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European Journal of Operational Research 177 (2007) 488–497
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Decision Support

The harmonic consistency index for the analytic hierarchy process

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Received 1 April 2005; accepted 13 October 2005
Available online 15 February 2006

Abstract

A new consistency measure, the harmonic consistency index, is obtained for any positive reciprocal matrix in the analytic hierarchy process. We show how this index varies with changes in any matrix element. A tight upper bound is provided for this new consistency measure when the entries of matrix are at most 9, as is often recommended. Using simulation, the harmonic consistency index is shown to give numerical values similar to the standard consistency index but it is easier to compute and interpret. In addition, new properties of the column sums of reciprocal matrices are obtained.

Keywords: Multiple criteria analysis; AHP; Consistency index; Reciprocal matrices

1. Introduction

The analytic hierarchy process (AHP) is a popular decision tool used to rank items based on subjective pairwise comparisons (Saaty, 1977, 1980, 1986). A decision maker is asked to state pairwise preferences among n items. Comparison of items i and j results in a positive number a_{ij} giving the strength of preference of item i over item j; the larger the number the greater the preference in favor of item i while a value of 1 indicates indifference. The strength of preference need only be obtained for the $n(n-1)/2$ pairs (i,j) with $i<j$. From those a_{ij} measures we can obtain the strength of preference a_{ji} of item j over i by setting $a_{ji}=1/a_{ij}$. The values can be displayed in a $n\times n$ matrix $A=(a_{ij})$ that has all positive values with 1's on the main diagonal and satisfies the property $a_{ji}=1/a_{ij}$. This type of matrix is called a reciprocal matrix (also called a symmetrically reciprocal

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doi:10.1016/j.ejor.2005.10.057

matrix by Fichtner, 1983). If $a_{ij} = a_{ik}a_{kj}$ for all i, j and $k = 1, \ldots, n$ then A is said to be consistent.

The goal of AHP analysis is to use the empirically obtained matrix A of pairwise comparisons to obtain implied relative preferences $w = (w_1, \ldots, w_n)^T$, called the priority vector. Suppose that each item i has a priority value w_i and that the decision maker uses these values to form the A matrix from the ratio $a_{ij} = w_i/w_j$. If so, the A matrix is consistent and any column will recover the w values, which are unique up to a positive scalar multiple. In that case column j will be of the form $(w_1/w_j, \ldots, w_n/w_j)^T$ which is proportional to w. In general, the responses of the decision maker will not be consistent. Given a reciprocal matrix A, not necessarily consistent, the central problem in AHP is to determine the vector w, that in some sense fits the responses in the A matrix. Saaty (1977) proposed the initial solution method based on solving an eigenvalue problem. One way is to choose w so that the matrix $W = (w_i/w_j)$ is close to the observed A using an appropriate metric. Two other popular methods adopt a specific error measures: the least-squares approach chooses w to minimize the sum of squared differences; the row geometric mean method minimizes the sum of squares of differences of the logarithms of the values. Gass and Rapcsák (2004) contains an excellent overview of the more common methods. Many other solution methods have been proposed in the AHP literature and their relative performance evaluated by simulation (e.g., Choo and Wedley, 2004).

We are not concerned here with the entire range of solution methods. Instead, we take the viewpoint that Saaty's method is appropriate and widely used. In addition, a simple column normalization procedure ("additive normalization") has received attention as a convenient approximation, which avoids computing eigenvectors. We concern ourselves with obtaining an appropriate consistency index to be used with the additive normalization procedure. This numerical value measures how close the decision maker is to making consistent decisions throughout the entire matrix. Saaty (1977) has produced the standard consistency index (CI) based on his eigenvector solution to the problem. Our new method can be viewed as an approximation to the CI but simpler to compute and to understand.

In Section 2, the eigenvector solution is discussed, Section 3 contains results on the properties of the column sums, Section 4 introduces the harmonic consistency index, Section 5 investigates how the harmonic consistency index changes with A, Section 6 obtains a bound on the harmonic consistency index in terms of a bound on A. Section 7 summarizes the results.

2. The standard solution

Perron's Theorem (e.g., Saaty, 1987) shows that any positive matrix A has a largest eigenvalue $\lambda_{\max}$ that is real and positive. The corresponding eigenvalue problem $Ax = \lambda_{\max}x$ has a solution x with $x_i > 0$ for all i which is unique to within multiplication by a scalar. Uniqueness will be obtained by requiring $\sum x_i = 1$. The eigenvalue $\lambda_{\max}$ always satisfies $\lambda_{\max} \geqslant n$ with $\lambda_{\max} = n$ if and only if consistency holds. The eigenvector solution was the original solution proposed to the AHP problem (Saaty, 1977). For any reciprocal matrix, not necessarily consistent, this vector x can be thought of as an estimate of a "true" priority vector. In this case, a matrix with (i,j) element given by x_i/x_j is an approximation to A. Saaty and Hu (1998) and Saaty (2003) argue that this is the only sensible solution. Only in the consistent case does a priority vector exist that recovers the A matrix exactly.

