Heuristic Rating Estimation Approach to The Pairwise Comparisons Method

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Abstract. The Heuristic Ratio Estimation *(HRE)* approach proposes a new way of using the pairwise comparisons matrix. It allows the assumption that the weights of some alternatives (herein referred to as concepts) are known and fixed, hence the weight vector needs to be estimated only for the other unknown values. The main purpose of this paper is to extend the previously proposed iterative *HRE* algorithm and present all the heuristics that create a generalized approach. Theoretical considerations are accompanied by a few numerical examples demonstrating how the selected heuristics can be used in practice.

1 Introduction

The first evidence of the usage of pairwise comparisons (herein abbreviated as PC) comes from *Ramon Llull* (the XIII century) [9,35], then the method was rediscovered in the XIX century by *Fechner* [12]. In the first half of the twentieth century it was developed by *Thurstone* [38]. The Analytic Hierarchy Process *(AHP)*, introduced by Saaty [33], was another important extension to the PC theory, providing handy methods for dealing with the large number of criteria. Many examples demonstrate the usefulness of the method [39,16,26,37]. Despite its long existence, research in the field of the PC research is still conducted. This is evidenced by the works discussing the strengths and weaknesses of the most popular *AHP* approach [11,31,3,34,2], and also by the works proposing the new *PC* paradigms, and exploring the new areas of applicability, such as the *Rough Set* theory approach [15], fuzzy *PC* relation handling [27,13,41,42], incomplete *PC* relation [6,14,22], data inconsistency reduction [24], non-numerical rankings [20] and others. A broader discussion of the *PC* method can be found in [36,17].

The newly proposed *HRE* approach [25] explores the use of the *PC* method in cases when some alternatives (herein referred to as concepts) have known and fixed priorities. Therefore, it divides the concepts into two sets - initially known elements C_K for which the weights are fixed and unknown elements C_U for which the weights need to be determined. Then, by iteratively averaging the available weights (initially only the weights of elements from C_K are available), subsequent propositions of the weight vector are computed.

The notion inherently integrated with the *PC* method is data inconsistency [8,7]. If the data are fully consistent then any single comparison provides enough information about the relative order and the intensity of preferences of two concepts. In such a case, after performing $n-1$ comparisons, the weights of all n concepts can be easily determined, provided that the n $-$ 1 comparisons involve all the concepts. Thus, any special way of data processing in order to derive the priorities of concepts is not needed. If the input data are inconsistent the best thing to be done is to propose a heuristic that, despite the data inconsistency, allows the weights of concepts to be calculated.

The presented work is a follow-up of [25]. It introduces the new ideas (the lack of reciprocity or the lack of data) and presents the concepts introduced in [25] in a more systematic and formal way. The *HRE* approach presented in this article includes four complementary heuristics that are useful for calculating weights when the reference set of initially known elements C_K is given (Sec. 3). Besides theoretical consideration the article examines the *HRE* weight derivation procedures on a few numerical examples (Sec. 4). The article is opened by two sections introducing the *PC* method (Sec. 1 and 2). A brief summary is provided in (Sec. 5). Additional explanations and definitions are placed in the appendices.

2 A pairwise comparisons method

Man always has to make choices. Therefore he/she always has to make comparisons. The best bet is when one (the better one) needs to be selected from a pair. People are accustomed to this type of comparison. In daily contact, in the market, where paying for a fruit everyone is trying to choose the heavier one. The relative weight of two fruits that look like they are the greatest can be easily estimated by comparing the weight of the fruit held in one hand with the weight of the fruit held in the other hand. Usually, making the right choice is possible without any additional tools indicating weight. In reality people have to compare much more complicated things than fruits. Often there is no way to make an accurate comparison. There is no 'weight' for the problem. Even worse, usually there are many different things that need to be compared. In such a case the PC approach comes to the rescue. It allows people to do what they do best - comparing pairs. The final synthesis of partial assessments is performed in accordance with predefined algorithms, such as the eigenvalue method or geometric mean method [19].

The input data to the *PC* method is a *(PC)* matrix $M = (m_{ij})$ and $m_{i,j} \in \mathbb{R}_+$ where $i, j \in \{1, \ldots, n\}$ represents partial assessments over the finite set of concepts $C \overset{df}{=} \{c_i \in \mathscr{C}, \, i \in \{1, \ldots, n\}\}$ where $\mathscr{C} \neq \emptyset$ is a universe of concepts. Let $\mu : C \nrightarrow \mathbb{R}_+$ be a partial function that assigns to some concepts from $C \subset \mathscr{C}$ positive values from $\mathbb{R}_+$. Thus, the value $\mu(c)$ represents the importance of c. The output of the *PC* method is the function μ defined for all $c \in C$. It introduces the total order in C and usually will be written in the form of a vector of weights $\mu \overset{df}{=} [\mu(c_1), \ldots \mu(c_n)]^T$ (see Fig. 1).

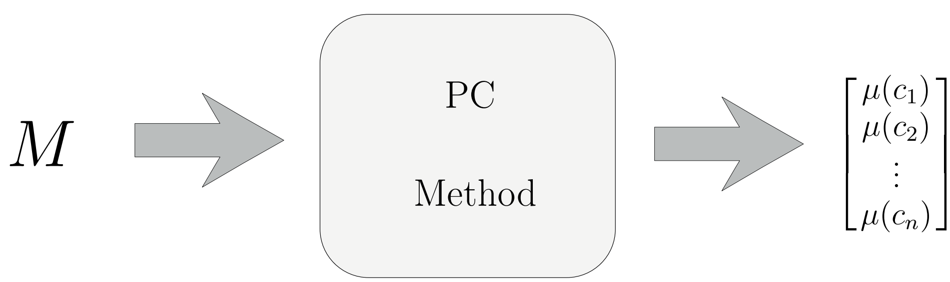

Fig. 1. PC Method input-output scheme

Concepts, originally referred to in the literature as subjective stimuli [38], alternatives [2] or activities [30], represent objects for which the relative importance indicators m_{ij} and m_{ji} need to be assessed.

It is assumed that, according to the best knowledge of experts, the importance of c_i equals m_{ij} of the importance of c_j i.e. $\mu(c_i) = m_{ij}\mu(c_j)$. The matrix M is said to be reciprocal if $\forall i, j \in \{1, \ldots, n\} : m_{ij} = \frac{1}{m_{ji}}$. This property reflects the intuition that if the relative importance ratio c_i to c_j is m_{ij} then the the importance ratio c_j to c_i should be $1/_{m_{ij}=m_{ji}}$. However, intuitive reciprocity may not always be met. The matrix M without reciprocity property is sometimes referred to in the literature as a generalized *PC* matrix [23].

Ideally M is also consistent i.e. $\forall i, j, k \in \{1, \ldots, n\} : m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$. Unfortunately, the knowledge stored in the *PC* matrix usually comes from different experts, the consistency condition may not be met. In such a case, reasoning using M may give ambiguous results. This leads to the data consistency (and inconsistency) concept formalized in the form of the inconsistency index. There are several different inconsistency indexes, including the *Eigenvector Method* [33], *Least Squares Method*, *Chi Squares Method* [7], *Koczkodaj's distance based inconsistency index* [21] and others. The most popular eigenvalue based approach [33] defines the consistency index (sometimes referred as the consistency ratio) as

$$CI = \frac{\lambda_{max} - n}{n - 1} \tag{1}$$

where $\lambda_{\max}$ is the principal eigenvalue of $n \times n$ matrix M. The iterative *HRE* algorithm [25] adopts the last of them as a convenient and easy to use 'gauge' of data inconsistency. Koczkodaj's inconsistency index $\mathscr{K}$ of $n \times n$ and $(n > 2)$ reciprocal matrix M is equal to:

$$\mathscr{K}(M) = \max_{i,j,k \in \{1,\ldots,n\}} \left\{ \min \left\{ \left| 1 - \frac{m_{ij}}{m_{ik} m_{kj}} \right|, \left| 1 - \frac{m_{ik} m_{kj}}{m_{ij}} \right| \right\} \right\} \quad (2)$$

where $i, j, k = 1, \ldots, n$ and $i \neq j \wedge j \neq k \wedge i \neq k$.

There are also several different methods of deriving the weights vector out of the matrix M [34,19]. Two the most popular are the eigenvector method [33] and the geometric mean method. According to the first one, the output μ (denoted as μ_{EV}) is the rescaled principal eigenvector of M, i.e.:

$$\mu_{EV} = \left[\frac{v_1}{s_{EV}}, \ldots, \frac{v_n}{s_{EV}} \right]^T \quad \text{where} \quad s_{EV} = \sum_{i=1}^{n} v_i \quad (3)$$

and $v = [v_1, \ldots, v_n]^T$ is the principal eigenvector of M. The second method [10] proposes the adoption of rescaled geometric means of rows of M as the output μ . Thus,

$$\mu_{GM} = \left[\frac{g_1}{s_{GM}}, \ldots, \frac{g_n}{s_{GM}} \right] \quad (4)$$

where

$$g_i = \left(\prod_{j=1}^{n} m_{ij} \right)^{1/n} \quad \text{and} \quad s_{GM} = \sum_{i=1}^{n} g_i \quad (5)$$

Other the priority deriving methods in the *AHP* approach can be found in [19,18,40].

3 The HRE Algorithm Approach

The *HRE* approach to the rating estimation in the pairwise comparisons method is based on a few intuitive heuristics. The first of them concerns dividing the set of concepts into known (reference) and unknown elements. Initially, μ is defined only for reference elements. Hence, only these elements can be used to estimate μ for unknown elements. With every subsequent step μ is specified for more and more elements. Thus, increasing the number of elements could be taken into account during calculations. The weights of initially known reference elements remain unchanged. Thus, the subsequent updates affect only unknown elements. In every step weights for unknown elements are determined as the arithmetic mean of determined values and the appropriate ratios (6). This iterative procedure forms an averaging with respect to the reference heuristics (a more detailed description in Sec. 3.1). Therefore, comparing with the eigenvalue based method, the *HRE* approach requires additional information about the reference elements (see Fig. 2).

Fig. 2. *HRE* approach input-output scheme. C_K means the non-empty set of reference concepts.

In fact, sometimes the *HRE* procedure is equivalent to finding a solution for some linear equation system. Hence, if this equation system has an admissible solution, then its solution can be adopted as the output of the *HRE* algorithm. If not, the weight vector needs to be determined with the help of the second heuristic (Sec. 3.2) as explained later in the work.

In general, the pairwise comparisons method assumes that the input matrix is reciprocal. It means that the ratio m_{ij} expressing the relative importance of c_i compared to c_j should be the inverse of m_{ji}. Unfortunately, this assumption may not always be met [24]. Since the situation when $m_{ij} \neq 1/m_{ji}$ is undesired, but possible in practice, the third heuristic proposes a simple method for calculating the new values $\widehat{m}_{ij}, \widehat{m}_{ji}$ so that they are mutually reciprocal and possibly close to the original m_{ij}, m_{ji}. The operation is called reciprocity restoration and is applied to any matrix M, which is processed by the *HRE* algorithm.

The fourth heuristic addresses the problem of incomplete data, where not all the ratios m_{ij} are known. Some of them can be recovered based on the assumed reciprocity. However, if both m_{ij} and its counterpart m_{ji} are unknown the reciprocity property does not help. In such a case, either the missing ratios are reconstructed [6,22] so that the standard methods can deal with the reconstructed matrix, or the procedure alone has to deal with the problem of missing matrix entries. The iterative *HRE* approach does not need the matrix reconstruction. During the course of the iterative procedure, the new ratio values are computed using only those defined concepts that are reachable due to the availability of an appropriate ratio. In other words, if some ratio is missing all the multiplications which use the missing ratio are excluded from the basic update formula (6).

3.1 Heuristics of averaging with respect to reference values

The iterative averaging approach presented in [25] assumes that the set of concepts $C = C_K \cup C_U$ and $C_K \cap C_U = \emptyset$, where C_K denotes concepts for which the actual value μ is initially known, and C_U contains concepts for which the value μ needs to be determined. The relation between different concepts in C is represented by M so that in the case of the fully consistent matrix it holds that $\mu(c_i)m_{ji} = \mu(c_j)$. Hence, for a known, complete matrix M and $c_i \in C_K, c_j \in C_U$ determining c_j boils down to the performance of a single multiplication. Since M is usually inconsistent, the *HRE* algorithm considers $m_{ji}\mu(c_i)$ as a sample of $\mu(c_j)$, where the expected value of $\mu(c_j)$ is the arithmetic mean of the values $m_{ji}\mu(c_i)$. Of course, not all the values $\mu(c_i)$ are defined at the very beginning, but only those for which $c_i \in C_K$. Hence, in the first step of the *HRE* procedure, the values $\mu(c_j)$ are estimated only on the basis of the initially known concepts. However, in the second step (assuming that M is complete) all the other values $\mu(c_i)$ computed during the first step (for $i \neq j$ and $c_i \in C_K \cup C_U$) can be used to determine $\mu(c_j)$. Thus, for every concept $c_j \in C_K$ the r'th subsequent estimation of $\mu_r(c_j)$ computed by the *HRE* iterative procedure (see [25]) meets the equation:

$$\mu_r(c_j) = \frac{1}{\left|C_j^{r-1}\right|} \sum_{c_i \in C_j^{r-1}} m_{ji}\mu_{r-1}(c_i) \tag{6}$$

where

$$C_j^{r-1} = \{c \in C : \mu_{r-1}(c) \text{ is known and } c \neq c_j\}, \text{ and } C_j^0 \overset{df}{=} C_K \tag{7}$$

and $\left|C_j^{r-1}\right|$ is the cardinality (number of elements) of C_j^{r-1} [25]. For simplicity, let us assume that $C_U = \{c_1, \ldots, c_k\}$ and $C_K = \{c_{k+1}, \ldots, c_n\}$. It turns out that the iterative procedure proposed in [25] follows the Jacobi iterative method for solving a linear equation system in the form[1]:

$$A\mu = b \tag{8}$$

where the matrix A is given as:

$$A = \begin{bmatrix} 1 & -\frac{1}{n-1}m_{1,2} & \cdots & -\frac{1}{n-1}m_{1,k} \\ -\frac{1}{n-1}m_{2,1} & 1 & \cdots & -\frac{1}{n-1}m_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{n-1}m_{k-1,1} & \cdots & \ddots & -\frac{1}{n-1}m_{k-1,k} \\ -\frac{1}{n-1}m_{k,1} & \cdots & -\frac{1}{n-1}m_{k,k-1} & 1 \end{bmatrix} \tag{9}$$

[1] The form of the linear equation system (8) is more thoroughly explained in Appendix C.

vector of constant terms is

$$b = \begin{bmatrix} \frac{1}{n-1}m_{1,k+1}\mu(c_{k+1}) + \ldots + \frac{1}{n-1}m_{1,n}\mu(c_n) \\ \frac{1}{n-1}m_{2,k+1}\mu(c_{k+1}) + \ldots + \frac{1}{n-1}m_{2,n}\mu(c_n) \\ \vdots \\ \frac{1}{n-1}m_{k,k+1}\mu(c_{k+1}) + \ldots + \frac{1}{n-1}m_{k,n}\mu(c_n) \end{bmatrix} \tag{10}$$

and values that need to be determined are denoted as:

$$\mu^T = [\mu(c_1), \ldots, \mu(c_k)] \tag{11}$$

The iteration matrix of the Jacobi method is given by:

$$B_J = D^{-1}(E + F) = I - D^{-1}A \tag{12}$$

The matrix D is the diagonal matrix of the diagonal entries of A, hence $D = D^{-1} = I$, whilst E is the lower triangular matrix of entries $e_{ij} = -\left(-\frac{1}{n-1}m_{i,j}\right) = -a_{ij}$, and F is the upper triangular matrix of entries $f_{ij} = -\left(-\frac{1}{n-1}m_{i,j}\right) = -a_{ij}$. Therefore, the update equation (6) can be written in the form:

$$\mu_r(c_i) = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1, j\neq i}^{k} a_{ij}\mu_{r-1}(c_j)\right] = b_i + \sum_{j=1, j\neq i}^{k} \frac{1}{n-1}m_{ij}\mu_{r-1}(c_j) \tag{13}$$

When the matrix A is strictly diagonally dominant by rows i.e. $|a_{ii}| > \Sigma_{j=1}^{n} |a_{ij}|$ for $i \neq j$ and $i = 1, \ldots, k$ then the Jacobi method is convergent [29][2]. In our case $a_{ii} = 1$ for $i = 1, \ldots, k$, hence the *HRE* procedure is convergent if

$$1 > \sum_{j=1, j\neq i}^{k} |a_{ij}| \tag{14}$$

for all $i = 1, \ldots, k$. Bearing in mind that $a_{ij} = -\frac{1}{n-1}m_{ij}$ let us note the *HRE* algorithm has a high chance to be convergent if the set C_U is relatively small (C_K is relatively large) and m_{ij} are not too large i.e. estimated values $\mu(c_j)$ for $j = 1, \ldots, k$ are similar. Both of these conditions are intuitive and, in practice, are likely to be satisfied. The first of them reflects the natural desire to provide the experts with rather more than the lower number of known reference concepts. The second corresponds to the common-sense observation that all the considered concepts should be similar to each other, because then it is easy to compare them.

The equation (8) could also be solved using direct methods. In such a case it has exactly one solution, if the determinant of A differs from 0, i.e.:

$$\det(A) \neq 0 \tag{15}$$

Unfortunately, it may turn out that this unique solution $\mu = (\mu(c_1), \ldots, \mu(c_k))$ is not in $\mathbb{R}^k_+$. For instance, some values $\mu(c_i)$ may be less than or equal 0. In such a case the iterative approach is not convergent (assuming that $\mu(c_j)$ for $c_j \in C_K$ are strictly positive, the values $\mu(c_i)$ for $i = 1, \ldots, k$ must also be strictly positive[3]). In such a case μ that meets (8) cannot be adopted as the *HRE* procedure output. Instead, the *HRE* procedure needs to be iterated a predetermined number of times and the result μ needs to be chosen following the minimizing estimation error heuristics (Sec. 3.2). In the presented approach, only those m_{ji} are determined by experts for which at least one of the two c_i, c_j comes from C_U. For two initially known concepts the value m_{ji} is just defined as $m_{ji} = {}^{\mu(c_j)}/_{\mu(c_i)}$. Hence, the matrix M is always consistent in the part relating to the known concepts i.e. $\forall c_i, c_j \in C_K : m_{ji}\mu(c_i) = \mu(c_j)$.

When the μ values are not initially known for any $c \in C$, i.e. $C_K = \emptyset$, then for an arbitrarily selected c_i the value $\mu(c_i)$ might be set by the experimenter to 1. In such a case the *HRE* procedure computes the relative order μ of concepts from C assuming that the weight $\mu(c_i)$ is a unit. Since c_i is treated as

[2] Note that the Jacobi method is convergent also for A strictly dominant by columns [1]

[3] Note that all the components of the right side of 13 are strictly positive.

the reference element it must be selected with special care. Their relationship with other concepts have a reference meaning, hence they should be highly reliable and well documented.

The final weight vector μ_{HRE} is synthesized by using k values determined by solving (8) and $n-k$ initially known reference values of concepts from C_K.

$$\mu_{HRE} = [\mu(c_1), \ldots \mu(c_k), \mu(c_{k+1}), \ldots, \mu(c_n)]^T \tag{16}$$

Thus the rescaled form of μ_{HRE} is:

$$\mu_{HREn} = \left[\frac{\mu(c_1)}{s_{HRE}}, \ldots, \frac{\mu(c_n)}{s_{HRE}}\right] \quad \text{where} \quad s_{HRE} = \sum_{i=1}^{n} \mu(c_i) \tag{17}$$

3.2 Heuristics of minimizing estimation error

The minimizing estimation error heuristics is proposed to deal with the case when it is impossible to uniquely determine $\mu(c_i)$ as the mean of $m_{ij}\mu(c_j)$ for $i \neq j$ (the vector μ cannot be determined by solving[4] (8)). In tests, it was noticed that the more often it happens, the higher the inconsistency. In such a case, rather than solving (8) someone may try to find μ that minimizes the average absolute estimation error, given as follows:

$$\widehat{e}_{\mu} = \frac{1}{|C_U|} \sum_{c \in C_U} e_{\mu}(c) \tag{18}$$

where

$$e_{\mu}(c_j) = \frac{1}{\left|C_j^{r-1}\right|} \sum_{c_i \in C_j^{r-1}} |\mu(c_j) - \mu(c_i) \cdot m_{ji}| \tag{19}$$

The problem of minimizing $\widehat{e}_{\mu}$ is discussed in (Appendix A). The preliminary *Monte Carlo* tests show that for the relatively small inconsistency (small $\mathscr{K}$) both: the minimizing estimation error heuristic (as defined above) and the averaging with respect to the reference values heuristic (Sec. 3.1) lead to very similar vectors μ. When the inconsistency index $\mathscr{K}$ rises then the solutions provided by these two heuristics become increasingly different. In general, it seems that the heuristic of averaging with respect to the reference values is more useful in practice. However, when the equation (8) does not have an admissible solution and there is an admissible μ minimizing (18), then the minimizing estimation error heuristic may be worth considering.

