A General Unified Framework for Pairwise Comparison Matrices in Multicriterial Methods

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In a multicriteria decision making context, a pairwise comparison matrix $A = (a_{ij})$ is a helpful tool to determine the weighted ranking on a set X of alternatives or criteria. The entry a_{ij} of the matrix can assume different meanings: a_{ij} can be a preference ratio (multiplicative case) or a preference difference (additive case) or a_{ij} belongs to [0, 1] and measures the distance from the indifference that is expressed by 0.5 (fuzzy case). For the multiplicative case, a consistency index for the matrix A has been provided by T.L. Saaty in terms of maximum eigenvalue. We consider pairwise comparison matrices over an abelian linearly ordered group and, in this way, we provide a general framework including the mentioned cases. By introducing a more general notion of metric, we provide a consistency index that has a natural meaning and it is easy to compute in the additive and multiplicative cases; in the other cases, it can be computed easily starting from a suitable additive or multiplicative matrix.

1. INTRODUCTION

A crucial step in a decision making process is the determination of a weighted ranking on a set $X = \{x_1, x_2, \ldots, x_n\}$ of alternatives with respect to criteria or experts. A way to determine the weighted ranking is to start from a relation
$\mathcal{A} : (x_i, x_j) \in X \times X \to a_{ij} = \mathcal{A}(x_i, x_j) \in G \subseteq R$ represented by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \tag{1.1}$$

that is called *pairwise comparison matrix* (PC matrix for short): a_{ij} expresses how much x_i is preferred to x_j and a condition of *reciprocity* is assumed in such way

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that the preference of x_i over x_j expressed by a_{ij} can be exactly read by means of the element a_{ji}. Under a suitable condition of *consistency*, X is totally ordered by $\mathcal{A}$ and there exists a vector $\underline{w}$, that perfectly represents the preferences over X. The reciprocity and consistency conditions depend on the different meaning given to the number a_{ij} as the following examples of PC matrices show.

1. **Multiplicative PC matrix**. $a_{ij} \in]0, +\infty[$ represents the preference ratio of x_i over x_j: $a_{ij} > 1$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < 1$ expresses the opposite preference and $a_{ij} = 1$ means that x_i and x_j are indifferent. Then, the condition of reciprocity is

   $$\textbf{mr)} \quad a_{ji} = \frac{1}{a_{ij}} \quad \forall\, i, j = 1, \ldots, n \qquad \textit{(multiplicative reciprocity)},$$

   so, $a_{ii} = 1$ for each $i = 1, 2, \ldots, n$. The consistency condition is given by

   $$\textbf{mc)} \quad a_{ik} = a_{ij}a_{jk} \quad \forall\, i, j, k = 1, \ldots, n \qquad \textit{(multiplicative consistency)}.$$

   The matrix $A = (a_{ij})$ is consistent if and only if there is a positive vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $\frac{w_i}{w_j} = a_{ij}$.
2. **Additive PC matrix**. $a_{ij} \in]-\infty, +\infty[$ represents the difference of preference between x_i and x_j: $a_{ij} > 0$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < 0$ expresses the opposite preference and $a_{ij} = 0$ means that x_i and x_j are indifferent. Then, the condition of reciprocity is

   $$\textbf{ar)} \quad a_{ji} = -a_{ij} \quad \forall\, i, j = 1, \ldots, n \qquad \textit{(additive reciprocity)},$$

   thus, $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$. The consistency condition is given by

   $$\textbf{ac)} \quad a_{ik} = a_{ij} + a_{jk} \quad \forall\, i, j, k = 1, \ldots, n \qquad \textit{(additive consistency)}.$$

   The matrix $A = (a_{ij})$ is consistent if and only if there is a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $w_i - w_j = a_{ij}$.
3. **Fuzzy PC matrix**. $a_{ij} \in [0, 1]$: $a_{ij} > 0.5$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < 0.5$ expresses the opposite preference and $a_{ij} = 0.5$ means that x_i and x_j are indifferent. Then, the condition of reciprocity is

   $$\textbf{fr)} \quad a_{ji} = 1 - a_{ij} \quad \forall\, i, j = 1, \ldots, n \qquad \textit{(fuzzy reciprocity)},$$

   thus, $a_{ii} = 0.5$ for all $i = 1, 2, \ldots, n$. The consistency condition is given by

   $$\textbf{fc)} \quad a_{ik} = a_{ij} + a_{jk} - 0.5 \quad \forall\, i, j, k = 1, \ldots, n \qquad \textit{(fuzzy consistency)}.$$

   The matrix $A = (a_{ij})$ is consistent if and only if there is a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $w_i - w_j = a_{ij} - 0.5$.

The multiplicative PC matrices play a basic role in the Analytic Hierarchy Process, a procedure developed by Saaty at the end of the 70s,[1,2] and widely used by governments and companies[2–4] in fixing their strategies. Saaty indicates a scale translating the comparisons expressed in verbal terms into the preference ratios a_{ij}. By applying this scale, a_{ij} may only take value in $S^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$. The assumption of the Saaty scale restricts the decision maker's possibility to be consistent: indeed if the decision maker expresses the following preference ratios $a_{ij} = 5$ and $a_{jk} = 3$ then he will not be consistent because $a_{ij}a_{jk} = 15 > 9$. The assumption of any limited and closed set of values presents the same drawback for each one of the considered PC matrices. In particular, under the assumption that $a_{ij} \in [0, 1]$, the consistency property **fc** cannot be respected, for instance, by a decision maker who claims $a_{ij} = 0.9$ and $a_{jk} = 0.8$, because $a_{ij} + a_{jk} - 0.5 = 1.7 - 0.5 > 1$.

A measure of closeness to the consistency for a multiplicative PC matrix has been provided by Saaty[2,5] in terms of the principal eigenvalue $\lambda_{\max}$:

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1} \qquad (\textit{consistency index}),$$

and the right eigenvector $\underline{w}_{\lambda_{\max}} = (w_1, w_2, \ldots, w_n)$ associated to $\lambda_{\max}$ has been considered as weighting vector. Saaty[5] shows that the more CI is close to 0, the more the ratios $\frac{w_i}{w_j}$ are close to the preference ratios a_{ij}: so enough small values of CI would ensure a good representation of the preferences over X by means of $\underline{w}_{\lambda_{\max}}$.

To get a weighted ranking, other methods have also been considered by scholars; for example, weighted rankings are obtained by applying the arithmetic or geometric mean operators to the rows of the multiplicative PC matrix.[2,6,7]

The consistency index CI has been questioned because it is not easy to compute, has not a simple and geometric meaning[8,9] and, in some cases, seems to be unfair.[10] Also, the methods used to provide a weighted ranking have been questioned: indeed they may indicate rankings that do not agree with the expressed preference ratios a_{ij}.[11–15]

The aim of the present study is to define a general context in which different approaches to a PC matrix can be unified and provide a meaningful consistency index suitable for each type of matrix. The definitions of reciprocity and consistency in the multiplicative or additive case imply only an operation and its inverse (the multiplication and the division for a multiplicative PC matrix, the addition and the difference for an additive PC matrix): so in the study the set G, on which the relation $\mathcal{A}$ takes its values, is embodied only with a commutative group operation $\odot$ and a total order $\leq$ compatible with the operation; G is not necessary a real subset. The reciprocity and consistency conditions are expressed in terms of the group operation $\odot$ and a notion of distance $d_{\mathcal{G}}$, linked to the abelian linearly ordered group $\mathcal{G} = (G, \odot, \leq)$, is introduced (see Section 3). The assumption of *divisibility* for $\mathcal{G}$ allows to introduce the mean $m_{\odot}(a_1, \ldots, a_n)$ of n elements (see Section 2.1) and associate a *mean vector* $\underline{w}_{m_{\odot}}$ to a PC matrix $A = (a_{ij})$ (see Section 5). By using the mean operator $m_{\odot}$ and the distance $d_{\mathcal{G}}$, a consistency index $I_{\mathcal{G}}(A)$ for the matrix

A is also provided (see Section 6). $I_{\mathcal{G}}(A)$ is equal to the identity element of $\odot$ if and only if $A = (a_{ij})$ is consistent (see Section 6) and, in this case, the mean vector $\underline{w}_{m_\odot}$ provides weights $w_1, w_2, \ldots, w_n$ for the alternatives perfectly agreeing with the entries a_{ij}: indeed it results $w_i \div w_j = a_{ij} \quad \forall i, j = 1, 2, \ldots, n$, where $\div$ is the inverse of $\odot$. Moreover, for $n = 3$, in case of inconsistency the closeness of the elements $w_i \div w_j$ to the entries a_{ij} of the PC matrix can be expressed in terms of the consistency index $I_{\mathcal{G}}(A)$ (see Section 6.1). In this way, the study generalizes the multiplicative and the additive cases and finds, for these cases, a consistency index easy to compute and naturally grounded on a notion of distance. Moreover, if G is a real open interval, then the consistency index can be obtained by computing the consistency index of a suitable multiplicative or additive PC matrix.