Since we will refer to column normalization frequently in this paper, let s_k be the sum of column k of the reciprocal matrix A. Divide each column k of A by s_k with the resulting matrix denoted by $\tilde{A}$. The matrix $\tilde{A}$ can be used to verify consistency of A as the following theorem shows. Saaty (1977) proved this result using eigenvectors, but we provide a direct proof in Appendix A.

Theorem 1. *An $n \times n$ reciprocal matrix A is consistent if and only if all columns of $\tilde{A}$ are identical.*

The reciprocals of the column sums $(s_1^{-1}, \ldots, s_n^{-1})$ appear on the diagonal of $\tilde{A}$ since the diagonal of A contains 1's. Furthermore, if A is consistent, then each column of $\tilde{A}$ will equal $(s_1^{-1}, \ldots, s_n^{-1})^T$.

Example 1

$$A_1 = \begin{bmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.3 & 0.3 & 0.3 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$$

and $s_1^{-1} + s_2^{-1} + s_3^{-1} = 1$.

The matrix A_1 is consistent since $a_{13} = a_{12}a_{23}$ and all columns of $\tilde{A}$ are identical.

In elementary expositions of the AHP, it is common to avoid mention of the eigenvalue problem and instead use a simpler method: the arithmetic mean within each row of $\tilde{A}$ (Anderson et al., 2004; Evans and Olson, 2003; Taylor, 2004; Winston and Albright, 2001). The origin of this approximation can be traced back to Saaty (1977, p. 239). This method was called additive normalization (AN) by Srdjevic (2005). The resulting vector will be fairly close to the eigenvector solution if the matrix A is close to consistent. In simulation experiments, the AN method has performed as well as any of the many methods that have been invented including Saaty's eigenvector method (Zahedi, 1986; Choo and Wedley, 2004; Srdjevic, 2005).

We note that the AN method, like the eigenvector method are deterministic methods; they take the data in the A matrix as given constants rather than observations on random variables. The lack of consistency in a decision maker's responses is assumed due to psychological limitations rather than statistical error.

Example 2. We now change a_{13}, and its reciprocal a_{31}, so that the matrix A_2 is not consistent:

$$A_2 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 3 \\ 1/4 & 1/3 & 1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0.571 & 0.600 & 0.500 \\ 0.286 & 0.300 & 0.375 \\ 0.143 & 0.100 & 0.125 \end{bmatrix}$$

and $s_1^{-1} + s_2^{-1} + s_3^{-1} = \dfrac{279}{280}$.

The AN solution is $(0.5572, 0.3202, 0.1226)^{\mathrm{T}}$. The eigenvector solution is $x = (0.5584, 0.3196, 0.1220)^{\mathrm{T}}$, $\lambda_{\max} = 3.0183$.

In Example 2 the AN solution and the eigenvector solution are virtually identical. This is due to the fact that the matrix A_2 is close to consistent as can be seen since $\lambda_{\max}$ is very close to 3. We will show in Section 4 that this similarity is not surprising as the AN method can be viewed as an approximation to the eigenvector method.

3. Properties of the column sums

In order to investigate the column sums of A, we will need three versions of Jensen's inequality (e.g., DeGroot, 1970) given in Proposition 1. The proofs are found in Appendix A.

Proposition 1

(a) *Assume,* $\sum p_i \leqslant 1$, *and* φ *is a concave function on the non-negative real line with* $\varphi(0) = 0$. *Then* $\sum p_i \varphi(x_i) \leqslant \varphi[\sum p_i x_i]$ *holds for all* $x_i > 0$.

(b) *If* $\sum p_i < 1$ *and* φ *is strictly increasing and concave with* $\varphi(0) = 0$ *then there does not exist* $x_1, \ldots, x_n$ *satisfying* $\sum p_i \varphi(x_i) = \varphi[\sum p_i x_i]$.

(c) *If* $\sum p_i = 1$, *all* $p_i > 0$ *and* φ *is strictly increasing and concave with* $x_1, \ldots, x_n$ *satisfying* $\sum p_i \varphi(x_i) = \varphi[\sum p_i x_i]$. *Then all* x_i *are equal.*

In the consistent case, $(s_1^{-1}, \ldots, s_n^{-1})$ is the solution given by any reasonable solution method. The next theorem shows that the sum over $(s_1^{-1}, \ldots, s_n^{-1})$ is at most 1. (All summations in this article without explicit indexing indicated are assumed to range from 1 to n.)