Certainly the search for the smallest $\widehat{e}_{\mu}$ makes sense if both: solving (8) and finding μ, which minimizes (18) fail. Then the intermediate *HRE* iteration result with the minimal absolute estimation error $\widehat{e}_{\mu_r}$ needs to be adopted as the output μ_{out} of the *HRE* procedure:

$$\mu_{out} = \left\{\mu_q : \widehat{e}_{\mu_q} = \min\{\widehat{e}_{\mu_1}, \ldots, \widehat{e}_{\mu_r}\}\right\} \tag{20}$$

Although r - the total number of iterations has to be arbitrarily set by an experimenter, in practice, it should be small enough (even one or two iterations may be useful).

3.3 Heuristics of reciprocity restoration

According to these heuristics, the input *PC* matrix M should be reciprocal to be processed by the *HRE* procedure. Hence, it should hold that $m_{ij} = \frac{1}{m_{ji}}$ for every two ratios m_{ij} and m_{ji} in M. Therefore, if the matrix M is not reciprocal, it should be transformed to a similar but reciprocal matrix. Let $\widehat{M} = [\widehat{m}_{ij}]$ be the new *PC* matrix obtained from $M = [m_{ij}]$ by replacing entry m_{ij} in M by the geometric mean of this entry and its (possibly reciprocal) counterpart i.e. $\widehat{m}_{ij} = \left(m_{ij}\frac{1}{m_{ji}}\right)^{1/2}$. It is easy to check that the new matrix $\widehat{M}$ is reciprocal. Moreover, if M is initially reciprocal then $\widehat{M} = M$. Therefore, every *PC* matrix M calculated by the *HRE* procedure should be preprocessed in order to restore the lost reciprocity property. If M is reciprocal the preprocessed matrix should be identical to M. If not, it is recommended to transform M into $\widehat{M}$ according to the definition given above. A similar approach to the lack of the reciprocity property has been discussed, for example, in [13]. The geometric mean properties have been discussed in [10].

[4] For the purpose of the *HRE* approach only $\mu \in \mathbb{R}_+^k$ are admissible.

3.4 Heuristics of missing data

Sometimes there may be a situation that not all indispensable ratios are defined. Then the resulting pairwise comparisons matrix M is incomplete and contains unknown values. In such a case the update equation (6) cannot include the products $m_{ji}\mu(c_i)$ where m_{ji} is not specified. Let us denote $m_{ji} = ?$ if m_{ji} is unspecified.

To handle this situation, the set of elements for which the μ_{r-1} values were known needs to be changed as follows:

$$C_j^{r-1} = \{c \in C : \mu_{r-1}(c) \text{ is known}, c \neq c_j \text{ and } c \neq c_i \text{ when } m_{ji} = ?\} \tag{21}$$

Although incomplete, M should be reciprocal. Hence the reciprocity restoration procedure needs to be extended to the case when some ratios are unknown. Thus, let us define:

$$\widehat{m}_{ij} = \begin{cases} \left(m_{ij}\frac{1}{m_{ji}}\right)^{1/2} & \text{where } m_{ji} \text{ and } m_{ij} \text{ are specified in } M \\ m_{ij} & \text{where } m_{ji} \text{ is unspecified in } M \\ \frac{1}{m_{ji}} & \text{where } m_{ij} \text{ is unspecified in } M \\ ? & \text{where } m_{ji} \text{ and } m_{ij} \text{ are unspecified in } M \end{cases} \tag{22}$$

The *HRE* algorithm equipped with the heuristics of missing data can handle matrices to which other methods might not be applicable[5]. The only limitation is the reachability of the unknown concepts understood as the condition that for each unknown concept $c_j \in C_U$ there must exist at least one concept $c_i \in C_K$ with known weight and a sequence of indices $i_1, i_2, \ldots, i_q$ such that $m_{ii_1} \neq ?, m_{i_1 i_2} \neq ?, \ldots, m_{i_q j} \neq ?$, where $i_1, i_2, \ldots, i_q \in \{1, 2, \ldots, n\}$. Therefore, the *HRE* procedure is able to propose the value $\mu_q(c_j)$ for $c_j \in C_U$ only if there is at least one $c_r \in C_U$ for which the product $m_{j,r}\mu_{q-1}(c_r)$ is known. In the case of an incomplete matrix the weights cannot be obtained by solving a linear equation system as shown in (8). In particular, due to the missing data, the set of known values C_j^{r-1} for j such that $c_j \in C_U$ may change for the second and subsequent iteration. Thus, the incomplete data requires an iterative approach when every subsequent value of $\mu(c_j)$ is estimated according to the update rule (6). If the procedure converges, a sufficiently accurate approximation might be adopted as the output. If not, the one with the smallest $\widehat{e}_\mu$ from the several initial iteration results needs to be adopted as the result of the procedure.

The missing data heuristics might be especially useful when a large number of different concepts should be compared with each other. In such a case the completion of all the ratios in the matrix M might be difficult, which may result in its incompleteness.

4 Numerical examples

Despite the fact that in the *HRE* approach the priorities of some concepts have to be initially known, the procedure might be used (for caution) in any case. However, this will require the adoption of arbitrarily selected elements as the reference concepts. The first numerical example (from [2]) demonstrates the case when the standard *PC* matrix is processed by the *HRE* algorithm and the arbitrary concept is chosen as the reference one. The second example shows a typical situation for *HRE*. There is a non-empty set C_K of the reference concepts and the set C_U consists of unknown elements. The third example addresses the problem of non-reciprocal matrices and demonstrates how the heuristic of reciprocity restoration works in practice. The last, fourth, example deals with an incomplete *PC* matrix. It uses an iterative version of the *HRE* procedure to derive the weight vector from M. It is designed to demonstrate how the iterative *HRE* procedure equipped with the missing data heuristic may support incomplete *PC* data sets.

Example 1 (Case of verbal judgements)

Let $c_1, \ldots, c_5$ be a set of concepts for which the following judgements were formulated by a person J: c_1 equally to moderately dominates c_2, c_1 moderately dominates c_3 , c_1 strongly dominates c_4, c_1 extremely dominates

[5] Various methods have their extensions to enable them to handle such cases, for example, the LSM extension can be found in [5].

c_5, c_2 equally to moderately dominates c_3, c_2 moderately to strongly dominates c_4, c_2 extremely dominates c_5, c_3 equally to moderately dominates c_4, c_3 very strongly dominates c_5, c_4 very strongly dominates c_5. Then, adopting the method of converting verbal judgements into numbers proposed in [32] the following *PC* matrix is obtained:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 3 & 5 & 9 \\ \frac{1}{2} & 1 & 2 & 4 & 9 \\ \frac{1}{3} & \frac{1}{2} & 1 & 2 & 8 \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{2} & 1 & 7 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{8} & \frac{1}{7} & 1 \end{bmatrix} \tag{23}$$

The rescaled eigenvector μ_{EV} (see 3) corresponding to the maximal eigenvalue of M is:

$$\mu_{EV} = \begin{bmatrix} 0.426\ 0.281\ 0.165\ 0.101\ 0.027 \end{bmatrix}^T \tag{24}$$

The geometric mean based weight vector (see 4) for M is:

$$\mu_{GM} = \begin{bmatrix} 0.424\ 0.284\ 0.169\ 0.098\ 0.026 \end{bmatrix}^T \tag{25}$$

As reported in [2] the ranking μ_{EV} does not meet the *Condition of Order Preservation* (herein abbreviated as *COP*, see Appendix B). In particular, since the value $m_{1,4} = 4$ is smaller than $m_{4,5} = 7$, *COP* also requires that $\frac{\mu_{EV}(c_1)}{\mu_{EV}(c_4)} < \frac{\mu_{EV}(c_4)}{\mu_{EV}(c_5)}$. It is easy to calculate that $\frac{\mu_{EV}(c_1)}{\mu_{EV}(c_4)} = 4.218$ and $\frac{\mu_{EV}(c_4)}{\mu_{EV}(c_5)} = 3.741$ which is in contradiction with the second *COP* postulate (58). It is easy to check that for μ_{GM} *COP* does not hold either. The eigenvalue based inconsistency index is low and equals $CI = 0.057$. In contrast, Koczkodaj's distance based inconsistency index is high[6] and equals $\mathscr{K}(M) = 0.743$.

To calculate the rank using the *HRE* approach when none of the concepts are initially known (i.e. $C_K = \emptyset$), it is necessary to choose some $c \in C_U$ and assign an arbitrary weight to it. Thus, based on our knowledge about the problem domain, let us assume that c_1 is a reference element ($C_K = \{c_1\}$ and $C_U = C_U \backslash \{c_1\}$) and set $\mu(c_1) = 1$. (It is easy to check that for a rescaled form of a weight vector μ the exact value assigned to $\mu(c_1)$ is not important). Then, after the first *HRE* iteration, the matrix A and vector b are determined[7],

$$A = \begin{bmatrix} 1 & -\frac{1}{n-1}m_{2,3} & -\frac{1}{n-1}m_{2,4} & -\frac{1}{n-1}m_{2,5} \\ -\frac{1}{n-1}m_{3,2} & 1 & -\frac{1}{n-1}m_{3,4} & -\frac{1}{n-1}m_{3,5} \\ -\frac{1}{n-1}m_{4,2} & -\frac{1}{n-1}m_{4,3} & 1 & -\frac{1}{n-1}m_{4,5} \\ -\frac{1}{n-1}m_{5,2} & -\frac{1}{n-1}m_{5,3} & -\frac{1}{n-1}m_{5,4} & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{n-1}m_{2,1}\mu(c_1) \\ \frac{1}{n-1}m_{3,1}\mu(c_1) \\ \frac{1}{n-1}m_{4,1}\mu(c_1) \\ \frac{1}{n-1}m_{5,1}\mu(c_1) \end{bmatrix} \tag{26}$$

so that the equation (8) takes the form:

$$\begin{bmatrix} 1 & -0.5 & -1 & -2.25 \\ -0.125 & 1 & -0.5 & -2 \\ -0.062 & -0.125 & 1 & -1.75 \\ -0.028 & -0.031 & -0.036 & 1 \end{bmatrix} \begin{bmatrix} \mu(c_2) \\ \mu(c_3) \\ \mu(c_4) \\ \mu(c_5) \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.083 \\ 0.05 \\ 0.028 \end{bmatrix} \tag{27}$$

(Note that $|C_U| = 4$ implies that the dimensions of matrix A are 4×4). Since $\det(A) \neq 0$ and $\mu(c_i) > 0$ for $i = 2, \ldots, 5$ then the rescaled vector μ_{HREn} obtained by solving (8) is adopted as the output of the *HRE* algorithm (an iterative procedure leads to the same solution).

$$\mu_{HREn} = \begin{bmatrix} 0.368\ 0.311\ 0.182\ 0.11\ 0.028 \end{bmatrix}^T \tag{28}$$

By examining all the possible cases, it is easy to check that the weight vector μ_{HREn} satisfies *COP*. It is noteworthy that all the three vectors: μ_{EV}, μ_{GM} and μ_{HREn} preserve the same order of elements and they differ only in intensities of preferences. Since the value $\mu(c_1)$ is chosen arbitrarily by an experimenter, the

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[6] The work [24] suggests that an acceptable threshold of inconsistency $\mathscr{K}(M)$, for most practical applications, turns out to be $1/3$.

[7] In practice, the matrix A can be obtained from the matrix M by removing the rows and columns corresponding to concepts from C_K, and multiplying the remaining values (except diagonal) by $-1/(n-1)$. The removed rows and columns form the vector b as shown in (10).

obtained result has only an ordinal meaning. As both vectors μ_{HREn} and μ_{HRE} carry the same (ordinal) information it is convenient to consider the rescaled vector μ_{HREn}.

Example 2 (Case with reference concept values)

The immediate inspiration for the second example is the scientific units evaluation in Poland. The proposed ranking algorithm [28] is based on the pairwise comparisons paradigm although it does not follow the *AHP* approach. The reference scientific units (as defined therein) are used to determine the scientific categories, and thereby funding levels.

Let $c_1, \ldots, c_5$ represent the hypothetical scientific units, where two of them c_2 and c_3 are the reference units for which the values $c_2, c_3 \in C_K$ are initially known and equal $\mu(c_2) = 5$ and $\mu(c_3) = 7$. The analysis of the scientific units c_1, c_4 and c_5 with respect to the criterion μ allows the formulation of the following pairwise comparisons matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & \frac{3}{5} & \frac{4}{7} & \frac{5}{8} & \frac{1}{2} \\ \frac{5}{3} & 1 & \frac{5}{7} & \frac{5}{2} & \frac{10}{3} \\ \frac{7}{4} & \frac{7}{5} & 1 & \frac{7}{2} & 4 \\ \frac{8}{5} & \frac{2}{5} & \frac{2}{7} & 1 & \frac{4}{3} \\ 2 & \frac{3}{10} & \frac{1}{4} & \frac{3}{4} & 1 \end{bmatrix} \tag{29}$$

The rescaled eigenvector μ_{EV} (see 3) and the rescaled geometric mean based vector μ_{GM} (see 4) for M are as follows:

$$\mu_{EV} = \begin{bmatrix} 0.12 \; 0.275 \; 0.356 \; 0.131 \; 0.118 \end{bmatrix}^T \tag{30}$$

and

$$\mu_{GM} = \begin{bmatrix} 0.113 \; 0.28 \; 0.359 \; 0.133 \; 0.114 \end{bmatrix}^T \tag{31}$$

The *HRE* approach requires the solution of the linear equation system for A and b as follows[8]:

$$A = \begin{bmatrix} 1 & -\frac{1}{n-1}m_{1,4} & -\frac{1}{n-1}m_{1,5} \\ -\frac{1}{n-1}m_{4,1} & 1 & -\frac{1}{n-1}m_{4,5} \\ -\frac{1}{n-1}m_{5,1} & -\frac{1}{n-1}m_{5,4} & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{n-1}m_{1,2}\mu(c_2) + \frac{1}{n-1}m_{1,3}\mu(c_3) \\ \frac{1}{n-1}m_{4,2}\mu(c_2) + \frac{1}{n-1}m_{4,3}\mu(c_3) \\ \frac{1}{n-1}m_{5,2}\mu(c_2) + \frac{1}{n-1}m_{5,3}\mu(c_3) \end{bmatrix} \tag{32}$$

hence, numerically:

$$\begin{bmatrix} 1 & -0.156 & -0.125 \\ -0.4 & 1 & -0.333 \\ -0.5 & -0.187 & 1 \end{bmatrix} \begin{bmatrix} \mu(c_1) \\ \mu(c_4) \\ \mu(c_5) \end{bmatrix} = \begin{bmatrix} 1.75 \\ 1.0 \\ 0.812 \end{bmatrix} \tag{33}$$

The not rescaled μ_{HRE} weight vector is:

$$\mu_{HRE} = \begin{bmatrix} 2.527 \; 5.0 \; 7.0 \; 2.88 \; 2.616 \end{bmatrix}^T \tag{34}$$

and after rescaling:

$$\mu_{HREn} = \begin{bmatrix} 0.126 \; 0.249 \; 0.349 \; 0.144 \; 0.13 \end{bmatrix}^T \tag{35}$$

The inconsistency indices are $CI = 0.07$ (AHP) and $\mathscr{K}(M) = 0.781$ (Koczkodaj).

It is easy to observe that in this hypothetical case the eigenvalue vector μ_{EV} also violates *COP*. That is because the ratio $m_{1,5} = \frac{1}{2} < 1$, whilst $\frac{\mu_{EV}(c_1)}{\mu_{EV}(c_5)} > 1$. The μ_{GM} and μ_{HRE} do not violate the first *COP* postulate (Appendix B). However, all the vectors μ_{EV}, μ_{GM} and μ_{HRE} do not meet the second *COP* postulate.

[8] note that $|C_U| = 3$ implies that the dimensions of matrix A are 3×3

Example 3 (Case of not reciprocal matrix)

The third example concerns a situation when the PC matrix is almost consistent but not reciprocal. Due to the lack of reciprocity, the use of the eigenvalue method as well as the geometric means method might be disputed (these methods are designed for reciprocal matrices [19]). Hence, the values μ_{EV} and μ_{GM} are computed just for testing the robustness and sensitivity of both methods to the incorrect data.

Let $c_1, \ldots, c_4$ represent four candidates for the position of a manager in some production company. As different examiners have been involved in the recruitment process, one examiner rated c_4 social skills twice as high as c_1, whilst another examiner, while comparing skills c_1 to c_4, ruled that both candidates are exactly on the same level. Assuming that in all other cases the recruitment committee has ruled that all other candidates present the same level of social skills, the PC matrix M representing the problem may appear as follows:

$$\mathsf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \tag{36}$$

An attempt to calculate the eigenvector based or geometric mean based rank leads to the following vectors:

$$\mu_{EV} = \begin{bmatrix} 0.236 & 0.236 & 0.236 & 0.292 \end{bmatrix}^T \tag{37}$$

$$\mu_{GM} = \begin{bmatrix} 0.239 & 0.239 & 0.239 & 0.284 \end{bmatrix}^T \tag{38}$$

For the purpose of the HRE algorithm the reciprocity property of M must be restored. Thus, according to the heuristics of reciprocity restoration M is transformed to $\widehat{M}$ in the form:

$$\widehat{\mathsf{M}} = \begin{bmatrix} 1 & 1 & 1 & 0.707 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1.414 & 1 & 1 & 1 \end{bmatrix} \tag{39}$$

then $\widehat{M}$ is processed following procedures formulated in (Sec. 3.1 and 3.2). Since C_k cannot be empty, then let us adopt c_1 as the reference element i.e. $C_K = C_K \cup \{c_1\}$ and $\mu(c_1) = 1$. Then, the matrix A and vector b can be determined,

$$A = \begin{bmatrix} 1 & -\frac{1}{n-1}\widehat{m}_{2,3} & -\frac{1}{n-1}\widehat{m}_{2,4} \\ -\frac{1}{n-1}\widehat{m}_{3,2} & 1 & -\frac{1}{n-1}\widehat{m}_{3,4} \\ -\frac{1}{n-1}\widehat{m}_{4,2} & -\frac{1}{n-1}\widehat{m}_{4,3} & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{n-1}\widehat{m}_{2,1}\mu(c_1) \\ \frac{1}{n-1}\widehat{m}_{3,1}\mu(c_1) \\ \frac{1}{n-1}\widehat{m}_{4,1}\mu(c_1) \end{bmatrix} \tag{40}$$

thus, to determine the vector μ_{HRE} the following linear equation system needs to be solved:

$$\begin{bmatrix} 1 & -0.333 & -0.333 \\ -0.333 & 1 & -0.333 \\ -0.333 & -0.333 & 1 \end{bmatrix} \begin{bmatrix} \mu(c_2) \\ \mu(c_3) \\ \mu(c_4) \end{bmatrix} = \begin{bmatrix} 0.333 \\ 0.333 \\ 0.471 \end{bmatrix} \tag{41}$$

The rescaled HRE weight vector is:

$$\mu_{HREn} = \begin{bmatrix} 0.227 & 0.25 & 0.25 & 0.273 \end{bmatrix}^T \tag{42}$$

Although all the tested methods rate the c_4 candidate higher than the others, only the HRE method rates c_1 below the average. Hence, only the HRE algorithm meets COP, i.e. $m_{4,1} > m_{4,2} \Rightarrow \frac{\mu(c_4)}{\mu(c_1)} > \frac{\mu(c_4)}{\mu(c_2)}$ is met only by μ_{HRE} (let us note that $\mu_{EV}(c_1) = \mu_{EV}(c_2)$ as well as $\mu_{GM}(c_1) = \mu_{GM}(c_2)$, thus $\mu_{EV}(c_4)/\mu_{EV}(c_1) = \mu_{EV}(c_4)/\mu_{EV}(c_2)$ and $\mu_{GM}(c_4)/\mu_{GM}(c_1) = \mu_{GM}(c_4)/\mu_{GM}(c_2)$). Although the eigenvalue and geometric mean methods have a problem with COP when M is not reciprocal, it should be noted that they have no problems with COP for $\widehat{M}$. This may suggest that the reciprocity restoration heuristic might be useful also for other weight derivation methods.

Example 4 (Case of incomplete matrix)

The fourth example represents situations where some data are missing. The known ratios representing the relative importance of concepts were placed into the matrix M. Question marks at the intersection of row i and column j in the matrix (43) mean unknown values m_{ij}. The immediate inspiration for this example was an observation of the meta analysis process in biochemistry [4] where the number and diversity of analyzed factors make drawing the final conclusions difficult or even impossible.

