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Theory and Methodology

Preference programming through approximate ratio comparisons

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Abstract

In the context of hierarchical weighting, this paper operationalizes interval judgments which allow the decision maker to enter ambiguous preference statements by indicating the relative importance of factors as intervals of values on a ratio scale. Through such judgments the decision maker can capture the subjective uncertainty in his preferences and thus avoid the often cumbersome elicitation of exact ratio estimates. After each new statement the interval judgments are synthesized into dominance relations on the alternatives by solving a series of linear programming problems. This leads to an interactive process of preference programming which provides more detailed results as the decision maker gradually enters a more specific preference description. Moreover, the overall effort of preference elicitation is smaller than in the analytic hierarchy process because the most preferred alternative can usually be identified before all possible comparisons between pairs of factors have been completed.

Keywords: Decision analysis; Analytic hierarchy process; Multiple criteria programming

1. Introduction

Methods of hierarchical weighting typically decompose the overall objectives of a problem into their lower level subobjectives until the resulting hierarchy provides a sufficiently detailed framework for the analysis. Within such a framework, the analytic hierarchy process (AHP) (Saaty, 1980) elicits preferences through pairwise comparisons in which the decision maker (DM) considers the relative importance of two factors at a time with respect to a common higher level criterion. For each comparison the DM indicates the intensity of preference of one factor over another as a point estimate on a ratio scale.

Several researchers have acknowledged the difficulties in eliciting exact ratio estimates. Van Laarhoven and Pedrycz (1983), Buckley (1985) and Boender et al. (1989) address this problem by suggesting fuzzy sets for the assessment and analysis of pairwise comparisons. Saaty and Vargas (1987), on the other hand, propose *interval judgments* which allow the DM to make approximate ratio statements as intervals of values on a ratio scale. Arbel (1989) interprets such judgments as linear constraints which at each criterion define a non-empty set of local priorities called the *feasible region*.

The present paper builds on Arbel's (1989,

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1991) approach by developing efficient algorithms for synthesizing interval judgments into dominance relations on the alternatives. These relations resemble those employed in multiattribute utility models (see, e.g. Bana e Costa, 1990; Hazen, 1986; Insua and French, 1991; Moskowitz et al., 1992; Weber, 1987), and because they are revised after each new preference statement, the DM is involved in an interactive decision support process which offers intermediate results even before most pairwise comparisons have been addressed.

The proposed approach to preference programming seems to have substantial practical potential due to the interactiveness of its decision support. It holds particular promise for the support of group decisions (Hämäläinen et al., 1992) since conflicting opinions can be modeled through composite intervals which contain the different views in the group. At the same time, the approach can be easily implemented into interactive decision aids such as INPRE (Salo and Hämäläinen, 1992c) because only linear programming is needed to compute the results.

This paper is organized as follows. Section 2 summarizes some of the earlier work on imprecise judgments in hierarchical weighting. Section 3 discusses dominance relations and develops algorithms for their computation. In Section 4, the consequences of the earlier judgments are analyzed to help the DM preserve the consistency of his judgments. Section 5 arranges the computations in the form of an algorithm, Section 6 addresses topics in sensitivity analysis, and Section 7 illustrates the methodology in the context of an energy policy problem.

2. Earlier work

In the framework of fuzzy analysis, van Laarhoven and Pedrycz (1983) propose triangular membership functions for the imprecise elicitation of pairwise comparisons and the computation of corresponding fuzzy weights. Buckley (1985) and Boender et al. (1989) extend these results to more general membership functions, and, like van Laarhoven and Pedrycz (1983), employ the logarithmic least squares method to compute the local priorities. Along with the many approximations involved, the main weakness of these approaches is perhaps the lack of clear-cut rules for converting the fuzzy weights into dominance results.

Following another line of investigation, Saaty and Vargas (1987) propose interval judgments for the AHP as a way to model the subjective uncertainty in the DM's preferences. With interval judgments, the DM can make statements such as 'the i-th subelement is at least weakly and at most strongly more important than the j-th subelement'. Using numerical counterparts for the verbal expressions of strength of preference gives the equivalent statement 'the i-th subelement is three to five times more important than the j-th subelement'; this is abbreviated as $I_{ij} = [l_{ij}, u_{ij}] = [3, 5]$.

In terms of interval judgments, the comparison matrix can be written as

$$\begin{pmatrix} 1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \cdots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \cdots & 1 \end{pmatrix}, \tag{1}$$

where the lower and upper bounds satisfy a reciprocity condition analogous to that of usual comparison matrices (Saaty and Vargas, 1987). That is, if the DM considers the i-th subelement to be more important than the j-th subelement ($l_{ij} = 1$), then the j-th subelement has to be less important than the i-th subelement ($u_{ji} = 1$). Extending this argument to the general case shows that $l_{ij}u_{ji} = 1$ for $i \neq j$.

Saaty and Vargas (1987) consider the derivation of local priorities from the matrix representation (1), but conclude that the problem of determining all the right principal eigenvectors of those reciprocal matrices whose elements belong to the intervals I_{ij} is computationally a relatively intractable task. In particular, the eigenvector is a nonlinear function of the matrix elements so that there are no straightforward techniques to determine exact bounds for its components. One possibility to approximate these bounds is to use Monte Carlo simulation to generate eigenvectors from samples of repreciprocal matrices whose elements belong to the intervals in (1).

Yoon (1988) studies how sensitive local priorities and final weights are to possible errors in the comparison matrix; however, in order to avoid complicated algebraic calculations, he approximates local priorities using the normalized row sum of the comparison matrix. Zahir (1991), on the other hand, characterizes perturbations in the right principal eigenvector due to uncertainty about the elements of the comparison matrix.

Instead of analyzing the properties of the interval matrix (1), Arbel (1989, 1991) interprets interval judgments as linear constraints on the local priorities. Emphasizing the definition of interval judgments, he points out that a given local priority vector $w=(w_1,\dots,w_n)$ is consistent with the judgment $I_{ij}=[l_{ij}, u_{ij}]$ only if it satisfies the constraints

$$l_{ij}w_j \le w_i \le u_{ij}w_j.$$

He then defines the *feasible region* as the set of those local priorities which satisfy all such constraints. Thus the feasible region can be written as

$$S=Q^n \cap \{w \mid l_{ij}w_j \le w_i \le u_{ij}w_j\}, \tag{2}$$

where

$$Q^n=\left\{(w_1,\dots,w_n) \mid w_i \ge 0,\ \sum_{i=1}^{n} w_i=1\right\}$$

and l_{ij}, u_{ij} correspond to those interval bounds that the DM has specified. According to this definition, the feasible region is well-defined even if some or even all of the bounds in (1) are missing, but it may become empty if the judgments are inconsistent. Fig. 1 shows the feasible region $S \in Q^3$ based on the judgments $I_{12}=[1, 2]$, $I_{13}=[1, 3]$.

Arbel and Vargas (1992) formulate maximization and minimization problems for establishing bounds for the components of right principal eigenvectors when the elements of a reciprocal matrix are constrained to the intervals in (1). They also characterize the alternatives' weight

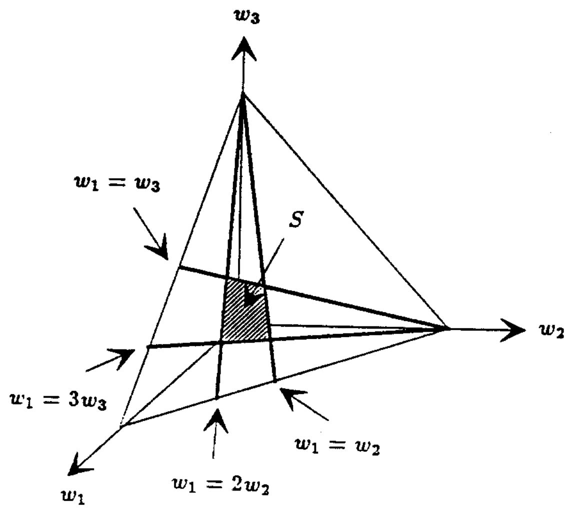

Fig. 1. A feasible region.

intervals as solutions to non-linear programs in which all the local priorities in the hierarchy are included as decision variables. As a result, the time needed to solve these optimization problems increases with the size of the hierarchy, and therefore this approach, which in the consistent case would produce weight intervals similar to ours, is unappealing for the development of interactive decision support.