Theorem 2. *Let* A_n *be a* $n \times n$ *reciprocal matrix with* s_j *the sum of column j. Then* $\sum s_j^{-1} \leqslant 1$.

Proof. The proof is by induction on n. If $n = 1$ then $s_1 = 1$ and the result holds. Now assume the result holds for n and show it holds for $n + 1$. Let A_{n+1} be any $(n + 1) \times (n + 1)$ reciprocal matrix and partition it by separating out the last row and column:

$$A_{n+1} = \left(\begin{array}{ccc|c} & A_n & & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \\ \hline 1/b_1 & \cdots & 1/b_n & 1 \end{array} \right)$$

The $\{b_i\}$ are arbitrary positive elements in column $n+1$ which also determine row $n+1$. Let $s_1,\ldots,s_n$ be the column sums of the submatrix A_n. Let R be the sum of the reciprocals of the column sums of A_{n+1}. Then we need to show that for all $b_i > 0$ we have $R \leqslant 1$, where

$$R = \sum [s_j + b_j^{-1}]^{-1} + [1+b]^{-1} \tag{1}$$

and $b = \sum b_i$. Let $p_j = s_j^{-1}$ so that $R = \sum p_j \left[\frac{b_j/p_j}{1+b_j/p_j}\right] + \frac{1}{1+b}$. Change the ariables from b_j to $x_j = b_j/p_j$. R now becomes $R = \sum p_j \frac{x_j}{1+x_j} + \frac{1}{1+\sum p_i x_i}$. $R \leqslant 1$ is equivalent to

$$\sum p_j \frac{x_j}{1+x_j} \leqslant \frac{\sum p_i x_i}{1+\sum p_i x_i}. \tag{2}$$

Consider the concave function $\varphi(x) = x/(1+x)$. This satisfies Proposition 1(a) and with $\sum p_j \leqslant 1$ from the induction hypothesis we conclude (2) holds for all $x_i > 0$. Therefore the proof by induction is complete. □

We now connect the value of $\sum s_j^{-1}$ with consistency. For motivation, we note that in Examples 1 and 2 we have $(s_1^{-1},\ldots,s_n^{-1})$ on the main diagonal of $\tilde{A}_1$ and $\tilde{A}_2$. In the consistent case of Example 1, the $\sum s_j^{-1} = 1$ and all columns of $\tilde{A}_1$ were $(s_1^{-1},\ldots,s_n^{-1})$. In Example 2, $\sum s_j^{-1} < 1$ and we had an inconsistent matrix.

Theorem 3. *The $n \times n$ reciprocal matrix A_n is consistent if and only if $\sum s_j^{-1} = 1$.*

Proof. ($\Rightarrow$) If A_n is consistent, then from the proof of Theorem 1 there exist positive weights $w_1,\ldots,w_n$ such that column j is given by $a_{ij} = w_i/w_j$ for $i = 1,\ldots,n$. Therefore, $s_j = \sum(w_i/w_j) = \frac{1}{w_j}\sum w_i$ and thus $\sum s_j^{-1} = 1$.

($\Leftarrow$) The proof is by induction on n. If $n = 1$ then, $s_1 = 1$ and the result holds. Now assume the result holds for n and show it holds for $n+1$. Let A_{n+1} be any $(n+1)\times(n+1)$ reciprocal matrix and partition it by separating out the last row and column as in the proof of Theorem 2. Let R be the sum of the reciprocals of the column sums of A_{n+1}. We assume $R = 1$ and want to show A_{n+1} is consistent. Proceeding as in the proof of Theorem 2, we have

$$\sum p_j \frac{x_j}{1+x_j} = \frac{\sum p_i x_i}{1+\sum p_i x_i}, \tag{3}$$

with $p_j = s_j^{-1}$. First suppose $\sum p_j < 1$. From Proposition 1(b) with $\varphi(x) = x/(1+x)$ we see that (3) cannot hold. From Theorem 2, the only other possibility is that $\sum p_j = 1$ and by the induction hypothesis we conclude the submatrix A_n is consistent. By Theorem 1, all columns of $\tilde{A}_n$ are the same and each column is $(s_1^{-1},\ldots,s_n^{-1})^{\mathrm{T}}$ from which we conclude column 1 of A_n can be written as $(1, s_1p_2,\ldots,s_1p_n)^{\mathrm{T}}$.

