (c_{ij}) = (w_i - w_j)$$

$$E = (e_{ij}) = (a_{ij} - c_{ij})$$

Then $A = C + E$ is a decomposition of A into its consistent and error components (for reasons that will become clear later in this section, the error component will be called totally inconsistent). In addition, $m(A) = \Sigma_{ij} e_{ij}^2$ is a measure of the error or amount of inconsistency of A with at least the following properties.

1. $m(A)$ is a continuous function of A.
2. $m(A) = 0$ if and only if A is consistent.
3. $m(-A) = m(A)$
4. $m(A_1) > m(A_2)$ if and only if the related error terms satisfy $\Sigma_{ij} e_{ij,1}^2 > \Sigma_{ij} e_{ij,2}^2$.
5. m may be naturally extended to the multiplicative case by defining $m_1(A) = m(\log A)$ for $A \in A^{\times}$.

Property 1 is a reasonable requirement of any measure as it is difficult to interpret discontinuities in this context. Property 2 enables us to distinguish between consistent and inconsistent matrices but is insufficient to determine the degree of inconsistency of inconsistent matrices. Property 3 is related to the fundamental theory-of-measurement issue of independence of scale inversion (Barzilai, 1996, 1997). Some form of Property 4 must be satisfied by any meaningful measure and Property 5 is needed to link the parallel structures of the multiplicative and additive problems. Using the measure $m(A)$ as our starting point, we can improve on it by noting its deficiences.

- $m_1(A)$ depends on the logarithm base we use.
- A statement of the type 'A is less consistent than B if $m(A) > m(B)$' does not appear to be meaningful when A and B are of different dimensions.
- A cut-off rule of the type 'A is close enough to being consistent if $m(A) \leq \alpha$' for some fixed positive constant α, independent of n, does not appear to be meaningful.

A natural way to construct a measure that preserves the properties of $m(A)$ and addresses its deficiencies is to consider the *relative error* of A defined by $\mathrm{RE}(A) = 0$ if $A = 0$ and for $A \neq 0$ by

$$\mathrm{RE}(A) = \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2}$$

Note that $\mathrm{RE}(A) = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} e_{ij}^2 / \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} a_{ij}^2 = \Sigma_{i=1}^{n-1}\Sigma_{j=i+1}^{n} e_{ij}^2 / \Sigma_{i=1}^{n-1}\Sigma_{j=i+1}^{n} a_{ij}^2$ since A and E are antisymmetric. We will see that this measure and a related measure of *relative consistency* (formally defined in Section 5) are mappings onto the interval [0, 1] regardless of n, the size of A. Using these measures, we may compare consistency of matrices of different dimensions and justify the use of cut-off rules accepting A as sufficiently consistent if, for example, its relative error satisfies $\mathrm{RE}(A) \leq 0.10$ or, equivalently, if its relative consistency satisfies $\mathrm{RC}(A) \geq 0.90$. In addition, the extension of these measures to the multiplicative case fits the algebraic structure of the problem and is independent of logarithm bases. The following theorem establishes the range of $\mathrm{RE}(A)$.

Theorem 1
The relative error of any $A \in A^+$ satisfies $0 \leq \mathrm{RE}(A) \leq 1$.

Proof
$\mathrm{RE}(A)$ is zero when A is consistent and cannot be negative since it is the ratio of sums of squares. To prove that $\mathrm{RE}(A) \leq 1$, we need to show that $\Sigma_{ij} e_{ij}^2 \leq \Sigma_{ij} a_{ij}^2$. Since the error component minimizes the sum of squares of errors, we have

$$\sum_{ij} e_{ij}^2 = \sum_{ij} [a_{ij} - (w_i^* - w_j^*)]^2 \leq \sum_{ij} [a_{ij} - (w_i - w_j)]^2$$

where w^* is the arithmetic mean which minimizes the error and w is any weight vector. Substituting $w = 0$, we see that $\mathrm{RE}(A)$ is bounded from above by unity. □

A matrix A will be called totally inconsistent if its relative error or inconsistency is maximal, i.e. $\mathrm{RE}(A) = 1$. We will show in the next section that any matrix $A \in A^+$ can be uniquely decomposed into consistent and totally inconsistent components: $A = C + E$ with $\mathrm{RE}(C) = 0$ and $\mathrm{RE}(E) = 1$. It follows that each inconsistent $A \in A^+$ can be mapped into a totally inconsistent matrix through this decomposition.

