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Generalization of a new definition of consistency for pairwise comparisons

Zbigniew Duszak [a], Waldemar W. Koczkodaj [b,*]

[a] Expert Systems Laboratory, Laurentian University, Sudbury, Ontario, Canada P3E 2C6

[b] Department of Computer Science, Laurentian University, Sudbury, Ontario, Canada P3E 2C6

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1. Consistency

The generalization of a new definition of consistency for pairwise comparisons (see [3]) is proposed. Making comparative judgments of intangible criteria (e.g. the degree of an environmental hazard or pollution factors) involves not only imprecise or inexact knowledge but also inconsistency in our own judgments. The improvement of knowledge elicitation by controlling the inconsistency of experts' judgments is not only desirable but absolutely necessary.

Checking consistency in the pairwise comparison method could be compared to checking that the divisor is not equal to 0. It simply does not make sense to divide anything by 0 and all proposed (heuristic) solutions to pairwise comparison models are based on an assumption that the given reciprocal matrix is consistent (see [5]). Can we not assume that the reciprocal matrix must be consistent? Requesting all the judgments to be consistent is not the answer. We know that most judgements are subjective and nearly always contain some type of bias.

The definition of consistency of a pairwise comparison matrix **A**, based on eigenvalues, was introduced by Saaty [5]. His consistency definition is given by the following formula,

$$cf = \lambda_A - \mathrm{order}(A)/(\mathrm{order}(A) - 1)\lambda_{\mathrm{random}},$$

where λ_A is the largest eigenvalue of the reciprocal matrix **A** and $\lambda_{\mathrm{random}}$ is the largest eigenvalue of randomly generated reciprocal matrix of the same order as matrix **A** (see [5]).

However, the above formula leads to some theoretical problems (see [6] and [3]). A new definition of consistency (see [3]) is based on one triad (A_i, A_j, A_k) of the comparisons matrix **A**. In this case, the pairwise comparisons matrix reduces to the following 3×3 basic reciprocal matrix **A3**,

$$\mathbf{A3}(A_i, A_j, A_k) = \begin{vmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{vmatrix},$$

where a expresses an expert's relative preference of criterion A_i, over A_j, b expresses preference of criterion A_i, over A_k, and c is a relative preference of

* Corresponding author. Partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant OGP 0036838 and by the Ministry of Northern Development and Mines through the Northern Ontario Heritage Fund Corporation. Email: waldemar@ramsey.cs.laurentian.ca.

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stimulus A_j, over stimulus A_k. Matrix $\mathbf{A3}(A_i, A_j, A_k)$ is consistent if and only if $b = ac$.

The new definition of consistency of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k$ is based on the following intuition: it is a measure of deviation from the nearest basic consistent reciprocal matrix. The interpretation of the consistency measure becomes more apparent when we reduce a basic reciprocal matrix to a vector of three coordinates $[a, b, c]$. We know that $b = ac$ holds for each consistent reciprocal matrix. Therefore, we can always produce three consistent reciprocal matrices (therefore three vectors) by computing one coordinate from the combination of the remaining two coordinates. These three vectors are: $[b/c, b, c]$, $[a, ac, c]$, and $[a, b, b/a]$. The inconsistency measure will be defined as the relative distance to the nearest consistent reciprocal matrix represented by one of these three vectors for a given metric. In the case of Euclidean (or Chebysheff) metric we have the following definition (see [3]):

Definition 1. Consistency measure $\mathbf{CM}(A_i, A_j, A_k)$ of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k)$ with the following ratios of comparisons $A_i/A_j = a$, $A_i/A_k = b$, $A_j/A_k = c$, where $a, b, c > 0$, is equal to

$$\begin{aligned}&\mathbf{CM}(A_i, A_j, A_k)\\ &\quad = \min\{|a - b/c|/a, |b - ac|/b, |c - b/a|/c\}.\end{aligned}$$

Note that the consistency measure is not a metric. It is a global matrix characteristic, which could be compared (in spirit) to the entropy of a probabilistic sample space. The definition of $\mathbf{CM}$ can be extended to the reciprocal matrices of any order. To achieve this we must first prove that Definition 1 does not depend on the order of criteria.

Lemma 2. *If $\mathbf{A3}(A_i, A_j, A_k)$ is a basic reciprocal matrix then, for each permutation $\tau : \mathbb{N}^3 \to \mathbb{N}^3$, $\mathbf{CM}(A_i, A_j, A_k) = \mathbf{CM}(A_{\tau(i)}, A_{\tau(j)}, A_{\tau(k)})$ where $\mathbb{N}$ denotes the set of all positive integers.*

Proof. Assume the following ratios, $A_i/A_j = a$, $A_i/A_k = b$, $A_j/A_k = c$ and $a, b, c > 0$; then:

$$\begin{aligned}&\mathbf{CM}(A_i, A_j, A_k) = \mathbf{CM}(a, b, c)\\ &\quad = \min\{|a - b/c|/a, |b - ac|/b, |c - b/a|/c\}\\ &\quad = \min\{|ac - b|/ac, |b - ac|/b, |b - ac|/ac\}\\ &\quad = \min\{|1 - b/ac|, |1 - ac/b|\}.\end{aligned}$$