We now proceed to find all solutions $\{x_i\}$ of (3). Since $\varphi(x) = x/(1+x)$ is strictly increasing and concave we apply Proposition 1(c) to conclude that the only solution of (3) is a constant solution $x_i = \kappa$ for all i. Therefore, with $b_j = x_j/s_j = \kappa p_j$, column $n+1$ of A_{n+1} is given by $(\kappa p_1,\ldots,\kappa p_n, 1)^{\mathrm{T}}$ and, since $p_1 + \cdots + p_n = 1$, when normalized the column is $(\kappa p_1,\ldots,\kappa p_n, 1)^T/(\kappa+1)$. The reciprocal of b_1 appears in element $(n+1,1)$ of A_{n+1}. Using the result from the previous paragraph, the first column of A_{n+1} becomes $(1, s_1p_2,\ldots,s_1p_n, s_1/\kappa)^{\mathrm{T}}$ and after normalization this is $(\kappa p_1,\ldots,\kappa p_n, 1)^{\mathrm{T}}/(\kappa+1)$. And the same is true for any of the first n columns of A_{n+1}. This is the same vector we found for the normalized column $n+1$. Therefore, all columns of $\tilde{A}_{n+1}$ are identical and by Theorem 1 we conclude A_{n+1} is consistent. □

4. The harmonic consistency index

Saaty (1980) defined a consistency index for a reciprocal matrix as $\mathrm{CI} = (\lambda_{\max} - n)/(n-1)$, $\mathrm{CI} \geqslant 0$ with $\mathrm{CI} = 0$ if and only if the matrix is consistent. Other consistency indices have been created: Monsuur (1996) used a transformation of $\lambda_{\max}$; Peláez and Lamata (2003) examined all triples of elements and used the determinant to measure consistency and Koczkodaj (1993) also required the analysis of all triples of elements. Another type of consistency measure is the distance from a specific consistent matrix. Crawford and Williams (1985) used the sum of squared deviations of the log of the elements of a matrix from the log of the matrix elements generated by the row geometric mean solution. See Aguarón and

Moreno-Jiménez (2003) for a discussion of additional consistency indices.

We now define an index based on Theorem 2 using the harmonic mean. Denote the harmonic mean of $s = (s_1, \ldots, s_n)$ by HM(s). The condition $\sum s_j^{-1} \leqslant 1$ of Theorem 2 is equivalent to $\mathrm{HM}(s) \geqslant n$. Like the CI, it is desirable to scale this quantity as a function of n. We could use $[\mathrm{HM}(s) - n]/(n-1)$ by analogy with CI but we choose a slight adjustment of this quantity, based on simulation results, to make it more comparable to the CI values (see Table 1). Therefore, we define the *harmonic consistency index*, by

$$\mathrm{HCI} = \frac{[\mathrm{HM}(s) - n](n+1)}{n(n-1)}. \tag{4}$$

Returning to Example 2, we use $\lambda_{\max} = 3.0183$ to compute CI = 0.0091, and the HM = 280/93 = 3.0107 from which it follows that HCI = 0.0072.

Fig. 1 shows a plot of CI versus HCI from a simulation of 500 4×4 matrices. The simulation procedure to construct a random matrix was as follows. We only need to specify the elements of the upper triangular portion of the A matrix and the other half will be filled in with reciprocals. Saaty (1977) has suggested that a maximum value of 9 be used in the A matrix. Using integer values 1 to 9 and their reciprocals, the upper triangular elements of A were randomly filled with elements from $\{1, 2, \ldots, 9, 1/2, 1/3, \ldots, 1/9\}$ (Tummala and Ling, 1998). The results showed that on average

Table 1
Random consistency index (RI)

	n										
	3	4	5	6	7	8	9	10	15	20	25
RI	0.525	0.882	1.115	1.252	1.341	1.404	1.452	1.484	1.583	1.630	1.654
HRI	0.550	0.859	1.061	1.205	1.310	1.381	1.437	1.484	1.599	1.650	1.675