An example of a totally inconsistent matrix is

$$X = \begin{pmatrix} 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 \end{pmatrix}$$

We will prove in Section 5 that $A \in A^+$ is totally inconsistent if and only if its row sums are all zero, but it is easy to be see by a direct computation that $\mathrm{RE}(X) = 1$. Reading the matrix X by rows, we see that it corresponds to ranking five objects A, B, C, D, E as follows:

D, E, A, B, C relative to A in row 1,
E, A, B, C, D relative to B in rwo 2,
A, B, C, D, E relative to C in row 3,
B, C, D, E, A relative to D in row 4,
C, D, E, A, B relative to E in row 5.

5. RELATIVE CONSISTENCY AND ORTHOGONAL DECOMPOSITION

Our derivation so far has been based on the classical idea of studying relative errors in an intuitive manner. We now prove a stronger result (Theorem 3 which implies Theorem 1) based on the algebraic properties of the decomposition defined in the previous section. Let $A = C + E$ be the decomposition of $A = (a_{ij}) \in A^+$ into its consistent and inconsistent components, i.e.

$$w_i = \frac{1}{n} \sum_{j=1}^{n} a_{ij}$$
$$C = (c_{ij}) = (w_i - w_j)$$
$$E = (e_{ij}) = (a_{ij} - c_{ij}).$$

We define the relative consistency of $A \neq 0$ as

$$\mathrm{RC}(A) = \frac{\sum_{ij} c_{ij}^2}{\sum_{ij} a_{ij}^2}$$

and $\mathrm{RC}(0) = 1$. The main result of this section is based on the classical *projection theorem* which we restate without proof (Gantmacher, 1959, Section 9.4; Luenberger, 1969, p. 51).

Theorem 2
Let A be an arbitrary vector in a Euclidean space R and let S be a subspace. Then A can be represented uniquely in the form $A = C + E$, where $C \in S$ and $E \perp S$. Furthermore, C and only C satisfies $\|A - C\| \leq \|A - X\|$ for all $X \in S$, where $\|A\|$ is the Euclidean norm of A.

We may rewrite any $n \times n$ matrix as an n^2-dimensional vector by ordering its elements by rows. In this n^2-dimensional Euclidean space the vectors corresponding to the consistent matrices in C^+ form a subspace. Combining the projection theorem with the minimization problem which characterizes the arithmetic mean, we see that the decomposition of A into its consistent and inconsistent components is an orthogonal decomposition, i.e. $C \perp E$ or $\Sigma_{ij} c_{ij} e_{ij} = 0$. Using this equation with $a_{ij} = c_{ij} + e_{ij}$, we obtain

$$\sum_{ij} a_{ij}^2 = \sum_{ij} c_{ij}^2 + 2\sum_{ij} c_{ij} e_{ij} + \sum_{ij} e_{ij}^2$$

and therefore

$$\sum_{ij} a_{ij}^2 = \sum_{ij} c_{ij}^2 + \sum_{ij} e_{ij}^2$$

(which is a restatement of the theorem of Pythagoras—see Gantmacher (1959, p. 244)). For $A \neq 0$ we then have

$$\frac{\sum_{ij} c_{ij}^2}{\sum_{ij} a_{ij}^2} + \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2} = 1$$

or

$$\mathrm{RC}(A) + \mathrm{RE}(A) = 1$$

which (regardless of the size n of the matrices) can be restated as $\cos^2(\alpha) + \sin^2(\alpha) = 1$, where α is the angle between the vectors A^* and C^* generated by rewriting the elements of the matrices A and C as n^2-dimensional vectors. (The analogy with measuring statistical coefficients of determination is clear.) Recalling that we defined $\mathrm{RC}(0) = 1$ and $\mathrm{RE}(0) = 0$, we have proved the following.

Theorem 3
For any $A \in A^+$, $\mathrm{RC}(A) + \mathrm{RE}(A) = 1$.

The following theorem provides an alternative characterization of totally inconsistent matrices.

Theorem 4
$0 \neq E \in A^+$ is totally inconsistent if and only if the row sums of E are all zero, i.e. $\Sigma_j e_{ij} = 0$ for all i.

Proof
If $w_i = (\Sigma_j e_{ij})/n = 0$, then $c_{ij} = w_i - w_j = 0$ and $e_{ij} = a_{ij}$ so that $\Sigma_{ij} e_{ij}^2 = \Sigma_{ij} a_{ij}^2$, implying $\mathrm{RE}(E) = 1$ and E is totally inconsistent.