In this approach, the definition of fuzzy consistency is modified in such way that the underlying operation is a group operation (see Proposition 4.2 and Remark 5.1) and the shown drawback, related to the possibility to build a consistent matrix, is removed.

2. ABELIAN LINEARLY ORDERED GROUPS

In this section, we recall some notions and properties related to abelian linearly ordered groups.

DEFINITION 2.1. *Let G be a nonempty set, $\odot : G \times G \to G$ a binary operation on G, $\leq$ a total weak order on G. Then $\mathcal{G} = (G, \odot, \leq)$ is an* abelian linearly ordered group*,* alo-group *for short, if and only if $(G, \odot)$ is an abelian group and*

$$a \leq b \Rightarrow a \odot c \leq b \odot c. \tag{2.1}$$

As an abelian group satisfies the cancellative law "$a \odot c = b \odot c \Leftrightarrow a = b$," Equation 2.1 is equivalent to the strict monotonicity of $\odot$ in each variable:

$$a < b \Leftrightarrow a \odot c < b \odot c. \tag{2.2}$$

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, we will indicate by:

- e the identity of $\mathcal{G}$,
- $x^{(-1)}$ the symmetric of $x \in G$ with respect to $\odot$,
- $\div$ the inverse operation of $\odot$ defined by $a \div b = a \odot b^{(-1)}$,
- $<$ the strict simple order defined by "$x < y \leftrightarrow x \leq y$ *and* $x \neq y$",
- $\geq$ and $>$ the opposite relations of $\leq$ and $<$, respectively.

Then

$$b^{(-1)} = e \div b, \quad (a \odot b)^{(-1)} = a^{(-1)} \odot b^{(-1)}, \quad (a \div b)^{(-1)} = b \div a; \tag{2.3}$$

moreover, assuming that G is no trivial, that is $G \neq \{e\}$, by Equation 2.2 we get

$$a < e \Leftrightarrow a^{(-1)} > e, \;\; a > e \Leftrightarrow e > a^{(-1)},$$

$$a \odot a > a \quad \forall a > e, \;\; a \odot a < a \quad \forall a < e. \tag{2.4}$$

If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then G is naturally equipped with the order topology induced by $\leq$ and $G \times G$ is equipped with the related product topology. We say that $\mathcal{G}$ is a *continuous* alo-group if and only if $\odot$ is continuous.

By definition, an alo-group $\mathcal{G}$ is a *lattice ordered group*,[16] that is there exists $a \vee b = max\{a, b\}$, for each pair $(a, b) \in G^2$. Nevertheless, by Equation 2.4, we get the following proposition.

PROPOSITION 2.1. *A nontrivial alo-group $\mathcal{G} = (G, \odot, \leq)$ has neither the greatest element nor the least element.*

Remark 2.1. By Proposition 2.1, neither the interval [0, 1] nor the Saaty set $S^* = \{1, 2, \ldots, 9, \frac{1}{2}, \frac{1}{3} \ldots, \frac{1}{9}\}$, embodied with the usual order $\leq$ on R, can be structured as linearly ordered group.

(n)-powers. Because of the associative property, the operation $\odot$ can be extended by induction to n-ary operation, $n > 2$, by setting

$$\bigodot_{i=1}^{n} x_i = \left(\bigodot_{i=1}^{n-1} x_i \right) \odot x_n. \tag{2.5}$$

Then, for a positive integer n, the *(n)- power* $x^{(n)}$ of $x \in G$ is defined by

$$\begin{cases} x^{(1)} = x \\ x^{(n)} = \bigodot_{i=1}^{n} x_i, \;\; x_i = x \forall i = 1, \ldots, n, \quad \text{for} \quad n \geq 2, \end{cases}$$

and verifies the following properties:

$$x^{(n)} \odot x^{(m)} = x^{(n+m)} = x^{(m)} \odot x^{(n)}, \quad (x^{(n)})^{(m)} = x^{(nm)} = (x^{(m)})^{(n)}, \tag{2.6}$$

$$x^{(n)} \odot y^{(n)} = (x \odot y)^{(n)}. \tag{2.7}$$

By the properties in Equations 2.2 and 2.4, we can get by induction

$$x < y \Leftrightarrow x^{(n)} < y^{(n)},$$

$$a^{(n)} > a \quad \forall a > e, \qquad a^{(n)} < a \quad \forall a < e. \tag{2.8}$$

We can extend the meaning of power $x^{(s)}$ to the case that s is a relative integer by setting

$$x^{(0)} = e \quad and \quad x^{(-n)} = \left(x^{(n)}\right)^{(-1)}. \tag{2.9}$$

By Equations 2.9 and 2.7, $x^{(n)} \odot x^{(-n)} = e = (x \odot x^{(-1)})^{(n)} = x^{(n)} \odot (x^{(-1)})^{(n)}$, so

$$x^{(-n)} = \left(x^{(-1)}\right)^{(n)}. \tag{2.10}$$

As a consequence, the properties in Equations 2.6 and 2.7 are satisfied for all integers m, n and, as particular case, we have

$$(a \div b)^{(n)} = \left(a \odot b^{(-1)}\right)^{(n)} = a^{(n)} \odot (b^{(n)})^{(-1)} = a^{(n)} \div b^{(n)}. \tag{2.11}$$

Isomorphism between alo-groups. An *isomorphism* between two alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ is a bijection $h : G \to G'$ that is both a lattice isomorphism and a group isomorphism, that is,

$$x < y \Leftrightarrow h(x) < h(y) \quad and \quad h(x \odot y) = h(x) \circ h(y). \tag{2.12}$$

Thus, $h(e) = e'$, where e' is the identity in $\mathcal{G}'$, and

$$h\left(x^{(-1)}\right) = (h(x))^{(-1)}. \tag{2.13}$$

By applying the inverse isomorphism $h^{-1} : G' \to G$, we get

$$h^{-1}(x' \circ y') = h^{-1}(x') \odot h^{-1}(y'), \quad h^{-1}\left(x'^{(-1)}\right) = (h^{-1}(x'))^{(-1)}. \tag{2.14}$$

By the associativity of the operations $\odot$ and $\circ$, the equality in Equation 2.12 can be extended by induction to the n-operation $\bigodot_{i=1}^{n} x_i$, so that

$$h\left(\bigodot_{i=1}^{n} x_i\right) = \bigcirc_{i=1}^{n} h(x_i), \quad h\left(x^{(n)}\right) = h(x)^{(n)}. \tag{2.15}$$

2.1. Divisible Alo-Group, (*n*)-Roots and Mean Operator

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. By properties in Equation 2.8, for every positive integer n and every $a \in G$ there exists at most a solution $x \in G$ of the equation $x^{(n)} = a$. So, if there exists a solution b of the equation $x^{(n)} = a$, then this is the only one. Hence, we give the following definition:

Definition 2.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. If $b^{(n)} = a$, then we say that b is the* (n)-root *of a and write $b = a^{(1/n)}$.*