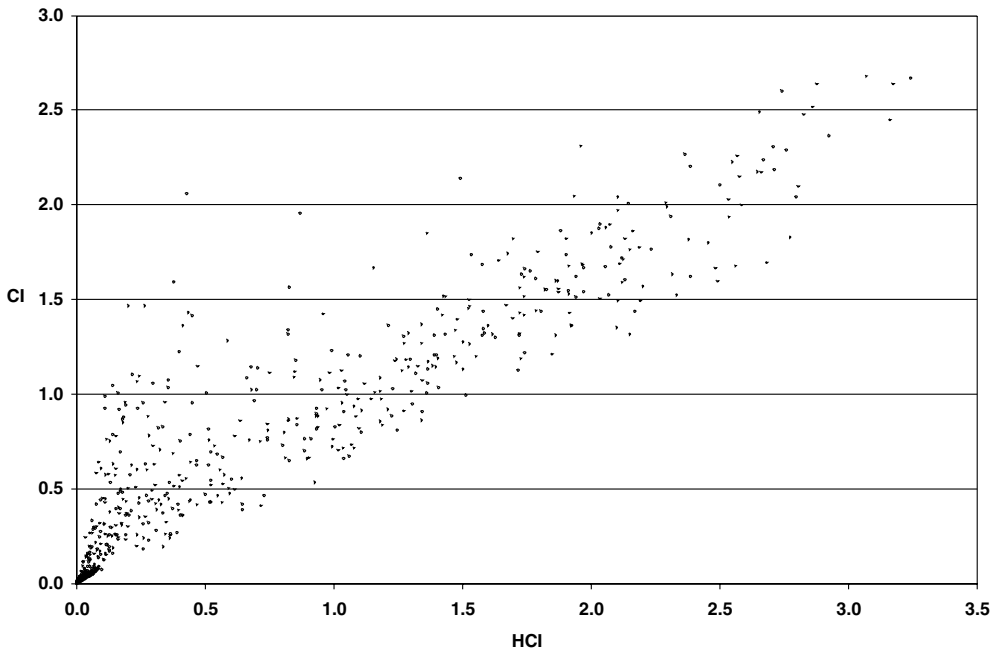

Fig. 1. The harmonic consistency index versus the consistency index for 500 simulated 4×4 A matrices.

CI and HCI were close and the correlation coefficient for $n = 4$ was 0.90.

Since CI tends to increase with matrix size, Saaty (1980, p. 21) suggested a method to try to remove the dependence on n. The same type of random matrices is used as above. The average of the CI values for the these random matrices for each n, called the random consistency index (RI), are given in Table 1 up to $n = 15$ (Aguarón and Moreno-Jiménez, 2003). Since values for size 20 and 25 are not available in the literature, we conducted a simulation to obtain the RI (accurate to within 0.004). The RI is used as a baseline measure. If we divide the CI for a problem by the appropriate RI from Table 1 the result is called the consistency ratio (CR). The rule-of-thumb generally used is that if a matrix has a CR of at most 0.10 (0.05 for $n = 3$ and 0.08 for $n = 4$) then it is sufficiently close to being consistent that the eigenvector priority vector is meaningful (Saaty, 1994).

We repeated this simulation process for the HCI. Generating random matrices and averaging the HCI gave what we call the harmonic random consistency index (HRI). Replications of up to 100,000 were made for each n to obtain accuracy 0.004. We see that the formula for HCI as defined in (4) is sufficiently accurate to track the CI as n changes. We should divide the HCI in a given problem by the value of HRI in Table 1 to remove the effect of matrix size. This results in the harmonic consistency ratio (HCR). From the results displayed in Fig. 1 and Table 1, we conclude that the HCI tracks the CI quite closely and therefore, pending detailed simulations, the HCR should be used with the same 0.10 rule-of-thumb as the CR.

HCI is recommended as a consistency measure if the AN solution method is used. The reason is that there is a natural connection between the HCI and the AN. To see this, first compute the matrix power A^k and then normalize any of its columns to sum to 1. As k increases, this column vector converges to the dominant eigenvector of A (Saaty, 1986, Theorem 14). This result can be stated as

Limiting result

Start with any n-vector $x_0 > 0$ and define $A^k x^0$, $k \geqslant 1$. Define the vector norm $\|y\| \equiv \sum |y_i|$. Then $A^k x_0 / \|A^k x_0\|$ converges to the dominant eigenvector and $\|A^k x_0\| / \|A^{k-1} x_0\|$ converges to $\lambda_{\max}$.

This is equivalent to the power method of linear algebra (Saaty, 1977; Reiter, 1990) which is the standard way to iteratively compute the dominant eigenvalue of a matrix and its corresponding eigenvector. The only difference is that after each iteration we normalize the x_k vector to sum to 1.

Power method

Start with any initial vector $x_0 > 0$, $\|x_0\| = 1$. Define a sequence of scalars $\lambda_1, \lambda_2, \ldots$ and vectors $x_1, x_2, \ldots$ with unit norm that satisfy the approximate eigenvalue problem $Ax_{k-1} = \lambda_k x_k$. This is accomplished by defining $x_k = Ax_{k-1} / \|Ax_{k-1}\|$ and $\lambda_k = \|Ax_{k-1}\|$. This provides an iterative method to solve the eigenvalue problem $Ax = \lambda_{\max} x$ since $x_k \to x$ and $\lambda_k \to \lambda_{\max}$ as k increases.

