Characterizations of Consistent Pairwise Comparison Matrices over Abelian Linearly Ordered Groups

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We consider the framework of pairwise comparison matrices over abelian linearly ordered groups. We introduce the notion of $\odot$-proportionality that allows us to provide new characterizations of the consistency, efficient algorithms for checking the consistency and for building a consistent matrix. Moreover, we provide a new consistency index. © 2010 Wiley Periodicals, Inc.

1. INTRODUCTION

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives or criteria. An useful tool to determine a weighted ranking on X is a *pairwise comparison matrix* (PCM)

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \tag{1}$$

where entry a_{ij} expresses how much the alternative x_i is preferred to alternative x_j. A condition of *reciprocity* is assumed for the matrix $A = (a_{ij})$ in such a way that the preference of x_i over x_j expressed by a_{ij} can be exactly read by using the element a_{ji}.

Under a suitable condition of *consistency* for $A = (a_{ij})$, X is totally ordered and the values a_{ij} can be expressed by means of the components w_i and w_j of a suitable vector $\underline{w}$, that is called *consistent vector* for the matrix $A = (a_{ij})$; then $\underline{w}$ provides the weights for the elements of X.

The shape of the reciprocity and consistency conditions depends on the different meaning given to the number a_{ij}, as the following well-known cases show.

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Multiplicative case: $a_{ij} \in]0, +\infty[$ is a preference ratio and the conditions of *multiplicative reciprocity* and *consistency* are given respectively by

$$a_{ji} = \frac{1}{a_{ij}} \quad \forall\, i, j = 1, \ldots, n,$$

$$a_{ik} = a_{ij} a_{jk} \quad \forall\, i, j, k = 1, \ldots, n.$$

A consistent vector is a positive vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $\frac{w_i}{w_j} = a_{ij}$ and so perfectly representing the preferences over X.

Additive case: $a_{ij} \in]-\infty, +\infty[$ is a preference difference and the conditions of *additive reciprocity* and *consistency* are expressed as follows:

$$a_{ji} = -a_{ij} \quad \forall\, i, j = 1, \ldots, n,$$

$$a_{ik} = a_{ij} + a_{jk} \quad \forall\, i, j, k = 1, \ldots, n.$$

A consistent vector is a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $w_i - w_j = a_{ij}$.

Fuzzy case: $a_{ij} \in [0, 1]$ measures the distance from the indifference that is expressed by 0.5; the conditions of *fuzzy reciprocity* and *consistency* are

$$a_{ji} = 1 - a_{ij} \quad \forall\, i, j = 1, \ldots, n,$$

$$a_{ik} = a_{ij} + a_{jk} - 0.5 \quad \forall\, i, j, k = 1, \ldots, n.$$

A consistent vector is a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $w_i - w_j = a_{ij} - 0.5$.

The multiplicative *PCMs* play a basic role in the analytic hierarchy process, a procedure developed by T.L. Saaty at the end of the 1970s.[12–14] In Refs. 2–5, and 9, properties of multiplicative PCMs are analyzed in order to determine a qualitative ranking on the set of the alternatives and find vectors representing this ranking. Additive and fuzzy matrices are investigated for instance by Barzilai[1] and Herrera-Viedma et al.[11]

In the case of a multiplicative PCM, Saaty suggests that the comparisons expressed in verbal terms have to be translated into preference ratios a_{ij} taking value in $S^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$. Let us stress that the assumption of the Saaty scale restricts the decision maker's possibility to be consistent: indeed, if the decision maker (DM) expresses the following preference ratios $a_{ij} = 5$ and $a_{jk} = 3$, then he will not be consistent because $a_{ij} a_{jk} = 15 > 9$. Similarly, for the fuzzy case, the assumption that $a_{ij} \in [0, 1]$, restricts the possibility to realize the fuzzy consistency: indeed, if the DM claims $a_{ij} = 0.9$ and $a_{jk} = 0.8$, then he will not be consistent because $a_{ij} + a_{jk} - 0.5 = 1.7 - 0.5 > 1$.

In order to unify the several approaches to PCMs and remove the above drawbacks, in Ref. 6 we introduce PCMs whose entries belong to an abelian linearly ordered group (*alo-group*) $\mathcal{G} = (G, \odot, \leq)$. In this way, the reciprocity and consistency conditions are expressed in terms of the group operation $\odot$ and the drawbacks related to the consistency condition are removed; in fact the consistency condition is expressed by $a_{ik} = a_{ij} \odot a_{jk}$, where $a_{ij} \odot a_{jk}$ is an element of G, for each choice of $a_{ij}, a_{jk} \in G$. As a nontrivial alo-group $\mathcal{G} = (G, \odot, \leq)$ has neither the greatest element nor the least element (see Ref. 6), the Saaty set S* and the interval [0, 1], embodied with the usual order $\leq$ on R, cannot be structured as alo-groups. Moreover:

- the assumption of *divisibility* for $\mathcal{G}$ allows us to introduce the notion of mean $m_{\odot}(a_1, \ldots, a_n)$ of n elements and associate a *mean vector* $\underline{w}_{m_{\odot}}$ to a PCM;
- the introduction of a notion of distance $d_{\mathcal{G}}$, linked to the operation $\odot$ in a divisible alo-group $\mathcal{G}$, allows us to provide a measure of consistency for a PCM over $\mathcal{G}$: indeed the consistency index $I_{\mathcal{G}}(A)$ is defined as mean of the distances $d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk})$, with $i < j < k$.

In Ref. 7, we analyze some properties of a consistent PCM and provide algorithms, to check whether or not a matrix is consistent and to build a consistent matrix by means of $n - 1$ comparisons. Moreover, we provide a new consistency index linked to the index $I_{\mathcal{G}}(A)$ introduced in Ref. 6 but easier to compute.

In this paper, we extend the previous results and introduce the abelian group $\mathcal{G}^n = (G^n, \odot_\times)$ and the notion of $\odot$-proportionality, that allows us to:

- provide new characterizations of a consistent PCM;
- introduce an equivalence relation on G^n and a bijection between the quotient set $G^n/\sim_{\odot}$ and the set of the consistent PCMs;
- provide more efficient algorithms to check the consistency;
- provide more efficient algorithms to build a consistent PCM.

The paper is organized as follows: Section 2 provides notations, definitions and some results useful in the next sections; starting from an alo-group $\mathcal{G} = (G, \odot, \leq)$, Section 3 introduces the abelian group $\mathcal{G}^n = (G^n, \odot_\times)$ and the notion of $\odot$-proportional vectors; Section 4 provides characterizations of the consistency in terms of $\odot$-proportionality of rows or columns and algorithms to build consistent PCMs; furthermore, this section links the consistent PCMs to the related consistent vectors and provides the way to build consistent PCMs starting from a vector; Section 5 provides further characterizations that allows us to build consistent PCMs and to check the consistency in a more efficient way; Section 6 gives the consistency index for a PCM; Section 7 provides concluding remarks and directions for future work.