Conversely, if E is totally inconsistent, then $\mathrm{RE}(E) = 1$ and, by Theorem 3, $\mathrm{RC}(E) = 0$, implying $\Sigma_{ij} c_{ij}^2 = 0$ and therefore $w_i = w_j$ for all i, j and $w_i = 0$ for all i since $\Sigma_i w_i = 0$. Finally, since $\Sigma_j e_{ij} = n w_i$ and $w_i = 0$, $\Sigma_j e_{ij} = 0$. □

The geometric interpretation of the projection theorem makes it clear that a non-zero matrix is totally inconsistent if and only if it is orthogonal to the subspace of all consistent matrices. The following theorem is the formal statement of this result.

Theorem 5
$0 \neq E \in A^+$ is totally inconsistent if and only if $E \perp C^+$, i.e. $E \perp C$ for all $C \in C^+$.

Proof
By Theorem 4, if $E = (e_{ij})$ is totally inconsistent, then $\Sigma_j e_{ij} = 0$ for all i and, since E is antisymmetric, $\Sigma_i e_{ij} = 0$ for all j as well. If C is a consistent matrix, then $C = (c_{ij}) = (w_i - w_j)$ for some weight vector w. To show that E is orthogonal to C, we calculate $\Sigma_{ij} (w_i - w_j) e_{ij} = \Sigma_{ij} w_i e_{ij} - \Sigma_{ij} w_j e_{ij} = 0$ since $\Sigma_{ij} w_i e_{ij} = \Sigma_i w_i \Sigma_j e_{ij} = 0$ and $\Sigma_{ij} w_j e_{ij} = \Sigma_j w_j \Sigma_i e_{ij} = 0$.

Conversely, if E is orthogonal to the subspace C^+, then $E \perp C$ for some $0 \neq C \in C^+$. Defining $A = C + E$, the projection theorem implies that $A = C + E$ is precisely the decomposition of A into its consistent and totally inconsistent components and therefore E is totally inconsistent. □

The following theorem is a corollary of the uniqueness—by the projection theorem—of the projection of A onto the subspace C^+.

Theorem 6
The decomposition of a consistent matrix $C \in C^+$ is given by $C = C + 0$. The decomposition of a totally inconsistent matrix E is given by $E = 0 + E$.

6. MULTIPLICATIVE CASE

The logarithmic isomorphism relating $A^\times$ and A^+ enables us to extend our results to the multiplicative case. Given $A \in A^\times$, define $L = (l_{ij})$ by $l_{ij} = \log_2 a_{ij}$ and define $\mathrm{RC}(A) = \mathrm{RC}(L)$ and $\mathrm{RE}(A) = \mathrm{RE}(L)$. This convenient notation is justified since $\mathrm{RC}(A)$ and $\mathrm{RE}(A)$ are independent of the logarithm base (any fixed base other than two may be used) and A cannot belong to both A^+ and $A^\times$. (In computing terms we may write a programme named RC which accepts an additive or multiplicative matrix A as its input and computes the relative consistency of this matrix. The programme first checks whether A is multiplicative by testing if $a_{11}=1$, in which case it replaces each a_{ij} with $\log_2 a_{ij}$. It then proceeds to compute the relative consistency of an additive matrix—either the input matrix or the one obtained by the above transformation from a multiplicative input matrix.)

The decomposition of $A=(a_{ij})\in A^{\times}$ into its consistent and inconsistent (or error) components is given by $A=C\times E$, where

$$w_i=\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}$$

$$C=(c_{ij})=(w_i \div w_j)$$

$$E=(e_{ij})=(a_{ij}\div c_{ij})$$

In analogy with the additive case, $A\in A^{\times}$ will be called totally inconsistent if its relative error or inconsistency is maximal, i.e. $\mathrm{RE}(A)=1$. The counterparts of Theorems 3 and 4 are now stated for the multiplicative case without proof.

Theorem 7
For any $A\in A^{\times}$, $\mathrm{RC}(A)+\mathrm{RE}(A)=1$.

Theorem 8
$1\neq E\in A^{\times}$ is totally inconsistent if and only if the row products of E are all ones, i.e. $\Pi_j e_{ij}=1$ for all i.