Let us consider the four drugs $c_1, \ldots, c_4$ with proven efficacy in controlling the disease X. Based on the available scientific articles, Dr H. came to the conclusion that c_1 and c_2 have similar efficacy, the same for c_3 and c_4. He also came across research showing that in some cases c_2 is two times more effective than c_3, and also c_4 fails three times more likely than c_1. Unfortunately, Dr. H. found no studies comparing the therapeutic effect of drugs in pairs (c_1, c_3) and (c_2, c_4). Therefore the PC matrix M (43) prepared by Dr H. looks like as follows:

$$\mathsf{M} = \begin{bmatrix} 1 & 1 & ? & ? \\ ? & 1 & 2 & ? \\ ? & ? & 1 & ? \\ \frac{1}{3} & ? & 1 & 1 \end{bmatrix} \tag{43}$$

The drug c_1 is very popular, so there are many studies on its efficacy. There are also some studies that compare efficacy of c_1 and c_2 but c_2 is less popular. Since c_1 is the most popular drug on X, and what follows, the relationship between c_1 and c_2 have been most extensively tested, then c_1 has been adopted as the reference concept, i.e. $C_K = C_K \cup \{c_1\}$ and $\mu(c_1) = 1$. The HRE procedure, applied to M (a reciprocity restoration included) converges to:

$$\mu_{HREn} = \begin{bmatrix} 0.369 \; 0.338 \; 0.154 \; 0.138 \end{bmatrix}^T \tag{44}$$

Thus, the most recommended cure for X is c_1, then c_2, c_3 and c_4. It should be noted that the proposed weights by the HRE algorithm are in line with COP. For example, if $m_{2,3} = 2 > 1$ then also $\frac{\mu_{HRE}(c_1)}{\mu_{HRE}(c_3)} = 2,396 > 1$, Similarly, $m_{4,1} = \frac{1}{3} < 1$ then also $\frac{\mu_{HRE}(c_4)}{\mu_{HRE}(c_1)} = 0,374 < 1$ etc. Due to the incompatible input matrix format, the eigenvalue method and the geometric mean method could not be used in this case[9].

5 Summary

The quality of the results achieved using the HRE approach is inextricably linked to input data quality. According to the popular adage "garbage in, garbage out", when data are bad even the best algorithm is not able to provide good output. In the case of heuristic algorithms, the domain of applicability depends on the adopted heuristics. Despite the promising results for different types (and different quality) of input data, the application area of the HRE approach has only been sketched. It is therefore necessary to conduct further research to better define assumed heuristics and the situations in which they may be most useful. In particular, relationships between different formulations of data inconsistency levels and the priority estimation quality seem to be very interesting.

The HRE approach presented in the article is based on the iterative HRE algorithm primarily formulated in [25]. The heuristics indicated are much more thoroughly analyzed in this work. In particular, the heuristic of averaging with respect to the reference value and the heuristic of minimizing estimation error are given in the general form as the linear equation system solving problems. The new useful heuristic of reciprocity restoration has been introduced and the incomplete PC matrix problem has been addressed. The presented theoretical considerations are accompanied by four numerical examples demonstrating different situations in which the proposed solution might be helpful. The HRE approach tries to complement other methods. It has been designed to help estimation of the relative order of concepts when a non-empty reference subset of concepts is known (or a set of such can be readily determined). Therefore, with this new application area, it may be of interest to a wide range of both researchers and practitioners.

[9] There are several approaches that address the problem of incomplete PC data. See e.g. [6,14,22]

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A About the heuristic of minimizing estimation error

From the point of view of the heuristics of minimizing estimation error, the best solution μ should minimize $\widehat{e}_{\mu}$

$$\widehat{e}_{\mu} = \frac{1}{|C_U|}\sum_{c\in C_U} e_{\mu}(c) \tag{45}$$

where

$$e_{\mu}(c_j) = \frac{1}{\left|C_j^{r-1}\right|}\sum_{c_i\in C_j^{r-1}} |\mu(c_j) - \mu(c_i)\cdot m_{ji}| \tag{46}$$

The problem by replacing the absolute difference $|\mu(c_j) - \mu(c_i)\cdot m_{ji}|$ by the squared difference $(\mu(c_j) - \mu(c_i)\cdot m_{ji})^2$ leads to the equivalent one of finding the μ minimizing function $f:\mathbb{R}_+^k \to \mathbb{R}$ given as:

$$\begin{aligned} f(\mu(c_1),\ldots,\mu(c_k)) = &\sum_{j\in I_U}\sum_{i\in I_U\setminus\{j\}} (\mu(c_j) - \mu(c_i)\cdot m_{ji})^2 \\ &+ \sum_{j\in I_U}\sum_{i\in I_K} (\mu(c_j) - \mu(c_i)\cdot m_{ji})^2 \end{aligned} \tag{47}$$

where I_U, I_K and I_C denote the sets of indices of elements from C_U, C_K and C correspondingly[10].

In order to determine the extremum of the function f, the following linear equation system needs to be solved:

$$\begin{bmatrix} \frac{\partial f}{\partial \mu(c_1)} \\ \vdots \\ \frac{\partial f}{\partial \mu(c_k)} \end{bmatrix} = 0 \tag{48}$$

where every single equation has the form:

$$\begin{aligned} \frac{\partial f}{\partial \mu(c_j)} = 2 \cdot \Bigg(& \sum_{i \in I_U \setminus \{j\}} (\mu(c_j) - \mu(c_i) \cdot m_{ji}) - \\ & - \sum_{i \in I_U \setminus \{j\}} (\mu(c_i) - \mu(c_j) \cdot m_{ij}) \cdot m_{ij} + \sum_{i \in I_K} (\mu(c_j) - \mu(c_i) \cdot m_{ji}) \Bigg) = 0 \end{aligned} \tag{49}$$

for $j \in I_U$. Hence, the above equation is equivalent to:

$$\left(n - 1 + \sum_{i \in I_U \setminus \{j\}} m_{ij}^2 \right) \mu(c_j) - \sum_{i \in I_U \setminus \{j\}} (m_{ji} + m_{ij}) \cdot \mu(c_i) + \sum_{i \in I_K} \mu(c_i) \cdot m_{ji} = 0 \tag{50}$$

Dividing both sides of (49) by $(n-1)$, and denoting $\frac{1}{n-1} \sum\limits_{i \in I_U \setminus \{j\}} m_{ij}^2 \overset{df}{=} S_j$ it is easy to observe that (50) turns into:

$$\begin{aligned} - \frac{m_{j1} + m_{1j}}{n-1} \mu(c_1) - \frac{m_{j2} + m_{2j}}{n-1} \mu(c_2) - \dots \\ + (1 + S_j) \mu(c_j) - \frac{m_{jk} + m_{kj}}{n-1} \mu(c_k) = b_j \end{aligned} \tag{51}$$

Thus, finding the extremum point of f boils down to solving the following equation:

$$E\mu = b \tag{52}$$

where:

$$E = \begin{bmatrix} 1 + S_1 & -\frac{m_{1,2}+m_{2,1}}{n-1} & \cdots & -\frac{m_{1,k}+m_{k,1}}{n-1} \\ -\frac{m_{2,1}+m_{1,2}}{n-1} & 1 + S_2 & \cdots & -\frac{m_{2,k}+m_{k,2}}{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{m_{k-1,1}+m_{1,k-1}}{n-1} & \cdots & \ddots & -\frac{m_{k-1,k}+m_{k,k-1}}{n-1} \\ -\frac{m_{k,1}+m_{1,k}}{n-1} & \cdots & -\frac{m_{k,k-1}+m_{k-1,k}}{n-1} & 1 + S_k \end{bmatrix} \tag{53}$$

and b is defined as in (10). It is easy to show that the *Hessian* matrix defined as:

$$H_{ij} = \left[\frac{\partial^2 f}{\partial \mu(c_j) \partial \mu(c_i)} \right] \tag{54}$$

equals:

$$H = 2(n-1)E \tag{55}$$

Therefore, if E is strictly diagonally dominant, then H is also strictly diagonally dominant. Since the diagonal entries of H are all positive, then H is positively definite [29, page 29]. Thus, for E strictly diagonally dominant the solution of (52) is the minimum of f.

---

[10] In particular it is assumed that $I_U = \{1, \dots, k\}$

B Condition of order preservation

Among the various criticisms raised at *AHP* and the eigenvalue method, a *Condition of Order Preservation* (*COP*) postulate [2] seems to be one of the more interesting. According to *COP,* the output of the weight calculation method should preserve the order as well as the intensity of preferences. In other words, *COP* is met by the weight vector μ if for every four concepts $c_1, \ldots, c_4 \in C$ such that c_1 dominates c_2 more than c_3 dominates c_4 in M i.e. $m_{1,2} > 1 \wedge m_{3,4} > 1 \wedge m_{1,2} > m_{3,4}$, the following two assertions are true:

1. Preservation of Order of Preference (*POP*)

$$\mu(c_1) > \mu(c_2) \tag{56}$$

$$\mu(c_3) > \mu(c_4) \tag{57}$$

2. Preservation of Order of Intensity of Preference (*POIP*)

$$\frac{\mu(c_1)}{\mu(c_2)} > \frac{\mu(c_3)}{\mu(c_4)} \tag{58}$$

COP does not depend on any concepts specific for the eigenvalue method. It reflects the natural desire that the final ranking should be consistent with the individual expert judgments. Thus, although *COP* was formulated with reference to the eigenvalue method, it might be used as a quality test for any priority deriving methods, including the *HRE* approach.

C Heuristics of averaging with respect to reference values - the form of the linear equation system

For simplicity, let us assume that $C_U = \{c_1, \ldots, c_k\}$, $C_K = \{c_{k+1}, \ldots, c_n\}$. The values μ for $c_j \in C_K$ are known, whilst the values μ for elements of C_U need to be estimated. The heuristics of averaging with respect to the reference values assumes that for every unknown $c_j \in C_U$ the value $\mu(c_j)$ should be estimated as the arithmetic mean of all the other values $\mu(c_i)$ multiplied by factor m_{ji}:

$$\mu(c_j) = \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} m_{ji}\mu(c_i) \tag{59}$$

Thus, during the second and subsequent iterations the algorithm shown in [25] calculates the new estimation value $\mu(c_i)$ for each unknown concepts $c_j \in C_U$ according to one of the following equations:

$$\begin{array}{l} \mu(c_1) = \frac{1}{n-1}(m_{2,1}\mu(c_2) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + m_{n,1}\mu(c_n)) \\ \mu(c_2) = \frac{1}{n-1}(m_{1,2}\mu(c_1) + m_{3,2}\mu(c_3) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + m_{n,2}\mu(c_n)) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu(c_k) = \frac{1}{n-1}\left(m_{1,k}\mu(c_1) + \ldots + m_{k-1,k}\mu(c_{k-1}) + m_{k+1,k}\mu(c_{k+1}) + \ldots + m_{n,k}\mu(c_n)\right) \end{array} \tag{60}$$

Since the values $\mu(c_{k+1}), \ldots, \mu(c_n)$ are known and constant ($c_{k+1}, \ldots, c_n$ are the reference concepts), so they can be grouped together. Let us denote:

$$b_j = \frac{1}{n-1} m_{k+1,j}\mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{n,j}\mu(c_n) \tag{61}$$

Thus, the linear equations system (60) could be written as:

$$\begin{array}{l} \mu(c_1) = \frac{1}{n-1} m_{2,1}\mu(c_2) + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{n-1} m_{k,1}\mu(c_k) + b_1 \\ \mu(c_2) = \frac{1}{n-1} m_{1,2}\mu(c_1) + \frac{1}{n-1} m_{3,2}\mu(c_3) + \ldots + \frac{1}{n-1} m_{k,2}\mu(c_k) + b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu(c_k) = \frac{1}{n-1} m_{1,k}\mu(c_1) + \ldots\ldots\ldots\ldots + \frac{1}{n-1} m_{k-1,k}\mu(c_{k-1}) + b_k \end{array} \tag{62}$$

It is easy to see that the linear equation system (62) forms the matrix equation (8) where A, b and μ are defined in (9), (10) and (11). Finding the solution of (8) is equivalent to determine the values $\mu(c_1), \ldots, \mu(c_k)$ with respect to the reference (known) concepts grouped in C_K.

Heuristic rating estimation - geometric approach

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Abstract. Heuristic Rating Estimation (HRE) is a newly proposed method supporting decisions analysis based on the use of pairwise comparisons. It allows that the ranking values of some alternatives (herein referred to as concepts) are initially known, whilst the ranks for the other concepts have yet to be estimated. To calculate the missing ranks it is assumed that the priority of every single concept can be determined as the weighted arithmetic mean of priorities of all the other concepts. It has been shown that the problem has admissible solution if the inconsistency of pairwise comparisons is not too high.

The proposed approach adopts the heuristics according to which to determine the missing priorities a weighted geometric mean is used. In this approach, despite an increased complexity, the solution always exists and their existence does not depend on the inconsistency of the input matrix. Thus, the presented approach might be appropriate for a larger number of problems than the previous method. The formal definition of the proposed geometric heuristics is accompanied by two numerical examples.

1 Introduction

The first written evidence about pairwise comparisons (PC) method dates back to the thirteenth century, when *Ramon Llull* from Majorca wrote a seminal piece "Artifitium electionis personarum" (The method for the elections of persons) about voting and elections [4, 3], followed by the two consecutive works being a practical study on the election processes[1]. Nowadays PC as a voting method is a way of deciding on the relative utility of alternatives used in decision theory [19] and other fields like economy [16], psychometrics and psychophysics [20] and so on. The PC theory is developed by many research teams representing different fields and approaches. One can point out some characteristic approaches like fuzzy PC relation developed by *Kacprzyk* et al. and *Mikhailov* [7, 15], data inconsistency reduction methods proposed by *Koczkodaj* and *Szarek* [10] and issue of incomplete PC relation by *Koczkodaj* and *Orłowski* [8] and *Bozoki* and *Rapcsak* [1], problem of non-numerical rankings addressed by *Janicki* and *Zhai* [6] or using PC in Data Envelopment Analysis [14].

Currently, the Heuristic Rating Estimation (HRE) method which enables the user to explicitly define the reference set of concepts, for which the ranking values are a priori known, is being developed [11, 12]. The base heuristics used in *HRE* proposes to determine the relative values of a single non–reference concept as a weighted arithmetic mean of all the other concepts. This proposition leads to the linear equation system defined by the matrix A and the strictly positive vector of constant terms b.

In this work, the authors show that using a geometric mean to determine the relative priorities of concepts instead of arithmetic one in some cases may be more convenient. The main benefit of the proposed solution stems from the guarantee of solution existence. Hence, unlike the original proposal, the ranking list can always be created. This guarantee is paid with the increase in computational complexity. The presented solution is accompanied by two numerical examples.

The presented work is a follow-up of research initiated in [11, 12]. It redefines the main heuristics of HRE and the method of calculating the solution. The HRE approach as proposed in the previous articles is briefly outlined in (Sec. 2). There are also a short summary of a few important properties of *M-matrices* (Sec. 2.3), which are essential to the properties of the presented method. The next section (Sec. 3) describes the proposed solution and discusses two important properties: solution existence (Sec. 3.2) and optimality (Sec. 3.3). Theoretical considerations are accompanied by two meaningful examples showing how the presented method can be used in practice (Sec 4). A brief summary is provided in (Sec. 5).

[1] see: The Augsburg Web Edition of Llull's Electoral Writings

2 Preliminaries

2.1 Basic concepts of pairwise comparisons method

The input to the *PC* method is the *PC* matrix $M=(m_{ij})$, where $m_{ij}\in\mathbb{R}_+$ and $i,j\in\{1,\ldots,n\}$. It expresses a quantitative relation R over the finite set of concepts $C\overset{df}{=}\{c_i\in\mathscr{C}$ and $i\in\{1,\ldots,n\}\}$ where $\mathscr{C}$ is a non empty universe of concepts, and $R(c_i,c_j)=m_{ij}$, $R(c_j,c_i)=m_{ji}$. The values m_{ij} and m_{ji} represent subjective expert judgment as to the relative importance, utility or quality indicators of concepts c_i and c_j. Thus, according to the best knowledge of experts should holds that $c_i=m_{ij}c_j$.

Definition 1. *A matrix M is said to be reciprocal if for all $i,j\in\{1,\ldots,n\}$ holds $m_{ij}=\frac{1}{m_{ji}}$, and M is said to be consistent if for all $i,j,k\in\{1,\ldots,n\}$ is $m_{ij}\cdot m_{jk}\cdot m_{ki}=1$.*

Since the data in the *PC* matrix represents subjective opinions of experts, thus they might be inconsistent. Hence, it may exist a triad m_{ij},m_{jk},m_{ki} of entries in M for which $m_{ik}\cdot m_{kj}\neq m_{ij}$. This leads to the situation in which the relative importance of c_i with respect to c_j is either $m_{ik}\cdot m_{kj}$ or m_{ij}. This observation underlies two related concepts: a priority deriving method that transform even an inconsistent matrix M into consistent priority vector, and an inconsistency index describing how far the matrix M is inconsistent. There are a number of priority deriving methods and inconsistency indexes [2, 5]. For the purpose of the article the *Koczkodaj's inconsistency index* is adopted.

Definition 2. *Koczkodaj's inconsistency index $\mathscr{K}$ of $n\times n$ and $(n>2)$ reciprocal matrix M is equal to*

$$\mathscr{K}(M)\overset{df}{=}\max_{i,j,k\in\{1,\ldots,n\}}\left\{\min\left\{\left|1-\frac{m_{ij}}{m_{ik}m_{kj}}\right|,\left|1-\frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\}\right\}\tag{1}$$

where $i,j,k=1,\ldots,n$ and $i\neq j\wedge j\neq k\wedge i\neq k$.

The result of the pairwise comparisons method is ranking - a function that assigns values to the concepts. Formally, it can be defined as follows.

Definition 3. *The ranking function for C (the ranking of C) is a function $\mu:C\to\mathbb{R}_+$ that assigns to every concept from $C\subset\mathscr{C}$ a positive value from $\mathbb{R}_+$.*

Thus, $\mu(c)$ represents the ranking value for $c\in C$. The μ function is usually defined as a vector of weights $\mu\overset{df}{=}[\mu(c_1),\ldots,\ \mu(c_n)]^T$. According to the most popular eigenvalue based approach proposed by *Saaty* [19] the final ranking μ_{ev} is determined as the principal eigenvector of the *PC* matrix M, rescaled so that the sum of all its entries is 1, i.e.

$$\mu_{ev}=\left[\frac{\mu_{max}(c_1)}{s_{ev}},\ldots,\frac{\mu_{max}(c_n)}{s_{ev}}\right]^T\ \text{and}\ s_{ev}=\sum_{i=1}^{n}\mu_{max}(c_i)\tag{2}$$

where μ_{ev} - the ranking function, $\mu_{max}\overset{df}{=}[\mu_{max}(c_1),\ldots,\ \mu_{max}(c_n)]^T$ - the principal eigenvector of M. Another popular approach proposes the rescaled geometric mean (GM) of rows of M as the ranking result, i.e.

$$\mu_{gm}=\left[\frac{p_1}{s_{gm}},\ldots,\frac{p_n}{s_{gm}}\right]^T\tag{3}$$

where

$$p_i=\left(\prod_{j=1}^{n}m_{ij}\right)^{\frac{1}{n}}\ \text{and}\ \ s_{gm}=\sum_{i=1}^{n}\left(\prod_{j=1}^{n}m_{ij}\right)^{\frac{1}{n}}\tag{4}$$

It can be shown that for the fully consistent matrix M both ranking vectors μ_{ev} and μ_{gm} are identical. A more completely overview including other methods can be found in [2, 5].

2.2 Pairwise comparisons method with the reference set

Usually when using the pairwise comparisons method the ranking values $\mu(c_1),\ldots,\mu(c_n)$ are initially unknown. Hence they are need to be determined by the priority deriving procedure. In some cases, however, there are concepts for which the priorities are known from elsewhere. Hence, the decision makers may have additional knowledge about the group of elements $C_K \subseteq C$ that allow them to determine $\mu(c)$ for C_K in advance.

For example, let c_1, c_2 and c_3 represent oil paintings that an auction house plans to put for auction. The sequence of paintings during the auction should correspond to their approximate valuation. In order to determine the indicative price of paintings the auction house asked experts to evaluate them in pairs taking into account that two other paintings from the same period of time were previously auctioned for $\mu(c_4)$ and $\mu(c_5)$.