To connect this with the AN method, define x_0 as a column n-vector with element i given by $(ns_i)^{-1}$. By defining $x_1 = Ax_0$ we obtain a vector x_1 which gives the averages in each row of the column normalized A matrix. Thus, x_1 is the AN solution vector. It will always sum to 1 since $(1, \ldots, 1)$ $A[(ns_1)^{-1} \cdots (ns_n)^{-1}]^{\mathrm{T}} = (s_1, \ldots, s_n)[(ns_1)^{-1} \ldots (ns_n)^{-1}]^{\mathrm{T}} = 1$. Since $\|x_0\| \neq 1$ we see that the AN solution is the first iteration using the limiting result above. Therefore AN is the initial step in an approximation of Saaty's eigenvector solution to the problem. Furthermore, for $k = 1$, $\|A^k x_0\| / \|A^{k-1} x_0\| = \|x_1\| / \|x_0\| = 1 / \|x_0\| = \left[\sum s_i^{-1} / n\right]^{-1} = \mathrm{HM}(s)$. Thus the initial approximation to $\lambda_{\max}$ is the harmonic mean of the column sums of the A matrix. Using the power method viewpoint we obtain the same conclusion: Now we start with a normalized with element i given by $\mathrm{HM}(s)/(ns_i)$. Then $x_1 = Ax_0 / \|Ax_0\|$ is still the AN solution since multiplying the previous x_0 by a scalar will cancel out in this ratio. Also, $\lambda_1 = \|Ax_0\| = \mathrm{HM}(s)(1, \ldots, 1)A[(ns_1)^{-1} \cdots (ns_n)^{-1}]^{\mathrm{T}} = \mathrm{HM}(s)$.

The AN method is a simple and quick approximation to Saaty's eigenvector method. The harmonic mean of the column sums is the corresponding approximation to $\lambda_{\max}$. Returning

to Example 2, we see $\mathrm{HM}(s) \approx \lambda_{\max}$ and the AN solution is very close to the eigenvector solution.

If we want a measure of consistency based on the AN, then the HCI is the natural measure. In addition, AN and HCI are easier to compute than the eigenvector and CI plus they have the added benefit of being easier to understand.

5. How the HCI changes as the A matrix changes

Aupetit and Genest (1993) investigate how the CI varies as an element of the A matrix changes. Of course, the corresponding reciprocal element is also forced to change. They showed that as any one element increases the CI will: (i) always increase, (ii) always decrease or (iii) decrease to a minimum and then increase. The latter case is the usual one as we would expect to locate a choice for the element of A, which minimizes the inconsistency relative to the rest of the matrix. If there were multiple local minima in the CI function it would cast doubt on using it as a meaningful measure of consistency. We now prove the analogous result for the HCI.

Theorem 4. *As a function of any one element of the $n \times n$ reciprocal A matrix, $n \geqslant 3$, the HCI has exactly one minimum and no maximum.*

Proof. Given any reciprocal matrix A with $s_1, \ldots, s_n$ as the column sums, we multiply the element in position (j,k) by any $\varepsilon > 0$. Let the initial value of a_{jk} be denoted a. The sum of the elements in column k becomes $a_{1k} + \cdots + a\varepsilon + \cdots + a_{nk}$ and this can be written as $s_k + a(\varepsilon - 1)$ where $s = (s_1, \ldots, s_n)$ are the original column sums. The value in position (k,j) now changes from $1/a$ to $1/a\varepsilon$. Let $s(\varepsilon) = (s_1(\varepsilon), \ldots, s_n(\varepsilon))$ denote the vector of column sums after the 2 cells are adjusted. The sum of column j becomes $s_j(\varepsilon) = s_j + 1/a\varepsilon - 1/a$. The partial derivative of $\mathrm{HM}(s(\varepsilon))$ with respect to ε is $\frac{\mathrm{HM}(s(\varepsilon))^2}{n}\left[\frac{\partial s_j(\varepsilon)}{\partial \varepsilon}\frac{1}{s_j^2(\varepsilon)} + \frac{\partial s_k(\varepsilon)}{\partial \varepsilon}\frac{1}{s_k^2(\varepsilon)}\right]$. Since we are only interested in the sign of this derivative we only need to examine the quantity in brackets. That simplifies to

$$\frac{a}{[s_k + a(\varepsilon - 1)]^2} - \frac{a}{[a\varepsilon s_j + 1 - \varepsilon]^2}.$$

The quantities inside the squared terms are always positive. Therefore, this expression is positive if and only if $\varepsilon > [s_k - (a+1)]/[as_j - (a+1)]$. Note that the numerator is always positive since $n \geqslant 3$ implies there is at least one other (positive) element in column k of A in addition to a and 1. Likewise the denominator is strictly positive since $s_j > 1 + 1/a$. Therefore there is exactly one stationary point as ε ranges over the positive reals:

$$\varepsilon^* = [s_k - (a+1)]/[as_j - (a+1)]. \tag{5}$$

This corresponds to a minimum in HM since the derivative is positive for all $\varepsilon > \varepsilon^*$ and negative for all $\varepsilon < \varepsilon^*$. □

The proof of the previous theorem shows that for any inconsistent reciprocal matrix A, there is a unique adjustment for each given element to make the consistency the best it can be (minimize the HCI) by using ε^*. Note that if $n = 2$ then ε^* might be 0 or undefined (infinite). That is, HM might increase or decrease for all ε just as $\lambda_{\max}$ does.