The method of Solymosi and Dombi (1986) converts ordinal preference statements into a feasible region from which the centroid vector is then chosen to represent the DM's preferences. Olson and Dorai (1992) observe that the centroid method often gives approximately similar results as the AHP even if it requires far fewer inputs. However, the centroid method may be somewhat unreliable because it does not allow the DM to enter ratio or other judgments that are more informative than ordinal comparisons.

The present paper focuses on the development of computationally efficient mechanisms for determining the preferred alternatives when the local priorities are constrained to the feasible sets (2). Furthermore, it extends earlier results on the derivation of weight intervals (Salo and Hämäläinen, 1992a) in that it proposes a less restrictive dominance criterion, considers the computation of ranges of criteria weights, and suggests an index for measuring the amount of imprecision in the DM's judgments.

The methodology of this paper can be regarded as a generalization of the conventional AHP. The main difference is that in addition to exact ratio comparisons the DM may specify interval judgments as well. This results in a natural extension of the AHP (Kress, 1991) as the consistency bounds of Section 4 help the DM to maintain the consistency of the preference model.

3. Processing interval judgments

This section considers two dominance concepts for synthesizing non-empty feasible regions into dominance relations for the alternatives. These concepts, following those used in imprecisely specified multiattribute utility models (see Bana e Costa, 1990; Hazen, 1986; Salo and Hämäläinen, 1992b; Weber, 1987), are based on an analysis of the weights that feasible parameter values assign to the alternatives. Computationally, the dominance results are determined from a decomposition scheme of linear programs.

3.1. Weight intervals and absolute dominance

Each combination of local priorities from the feasible regions gives a unique weight to each alternative. Therefore, as the local priorities are allowed to vary over the feasible regions, every alternative receives an interval of weights. Like the weights of the alternatives in the conventional AHP, these weight intervals convey information about which alternatives are preferred to others.

More specifically, if $V(x)$, the weight interval of alternative x, lies above that of alternative y (i.e. $r > s$ whenever $r \in V(x)$, $s \in V(y)$), then any feasible combination of local priorities assigns to x a weight greater than that of y. In such a situation x is said to dominate y according to the *absolute dominance* criterion, which is defined by

$$x \succ_A y \Leftrightarrow \min_{r \in V(x)} r > \max_{s \in V(y)} s. \tag{3}$$

Clearly, if $x \succ_A y$, then x is preferred to y because for all feasible local priorities the alternative x has the higher weight. The absolute dominance relation can be displayed to the DM conveniently through the weight intervals (see Fig. 10 in Section 7).

In principle, it is possible to approximate weight intervals by applying interval arithmetic to process bounds for the components of the local priorities (see, e.g. Moore, 1966). However, this approach is unsatisfactory because the components of the local priority vector, which add up to one, are not independent. For instance, if in a hierarchy of two alternatives and three criteria all feasible regions are convex combinations of the points $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$, then according to interval arithmetic the weight intervals for the alternatives are bounded from above by

$$\tfrac{3}{4} \times \tfrac{3}{4} + \tfrac{3}{4} \times \tfrac{3}{4} = \tfrac{9}{8} > 1.$$

This meaningless result demonstrates that interval arithmetic has to be rejected.

Instead, tight bounds for the weights of the alternatives can be found by solving optimization problems in which the alternative's weight is maximized/minimized subject to the DM's statements. By taking advantage of the principle of hierarchical composition, which guarantees that the weights of upper level criteria are independent of the judgments on the lower levels, these problems can decomposed into a series of linear programming problems over the feasible regions.

This decomposition is presented in the following notation which has been adopted from Saaty (1980). The hierarchy $\boldsymbol{H} = \boldsymbol{C} \cup \boldsymbol{A}$ consists of a set of criteria $\boldsymbol{C}$ and the set of alternatives $\boldsymbol{A}$. For any criterion $x \in \boldsymbol{C}$ the set x^-, which cannot be empty, contains those elements of $\boldsymbol{H}$ which are structured directly under x. Conversely, for any element x the set x^+ consists of the criteria $y \in \boldsymbol{C}$ such that x is structured directly under y.

The hierarchy $\boldsymbol{H}$ can be partitioned into levels $L_1, L_2, \ldots, L_h$ such that L_h is the set of alternatives, L_1 consists of the topmost criterion b, and for any other criterion $x \in L_i$ the criteria in x^+ belong to L_{i-1}. If x is an alternative then x^+ is the set of criteria y for which $y^- \subset \boldsymbol{A}$. For such an x, the set x^+ contains the criteria in L_{h-1}, but it can also contain criteria on the higher levels of the hierarchy.

The feasible region at criterion y is denoted by S_y. If $w^y \in S_y$ and $x \in y^-$, when w_x^y is the component of w^y that corresponds to x (in the sequel the superscript is usually omitted). By definition the weight of the topmost element is one, i.e. $v(b) = 1$. For the other elements the weights are derived from the feasible local priorities $w^y \in S_y$, $y \in \boldsymbol{C}$ recursively by

$$v(x) = \sum_{y \in x^+} v(y) w_x^y. \tag{4}$$

Theorem 1 decomposes the computation of weight intervals into a sequence of linear programming problems over the feasible regions. Intuitively, this theorem is based on the following argument which resembles the principle of optimality in dynamic programming. First, if the alternative x is structured under the criterion y, then y gives the largest share of its weight to x when the component w_x^y attains its maximum over the feasible region S_y. At the same time, this maximum $\bar{v}_y(x)$, called the *absolute upper bound*, is the highest weight that x can have in the subhierarchy of $\boldsymbol{H}$ rooted at y (see Fig. 2).

Next, take a criterion $z \in y^+$. The weight that x receives from z comes through the criteria in z^-. For these criteria, including y, the absolute upper bounds indicate how much of their weight they can give to x. These absolute bounds can therefore be employed at z as the coefficients of a linear maximization problem the solution of which gives the maximum for the weight of x in the subhierarchy rooted at z. The resulting linear program at z can be viewed as the problem of allocating weight among the already solved subhierarchies.

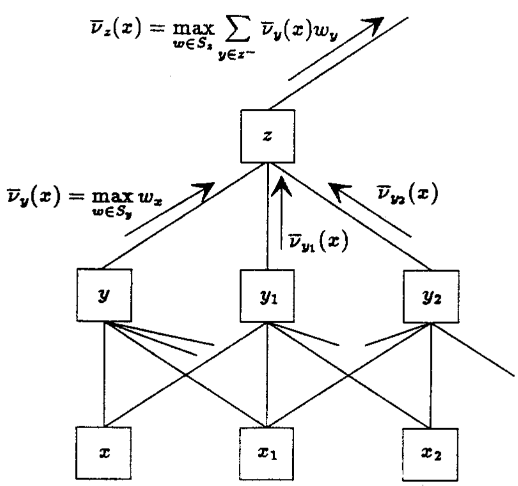

Fig. 2. Computation of weight intervals.

The above procedure can now be repeated until at the topmost element b the absolute bound $\bar{v}_b(x)$ equals the maximum for the weight of x in the entire hierarchy. In the same way, the minimum for the weight of x can be found by computing the *absolute lower bounds* $\underline{v}_y(x)$ which are equal to the smallest weights that x can receive in the subhierarchies rooted at $y \in \boldsymbol{C}$.