The situation as described above prompted the first author [11, 12] to propose a *Heuristic Rating Estimation (HRE)* model. According to *HRE* the set of concepts C is composed of unknown concepts $C_U = \{c_1,\ldots,c_k\}$ and known (reference) concepts $C_K = \{c_{k+1},\ldots,c_n\}$, where $C_U, C_K \neq \emptyset$ and $C_U \cap C_K = \emptyset$. The values $\mu(c_i)$ for $c_i \in C_K$ are known, whilst the values $\mu(c_j)$ for elements $c_j \in C_U$ need to be calculated. Following the heuristics of *averaging with respect to the reference values* [12] solution proposed by HRE is to adopt as $\mu(c_j)$, for every $c_j \in C_U$, the arithmetic mean of all the other values $\mu(c_i)$ multiplied by factor m_{ji}:

$$\mu(c_j) = \frac{1}{n-1} \sum_{i=1, i\neq j}^{n} m_{ji}\mu(c_i) \tag{5}$$

If the experts judgments gathered in the matrix M were fully consistent (Def. 1), then every component of the sum (5) in the form $m_{ji}\mu(c_i)$ would equal $\mu(c_j)$. Because, it is generally not, then every component is only an approximation of $\mu(c_j)$. Thus, the arithmetic mean of the individual approximations has been adopted as the most probable value of $\mu(c_j)$. To determine unknown values $\mu(c_j)$ for $c_j \in C_U$ the problem formalised as (5) can be written down as the linear equation system $A\mu = b$, where:

$$A = \begin{bmatrix} 1 & \cdots & -\frac{1}{n-1}m_{1,k} \\ -\frac{1}{n-1}m_{2,1} & \cdots & -\frac{1}{n-1}m_{2,k} \\ \vdots & \ddots & \vdots \\ -\frac{1}{n-1}m_{k,1} & \cdots & 1 \end{bmatrix} \tag{6}$$

and

$$b = \begin{bmatrix} \frac{1}{n-1}\sum_{i=k+1}^{n} m_{1,i}\mu(c_i) \\ \frac{1}{n-1}\sum_{i=k+1}^{n} m_{2,i}\mu(c_i) \\ \vdots \\ \frac{1}{n-1}\sum_{i=k+1}^{n} m_{k,i}\mu(c_i) \end{bmatrix} \tag{7}$$

The solution $\mu = \left[\mu(c_1),\ldots,\mu(c_k)\right]^T$ determines the values of μ for elements from C_U. Together with known $\mu(c_{k+1}),\ldots,\mu(c_n)$ the vector μ forms the complete result list, which after sorting can be used to build ranking. Although the values $\mu(c)$ for $c \in C$ are called priorities, they usually have a specific meaning. In the case of previously mentioned example they represent the expected price of paintings.

According (Def. 3) the ranking results must be strictly positive, hence only strictly positive vectors μ are considered as feasible. It can be shown that the equation $A\mu = b$ has a feasible solution if A is strictly diagonally dominant by rows [12]. It has recently been shown that the equation has a feasible solution when the inconsistency index $\mathscr{K}(M)$ is not to high [13].

2.3 M-matrices

Very often the real life problem can be reduced to the linear equation system $A\mu = b$, where the matrix A has some special structure. Frequently the matrix A has positive diagonal and nonpositive off-diagonal entries. Due to their importance to the practice this type of matrix was especially thoroughly studied by researchers [17, 18]. To define it formally a few more notions and definitions are needed.

Let $\mathscr{M}_{\mathbb{R}}(n)$ be a set of $n \times n$ matrices over $\mathbb{R}$, and $\mathscr{M}_{\mathbb{Z}}(n)$ the set of all $A = [a_{ij}] \in \mathscr{M}_{\mathbb{R}}(n)$ with $a_{ij} \leq 0$ if $i \neq j$ and $i, j \in \{1,\ldots,n\}$. Furthermore, assume that for every matrix $A \in \mathscr{M}_{\mathbb{R}}(n)$ and vector $b \in \mathbb{R}^n$ the notation $A \geq 0$ and

$b \geq 0$ will mean that every m_{ij} and b_k are non-negative and neither A nor b equals 0. The spectral radius of A is defined as $\rho(A) \stackrel{df}{=} \max\{|\lambda| : \det(\lambda I - A) = 0\}$.

Definition 4. *An $n \times n$ matrix that can be expressed in the form $A = sI - B$ where $B = [b_{ij}]$ with $b_{ij} \geq 0$ for $i, j \in \{1, \ldots, n\}$, and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B, is called* M-matrix.

Following [17] some of the *M-matrix* properties are recalled below in the form of the Theorem 1.

Theorem 1. *For every $A \in \mathscr{M}_{\mathbb{Z}}(n)$ each of the following conditions is equivalent to the statement: A is a nonsingular* M-matrix.

1. A is inverse positive. That is, A^{-1} exists and $A^{-1} \geq 0$
2. There exists a positive diagonal matrix D such that AD has all positive row sums.

It is worth to note that for every matrix equation in the form $A\mu = b$, where A is a nonsingular M-matrix, holds $\mu = A^{-1}b$. Since $A^{-1} \geq 0$, thus, $b > 0$ implies that also $\mu > 0$.

3 HRE - geometric approach

3.1 Heuristics of the geometric averaging with respect to the reference values

Most often the pairwise comparisons method is used to transform the *PC* matrix into the ranking list of mutually compared concepts. During the transformation to each concept a priority is assigned. Therefore, this transformation is often called a priority deriving method. There are many priority deriving methods. Besides the eigenvalue based method (2), where the ranking values $\mu(c_i)$ are approximated as the arithmetic means of $m_{ij} \cdot \mu(c_j)$, also the geometric mean of rows is used (3). This may suggest that also for the ranking problem with the reference set [12], the arithmetic mean (5) might be replaced by the geometric mean. This observation prompted the author to formulate and investigate *the geometric averaging with respect to the reference values heuristics*. According to this proposition to determine the unknown values $\mu(c_j)$ for $c_j \in C_U$ the following non-linear equation is used:

$$\mu(c_j) = \left(\prod_{i=1, i \neq j}^{n} m_{ji} \mu(c_i) \right)^{\frac{1}{n-1}} \tag{8}$$

After rising both sides to the $n-1$ power the geometric averaging heuristics equation (8) leads to the non-linear equation system in the form:

$$\begin{aligned} \mu^{n-1}(c_1) &= m_{1,2}\mu(c_2) \cdot \ldots\ldots\ldots\ldots \cdot m_{1,n}\mu(c_n) \\ \mu^{n-1}(c_2) &= m_{2,1}\mu(c_1) \cdot m_{2,3}\mu(c_3) \cdot \ldots \cdot m_{2,n}\mu(c_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu^{n-1}(c_k) &= m_{k,1}\mu(c_1) \cdot \ldots\ldots\ldots \cdot m_{k,n-1}\mu(c_{n-1}) \end{aligned} \tag{9}$$

Of course, since the ranking values for $c_{k+1}, \ldots, c_n \in C_K$ make the reference set where the values $\mu(c_j)$ are known and fixed, some products in the form $m_{ji}\mu(c_i)$ are initially known constants. Let us denote:

$$g_j = \prod_{i=k+1}^{n} m_{ji} \mu(c_i) \tag{10}$$

for $j = 1, \ldots, k$ as the constant part of each equation (9). Thus, the non-linear equation system can be written as:

$$\begin{aligned} \mu^{n-1}(c_1) &= m_{1,2}\mu(c_2) \cdot \ldots\ldots\ldots\ldots \cdot m_{1,k}\mu(c_k) \cdot g_1 \\ \mu^{n-1}(c_2) &= m_{2,1}\mu(c_1) \cdot m_{2,3}\mu(c_3) \cdot \ldots \cdot m_{2,k}\mu(c_k) \cdot g_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu^{n-1}(c_k) &= m_{k,1}\mu(c_1) \cdot \ldots\ldots\ldots \cdot m_{k,k-1}\mu(c_{k-1}) \cdot g_k \end{aligned}$$

Hence $\mu(c_j)$, m_{ij}, $g_j \in \mathbb{R}_+$, let us denote $\log_\xi \mu(c_j) \overset{df}{=} \widehat{\mu}(c_j)$, $\widehat{m}_{ij} \overset{df}{=} \log_\xi m_{ij}$ and $\widehat{g}_j \overset{df}{=} \log_\xi g_j$ for some $\xi \in \mathbb{R}_+$. It is easy to see that the above non-linear equation system is equivalent to the following one:

$$
\begin{array}{l}
(n-1)\widehat{\mu}(c_1) = \widehat{m}_{1,2} + \widehat{\mu}(c_2) + \ldots\ldots\ldots + \widehat{m}_{1,k} + \widehat{\mu}(c_k) + \widehat{g}_1 \\
(n-1)\widehat{\mu}(c_2) = \widehat{m}_{2,1} + \widehat{\mu}(c_1) + \ldots\ldots\ldots + \widehat{m}_{2,k} + \widehat{\mu}(c_k) + \widehat{g}_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(n-1)\widehat{\mu}(c_k) = \widehat{m}_{k,1} + \widehat{\mu}(c_1) + \ldots + \widehat{m}_{k,k-1} + \widehat{\mu}(c_{k-1}) + \widehat{g}_k
\end{array}
\tag{11}
$$

By grouping all the constant terms on the right side of each above equation we obtain the linear equation system

$$
\begin{array}{l}
(n-1)\widehat{\mu}(c_1) - \sum_{i=2}^{k} \widehat{\mu}(c_i) \quad = b_1 \\
(n-1)\widehat{\mu}(c_2) - \sum_{i=1, i\neq 2}^{k} \widehat{\mu}(c_i) = b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(n-1)\widehat{\mu}(c_k) - \sum_{i=1}^{k-1} \widehat{\mu}(c_i) \quad = b_k
\end{array}
\tag{12}
$$

where $b_i \overset{df}{=} \sum_{j=1, j\neq i}^{k} \widehat{m}_{1,j} + \widehat{g}_i$ for $i = 1, \ldots, k$, which can be easily written down in the matrix form

$$\widehat{A}\widehat{\mu} = b \tag{13}$$

where:

$$
\widehat{A} = \begin{bmatrix} (n-1) & -1 & \cdots & -1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \cdots & (n-1) \end{bmatrix},
\tag{14}
$$

$$
\widehat{\mu} = \begin{bmatrix} \widehat{\mu}(c_1) \\ \widehat{\mu}(c_2) \\ \vdots \\ \widehat{\mu}(c_k) \end{bmatrix}, \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}
\tag{15}
$$

Therefore, the solution $\widehat{\mu}$ of the linear equation system (13) automatically provides the solution to the original non-linear problem as formulated in (9). Indeed the ranking vector μ can be computed following the formula:

$$\mu = \left[\xi^{\widehat{\mu}(c_1)}, \ldots, \xi^{\widehat{\mu}(c_k)}\right]^T \tag{16}$$

Importantly, as it is shown below a feasible solution of (13) always exists. Hence, the heuristics of the averaging with respect to the geometric mean always provides the user an appropriate ranking function.

3.2 Existence of solution

The form of $\widehat{A}$ is specific. The positive diagonal and the negative off-diagonal real entries cause that $\widehat{A} \in \mathscr{M}_{\mathbb{Z}}(k)$ (see Sec. 2.3). Let us put:

$$D = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

and $D \in \mathscr{M}_{\mathbb{R}}(k)$. Of course D is positively dominant matrix. Thus, the product $\widehat{A} \cdot D = \widehat{A}$. The sum of each row in $\widehat{A}$ equals

$$(n-1) + \sum_{i=1}^{k-1}(-1) = n-k$$

Since C_K is nonempty, thus its cardinality $|C_K| = n-k$ is greater than 0. This means that the sum of each row of $\widehat{A} \cdot D$ is positive. Hence, due to the Theorem 1, $\widehat{A}$ is a nonsingular M-matrix (Def. 4). Thus, $\widehat{A}^{-1}$ exists (i.e. $\widehat{\mu} = \widehat{A}^{-1} b$) and always the equation (13) has a solution in $\mathbb{R}^k$. Due to the form of the solution of the main problem (16) μ is a vector in $\mathbb{R}^k_+$, i.e. every its entry is strictly positive. In other words unlike the original proposition [12] the heuristics of the geometric averaging with respect to the reference values always provides a feasible ranking result to the user.

3.3 Optimality condition

One of the reasons for introducing the geometric mean method (3) is minimizing the multiplicative error e_{ij} [5] defined as:

$$m_{ij} = \frac{p_i}{p_j} e_{ij} \tag{17}$$

In the case of the geometric averaging heuristics the multiplicative error equation takes the form:

$$m_{ij} = \frac{\mu(c_i)}{\mu(c_j)} e_{ij} \tag{18}$$

The multiplicative error is commonly accepted to be log normal distributed (in the same way the additive error would be assumed to be normally distributed). Let $e : \mathbb{R}^n_+ \to \mathbb{R}$ be the sum of multiplicative errors (see [5]) defined as follow:

$$e(\mu(c_1), \ldots, \mu(c_n)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln(m_{ij}) - \ln\left(\frac{\mu(c_i)}{\mu(c_j)} \right) \right)^2 \tag{19}$$

As it is shown in the Theorem below very often the heuristics (8) is optimal with respect to the value of multiplicative error function e.

Theorem 2. *The geometric averaging with respect to the reference values heuristics minimizes the sum of multiplicative errors* $e(\mu(c_1), \ldots, \mu(c_n))$ *if*

$$\mu(c_i) < (n-1) \sum_{j=1, j \neq i}^{n} \mu(c_j) \tag{20}$$

for $i = 1, \ldots, n$.

Proof. To determine the minimum of (19) let us forget for a moment that $\mu(c_{k+1}), \ldots, \mu(c_n)$ are constants (the reference values), and let us treat them as any other arguments of e. In order to determine the minimum of (19) the first derivative need to be calculated. Thus,

$$\frac{\partial e}{\partial \mu(c_i)} = \frac{1}{\mu(c_i)} \left(\sum_{r=1, r \neq i}^{n} 4(n-1) \ln \mu(c_i) - 4 \sum_{j=1, j \neq i}^{n} \ln \mu(c_j) + 2 \sum_{r=1, r \neq i}^{n} \ln(m_{ri}) - 2 \sum_{j=1, j \neq i}^{n} \ln(m_{ij}) \right) \tag{21}$$

for $i = 1, \ldots, n$. Due to the reciprocity of M, i.e. $m_{ij} = 1/m_{ji}$, the equation (21) can be written as:

$$\frac{\partial e}{\partial \mu(c_i)} = -4 \left(\frac{\sum_{j=1, j \neq i}^{n} (\ln \mu(c_j) + \ln(m_{ij})) - (n-1) \ln \mu(c_i)}{\mu(c_i)} \right) \tag{22}$$

The function e reaches the minimum if $\partial e / \partial \mu(c_i) = 0$. This leads to the postulate that

$$\sum_{j=1, j \neq i}^{n} (\ln \mu(c_j) + \ln(m_{ij})) - (n-1) \ln \mu(c_i) = 0 \tag{23}$$

for $i = 1, \ldots, n$. Thus,

$$\ln \mu(c_i) = \frac{1}{n-1} \left(\sum_{j=1, j \neq i}^{n} \ln m_{ij} \mu(c_j) \right) \tag{24}$$

which is directly equivalent to (8). In other words any solution to the equation system (9) is a good candidate to be a minimum of (19). It remains to settle the matrix H of second derivative of e. When H is positive definite then the solution of (9) actually minimizes the function e. As a result of further differentiation is determined that the diagonal elements of H are

$$\frac{\partial^2 f}{\partial \mu(c_i) \partial \mu(c_i)} = \frac{4(n-1)}{\mu^2(c_i)} - \frac{1}{\mu(c_i)} \frac{\partial f}{\partial \mu(c_i)} \tag{25}$$

Proof. where $i = 1, \dots, n$, and the other elements for which $i \neq j$ and $i, j = 1, \dots, n$ take the form:

$$\frac{\partial^2 f}{\partial \mu(c_i) \partial \mu(c_j)} = -\frac{4}{\mu(c_i)\mu(c_j)} \tag{26}$$

Since the matrix H is considered for e in the point $\left(\mu(c_1), \dots, \ \mu(c_n)\right)$ such that (8) holds, thus the first derivative of e is 0. Therefore, the Hessian matrix H takes the form:

$$H = \begin{bmatrix} \frac{4(n-1)}{\mu^2(c_1)} & -\frac{4}{\mu(c_1)\mu(c_2)} & \cdots & -\frac{4}{\mu(c_1)\mu(c_n)} \\ \vdots & \frac{4(n-1)}{\mu^2(c_2)} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{4}{\mu(c_n)\mu(c_1)} & -\frac{4}{\mu(c_n)\mu(c_2)} & \cdots & \frac{4(n-1)}{\mu^2(c_n)} \end{bmatrix} \tag{27}$$

According to [18, p. 29] if H is strictly diagonally dominant by rows, symmetric, and with positive diagonal entries then it is also positive definite. To meet the first strict diagonal dominance criterion (other are satisfied) it is required that:

$$\left| \frac{n-1}{\mu^2(c_i)} \right| > \sum_{j=1, j\neq i}^{n} \left| -\frac{1}{\mu(c_i)\mu(c_j)} \right| \tag{28}$$

for $i = 1, \dots, n$. Thus,

$$\mu^2(c_i) < (n-1)\mu(c_i) \sum_{j=1, j\neq i}^{n} \mu(c_j) \tag{29}$$

Since every $\mu(c_i) > 0$, then it is easy to verify that the above equation is equivalent to the desired condition (20).

4 Numerical examples

The HRE method can be useful in many situations in which, based on the expert subjective opinions and the actual data, the new concepts, objects or entities need to be assessed. In order to show how the method may work in practice the following two numerical examples are presented. The first one, more abstract, discusses the method for solving the non-linear equation system. The second one, more complex, tries to put the method into the actual business context, where it can be successfully used.

In both examples the set of concepts consists of C_K - the reference (known) and C_U - the initially unknown elements. To solve an intermediate linear equation system (13) the Gaussian elimination method is used.

