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The new triad based inconsistency indices for pairwise comparisons

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Abstract

Pairwise comparisons are widely recognized method supporting decision making process based on the subjective judgments. The key to this method is the notion of inconsistency that has a significant impact on the reliability of results. Inconsistency is expressed by means of inconsistency indices. Depending on their construction, such indices may pay attention to different aspects of the set of pairwise comparisons.

The family of indices proposed in this article tries to combine the advantages coming from different indices, thereby increases the expressiveness of the family elements. The newly introduced notion of equivalence can help in comparing the indices and identifying their common properties.

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1. Introduction

Pairwise comparisons is a method of comparing objects in pairs in order to decide their importance. It is especially useful when intangible attributes of objects need to be compared. Due to their intangibility and abstractness objects (alternatives) need to be compared by human experts. Since, it is easier for a human to compare two objects at a time, first individual comparisons are made, then the result is synthesized as the ordered (ranking) list of objects. The list is used by decision makers to make a decision on objects.

Pairwise comparisons play an important role in well-known decision analysis methods such as *AHP* (Analytic Hierarchy Process) and its generalization *ANP* (Analytic Network Process)[23]. The effectiveness of the method was confirmed several times in practice[10].

A desirable property of a set of comparisons is consistency. In other words if there are two experts opinions that the object a is two times more important than b, and b is three times more important than c, there should be also an opinion according to which a is six times more important than c. If the opinion on the pair (a, c) is different, this means there is a lack of consistency (existence of inconsistency). Since objects are compared in pairs by fallible humans, very often with regard to the abstract, intangible criteria, an inconsistency in judgments happens very often. If experts have a deep insight into the problem domain and act rationally, one can expect that the inconsistency is not to high, i.e. the

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judgment as regards the pair (a, c) does not differ to much from what could be inferred from two other comparisons (a, b) and (b, c). If it is high either the experts are not competent enough, they do not act rationally or the problem (criterion) is so complex and difficult that even two essentially different judgments about the same thing are justified. From the perspective of a person who has to make a decision based on the highly inconsistent expert judgments, each of these situations is bad. The decision becomes impossible. Therefore from the decision maker point of view it is important to have a possibly consistent set of comparative opinions.

Since the inconsistency (or consistency) is so important, the question arises how to represent it in the appropriate and meaningful way. The search for answers led to define a number of inconsistency (consistency) indices including the most known *Saaty's* Eigenvalue Based Index[22], Geometric Consistency Index proposed *Aguarï¿œn* and *Moreno-Jimenï¿œz*[1], *Barzilai's* Relative Error[2], *Pelï¿œez* and *Lamata's* index[20], *Koczkodaj's* inconsistency index[13] and others. Most of the proposed indices try to answer the question about the average level of inconsistency. Since every single disturbance contributes to the average, usually a single, local improvement in inconsistency between three different objects entails decreasing an inconsistency index. Thus, in the case of a change the customer quickly gets a useful feedback. On the other hand, it may happen that, despite the fact that the average inconsistency is relatively low, there are a small subset of comparisons for which inconsistency is high. The index defined by *Koczkodaj*[13] informs about the maximal local inconsistency. Therefore, it is able to protect the client against the local large disturbances of data. Whilst focusing on the worst local disturbance it does not take into account the average inconsistency. Thus, a number of changes (for better and for worse) may pass unnoticed.

The aim of the paper is to define a family of inconsistency indices that combine both the above-mentioned characteristics - sensitivity to any local changes and taking into account the worst local inconsistency spot. As a starting point the inconsistency introduced by a triad of objects, which is used to define more complex concepts including *Koczkodaj's* index, was chosen. This provides a basis for defining a family of inconsistency indices that, on the one hand, reflects the level of an average inconsistency but, on the other hand, provides information about the worst local disturbance.

2. Preliminaries

The first use of the set of pairwise comparisons for further result synthesis is attributed to *Llull* - a medieval scholar, mathematician, philosopher, alchemist[7,24]. Again, the use of this approach can be found in the *Condorcet's Theory of Voting*[27]. The pairwise comparisons method began to be regularly studied and used in the twentieth century. A number of works including *Thurstone*[26], *Bradley* et al.[4] or *Saaty*[22] significantly influenced on the development of the method. The latter introduced a hierarchy, which provides an easy way to handle the large number of criteria.

In time, the method has become more and more popular. The increased interest among researchers and practitioners has resulted in a number of publications dealing with the theory and practice of the method[12,21,14]. The result of scientific explorations are, for example, the *Rough Set* approach[9], fuzzy *PC* relation handling[19], incomplete *PC* relation[8], non-numerical rankings[11] and a ranking with a reference set[17,18]. A lot of research is devoted to the problem of inconsistency measuring. In works[16,6,3,5] authors analyze various properties of different inconsistency indices trying to find important regularities in their behavior.

The input data to the pairwise comparison method is a matrix $M = (m_{ij}) \wedge m_{ij} \in \mathbb{R}_+ \wedge i, j \in \{1, \ldots, n\}$ that contains the numerical values of expert judgments, so that m_{ij} means the relative importance of object c_i with respect to the object c_j.