Definition 2.3. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, $\mathcal{G}$ is* divisible *if and only if for each positive integer n and each $a \in G$ there exists the (n)-root of a.*

Proposition 2.2. *The (n)-root verifies the following properties:*

$$(a \odot b)^{(\frac{1}{n})} = a^{(\frac{1}{n})} \odot b^{(\frac{1}{n})}, \quad \left(a^{(-1)}\right)^{(\frac{1}{n})} = \left(a^{(\frac{1}{n})}\right)^{(-1)}, \tag{2.16}$$

$$a < b \Rightarrow a^{(1/n)} < b^{(1/n)}. \tag{2.17}$$

Proof. By Equation 2.7, $(a^{(\frac{1}{n})} \odot b^{(\frac{1}{n})})^{(n)} = (a^{(\frac{1}{n})})^n \odot (b^{(\frac{1}{n})})^n = a \odot b$ and so the first equality in Equation 2.16 is achieved. The second equality is also achieved, since $e = (a \odot a^{(-1)})^{(\frac{1}{n})} = a^{(\frac{1}{n})} \odot (a^{(-1)})^{(\frac{1}{n})}$. Finally, Equation 2.17 follows from Equation 2.8. ■

Definition 2.4 *Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible alo-group. Then, the $\odot$-* mean *$m_{\odot}(a_1, a_2, \ldots, a_n)$ of the elements $a_1, a_2, \ldots, a_n$ of G is defined by*

$$m_{\odot}(a_1, a_2, \ldots, a_n) = \begin{cases} a_1 & \text{for} \quad n = 1, \\ (\bigodot_{i=1}^{n} a_i)^{(1/n)} & \text{for} \quad n \geq 2. \end{cases}$$

In the sequel, for sake of simplicity, we say *mean* instead of $\odot$- *mean*.

Proposition 2.3. *Let $h : G \to G'$ be an isomorphism between the alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$. Then, $\mathcal{G}$ is divisible if and only if $\mathcal{G}'$ is divisible. Moreover, under the assumption of divisibility:*

$$m_{\odot}(x_1, x_2, \ldots, x_n) = h^{-1}\big(m_{\circ}(h(x_1), h(x_2), \ldots, h(x_n))\big) \tag{2.18}$$

$$m_{\circ}(y_1, y_2, \ldots, y_n) = h\big(m_{\odot}(h^{-1}(y_1), h^{-1}(y_2), \ldots, h^{-1}(y_n))\big). \tag{2.19}$$

Proof. Let us set, for $x, x_i, a \in G$: $y = h(x)$, $y_i = h(x_i)$ and $b = h(a)$. By Equation 2.15, $x^{(n)} = a \Leftrightarrow y^{(n)} = b$, and so $\mathcal{G}$ is divisible if and only if $\mathcal{G}'$ is divisible.

Assume now that $\mathcal{G}$ and $\mathcal{G}'$ are divisible. Then, by Equation 2.15,

$$x^{(n)} = \bigodot_{i=1}^{n} x_i \Leftrightarrow h(x^{(n)}) = h\left(\bigodot_{i=1}^{n} x_i\right) \Leftrightarrow (h(x))^{(n)} = \bigcirc_{i=1}^{n} h(x_i).$$

Hence, $x = m_{\odot}(x_1, \ldots, x_n)$ if and only if $h(x) = m_{\circ}(h(x_1), \ldots, h(x_n))$ and Equation 2.18 is achieved. Equation 2.19 follows from Equation 2.18. ■

3. $\mathcal{G}$-METRIC

Following Ref. 17, we give the following definition of norm:

DEFINITION 3.1. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the function:*

$$||\cdot|| : a \in G \to ||a|| = a \vee a^{(-1)} \in G \tag{3.1}$$

*is a $\mathcal{G}$-*norm*, or a* norm *on $\mathcal{G}$.*

PROPOSITION 3.1. *The $\mathcal{G}$-norm satisfies the properties:*

1. $||a|| = ||a^{(-1)}||$;
2. $a \leq ||a||$;
3. $||a|| \geq e$;
4. $||a|| = e \Leftrightarrow a = e$;
5. $||a^{(n)}|| = ||a||^{(n)}$;
6. $||a \odot b|| \leq ||a|| \odot ||b||$ (triangle inequality).

Proof. Items 1, 2, 3, 4 follow immediately from Definition 3.1. Item 5 follows by Equation 2.8 for which $a = x \vee x^{(-1)}$ if and only if $a^{(n)} = x^{(n)} \vee x^{(-1)^{(n)}}$. By Equation 2.1 and item 2, $a \odot b \leq ||a|| \odot ||b||$; so by item 1 and the second equality in Equation 2.3, the triangle inequality follows. ■

DEFINITION 3.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d : (a, b) \in G^2 \to d(a, b) \in G$$

*is a $\mathcal{G}$-*metric *or $\mathcal{G}$-*distance *if and only if:*

1. $d(a, b) \geq e$;
2. $d(a, b) = e \Leftrightarrow a = b$;
3. $d(a, b) = d(b, a)$;
4. $d(a, b) \leq d(a, c) \odot d(b, c)$.

PROPOSITION 3.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d_{\mathcal{G}} : (a, b) \in G^2 \to d_{\mathcal{G}}(a, b) = ||a \div b|| \in G \tag{3.2}$$

is a $\mathcal{G}$-distance.

Proof. The conditions 1, 2, and 3 are verified by $d_{\mathcal{G}}$ as consequence of the properties 3, 4, and 1 of the $\mathcal{G}$-norm and the equality $(a \div b)^{(-1)} = b \div a$. By applying the triangle inequality of the $\mathcal{G}$-norm, we get

$$||a \div b|| = ||a \odot c^{(-1)} \odot c \odot b^{(-1)}|| = ||(a \div c) \odot (c \div b)|| \leq ||a \div c|| \odot ||c \div b||;$$

thus, also the condition 4 is verified. ∎

PROPOSITION 3.3. *Let $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ be alo-groups and h: $G \to G'$ an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Then*

$$d_{\mathcal{G}'}(a', b') = h(d_{\mathcal{G}}(h^{-1}(a'), h^{-1}(b'))), \quad d_{\mathcal{G}}(a, b) = h^{-1}(d_{\mathcal{G}'}(h(a), h(b))). \tag{3.3}$$

Proof. By definition of $\mathcal{G}$-distance and properties in Equations 2.12 and 2.14:

$$h^{-1}(d_{\mathcal{G}'}(a', b')) = h^{-1}\big((a' \circ (b')^{(-1)}) \vee (b' \circ (a')^{(-1)})\big)$$

$$= \big(h^{-1}(a') \odot (h^{-1}(b'))^{(-1)}\big) \vee \big(h^{-1}(b') \odot (h^{-1}(a'))^{(-1)}\big) = d_{\mathcal{G}}\big(h^{-1}(a'), h^{-1}(b')\big)$$

and the first equality in Equation 3.3 is achieved. The second one is achieved in an analogous way. ∎

4. CONTINUOUS ALO-GROUPS OVER A REAL INTERVAL

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R. If G is a proper interval of R then, by Proposition 2.1, it is an open interval.

Let Q be the set of the rational numbers, Q^+ the set of the positive rational numbers, $+$ the usual addition and $\cdot$ the usual multiplication on R. Then, we provide the following examples of real alo-groups.

Example 1. $\mathcal{R} = (R, +, \leq)$ and $\mathcal{Q} = (Q, +, \leq)$ are continuous alo-groups with: $e = 0$, $x^{(-1)} = -x$, $x^{(n)} = nx$, $x \div y = x - y$; the norm $||a|| = |a| = a \vee (-a)$ generates the usual distance over R (resp. Q):

$$|a - b| = (a - b) \vee (b - a).$$

$\mathcal{R}$ and $\mathcal{Q}$ are both divisible: the (n)-root $x^{(n)}$ of a is the solution of $nx = a$ that is usually indicated with the symbol a/n. The mean $m_+(a_1, a_2, \ldots, a_n)$ is the arithmetic mean: $\frac{\sum_i a_i}{n}$.