4.1 Example I (Scientific entities assessment)

Let $c_1, \dots, c_5$ represent the scientific entities[2], where two of them $c_2, c_3 \in C_K$ are the reference entities. Their values were arbitrarily set by experts to $\mu(c_2) = 5$ and $\mu(c_3) = 7$. The analysis of the scientific achievements of the entities c_1, c_4 and c_5 leads to the following PC matrix:

$$M = \begin{bmatrix} 1 & \frac{3}{5} & \frac{4}{7} & \frac{5}{8} & \frac{5}{9} \\ \frac{5}{3} & 1 & \frac{5}{7} & \frac{5}{2} & \frac{10}{3} \\ \frac{7}{4} & \frac{7}{5} & 1 & \frac{7}{2} & 4 \\ \frac{8}{5} & \frac{2}{5} & \frac{2}{7} & 1 & \frac{4}{3} \\ \frac{9}{5} & \frac{3}{10} & \frac{1}{4} & \frac{3}{4} & 1 \end{bmatrix} \tag{30}$$

To calculate the rank using HRE with the geometric averaging heuristics, the following system of non-linear equations (compare with 9) need to be solved:

$$\begin{aligned} \mu(c_1) &= \left(m_{1,2}\mu(c_2) \cdot \dots\dots\dots\dots \cdot m_{1,5}\mu(c_5)\right)^{\frac{1}{4}} \\ \mu(c_4) &= \left(m_{4,1}\mu(c_1) \cdot \dots \cdot m_{4,3}\mu(c_1) \cdot m_{4,5}\mu(c_5)\right)^{\frac{1}{4}} \\ \mu(c_5) &= \left(m_{5,1}\mu(c_1) \cdot \dots\dots\dots\dots \cdot m_{5,4}\mu(c_4)\right)^{\frac{1}{4}} \end{aligned} \tag{31}$$

[2] Actually the official ranking of the scientific entities in Poland compares the entities in pairs [9].

thus, after rising both sides of the equations to the power,

$$
\begin{aligned}
\mu^4(c_1) &= m_{1,2}\mu(c_2)\cdot \dots\dots\dots\dots \cdot m_{1,5}\mu(c_5)\\
\mu^4(c_4) &= m_{4,1}\mu(c_1)\cdot\ldots\cdot m_{4,3}\mu(c_1)\cdot m_{4,5}\mu(c_5)\\
\mu^4(c_5) &= m_{5,1}\mu(c_1)\cdot \dots\dots\dots\dots \cdot m_{5,4}\mu(c_4)
\end{aligned}
\tag{32}
$$

Substituting the logarithm of both sides of the equations, we get the following system:

$$
\begin{aligned}
4\lg\mu(c_1) &= \lg\left(m_{1,2}\mu(c_2)\cdot \dots\dots\dots\dots \cdot m_{1,5}\mu(c_5)\right)\\
4\lg\mu(c_4) &= \lg\left(m_{4,1}\mu(c_1)\cdot\ldots\cdot m_{4,3}\mu(c_1)\cdot m_{4,5}\mu(c_5)\right)\\
4\lg\mu(c_5) &= \lg\left(m_{5,1}\mu(c_1)\cdot \dots\dots\dots\dots \cdot m_{5,4}\mu(c_4)\right)
\end{aligned}
\tag{33}
$$

which leads to the intermediate, linear logarithmic equation system:

$$
\begin{aligned}
4\lg\mu(c_1)-\lg\mu(c_4)-\lg\mu(c_5) &= b_1\\
-\lg\mu(c_1)+4\lg\mu(c_4)-\lg\mu(c_5) &= b_4\\
-\lg\mu(c_1)-\lg\mu(c_4)+4\lg\mu(c_5) &= b_5
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
b_1 &\stackrel{df}{=} \lg\left(m_{1,2}\mu(c_2)m_{1,3}\mu(c_3)m_{1,4}m_{1,5}\right)\\
b_4 &\stackrel{df}{=} \lg\left(m_{4,1}m_{4,2}\mu(c_2)m_{4,3}\mu(c_3)m_{4,5}\right)\\
b_5 &\stackrel{df}{=} \lg\left(m_{5,1}m_{5,2}\mu(c_2)m_{5,3}\mu(c_3)m_{5,4}\right)
\end{aligned}
\tag{35}
$$

Then, according to the procedure proposed in (Sec. 3.1) the linear equation system (13) where the unknown values $\hat{\mu}(c_i)\stackrel{df}{=}\lg\left(\mu(c_i)\right)$ for $i=1,4,5$ takes the form:

$$
\begin{bmatrix} n-1 & -1 & -1\\ -1 & n-1 & -1\\ -1 & -1 & n-1 \end{bmatrix}
\begin{bmatrix} \widehat{\mu}(c_1)\\ \widehat{\mu}(c_4)\\ \widehat{\mu}(c_5) \end{bmatrix}
=
\begin{bmatrix} b_1\\ b_4\\ b_5 \end{bmatrix}
\tag{36}
$$

hence, numerically:

$$
\begin{bmatrix} 4 & -1 & -1\\ -1 & 4 & -1\\ -1 & -1 & 4 \end{bmatrix}
\begin{bmatrix} \widehat{\mu}(c_1)\\ \widehat{\mu}(c_4)\\ \widehat{\mu}(c_5) \end{bmatrix}
=
\begin{bmatrix} 0.62\\ 0.949\\ 0.537 \end{bmatrix}
\tag{37}
$$

Solving the linear equation system provides us with $\widehat{\mu}(c_1)=0.335$, $\widehat{\mu}(c_4)=0.4$ and $\widehat{\mu}(c_5)=0.318$ which leads to the desired result $10^{\widehat{\mu}(c_1)}=2.16$, $10^{\widehat{\mu}(c_4)}=2.514$ and $10^{\widehat{\mu}(c_5)}=2.08$. The non-scaled weight vector μ supplemented by the known values $\mu(c_2)=5$ and $\mu(c_3)=7$ takes the form:

$$
\mu=[2.16,5,7,2.514,2.08]^T
\tag{38}
$$

and after rescaling:

$$
\mu_n=[0.115,0.267,0.373,0.134,0.111]^T
\tag{39}
$$

Note that $|C_U|=3$ implies that the dimensions of matrix $\hat{A}$ are 3×3, moreover $\det(\hat{A})\neq 0$ and $\mu(c_i)>0$ for $i=1,4,5$ (see sec. 3.2) .

4.2 Example II (Choosing the best TV show)

Certain TV broadcaster wants to produce a new entertainment TV show in one of the European countries. It considering a purchase the license for one of the five entertainment shows produced in the United States. So far in Europe three similar programs were broadcasted. Through the market research there are known approximate size of their European audience. They are respectively $5,500,000$, $4,500,000$ and $4,950,000$ persons for programs c_6, c_7 and c_8 correspondingly. The production costs of these programs are similar. In order to select possibly the most profitable TV show the station hires a few seasoned media experts. During the expert panel they prepared the following *PC* matrix M representing a relative attractiveness of all the considered programs.

$$M=\begin{bmatrix} 1 & 0.8 & 1.333 & 0.7 & 0.5 & 0.6 & 0.75 & 0.667 \\ 1.25 & 1 & 1.667 & 0.875 & 0.625 & 0.75 & 0.9 & 0.833 \\ 1.333 & 0.6 & 1 & 0.933 & 0.667 & 0.8 & 0.978 & 0.889 \\ 1.429 & 1.143 & 1.071 & 1 & 0.714 & 0.857 & 1.05 & 0.952 \\ 2 & 1.6 & 1.5 & 1.4 & 1 & 1.2 & 1.467 & 1.333 \\ 1.667 & 1.333 & 1.25 & 1.167 & 0.833 & 1 & 1.222 & 1.111 \\ 1.333 & 1.111 & 1.023 & 0.952 & 0.682 & 0.818 & 1 & 0.909 \\ 1.5 & 1.2 & 0.382 & 1.05 & 0.75 & 0.9 & 1.1 & 1 \end{bmatrix} \tag{40}$$

In the matrix M every entry m_{ij} corresponds to the ratio describing attractiveness of the *TV* show c_i with respect to the attractiveness of *TV* show c_j. Since the values of attractiveness for c_6, c_7 and c_8 are known (they are approximated by the number of people watching the given TV show), thus the appropriate ratios m_{ij} for $i, j = 6,7,8$ are not the subject of the expert judgment. Instead, they are calculated based on data from the market research. For example:

$$m_{6,7} = \frac{\mu(c_6)}{\mu(c_7)} = \frac{5,100,000}{4,500,000} = 1.222 \tag{41}$$

or

$$m_{6,8} = \frac{\mu(c_6)}{\mu(c_8)} = \frac{5,100,000}{4,950,000} = 1.111 \tag{42}$$

The other entries of M represent the subjective judgements of experts.

Similarly as before, to find a solution with the help of HRE supported by the geometric averaging heuristics, the system of equations (9) must be solved. The desired values $\mu(c_i)$ for $i = 1\ldots,5$ will be derived from the formula $\widehat{\mu}(c_i) = \log \mu(c_i)$. Because $|C_U| = 5$, the dimensions of matrix $\widehat{A}$ are 5×5. The linear equation system need to be solved is as follows:

$$\begin{bmatrix} n-1 & -1 & -1 & -1 & -1 \\ -1 & n-1 & -1 & -1 & -1 \\ -1 & -1 & n-1 & -1 & -1 \\ -1 & -1 & -1 & n-1 & -1 \\ -1 & -1 & -1 & -1 & n-1 \end{bmatrix} \begin{bmatrix} \widehat{\mu}(c_1) \\ \widehat{\mu}(c_2) \\ \widehat{\mu}(c_3) \\ \widehat{\mu}(c_4) \\ \widehat{\mu}(c_5) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \tag{43}$$

where

$$\begin{aligned} b_1 &\overset{df}{=} \lg\left(m_{1,2}m_{1,3}m_{1,4}m_{1,5}m_{1,6}\mu(c_6)m_{1,7}\mu(c_7)m_{1,8}\mu(c_8)\right) \\ b_2 &\overset{df}{=} \lg\left(m_{2,1}m_{2,3}m_{2,4}m_{2,5}m_{2,6}\mu(c_6)m_{2,7}\mu(c_7)m_{2,8}\mu(c_8)\right) \\ b_3 &\overset{df}{=} \lg\left(m_{3,1}m_{3,2}m_{3,4}m_{3,5}m_{3,6}\mu(c_6)m_{3,7}\mu(c_7)m_{3,8}\mu(c_8)\right) \\ b_4 &\overset{df}{=} \lg\left(m_{4,1}m_{4,2}m_{4,3}m_{4,5}m_{4,6}\mu(c_6)m_{4,7}\mu(c_7)m_{4,8}\mu(c_8)\right) \\ b_5 &\overset{df}{=} \lg\left(m_{5,1}m_{5,2}m_{5,3}m_{5,4}m_{5,6}\mu(c_6)m_{5,7}\mu(c_7)m_{5,8}\mu(c_8)\right) \end{aligned} \tag{44}$$

hence, (43) numerically:

$$\begin{bmatrix} 7 & -1 & -1 & -1 & -1 \\ -1 & 7 & -1 & -1 & -1 \\ -1 & -1 & 7 & -1 & -1 \\ -1 & -1 & -1 & 7 & -1 \\ -1 & -1 & -1 & -1 & 7 \end{bmatrix} \begin{bmatrix} \widehat{\mu}(c_1) \\ \widehat{\mu}(c_2) \\ \widehat{\mu}(c_3) \\ \widehat{\mu}(c_4) \\ \widehat{\mu}(c_5) \end{bmatrix} = \begin{bmatrix} 19.137 \\ 19.895 \\ 19.627 \\ 20.118 \\ 21.286 \end{bmatrix} \tag{45}$$

The intermediate result vector is:

$$\widehat{\mu} = [6.561, 6.656, 6.623, 6.684, 6.83]^T \tag{46}$$

Hence, following the rule $\mu(c_i) = \xi^{\widehat{\mu}(c_i)}$, where $\xi = 10$ is the logarithm base, the final result vector is calculated.

$$\mu = \begin{bmatrix} 3,643,307 \\ 4,530,955 \\ 4,196,128 \\ 4,831,326 \\ 6,761,938 \end{bmatrix} \tag{47}$$

Thus, according to the expert judgments and the market research the TV show number 5 (denoted as c_5) has a chance to gather in front of TVs near 6.8 million people, whilst the second one in line “only” 4.8 million of people. Based on this estimate the board of directors representing the broadcaster has decide to recommend the purchase of the license for the fifth presented TV show.

5 Summary

The presented geometric HRE approach is another solution to the problem of rankings with the reference set. It proposes to use a geometric mean instead of arithmetic one used in [11, 12]. The advantage of this approach is the robustness of the procedure. As has been shown in (Sec. 3.2) the proposed solution works for arbitrary set of input data producing admissible vector of weights. The resulted ranking very often turns out to be optimal in sense of the magnitude of multiplicative errors. According to the formulated and proven condition (Sec. 3.3), this happens when the differences between the resulted priorities are not too large.

The *HRE* approach may be useful in many different situations including, ranking creation, valuation of goods and services, risk assessment and others. Due to the lack of restrictions on the input *PC* matrix (method with the geometric mean always produces an admissible result), the scope of the applicability of the HRE method increases. Thus, the presented method covers cases which can not always be dealt with using the arithmetic mean heuristics.

Despite the encouraging results, much remains to be done. In particular, the role of the inconsistency in the input matrix M should be more deeply investigated. Of course, the more studied examples, the better. Thus, further development of the method will be particularly focused on the study and analysis of use cases.

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INCOMPLETE PAIRWISE COMPARISONS IN THE ANALYTIC HIERARCHY PROCESS

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Abstract—The Analytic Hierarchy Process is a decision-analysis tool which was developed by T. L. Saaty in the 1970s and which has been applied to many different decision problems in corporate, governmental and other institutional settings. The most successful applications have come about in group decision-making sessions, where the group structures the problem in a hierarchical framework and pairwise comparisons are elicited from the group for each level of the hierarchy. However, the number of pairwise comparison necessary in a real problem often becomes overwhelming. For example, with 9 alternatives and 5 criteria, the group must answer 190 questions. This paper explores various methods for reducing the complexity of the preference eliciting process. The theory of a method based upon the graph-theoretic structure of the pairwise comparison matrix and the gradient of the right Perron vector is developed, and simulations of a series of random matrices are used to illustrate the properties of this approach.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP) was developed in the 1970s by T. L. Saaty [1] and over the years, has proven to be a very effective decision-analysis tool. Numerous application of this technique have included forecasting (inter- and intra-regional migration patterns, stock-market fluctuations etc.), investment decisions (portfolio selection, computer investment etc.) and socio-economic planning issues (transportation planning in Sudan, energy planning etc.). The essential ingredients in the AHP which lead to successful applications are the ability to incorporate "intangibles" into the decision-making process and its ease of use. In particular, applications of this technique to group decision making have proven to be most fruitful. In this type of situation, the group structures the problem in a hierarchical fashion, placing the overall objective of the decision at the top of the hierarchy and the criteria, subcriteria and decision alternatives on each descending level of the hierarchy. Once the group is satisfied with the problem structure, pairwise comparisons are elicited for each level of hierarchy in order to obtain the weights for each level with respect to one element in the next highest level in the hierarchy. For example, if the group is to choose one of four automobiles and three criteria are deemed to be important (style, handling, maintenance costs), then each criteria would be compared with all other criteria in a pairwise fashion with respect to the goal of purchasing the best car. Next, the automobiles would be compared according to each criteria. Finally, an overall weighting of the automobiles is obtained by synthesizing the weight from each level of the hierarchy; the book by Saaty [1] presents the theory of this process in detail.

The two advantages which the AHP has over other multi-criteria methods in this group setting are the ease of use and the ability to handle inconsistencies in judgments. People, acting unilaterally, are rarely consistent in their judgments. Thus, how can one ever expect a group to be consistent? The AHP does not force an individual or a group to be perfectly consistent when making pairwise comparisons, but incorporates the inconsistencies into the process decision maker a measure of the inconsistency in his/her/their judgments. The ability to handle inconsistency is a major point of the second advantage—the ease of use. Methods such as multi-attribute utility theory (MAUT) elicit transitive preferences at the cost of using complex eliciting mechanisms. The experience with the AHP supports Saaty's [1] claim that pairwise comparisons are somewhat "natural"; i.e. individuals or groups quickly become comfortable with the pairwise comparison mechanism and find it easy to use. By not forcing consistency of preferences, the AHP leads to a useful and usable decision-analysis tool.

The major drawback in the use of the AHP in either an individual or group decision process is the amount of work required to make all of the necessary pairwise comparisons. For example, if we have a problem of comparing 9 alternatives according to 5 criteria, a total of 190 pairwise comparison must be made. In realistic problems, this number is often quite higher. Thus, one comes to an important philosophical question concerning a decision-analysis tool: should the tool run the decision process or should the tool be considered to be a part of the process and not the process itself. It is the contention of this paper that, especially in group decision making, the latter must be the case. The structuring of the problem and the debate which precedes each pairwise comparison are vital aspects of the process which should not be curtailed due to time pressures arising from the need to complete all pairwise comparisons. Therefore, the purpose of this paper is to present a method for reducing the number of pairwise comparisons which must be made in an AHP session and thus, enable the group to focus on the debate and not the laborious task of completely filling in every comparison matrix.

There is another purpose for the development of a method to deal with incomplete pairwise comparisons. The AHP is based on the fact that pairwise comparisons are made on a ratio scale. Typically, the scale is bounded and the scale 1–9 is used, although any other scale could be used in this method [2]. Corresponding to this scale is a verbal description of the intensity of preference (equal, weak, moderate, strong, very strong, absolute). It is the intention of this research to lay the foundation for the development of a system with which the decision maker (a) only responds verbally and (b) is asked questions by the computer. Part (a) has already been implemented in systems such as *Expert Choice*. This paper presents the mathematical foundation for part (b); i.e. the development of an expert system-type implementation of the AHP. This system should be able to guide the decision maker in making the appropriate (i.e. important) judgments and to suggest that the decision maker stop making judgments after a certain number have already been made. Thus, a theory for dealing with incomplete pairwise comparisons must be developed in order to attain this expert system-like implementation of the AHP.

The remainder of this paper is structured as follows: Section 2 reviews the various methods which have been suggested for synthesizing a set of pairwise comparisons (least-squares, logarithmic least-squares and the eigenvector method), presents an argument for the use of the eigenvector method and discusses the problem of reducing the number of comparisons made in this method. Section 3 then presents the details of the method to deal with incomplete pairwise comparisons, Section 4 presents the results of a series of simulations using the proposed method and conclusions are drawn in Section 5.

2. SYNTHESIZING PAIRWISE COMPARISONS

Consider the problem of comparing a set of n alternatives with respect to a single criterion. Let $\mathbb{A} = (a_{ij})$ be the matrix of pairwise comparisons arising from this process, where

$$a_{ij} > 0 \quad \text{for } i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n,$$

and

$$a_{ji} = 1/a_{ij} \quad \text{for } i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n,$$

and $n = |\mathbb{A}|$. Thus, $\mathbb{A}$ is a positive reciprocal matrix of size n. Three methods have been suggested for synthesizing the set of pairwise comparisons to obtain a vector of attribute weights, $\mathbf{w} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$: least-squares (LS), logarithmic least-squares (LLS) and the eigenvector method (EM). The LS method [3] minimizes the Euclidean metric

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - w_i/w_j)^2 \tag{1}$$

to obtain the attribute weights $\mathbf{w}$, and the LLS method [4] minimizes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} [\ln(a_{ij}) - \ln(w_i/w_j)]^2. \tag{2}$$

The EM, or Saaty's method, sets the attribute weights equal to the right principal eigenvector or right Perron vector of the matrix $\mathbb{A}$:

$$\mathbb{A}\mathbf{w} = \lambda_{\max}\mathbf{w}, \tag{3}$$

where $\lambda_{\max}$ is the principal eigenvector or Perron root of $\mathbb{A}$.

Which method should be used? Saaty and Vargas [5] have recently shown that the EM is the only one of the three methods which has desirable rank preservation properties. A positive reciprocal matrix $\mathbb{A} = (a_{ij})$ is said to be consistent if $a_{ij}\, a_{jk} = a_{ik}\ \forall\, i,j,k$. That is, $\mathbb{A}$ is consistent if all paths between any two vertices in the fully connected, directed graph corresponding to $\mathbb{A}$ have the same intensity, where the intensity of a path is defined to be the multiplication of all the arc intensities:

$$a_{ij} = a_{i,i1}a_{i1,i2}\cdots a_{ir,j}.$$

In the case of a consistent matrix, Saaty and Vargas [5] show that all methods yield the same attribute weights $\mathbf{w}$. This result is intuitive in that if $\mathbb{A}$ is consistent, only the $(n-1)$ judgments making up the top row of $\mathbb{A}$ (note $a_{ii} = 1\ \forall\, i$) are needed since all other matrix entries can be derived from the relation $a_{ij} = a_{i1}a_{1j}$. It is when the judgments are inconsistent that the methods diverge. In this case, the top triangular portion of the matrix, $n(n-1)/2$ judgments, must be completed. In this case, the EM is the only method which fully captures the rank ordering inherent in the data [5]. This result can also be intuited by the graph-theoretic interpretation of the EM. Saaty [1] has shown that the right Perron vector is just the average of the intensities of all paths starting at a particular alternative; i.e. the eigenvector is just the average dominance of an alternative over the other $n-1$ alternatives (see the example in Ref. [2] for a more complete presentation of this interpretation). Thus, the EM is an averaging process. The LS and LLS methods, on the other hand, use the implicit assumption that a decision maker minimizes some Euclidean measure of his inconsistency, equation (1) or (2), when choosing attribute weights. It is the author's belief that an averaging process is much more "natural" than imposing some metric and optimizing behavior on the problem. It is this EM which is at the heart of the AHP and which will be the subject of this paper.

If a decision maker were perfectly consistent, then only $(n-1)$ judgments must be elicited and $\lambda_{\max} \doteq n$ [1]. However, any inconsistencies would necessitate the completion of the top triangular portion of the matrix—$n(n-1)/2$ judgments. In this case, $\lambda_{\max} > n$ [1]. Thus, the index

$$\text{C.I.} = (\lambda_{\max} - n)/n \tag{4}$$

has been suggested by Saaty [1] as a measure of the inconsistency of the judgments. Typically, if C.I. $\leqslant 0.1$ the judgments are taken as acceptable and if C.I. > 0.1, the decision maker is urged to reconsider his or her judgments (see Ref. [6] for a description of this process and the suggestion of a new method by which the reconsideration of judgments takes place).

Since it is unknown whether or not a decision maker will be consistent, all $n(n-1)/2$ judgments must be elicited. Thus, the eigenvector method includes a great deal of redundancy in the sense that $n(n-1)/2$ judgments are elicited instead of the minimum number $(n-1)$. This redundancy plays a useful role in that a decision maker can incorrectly answer one pairwise comparison, but the final attribute weights will not be greatly affected due to the redundancy and the averaging effect of the EM. Therefore, one would not want to make only $(n-1)$ pairwise comparison since a certain amount of redundancy is necessary to "correct" any errors in the judgments. However, the completion of all $n(n-1)/2$ judgments is a laborious task. It is the purpose of this paper to explore ways by which a decision maker has some redundancy in his judgments, but does not need to make the complete set of pairwise comparisons. Hence, the remainder of this paper is devoted to the concept of *incomplete* pairwise comparison in the eigenvector and thus the AHP method.