An example of a totally inconsistent multiplicative matrix is

$$Y=\begin{bmatrix} 1 & 2 & 4 & \frac14 & \frac12 \\ \frac12 & 1 & 2 & 4 & \frac14 \\ \frac14 & \frac12 & 1 & 2 & 4 \\ 4 & \frac14 & \frac12 & 1 & 2 \\ 2 & 4 & \frac14 & \frac12 & 1 \end{bmatrix}$$

Note that $Y=(y_{ij})=(2^{x_{ij}})$, where $X=(x_{ij})$ is the totally inconsistent additive matrix defined in Section 4, $\mathrm{RE}(Y)=1$ and $\mathrm{RC}(Y)=0$.

The multiplicative decomposition is unique and is given for $C\in C^{\times}$ by $C=C\times 1$, while the decomposition of a multiplicative totally inconsistent matrix E is given by $E=1\times E$, where '1' is the symbol for the unit matrix with ones in all positions.

7. NUMERICAL EXAMPLES

Define the multiplicative matrix M and the additive matrix A as

$$M=\begin{pmatrix} 1 & 2 & \frac{1}{16} \\ \frac12 & 1 & 128 \\ 16 & \frac{1}{128} & 1 \end{pmatrix}$$

$$A=\begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

To compute the relative consistency of M, we first need to compute $A=(a_{ij})=(\log_2 m_{ij})$ as above. Note that any positive number rather than two can serve as the logarithm base since $\mathrm{RC}(kA)=\mathrm{RC}(A)$. Since A is antisymmetric, $\Sigma_{ij}\, a_{ij}^2=2(1^2+4^2+7^2)=132$. The row arithmetic mean vector for A is given by $w=(-1,\ 2,\ -1)$. Next we compute the consistent component of A.

$$C_A=(c_{ij})=(w_i-w_j)=\begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix}$$

We can now compute $\Sigma_{ij}\, c_{ij}^2=2(3^2+0^2+3^2)=36$ and

$$\mathrm{RC}(M)=\mathrm{RC}(A)=\frac{\sum_{ij} c_{ij}^2}{\sum_{ij} a_{ij}^2}=\frac{36}{132}=\frac{3}{11}=0.2727$$

indicating that the level of consistency of M and A is very low. The error component of A is computed by $E_A=A-C_A$ and the decomposition of A is given by

$$A=\begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}=C_A+E_A$$

$$=\begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 4 & -4 \\ -4 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix}$$

Note that in accordance with Theorem 4 the row sums of E_A are all zero. To demonstrate Theorem 3, note that $\Sigma_{ij} e_{ij}^2 = 2(3 \times 4^2) = 96$,

$$\mathrm{RE}(M) = \mathrm{RE}(A) = \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2} = \frac{96}{132} = \frac{8}{11} = 0.7273$$

indicating a high level of inconsistency, and $\mathrm{RC}(M) + \mathrm{RE}(M) = \mathrm{RC}(A) + \mathrm{RE}(A) = 1$. Finally, the decomposition of M can be computed from that of A:

$$M = \begin{pmatrix} 1 & 2 & \frac{1}{16} \\ \frac{1}{2} & 1 & 128 \\ 16 & \frac{1}{128} & 1 \end{pmatrix} = C_M \times E_M$$

$$= \begin{pmatrix} 1 & \frac{1}{8} & 1 \\ 8 & 1 & 8 \\ 1 & \frac{1}{8} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 16 & \frac{1}{16} \\ \frac{1}{16} & 1 & 16 \\ 16 & \frac{1}{16} & 1 \end{pmatrix}$$

8. AHP MEASURES

As mentioned above, in the additive case, Saaty (1983) proposes $(\Sigma_{ij} a_{ij})/n^2$ as a measure of consistency. This is not a meaningful measure since for any $A \in A^+$ the condition $a_{ij} = -a_{ji}$ implies $\Sigma_{ij} a_{ij} = 0$. Note also that the denominator of any reasonable variant of this measure should be $n(n-1)$ to reflect the constraint that the diagonal of matrices in the set A^+ is always zero, while the numerator should be an appropriate function of the errors, such as their absolute values, squares, etc.