Example 2. $]0,+\infty[= (]0, +\infty[, \cdot, \leq)$ and $\mathcal{Q}^+ = (Q^+, \cdot, \leq)$ are continuous alo-groups with: $e = 1$, $x^{(-1)} = x^{-1} = 1/x$, $x^{(n)} = x^n$, $x \div y = \frac{x}{y}$ and $||a|| = a \vee a^{-1}$; so $d_{]0,+\infty[}(a, b)$ and $d_{\mathcal{Q}^+}(a, b)$ are both given by

$$\left|\left|\frac{a}{b}\right|\right| = \frac{a}{b} \vee \frac{b}{a} \in [1, +\infty[.$$

The alo-group **]0,+∞[** is divisible and the (n)-root of a is $x = \sqrt[n]{a}$. The mean $m_{\cdot}(a_1, \ldots, a_n)$ is the geometric mean: $\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$.
The alo-group $\mathcal{Q}^+$ is not divisible: indeed $x^2 = 2$ has not solution in Q^+.

Let us consider the condition:

I) *G is a proper open interval of R and $\leq$ the total order on G inherited from the usual order on R.*

The following result of Aczél will be helpful to show that, under the condition **I**, a continuous real alo-group $\mathcal{G} = (G, \odot, \leq)$ can be built starting from the real alo-group $\mathcal{R}$ or the real alo-group **]0,+∞[**.

THEOREM 4.1. *Ref. 18. Under the assumption **I**, let $\odot$ be a binary operation over G. Then $\odot$ is a continuous, associative and cancellative operation if and only if there exists a continuous and strictly monotonic function $\phi : J \to G$ such that:*

$$x \odot y = \phi(\phi^{-1}(x) + \phi^{-1}(y)) \tag{4.1}$$

and J is R or one of real intervals $]-\infty, \gamma[$, $]-\infty, \gamma]$, $]\delta, +\infty[$, $[\delta, +\infty[$. The function ϕ in Equation 4.1 is unique up to a linear transformation of the variable (that is $\phi(x)$ may by replaced by $\phi(Cx)$, $C \neq 0$, but by no other function.)

COROLLARY 4.1. *Under the assumption **I**, let $\odot$ be a continuous, associative and cancellative operation over G. Then, $\odot$ is commutative and strictly increasing in each variable.*

Proof. *By Equation 4.1, commutativity of the addition and strict monotonicity of ϕ.* ■

THEOREM 4.2. *Under the assumption **I**, the following assertions are equivalent:*

1. *$\mathcal{G} = (G, \odot, \leq)$ is a continuous alo-group;*
2. *there exists a continuous and strictly increasing function $\phi : R \to G$ verifying the equality in Equation* 4.1;

3. *there exists a continuous and strictly increasing function* $\psi :]0, +\infty[\to G$ *verifying the equality*

$$x \odot y = \psi(\psi^{-1}(x) \cdot \psi^{-1}(y)). \tag{4.2}$$

Proof. $1 \Leftrightarrow 2$. By Theorem 4.1, Corollary 4.1, and Equation 2.2, $\mathcal{G}$ is a continuous alo-group if and only if there exists a continuous and strictly monotonic function $\phi : J \to G$ defined on a proper interval J of R and verifying the equality in Equation 4.1; this function can be chosen strictly increasing because it is unique up to a linear transformation of the variable. So, in order to prove the equivalence between item *1* and *2* it is enough to prove that the domain J of the function ϕ in Equation 4.1 coincides with R. To this purpose we observe that, by Equation 4.1,

$$x = x \odot e \Leftrightarrow \phi^{-1}(x) = \phi(x)^{-1} + \phi^{-1}(e) \Leftrightarrow \phi^{-1}(e) = 0,$$

thus $0 \in J$ and

$$x \odot x^{(-1)} = e \Leftrightarrow \phi^{-1}(x) + \phi^{-1}\left(x^{(-1)}\right) = \phi^{-1}(e) = 0 \Leftrightarrow \phi^{-1}\left(x^{(-1)}\right) = -\phi^{-1}(x);$$

so, if $a = \phi^{-1}(x) \in J$ then also $-a = \phi^{-1}(x^{(-1)}) \in J$. By Theorem 4.1, the equality $J = R$ follows.

$2 \Leftrightarrow 3$. Assume the assertion *2* is true. Then, by composing ϕ on the function $h : x \in]0, +\infty[\to log(x) \in R$, we get:

$$\psi : x \in]0, +\infty[\to \phi(\log(x)) \in G,$$

that is a bijection between $]0, +\infty[$ and G. Moreover $\psi^{-1}(y) = \exp(\phi^{-1}(y)$ and $\psi(\psi^{-1}(x) \cdot \psi^{-1}(y)) = \phi(\log(\exp(\phi^{-1}(x)) \cdot \exp(\phi^{-1}(y)))) = \phi(\phi^{-1}(x) + \phi^{-1}(y)) = x \odot y$. The implication $2 \Rightarrow 3$ is achieved. The reverse implication can be proved by an analogous reasoning. ■

COROLLARY 4.2. *Under the assumption* ***I****, a continuous alo-group* $\mathcal{G} = (G, \circ, \leq)$ *is isomorphic to* $\mathcal{R}$ *and to* $]0,+\infty[$ *and is divisible; moreover, if* ϕ *and* ψ *are the functions in items 2 and 3 of Theorem 4.2, then*

$$m_{\odot}(a_1, a_2, \ldots, a_n) = \phi\left(\frac{1}{n}\sum_{i=1}^{n}\phi^{-1}(a_i)\right) = \psi\left(\prod_{i=1}^{n}\psi^{-1}(a_i)\right)^{\frac{1}{n}},$$

$$d_{\mathcal{G}}(a, b) = \phi(d_{\mathcal{R}}(\phi^{-1}(a), \phi^{-1}(b))) = \psi(d_{]\mathbf{0},+\infty[}(\psi^{-1}(a), \psi^{-1}(b))).$$

Proof. The functions ϕ and ψ in items 2 and 3 of Theorem 4.2 are obviously isomorphisms between $\mathcal{R}$ and $\mathcal{G}$ and between **]0,+∞[** and $\mathcal{G}$, respectively; so, $\mathcal{G}$ is divisible by Proposition 2.3, and the equalities involving $m_{\odot}(a_1, a_2, \ldots, a_n)$ and $d_{\mathcal{G}}(a, b)$ follow by Propositions 2.3 and 3.3. ■

By applying Theorem 4.2, we provide, in the following propositions, two examples of continuous real alo-groups over a limited interval of R.

PROPOSITION 4.1. *Let* $\oplus :]-1,1[^2 \to]-1,1[$ *be the operation defined by*

$$x \oplus y = \frac{(1+x)(1+y)-(1-x)(1-y)}{(1+x)(1+y)+(1-x)(1-y)} \tag{4.3}$$

and $\leq$ *the order inherited by the usual order in* R*. Then* $\mathbf{]-1,1[} = (]-1,1[, \oplus, \leq)$ *is a continuous alo-group and it is* $e = 0$*,* $x^{(-1)} = -x$ *for each* $x \in]-1,1[$.