For each permutation of indexes i, j, k we have:

$$\begin{aligned}&\mathbf{CM}(A_i, A_k, A_j) = \mathbf{CM}(b, a, 1/c)\\ &\quad = \min\{|1 - ac/b|, |1 - b/ac|\},\\ &\mathbf{CM}(A_k, A_i, A_j) = \mathbf{CM}(1/b, 1/c, a)\\ &\quad = \min\{|1 - b/ac|, |1 - ac/b|\},\\ &\mathbf{CM}(A_k, A_j, A_i) = \mathbf{CM}(1/c, 1/b, 1/a)\\ &\quad = \min\{|1 - ac/b|, |1 - b/ac|\},\\ &\mathbf{CM}(A_j, A_i, A_k) = \mathbf{CM}(1/a, c, b)\\ &\quad = \min\{|1 - ac/b|, |1 - b/ac|\},\\ &\mathbf{CM}(A_j, A_k, A_i) = \mathbf{CM}(c, 1/a, 1/b)\\ &\quad = \min\{|1 - b/ac|, |1 - ac/b|\}.\end{aligned}$$

The above equations complete the proof. □

As an immediate consequence of the above lemma we have obtained the following, equivalent formula for calculating the consistency measure of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k)$.

Definition 3. Consistency measure $\mathbf{CM}(A_i, A_j, A_k)$ of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k)$ with ratios of comparisons $A_i/A_j = a$, $A_i/A_k = b$, $A_j/A_k = c$, $a, b, c > 0$ is equal to

$$\begin{aligned}&\mathbf{CM}(A_i, A_j, A_k)\\ &\quad = \mathbf{CM}(a, b, c) = \min\{|1 - b/ac|, |1 - ac/b|\}.\end{aligned}$$

We can proceed now to the general case of any $n \times n$ reciprocal matrix.

Lemma 4. *The number of all possible triads of the $n \times n$ comparison matrix is equal to $n(n-1)(n-2)/3!$.*

Proof. To have inconsistent judgments we must have at least three criteria to be compared. Consequently we may assume that all indexes i, j, k must be pairwise different. According to Lemma 2, we may calculate inconsistencies only for triads with indexes holding the property $1 \leqslant i < j < k \leqslant n$. It is known that the number of such indexes equals the number of all three element subsets of the set $\{1, 2, \ldots, n\}$. By

Newton's formula it is equal $n!/(n-3)!3!$, which is $n(n-1)(n-2)/3!$. $\square$

Both of the above lemmas give the following definition of the consistency measure of any $n \times n$ pairwise comparison matrix $\mathbf{A}$.

Definition 5. Consistency measure $\mathbf{CM}(\mathbf{A})$ of an n by n $(n > 2)$ reciprocal matrix $\mathbf{A}$ is given by the following formula:

$$\mathbf{CM}(\mathbf{A}) = \max\{\mathbf{CM}(A_i, A_j, A_k) \mid 1 \leqslant i < j < k \leqslant n)\}.$$

Definition 5 gives the opportunity to design an algorithm for reducing the inconsistency of the expert's judgments. It should be seen as a technique for data validation in the knowledge acquisition process. The consistency measure of a comparison matrix is the measure of the validity of knowledge. To"improve" the quality of the knowledge, experts, with the help of computer software, compute the consistency of their judgments. The program displays the triad with the largest inconsistency, so the experts have a possibility to revise their preferences. The important point is that the system does not force the experts to change their judgment. Instead, the computer program flags the most critical spot in the set of judgments.

2. Applications

The authors worked in conjuction with the mining rehabilitation experts from the Provincial Ministry of Northern Development and Mines. An expert system is implemented for rehabilitation problems connected with abandoned mines in Ontario. The system will assist in semistructered decision situations. The main goal of this system is to provide management with the most comprehensive and most updated information necessary to make consistent decisions. Four major objectives of the system are:

- to develop a practical tool for priority setting and decision making procedures in coping with abandoned mine issues in Ontario,
- to ensure the most effective use of public funds allocated for mine rehabilitation work,
- to protect public interest for Ontario residents in public safety, public health, environmental concerns, social concerns and economic concerns,
- to find the way to link, in many cases contradictory, technical and socio-economic factors included in the decision making process.

In building our expert system, the process of knowledge acquisition is based on the pairwise comparisons method. The consistency measure described in this paper is the basic tool in the knowledge validation process (see [1] and [2] for details).

The authors are also involved in another project for a university library. The purpose is to help librarians select the most appropriate CD-ROM collection. The system will prioritize the CD-ROM titles according to the library policies and the librarians' preferences.

3. Conclusions

Weights, reflecting the relative importance of the objectives concerned are a valuable piece of information. There exist various ways of formulating priorities: trade-off methods, ratings, rankings, verbal statements, pairwise comparisons. Voogd (see [7]) reports that the ranking method is preferable by the majority of interviewed experts. There is no tool, however, for knowledge validation in this case. Under some circumstances it might be attractive to use a method that enables decision makers to express their priorities in a more refined way. Therefore, the pairwise comparisons method is proposed with the consistency measure as a knowledge elicitation technique. This approach, which we call a consistency-driven knowledge acquisition, improves the problem understanding and enhances quality of the knowledge acquired for the design of an expert system. The generalized definition is simplified for a more efficient computability and extended to a reciprocal matrix of any order.

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