2. PRELIMINARIES

Let G be a nonempty set provided with a total weak order $\leq$ and a binary operation $\odot : G \times G \to G$. Then $\mathcal{G} = (G, \odot, \leq)$ is called *alo-group*, if and only if

$(G, \odot)$ is an abelian group and the the following implication holds:

$$a \leq b \Rightarrow a \odot c \leq b \odot c.$$

The above implication is equivalent to

$$a < b \Rightarrow a \odot c < b \odot c,$$

where $<$ is the strict simple order associated to $\leq$.
If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then G is naturally equipped with the order topology induced by $\leq$ and the abelian group $G \times G$ is equipped with the related product topology. $\mathcal{G}$ is called a *continuous* alo-group if and only if the function $\odot$ is continuous.
Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group, then we assume that e denotes the *identity* of $\mathcal{G}$, $x^{(-1)}$ the *symmetric* of $x \in G$ with respect to $\odot$, $\div$ the *inverse operation* of $\odot$ defined by

$$a \div b = a \odot b^{(-1)}.$$

PROPOSITION 1. *It results:*

i) $a = b \odot c$ *if and only if* $c = b^{(-1)} \odot a$;
ii) $a = b \odot c$ *if and only if* $a^{(-1)} = b^{(-1)} \odot c^{(-1)}$;
iii) $(a \div c)^{(-1)} = c \div a$.

Proof. Because of the associativity and the cancellation property of a commutative group operation, we have: $a = b \odot c \Leftrightarrow b^{-1} \odot a = b^{-1} \odot (b \odot c) \Leftrightarrow b^{-1} \odot a = c$. Moreover, $a = b \odot c \Leftrightarrow a^{(-1)} = (b \odot c)^{(-1)} \Leftrightarrow a^{(-1)} = b^{(-1)} \odot c^{(-1)}$. Finally, by applying (ii), we get $(a \div c)^{(-1)} = (a \odot c^{(-1)})^{(-1)} = a^{(-1)} \odot c = c \div a$. ∎

For a positive integer n, the (n)-*power* $x^{(n)}$ of $x \in G$ is defined as follows:

$$x^{(1)} = x$$

$$x^{(n)} = \bigodot_{i=1}^{n-1} x_i \odot x_n = \bigodot_{i=1}^{n} x_i, \qquad x_i = x \quad i = 1, \ldots, n, \; n \geq 2.$$

If $b^{(n)} = a$, then we say that b is the (n)-*root* of a and write $b = a^{(1/n)}$.

$\mathcal{G}$ is *divisible* if and only if for each positive integer n and each $a \in G$ there exists the (n)-root of a.

DEFINITION 1. *Let* $\mathcal{G} = (G, \odot, \leq)$ *be a divisible alo-group. Then, the* $\odot$-mean $m_{\odot}(a_1, a_2, \ldots, a_n)$ *of the elements* $a_1, a_2, \ldots, a_n$ *of* G *is defined by*

$$m_{\odot}(a_1, a_2, \ldots, a_n) = \begin{cases} a_1 & n = 1, \\ \left(\bigodot_{i=1}^{n} a_i\right)^{(1/n)} & n \geq 2. \end{cases}$$

DEFINITION 2. *An* isomorphism *between two alo-groups* $\mathcal{G} = (G, \odot, \leq)$ *and* $\mathcal{G}' = (G', \circ, \leq)$ *is a bijection* $h : G \to G'$ *that is both a lattice isomorphism and a group isomorphism, that is,*

$$x < y \Leftrightarrow h(x) < h(y)$$

$$h(x \odot y) = h(x) \circ h(y).$$

PROPOSITION 2.[6] *Let* $h : G \to G'$ *be an isomorphism between the alo-groups* $\mathcal{G} = (G, \odot, \leq)$ *and* $\mathcal{G}' = (G', \circ, \leq)$. *Then* $\mathcal{G}$ *is divisible if and only if* $\mathcal{G}'$ *is divisible and, under the assumption of divisibility*

$$m_{\circ}(y_1, y_2, \ldots, y_n) = h\big(m_{\odot}(h^{-1}(y_1), h^{-1}(y_2), \ldots, h^{-1}(y_n))\big).$$

2.1. The Notion of Distance

The *norm* of an element $a \in G$ is defined by setting

$$||a|| = a \vee a^{(-1)}. \tag{2}$$

Let us consider the operation

$$d_{\mathcal{G}} : (a, b) \in G^2 \to ||a \div b|| \in G. \tag{3}$$

In Ref. 6, we prove that $d_{\mathcal{G}}$ verifies the conditions:

1. $d_{\mathcal{G}}(a, b) \geq e$;
2. $d_{\mathcal{G}}(a, b) = e \Leftrightarrow a = b$;
3. $d_{\mathcal{G}}(a, b) = d_{\mathcal{G}}(b, a)$;
4. $d_{\mathcal{G}}(a, b) \leq d_{\mathcal{G}}(a, c) \odot d_{\mathcal{G}}(b, c)$,

so, we provide the following definition.

DEFINITION 3. *The operation* $d_{\mathcal{G}}$ *in (3) is a* $\mathcal{G}$-metric *or* $\mathcal{G}$-distance.

PROPOSITION 3.[6] *Let $h : G \to G'$ be an isomorphism between the alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$. Then,*

$$d_{\mathcal{G}'}(a', b') = h(d_{\mathcal{G}}(h^{-1}(a'), h^{-1}(b'))).$$

2.2. Real Alo-groups

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R

Under the above assumption for G and $\leq$, examples of real divisible and continuous alo-groups are the following:

Multiplicative alo-group. $]0, +\infty[$ $= (]0, +\infty[, \cdot, \leq)$, where $\cdot$ is the usual multiplication on R. Then, $e = 1$, $x^{(-1)} = 1/x$, $x^{(n)} = x^n$ and $x \div y = \frac{x}{y}$. So

$$d_{]0,+\infty[}(a, b) = \frac{a}{b} \vee \frac{b}{a}$$

and

$$m_{\cdot}(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$

is the geometric mean.

Additive alo-group. $\mathcal{R} = (R, +, \leq)$, where $+$ is the usual addition on R. Then, $e = 0$, $x^{(-1)} = -x$, $x^{(n)} = nx$, $x \div y = x - y$. So

$$d_{\mathcal{R}}(a, b) = |a - b| = (a - b) \vee (b - a)$$

and

$$m_{+}(a_1, \ldots, a_n) = \frac{\sum_i a_i}{n}$$

is the arithmetic mean.