Theorem 1. *Let $x \in \boldsymbol{A}$ be a decision alternative. For $y \in \boldsymbol{C}$ such that $y^- \subset \boldsymbol{A}$ define the absolute bounds*

$$\bar{v}_y(x) = \max_{w \in S_y} w_x, \tag{5}$$

$$\underline{v}_y(x) = \min_{w \in S_y} w_x. \tag{6}$$

Proceed successively from level L_{h-2} upwards by recursively defining the absolute bounds

$$\bar{v}_y(x) = \max_{w \in S_y} \sum_{z \in y^-} \bar{v}_z(x) w_z, \tag{7}$$

$$\underline{v}_y(x) = \min_{w \in S_y} \sum_{z \in y^-} \underline{v}_z(x) w_z \tag{8}$$

for the criteria $y \in L_i$, $1 \le i < h-1$ ($y^- \not\subset \boldsymbol{A}$). Then $V(x) = [\underline{v}_b(x), \bar{v}_b(x)]$ is the set of weights assigned to x by feasible combinations of local priorities.

Proof. See the Appendix.

The recursive structure of Theorem 1 affords a constructive algorithm for finding the weight intervals for the alternatives. Once the absolute bounds at some level L_i of the hierarchy have been computed, the problems (7)–(8) can be solved at the adjacent higher level criteria by using the absolute bounds of level L_i. Since at each criterion two linear programs must be solved for every alternative, the total number of linear programs is $2|C||A|$ (here $|\cdot|$ denotes the number of elements in a set).

As the local priorities vary over the feasible regions, the Eq. (4) assigns weight intervals to the different criteria as well. Such intervals help the DM identify which criteria are the most important ones, and they, too, can be computed by solving a series of linear programs. Specifically, the weight intervals for the criteria in L_i can be found by considering the levels L_1 through L_i of the hierarchy; this subhierarchy is then processed with Theorem 1 by treating the elements of L_i as if they were alternatives.

3.2. Pairwise dominance

Even if the weight intervals of alternatives x and y overlap, it can happen that for all fixed combinations of feasible local priorities the weight of x is higher than that of y. In such a situation the DM cannot tighten the constraints on the feasible regions so that y would receive a weight higher than or equal to the weight of x. Formally this criterion for determining preferred alternatives is defined through the *pairwise dominance* relation $\succ_{\mathrm{P}}$,

$$x \succ_{\mathrm{P}} y \quad \Leftrightarrow \quad \min[v(x) - v(y)] > 0, \tag{9}$$

where the minimization is taken over all the feasible regions in the hierarchy. This relation, too, is transitive, for if $x \succ_{\mathrm{P}} y$ and $y \succ_{\mathrm{P}} z$, then

$$\begin{aligned} 0 &< \min[v(x) - v(y)] + \min[v(y) - v(z)] \\ &\le \min[v(x) - v(y) + v(y) - v(z)] \\ &= \min[v(x) - v(z)]. \end{aligned}$$

There are several reasons for using pairwise dominance in the choice of preferred alternatives. First, the iterative elicitation of interval judgments can be viewed as a process of gradually eliminating the local priorities that are incompatible with the DM's preferences. Thus, the DM's 'true' preference vectors are contained somewhere within the feasible regions. But if for all combinations of feasible priorities x has the higher weight, then this must be the case for the DM's true preference vector as well.

Second, the interval judgments can be seen as constraints on the probability distributions which can be thought to represent the DM's local preferences. In a natural way, these distributions define weight distributions in which only those weights that result from some combination of feasible priorities have positive probabilities (Saaty and Vargas, 1987). In this context, pairwise dominance of x over y means that for all possible outcomes the alternative x has more weight than y and is therefore preferred to y.

If the precise form of the distributions were known, then also other dominance concepts, based, e.g. on the comparison of expected weights, could be employed. However, the elicitation and the synthesis of such distributions would be more time-consuming than the analysis of interval judgments.

Pairwise dominance among the alternatives can be graphically displayed as a *domination digraph* (see Fig. 10), in which the alternatives correspond to the nodes and the relation $x \succ_{\mathrm{P}} y$ is shown as an arc from x to y (Sage and White, 1984). In this graph, the non-dominated alternatives have no incoming arcs so that the most preferred alternative is found when only one such node remains.

Since absolute dominance in (3) clearly implies (9), pairwise dominance need be computed only for alternatives which have overlapping weight intervals. More specifically, the possible pairwise dominance of x over y must be checked only if the absolute bounds satisfy the inequalities

$$\bar{v}_b(x) > \bar{v}_b(y) \ge \underline{v}_b(x) > \underline{v}_b(y).$$

The transitivity of the relation $\succ_{\mathrm{P}}$ can be exploited to further reduce the number of pairs for which pairwise dominance has to be determined. If there are only two alternatives the two relations coincide, because in this case the relation $x \succ_{\mathrm{P}} y$ gives the inequality $v(x) > v(y)$ which together with $v(x) + v(y) = 1$ implies that $v(y) < \frac{1}{2} < v(x)$; hence $x \succ_{\mathrm{A}} y$.

The following argument demonstrates that pairwise dominance, too, can be computed from a series of linear programming problems. If $z \in L_{h-1}$, then alternative x dominates y in the subhierarchy rooted at z only if $w_x - w_y > 0$ for all $y \in S_z$. The minimum for this difference,

$$\pi_z(x, y) = \min_{w \in S_z} (w_x - w_y),$$

is called the *pairwise bound* for the weight difference of x and y at criterion z (see Fig. 3).

If $t \in z^+$, then x dominates y with respect to $t \in z^+$ only if

$$\sum_{z \in t^-} w_z^t (w_x^z - w_y^z) > 0$$

for all feasible priorities $w^t \in S_t$, $w^z \in S_z$. Since the coefficients w_z^t are non-negative, the minimum of this sum is attained when the differences $w_x^z - w_y^z$ are replaced by the pairwise bounds for the criteria in t^-. For t the pairwise bound is therefore obtained as a solution to a linear programming problem in which the lower level pairwise bounds appear as coefficients for the components of w^t. In this way the bounds can be propagated upwards until the topmost element b is reached.

Theorem 2. *Fix x, $y \in A$. For $z \in C$ such that $z^- \subset A$ define the pairwise bounds*

$$\pi_z(x, y) = \min_{w \in S_z} (w_x - w_y). \tag{10}$$

Proceed successively from level L_{h-2} upwards by recursively defining the pairwise bounds

$$\pi_z(x, y) = \min_{w \in S_z} \sum_{t \in z^-} \pi_t(x, y) w_t \tag{11}$$

for the criteria $z \in L_i$, $1 \le i < h-1$ ($z^- \not\subset A$). Then x dominates y, i.e. $x \succ_P y$, if and only if $\pi_b(x, y) > 0$.

Proof. See the Appendix.

Due to the hierarchical composition, changes in the feasible region at criterion x affect the absolute and pairwise bounds only at those criteria y under which x is either directly or indirectly structured. Therefore updated results can be computed by first revising the bounds at x and then at x^+, thus moving progressively from x towards the topmost element.

The validity of Theorems 1 and 2 does not depend on the particular structure of the feasible

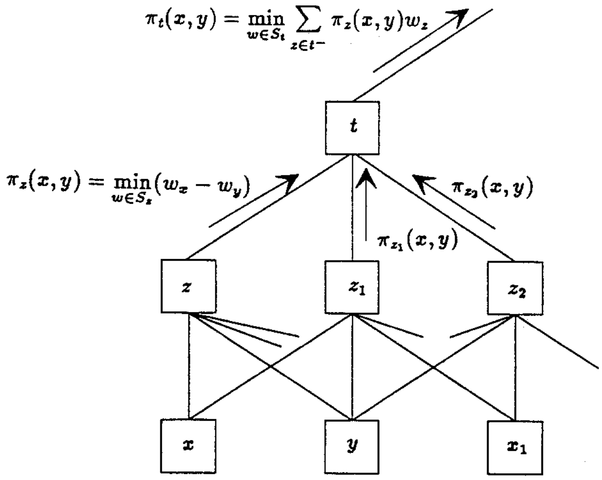

Fig. 3. Computation of pairwise dominance.

regions, provided that these regions are convex and closed sets. As a result even other types of constraints, such as bounds on the components of the local priorities (e.g. $0.25 \le w_x \le 0.80$), could be employed to characterize the DM's preferences and the dominance relations could still be computed as before. In the sequel, however, it is assumed that the DM describes his preferences through interval judgments. Other types of constraints are excluded in favor of pairwise comparisons which have been found an efficient approach to preference assessment both in the AHP and ratio-based techniques of multiattribute value measurement.