Proof. The function $g : t \in]0,+\infty[\to \frac{t-1}{t+1} \in]-1,1[$, is a bijection between $]0,+\infty[$ and $]-1,1[$, that is continuous and strictly increasing. For $a, b \in]0,+\infty[$ and $x = g(a),\ \ y = g(b)$, we get

$$g(a) \oplus g(b) = \frac{\left(1+\frac{a-1}{a+1}\right)\left(1+\frac{b-1}{b+1}\right)-\left(1-\frac{a-1}{a+1}\right)\left(1-\frac{b-1}{b+1}\right)}{\left(1+\frac{a-1}{a+1}\right)\left(1+\frac{b-1}{b+1}\right)+\left(1-\frac{a-1}{a+1}\right)\left(1-\frac{b-1}{b+1}\right)} = \frac{ab-1}{ab+1} = g(a \cdot b).$$

Thus, $x \oplus y = g(g^{-1}(x) \cdot g^{-1}(y))$, and Equation 4.2 in Theorem 4.2 is verified with $\psi = g$. Finally, it is easy to verify that $x \oplus 0 = x$ and $x \oplus (-x) = 0$. ■

PROPOSITION 4.2. *Let* $\otimes :]0,1[^2 \to]0,1[$ *be the operation defined by*

$$x \otimes y = \frac{xy}{xy+(1-x)(1-y)}, \tag{4.4}$$

and $\leq$ *the order inherited by the usual order in* R*. Then* $\mathbf{]0,1[} = (]0,1[, \otimes, \leq)$ *is a continuous alo-group and it is* $e = 0,5$ *and* $x^{(-1)} = 1 - x$ *for each* $x \in]0,1[$.

Proof. The function

$$v : t \in]0,+\infty\Big[\to \frac{t}{t+1} \in \Big]0,1[, \tag{4.5}$$

is a bijection between $]0,+\infty[$ and $]0,1[$ that is continuous and strictly increasing. For $a, b \in]0,+\infty[$ and $x = v(a),\ \ y = v(b)$, we get:

$$v(a) \otimes v(b) = \frac{\frac{a}{a+1}\frac{b}{b+1}}{\frac{a}{a+1}\frac{b}{b+1}+\left(1-\frac{a}{a+1}\right)\left(1-\frac{b}{b+1}\right)} = \frac{ab}{ab+1} = v(a \cdot b).$$

Thus, $x \otimes y = v(v^{-1}(x) \cdot v^{-1}(y))$, and Equation 4.2 in Theorem 4.2 is verified with $\psi = v$. Finally, it is easy to verify that $x \otimes 0.5 = x$ and $x \otimes (1-x) = 0.5$. ■

Let $\mathcal{R} = (R, +, \leq)$ and $]\mathbf{0},+\infty[= (]0, +\infty[, \cdot, \leq)$ be the alo-groups in Examples 1 and 2 and $]\mathbf{0}, \mathbf{1}[$ the alo-group in Proposition 4.2. Then, will call:

- $\mathcal{R}$ the *additive (real) alo-group*,
- $]\mathbf{0},+\infty[$ the *multiplicative (real) alo-group*,
- $]\mathbf{0}, \mathbf{1}[$ the *fuzzy (real) alo-group*.

Isomorphisms between $]\mathbf{0},+\infty[$ and $\mathcal{R}$ are

$$h : x \in]0, +\infty[\to \log x \in R, \quad h^{-1} : y \in R \to \exp(y) \in]0, +\infty[. \tag{4.6}$$

Isomorphisms between $]\mathbf{0},+\infty[$ and $]\mathbf{0}, \mathbf{1}[$ are the function v in Equation 4.5 and its inverse:

$$v^{-1} : y \in]0, 1\left[\to \frac{y}{1-y} \in \right]0, +\infty[. \tag{4.7}$$

5. PAIRWISE COMPARISON MATRICES OVER A DIVISIBLE ALO-GROUP

In this section and in the next one, $\mathcal{G} = (G, \odot, \leq)$ denotes a divisible alo-group. A pairwise comparison system over $\mathcal{G}$ is a pair $(X, \mathcal{A})$ constituted by a set $X = \{x_1, \dots, x_n\}$ and a relation $\mathcal{A} : (x_i, x_j) \in X^2 \to a_{ij} = \mathcal{A}(x_i, x_j) \in G$, represented by means of the PC matrix in Equation 1.1, with entries in G. In the context of an evaluation problem, the element a_{ij} can be interpreted as a measure on $\mathcal{G}$ of the preference of x_i over x_j: $a_{ij} > e$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < e$ expresses the opposite preference and $a_{ij} = e$ means that x_i and x_j are indifferent. Then $A = (a_{ij})$ is assumed to be *reciprocal* with respect to the operation $\odot$, that is,

$\mathbf{r}_{\odot}$) $\quad a_{ji} = a_{ij}^{(-1)} \quad \forall\, i, j = 1, \dots, n$ *(reciprocity)*,

so $a_{ii} = e$ for each $i = 1, 2, \dots, n$ and $a_{ij} \odot a_{ji} = e$ for $i, j \in \{1, 2, \dots, n\}$.

In the sequel, $\mathcal{PC}_n(\mathcal{G})$ will denote the set of the reciprocal PC matrices of order $n \geq 3$ over $\mathcal{G}$. Then, a matrix of $\mathcal{PC}_n(]\mathbf{0}, +\infty[)$ is a *multiplicative* PC matrix, a matrix of $\mathcal{PC}_n(\mathcal{R})$ is an *additive* PC matrix. In this context, a *fuzzy* PC matrix is a matrix belonging to $\mathcal{PC}_n(]\mathbf{0}, \mathbf{1}[)$.

If $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ then we will denote by

- $\underline{a}_i$ the $i-th$ row of A: $\underline{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$;
- $\underline{a}^j$ the $j-th$ column of A: $\underline{a}^j = (a_{1j}, a_{2j}, \dots, a_{nj})$;
- $m_{\odot}(\underline{a}_i)$ the mean $m_{\odot}(a_{i1}, a_{i2}, \dots, a_{in})$;
- $\underline{w}_{m_{\odot}}(A)$ the mean vector $(m_{\odot}(\underline{a}_1), m_{\odot}(\underline{a}_2), \cdots, m_{\odot}(\underline{a}_n))$;
- ρ_{ijk} the element $a_{ik} \div (a_{ij} \odot a_{jk})$ of G.

Hence

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = ||\rho_{ijk}||. \tag{5.1}$$

Because of the assumption $\mathbf{r}_{\odot}$) the equality $a_{ik} = a_{ij} \odot a_{jk}$ does not depend on the considered order of the indexes i, j, k, that is,

$$a_{ik} = a_{ij} \odot a_{jk} \Leftrightarrow a_{ij} = a_{ik} \odot a_{kj} \Leftrightarrow a_{jk} = a_{ji} \odot a_{ik} \Leftrightarrow a_{ji} = a_{jk} \odot a_{ki} \Leftrightarrow \ldots . \tag{5.2}$$

So the following definition is well done.

DEFINITION 5.1. *Let $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$. Then*

1. $A = (a_{ij})$ is consistent with respect to the $3 - subset$ $\{x_i, x_j, x_k\}$ of X *if and only if* $a_{ik} = a_{ij} \odot a_{jk}$;
2. $A = (a_{ij})$ is consistent *if and only if it is consistent with respect to each* $3 - subset$ $\{x_i, x_j, x_k\}$ *of* X*, that is*

$$\boldsymbol{c}_{\odot}) \quad a_{ik} = a_{ij} \odot a_{jk} \quad \forall i, j, k \qquad \textit{(consistency)}.$$

Remark 5.1. In our context, a fuzzy PC matrix is defined over $]\mathbf{0}, \mathbf{1}[$ (see Remark 2.1); then the condition of fuzzy consistency, by Definition 5.1, becomes:

$$\boldsymbol{c}_{\otimes}) \quad a_{ik} = \frac{a_{ij}a_{jk}}{a_{ij}a_{jk} + (1 - a_{ij})(1 - a_{jk})} \quad \forall i, j, k.$$

PROPOSITION 5.1. *The property of consistency is equivalent to each one of the following conditions:*

$$\boldsymbol{c'}_{\odot}) \quad a_{ik} \div a_{jk} = a_{ij} \quad \forall i, j, k;$$

$$\boldsymbol{c''}_{\odot}) \quad \rho_{ijk} = e \quad \forall i, j, k.$$

Proof. By Definition 5.1 and the meanings of $\div$ and ρ_{ijk}. ∎

PROPOSITION 5.2. $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ *is consistent if and only if*

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = e \qquad \forall i, j, k. \tag{5.3}$$

Proof. By Equation 5.1 and Proposition 5.1. ∎

Remark 5.2. Because of the equivalences in Equation 5.2 in checking the conditions $\boldsymbol{c}_{\odot}$), $\boldsymbol{c}'_{\odot}$), $\boldsymbol{c}''_{\odot}$) and Equation 5.2, we can limit ourselves to the case $i < j < k$.