Definition 1. *A matrix M is said to be reciprocal if for all* $i, j \in \{1, \ldots, n\} : m_{ij} = 1/m_{ji}$, *and M is said to be consistent if for all* $i, j, k \in \{1, \ldots, n\} : m_{ij} \cdot m_{jk} = m_{ik}$.

Very often it is assumed that the matrix M is reciprocal, which is in line with the natural intuition according to which if c_i is two times larger than c_j, thus also c_j is two times smaller than c_i. Of course, there are exceptions to this rule[15]. On the other hand, the matrices M, as containing subjective opinions of humans, are usually inconsistent. Thus, there are triads in M in the form m_{ij}, m_{jk}, m_{ik} for which $m_{ij} \cdot m_{jk} \neq m_{ik}$. Moreover, the more consistent triad m_{ij}, m_{jk}, m_{ik} the closer $m_{ij} \cdot m_{jk}$ and m_{ik}. This leads to the observation that along with the decreasing inconsistency in the triad m_{ij}, m_{jk}, m_{ik} the ratio $m_{ij} \cdot m_{jk}/m_{ik}$ tends to 1. It is easy to see that the following is always true:

$$\frac{m_{ij}m_{jk}}{m_{ik}} \leq 1 \quad \text{or} \quad \frac{m_{ik}}{m_{ij}m_{jk}} \leq 1 \tag{1}$$

Let the distance between the smaller ratio out of the two defined above and 1 be the triad inconsistency. Formally:

Definition 2. *The triad inconsistency is:*

$$\mathscr{K}(t) \overset{df}{=} min\left\{\left|1 - \frac{m_{ij}}{m_{ik}m_{kj}}\right|, \left|1 - \frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\} \tag{2}$$

where t is the triad (m_{ik}, m_{ij}, m_{kj}) and $i < k < j$.

The above definition allows for easy formulation of the Koczkodaj's inconsistency index [13]. Let T be a set of triads in the form $t = (m_{ik}, m_{ij}, m_{kj})$ where $i < k < j$, composed from the entries of M. Thus,

Definition 3. The *Koczkodaj* inconsistency index is

$$\mathscr{K}(M) \overset{df}{=} \max_{t \in T} \mathscr{K}(t) \tag{3}$$

It is easy to see that (Def. 3) detects the worst case of triad inconsistency. Thus, the changes in other triads, except one with the maximal inconsistency, are not reflected in the value of $\mathscr{K}(M)$.

According to [20,6] one can define the *Pelï¿œez-Lamata* triad inconsistency as follows:

$$\mathscr{PL}(t) \overset{df}{=} \frac{m_{ij}}{m_{ik}m_{kj}} + \frac{m_{ik}m_{kj}}{m_{ij}} - 2 \tag{4}$$

where t is the triad (m_{ik}, m_{ij}, m_{kj}) and $i < k < j$. This leads to the definition of the index for the entire matrix M.

Definition 4. *The Pelï¿œez-Lamata inconsistency index is*

$$\mathscr{PL}(M) \overset{df}{=} \frac{6 \sum\limits_{t \in T} \mathscr{PL}(t)}{n(n-1)(n-3)}. \tag{5}$$

3. The new inconsistency indices

The triad inconsistency (Def. 2) can also be used to define inconsistency indices that will have opposite properties than (Def. 3). That is, every improvement of a triad inconsistency will entail decreasing the index, although it may happen that even the high triad inconsistency remains undetected when the rest of the triads are sufficiently consistent. The natural candidates for such indices are the average values of the triad inconsistencies.

Before defining them let us look closer to $\mathscr{K}(t)$. It is easy to see that $\mathscr{K}(t) < 1$. That is because $|1 - x| < 1$ for $x \in (0, 1]$, and $|1 - \frac{1}{x}| < 1$ for $x > 1$. On the other hand, a single triad (m_{ik}, m_{ij}, m_{kj}) corresponds to the one set $\{i, k, j\}$, where $i \neq j$, $j \neq k$, $i \neq k$, $i, j, k \in \{1, \ldots, n\}$ and n is the number of objects for comparison. Such sets could be selected in $\binom{n}{3}$ ways thus, the total number of distinct triads is $\binom{n}{3}$. This observation leads to the following estimation:

$$\sum_{t \in T} \mathscr{K}(t) < \binom{n}{3} = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{6} \tag{6}$$

where M is $n \times n$. Let the new inconsistency indices be

$$I_1(M) \overset{df}{=} \frac{6 \sum_{t \in T} \mathscr{K}(t)}{n(n-1)(n-2)} \tag{7}$$

and

$$I_2(M) \overset{df}{=} \frac{6 \sqrt{\sum_{t \in T} \mathscr{K}^2(t)}}{n(n-1)(n-2)} \tag{8}$$

Since $\mathscr{K}(t) < \mathscr{K}(M)$ for each $t \in T$, it is easy to see that

$$0 \leq I_2(M) \leq I_1(M) \leq \mathscr{K}(M) < 1. \tag{9}$$

In particular,

Remark 1. If M is consistent, then

$$I_2(M) = I_1(M) = \mathscr{K}(M) = 0. \tag{10}$$

The attentive reader may notice some similarity between the $I_1(M), I_2(M)$ inconsistency indices and the idea coming from *Pelï¿œez-Lamata*[20]. As it will be shown later, this similarity is superficial and indices I_1 and I_2 cannot be easily replaced by $\mathscr{PL}(M)$.