As a footnote to this section, the only other authors to attempt to deal with the problem of using the AHP with a large number of alternatives are Weiss and Rao [7]. Their approach is essentially a factorial design of the comparisons. The approach which is detailed in this paper is to create a "real-time" or expert system-like method and thus, is fundamentally different from the Weiss and Rao methodology.

3. INCOMPLETE PAIRWISE COMPARISONS

It is obvious that if all the pairwise comparisons are not made, then the LS and LLS methods can be easily generalized to this situation by restricting the indices on the summations in equations (1) and (2), respectively. One must only be sure to have at least one nonzero entry in each row of the pairwise comparison matrix $\mathbb{A} = (a_{ij})$; i.e. one must be sure to create at least a spanning tree in the directed graph $D(\mathbb{A})$ associated with the matrix $\mathbb{A}$. Thus, the graph $D(\mathbb{A})$ is no longer fully connected as is the case when all $n(n-1)/2$ comparisons are made, but it must at least be *connected* (see, for example, Ref. [8] for the definition of these graph-theoretic concepts). Given comparisons of this type, both the LS and LLS methods can be used. However, it was argued in the previous section that these methods are inferior to the EM and thus, a generalization to the EM to deal with incomplete comparisons is necessary.

Given a set of pairwise comparisons, not necessarily complete, which constitute a reciprocal matrix $\mathbb{A} = (a_{ij})$, the directed graph corresponding to the positive elements in $\mathbb{A}$ is a reflexive graph. Furthermore, it will be assumed that this graph is always connected. In this situation, what is the natural way to derive the attribute weights $\mathbf{w}$? Consider a matrix element $a_{ij} = 0$; i.e. a pairwise comparison which has not yet been made. For a reflexive connected graph there must exist at least one path from i to j. Thus, a natural way to fill in the missing matrix element would be to take the average of the intensities of all the possible elementary paths connecting i and j. That is, the judgment a_{ij} is the average of all the possible ways in which i and j can be judged by considering their relationship with intermediate attributes or nodes. If the incomplete judgments in $\mathbb{A}$ were perfectly consistent, then every elementary path from i to j must have the same intensity. With the presence of inconsistencies the intensity of each path may differ. In this case, an average of these path intensities must be taken. This average is *not* the arithmetic mean however. Aczél and Saaty [9] have proven that to synthesize group judgments, the *geometric mean* must be used in order to preserve the reciprocal property—if the synthesis of the judgments yield $a_{ij} = \alpha$, then the synthesis of the reciprocal of the judgments should yield $a_{ji} = 1/\alpha$. Since one can treat each path intensity as a separate judgment in a set of group judgments, the geometric mean of the path intensities must be used to synthesize this information to yield a_{ij}. Therefore, given a set of incomplete comparison which form a connected graph $D(\mathbb{A})$, the missing matrix elements in the top triangular portion of $\mathbb{A}$ are found by taking the geometric mean of the intensities of all the elementary paths connecting the two attributes in $D(\mathbb{A})$. The lower triangular position of this matrix is then calculated by the reciprocal property $a_{ji} = 1/a_{ij}$. Given the updated matrix $\mathbb{A}$, the weights can then be derived by the standard EM.

Say that one starts the process by eliciting n pairwise comparisons in such a way that the graph $D(\mathbb{A})$ is connected. Thus, some redundancy is included by asking one more question than the minimum, $n-1$. By following the procedure outlined above, a vector of attribute weights $\mathbf{w}$ can be derived. One could of course stop the process at this point and consider $\mathbf{w}$ to be the final vector of weights. However, it may be the case that either the decision maker is unhappy or uncomfortable with the current ranking in $\mathbf{w}$ or that the decision maker was highly inconsistent in the current set of comparisons. In either case, more comparisons need to be elicited. Thus, the question arises as to how to select the next comparison to be made. Of course, the decision maker may know which judgment is best in the sense that he or she is most confident in its value, in which case this comparison should be made next. However, it is more often the case that the decision maker must be guided to the next comparison. It is intuitively obvious in this case that the next question should be the one which has the greatest impact on the weighting $\mathbf{w}$; i.e. the next question should be the one which in some way is related to the largest absolute gradient of $\mathbf{w}$ with respect to the unknown matrix elements. The choice of such a question will be detailed in a moment but first, formulas for the gradient of $\mathbf{w}$ with respect to a matrix element a_{ij} must be derived.

Consider the class of positive reciprocal (square) matrices $\mathbb{A} = (a_{ij})$

$$\Lambda^{n,n} = \{\mathbb{A} = (a_{ij}) \in R^{n,n} \mid a_{ij} > 0 \quad \forall\, 1 \leqslant i,j \leqslant n; a_{ji} = 1/a_{ij} \quad \forall\, 1 \leqslant i,j \leqslant n\},$$

and consider the following eigenvector problem:

$$\mathbb{A}\mathbf{x}(\mathbb{A}) = r(\mathbb{A})\mathbf{x}(\mathbb{A}) \tag{5}$$

where $r(\mathbb{A}) = \lambda_{\max}$ is the Perron root or principal eigenvalue of $\mathbb{A}$ and $\mathbf{x}(\mathbb{A})$ is the right Perron vector of $\mathbb{A}$. The author [6] has recently proven the following results on the derivatives of $r(\mathbb{A})$ with respect to a matrix element in the upper triangular portion of $\mathbb{A}$.

Lemma 1

Let $\mathbb{A} \in \Lambda^{n,n}$. Then for $j > i$, $\partial\mathbb{A}/\partial_{ij}$ is an $n \times n$ matrix of the form

$$\left[\frac{\partial \mathbb{A}}{\partial_{ij}}\right]_{kl} = \begin{cases} 1 & \text{if } k = i, l = j \\ -1/(a_{ij})^2 & \text{if } k = j, l = i \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2

Let $\mathbb{A} \in \Lambda^{n,n}$. Then $\mathbb{D}_r^{\mathbb{A}}$ is an $n \times n$ upper triangular matrix of the form

$$\begin{aligned} \mathbb{D}_r^{\mathbb{A}} &= \left[\frac{\partial r(\mathbb{A})}{\partial_{ij}}\Big|_{j>i}\right] \\ &= [[\mathbf{y}(\mathbb{A})_i \mathbf{x}(\mathbb{A})_j] - [\mathbf{y}(\mathbb{A})_j \mathbf{x}(\mathbb{A})_i]/[a_{ij}]^2|_{j>i}], \end{aligned} \tag{6}$$

where $\mathbf{x}(\mathbb{A})$ and $\mathbf{y}(\mathbb{A})$ are, respectively, the right and left Perron vectors of $\mathbb{A}$ and $\mathbf{y}(\mathbb{A})^{\mathrm{T}}\mathbf{x}(\mathbb{A}) = 1$.

Using these results, the following theorem on the derivatives of $\mathbf{x}(\mathbb{A})$ with respect to a matrix element in the upper triangular portion of $\mathbb{A}$ can be proven.

Theorem

Let $\mathbb{A} \in \Lambda^{n,n}$ and let $r(\mathbb{A})$, $\mathbf{x}(\mathbb{A})$, $\mathbf{y}(\mathbb{A})$ denote respectively, the Perron root and right and left vectors of $\mathbb{A}$. Then

$$\begin{aligned} \mathbb{D}_{\mathbf{x}}^{\mathbb{A}} &= \left[\frac{\partial \mathbf{x}(\mathbb{A})}{\partial_{ij}}\Big|_{j>i}\right] \\ &= \left[\begin{bmatrix} \tilde{\mathbb{A}} - r(\mathbb{A})\tilde{\mathbb{I}} \\ \mathbf{e} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbb{D}}_r^{\mathbb{A}}\mathbf{x}(\mathbb{A}) - \tilde{\mathbf{Z}}(\mathbb{A}) \\ 0 \end{bmatrix}\right], \end{aligned} \tag{7}$$

where $\mathbb{I}$ is the $n \times n$ identity matrix, $\mathbf{e}$ is an n-dimensional row vector of ones, $\mathbf{Z}(\mathbb{A}) = (\mathbf{Z}_k)$ is an n-dimensional column vector defined as follows:

$$\mathbf{Z}_k = \begin{cases} \mathbf{x}_j(\mathbb{A}) & \text{if } k = i \\ -\mathbf{x}_i(\mathbb{A})/(a_{ij})^2 & \text{if } k = j \\ 0 & \text{otherwise;} \end{cases}$$

and "$\sim$" denotes the matrix or vector with its last row deleted.

Proof. It is well-known that the Perron root of a matrix $\mathbb{A} \in \Lambda^{n,n}$ is simple and thus, there exists a neighborhood $N_{\mathbb{A}}$ of $\mathbb{A}$ in $R^{n,n}$ such that each $\mathbb{B} \in N_{\mathbb{A}}$ has a simple eigenvalue $\lambda(\mathbb{B})$ and such that for $\mathbb{B} \in N_{\mathbb{A}} \cap \Lambda^{n,n}$ we have $\lambda(\mathbb{B}) = r(\mathbb{B})$ [6]. Furthermore, $\lambda(\cdot)$ is analytic as a function of the n^2 entries of the elements in $N_{\mathbb{A}}$ and thus, the partial derivatives of all orders of $\lambda(\cdot)$ with respect to the n^2 matrix elements must exist and be well-defined. For each $\mathbb{B} \in N_{\mathbb{A}}$, let $\mathbf{x}(\mathbb{B})$ be the right eigenvector of $\mathbb{B}$ corresponding to $\lambda(\mathbb{B})$:

$$\mathbb{B}\mathbf{x}(\mathbb{B}) = \lambda(\mathbb{B})\mathbf{x}(\mathbb{B}), \tag{8}$$

for which

$$\mathbf{ex}(\mathbb{B}) = 1, \tag{9}$$

where $\mathbf{e}$ is a row vector of ones. Thus, $\mathbf{x}(\cdot)$ is analytic as a function of each of the elements of $N_{\mathbb{A}}$, and the partial derivatives $\partial\mathbf{x}(\mathbb{B})/\partial_{ij}$ of $\mathbf{x}(\cdot)$ at $\mathbb{B}$ with respect to the matrix elements must exist and be well-defined. Now let $\mathbb{B} = (b_{ij}) \in N_{\mathbb{A}}$ and consider the following:

$$\mathbb{B}\mathbf{x}(\mathbb{B}) = r(\mathbb{B})\mathbf{x}(\mathbb{B}). \tag{10}$$

Differentiating equation (10) on both sides by the (i,j)th entry $1 \leqslant i, j \leqslant n, j > i$, one obtains

$$\frac{\partial\mathbb{B}}{\partial_{ij}}\mathbf{x}(\mathbb{B}) + \mathbb{B}\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} = \frac{\partial r(\mathbb{B})}{\partial_{ij}}\mathbf{x}(\mathbb{B}) + r(\mathbb{B})\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}}. \tag{11}$$

Using the results of Lemmas 1 and 2, this equation can be rewritten as

$$\begin{aligned}[\mathbb{B} - r(\mathbb{B})\mathbb{I}]\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} &= \frac{\partial r(\mathbb{B})}{\partial_{ij}}\mathbf{x}(\mathbb{B}) - \frac{\partial\mathbb{B}}{\partial_{ij}}\mathbf{x}(\mathbb{B}) \\ &= \mathbb{D}_r^{\mathbb{B}}\mathbf{x}(\mathbb{B}) - \mathbf{Z}(\mathbb{B}),\end{aligned} \tag{12}$$

where $\mathbf{Z}(\mathbb{B}) = (\mathbf{Z}_k)$ is an n-dimensional column vector of the form

$$\mathbf{Z}_k = \begin{cases} \mathbf{x}_j(\mathbb{B}) & \text{if } k = i \\ -\mathbf{x}_i(\mathbb{B})/(b_{ij})^2 & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the matrix $[\mathbb{B} - r(\mathbb{B})\mathbb{I}]$ is singular. However, since

$$\mathbf{ex}(\mathbb{B}) = 1,$$

it follows that

$$\mathbf{e}\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} = 0. \tag{13}$$

Deleting the last row in $[\mathbb{B} - r(\mathbb{B})\mathbb{I}]$, $\mathbb{D}_r^{\mathbb{B}}$ and $\mathbf{Z}(\mathbb{B})$ and adding the row vector $\mathbf{e}$ to the l.h.s. of equation (12) and 0 to the r.h.s. yields

$$\begin{bmatrix} \tilde{\mathbb{B}} - r(\mathbb{B})\tilde{\mathbb{I}} \\ \mathbf{e} \end{bmatrix}\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} = \begin{bmatrix} \tilde{\mathbb{D}}_r^{\mathbb{B}}\mathbf{x}(\mathbb{B}) - \tilde{\mathbf{Z}}(\mathbb{B}) \\ 0 \end{bmatrix}, \tag{14}$$

where "$\tilde{}$" denotes the same vector or matrix with its last row deleted. The matrix on the l.h.s. equation (14) will now be nonsingular and, by letting $\mathbb{B} = \mathbb{A}$, the conclusion of the theorem is obtained.† Q.E.D.

The above theorem gives one a means by which the gradients of the right Perron vector and hence the attribute weights $\mathbf{w}$ can be easily calculated from the right and left Perron vectors. How does one now use this information to guide the decision maker to the next comparison, and how

† This formula is an extension of the type of results obtained by Wilkinson [10] and Vargas [11] on perturbation to essentially nonnegative and positive reciprocal matrices.

is this information used to devise stopping rules i.e. rules for terminating the pairwise comparisons before all $n(n-1)/2$ comparisons are made? These two questions will now be addressed.

The logical choice of the next question would be the cell entry which has the greatest impact on the attribute weights; i.e. the comparison with the largest absolute gradient of the right Perron vector. Obviously, one would not want to ask a question which had little influence on the weights. Thus, the choice of the next comparison (i,j) by the rule

$$(i,j) = \operatorname*{argmax}_{(k,l)\in Q} (\|\partial \mathbf{x}(A)/\partial_{kl}\|_{\infty}), \tag{15}$$

where Q is the set of unanswered comparisons and $\|\cdot\|_{\infty}$ denotes the L_{∞} or Tchebyshev norm, will direct one to the most important question. Of course, one should not force the decision maker to choose this comparison and not consider any others, but one should present the decision maker with a ranking of the unanswered comparison in terms of equation (15) and allow him to select the next comparison. This ranking, however, is vital for the decision maker since it gives him information on the importance of the remaining comparisons.

The next issue involves the decision to stop making pairwise comparisons. There are three possible ways which this decision can be made. The first is to let the decision maker decide whether or not to continue with the questioning. In fact, this option is always available under the other two stopping rules. The second rule would state that if the maximum absolute difference in the attribute weights from one question to the next is $\leqslant \alpha\%$, where α is a given constant, then one should stop since the new comparison did not have a major influence on your weighting. Formally, if $\mathbf{w}^k$ and $\mathbf{w}^{k+1}$ are, respectively, the attribute weights after k and $k+1$ comparisons have been made and

$$l = \operatorname*{argmax}_{1\leqslant i\leqslant n} |w_i^{k+1} - w_i^k|/w_i^k, \tag{16}$$

then the procedure would stop at $k+1$ comparisons if

$$\frac{|w_l^{k+1} - w_l^k|}{w_l^k} \leqslant \alpha. \tag{17}$$

This rule is very "liberal" in the sense that further questioning may drastically alter the weights. However, the decision maker always has a veto power and hence this rule may work well in practice.

The third stopping rule is very conservative in the sense that comparisons will continue to be made until one is sure that ordinal rank will not be reversed. The weights $\mathbf{w}$ are cardinal rankings of the alternatives which, of course, create an ordinal ranking of the alternatives. By answering more questions, the cardinal ranking in $\mathbf{w}$ may be slightly altered but the ordinal ranking could remain the same. The third stopping criterion states that the next question derived from the gradient procedure just described will only be asked if it appears that the ordinal ranking could be reversed. More precisely, consider a current value a_{ij} of the (i,j)th question which has just been chosen as the next question to be asked. Let $U_{ij} = \max(1, \bar{a}_{ij} - a_{ij})$, where $\bar{a}_{ij}$ is the largest path intensity in the set of all elementary paths connecting i and j, and let $L_{ij} = \max(1, a_{ij} - \underline{a}_{ij})$, where $\underline{a}_{ij}$ is the smallest path intensity. If the decision maker was perfectly consistent up to the current comparison, then $\bar{a}_{ij} = a_{ij} = \underline{a}_{ij}$. However, one cannot be sure that the decision maker will not be at least slightly inconsistent in the next question and thus, a perturbation of 1 is introduced. For example, if the current value of $a_{ij} = 6$ and $\bar{a}_{ij} = 9, \underline{a}_{ij} = 5.4$, then $U_{ij} = 3$ and $L_{ij} = 1$. One cannot assume that perfect consistency in a subset of comparisons is a valid criterion for stopping the process since there is always the possibility that perturbations could occur. By choosing L_{ij} and U_{ij} in the way which is described above, one allows for these perturbations. Given these upper and lower bounds on the possible deviation from a_{ij}, let us define $P(\mathbf{w})$ to be a function which returns the ordinal ranking inherent in the cardinal ranking $\mathbf{w}$; i.e. $P: R^n \to Z^n$ where Z^n is the n-dimensional space of natural numbers. For example, if $\mathbf{w} = (0.15,\ 0.3,\ 0.2,\ 0.35)^T$ then $P(\mathbf{w}) = (4, 2, 3, 1)^T$. Using this function, three ranking can be defined:

$$P_1 = P(\mathbf{w}),$$

$$P_2 = P\left(w + \frac{\partial w}{\partial_{ij}}(a_{ij} + U_{ij})\right)$$

and

$$P_3 = P\left(w + \frac{\partial w}{\partial_{ij}}(a_{ij} - L_{ij})\right).$$

Ranking P_1 is the current ordinal ranking, and rankings P_2 and P_3 are the approximations to the ordinal rankings which would occur if the (i,j)th comparison achieved its maximum and minimum deviation, respectively. If $P_1 = P_2 = P_3$, then it is likely that the next comparison will not alter the ordinal ranking inherent in $\mathbf{w}$ and hence, the procedure may be terminated. This ordinal rank reversal criterion is very conservative in the sense that two alternatives may have low but almost equivalent weights and this criterion would not terminate the comparisons in these circumstances. Alternatives with low weights are not important and thus one would like to ignore a possible rank reversal in this situation. However, the criterion described above would force the eliciting process to continue. Therefore, one must either consider using this stopping criterion, a stopping criterion such as the above mentioned $\alpha\%$ rule and making it possible for the decision maker to decide to stop, or continue this process, or some combination of these three rules; which rule is best becomes a purely empirical question which will be explored in future research.

In the actual implementation of the procedure outlined above, computational considerations call for a modification to this method. For an incomplete comparison method to be useful to a decision maker, the computation of the eigenvector and derivatives after each question must be done quickly. The most computationally burdensome task in this step is the computation of all the elementary paths between two specified vertices in $D(\mathbb{A})$. As Carré [8] points out, this problem can be solved via a backtracking algorithm. However, as the number of completed comparison grows, the number of elementary paths grows exponentially. Thus, the determination of all elementary paths becomes extremely difficult. Due to the computational complexity of this task, a simplification will be made. Instead of finding all elementary paths, a sample of random spanning trees will be used to calculate a_{ij}, $\bar{a}_{ij}$ and $\underline{a}_{ij}$. Finding the shortest spanning tree in a graph is extremely easy, so a procedure in which arc costs are randomly derived will be used in conjunction with a shortest spanning tree algorithm to derive a sample of elementary paths.

In summary, the steps used in the incomplete pairwise comparison method are:

Step 0. Have the decision maker provide n judgments which form a connected graph $D(\mathbb{A})$.

Step 1. Using the completed pairwise comparisons, derive the missing comparison by taking the geometric mean of intensities of a sample of random spanning trees. Calculate the weight $\mathbf{w}$.

Step 2. Calculate the derivatives of $\mathbf{w}$ with respect to the missing matrix elements and select the next question according to equation (15).

Step 3. If this question meets the appropriate stopping criteria (subjective assessment, $\alpha\%$ rule, ordinal rank rule etc.), stop; else elicit this comparison and return to Step 1.

As a final comment on this section, the results of the theorem can also be used in a sensitivity analysis at the end of the process. By being easy to compute, the derivatives of $\mathbf{w}$ with respect to the matrix elements can be quickly used to guide to decision maker in revising any judgments which were made during the course of this process.