Fuzzy alo-group. $]0,1[$ $= (]0, 1[, \otimes, \leq)$, where $\otimes :]0, 1[^2 \to]0, 1[$ is the operation defined by

$$x \otimes y = \frac{xy}{xy + (1 - x)(1 - y)}.$$

Then, $e = 0.5$, $x^{(-1)} = 1 - x$, $x \div y = \frac{x(1-y)}{x(1-y)+(1-x)y}$. So

$$d_{]0,1[}(a, b) = \frac{a(1 - b)}{a(1 - b) + (1 - a)b} \vee \frac{b(1 - a)}{b(1 - a) + (1 - b)a}.$$

We will compute the fuzzy mean $m_{\otimes}(a_1, \ldots, a_n)$ in the sequel of this paper.

Remark 1. Our choice of the operation structuring the ordered interval $]0, 1[$ as an alo-group wants to obey the requests: 0,5 is the identity element and $1 - x$ is the symmetric of x. In this way, the condition of reciprocity for a PCM over a fuzzy alo-group is given again by $a_{ji} = 1 - a_{ij}$, as defined in Section 1.

By setting $G =]0, 1[$ and

$$\psi : t \in]0, +\infty[\to \frac{t}{t+1} \in]0, 1[, \tag{4}$$

that is a continuous and strictly increasing function between $]0, +\infty[$ and $]0, 1[$, we get

$$x \otimes y = \psi(\psi^{-1}(x) \cdot \psi^{-1}(y)).$$

Thus $(G, \otimes, \leq)$ is a continuous alo-group as a consequence of the following result of Ref. 6:

THEOREM 1. *Let G be a proper open interval of R and $\leq$ the total order on G inherited from the usual order on R, then the following assertions are equivalent:*

1. *$\mathcal{G} = (G, \odot, \leq)$ is a continuous alo-group;*
2. *there exists a continuous and strictly increasing function $\psi :]0, +\infty[\to G$ verifying the equality*

$$x \odot y = \psi(\psi^{-1}(x) \cdot \psi^{-1}(y)).$$

Moreover, $\otimes$ verifies our requests about the identity and the symmetric of an element of the group.

Remark 2. The operation $\otimes$ is the restriction to $]0, 1[^2$ of the uninorm (see Ref. 10):

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\}; \\ \frac{xy}{xy + (1-x)(1-y)}, & \text{otherwise.} \end{cases}$$

The multiplicative, the additive and the fuzzy alo-groups are isomorphic; in fact the bijection

$$h : x \in]0, +\infty[\to \log x \in R$$

is an isomorphism between **]0,+∞[** and $\mathcal{R}$ and ψ in (4) is an isomorphism between **]0,+∞[** and **]0,1[**. So, by Proposition 2, the mean $m_{\otimes}(a_1, \ldots, a_n)$ related to the fuzzy alo-group can be computed, by means of the function in (4), as follows:

$$m_{\otimes}(a_1, \ldots, a_n) = \psi\left(\left(\prod_{i=1}^{n} \psi^{-1}(a_i)\right)^{\frac{1}{n}}\right).$$

2.3. PCMs over an Alo-group

Let $X = \{x_1, x_2, \ldots x_n\}$ a set of alternatives, $A = (a_{ij})$ in (1) the related PCM. We claim that $A = (a_{ij})$ is a PCM over the alo-group $\mathcal{G}$ if and only $a_{ij} \in G,\ \forall i, j \in \{1, \ldots, n\}$. We will denote by

1. $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ the rows of A;
2. $\underline{a}^1, \underline{a}^2, \ldots, \underline{a}^n$ the columns of A.

If $\mathcal{G} = (G, \odot, \leq)$ is a divisible alo-group, then we consider the *mean vector* $\underline{w}_{m_\odot}(A)$, associated to A, that is defined as follows:

$$\underline{w}_{m_\odot}(A) = (m_\odot(\underline{a}_1), m_\odot(\underline{a}_1), \ldots, m_\odot(\underline{a}_n)). \tag{5}$$

From now on, we assume that $A = (a_{ij})$ is *reciprocal* with respect to $\odot$, that is,

$$a_{ji} = a_{ij}^{(-1)} \quad \forall\, i, j = 1, \ldots, n, \tag{6}$$

so

$$a_{ii} = e \quad \text{and} \quad a_{ij} \odot a_{ji} = e \ \ \forall i, j \in \{1, 2, \ldots, n\}. \tag{7}$$

DEFINITION 4.[6] *$A = (a_{ij})$ is a* consistent *PCM with respect to $\odot$, if and only if*

$$a_{ik} = a_{ij} \odot a_{jk} \quad \forall i, j, k.$$

In Ref. 6, we provide the following characterization.

PROPOSITION 4. *$A = (a_{ij})$ is a consistent PCM with respect to $\odot$, if and only if:*

$$a_{ik} = a_{ij} \odot a_{jk} \quad \forall\, i, j, k : i < j < k. \tag{8}$$

2.4. A Consistency Index for a PCM

Let T be the set $\{(a_{ij}, a_{jk}, a_{ik}), i < j < k\}$ and $n_T = |T|$. From Proposition 2.3., we derive the following.

PROPOSITION 5. *$A = (a_{ij})$ is a* consistent *matrix with respect to $\odot$, if and only if*

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = e \quad \forall\, i, j, k : i < j < k.$$

So, by Proposition 5, $A = (a_{ij})$ is inconsistent if and only if $d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) > e$ for some triple $(a_{ij}, a_{jk}, a_{ik}) \in T$. Thus, in Ref. 6, we provide the following definition of consistency index.

DEFINITION 5. *The* consistency index *of A is given by*

$$I_{\mathcal{G}}(A) = \begin{cases} d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) & n = 3, \\ (\bigodot_T d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}))^{(\frac{1}{n_T})} & n > 3. \end{cases}$$

with

$$n_T = \frac{n(n-2)(n-1)}{6}.$$

PROPOSITION 6. *$I_{\mathcal{G}}(A) \geq e$ and A is consistent if and only if $I_{\mathcal{G}}(A) = e$.*

3. THE ABELIAN GROUP $\mathcal{G}^N = (G^N, \odot_\times)$ AND THE NOTION OF $\odot$-PROPORTIONALITY

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group and

$$G^n = \underbrace{G \times G \times \ldots \times G}_{n}, \tag{9}$$

the cartesian product where elements are the vectors $\underline{w} = (w_1, w_2, \ldots w_n)$, $w_i \in G$, $\forall i = 1, \ldots, n$.
We consider G^n embodied with the binary operation

$$\odot_\times : (\underline{w}, \underline{v}) \to \underline{w} \odot_\times \underline{v} = (w_1 \odot v_1, w_2 \odot v_2, \ldots w_n \odot v_n). \tag{10}$$

The operation $\odot_\times$ is associative and commutative, has identity element equal to $\underline{e} = (e, e, \ldots, e)$, and, for each $\underline{w} = (w_1, w_2, \ldots w_n) \in G^n$, $\underline{w}^{(-1)} = (w_1^{(-1)}, w_2^{(-1)}, \ldots, w_n^{(-1)})$ is the symmetric element of $\underline{w}$. Thus, $\mathcal{G}^n = (G^n, \odot_\times)$ is an abelian group.