4. Support for refining interval judgments

To prevent the feasible regions from becoming empty, the DM needs guidance in the specification of new comparisons and tightening of earlier judgments. For this purpose, this section summarizes some results from Salo and Hämäläinen (1992b) where interval judgments are analyzed under the assumption

$$x \in z^- \quad \Rightarrow \quad \exists w \in S_z \quad \text{such that } w_x > 0. \tag{12}$$

This assumption can be justified by noting that if it did not hold, x would not receive any weight from z because the x-component of w^z would be zero for all feasible local priorities, meaning that x would be irrelevant to z. Yet is worth noting that (12) does not require all feasible priorities to have positive x-components.

At criterion z the impact of the DM's earlier judgments can be characterized by the *consistency intervals* $\hat{I}_{xy} = [\hat{l}_{xy}, \hat{u}_{xy}]$. The bounds of these intervals are defined by

$$\hat{u}_{xy} = \max_{w \in S_z} w_x / w_y \tag{13}$$

$$\hat{l}_{xy} = 1/\hat{u}_{yx}, \tag{14}$$

where x and y are subelements of z, and the ratio in (13) is taken to be ∞ if $w_x > 0$, $w_y = 0$ and 0 if $w_x = w_y = 0$. From (12) it follows that $w_x > 0$ for some $w \in S_z$ so that $\hat{u}_{xy} > 0$. The consistency bounds can be computed e.g. with linear fractional algorithms (see Bazaraa and Shetty, 1979) or the algorithm of Potter and Anderson (1980). Alternatively, they can be found by inspecting the extreme points of the feasible region. In Fig. 1, for instance, the interval $\hat{I}_{23} = [\frac{1}{2}, 3]$ shows the consistent ratios for the relative importance of the second and the third subelements that have not been compared yet.

The DM can tighten the constraints on the feasible regions either through new comparisons or by narrowing the bounds of the earlier interval judgments. Both modifications can be modeled by assuming that the DM changes the interval I_{xy} to $I'_{xy} \subset I_{xy}$ (here $\subset$ denotes proper set inclusion) and that after this change the modified feasible region becomes $S'_z \subseteq S_z$. In such a situation, the modified feasible region inherits the property (12) and becomes a proper subset of the earlier feasible region only if the intersection of the intervals I'_{xy} and $\hat{I}_{xy}$ is non-empty and a proper subset of $\hat{I}_{xy}$. In this way, the consistency intervals help the DM see the impact of the earlier judgments and allow him to effectively refine the preference description (see Fig. 7).

If the DM does want to enter an inconsistent judgment, he needs to relax earlier judgments until the feasible region becomes large enough to contain local priorities which satisfy the constraints of such a judgment. For instance, if the DM would like to enter a new lower bound l'_{xy} larger than $\hat{u}_{xy}$, he should first relax some of the upper bounds (these can be identified with the algorithm of Potter and Anderson, 1980) to increase the value of $\hat{u}_{xy}$.

During the analysis, the DM may wish to locate the criteria at which the characterization of his preferences is less precise than elsewhere. To this end, the *ambiguity index* is defined from the consistency intervals through the formula

$$\mathrm{AI}(S_z) = \frac{1}{n(n-1)} \sum_{x,y \in z^-} \frac{\hat{u}_{xy} - \hat{l}_{xy}}{(1+\hat{u}_{xy})(1+\hat{l}_{xy})}, \tag{15}$$

where the ratio is taken to be $1/(1+\hat{l}_{xy})$ when $\hat{u}_{xy} = \infty$. Because of the reciprocity property of the bounds in (13)–(14), the terms corresponding to the intervals $\hat{I}_{xy}$, $\hat{I}_{yx}$ are equal.

To each criterion the ambiguity index assigns a value in the zero to one range. The index has the value zero only if the feasible region consists of a single local priority vector, and attains the maximum of one only if the DM has not entered any preference statements. The ambiguity index is also monotonous since the feasible region S_z has a greater value than any of its proper subsets. The feasible region in Fig. 1, for instance, has the value

$$\mathrm{AI}(S) = \frac{2}{3 \times 2}\left(\frac{3-1}{4 \times 2} + \frac{2-1}{3 \times 2} + \frac{3-\frac{1}{2}}{4 \times \frac{3}{2}}\right) = \frac{5}{18} = 0.28.$$

5. Computational issues

The computations of Sections 3 and 4 can be sequenced following the algorithm of Fig. 4. In this diagram, the two loops indicate that the consistency intervals can be computed in parallel with the dominance relations. However, if the computer environment does not permit parallelism, it may be advisable to start by updating the consistency intervals and the ambiguity indexes; these help the DM make the next judgment while updated dominance results are being revised.

The DM can perform the pairwise comparisons either a top-down or bottom-up fashion, or in some other order that he finds convenient. In practice, the DM may prefer to start with the comparisons that are easiest to make and move towards the more difficult judgments. During the analysis the DM can identify which judgments would tighten the results most by examining the ambiguity indexes and the intervals of criteria weights.

The algorithm in Fig. 4 has been implemented into a mouse-driven program called INPRE (Salo

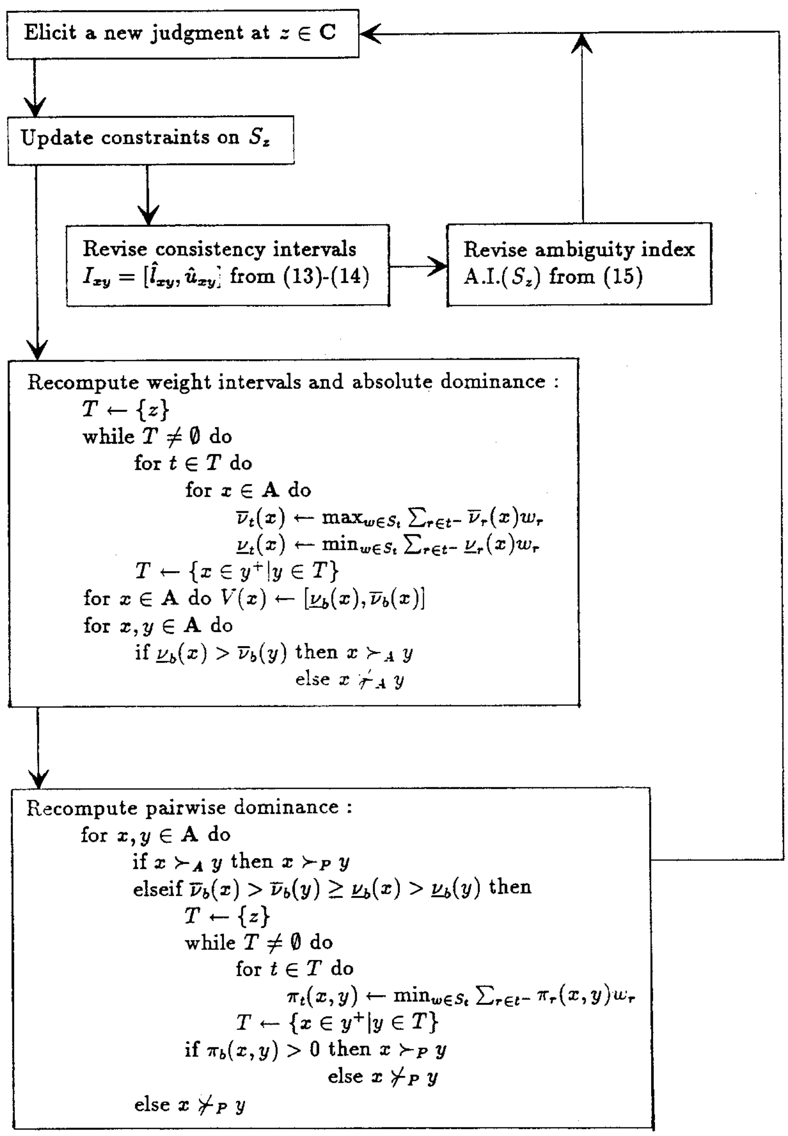

Fig. 4. An algorithm for revising the results.

and Hämäläinen, 1992c). For hierarchies of approximately dozen factors, INPRE computes revised results within a second or two on an AT-compatible microcomputer. This demonstrates that preference programming can indeed be implemented into functional decision support tools.