Table 1

Distance from Philadelphia	1	2	3	4	5	6	w
1. Cairo	1	1/3	8	3	3	7	0.2619
2. Tokyo		1	9	3	3	9	0.3975
3. Chicago			1	1/6	1/5	2	0.0334
4. San Francisco				1	1/3	6	0.1164
5. London					1	6	0.1642
6. Montreal						1	0.0266

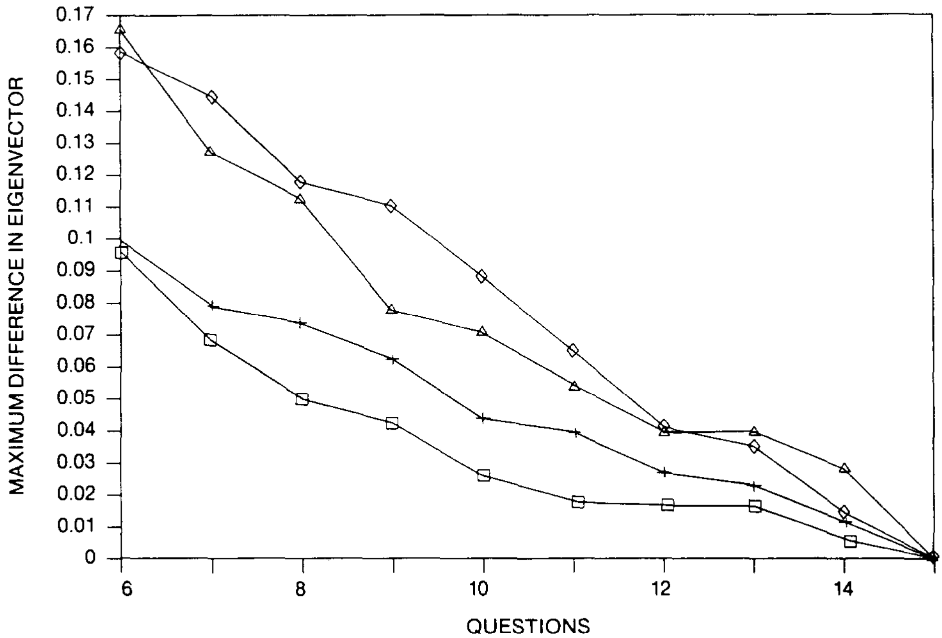

Fig. 1. Results of matrices of size $N = 6$.

4. NUMERICAL RESULTS

In order to obtain some insight as to the possible benefits which the proposed method might yield, a series of numerical experiments were performed. First, consider the example in Saaty [1] of using the eigenvector method to predict the relative distance of a set of cities from Philadelphia (see Table 1). Consider the situation where the first 6 comparisons are (1, 2), (1, 3), (2, 5), (3, 6), (4, 5) and (5, 6). The process described in Section 3 yields the results presented in Table 2 (there are 15 comparisons in total). The 5% stopping rule would say to stop at question 6 while the rank order criterion would stop the procedure at question 10. In either case, it is obvious from the table that not all 15 questions are necessary. Thus, the procedure outlined in this paper can yield significant time savings in this example.

In order to test this procedure more thoroughly, a simulation of 50 random matrices of size 6, 7, 8 and 9 was performed; Figs 1–4 show the results of this experiment. In these figures, the maximum difference in the eigenvector is defined as $\|\mathbf{w}^k - \mathbf{w}^*\|_\infty$, where $\mathbf{w}^k$ is the eigenvector after

Table 2

Question	w_1	w_2	w_3	w_4	w_5	w_6
6	0.2339	0.4612	0.0382	0.0659	0.1734	0.0273
7	0.2237	0.4649	0.0394	0.0688	0.1757	0.0275
8	0.2863	0.4504	0.0407	0.0474	0.1464	0.0288
9	0.2769	0.4435	0.0321	0.0785	0.1433	0.0257
10	0.2200	0.4004	0.0349	0.1361	0.1771	0.0315
11	0.2684	0.3855	0.0325	0.1254	0.1641	0.0240
12	0.2694	0.4003	0.0338	0.1071	0.1622	0.0273
13	0.2686	0.3991	0.0335	0.1104	0.1622	0.0262
14	0.2676	0.3970	0.0330	0.1149	0.1623	0.0251
15	0.2619	0.3975	0.0334	0.1164	0.1642	0.0266

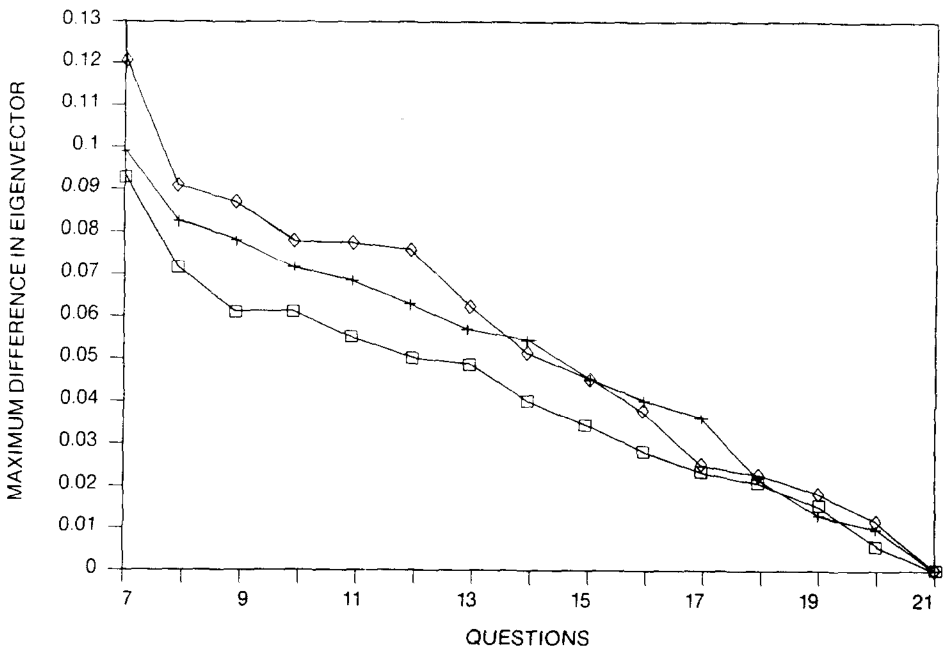

Fig. 2. Results of matrices of size $N = 7$.

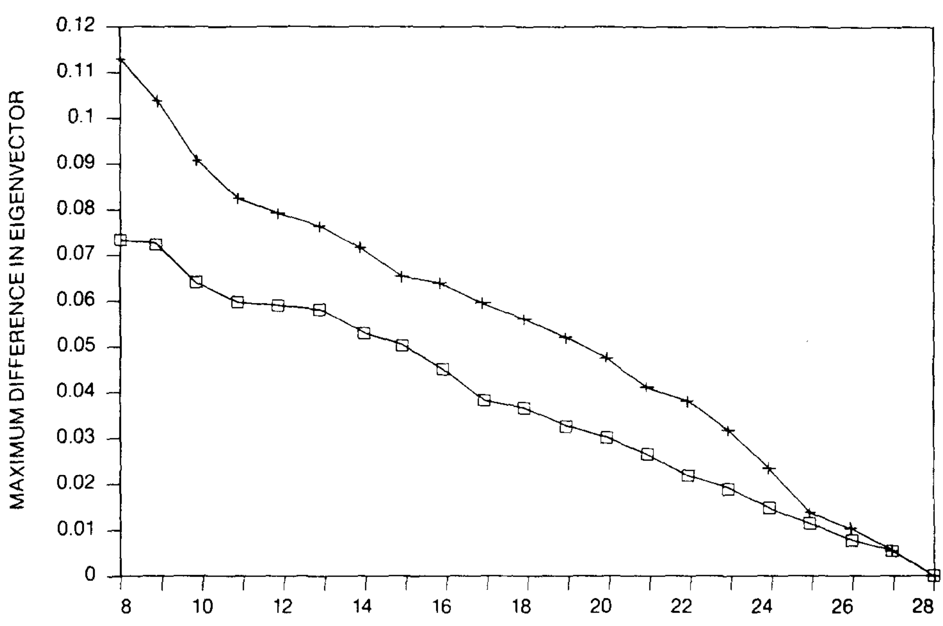

Fig. 3. Results of matrices of size $N = 8$.

question k has been answered and $\mathbf{w}^*$ is the eigenvector when the matrix is complete. As one can see, the errors increase with increasing inconsistency (C.I.) as expected. Also, the errors tend to fall more rapidly at the beginning of the process and tend to fall very slowly as k approaches $n(n-1)/2$. This result is also expected since the process is first choosing those questions with the greatest impact on the eigenvector. Therefore, as k approaches $n(n-1)/2$, the comparison tend to become less and less useful which confirms the belief that it is not worthwhile to make *all* the comparisons in the EM. Finally, the average number of questions which need to be asked under the 5% and ordinal rank stopping criteria for the various size matrices are as given in Table 3. One can immediately see how conservative the ordinal ranking rule is in practice and the liberality of the

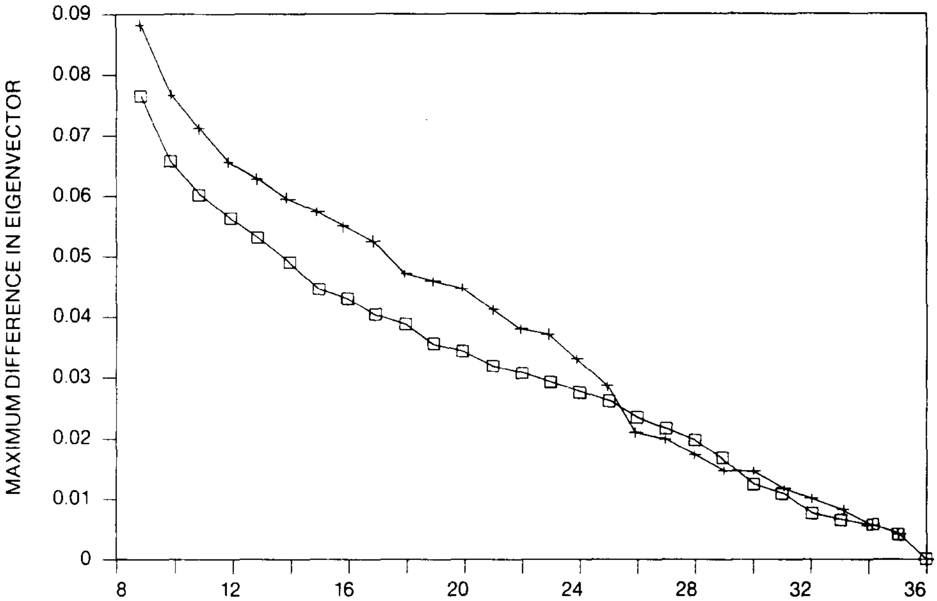

Fig. 4. Results of matrices of size $N = 9$.

Table 3

n	5% Rule	Percentage of total No. of questions	Ordinal ranking rule	Percentage of total No. of questions
6	10.90	72.67	12.60	84.00
7	12.24	58.29	17.18	81.81
8	13.56	48.43	22.78	81.36
9	14.36	39.89	30.68	85.22

α% rule. Also, there is a dramatic decrease in the percentage of questions which must be answered under the 5% rule. In practice, these results may not be as striking, but these results clearly indicate that the incomplete pairwise comparison process described in this paper can substantially reduce the work involved in the EM.

5. CONCLUSION

This paper has developed a method by which substantial time savings in using the AHP can be achieved. These time savings are important in that they simplify the work involved in making the pairwise comparisons and therefore, give the individual or group of decision makers more time to debate certain judgments and create different hierarchical structures for the problem which can then be compared and synthesized. Thus, the incomplete pairwise comparison method presented in this paper helps remove the decision-analysis tool as the primary focus of the decision process and puts it in being an aid to the process.

At least two research items emerge from this study. The first is to answer the question of how well this process will work in practice. A series of empirical experiments must be performed in order to validate and refine the proposed method. The second involves the hierarchical structure. The AHP currently asks the decision maker to make pairwise comparisons of one criteria at a time. The method described in this paper simply reduces the amount of work needed under each criteria. There is another way of looking at the process. Instead of comparing all alternatives under each criteria, one could compare two alternatives under all criteria. Both a theoretical question as to how the method proposed in this paper can deal with this reversal in the comparison process, especially when there are more than two levels in the hierarchy and an empirical question of the ability of a decision maker to congnitively process information in this new frame of reference remain to be answered.

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Decision Aiding

The geometric consistency index: Approximated thresholds

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Abstract

Crawford and Williams [Journal of Mathematical Psychology 29 (1985) 387] suggested for the Row Geometric Mean Method (RGMM), one of the most extended AHP's priorization procedure, a measure of the inconsistency based on stochastic properties of a subjacent model. In this paper, we formalize this inconsistency measure, hereafter called the Geometric Consistency Index (GCI), and provide the thresholds associated with it. These thresholds allow us an interpretation of the inconsistency tolerance level analogous to that proposed by Saaty [Multicriteria Decision Making: The Analytic Hierarchy Process, New York, 1980] for the Consistency Ratio (CR) used with the Right Eigenvector Method in Conventional-AHP.

Keywords: Multicriteria; AHP; Geometric mean; Consistency; Thresholds

1. Introduction

The Analytic Hierarchy Process (AHP) is a multicriteria decision making (MCDM) technique proposed by Saaty (1977, 1980), that integrates pairwise comparison ratios into a ratio scale. One advantage of this MCDM tool is that it allows us to measure the consistency of the decision maker when eliciting the judgements in a formal and elegant way.

Defining the consistency of a positive reciprocal pairwise comparison matrix, $A = (a_{ij})$, as the cardinal transitivity between the judgements, that is to say, $a_{ij}a_{jk} = a_{ik}$, $i, j, k = 1, \dots, n$, Saaty suggested that the inconsistency in Conventional-AHP, where the Right Eigenvector Method (EVM) is used as priorization procedure, can be captured by a single number ($\lambda_{\max} - n$) which reflects the deviations of all a_{ij} from the estimated ratio of priorities ω_i/ω_j.

In this case, to provide a measure independent of the order of the matrix, n, Saaty proposed the use of the Consistency Ratio (CR). This is obtained by taking the ratio between $\lambda_{\max} - n$ to its expected value over a large number of positive reciprocal matrices of order n, whose entries are randomly chosen in the set of values $\{1/9, \dots, 1, \dots, 9\}$. For this consistency measure, he proposed a 10% threshold for the CR (5% and 8% for the 3 by 3 and 4 by 4 matrices, respectively) to accept the estimation of ω (Saaty, 1994). When the CR is greater than 10%, then, in order to improve

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the consistency, most inconsistency judgements, that is to say, those with a greater difference between a_{ij} and ω_i/ω_j, are usually modified and a new ω derived.

There are many other priorization procedures in the literature, but only a few of them present their corresponding indicators to evaluate the inconsistency. Furthermore, when these consistency indexes have been proposed (Crawford and Williams, 1985; Harker, 1987; Golden and Wang, 1989; Wedley, 1991; Takeda, 1993; Takeda and Yu, 1995; Monsuur, 1996; Escobar and Moreno-Jiménez, 1997; Aguarón, 1998), they lack a meaningful interpretation due to the absence of the corresponding thresholds. Obviously, if the priorization procedure is not the EVM, the Saaty approach to evaluate the consistency is not appropriate, by construction, and new consistency measures, related to the priorization procedure, are required.

Recently, and despite the strong defense of the EVM presented by the Saaty school (Saaty, 1990; Vargas, 1994, 1997), the use of the Row Geometric Mean Method (RGMM), or Logarithmic Least Squares Method, as a priorization procedure in AHP has significantly increased (Ramanathan, 1997; Van den Honert, 1998; Levary and Wan, 1999) due fundamentally to its psychological (Gescheider, 1985; Lootsma, 1993; Barzilai and Lootsma, 1997; Brugha, 2000) and mathematical (Narasimhan, 1982; Jensen, 1984; Budescu, 1984; Barzilai, 1997; Aguarón and Moreno-Jiménez, 2000; Escobar and Moreno-Jiménez, 2000; Brugha, 2000) properties.

Crawford and Williams (1985) justified the RGMM by means of two different approaches: (1) the minimization of the log quadratic distance of errors (Logarithmic Least Squares Method); and (2) the maximum likelihood estimator of the priorities. The first is a deterministic approach and the second a stochastic one, where a multiplicative model for the perturbations has been supposed ($a_{ij} = (\omega_i/\omega_j)\pi_{ij}$, with π_{ij} independent and log-normal distributions with zero mean and constant variance $\pi_{ij} \sim \text{Lognormal}(0, \sigma)$).

For this priorization procedure (RGMM), Crawford and Williams suggested that the estimator of the variance of the perturbations can be used as a measure of the consistency, where the lower the value, the better the consistency of the judgements. In what follows assuming the proposal of Crawford and Williams, and without entering into the analysis of the validity of the CR as a consistency measure in AHP, we calculate the thresholds that make this measure, called the Geometric Consistency Index (GCI), operative and that allow us to fix a tolerance level with an interpretation analogous to that considered for Saaty's CR.

The paper has been structured as follows: Section 2 presents the two consistency measures considered in this paper (CR and GCI); Section 3 establishes a theoretical relation between the CR and the GCI, the validity of which is tested through a regression analysis; finally, Section 4 closes the paper with some comments about the GCI thresholds.

2. Consistency measures. The Geometric Consistency Index (GCI)

In the Conventional-AHP (Saaty, 1980), the priorities (ω_i, $i = 1, \ldots, n$) are obtained by solving the eigenvector problem

$$A\omega = \lambda_{\max}\omega \sum_{i=1}^{n} \omega_i = 1, \tag{1}$$

where A is a positive pairwise comparison matrix of order n, $\lambda_{\max}$ is the principal eigenvalue of A and ω is the priority vector.

For this priorization procedure, the EVM, Saaty (1980) proposed a measure of the inconsistency in judgements, called the Consistency Index (CI), that is given by

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1}, \tag{2}$$

where $\lambda_{\max}$ is the principal eigenvalue of the judgement matrix and n is its order.

When the reciprocal comparison matrix is consistent, $\lambda_{\max} = n$, and the CI is equal to zero; otherwise, its value is positive. To overcome the order dependency of the CI, Saaty proposed a normalized measure, called the CR, that is given by

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}(n)}, \tag{3}$$

where $\mathrm{RI}(n)$ is the Random (Consistency) Index for matrices of order n. This term is defined as the expected value of the CI corresponding to matrices of order n ($\mathrm{RI} = E[\mathrm{CI}(n)]$), when the judgements are simulated in the set $\{1/9, \ldots, 1, \ldots 9\}$ and the EVM is used as the priorization procedure.

The CR gives a measure of where the judgements in the pairwise comparison matrix lie between totally consistent and totally random. When $\mathrm{CR} = 1$, then $\mathrm{CI} = E[\mathrm{CI}(n)]$ and the judgements are totally random (low precision). High values of CR reflect more inconsistency and thus we are interested in values of CR as low as possible. To accept the consistency of the matrix, Saaty (1980) suggested as a rule of thumb a threshold of 10 percent or less ($\mathrm{CR} \leqslant 0.1$). More recently, Saaty (1994) suggested thresholds of 5% and 8% for 3 by 3 and 4 by 4 matrices, respectively.

Lemma 1. *The CI proposed by Saaty for the EVM can be expressed as an average of the differences between the errors and unity, that is to say*

$$\mathrm{CI} = \frac{1}{n(n-1)} \sum_{i \neq j}^{n} (e_{ij} - 1), \tag{4}$$

where the errors are $e_{ij} = a_{ij}\omega_j/\omega_i$.

Proof. Immediately from definition of CI. □

In this paper we consider the priorization procedure known as the RGMM, where the priorities (without the normalization factor) are given by

$$\omega_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}. \tag{5}$$

For the RGMM, Crawford and Williams (1985) suggested the use of an unbiased estimator of the variance of the perturbations as a measure of the consistency:

$$s^2 = S/\mathrm{d.f.} = \frac{2\sum_{i<j} (\log a_{ij} - \log \omega_i/\omega_j)^2}{(n-1)(n-2)}. \tag{6}$$

The numerator (S) is the squared distance between the log of the judgements a_{ij} and the log of the ratios (ω_i/ω_j), and the denominator (d.f.) is the degrees of freedom that are given as the difference between the judgements included $(n(n-1)/2)$ and the estimated parameters $(n-1)$.

From a deterministic point of view, the smaller the s^2, the shorter the distance between the judgements a_{ij} and the ratios ω_i/ω_j. From a stochastic one, the smaller the s^2, the smaller will be the variance of the perturbations π_{ij} and the better will be the fit between the judgements and the priorities vector ω.

Next, we consider the measure of consistency proposed by Crawford and Williams (1985) which, in what follows, we call the GCI.

Definition 1. Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, and the vector of priorities, ω, obtained by the RGMM, let us define the GCI as

$$\mathrm{GCI} = \frac{2}{(n-1)(n-2)} \sum_{i<j} \log^2 e_{ij}, \tag{7}$$

where $e_{ij} = a_{ij}w_j/w_i$ is the error obtained when the ratio ω_i/ω_j is approximated by a_{ij}.