Let $\div_\times$ be the inverse operation of $\odot_\times$, defined as follows: $\underline{w} \div_\times \underline{v} = \underline{w} \odot_\times \underline{v}^{(-1)}$. Then

$$\underline{w} \div_{\times} \underline{v} = (w_1 \div v_1, w_2 \div v_2, \dots w_n \div v_n). \tag{11}$$

In the following, for simplicity of notation, we will use $\odot$ for $\odot_{\times}$ and $\div$ for $\div_{\times}$.

PROPOSITION 7. *It results:*

i) $\underline{w} = \underline{t} \odot \underline{v}$ *if and only if* $\underline{v} = \underline{t}^{(-1)} \odot \underline{w}$;
ii) $\underline{w} = \underline{t} \odot \underline{v}$ *if and only if* $\underline{w}^{(-1)} = \underline{t}^{(-1)} \odot \underline{v}^{(-1)}$;
iii) $(\underline{w} \div \underline{v})^{(-1)} = \underline{v} \div \underline{w}$.

Proof. From Proposition 1. ■

DEFINITION 6. *Let $c \in G$ and $\underline{w} = (w_1, w_2, \dots w_n) \in G^n$. Then the $\odot$-composition of c and $\underline{w}$ is the vector $\underline{c} \odot \underline{w}$, where $\underline{c} = (c, c, \dots, c) \in G^n$; that is,*

$$c \odot \underline{w} = (c \odot w_1, c \odot w_2, \dots, c \odot w_n).$$

Then, $c \div \underline{w}$ denotes the composition $c \odot \underline{w}^{(-1)}$; so

$$c \div \underline{w} = (c \div w_1, \dots, c \div w_n) = (c \odot w_1^{(-1)}, \dots, c \odot w_n^{(-1)}).$$

Similarly, $\underline{w} \div c$ denotes the vector

$$\underline{w} \div c = (w_1 \div c, \dots, w_n \div c) = (w_1 \odot c^{(-1)}, \dots, w_n \odot c^{(-1)}).$$

From Proposition 7, we derive

$$\underline{w} = c \odot \underline{v} \Leftrightarrow \underline{v} = c^{(-1)} \odot \underline{w}. \tag{12}$$

Hence we provide the following definition.

DEFINITION 7. *The vectors $\underline{w}$ and $\underline{v}$ are $\odot$-proportional if and only if there exists $c \in G$ such that $\underline{w} = c \odot \underline{v}$.*

PROPOSITION 8. *The vectors $\underline{w}$ and $\underline{v}$ are $\odot$-proportional if and only the vectors $\underline{w}^{(-1)}$ and $\underline{v}^{(-1)}$ are also $\odot$-proportional.*

Proof. From Proposition 7, $\underline{w} = c \odot \underline{v}$ if and only if $\underline{w}^{(-1)} = c^{(-1)} \odot \underline{v}^{(-1)}$. ■

PROPOSITION 9. *The vectors $\underline{w}$ and $\underline{v}$ are $\odot$-proportional if and only if*

$$w_i \div w_j = v_i \div v_j \ \forall i, j = 1, \ldots n. \tag{13}$$

Proof. Let $\underline{w}$ and $\underline{v}$ be $\odot$-proportional vectors. Then there exists $c \in G$ such that $w_i = c \odot v_i \forall i = 1, \ldots, n$. Thus, $w_i \div w_j = (c \odot v_i) \odot (c \odot v_j)^{-1} = (c \odot v_i) \odot (c^{-1} \odot v_j^{-1}) = v_i \div v_j$ for each $i, j = 1, \ldots, n$.
Let (13) be verified, then there exists $c \in G$ such that $w_j \div v_j = c \ \forall j = 1, \ldots, n$. So $\underline{w} = c \odot \underline{v}$. ■

Example 1. Let us consider the multiplicative alo-group **]0, +∞[**.
The vectors $\underline{w} = (1, 2, 3)$ and $\underline{v} = (\frac{1}{2}, 1, \frac{3}{2})$ are $\cdot$-proportional because $\underline{v} = \frac{1}{2} \cdot \underline{w}$. Then

$$\frac{w_1}{w_2} = \frac{v_1}{v_2} = \frac{1}{2}, \quad \frac{w_1}{w_3} = \frac{v_1}{v_3} = \frac{1}{3}, \quad \frac{w_2}{w_3} = \frac{v_2}{v_3} = \frac{2}{3}.$$

Example 2. Let us consider the additive alo-group $\mathcal{R}$.
The vectors $\underline{w} = (-2, 0, 1)$ and $\underline{v} = (0, 2, 3)$ are $+$-proportional because $\underline{v} = 2 + \underline{w}$.
Then

$$w_1 - w_2 = v_1 - v_2 = -2, \quad w_1 - w_3 = v_1 - v_3 = -3,$$
$$w_2 - w_3 = v_2 - v_3 = -1.$$

Example 3. Let us consider the fuzzy alo-group **]0,1[**.
The vectors $\underline{w} = (0.5, 0.6, 0.4)$ and $\underline{v} = (0.4, 0.5, 0.3077)$ are $\otimes$-proportional because $\underline{v} = 0.4 \otimes \underline{w}$.
Then

$$w_1 \otimes w_2^{(-1)} = v_1 \otimes v_2^{(-1)} = 0.4, \quad w_2 \otimes w_1^{(-1)} = v_2 \otimes v_1^{(-1)} = 0.6,$$
$$w_1 \otimes w_3^{(-1)} = v_1 \otimes v_3^{(-1)} = 0.6, \quad w_3 \otimes w_1^{(-1)} = v_3 \otimes v_1^{(-1)} = 0.4,$$
$$w_2 \otimes w_3^{(-1)} = v_2 \otimes v_3^{(-1)} = 0.6923, \quad w_3 \otimes w_2^{(-1)} = v_3^{(-1)} \otimes v_2^{(-1)} = 0.3077.$$

PROPOSITION 10. *Given $\underline{w} = (w_1, w_2, \ldots, w_n)$ and $\underline{v} = (v_1, v_2, \ldots, v_n)$ in G^n, the relation $\sim_{\odot}$, defined as*

$$\underline{w} \sim_{\odot} \underline{v} \Leftrightarrow \exists c \in G : \underline{w} = c \odot \underline{v}$$

is an equivalence relation.