In the multiplicative case, Saaty (1980, p. 21) defines the consistency index of A by

$$\mathrm{CI}(A) = \frac{\lambda_{\max} - n}{n - 1}$$

where $\lambda_{\max}$ is the principal eigenvalue of A and n is its size. He then uses an average random index $\mathrm{ARI}(n)$ (the average consistency index of a sample of randomly generated reciprocal matrices of size n) to define the consistency ratio of A by

$$\mathrm{CR}(A) = \frac{\mathrm{CI}(A)}{\mathrm{ARI}(n)}$$

Both $\mathrm{CI}(A)$ and $\mathrm{CR}(A)$ are heuristics with poorly understood properties and justification. AHP consistency analysis tends to centre on properties of the CI measure, although the CR measure is the one used almost exclusively in practice and the role of the randomization process used to derive the CR measure is obscure. The critical issue concerning these measures is that the statement that 'the closer $\lambda_{\max}$ is to n to the more consistent is the result' (Saaty, 1980, p. 21) is not justified anywhere in the AHP literature. Stated differently, it is clear, but of little value in measuring consistency, that $\mathrm{CI}(A)$ is zero if and only if A is consistent, but it is not clear at all in what sense $\mathrm{CI}(A_1) < \mathrm{CI}(A_2)$ corresponds to A_1 being more consistent that A_2 other than in circularly stating that the consistency index of A_1 is smaller than that of A_2.

9. CONSISTENCY OF A HIERARCHY

Saaty (1980, Section 4-5) proposes a measure of consistency for an entire hierarchy as follows.

> What we do is to multiply the index of consistency obtained from a pairwise comparison matrix by the priority of the property with respect to which the comparison is made and add all the results for the entire hierarchy. This is then compared with the corresponding index obtained by taking randomly generated indices, weighting them by the priorities and adding. The ratio should be in the neighborhood of 0.10 in order not to cause concern for improvements with the actual operation and in the judgments.

No basis is given for applying the operations of addition and multiplication of weights and consistency indexes of individual matrices for the purpose of this computation and the properties of such hierarchy consistency measures are unknown. Furthermore, similar aggregation rules for the computation of AHP weights have already been shown to be invalid (Barzilai and Golany, 1994). Note that an AHP hierarchy is a collection of multiplicative comparison matrices related to one another by their position in the hierarchy. Although the position of a given matrix in the hierarchy affects its contribution to the *overall weights*, the position in the hierarchy does not affect the contribution to the *overall inconsistency* of the hierarchy. In other words, two hierarchies with the same collection of matrices in different positions display the same level of inconsistency. Note also the discussion in the next section on the independence between consistency and acceptability of derived or projected weights.

Let A_i, $i \in S$, be a collection of either additive or multiplicative matrices. The natural way of measuring the relative consistency of this collection of matrices is to compute for each matrix A_i its consistent component C_i and produce the corresponding n_i^2-dimensional vectors A_i^* and C_i^*. (Note that it is sufficient to take the upper diagonal of these matrices as we do in the example below.) These vectors are then concatenated to produce the vectors $A^{**} = (A_1^*, A_2^*, A_3^*, \ldots)$ and $C^{**} = (C_1^*, C_2^*, C_3^*, \ldots)$. Finally, the relative consistency of the hierarchy $\mathrm{RC}(H)$ is given by

$$\mathrm{RC}(H) = \frac{\sum_i (c_i^{**})^2}{\sum_i (a_i^{**})^2}$$

Note that the vectors A_i^* and C_i^* may be concatenated in any order provided that the same order applies to both A^{**} and C^{**}. In the additive case the error vector may now be defined as $E^{**} = A^{**} - C^{**}$. The results of Sections 5 and 6 carry over and are summarized without proof in the following theorem which has an obvious multiplicative version.

Theorem 9
For an additive collection of matrices H, the vectors A^{**}, C^{**} and E^{**} satisfy $A^{**} = C^{**} + E^{**}$ and $C^{**} \perp E^{**}$. In addition, $\mathrm{RC}(H) + \mathrm{RE}(H) = 1$.

As an example, consider the following collection of two matrices:

$$A_1 = \begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{pmatrix} = C_{A_1} + E_{A_1}$$

$$= \begin{pmatrix} 0 & -3 & 0 \\ 3 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 & -4 \\ -4 & 0 & 4 \\ 4 & -4 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & -1 & 10 \\ 1 & 0 & -4 \\ 10 & 4 & 0 \end{pmatrix} = C_{A_2} + E_{A_2}$$

$$= \begin{pmatrix} 0 & 4 & 5 \\ -4 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -5 & 5 \\ 5 & 0 & -5 \\ -5 & 5 & 0 \end{pmatrix}$$

We have $A^{**} = (1, -4, 7, -1, 10, -4)$, $C^{**} = (-3, 0, 3, 4, 5, 1)$ and $E^{**} = (4, -4, 4, -5, 5, -5)$. The inner product of the vectors C^{**} and E^{**} is zero, verifying that they are indeed orthogonal, $\Sigma_i (a_i^{**})^2 = 183$, $\Sigma_i (c_i^{**})^2 = 60$, $\Sigma_i (e_i^{**})^2 = 123$, $\mathrm{RC}(H) = 60/183$ and $\mathrm{RE}(H) = 123/183$ with $\mathrm{RC}(H) + \mathrm{RE}(H) = 1$.