6. Sensitivity analysis

The absolute and pairwise bounds for the dominance relations are determined as solutions to linear programs over the feasible regions. Therefore, in order to estimate the sensitivity of these relations to individual judgments, it is of interest to identify those bounds in (1) for which a linear functional of the form

$$g_z(w) = \sum_{t \in z^-} \alpha_t w_t^z, \quad w \in S_z, \tag{16}$$

no longer attains its maximum

$$\bar{g}_z = \max_{w \in S_z} g_z(w)$$

if such a bound is tightened by some small amount (the minimization problems (6), (8), (10)–(11) can be adapted to this formulation by converting them into maximization problems by reversing signs). Due to the reciprocity of the bounds l_{xy}, u_{xy}, there is no loss of generality in restricting the attention to the case where the upper bound u_{xy} is modified to $u'_{xy} < u_{xy}$.

If $u_{xy} > \hat{u}_{xy}$, then by the results of Section 4 the bound u_{xy} can be tightened to $\hat{u}_{xy}$ without affecting the feasible region at all. Thus the bound u_{xy} must be equal to $\hat{u}_{xy}$ if after any $u'_{xy} < u_{xy}$ the maximum of $g_z(\cdot)$ over the modified feasible region,

$$S'_z = S_z \cap \{w_x \le u'_{xy} w_y\},$$

is going to be smaller than the maximum over S_z.

Proposition 1. *Assume $\hat{l}_{xy} < u_{xy} = \hat{u}_{xy} \le \infty$. Then $u'_{xy} < u_{xy}$ implies*

$$\max_{w \in S'_z} g_z(w) < \max_{w \in S_z} g_z(w)$$

if and only if $w_x/w_y = \hat{u}_{xy}$ for all extreme points w of S_z such that $g_z(w) = \max_{w \in S_z} g_z(w)$.

Proof. See the Appendix.

Proposition 1 gives necessary and sufficient conditions for identifying the bounds u_{xy} the tightening of which decreases the maximum of $g_z(\cdot)$ over the feasible region. Nevertheless, the maximum of the functional $g_b(\cdot)$, whose coefficients come from the adjacent lower level, can still remain unchanged. This is possible if on the upper levels there exists a feasible combination of local priorities which does not give any weight to z.

7. Example

This section illustrates the application of preference programming in the context of the hierarchy which Hämäläinen (1988, 1990, 1991) employed to help Parliamentarians compare energy production alternatives. In this hierarchy, shown in Fig. 5, the overall benefit of the society has been decomposed into its economic, environmental and political dimensions. The three alternatives in the problem are: no big power plant (A_1), two big coal power plants (A_2) and a nuclear power plant (A_3).

The preference statements in this example approximate those of Mr. Vennamo who at the time of the energy policy decision was the minister of finance and the leader of the rural party (see Hämäläinen, 1990). The interval judgments have been chosen so that the feasible regions contain Mr. Vennamo's local priorities in the conventional AHP analysis. In part, this choice demonstrates how inconsistencies in exact preference statements can be imbedded into interval judgments.

The criteria are numbered consecutively from left to right starting from the top of the hierarchy; hence C_1 refers to the overall benefit of the society and C_4 represents political factors. In the subscripts the elements of the hierarchy are indicated by their indexes so that S_i, for instance, is the feasible region at criterion C_i. If C_j (or A_j) has been structured under C_i and $w \in S_i$ is a feasible local priority vector at C_i, then w_j is the share of the weight of C_i that goes to C_j (or A_j).

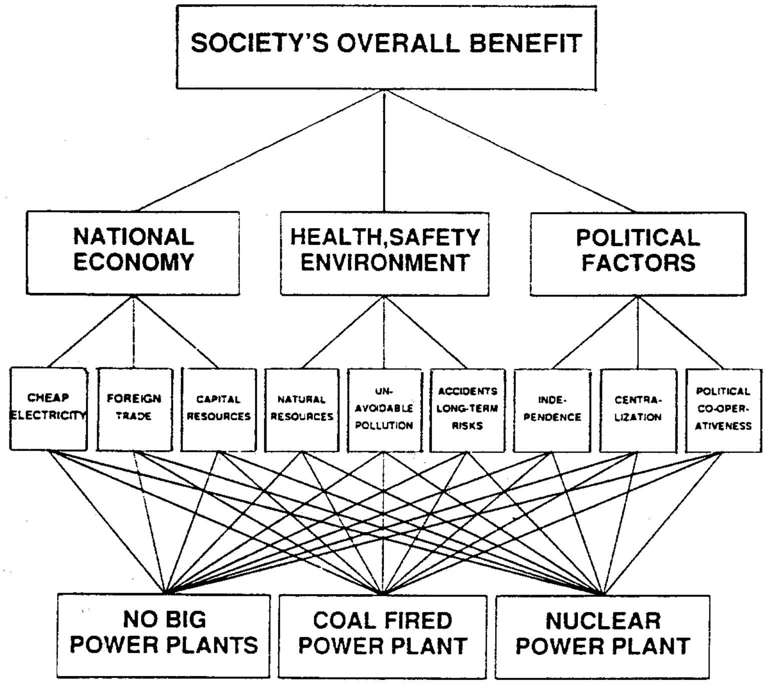

Fig. 5. The energy decision hierarchy.

At the lowest level criteria Mr. Vennamo's judgments were converted into feasible regions by forming the convex hulls of the priority vectors defined by pairs of ratio comparisons (see Salo, 1993). At criterion capital resources, for instance, the pairwise comparisons of the matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 7 & 4 \\ \frac{1}{7} & 1 & \frac{1}{5} \\ \frac{1}{4} & 5 & 1 \end{pmatrix} \end{array}. \tag{17}$$

correspond to three linear equality constraints which do not have a common solution due to the inconsistency of the matrix (17). Yet any two constraints do define a unique local priority vector: the entries a_{12}, a_{13}, for example, correspond to the constraints $w_1 = 7w_2$ and $w_1 = 4w_3$ which together with the normalization requirement give the vector

$$\tfrac{1}{39}(28, 4, 7) = (0.72, 0.10, 0.18).$$

From such vectors, the feasible region in Fig. 6 was defined as the set of local priorities $w =$

Table 1
Interval judgments at third level criteria

	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
I_{21}	$[-, \frac{1}{3}]$	$[-, 3]$	$[-, \frac{1}{7}]$	[1, '1]	$[-, 5]$	$[-, \frac{1}{7}]$	$[\frac{1}{5}, -]$	$[-, \frac{1}{8}]$	$[\frac{1}{5}, -]$
I_{31}	$[\frac{1}{5}, -]$	$[\frac{1}{5}, -]$	$[\frac{1}{4}, -]$	$[1, 1]$	$[\frac{1}{2}, -]$	$[\frac{1}{2}, -]$	$[-, \frac{1}{5}]$	$[\frac{1}{7}, -]$	$[-, \frac{1}{4}]$
i_{32}	$[-, 1]$	$[-, \frac{1}{6}]$	$[-, 5]$	$[-, -]$	$[-, \frac{1}{5}]$	$[-, 6]$	$[\frac{1}{3}, -]$	$[-, 3]$	$[1, -]$

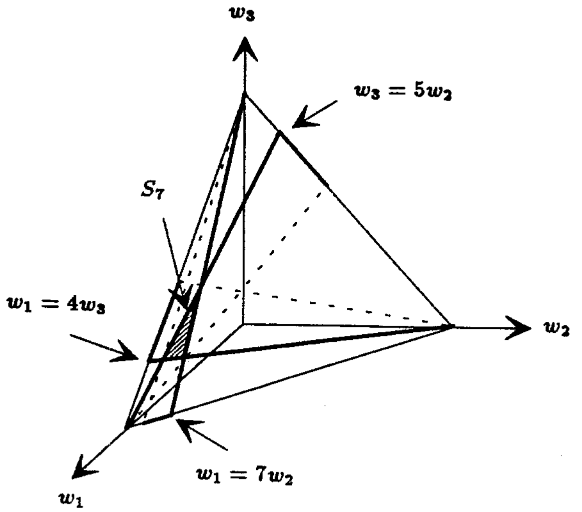

Fig. 6. The feasible region at criterion capital resources.