Proof. As $\underline{w} = e \odot \underline{w}$ then $\underline{w} \sim_{\odot} \underline{w}$. So $\sim_{\odot}$ is a reflexive relation.
Because of the equivalence (12), $\sim_{\odot}$ is a symmetric relation.
If $\underline{w} = c_1 \odot \underline{v}$ and $\underline{v} = c_2 \odot \underline{u}$, then, by associativity property, $\underline{w} = (c_1 \odot c_2) \odot \underline{u}$.
Thus $(\underline{w} \sim_{\odot} \underline{v},\ \ \underline{v} \sim_{\odot} \underline{u}) \Rightarrow \underline{w} \sim_{\odot} \underline{u}$ and $\sim_{\odot}$ is a transitive relation. ■

We denote by $[\underline{w}]$ the $\odot$-equivalence class of $\underline{w}$, that is,

$$[\underline{w}] = \{\underline{v} \in G^n : \underline{w} \sim_{\odot} \underline{v}\}$$

and with $G^n / \sim_{\odot}$ the quotient set of G^n by $\sim_{\odot}$.

4. CONSISTENT PCMS: FIRST CHARACTERIZATIONS

In this section, $\mathcal{G} = (G, \odot, \leq)$ is a divisible alo-group and $A = (a_{ij})$ in (1) is a PCM over $\mathcal{G}$. By assumption, A is *reciprocal* with respect to $\odot$, so by (6) we get

$$\underline{a}^i = \underline{a}_i^{(-1)} \quad \forall\, i = 1, \ldots, n. \tag{14}$$

By means of the following proposition we reformulate the condition of consistency given in Definition 4.

PROPOSITION . *$A = (a_{ij})$ is a consistent PCM with respect to $\odot$, if and only if*

$$\underline{a}_i = a_{ij} \odot \underline{a}_j \quad \forall i, j = 1, 2, \ldots, n, \tag{15}$$

or, equivalently

$$\underline{a}^i = a_{ij}^{(-1)} \odot \underline{a}^j \quad \forall i, j = 1, 2, \ldots, n. \tag{16}$$

Proof. The equalities (15) follow from Definitions 4 and 6; the equalities (16) follow from the equalities (14), (15) and item (ii) of Proposition 7. ■

By the reciprocity property, the consistency conditions (15) can be written as follows:

$$\underline{a}_i = a_{ji}^{(-1)} \odot \underline{a}_j \quad \forall i, j = 1, 2, \ldots, n. \tag{17}$$

The conditions (16) and (17) characterize the consistent PCMs and allows us to build a consistent PCM starting from a column or a row, respectively.

By applying (17), we provide Algorithm 1 for building a consistent PCM starting from $n-1$ comparisons of the row $\underline{a}_{i^*}$; this algorithm generalizes and improves the algorithm proposed in Ref. 7.

Algorithm 1: BUILD A CONSISTENT PCM FROM THE SET $\{a_{i^*i} : i \neq i^*\}$

$a_{i^*i^*} = e$
for $i = 1 \ldots n$ **do**
  if $i \neq i^*$ **then**
    $\underline{a}_i = a_{i^*i}^{(-1)} \odot \underline{a}_{i^*}$
  end if
end for

Example 4. Let us consider the multiplicative alo-group $\mathbf{]0,+\infty[}$.
Let $\{x_1, x_2, x_3\}$ be a set of alternatives. We suppose that the DM expresses the following preference ratios: $a_{12} = 2$ and $a_{13} = 3$. Thus, $i^* = 1$ and the rows of the related consistent PCM are the following:

$$\underline{a}_1 = (1, 2, 3),$$

and

$$\underline{a}_2 = \frac{1}{2} \cdot \underline{a}_1 = \left(\frac{1}{2}, 1, \frac{3}{2}\right),$$

$$\underline{a}_3 = \frac{1}{3} \cdot \underline{a}_1 = \left(\frac{1}{3}, \frac{2}{3}, 1\right).$$

Example 5. Let us consider the additive alo-group $\mathcal{R}$.
Let $\{x_1, x_2, x_3\}$ be a set of alternatives. We suppose that the DM expresses the following preference differences: $a_{21} = 2$ and $a_{23} = 3$. Thus, $i^* = 2$ and the rows of the related consistent PCM are the following:

$$\underline{a}_2 = (2, 0, 3),$$

and

$$\underline{a}_1 = -2 + \underline{a}_2 = (0, -2, 1)$$

$$\underline{a}_3 = -3 + \underline{a}_2 = (-1, -3, 0).$$

Example 6. Let us consider the fuzzy alo-group **]0,1[**.
Let $\{x_1, x_2, x_3\}$ be a set of alternatives. We suppose that the DM expresses the following fuzzy preferences: $a_{12} = 0.4$ and $a_{13} = 0.6$. Thus, $i^* = 1$ and the rows of the related consistent PCM are the following:

$$\underline{a}_1 = (0.5, 0.4, 0.6),$$

and

$$\underline{a}_2 = 0.6 \otimes \underline{a}_1 = (0.6, 0.5, 0.69),$$

$$\underline{a}_3 = 0.4 \otimes \underline{a}_1 = (0.4, 0.31, 0.5).$$

The following theorem gives a characterization of a consistent PCM in terms of $\odot$-proportional vectors. This characterization allows us to check in an easy way the property of consistency.

THEOREM 2. *The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij})$ is a consistent PCM;*
2. *for every choice of $i, j = 1, 2, \ldots, n$ the rows $\underline{a}_i$ and $\underline{a}_j$ are $\odot$-proportional vectors; so for a fixed index i^*, $[\underline{a}_i] = [\underline{a}_{i^*}]$ for $i = 1, 2, \ldots, n$;*
3. *for every choice of $i, j = 1, 2, \ldots, n$ the columns $\underline{a}^i$ and $\underline{a}^j$ are $\odot$-proportional vectors; so for a fixed index i^*, $[\underline{a}^i] = [\underline{a}^{i^*}]$ for $i = 1, 2, \ldots, n$.*

Proof. 1. $\Leftrightarrow$ 2. If $A = (a_{ij})$ is a consistent PCM the item 2 follows immediately from the condition of consistency as formulated in (17). Alternatively, if item 2 is verified, then, for every choice of $i, j = 1, 2, \ldots, n$, there exists an element $c_{ij} \in G$ such that $\underline{a}_i = c_{ij} \odot \underline{a}_j$, so that $a_{ik} = c_{ij} \odot a_{jk}$ for each $k = 1, 2, \ldots, n$. By choosing $k = j$, we get $a_{ij} = c_{ij} \odot e = c_{ij}$; thus the condition (15) is verified and $A = (a_{ij})$ is a consistent PCM.
2 $\Leftrightarrow$ 3. From Proposition 8 or reasoning as above starting from equality (16). ■

Remark 3. We stress that, because of the assumption of reciprocity, the $\odot$-proportionality of the rows (resp. of the columns) is a necessary and sufficient condition for the consistency. Moreover, the condition of reciprocity allows us to determine the constants of $\odot$-proportionality by means of one of the equalities (15), (16) or (17).