10. CONSISTENCY AND REVISION OF JUDGEMENTS

Section 3-5 in Saaty (1980) entitled 'Revising Judgments', opens with the following question.

> Assume that the consistency index is sufficiently large to warrant judgmental revision. When should it be made?

The underlying presumption here is that the projected weights should be accepted if the comparison matrix is sufficiently consistent and rejected otherwise. If the weights are rejected, the matrix is to be revised through an iterative procedure which converges to a consistent matrix.

The revision process leaves it unclear whether weights generated through this process are expected to be as close to the originally derived weights or as far away from these weights and on what basis. Another presumption that seems to underpin this proposal is that a convergent procedure is preferable to one that terminates in one step (i.e. replacing the input judgements with (w_i/w_j) which is consistent and does not involve a distortion of the decision maker's input judgements). To illustrate the difficulties with this proposal, consider the following matrices:

$$S_0 = \begin{pmatrix} 1 & 2.299 & 9.226 & 2.497 & 1.351 & 2.306 \\ 0.435 & 1 & 4.012 & 1.086 & 0.588 & 1.003 \\ 0.108 & 0.249 & 1 & 0.271 & 0.146 & 0.250 \\ 0.401 & 0.921 & 3.695 & 1 & 0.541 & 0.924 \\ 0.740 & 1.702 & 6.828 & 1.848 & 1 & 1.707 \\ 0.434 & 0.997 & 4.000 & 1.082 & 0.586 & 1 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 11.497 & 1.845 & 2.497 & 6.756 & 0.461 \\ 0.087 & 1 & 20.062 & 0.217 & 0.588 & 5.015 \\ 0.542 & 0.050 & 1 & 1.353 & 0.029 & 0.250 \\ 0.401 & 4.605 & 0.739 & 1 & 2.706 & 0.185 \\ 0.148 & 1.702 & 34.139 & 0.370 & 1 & 8.534 \\ 2.168 & 0.199 & 4.000 & 5.412 & 0.117 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 57.485 & 0.369 & 2.497 & 33.781 & 0.092 \\ 0.017 & 1 & 100.309 & 0.043 & 0.588 & 25.075 \\ 2.710 & 0.010 & 1 & 6.765 & 0.006 & 0.250 \\ 0.401 & 23.026 & 0.148 & 1 & 13.531 & 0.037 \\ 0.030 & 1.702 & 170.694 & 0.074 & 1 & 42.671 \\ 10.840 & 0.040 & 4.000 & 27.062 & 0.023 & 1 \end{pmatrix}$$

The matrix S_0 is consistent. For consistent matrices the (normalized) geometric mean and eigenvector are identical and these are given in this case by $gm = ev = (0.321,\ 0.140,\ 0.035,\ 0.128,\ 0.237,\ 0.139)$. In fact, to obtain the numbers in S_0, we computed in double precision (ev_i/ev_j), where the vector ev is the eigenvector solution for the School Example (Saaty, 1980, Table 1-2, p. 26).

The matrix S_1 is very inconsistent. Its principal eigenvalue, consistency index and consistency ratio are 12.40, 1.28 and 1.03 respectively. Its relative consistency and relative error are given by $\mathrm{RC}(S_1) = 0.36$ and $\mathrm{RE}(S_1) = 0.64$. For S_1 the geometric mean and eigenvector are again identical and are given as before by $gm = ev = (0.321,\ 0.140,\ 0.035,\ 0.128,\ 0.237,\ 0.139)$. The matrix S_1 was constructed as $S_1 = S_0 \times E_S$, where

$$E_S = \begin{pmatrix} 1 & 5 & 0.2 & 1 & 5 & 0.2 \\ 0.2 & 1 & 5 & 0.2 & 1 & 5 \\ 5 & 0.2 & 1 & 5 & 0.2 & 1 \\ 1 & 5 & 0.2 & 1 & 5 & 0.2 \\ 0.2 & 1 & 5 & 0.2 & 1 & 5 \\ 5 & 0.2 & 1 & 5 & 0.2 & 1 \end{pmatrix}$$

E_S is totally inconsistent since its row products are all ones. Recalling that S_0 is consistent, we see that $S_1 = S_0 \times E_S$ is the orthogonal decomposition of S_1, implying that S_1 and S_0 have the same consistent components and therefore the same geometric mean solution. The eigenvector solution is unchanged as well because the row sums of E_S are constant.