Similarly to the definition of strong equivalence of metrics[25] let us introduce the notion of equivalence of indices.

Definition 5. *Two inconsistency indices I and I' are called equivalent if there exist positive constants α,β such that for every $n \times n$ pairwise comparisons matrix M holds*

$$\alpha I(M) \leq I'(M) \leq \beta I(M). \tag{11}$$

One can easily notice that

$$\mathscr{K}(M) = \sqrt{\mathscr{K}^2(M)} \leq \sqrt{\sum_{t \in T} \mathscr{K}^2(t)}, \tag{12}$$

thus,

$$\mathscr{K}(M) \leq \binom{n}{3} I_2(M). \tag{13}$$

(9) and (13) imply that $\mathscr{K}(M)$, $I_1(M)$ and $I_2(M)$ are equivalent.

Notice, that definitions of $\mathscr{PL}(M)$, $I_1(M)$ and $I_2(M)$ look similar. However, for a given triad $t \in T$ expression $m_{ij}/m_{ik}m_{kj}$ is unbounded, so the index $\mathscr{PL}(M)$ may be arbitrary large. Thus, it is not equivalent to $\mathscr{K}(M)$, $I_1(M)$ or $I_2(M)$.

When we take the convex combinations of these indices we may obtain the whole family of indices:

Definition 6. *Let $\alpha, \beta, \alpha + \beta \in (0, 1)$. Put*

$$I_\alpha(M) = \alpha \mathscr{K}(M) + (1 - \alpha) I_1(M), \tag{14}$$

$$I_{\alpha,\beta}(M) = \alpha \mathscr{K}(M) + \beta I_1(M) + (1 - \alpha - \beta) I_2(M). \tag{15}$$

Of course,

$$0 \leq \alpha \mathscr{K}(M) < I_{\alpha,\beta}(M) < I_\alpha(M) \leq \mathscr{K}(M) < 1, \tag{16}$$

for each $\alpha, \beta, \alpha + \beta \in (0, 1)$, so $\mathscr{K}(M)$, $I_\alpha(M)$ and $I_{\alpha,\beta}(M)$ are also equivalent. In other words, most theorems proven for $\mathscr{K}(M)$ also hold for $I_\alpha(M)$ ($I_{\alpha,\beta}(M)$) and some α sufficiently close to 1 (and β close to 0).

Example. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 60 & 1 \\ 1 & 1 & 2 & \frac{1}{2} \\ \frac{1}{60} & \frac{1}{2} & 1 & \frac{1}{4} \\ 1 & 2 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 3 & \frac{5}{8} \\ 1 & 1 & 2 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{8}{5} & 2 & 4 & 1 \end{bmatrix}.$$

The rounded results of their indices' calculations are given in the table:

M	$\mathscr{K}(M)$	$\mathscr{PL}(M)$	$I_1(M)$	$I_2(M)$	$I_{\frac{1}{2}}(M)$	$I_{\frac{1}{2},\frac{1}{3}}(M)$
A	0.97	10.4	0.6	0.36	0.78	0.74
B	0.33	0.06	0.18	0.11	0.25	0.24

Hence,

$$0 < I_2(A) < I_1(A) < I_{\frac{1}{2},\frac{1}{3}}(A) < I_{\frac{1}{2}}(A) < \mathscr{K}(A) < 1 < \mathscr{PL}(A)$$

and

$$0 < \mathscr{PL}(B) < I_2(B) < I_1(B) < I_{\frac{1}{2},\frac{1}{3}}(B) < I_{\frac{1}{2}}(B) < \mathscr{K}(B) < 1.$$

4. Conclusion

Although the indices I_1 and I_2 are the opposite of *Koczkodaj's* index because they neglect the maximal triad inconsistency and follow every single change of M, the indices $I_\alpha(M)$ and $I_{\alpha,\beta}(M)$ combine properties of their predecessors. In the case of the local growth of $\mathscr{K}(t)$ for some $t \in T$ they are growing, and reversely, if $\mathscr{K}(t)$ drops down for some fixed $t \in T$ (and $\mathscr{K}(r)$ for $r \neq t$ and $r \in T$ is not rising) they are decreasing. On the other hand, these indices will never be smaller than $\alpha\mathscr{K}(M)$. Thus, they provide a guarantee that even a single but large triad inconsistency will not be ignored.

The parameters α, β and can be treated as the priority coefficients that determine the importance of the individual sub-indices. Thus, depending on the situation the customer can increase α at the expense of other factors, thereby increasing the importance of the maximal triad inconsistency, or decrease α assigning the average inconsistency of greater importance.

The concept of equivalence of inconsistency indices can also be used in the context of other indices than concerned. In particular it seems to be interesting to examine the equivalence classes introduced by this concept. Identification of mutual relationships between these classes can be interesting areas of further investigations.

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