Example 7. Let

$$A = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{3}{4} \\ 2 & \frac{4}{3} & 1 \end{pmatrix}$$

be a PCM over the multiplicative alo-group $]\mathbf{0}, +\infty[$. A is consistent because $\underline{a}^2, \underline{a}^3 \in [\underline{a}^1]$. Indeed, $\underline{a}^2 = \frac{2}{3} \odot \underline{a}^1$ and $\underline{a}^3 = \frac{1}{2} \odot \underline{a}^1$.
The matrix

$$B = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{1}{4} \\ 2 & 4 & 1 \end{pmatrix}$$

over $]\mathbf{0}, +\infty[$ is not consistent because $\frac{a_{21}}{a_{31}} = \frac{3}{4}$ is not equal to $\frac{a_{22}}{a_{32}} = \frac{1}{4}$, so for the column vectors $\underline{a}^1$ and $\underline{a}^2$ the condition (13) in Proposition 9 is not verified and $\underline{a}^2 \notin [\underline{a}^1]$.

COROLLARY 1. *The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij})$ is a consistent PCM;*
2. *for every choice of $i < n$, the rows $\underline{a}_i$ and $\underline{a}_{i+1}$ are $\odot$-proportional vectors,*
3. *for every choice of $i < n$, the columns $\underline{a}^i$ and $\underline{a}^{i+1}$ are $\odot$-proportional vectors.*

Proof. From Theorem 2 and the transitivity of the relation $\sim_\odot$. ■

Remark 4. As we stressed in Remark 3 the constants of $\odot$-proportionality between two rows (resp. columns) are computed by means of (17) (resp. (16)), so

$$\underline{a}_{i+1} = a_{i\,i+1}^{(-1)} \odot \underline{a}_i \quad \forall i < n;$$

$$\underline{a}^{i+1} = a_{i+1\,i}^{(-1)} \odot \underline{a}^i \quad \forall i < n.$$

4.1. Consistent Vectors and Consistent Matrices

DEFINITION 8.[6] *A vector $\underline{w} = (w_1, \ldots, w_n)$, with $w_i \in G$, is a* consistent *vector for $A = (a_{ij})$ if and only if*

$$w_i \div w_j = a_{ij} \;\forall\, i, j = 1, 2, \ldots, n.$$

PROPOSITION 12.[6] *$A = (a_{ij})$ is a consistent PCM if and only if there exists a consistent vector $\underline{w} = (w_1, w_2, \ldots, w_n)$, $w_i \in G$.*

PROPOSITION 13. *Let $A = (a_{ij})$ be a consistent PCM and $\underline{w}$ a consistent vector for $A = (a_{ij})$. A vector $\underline{v}$ is consistent for $A = (a_{ij})$ if and only if $\underline{v} \in [\underline{w}]$.*

Proof. From Definition 8 and Proposition 9. ■

We stress that the condition of consistency in Definition 4 can be written as

$$a_{ik} \div a_{jk} = a_{ij} \quad \forall i, j, k,$$

thus, in a consistent PCM, each column is a consistent vector. In Ref. 6 the following result is provided.

PROPOSITION 14. *The following assertions related to $A = (a_{ij})$ are equivalent:*

i) $A = (a_{ij})$ is a consistent PCM;
ii) each column $\underline{a}^k$ is a consistent vector;
iii) the mean vector $\underline{w}_{m_\odot} = (m_\odot(\underline{a}_1), \ldots, m_\odot(\underline{a}_n))$ is a consistent vector.

4.2. Bijection between $G^n/\sim_\odot$ and the Set of Consistent PCMs

THEOREM 3 . *Let CM be the set of the consistent PCM, then the relation:*

$$F : [\underline{v}] \in G^n/\sim_\odot \to A_v = \begin{pmatrix} v_1 \div v_1 & v_1 \div v_2 & \ldots & v_1 \div v_n \\ v_2 \div v_1 & v_2 \div v_2 & \ldots & v_2 \div v_n \\ \ldots & \ldots & \ldots & \ldots \\ v_n \div v_1 & v_n \div v_2 & \ldots & v_n \div v_n \end{pmatrix} \in CM$$

is a bijective function.

Proof. The function F is well defined because if $\underline{w} \in [\underline{v}]$ then, by Proposition 9, $w_i \div w_j = v_i \div v_j \;\forall i, j = 1, \ldots, n$ and so $A_{\underline{v}} = A_{\underline{w}}$.

Moreover,

- if $[\underline{v}] \neq [\underline{w}]$ then, by Proposition 9, $\exists i, j : w_i \div w_j \neq v_i \div v_j$, as a consequence $A_{\underline{v}} \neq A_{\underline{w}}$; so F is injective;
- if A is a consistent PCM, then, by Proposition 12, there exists a consistent vector $\underline{v}$; by Definition 8 and Proposition 13, $F([\underline{v}]) = A$. Thus F is surjective.

■

Theorem 3 allows us to build a consistent PCM starting form a vector; so we provide Algorithm 2.

Algorithm 2: BUILD A CONSISTENT PCM FROM A VECTOR $\underline{v} = (v_1, \ldots v_n)$

for $i = 1 \ldots n$ **do**
  $\underline{a}_i = v_i \div \underline{v}$
end for

Similarly, we can build the PCM for columns, by setting $\underline{a}^i = \underline{v} \div v_i$.

Example 8. Let us consider the multiplicative alo-group $]0, +\infty[$.
Let us assume $\underline{v} = (2, 3, 4)$. Let us set

$$a_1 = v_1/\underline{v} = 2/(2, 3, 4) = \left(1, \frac{2}{3}, \frac{1}{2}\right)$$

$$a_2 = v_2/\underline{v} = 3/(2, 3, 4) = \left(\frac{3}{2}, 1, \frac{3}{4}\right)$$

$$a_3 = v_3/\underline{v} = 4/(2, 3, 4) = \left(2, \frac{4}{3}, 1\right).$$

So $\underline{v}$ and each vector in $[\underline{v}]$ generates the consistent PCM

$$A = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{3}{4} \\ 2 & \frac{4}{3} & 1 \end{pmatrix}.$$

Example 9. Let us consider the additive alo-group $\mathcal{R}$.
The vector $\underline{v} = (-2, 1, 2)$ and each vector in $[\underline{v}]$ generates the consistent PCM

$$A = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & -1 \\ 4 & 1 & 0 \end{pmatrix}.$$

Example 10. Let us consider the fuzzy alo-group **]0,1[**.
The vector $\underline{v} = (0.3, 0.4, 0.7)$ and each vector in $[\underline{v}]$ generates the consistent PCM

$$A = \begin{pmatrix} 0.5 & 0.39 & 0.155 \\ 0.61 & 0.5 & 0.22 \\ 0.845 & 0.78 & 0.5 \end{pmatrix}.$$

In order to give a simple way to check the consistency of a PCM we provide the following proposition.