(w_1, w_2, w_3) which satisfy the inequalities $w_1 \geq 7w_2$, $w_1 \leq 4w_3$ and $w_3 \leq 5w_2$. In the representation (1) these constraints can be written as

$$\begin{array}{c c} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left(\begin{array}{ccc} 1 & [7, -] & [-, 4] \\ [-, \frac{1}{7}] & 1 & [\frac{1}{5}, -] \\ [\frac{1}{4}, -] & [-, 5] & 1 \end{array} \right) \end{array}, \tag{18}$$

where the dashes indicate the entries that do not impose constraints on the feasible region. A similar analysis of the other comparison matrices gave the intervals in Table 1.

For the feasible region in Fig. 6, the consistency bounds of (13) and (14) are

$$\begin{array}{c c} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left(\begin{array}{ccc} 1 & [7, 20] & [\frac{7}{5}, 4] \\ [\frac{1}{20}, \frac{1}{7}] & 1 & [\frac{1}{5}, \frac{4}{7}] \\ [\frac{1}{4}, \frac{5}{7}] & [\frac{7}{4}, 5] & 1 \end{array} \right) \end{array}. \tag{19}$$

In view of Section 4, these bounds show that the feasible region would not change if the DM were to state that the first alternative is better than the third ($l'_{13} = 1 < \frac{7}{5} = \hat{l}_{13}$), and that the DM would be inconsistent if he preferred the second alternative to the third ($\hat{u}_{23} = \frac{4}{7} < 1 = l'_{23}$). From (15) and (19) the ambiguity index of the feasible region in Fig. 6 is found as

$$\mathrm{AI}(S_7) = \frac{2}{3 \times 2} \left(\frac{20 - 7}{(20+1) \times (7+1)} + \frac{4 - \frac{7}{5}}{(4+1) \times (\frac{7}{5}+1)} + \frac{\frac{4}{7} - \frac{1}{5}}{(\frac{4}{7}+1) \times (\frac{1}{5}+1)} \right) = 0.16.$$

For each pairwise comparison the interval judgments I and the consistency intervals $\hat{I}$ contain up to four different bounds some of which may not be integers in the one to nine range. Instead of the two separate matrix representations these bounds can be displayed together as

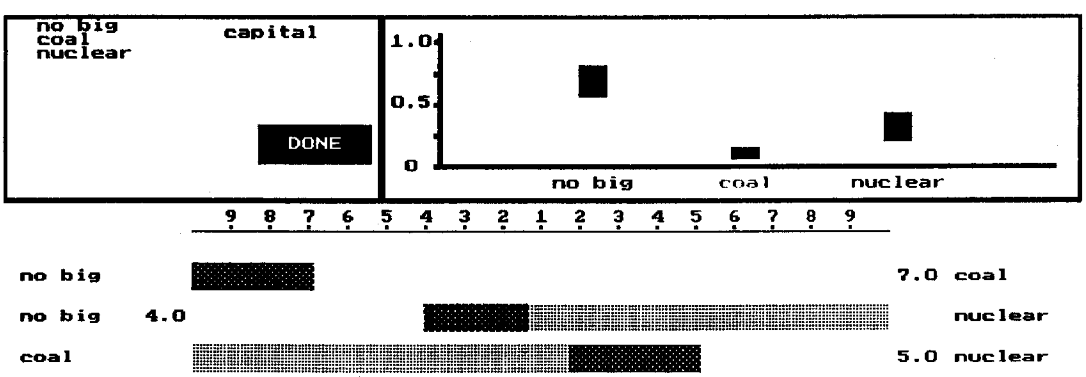

Fig. 7. Interval judgments and consistency bounds.

Table 2
Absolute bounds $\bar{v}$ and $\underline{v}$ at third level criteria

Alt.	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
$\bar{v}$;									
A_1	0.71	0.42	0.77	0.33	0.25	0.63	0.79	0.84	0.71
A_2	0.22	0.71	0.10	0.33	0.77	0.09	0.33	0.10	0.17
A_3	0.20	0.11	0.38	0.33	0.14	0.43	0.14	0.25	0.17
$\underline{v}$:									
A_1	0.60	0.22	0.54	0.33	0.14	0.50	0.56	0.67	0.67
A_2	0.14	0.50	0.04	0.33	0.63	0.05	0.14	0.04	0.14
A_3	0.13	0.05	0.18	0.33	0.08	0.30	0.05	0.11	0.14

horizontal bars. For instance, in Fig. 7 the intervals I in (18) are shown as the outer, shaded parts of the bars, and the inner darker parts correspond to the consistency intervals $\hat{I}$ in (19).

7.1. Phase one of preference analysis

From the interval judgments of Table 1, the weight intervals for the alternatives are found by first computing the absolute bounds at the third level of the hierarchy (see Table 2). For example, solving (5)–(6) over the feasible region in Fig. 6 gives the bounds

$$\bar{v}_7(A_1) = \max_{w \in S_7} w_1 = 0.77,$$

$$\underline{v}_7(A_1) = \min_{w \in S_7} w_1 = 0.54,$$

$$\bar{v}_7(A_2) = \max_{w \in S_7} w_2 = 0.10,$$

$$\underline{v}_7(A_2) = \min_{w \in S_7} w_2 = 0.04,$$

$$\bar{v}_7(A_3) = \max_{w \in S_7} w_3 = 0.38,$$

$$\underline{v}_7(A_3) = \min_{w \in S_7} w_3 = 0.18.$$

Because no judgments have been made on the higher levels of the hierarchy, the feasible regions of the first and second level criteria contain the local priority vectors (1, 0, 0), (0, 1, 0), (0, 0, 1). As a result, the bounds for the alternatives' weight intervals

$$V(A_1) = [0.14, 0.84], \tag{20a}$$

$$V(A_2) = [0.04, 0.77], \tag{20b}$$

$$V(A_3) = [0.05, 0.43], \tag{20c}$$

are found as the maximum and minimum elements on the rows of Table 2. Since intervals overlap absolute dominance does not hold for any pair of alternatives. Solving the minimization problem (10) to check the possible pairwise dominance of the first alternative over the second and the third gives the pairwise bounds of Table 3. The zeros under the column C_8 show that the first alternative is no better than the others with respect to the criterion natural resources. But since C_8 can receive all the weight from the topmost criterion at this point, pairwise dominance does not hold yet.

7.2. Phase two of preference analysis

Suppose that at the second level criteria the DM makes the preference statements

Table 3
Pairwise dominance π at third level criteria

Alt.	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
A_1, A_2	0.40	−0.48	0.46	0.00	−0.62	0.43	0.22	0.58	0.50
A_1, A_3	0.40	0.11	0.15	0.00	0.00	0.07	0.44	0.42	0.50

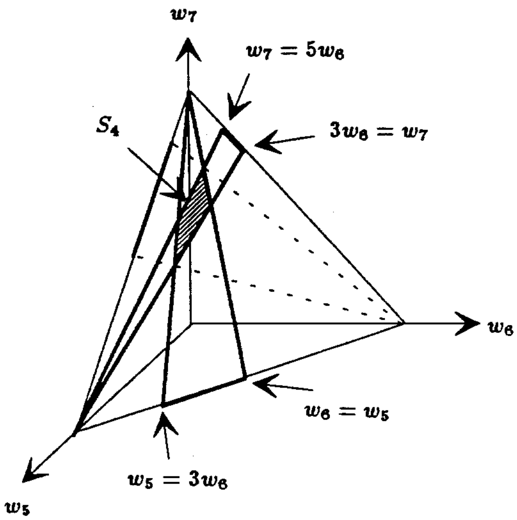

Fig. 8. The feasible region at criterion economic factors.