6. The maximum HCI if $1/S \leqslant a_{ij} \leqslant S$

Saaty (1980) and others have suggested that the response of decision makers be limited to a maximum value of 9, for psychological reasons. Aupetit and Genest (1993) relied upon classical matrix theory results to find the maximum value of $\lambda_{\max}$ and hence CI under this assumption. We will obtain the same upper bound for the HM as was found for $\lambda_{\max}$. Assume S is the maximum value for the elements of A. Consider this form of a reciprocal matrix:

$$A = \begin{bmatrix} 1 & S & 1/S & S & \cdots \\ 1/S & 1 & S & 1/S & \\ S & 1/S & 1 & S & \\ 1/S & S & 1/S & 1 & \\ \vdots & & & & \ddots \end{bmatrix}. \tag{6}$$

We will prove that HCI assumes its maximum value for this matrix. Aupetit and Genest (1993) showed λ_{max} and CI also achieved their maximum values for this same matrix. From a graph theoretic viewpoint, Genest et al. (1993) showed this matrix is 'maximally intransitive' in the sense that it has the maximum number of circular triads (cycles of length 3 where $a_{ij} > 1$, $a_{jk} > 1$, $a_{ki} > 1$). Vargas (1980) relates the definition of CI to cycles of various lengths.

Theorem 5. *Let A be any reciprocal matrix with bounded elements $1/S \leqslant a_{ij} \leqslant S$. The harmonic mean of the column sums satisfies. $n \leqslant \mathrm{HM}(s) \leqslant 1 + 0.5(n-1)(S + 1/S)$. Therefore $\mathrm{HCI} \leqslant (S + 1/S - 2)(n+1)/(2n)$.*

Proof. First we demonstrate that the maximum value of the HM will be achieved with an A matrix having only S and $1/S$ elements off the diagonal. This follows from Theorem 3 which shows that HM must assume its maximum on the boundary of the interval $[1/S, S]$. Thus we can always replace any element a, $1/S < a < S$, by either S or $1/S$ and thereby increase the HM.

Consider any reciprocal matrix consisting of only S and $1/S$ off diagonal elements. Since each S is paired with an $1/S$ element, we will have $n(n-1)/2$ S elements and the same number of $1/S$ elements in the entire matrix. Therefore, the sum of all the elements is given by $n + 0.5n(n-1)(S + 1/S)$ so the column average is $\sum s_i/n = 1 + 0.5(n-1)(S + 1/S) \equiv B$. We now use the well-known harmonic mean-arithmetic mean inequality to obtain $\mathrm{HM}(s) \leqslant B$. This provides an upper bound for all n.

Now assume n is odd and consider the matrix in (6). By direct computation all column sums are equal to B and thus the HM will be equal to B. So this matrix for any odd n maximizes HM. By Proposition 1(c), the only kind of matrix for which the HM will reach this maximal value will have column sums that are identical (and equal to B). This implies an equal number of S and $1/S$ elements in each column (and row). This suggests matrix (6) is not unique in maximizing HM as can be easily seen by example. In addition, if n is even it will not be possible to find a matrix with $\mathrm{HM} = B$. The closest we can get to equal column sums will be $n/2$ of the S elements and $n/2 - 1$ of the $1/S$ or vice-versa. (Half the columns will be of each type.) We can easily compute the harmonic mean for that type of matrix. For n even, this will provide a tight upper bound for HM that is slightly less than the previous bound of B:

$$B - (S - 1/S)^2/(4B). \qquad \square \tag{7}$$

It is reasonable that any measure of inconsistency should reach its maximum value at the maximally intransitive matrix (6). CI and HCI have passed this test. In fact, for odd n, λ_{max} and HM assume the same value, B, for this matrix.

7. Summary

New results have been obtained concerning properties of reciprocal matrices. In particular, since $\sum s_j^{-1} \leqslant 1$, we were able to define a new consistency measure for a reciprocal matrix, the harmonic consistency index (HCI) and the companion harmonic consistency ratio (HCR). Simulation results indicate HCI has values similar to the standard CI measure as illustrated in Fig. 1 and Table 1. Fig. 1 shows that while HCI and CI are similar on average, HCI is far from a simple transformation of CI. Since the CI and CR are so commonly used, we scaled HCI and HCR so that, on average, HCI $\approx$ CI and HCR $\approx$ CR. This should simplify the use and interpretation of HCI and HCR values.

Theorems 4 and 5 demonstrated how HCI varies as matrix elements change. Somewhat surprisingly, HM and λ_{max} satisfied the same bounds.