PROPOSITION 15. *$A = (a_{ij})$ is a consistent PCM if and only if $A = F[\underline{a}^i]$ for some $i = 1, 2, \ldots, n$.*

Proof. From Theorem 3, Definition 8 and Proposition 14. ■

For instance, the matrix A in Example 7 is consistent as $A = F(\underline{a}^1)$.

5. FURTHER CHARACTERIZATIONS OF CONSISTENT PCMS

In Section 4, we have provided characterizations for building a consistent PCM, starting from comparisons contained in a row (resp. a column) of the built PCM or from a vector that represents a consistent vector for the built PCM, and for checking the consistency by checking each entry of the given PCM.

In this section, we provide new characterizations of the consistency that allows us to build a consistent PCM from a different set of comparisons and to check the consistency, by checking only a minimum number of entries of the given PCM.

5.1. Useful Characterizations for Building a Consistent PCM

PROPOSITION 16. *The following assertions are equivalent:*

1. *A is a consistent PCM with respect to $\odot$;*
2. $a_{ik} = a_{i\,i+1} \odot a_{i+1\,k} \quad \forall i, k : i < k$;
3. $a_{ik} = a_{i\,i+1} \odot a_{i+1\,i+2} \odot \ldots \odot a_{k-1\,k} \quad \forall i, k : i < k$.

Proof. $1 \Rightarrow 2$. It is straightforward because of Proposition 4.

$2 \Rightarrow 3$. By 2:

$$a_{ik} = a_{i\,i+1} \odot a_{i+1\,k}$$

$$a_{i+1\,k} = a_{i+1\,i+2} \odot a_{i+2\,k}$$

$$\vdots$$

$$a_{k-2\,k} = a_{k-2\,k-1} \odot a_{k-1\,k}$$

Thus, by associativity of $\odot$, 3 is achieved.

$3 \Rightarrow 1$. By Proposition 4, it is enough to prove that $3 \Rightarrow (8)$. Let $i < j < k$. By 3, we have that

$$a_{ik} = a_{i\,i+1} \odot \ldots a_{j-1\,j} \odot a_{j\,j+1} \ldots \odot a_{k-1\,k};$$

so, by associativity of $\odot$ and applying again 3, we have that

$$a_{ik} = (a_{i\,i+1} \odot a_{i+1\,i+2} \odot \ldots \odot a_{j-1\,j})$$

$$\odot(a_{j\,j+1} \odot a_{j+1\,j+2} \odot \ldots \odot a_{k-1\,k}) = a_{ij} \odot a_{jk}.$$

■

Proposition 16 generalizes to PCMs defined on alo-group a result provided in Ref. 8 for the fuzzy case.

By Proposition 16, the following corollary follows.

COROLLARY 2. *$A = (a_{ij})$ is a consistent PCM with respect to $\odot$, if and only if:*

$$d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k}) = e \quad \forall\, i, k : i < k.$$

We provide Algorithm 3 to build a consistent PCM starting from $a_{12}, a_{23}, \ldots a_{n-1\,n}$.

Algorithm 3: BUILDING A CONSISTENT PCM STARTING FROM $a_{12}, a_{23}, \ldots a_{n-1\,n}$

for $i = 1, \ldots n-2$ **do**
  $a_{i\,i+2} = a_{i\,i+1} \odot a_{i+1\,i+2}$
  $temp = a_{i\,i+2}$
  for $k = i+3 \ldots n$ **do**
    $a_{i\,k} = temp \odot a_{k-1\,k}$
    temp $= a_{i\,k}$
  end for
end for

for $i = 2, \ldots n$ **do**
  for $j = 1, \ldots i - 1$ **do**
    $a_{i\,j} = a_{j\,i}^{(-1)}$
  end for
end for
for $i = 1, \ldots, n$ **do**
  $a_{ii} = e$
end for

Example 11. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives. We suppose that the DM expresses the following preference ratios (multiplicative case): $a_{12} = 2$, $a_{23} = 2$, $a_{34} = \frac{5}{4}$ and $a_{45} = \frac{6}{5}$. By means of Algorithm 3, we obtain

$$a_{13} = a_{12} \cdot a_{23} = 2 \cdot 2 = 4,$$

$$a_{14} = a_{12} \cdot a_{23} \cdot a_{34} = a_{13} \cdot a_{34} = 5,$$

$$a_{15} = a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{45} = a_{14} \cdot a_{45} = 6,$$

$$a_{24} = a_{23} \cdot a_{34} = \frac{5}{2},$$

$$a_{25} = a_{23} \cdot a_{34} \cdot a_{45} = a_{24} \cdot a_{45} = 3,$$

$$a_{35} = a_{34} \cdot a_{45} = \frac{3}{2}.$$

Computing the symmetric elements, we have

$$a_{21} = \frac{1}{2},$$

$$a_{31} = \frac{1}{4},\ a_{32} = \frac{1}{2},$$

$$a_{41} = \frac{1}{5},\ a_{42} = \frac{2}{5},\ a_{43} = \frac{4}{5},$$

$$a_{51} = \frac{1}{6},\ a_{52} = \frac{1}{3},\ a_{53} = \frac{2}{3},\ a_{54} = \frac{5}{6}.$$

Finally,

$$a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1.$$

Thus,

$$A = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ \frac{1}{2} & 1 & 2 & \frac{5}{2} & 3 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{5}{4} & \frac{3}{2} \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} & 1 & \frac{6}{5} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{5}{6} & 1 \end{pmatrix}.$$

Example 12. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives. We suppose that the DM expresses the following preferences (fuzzy case): $a_{12} = 0.6, a_{23} = 0.609, a_{34} = 0.632$ and $a_{45} = 0.692$. By means of Algorithm 3, we obtain

$$A = \begin{pmatrix} 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.4 & 0.5 & 0.609 & 0.727 & 0.857 \\ 0.3 & 0.391 & 0.5 & 0.632 & 0.794 \\ 0.2 & 0.273 & 0.368 & 0.5 & 0.692 \\ 0.1 & 0.143 & 0.206 & 0.308 & 0.5 \end{pmatrix}.$$

5.2. Useful Characterizations for Checking the Consistency

In the following, if $\underline{w} = (w_1, \dots, w_n) \in G^n$, then $\underline{w}[l : m]$, with $1 \leq l \leq m \leq n$, will denote the vector $(w_l, \dots, w_m) \in G^{m-l+1}$.