• W.r.t. economic factors, capital resources are three to five times more important than foreign trade, and cheap electricity is up to three times more important than foreign trade.

• W.r.t. to environmental factors, accidents are up to three times more important than pollution, and at least five times more important than the use of natural resources.

• W.r.t. political factors, centralization is up to two times more important than independence, and four to six times more important than co-operation.

In the computation of revised weight intervals, the absolute bounds are updated first at the second level criteria (see Table 4) and then at the topmost criterion. For example, at criterion economic factors the above interval judgments $I_{76} = [3, 5]$, $I_{56} = [1, 3]$ impose the constraints $3w_6 \le w_7 \le 5w_6$ and $w_6 \le w_5 \le 3w_6$. Thus the feasible region S_2 has the extreme points (0.14, 0.14, 0.71), (0.20, 0.20, 0.60), (0.43, 0.14, 0.43), (0.33, 0.11, 0.56), which together with Table 2 give the absolute bounds

$$\bar{v}_2(A_1) = \max_{w \in S_2} \sum_{C_i \in C_2^-} \bar{v}_i(A_1) w_i$$
$$= 0.71 \times 0.14 + 0.42 \times 0.14 + 0.77 \times 0.71$$
$$= 0.71,$$

$$\underline{v}_2(A_1) = \min_{w \in S_2} \sum_{C_i \in C_2^-} \underline{v}_i(A_1) w_i$$
$$= 0.60 \times 0.20 + 0.22 \times 0.20 + 0.54 \times 0.60$$
$$= 0.49.$$

Since no judgments have been made at the topmost criterion, the solutions to (7)–(8) at the top of the hierarchy are the maximum and minimum entries in the rows of Table 4, i.e.

$$V(A_1) = [0.32, 0.81], \tag{21a}$$
$$V(A_2) = [0.08, 0.43], \tag{21b}$$
$$V(A_3) = [0.09, 0.36]. \tag{21c}$$

Combining the extreme points of S_2 with the second row of Table 3 gives the revised pairwise bound

$$\pi_2(A_1, A_3)$$
$$= \min_{w \in S_2} \sum_{C_i \in C_2^-} \pi_i(A_1, A_3) w_i$$
$$= 0.40 \times 0.14 + 0.11 \times 0.14 + 0.15 \times 0.71$$
$$= 0.18 > 0.$$

The other second level pairwise bounds in Table 5 are positive, too, and consequently the big power plant alternative is preferred to the nuclear power plant. On the other hand, the negative pairwise bound

$$\pi_1(A_1, A_2) = \pi_3(A_1, A_2) = -0.09$$

indicates that the second alternative can receive

Table 4
Absolute bounds $\bar{v}$ and $\underline{v}$ at second level criteria

Alt.	$\bar{v}$			$\underline{v}$		
	C_2	C_3	C_4	C_2	C_3	C_4
A_1	0.71	0.54	0.81	0.49	0.32	0.62
A_2	0.25	0.43	0.21	0.12	0.20	0.08
A_3	0.32	0.36	0.21	0.14	0.19	0.09

Table 5
Pairwise bounds π at second level criteria

Alt.	c_2	C_3	C_4
A_1, A_2	0.25	−0.09	0.41
A_1, A_3	0.18	0.03	0.43

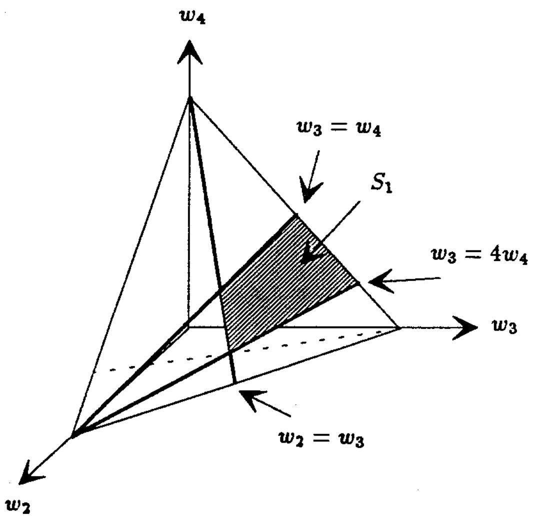

Fig. 9. The feasible region at the topmost criterion.

more weight than the first so that pairwise dominance does not hold for these alternatives.

7.3. Phase three of preference analysis

Finally, assume that the DM considers environmental factors to be more important than economic factors, and up to four times more important than political factors. These statements define the feasible region in Fig. 9 which is characterized by the linear constraints $w_2 \le w_3$ and $w_4 \le w_3 \le 4w_4$ and the extreme points (0.00, 0.50, 0.50), (0.00, 0.80, 0.20), (0.33, 0.33, 0.33), (0.44, 0.44, 0.11).

The revised weight intervals are computed by using the entries of Table 4 as coefficients in the linear programs (7)–(8) over the modified feasible region at the topmost criterion. For example, the bounds for the weight of the first alternative are

$$\bar{v}_1(A_1) = \max_{w \in S_1} \sum_{C_i \in C_1^-} \bar{v}_i(A_1) w_i$$
$$= 0.71 \times 0.33 + 0.54 \times 0.33 + 0.81 \times 0.33$$
$$= 0.69,$$
$$\underline{v}_1(A_1) = \min_{w \in S_1} \sum_{C_i \in C_1^-} \underline{v}_i(A_1) w_i$$
$$= 0.49 \times 0.00 + 0.32 \times 0.80 + 0.62 \times 0.20$$
$$= 0.38,$$

and a similar analysis of the two other alternatives gives the weight intervals

$$V(A_1) = [0.38, 0.69],$$
$$V(A_2) = [0.13, 0.38],$$
$$V(A_3) = [0.14, 0.33].$$

The weight interval $V(A_1)$ lies above $V(A_3)$ and thus the first alternative dominates the third according to both dominance concepts. Although the intervals $V(A_1)$ and $V(A_2)$ overlap, the first, no big power plant alternative dominates the second one as well, because minimizing (11) over S_1 using the first row of Table 5 gives the positive pairwise bound

$$\pi_1(A_1, A_2)$$
$$= \min_{w \in S_1} \sum_{C_i \in C_1^-} \pi_i(A_1, A_2) \omega_i$$
$$= 0.25 \times 0.00 - 0.09 \times 0.80 + 0.41 \times 0.20$$
$$= 0.01 > 0. \tag{22}$$

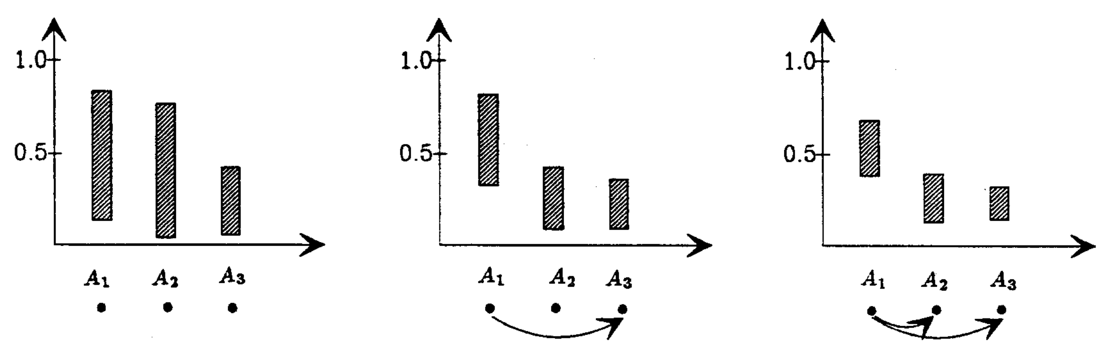

Fig. 10. Changes in weight intervals and pairwise dominance.

Fig. 10 shows the weight intervals and the pairwise dominance relation in the three phases of preference elicitation.