DEFINITION 5.2. *A vector* $\underline{w} = (w_1, w_2, \cdots, w_n)$*,* $w_i \in G$*, is* consistent *with respect to* $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ *if and only if*

$$w_i \div w_j = a_{ij} \quad \forall i, j = 1, 2, \ldots, n. \tag{5.4}$$

Remark 5.3. By Equation 5.4 and the equivalences ($w_i > w_j \Leftrightarrow w_i \div w_j > e$) and ($w_i = w_j \Leftrightarrow w_i \div w_j = e$), we get that $w_i > w_j \Leftrightarrow a_{ij} > e \quad and \quad w_i = w_j \Leftrightarrow a_{ij} = e$. Thus, the weights assigned to the alternatives by a consistent vector $\underline{w}$ agree with the preferences expressed by the entries a_{ij} of the PC matrix.

PROPOSITION 5.3. *$A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ is consistent if and only if there exists a consistent vector $\underline{w} = (w_1, w_2, \cdots, w_n)$, $w_i \in G$.*

Proof. Let $A = (a_{ij})$ be consistent. Then by **c'**$_\odot$, $a_{ij} = a_{ik} \div a_{jk}$; so the equalities in Equation 5.4 are verified by $\underline{w} = \underline{a}^k$. Viceversa, if $\underline{w}$ is a consistent vector, then $a_{ij} \odot a_{jk} = (w_i \div w_j) \odot (w_j \div w_k) = w_i \odot w_j^{(-1)} \odot w_j \odot w_k^{(-1)} = w_i \odot w_k^{(-1)} = a_{ik}$. ■

PROPOSITION 5.4. *The following assertions related to $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ are equivalent:*

i) $A = (a_{ij})$ is consistent;
ii) each column $\underline{a}^k$ is a consistent vector;
iii) the mean vector $\underline{w}_{m_\odot}$ is a consistent vector.

Proof. $i) \Leftrightarrow ii)$ because of Proposition 5.1, condition **c'**$_\odot$. $i) \Leftrightarrow iii)$. The implication $iii) \Rightarrow i)$ follows by Proposition 5.3. Under the assumption i) let us apply Equations 2.11 and 2.3 to get

$$(m_\odot(\underline{a}_i) \div m_\odot(\underline{a}_j))^{(n)} = m_\odot(\underline{a}_i)^{(n)} \div m_\odot(\underline{a}_j)^{(n)}$$

$$= (a_{i1} \odot a_{i2} \odot \cdots \odot a_{in}) \odot (a_{j1} \odot a_{j2} \odot \cdots \odot a_{jn})^{(-1)}$$

$$= (a_{i1} \odot a_{1j}) \odot (a_{i2} \odot a_{2j}) \odot \cdots \odot (a_{in} \odot a_{nj}) = a_{ij}^{(n)}.$$

So $\underline{w}_{m_\odot}$ verifies Equation 5.4 and the implication $i) \Rightarrow iii)$ is achieved. ■

PROPOSITION 5.5. *Let $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ be divisible alo-groups and $h : G \to G'$ an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Then*

$$H : A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G}) \to H(A) = A' = (h(a_{ij}))$$

is a bijection between $\mathcal{PC}_n(\mathcal{G})$ and $\mathcal{PC}_n(\mathcal{G}')$ that preserves the consistency, that is A is consistent if and only if A' is consistent.

Proof. H is an injection because h is an injective function. By applying h to the entries of the matrix $A = (a_{ij})$, we get the matrix $A' = (h(a_{ij}))$, that is reciprocal too, because of the equality in Equation 2.13: so $H(A) = (h(a_{ij})) \in \mathcal{PC}_n(\mathcal{G}')$. Moreover, by the equality in Equation 2.12, if $A = (a_{ij})$ is consistent, then the transformed $A' = H(A)$ is consistent too.

Viceversa, if $A' = (a'_{ij}) \in \mathcal{PC}_n(\mathcal{G}')$, by applying h^{-1} to the entries of A', we get the matrix $A = (h^{-1}(a'_{ij}))$ that belongs to $\mathcal{PC}_n(\mathcal{G})$ and, by Equation 2.14, is consistent if and only if A' is consistent too. ■

Under the hypotheses of Proposition 5.5, we say that $A' = (h(a_{ij}))$ is the *transformed* of A by means of h, and $A = (h^{-1}(a'_{ij})) = H^{-1}(A')$ is the *transformed* of A' by means of h^{-1}. By Proposition 2.3, if $A' = (h(a_{ij}))$, then the mean vector $\underline{w}_{m_\odot}(A)$ is transformed, by means of h, in the mean vector $\underline{w}_{m_\circ}(A')$. Viceversa h^{-1} transforms the mean vector $\underline{w}_{m_\circ}(A')$ in the mean vector $\underline{w}_{m_\odot}(A)$.

6. A CONSISTENCY INDEX

Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible alo-group and $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$. By Definition 5.1, $A = (a_{ij})$ is *inconsistent* if and only if it is inconsistent in at least one $3 -$ *subset* $\{x_i, x_j, x_k\}$. The closeness to the consistency depends on the degree of consistency with respect to each $3 -$ *subset* $\{x_i, x_j, x_k\}$ and can be measured by an average of these degrees. So, in order to define a consistency index for $A = (a_{ij})$, we first consider the case that X has only three elements.

6.1. Consistency Index in the Case $n = 3$

Let X be the set $\{x_1, x_2, x_3\}$ and the relation $\mathcal{A}$ on X represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{PC}_3(\mathcal{G}). \tag{6.1}$$

By Proposition 5.2, $A = (a_{ij})$ and Remark 5.2 is inconsistent if and only if $d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) > e$. It is natural to say that the more A is inconsistent the more $d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23})$ is far from e. So, we give the following definition:

DEFINITION 6.1. *The* consistency index *of the matrix in Equation* 6.1 *is given by*

$$I_{\mathcal{G}}(A) = ||\rho_{123}|| = d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}). \tag{6.2}$$

As particular cases, we get

- if $A \in \mathcal{PC}_3(]\mathbf{0},+\infty[)$ then
$$I_{]\mathbf{0},+\infty[}(A) = \frac{a_{13}}{a_{12} \cdot a_{23}} \vee \frac{a_{12} \cdot a_{23}}{a_{13}} \in [1, +\infty[\tag{6.3}$$
and A is consistent if and only if $I_{]\mathbf{0},+\infty[}(A) = 1$;
- if $A \in \mathcal{PC}_3(\mathcal{R})$, then
$$\begin{aligned} I_{\mathcal{R}}(A) &= |a_{13} - a_{12} - a_{23}| \\ &= (a_{13} - a_{12} - a_{23}) \vee (a_{12} + a_{23} - a_{13}) \in [0, +\infty[\end{aligned} \tag{6.4}$$
and A is consistent if and only if $I_{\mathcal{R}}(A) = 0$;

- if $A \in \mathcal{PC}_3(]\mathbf{0,1}[)$, then

$$I_{]\mathbf{0,1}[}(A)$$

$$= \frac{a_{13}(1-a_{12}\otimes a_{23})}{a_{13}(1-a_{12}\otimes a_{23})+(1-a_{13})(a_{12}\otimes a_{23})} \vee \frac{(a_{12}\otimes a_{23})(1-a_{13})}{(a_{12}\otimes a_{23})(1-a_{13})+(1-a_{12}\otimes a_{23})a_{13}}$$

$$= \frac{a_{13}(1-a_{12})(1-a_{23})}{a_{13}(1-a_{12})(1-a_{23})+(1-a_{13})a_{12}a_{23}} \vee \frac{a_{12}a_{23}(1-a_{13})}{a_{12}a_{23}(1-a_{13})+(1-a_{12})(1-a_{23})a_{13}}; \tag{6.5}$$

and A is consistent if and only if $I_{]\mathbf{0,1}[}(A) = 0.5$.