PROPOSITION 17. *The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij})$ is a consistent PCM;*
2. *the sub-rows $\underline{a}_i[i+2 : n]$ and $\underline{a}_{i+1}[i+2 : n]$ are $\odot$-proportional vectors for $i = 1 \dots n-2$, that is*

$$a_{i+1\,k} = a_{i\,i+1}^{(-1)} \odot a_{ik} \quad \forall i = 1, \dots n-2 \;\forall k = i+2, \dots n;$$

3. *the sub-columns $\underline{a}^i[1 : i-1]$ and $\underline{a}^{i+1}[1 : i-1]$ are $\odot$-proportional vectors for $i = 2, \dots n-1$, that is,*

$$a_{k\,i} = a_{i\,i+1}^{(-1)} \odot a_{k\,i+1} \quad \forall i = 2, \dots n-1 \;\forall k = 1, \dots i-1.$$

Proof. By assumption of reciprocity for $A = (a_{ij})$, Corollary 1 and Remark 4. ■

In order to check whether or not a PCM is consistent, we use item *2* of Proposition 17 and we provide Algorithm 4. In Algorithm 4, we assume that

- i is the index of the rows of A, it is initialized to $i = 1$;
- k is the index of the columns of A, it is initialized to $k = i + 2$;
- n is the order of A;
- *ConsistentMatrix* is a boolean variable and the algorithm returns $ConsistentMatrix = true$ if and only if the matrix is consistent. It is initialized to $true$, but $ConsistentMatrix = false$ is immediately returned when an inconsistent triple $(a_{ik}, a_{i\,i+1}, a_{i+1\,k})$ occurs.

Similar algorithm can be provided if we use item *3* of Proposition 17.

Algorithm 4: CHECKING CONSISTENCY

```
i = 1;
ConsistentMatrix = true;
while i ≤ n − 2 and ConsistentMatrix do
  k = i + 2;
  while k ≤ n and ConsistentMatrix do
    if a_{i+1 k} ≠ a_{i i+1}^{(−1)} ⊙ a_{ik} then
      ConsistentMatrix = false;
    end if
    k = k + 1;
  end while
  i = i + 1;
end while
return ConsistentMatrix;
```

Example 13. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives and

$$A = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ \frac{1}{2} & 1 & 2 & \frac{5}{3} & 3 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{5}{4} & \frac{3}{2} \\ \frac{1}{5} & \frac{3}{5} & \frac{4}{5} & 1 & \frac{6}{5} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{5}{6} & 1 \end{pmatrix}$$

the related multiplicative PCM. In order to check whenever A is consistent we have to check the following equalities:

$$\underline{a}_2[3:5] = a_{12}^{-1} \cdot \underline{a}_1[3:5] = \frac{1}{2} \cdot \underline{a}_1[3:5]$$

$$\underline{a}_3[4:5] = a_{23}^{-1} \cdot \underline{a}_2[4:5] = \frac{1}{2} \cdot \underline{a}_2[4:5]$$

and

$$\underline{a}_4[5:5] = a_{34}^{-1} \cdot \underline{a}_3[5:5] = \frac{4}{5} \cdot \underline{a}_3[5:5].$$

We stress that, for instance, $a_{24} \neq \frac{1}{2} \cdot a_{14}$; so the PCM is not consistent.

6. A NEW CONSISTENCY INDEX

In Section 2.4, we have introduced a consistency index $I_{\mathcal{G}}(A)$. At the light of Corollary 2, it is reasonable to define a new consistency index, considering only the distances $d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k})$, with $i < k-1$; of course, if $i = k-1$ then $d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k}) = e$. Let T^* be the set $\{(a_{i\,i+1}, a_{i+1\,k}, a_{ik}), i < k-1\}$ and $n_{T^*} = |T^*|$, then we consider the following index:

$$I_{\mathcal{G}}^*(A) = \begin{cases} d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) & n = 3, \\ \left(\bigodot_{T^*} d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k})\right)^{\left(\frac{1}{n_{T^*}}\right)} & n > 3. \end{cases}$$

with

$$n_{T^*} = \frac{(n-2)(n-1)}{2}.$$

From Corollary 2, we have the following.

Proposition 18. *$I_{\mathcal{G}}^*(A) \geq e$ and A is consistent if and only if $I_{\mathcal{G}}^*(A) = e$.*

As for $n > 3$ it results $n_{T^*} < n_T$, the index $I_{\mathcal{G}}^*(A)$ is more easy to compute than the consistency index $I_{\mathcal{G}}(A)$, thus, we provide the following definition.

Definition 9. *A* consistency index *of A is given by $I_{\mathcal{G}}^*(A)$.*

Proposition 19. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij}))$ the transformed of $A = (a_{ij})$ by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}}^*(A) = h^{-1}(I_{\mathcal{G}'}^*(A'))$.*

Example 14. Let us consider

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.4 \\ 0.7 & 0.5 & 0.1 & 0.2 \\ 0.6 & 0.9 & 0.5 & 0.8 \\ 0.6 & 0.8 & 0.2 & 0.5 \end{pmatrix}$$

that is a PCM over the fuzzy alo-group **]0,1[**. By applying the function ψ^{-1}, with ψ in 4, to the entries of A, we get the matrix

$$A' = \begin{pmatrix} 1 & \frac{3}{7} & \frac{2}{3} & \frac{2}{3} \\ \frac{7}{3} & 1 & \frac{1}{9} & \frac{1}{4} \\ \frac{3}{2} & 9 & 1 & 4 \\ \frac{3}{2} & 4 & \frac{1}{4} & 1 \end{pmatrix}.$$

A' is a PCM over the multiplicative alo-group $\mathbf{]0,+\infty[}$ and its consistency index is

$$I^*_{\mathbf{]0,+\infty[}}(A') = \sqrt[3]{I^*_{\mathbf{]0,+\infty[}}(A'_{123}) \cdot I^*_{\mathbf{]0,+\infty[}}(A'_{124}) \cdot I^*_{\mathbf{]0,+\infty[}}(A'_{234})}$$

$$= \sqrt[3]{14 \cdot \frac{56}{9} \cdot \frac{16}{9}} = 5.37.$$

Applying Proposition 19, we can compute the consistency index of A by means of the isomorphism ψ in 4:

$$I^*_{\mathbf{]0,1[}}(A) = \psi(I^*_{\mathbf{]0,+\infty[}}(A')) = \frac{5.37}{6.37} = 0.84.$$

7. CONCLUSION AND FUTURE WORK

We consider PCMs over an alo-group $\mathcal{G} = (G, \odot)$; in this framework, the several approaches to PCMs are unified and the drawbacks linked to the possibility of the DM to be consistent are removed. Moreover,

- we introduce the abelian group $\mathcal{G}^n = (G^n, \odot_\times)$ and the notion of $\odot$-proportionality;
- we provide new characterizations of a consistent PCM;
- we introduce an equivalence relation $\sim_\odot$ on G^n and a bijection between the quotient set $G^n/\sim_\odot$ and the set of the consistent PCMs;
- we provide efficient algorithms to check the consistency and to build a consistent PCM;
- we define a new consistency index.

Following the results in Refs. 2, 4, 5, and 9 for the multiplicative case, our future work will be directed to investigate, in the general context of the PCMs over alo-groups, conditions that allow us to obtain an evaluation vector $\underline{w} = (w_1, \ldots, w_n) \in G^n$, and able to represent the actual ranking of the alternatives at different levels.

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