In the last phase the coal plant alternative A_2 achieves its largest weight 0.38 only at the extreme point (0.00, 0.80, 0.20) of the feasible region S_1. Thus, by Proposition 1, the maximum weight for the coal plant decreases if the DM goes on to tighten the bound $u_{34} = 4$. In fact, if this bound is set to three, the revised weight intervals become $V(A_1) = [0.39, 0.69]$ and $V(A_2) = [0.13, 0.37]$; thus the second alternative becomes absolutely dominated by the first one.

In this example the most preferred alternative was found even though many of the interval judgments were left unspecified. For instance, no judgments about the relative importance of economic and political factors were made at the topmost criterion, and at the lowest criterion level only one of the bounds was specified for most interval judgments. This illustrates that preference programming can indeed reduce the effort of preference elicitation.

8. Conclusion

The proposed approach to preference programming is based on computationally efficient mechanisms for processing interval-valued ratio comparisons in hierarchical weighting. The DM's statements are interpreted as linear constraints which after each new preference statement are processed into revised results about nondominated alternatives. These results become gradually more informative as the DM specifies the preference model in increasing detail. The consequences of the earlier judgments are also presented throughout the analysis to help the DM enter statements that lead to more conclusive results.

In comparison to the conventional AHP, the interactive decision support derived from preference programming supplies information about the desirability of the alternatives already in the early stages of the analysis. This feature is likely to reduce the amount of comparison effort since the preferred option can typically be found before all the pairwise comparisons have been completed. On the other hand, in a group setting the DMs can profit from the interval description by specifying aggregate statements that are loose enough to contain their individual judgments. In view of these advantages, it appears that the interactive analysis of interval judgments, as described in this paper, holds substantial promise for real-life applications.

9. Appendix

Proof of Theorem 1. $V(x)$ is a closed set because $v(x)$ is a continuous function of the local priorities w^y in the compact sets S_y, $y \in \boldsymbol{C}$. To prove that it is convex, choose $\overline{w}^y$ from S_y such that $v(x)$ is maximized and $\underline{w}^y$ such that $v(x)$ is minimized. For $z \in \{x\} \cup \boldsymbol{C}$ define the functions $f_z : [0, 1] \to [0, 1]$ recursively by $f_b(t) = 1$ and

$$f_z(t) = \sum_{y \in z^+} \left[t\overline{w}_z^y + (1-t)\underline{w}_z^y \right] f_y(t).$$

Fix

$$r \in \left[\min_{p \in V(x)} p, \ \max_{q \in V(x)} q \right] = \left[f_x(0), f_x(1) \right].$$

Since f_x is continuous there exists $t^r \in [0, 1]$ such that $f_x(t^r) = r$. The local priorities

$$t^r \overline{w}^y + (1 - t^r) \underline{w}^y,$$

which by the convexity of S_y are feasible, give the weight r to x.

To prove that the bounds $\overline{v}_b(x)$, $\underline{v}_b(x)$ are tight, fix $w^y \in S_y$, $y \in \boldsymbol{C}$. Then for $1 < i \le h - 1$,

$$\begin{aligned} \sum_{y \in L_i} v(y) \overline{v}_y(x) &= \sum_{y \in L_i} \left[\sum_{z \in y^+} v(z) w_y^z \right] \overline{v}_y(x) \\ &= \sum_{\substack{z \in L_{i-1} \\ z^- \not\subset A}} v(z) \left[\sum_{y \in z^-} \overline{v}_y(x) w_y^z \right] \\ &\le \sum_{\substack{z \in L_{i-1} \\ z^- \not\subset A}} v(z) \overline{v}_z(x). \end{aligned}$$

Applying this inequality repeatedly gives

$$v(x) = \sum_{y \in x^+} v(y) w_x^y$$

$$\le \sum_{y \in L_{h-1}} v(y)\bar{v}_y(x) + \sum_{\substack{y \in x^+ \\ L(y)<h-1}} v(y)\bar{v}_y(x)$$

$$\le \sum_{\substack{y \in L_{h-2} \\ y^- \not\subset A}} v(y)\bar{v}_y(x) + \sum_{\substack{y \in x^+ \\ L(y)<h-1}} v(y)\bar{v}_y(x)$$

$$= \sum_{y \in L_{h-2}} v(y)\bar{v}_y(x) + \sum_{\substack{y \in x^+ \\ L(y)<h-2}} v(y)\bar{v}_y(x)$$

$$\vdots$$

$$\le \sum_{y \in L_1} v(y)\bar{v}_y(x) = v(b)\bar{v}_b(x) = \bar{v}_b(x).$$

For those local priorities which maximize (5) and (7) the above inequalities become equalities, and thus

$$\max_{r \in V(x)} r = \bar{v}_b(x).$$

The proof for the lower bound is similar. □

Proof of Theorem 2. Fix $w^z \in S_z$, $z \in C$. Then for $1 < i \le h-1$,

$$\sum_{z \in L_i} v(z)\pi_z(x, y)$$

$$= \sum_{z \in L_i} \left[\sum_{t \in z^+} v(t) w_z^t \right] \pi_z(x, y)$$

$$= \sum_{\substack{t \in L_{i-1} \\ t^+ \not\subset A}} v(t) \left[\sum_{z \in t^-} \pi_z(x, y) w_z^t \right]$$

$$\ge \sum_{\substack{t \in L_{i-1} \\ t^- \not\subset A}} v(t)\tau_t(x, y).$$

Thus

$$v(x) - v(y)$$

$$= \sum_{z \in x^+} v(z) w_x^z - \sum_{z \in y^+} v(z) w_y^z$$

$$\ge \sum_{\substack{z \in C \\ z^- \subset A}} v(z)\pi_z(x, y)$$

$$= \sum_{z \in L_{h-1}} v(z)\pi_z(x, y) + \sum_{\substack{L(z)<h-1 \\ z^- \subset A}} v(z)\pi_z(x, y)$$

$$\ge \sum_{\substack{z \in L_{h-2} \\ z^- \not\subset A}} v(z)\pi_z(x, y) + \sum_{\substack{L(z)<h-1 \\ z^- \subset A}} v(z)\pi_z(x, y)$$

$$= \sum_{z \in L_{h-2}} v(z)\pi_z(x, y) + \sum_{\substack{L(z)<h-2 \\ z^+ \subset A}} v(z)\pi_z(x, y)$$

$$\vdots$$

$$\ge \sum_{z \in L_1} v(z)\pi_z(x, y) = \pi_b(x, y).$$

For those local priorities which minimize (10)–(11) the above inequalities become equalities. This completes the proof. □

Proof of Proposition 1. $\Rightarrow$: If $\exists\, w \in \text{ext } S$ such that $g_z(w) = \bar{g}_z$ and $w_x/w_y < \hat{u}_{xy}$, then the constraint u_{xy} can be tightened to some $u'_{xy} < u_{xy}$ so that $w_x \le u'_{xy} w_y$. But then $w \in S'$ and

$$\max_{w \in S'_z} g_z(w) = \bar{g}_z.$$

$\Leftarrow$: Let $\hat{l}_{xy} \le u'_{xy} < \hat{u}_{xy}$ and assume that $\exists\, w \in S' \subseteq S$ such that $g_z(w) = \bar{g}_z$. Since S is a polytope $\exists\, \lambda_k > 0$, $w^k \in \text{ext } S$ such that

$$\sum_{k=1}^{K} \lambda_k = 1 \quad \text{and} \quad w = \sum_{k=1}^{K} \lambda_k w^k.$$

From the linearity of $g_z(\cdot)$ and $g_z(w^k) \le \bar{g}_z$ it follows that $g_z(w^k) = \bar{g}_z$. By assumption $w_x^k/w_y^k = \hat{u}_{xy}$ so that $w_x^k > u'_{xy} w_y^k$ for both finite and infinite $\hat{u}_{xy}$. Multiplying these inequalities by λ_k and summing them gives $w_x > u'_{xy} w_y$, which implies $w \notin S'$, a contradiction. □

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