The following proposition shows that the more $I_{\mathcal{G}}(A)$ is close to e the more the mean vector $\underline{w}_{m_\odot}$ is close to be a consistent vector.

PROPOSITION 6.1. *Let $\underline{w}_{m_\odot} = (w_1, w_2, w_3)$ be the mean vector associated to the matrix in Equation* 6.1 *and $\rho = \rho_{123}$. Then*

$$d_{\mathcal{G}}(w_i \div w_j, a_{ij}) = ||\rho||^{\frac{1}{3}} \quad \forall i \neq j.$$

Proof. By definition of ρ

$$a_{13} = \rho \odot a_{12} \odot a_{23}, \quad a_{21} = \rho \odot a_{23} \odot a_{31} \quad \textit{and} \quad a_{32} = \rho \odot a_{12} \odot a_{31}.$$

By the above inequalities and the equality $a_{ii} = e$, we get

- $w_1 = (a_{11} \odot a_{12} \odot a_{13})^{(\frac{1}{3})} = ({a_{12}}^2 \odot \rho \odot a_{23})^{(\frac{1}{3})}$;
- $w_2 = (a_{21} \odot a_{22} \odot a_{23})^{(\frac{1}{3})} = (\rho \odot {a_{23}}^{(2)} \odot a_{31})^{(\frac{1}{3})}$;
- $w_3 = (a_{31} \odot a_{32} \odot a_{33})^{(\frac{1}{3})} = (\rho \odot {a_{31}}^{(2)} \odot a_{12})^{(\frac{1}{3})}$.

Thus:

1. $w_1 \div w_2 = (a_{12}^3 \odot \rho)^{(\frac{1}{3})} = a_{12} \odot \rho^{(\frac{1}{3})}$;
2. $w_2 \div w_3 = (\rho \odot {a_{23}}^{(2)} \odot a_{31} \odot a_{13} \odot a_{23})^{(\frac{1}{3})} = a_{23} \odot \rho^{(\frac{1}{3})}$;
3. $w_3 \div w_1 = (\rho \odot {a_{31}}^{(2)} \odot a_{12} \odot a_{21} \odot a_{31})^{(\frac{1}{3})} = a_{31} \odot \rho^{(\frac{1}{3})}$.

By item 1, we get: $(w_1 \div w_2) \div a_{12} = \rho^{(\frac{1}{3})}$ and $a_{12} \div (w_1 \div w_2) = (\rho^{(-1)})(\frac{1}{3})$, thus $d_{\mathcal{G}}(w_1 \div w_2, a_{12}) = ||\rho||^{(\frac{1}{3})}$.
By item 2, we get: $(w_2 \div w_3) \div a_{23} = \rho^{(\frac{1}{3})}$ and $a_{23} \div (w_2 \div w_3) = (\rho^{(-1)})(\frac{1}{3})$, thus $d_{\mathcal{G}}(w_2 \div w_3, a_{23}) = ||\rho||^{(\frac{1}{3})}$.
By item 3, $(w_3 \div w_1) \div a_{31} = \rho^{(\frac{1}{3})}$ and $a_{31} \div (w_3 \div w_1) = (\rho^{(-1)})(\frac{1}{3})$, thus $d_{\mathcal{G}}(w_3 \div w_1, a_{31}) = ||\rho||^{(\frac{1}{3})}$. ■

PROPOSITION 6.2. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij})) \in \mathcal{PC}_3(\mathcal{G}')$ the transformed of the matrix in Equation* 6.1*, by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}'}(A') = h(I_{\mathcal{G}}(A))$.*

Proof. By the equality $h(a_{12}) \circ h(a_{23}) = h(a_{12} \odot a_{23})$ and Proposition 3.3,

$$I_{\mathcal{G}'}(A') = d_{\mathcal{G}'}(h(a_{13}), h(a_{12}) \circ h(a_{23})) = h(d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23})) = h(I_{\mathcal{G}}(A)).$$

■

COROLLARY 6.1. *Under the assumption **I**, let $\mathcal{G}$ be a continuous alo-group and $\phi : R \to G$ and $\psi :]0, +\infty[\to G$ the functions in items 2 and 3 of Theorem 4.2. Then, the consistency index of the matrix in Equation* 6.1 *is*

$$I_{\mathcal{G}}(A = (a_{ij})) = \phi(I_{\mathcal{R}}(A' = (\phi^{-1}(a_{ij})))) = \psi(I_{]\mathbf{0},+\infty[}(A'' = (\psi^{-1}(a_{ij})))). \quad (6.6)$$

Proof. By Proposition 6.2 or Corollary 4.2. ■

COROLLARY 6.2. *Let v be the isomorphism in Equation 4.5 between* $]\mathbf{0}, +\infty[$ *and* $]\mathbf{0,1}[$, *v^{-1} the inverse isomorphism in Equation 4.7 and $A' = (a'_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$ the transformed of $A = (a_{ij}) \in \mathcal{PC}_3(]\mathbf{0}, \mathbf{1}[)$ by means of v^{-1}. Then*

$$A' = \left(\frac{a_{ij}}{1 - a_{ij}}\right) \quad \textit{and} \quad I_{]]\mathbf{0},\mathbf{1}[}(A) = v(I_{]\mathbf{0},+\infty[}(A')). \quad (6.7)$$

Proof. By Corollary 6.1. ■

For an example related to the equalities in Equation 6.7 see Example 8.1.

6.2. Consistency Index in the Case $n > 3$

Let $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$, $n > 3$. Then, we will denote by

- T the set of the $3 -$ *subset* $\{x_i, x_j, x_k\}$ of X;
- $n_T = \frac{n!}{3!(n-3)!}$ the cardinality of T.

Of course, n_T is also the cardinality of the set $T(A) = \{(a_{ij}, a_{jk}, a_{ik}), i < j < k\}$. For i, j, k, with $i < j < k$,

$$A_{ijk} = \begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix}$$

is a submatrix of A related to the $3 -$ *subset* $\{x_i, x_j, x_k\}$ and $I_{\mathcal{G}}(A_{ijk}) = ||\rho_{ijk}|| = d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk})$ is its consistency index. By item 2 of Definition 5.1 and Remark 5.2, a consistency index of A has to be expressed in terms of the consistency indices $I_{\mathcal{G}}(A_{ijk})$. Hence, we set

DEFINITION 6.2. *The consistency index of $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ is given by*

$$I_{\mathcal{G}}(A) = \left(\bigodot_{i<j<k} I_{\mathcal{G}}(A_{ijk}) \right)^{(1/n_T)} = \left(\bigodot_{i<j<k} d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) \right)^{(1/n_T)} . \qquad (6.8)$$

PROPOSITION 6.3. *If $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ then $I_{\mathcal{G}}(A_{ijk}) \geq e$ and $A = (a_{ij})$ is consistent if and only if $I_{\mathcal{G}}(A) = e$.*

Proof. As $\odot$ is increasing with respect to each variable, the statement follows by the property 1 of a $\mathcal{G}$-distance (see Definition 3.2) and by Proposition 5.2. ■

As particular cases, we get

- if $A \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$, then $I_{]\mathbf{0},+\infty[}(A) = \left(\prod_{i<j<k} I_{]\mathbf{0},+\infty[}(A_{ijk}) \right)^{\frac{1}{n_T}} \geq 1$ and A is consistent if and only if $I_{]\mathbf{0},+\infty[}(A) = 1$;
- if $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{R})$, then $I_{\mathcal{R}}(A) = \frac{1}{n_T} \sum_{i<j<k} I_{\mathcal{R}}(A_{ijk}) \geq 0$, and A is consistent if and only if $I_{\mathcal{R}}(A) = 0$;
- if $A \in \mathcal{PC}_n(]\mathbf{0}, \mathbf{1}[)$, then $I_{]\mathbf{0},\mathbf{1}[}(A) = (\bigotimes_{i<j<k} I_{]\mathbf{0},\mathbf{1}[}(A_{ijk}))^{(\frac{1}{n_T})} \in [0.5, 1[$ and A is consistent if and only if $I_{]\mathbf{0},\mathbf{1}[}(A) = 0.5$.