The authors are not advocating use of the AN method; other procedures may be preferred. The authors view three approaches as superior in both a practical and theoretical sense: the standard eigenvector approach, the row geometric mean (Crawford and Williams, 1985) and the singular value decomposition (Gass and Rapcsák, 2004). The AN method has gained favor as a simple but effective solution method since it only requires column normalization and row averages. The simple nature of AN has led to its increased use,

especially as a pedagogical tool. However, the CI method is then used along with the AN method to measure consistency. Therefore computation of the dominant eigenvalue is still required which defeats the purpose of using a simple approach. When we showed that the AN is an approximation of the eigenvalue method (via the power method), the harmonic mean arose as an approximation to $\lambda_{\max}$. Thus the HCI provides a natural and appropriate consistency measure when using AN that is simple to obtain and interpret.

Additional theoretical and applied analysis concerning the properties of the harmonic consistency index is in progress.

Appendix A

This appendix contains the proofs of Theorem 1 and Proposition 1.

Proof of Theorem 1. ($\Rightarrow$) If we have consistency, then any column of A can be obtained from column 1: $a_{ij} = a_{i1}a_{1j}$ for all i and j. Summing both sides over i gives $s_j = s_1 a_{1j}$. By definition, $\tilde{a}_{ij} = a_{ij}/s_j = a_{i1}a_{1j}/s_j = a_{i1}/s_1 = \tilde{a}_{i1}$. So all columns of $\tilde{A}$ are the same.

($\Leftarrow$) Let $p_k = s_k^{-1}$. Let w be any of the identical columns of $\tilde{A}$. Then $w_i = a_{ij}/s_j = a_{ij}p_j$ which implies $w_i = p_i$ since $a_{ii} = 1$. Thus $a_{ij} = p_i/p_j$ and $a_{ik}a_{kj} = (p_i/p_k)(p_k/p_j) = p_i/p_j = a_{ij}$. □

Proof of Proposition 1. (a) Let $x^* = \sum p_i x_i$. By properties of a concave function the tangent line to the curve at the point $x = x^*$ will satisfy $\varphi(x) \leqslant \alpha + \beta x$ for all x. The curve and the line meet at $x^* : \varphi(x^*) = \alpha + \beta x^*$. Note that $\varphi(0) = 0$ implies $\alpha \geqslant 0$. Therefore $\varphi(x_i) \leqslant \alpha + \beta x_i$ implies $\sum p_i\varphi(x_i) \leqslant \alpha\sum p_i + \beta\sum p_i x_i \leqslant \alpha + \beta x^*$ since $\alpha \geqslant 0$ and $\sum p_j \leqslant 1$. Therefore we conclude $\sum p_i\varphi(x_i) \leqslant \varphi(x^*)$.

(b) $x^* = \sum p_i x_i$. By properties of a strictly increasing concave function the tangent line to the curve at the point $x = x^*$ will satisfy $\varphi(x) < \alpha + \beta x$ for all $x \neq x^*$. The curve and the line meet at x^* so that $\varphi(x^*) = \alpha + \beta x^*$. Note that $\varphi(0) = 0$ implies $\alpha > 0$. Thus $\sum p_i\varphi(x_i) \leqslant \alpha\sum p_i + \beta\sum p_i x_i < \alpha + \beta x^*$ since $\alpha > 0$ and $\sum p_i < 1$. Therefore we conclude $\sum p_i\varphi(x_i) < \varphi(x^*)$.

(c) Partition $\{1,\ldots,n\}$ into the set $E = \{i : x_i = x^*\}$ and its complement. As in the proof of (b), $\varphi(x_i) < \alpha + \beta x_i$ for $i \notin E$ and $\varphi(x_i) = \alpha + \beta x_i$ for $i \in E$. Therefore, we can write $\sum p_i\varphi(x_i) = \sum_{i\in E} p_i\varphi(x_i) + \sum_{i\notin E} p_i\varphi(x_i)$ where $\sum_{i\notin E} p_i\varphi(x_i) < \alpha\sum_{i\notin E} p_i + \beta\sum_{i\notin E} p_i x_i$ while $\sum_{i\in E} p_i\varphi(x_i) = \alpha\sum_{i\in E} p_i + \beta\sum_{i\in E} p_i x_i$. The sum of the right hand sides of the previous two relations reduces to $\varphi(x^*)$. Therefore if $E = \{1,\ldots,n\}$ then $\sum p_i\varphi(x_i) = \varphi(x^*)$ while if even one $i \notin E$, then $\sum p_i\varphi(x_i) < \varphi(x^*)$. So we conclude $\sum p_i\varphi(x_i) = \varphi(x^*)$ implies all the x_i must be the same value. (Note that we know all the p_i are positive. If only $p_i \geqslant 0$ we would conclude all the $x_i = x^*$ except for x_i with $p_i = 0$.) □

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