The matrix S_2 is more inconsistent that S_1. Its principal eigenvalue, consistency index and consistency ratio are 52.08, 9.22 and 7.43 respectively, $\mathrm{RC}(S_2) = 0.12$ and $\mathrm{RE}(S_2) = 0.88$. For S_2 the geometric mean and eigenvector are again identical and unchanged from the values above. The matrix S_2 was constructed as $S_2 = S_0 \times E_S^2$, where the squares in E_S^2 are computed componentwise.

In general, define $S_k = S_0 \times E_S^k$, where the powers in E_S^k are computed componentwise. The geometric mean and eigenvector of S_k for any real $k \geq 0$ are given by $gm = ev = (0.321,\ 0.140,\ 0.035,\ 0.128,\ 0.237,\ 0.139)$, while the inconsistency of S_k increases as k does ($\lambda \to \infty$ and $\mathrm{RC}(S_k) \to 0$). Since the projected weights of S_k for any $k \geq 0$ are constant, they are independent of its level of consistency. Clearly, the properties of this set of matrices have nothing to do with whether the weights are derived using the eigenvector or geometric mean and what consistency measures are used. The projected weights for these matrices are derived directly from the decision maker's undistorted input and are a true representation of the decision maker's preferences whether or not the decision maker is consistent. Regardless of the value of k, the decision analyst should remove whatever level of inconsistency present in the decision maker's input without injecting into it distorted or revised judgements—which is exactly what we do when we compute the projected weights.

Decomposing the input comparison matrix A into its components $A = C \times E$, we see that our aim is to reduce inconsistency by driving E as close as possible to '1'—the multiplicative group identity matrix. While revising E may appear desirable, distorting C in the process is not justified. However, if C is not to be distorted, no revision process is needed to determine the decision maker's preferences. We close this section by noting the last paragraph of Section 3-5 in Saaty (1980), which states the following.

> We caution against excessive use of this process of forcing the values of judgments to improve consistency. It distorts the answer. One would rather have naturally improved judgments arising from experience.

Forcing the values of judgements to improve consistency distorts the answer regardless of the level of consistency of A. In our opinion the projected weights should be presented to the decision maker as feedback from the analysis. If the decision maker can confirm that the matrix C is indeed an acceptable reflection of his/her preferences, no revision of judgements is necessary regardless of his/her level of consistency. Otherwise, a revision of the judgements is justified.

11. CONCLUSIONS

Based on our analysis of the mathematical foundations of the AHP, we constructed simple measures of consistency of additive and multiplicative pairwise comparison matrices. These measures, the *relative consistency* and *relative error*, are extended in a natural way to hierarchies or arbitrary collections of comparison matrices. They are derived from and fit the algebraic structure of the problem, are easy to compute and have clear and simple algebraic and geometric meaning, interpretation and properties. The correspondence between these measures in the additive and multiplicative cases reflects the same correspondence which underpins the algebraic structure of the problem and relates naturally to the corresponding optimization models and axiom systems.

The *relative consistency* and *relative error* are related to one another by the theorem of Pythagoras through the decomposition of comparison matrices into their consistent and error components. Since these consistency measures map comparison matrices of any size onto the common scale [0, 1], they make it possible to compare consistency of comparison matrices and collections of comparison matrices of diverse dimensions.

It is important to re-emphasize that for the *relative consistency* measure the fundamental comparison '$\mathrm{RC}(A_1) > \mathrm{RC}(A_2)$ if and only if A_1 is more consistent than A_2' and cut-off rules of the type 'A is sufficiently consistent if $\mathrm{RC}(A) \geq 0.90$' are meaningful and easy to understand and interpret. Finally, the insight gained from our analysis leads us to conclude that, by itself, inconsistency is not a sufficient reason to require the decision maker to revise his/her judgements.

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