PROPOSITION 6.4. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij})) \in \mathcal{PC}_n(\mathcal{G}')$ the transformed of $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}'}(A') = h(I_{\mathcal{G}}(A))$.*

Proof. By Propositions 6.2 and Proposition 2.3. ■

COROLLARY 6.3. *Under the assumption* ***I****, let $\mathcal{G}$ be a continuous alo-group and $\phi : R \to G$ and $\psi :]0, +\infty[\to G$ the functions in items 2 and 3 of Theorem 4.2. Then, the consistency index of $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ verifies the equalities in Equation* 6.6.

COROLLARY 6.4. *Let $A' = (a'_{ij}) \in \mathcal{PC}_n(\mathcal{R})$ be the transformed of $A = (a_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$, by means of the isomorphism h between* **]0,+∞[** *and $\mathcal{R}$, given in Equation* 4.6. *Then, $A' = (log(a_{ij}))$, $A = (\exp(a'_{ij}))$ and*

$$I_{\mathcal{R}}(A') = log(I_{]\mathbf{0},+\infty[}(A)), \quad I_{]\mathbf{0},+\infty[}(A) = \exp(I_{\mathcal{R}}(A')).$$

COROLLARY 6.5. *Let $A' = (a'_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$ be the transformed of $A = (a_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, \mathbf{1}[)$, by means of the isomorphism v^{-1} in Equation* 4.7. *Then, the equalities in Equation 6.7 hold.*

For examples related to the above corollaries see Examples 8.2 and 8.3.

7. CONCLUSION AND FUTURE WORK

We have defined a general context in which different approaches to pairwise comparison matrices can be unified. We have also provided a meaningful consistency index suitable for each kind of matrix, naturally linked to a notion of distance and easy to compute in the additive and multiplicative case; in the other cases, this index is the transformed of the consistency index of a suitable multiplicative matrix or a suitable additive matrix.

Following the results in Refs. 13–15, 19 for the multiplicative case, our future work will be directed to investigate, in the new general context, the following problems related to a pairwise comparison matrix:

- to determine the conditions on a PC matrix inducing a qualitative ranking (*actual ranking*) on the set X;
- to individuate the conditions ensuring the existence of vectors representing the actual ranking at different levels.

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APPENDIX

In this section, we provide examples of computing consistency indices; multiplicative, additive, and fuzzy cases are considered. In Examples 8.1 and 8.2, we verify the relationship in Corollary 6.2 and in Corollary 6.4. In Example 8.3, we apply the Corollary 6.5.

Example 8.1. Let us consider the matrix

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.1 \\ 0.6 & 0.9 & 0.5 \end{pmatrix} \in \mathcal{PC}_3(]\mathbf{0}, \mathbf{1}[);$$

then, by Equation 6.5,

$$I_{]\mathbf{0},\mathbf{1}[}(A) = \frac{0.4 \cdot 0.7 \cdot 0.9}{0.4 \cdot 0.7 \cdot 0.9 + 0.3 \cdot 0.1 \cdot 0.6} \vee \frac{0.3 \cdot 0.1 \cdot 0.6}{0.4 \cdot 0.7 \cdot 0.9 + 0.3 \cdot 0.1 \cdot 0.6}$$

$$= 0.9\overline{3} \vee 0.0\overline{6} = 0.9\overline{3}.$$

By applying the isomorphism v^{-1} in Equation 4.7 to the entries of A, we get

$$A' = \begin{pmatrix} 1 & \frac{3}{7} & \frac{2}{3} \\ \frac{7}{3} & 1 & \frac{1}{9} \\ \frac{3}{2} & 9 & 1 \end{pmatrix} \in \mathcal{PC}_3(]\mathbf{0}, +\infty[)$$

which consistency index, by Equation 6.3, is $I_{]\mathbf{0},+\infty[}(A') = 14 \vee \frac{1}{14} = 14$.
Let v be the isomorphism in Equation 4.5, then in accordance with Corollary 6.2, $I_{]\mathbf{0},\mathbf{1}[}(A) = v(I_{]\mathbf{0},+\infty[}(A')) = v(14) = \frac{14}{15} = 0.9\overline{3}$.

Example 8.2. Let us consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{7} & \frac{1}{7} & \frac{1}{5} \\ 7 & 1 & \frac{1}{2} & \frac{1}{3} \\ 7 & 2 & 1 & \frac{1}{9} \\ 5 & 3 & 9 & 1 \end{pmatrix} \in \mathcal{PC}_4(]\mathbf{0}, +\infty[);$$

then

$$I_{]\mathbf{0},+\infty[}(A) = \sqrt[4]{I_{]\mathbf{0},+\infty[}(A_{234}) \cdot I_{]\mathbf{0},+\infty[}(A_{134}) \cdot I_{]\mathbf{0},+\infty[}(A_{124}) \cdot I_{]\mathbf{0},+\infty[}(A_{123})}$$

$$= \sqrt[4]{6 \cdot 12.6 \cdot 4.2 \cdot 2} = 5.02.$$

Let h be the isomorphism in Equation 4.6 between $]\mathbf{0},+\infty[$ and $\mathcal{R}$. By applying h to the entries of A, we get

$$A' = \begin{pmatrix} 0 & -ln7 & -ln7 & -ln5 \\ ln7 & 0 & -ln2 & -ln3 \\ ln7 & ln2 & 0 & -ln9 \\ ln5 & ln3 & ln9 & 0 \end{pmatrix} \in \mathcal{PC}_4(\mathcal{R})$$

which consistency index is

$$I_{\mathcal{R}}(A') = \frac{I_{\mathcal{R}}(A'_{234}) + I_{\mathcal{R}}(A'_{134}) + I_{\mathcal{R}}(A'_{124}) + I_{\mathcal{R}}(A'_{123})}{4}$$

$$= \frac{1.7917 + 2.5336 + 1.4350 + 0.6931}{4} = 1.6134.$$

In accordance with Corollary 6.4, $I_{\mathcal{R}}(A') = log(I_{]\mathbf{0},+\infty[}(A)) = \log(5.02) = 1.6134$.

Example 8.3. Let us consider the matrix

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.4 \\ 0.7 & 0.5 & 0.1 & 0.2 \\ 0.6 & 0.9 & 0.5 & 0.8 \\ 0.6 & 0.8 & 0.2 & 0.5 \end{pmatrix} \in \mathcal{PC}_4(]\mathbf{0},\mathbf{1}[)$$

By applying the function v^{-1} in Equation 4.7 to the entries of A, we get

$$A' = \begin{pmatrix} 1 & \frac{3}{7} & \frac{2}{3} & \frac{2}{3} \\ \frac{7}{3} & 1 & \frac{1}{9} & \frac{1}{4} \\ \frac{3}{2} & 9 & 1 & 4 \\ \frac{3}{2} & 4 & \frac{1}{4} & 1 \end{pmatrix} \in \mathcal{PC}_4(]\mathbf{0},+\infty[)$$

which consistency index is

$$I_{]\mathbf{0},+\infty[}(A') = \sqrt[4]{I_{]\mathbf{0},+\infty[}(A'_{234}) \cdot I_{]\mathbf{0},+\infty[}(A'_{134}) \cdot I_{]\mathbf{0},+\infty[}(A'_{124}) \cdot I_{]\mathbf{0},+\infty[}(A'_{123})}$$

$$= \sqrt[4]{\frac{16}{9} \cdot 4 \cdot \frac{56}{9} \cdot 14} = 4.9888.$$

Let v be the isomorphism in Equation 4.5, then, by Corollary 6.5,

$$I_{]\mathbf{0},\mathbf{1}[}(A) = v(I_{]\mathbf{0},+\infty[}(A')) = \frac{4.9888}{5.9888} = 0.833.$$