In an interpretation analogous to that given in Lemma 1 for the CI, the GCI can be seen as an average of the squared difference between the log of the errors and the log of unity:

$$\mathrm{GCI} = \frac{2}{(n-1)(n-2)} \sum_{i<j} (\log e_{ij} - \log 1)^2.$$

In this case, the reciprocal property of the errors is considered in the proposed measure, because the errors $e_{ij} > 1$ and $e_{ij} < 1$ contribute with the same amount.

3. Approximated thresholds for the Geometric Consistency Index

As Barzilai (1996) indicates, the value of s^2 (GCI) can be considered as a measure of the goodness of fit. However, the range of values that will give it the operative character required by a measure of these characteristics remains to be established (also see Golden and Wang, 1989). One way of making this measure operative would be to normalize it in a way analogous to that carried out with Saaty's consistency ratio; that is to say, to divide the value that measures the log quadratic distance between the errors e_{ij} and unity (s^2) by its expected value when the judgements are simulated in the interval $\{1/9, \dots, 1, \dots, 9\}$.

However, as we will demonstrate in Lemma 2, the expected value of s^2 is a constant. Therefore, we will follow an indirect procedure to obtain the thresholds associated with the GCI, which allows for an interpretation of the inconsistency tolerance level analogous to that proposed by Saaty for the EVM (CR $\leqslant 0.1$).

Lemma 2. *If the judgements of a pairwise comparisons matrix follow independent, reciprocal and identical distributions, the mean of the GCI is given by*

$$E[\mathrm{GCI}] = \mathrm{Var}(\log a_{ij}). \tag{8}$$

Proof. See Appendix A. □

The indirect procedure consists in establishing the thresholds for the new measure on the basis of its relationship with the CR in its band where we can accept the estimation of the priorities as being good (CR $\leqslant 0.1$). Note, however, that the study has in fact been made in a broader range to guarantee the validity of the conclusions (CR $\leqslant$ 0.2).

In this sense, we first present a theoretical relation between the GCI and the CR that is valid for small errors and that will allow us to estimate the corresponding thresholds.

Theorem 1. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \dots, n$, and the vector of priorities, ω, obtained by the RGMM, it holds that*

$$\mathrm{GCI} = \frac{2n}{n-2}\mathrm{CI} + \mathrm{o}(\varepsilon^3), \tag{9}$$

where $\varepsilon = \max_{ij}\{|\log e_{ij}|\}$ and $e_{ij} = a_{ij}\omega_j/\omega_i$.

Proof. See Appendix A. □

Corollary 1. *Under the conditions of Theorem* 1, *it holds that*

$$\mathrm{GCI} = k(n)\mathrm{CR} + \mathrm{o}(\varepsilon^3), \tag{10}$$

where

$$k(n) = \frac{2n}{n-2}E[\mathrm{CI}(n)].$$

Proof. Immediate on the basis of the earlier theorem and $\mathrm{CR} = \mathrm{CI}/\mathrm{RI}(n)$. □

Using this result and removing the term $\mathrm{o}(\varepsilon^3)$, the relationship, $k(n)$, between GCI and CR is given in Table 1, where the $\mathrm{RI}(n) = E[\mathrm{CI}(n)]$ have been obtained through the simulation of 100,000 matrices for each order (n), where the judgements belong to the set of values $\{1/9, \dots, 1, \dots, 9\}$. For $n = 3$, the value of RI(3) = 0.525 is a rounded value of the exact one, RI(3) = 0.5245, obtained by enumerating all possible combinations of judgements.

This result makes clear that if the judgements matrices $A = (a_{ij})$ are close to consistency (small errors), then the two measures, that is to say, the CR of Saaty and the earlier-mentioned GCI, are proportional. We tested this using a regression analysis based on the values of the CR and the GCI obtained from a simulation study where nearly 1,200,000 matrices $A = (a_{ij})$ were generated around the identity (near consistency).

In general terms, it can be noted that the behaviour of the two measures is similar (see Fig. 1, corresponding to $n = 4$, as an example) for the different values of n (high values of CR provide high values of GCI). Nevertheless, it is interesting

Table 1
Values of RI(n) and $k(n)$ for $n = 3, \dots, 16$

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
RI(n)	0.525	0.882	1.115	1.252	1.341	1.404	1.452	1.484	1.513	1.535	1.555	1.570	1.583	1.595
$k(n)$	3.147	3.526	3.717	3.755	3.755	3.744	3.733	3.709	3.698	3.685	3.674	3.663	3.646	3.646

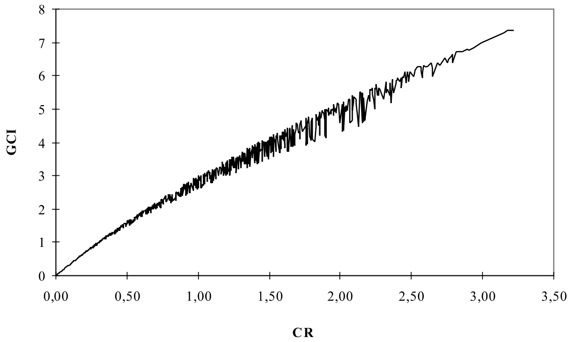

Fig. 1. Relation between GCI and CR for $n = 4$.

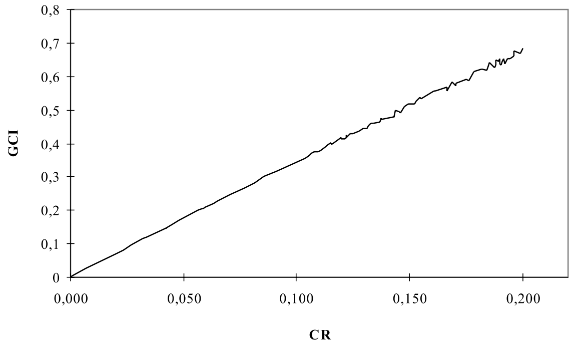

Fig. 2. Relation between GCI and CR for $n = 4$ and CR < 0.2.

to carry out a more detailed analysis of the behaviour of the two indices when the CR takes values that are considered as acceptable (CR $\leqslant$ 0.1). Moreover, in what follows, we will consider a broader range (CR $\leqslant$ 0.2).

Here (Fig. 2), we can see that for low values of Saaty's consistency ratio the relationship is practically linear, as guaranteed by the earlier theorem. This fit is particularly significant when CR is below 0.1. In order to test this fit, we have calculated the regression line of GCI over CR for different intervals of CR. Table 2 presents the number of available observations (N), the slope of the regression line (m), and the goodness of fit (r^2) for different orders of the judgement matrix n and four different consistency intervals (0–0.01, 0–0.05, 0–0.10, 0–0.20).

Note that in all cases the goodness of fit is very high, in such a way that we can accept an almost linear relation between the two measures. Nevertheless, as the range considered for the inconsistency increases, the slopes estimated through regression decrease, due to the small concavity of the relation. The maximum differences between these values and the approximated ones obtained from Corollary 1 have been of 1% for CR $\leqslant$ 0.01, 2.5% for CR $\leqslant$ 0.05, 4.7% for CR $\leqslant$ 0.1 and 7.9% for CR $\leqslant$ 0.2. Therefore, and despite the fact that the approximated values overestimate the true ones, we can consider the thresholds of Table 3 as corresponding with Saaty's consistency ratio. For the CR $= 0.1$, the associated GCI are: GCI $=$ 0.3147 for $n = 3$; GCI $= 0.3526$ for $n = 4$ and GCI $= 0.370$ for $n > 4$.

Table 2
Estimations of the regression coefficient (m) between GCI and CR (GCI $= m$CR)

n	0–0.01			0–0.05			0–0.10			0–0.20		
	N	m	r^2	N	m	r^2	N	m	r^2	N	m	r^2
3	13,934	3.147	1.000	34,177	3.137	1.000	45,142	3.128	1.000	55,616	3.111	1.000
4	13,599	3.518	1.000	22,857	3.477	1.000	39,063	3.436	1.000	56,023	3.364	0.999
5	14,259	3.702	1.000	18,273	3.637	0.999	36,369	3.570	0.999	58,114	3.466	0.997
6	14,313	3.739	1.000	15,676	3.664	0.999	34,161	3.584	0.998	59,750	3.470	0.996
7	14,166	3.737	1.000	14,404	3.662	0.999	32,669	3.578	0.998	60,578	3.460	0.996
8	13,769	3.726	1.000	13,565	3.653	0.999	31,936	3.570	0.998	61,144	3.449	0.997
9	13,563	3.715	1.000	13,037	3.643	0.999	31,377	3.563	0.999	61,489	3.439	0.997
10	13,240	3.693	1.000	12,605	3.623	0.999	30,941	3.543	0.999	61,588	3.421	0.997
11	12,925	3.681	1.000	12,303	3.613	0.999	30,701	3.536	0.999	61,621	3.413	0.997
12	12,550	3.668	1.000	11,934	3.603	0.999	30,479	3.526	0.999	61,617	3.404	0.998
13	12,275	3.659	1.000	11,589	3.596	0.999	30,304	3.519	0.999	61,597	3.398	0.998
14	12,002	3.648	1.000	11,333	3.588	0.999	30,291	3.511	0.999	61,524	3.391	0.998
15	11,737	3.639	1.000	11,126	3.581	0.999	30,248	3.504	0.999	61,478	3.384	0.998
16	11,549	3.632	1.000	10,813	3.576	0.999	30,162	3.498	0.999	61,462	3.380	0.998

Table 3
Approximated thresholds

CR	0.01	0.05	0.1	0.15
GCI ($n = 3$)	0.0314	0.1573	0.3147	0.4720
GCI ($n = 4$)	0.0352	0.1763	0.3526	0.5289
GCI ($n > 4$)	~0.037	~0.185	~0.370	~0.555

The interpretation of the GCI, the inconsistency measure used for the RGMM, is analogous to that proposed by Saaty for the CR used with the EVM. When the value of the GCI is greater than the corresponding threshold, the most inconsistent judgement (that with larger e_{ij}) has to be modified in the sense of approximating it (a_{ij}) to ω_i/ω_j.

4. Conclusions

In recent years there has been a move towards using geometric mean synthesis of AHP-type scores, for example, the RGMM. This priorization procedure provides estimations that are very close to the priorities of the traditional EVM. Moreover, it presents more desirable analytical properties and requires less computational effort.

In this paper, we have formalised the inconsistency measure proposed for the RGMM by Crawford and Williams (1985), calling it the GCI.

Following an indirect method, due to the GCI's independence order for this measure, we have computed thresholds that allow us an interpretation of the inconsistency level which is analogous to that proposed by Saaty for the EVM (see Table 3).

To obtain these thresholds, we have proved an analytical relation between the GCI and the CR that is valid for small inconsistencies, but that slightly overestimates these values, as we have seen through a regression analysis. The approximated thresholds,

$$k(n) = \frac{2n}{n-2}\mathrm{RI}(n),$$

computed from this relation are given in Table 1, where the values of the $\mathrm{RI}(n)$ used here have been obtained through the simulation of 100,000 matrices for each order (to the best of our knowledge, the most complete study in this sense).

Finally, we should highlight that, assuming the small discrepancies obtained for $\mathrm{CR} \leqslant 0.1$, the practical values (associated rounded values) of the GCI corresponding to the usual value of the CR = 10% are: GCI = 0.31 for $n = 3$; GCI = 0.35 for $n = 4$ and GCI = 0.37 for $n > 4$.

From a practical point of view, the interpretation of the *GCI* is analogous to that proposed by Saaty for the Consistency Ratio used with the Eigenvector Method in Conventional AHP. In our case, if we use the Row Geometric Mean Method as the priorization procedure in AHP and the judgements of the $(n \times n)$ pairwise comparison matrix belong to the fundamental scale of Saaty ($\{1/9, \ldots, 9\}$), then, when the value of the $GCI(n)$ is greater than its corresponding threshold (for example, $GCI > 0.1573$ for $n = 3$), the most inconsistent judgement would have to be modified and a new priority vector calculated. Otherwise, the estimations of the priorities given by the RGMM are accepted.

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Appendix A

Proof of Lemma 2

$$E[\mathrm{GCI}] = E\left[\frac{2}{(n-1)(n-2)}\sum_{i<j}\log^2 e_{ij}\right] = \frac{2}{(n-1)(n-2)}\sum_{i<j}E\left[\log^2 e_{ij}\right]. \tag{A.1}$$

The terms $\log^2 e_{ij}$ can be expressed as

$$\log^2 e_{ij} = \log^2\left[a_{ij}\frac{\omega_j}{\omega_i}\right] = \log^2\left[a_{ij}\left(\prod_{k=1}^{n}\frac{a_{jk}}{a_{ik}}\right)^{1/n}\right] = \log^2\left[a_{ij}^{1-2/n}\left(\prod_{k\neq i,j}\frac{a_{jk}}{a_{ik}}\right)^{1/n}\right].$$

As all these terms have an identical distribution, the expression (A.1) holds as

$$E[\mathrm{GCI}] = \frac{2}{(n-1)(n-2)}\frac{n(n-1)}{2}E\left[\log^2 e_{ij}\right] = \frac{n}{n-2}E\left[\log^2 e_{rs}\right] \tag{A.2}$$

for any r,s. To calculate the expected value of $\log^2 e_{rs}$, we operate as follows:

$$\log^2 e_{rs} = \log^2\left[a_{rs}^{1-2/n}\left(\prod_{k\neq r,s}\frac{a_{sk}}{a_{rk}}\right)^{1/n}\right] = \left[\log a_{rs}^{1-2/n} + \sum_{k\neq r,s}\log a_{sk}^{1/n} - \sum_{k\neq r,s}\log a_{rk}^{1/n}\right]^2.$$

When developing the square of this parenthesis, we have two kinds of terms: those that include a square logarithm (i.e. $\log^2 a_{rs}$) and those that include the product of two logarithms (i.e. $\log a_{rs}\log a_{rk}$). In the second case, as the judgements a_{ij} are reciprocal and independent, the expected value is zero, so we only consider the terms in $\log^2$:

$$E[\log^2 e_{rs}] = E\left[\log^2 a_{rs}^{1-2/n} + \sum_{k\neq r,s}\log^2 a_{sk}^{1/n} + \sum_{k\neq r,s}\log^2 a_{rk}^{1/n}\right] = E\left[\left(\frac{n-2}{n}\right)^2\log^2 a_{rs} + \sum_{k\neq r,s}\frac{1}{n^2}\log^2 a_{sk} + \sum_{k\neq r,s}\frac{1}{n^2}\log^2 a_{rk}\right].$$

Because all the judgements have the same distribution, the expected values of the terms $\log^2 a_{ij}$ coincide with that of $\log^2 a_{rs}$, so it holds that

$$E[\log^2 e_{rs}] = \left[\left(\frac{n-2}{n}\right)^2 + 2\frac{n-2}{n^2}\right]E[\log^2 a_{rs}] = \frac{n-2}{n}E[\log^2 a_{rs}]. \tag{A.3}$$

Using this result in expression (A.2), and taking into account that the expected value of the log of the reciprocal distributions is zero, we have

$$E[\mathrm{GCI}] = E[\log^2 a_{rs}] = \mathrm{Var}(\log a_{rs}). \qquad \square$$

Proof of Theorem 1. Let us consider the matrix $E = (e_{ij})$ with $e_{ij} = a_{ij}\omega_j/\omega_i$ where $\omega = (\omega_i)$ is the weights vector obtained by applying the row geometric mean method. For such a matrix it holds that

$$\prod_{j=1}^{n} e_{ij} = 1, \quad i = 1,\ldots,n. \tag{A.4}$$

Taking $\varepsilon_{ij} = \log e_{ij}$, we have that

$$\sum_{j=1}^{n}\varepsilon_{ij} = 0, \quad i = 1,\ldots,n. \tag{A.5}$$

If the matrix is consistent, the errors e_{ij} have the value one and the values of the ε_{ij} are null. If the inconsistency is small, by continuity (Saaty, 1980), the ε_{ij} will be found relatively close to zero. We take $\varepsilon = \max_{ij}\{|\varepsilon_{ij}|\}$.

Let $v = (v_i)$, $i = 1,\ldots,n$, be the priorities vector obtained by applying the right eigenvector method on the matrix E. We know that in the consistent case the two methods coincide, in such a way that, if the inconsistency is low it can be considered that the vector v will be close to that obtained by the geometric mean (equal to the unit vector as a consequence of expressions (A.4)). Let us take $v_i = 1 + d_i$ with $\sum_i d_i = 0$. With this we can verify

$$Ev = \lambda_{\max} v \tag{A.6}$$

developing

$$\begin{pmatrix} 1 & e_{12} & \cdots & e_{1n} \\ e_{21} & 1 & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1+d_1 \\ 1+d_2 \\ \vdots \\ 1+d_n \end{pmatrix} = \lambda_{\max}\begin{pmatrix} 1+d_1 \\ 1+d_2 \\ \vdots \\ 1+d_n \end{pmatrix} \tag{A.7}$$

and, therefore,

$$\sum_{j=1}^{n} e_{ij}(1+d_j) = \lambda_{\max}(1+d_i). \tag{A.8}$$

On the basis of this expression, we have

$$\lambda_{\max}(1+d_i) \leqslant \max_{ij} e_{ij}\sum_{j=1}^{n}(1+d_j) = \mathrm{e}^{\varepsilon} n \tag{A.9}$$

and, therefore, ($\lambda_{\max} \geqslant n$)

$$d_i \leqslant \frac{e^{\varepsilon} n}{\lambda_{\max}} - 1 \leqslant e^{\varepsilon} - 1. \qquad (A.10)$$

As $\sum_j d_j = 0$, $d_i = -\sum_{j \neq i} d_j$ and, on the basis of the above inequality, we have that $d_i \geqslant -(n-1)(e^{\varepsilon}-1)$. If the values of d_i are delimited between $-(n-1)(e^{\varepsilon}-1)$ and $e^{\varepsilon}-1$, then they are, at the very most, of the order $e^{\varepsilon}-1 = o(\varepsilon)$. This shows us that for small inconsistencies, the differences between the two priorities vectors are of the same order as the errors e_{ij} of the matrix.

If we add the identities (A.8) in i, we obtain

$$\sum_{i,j} e_{ij}(1+d_j) = n\lambda_{\max}. \qquad (A.11)$$

Taking into account that $e_{ij} = e^{\varepsilon_{ij}}$ and using the development in powers,

$$\begin{aligned} n\lambda_{\max} &= \sum_{ij}\left(1+\varepsilon_{ij}+\frac{1}{2}\varepsilon_{ij}^2+\frac{1}{6}\varepsilon_{ij}^3+o(\varepsilon_{ij}^3)\right)(1+d_j) \\ &= \sum_{ij}\left(1+\varepsilon_{ij}+\frac{1}{2}\varepsilon_{ij}^2+\frac{1}{6}\varepsilon_{ij}^3+d_j+d_j\varepsilon_{ij} \right. \\ &\qquad \left. +\frac{1}{2}d_j\varepsilon_{ij}^2+\frac{1}{6}d_j\varepsilon_{ij}^3\right)+o(\varepsilon^3) \\ &= n^2+\sum_{ij}\varepsilon_{ij}+\frac{1}{2}\sum_{ij}\varepsilon_{ij}^2+\frac{1}{6}\sum_{ij}\varepsilon_{ij}^3+\sum_{ij}d_j \\ &\qquad +\sum_{ij}d_j\varepsilon_{ij}+\frac{1}{2}\sum_{ij}d_j\varepsilon_{ij}^2+o(\varepsilon^3), \end{aligned}$$

where we have taken into account that the terms $d_j\varepsilon_{ij}^3$ are of order $o(\varepsilon^3)$. Furthermore,

$$\sum_{ij} d_j\varepsilon_{ij} = \sum_j\left(d_j\sum_i \varepsilon_{ij}\right) = 0.$$

Additionally, $\varepsilon_{ij} = -\varepsilon_{ji}$, so that $\sum_{ij}\varepsilon_{ij}^3 = 0$. With all this, we have that

$$\begin{aligned} n\lambda_{\max} &= n^2+0+\frac{1}{2}\sum_{ij}\varepsilon_{ij}^2+0+0+0 \\ &\qquad +\frac{1}{2}\sum_{ij}d_j\varepsilon_{ij}^2+o(\varepsilon^3) \end{aligned}$$

from where

$$\lambda_{\max} = n+\frac{1}{2n}\sum_{ij}\varepsilon_{ij}^2+\frac{1}{2n}\sum_{ij}d_j\varepsilon_{ij}^2+o(\varepsilon^3). \qquad (A.12)$$

We can now see that the values d_i are of order $o(\varepsilon)$. Returning to expression (A.8)

$$\begin{aligned} \lambda_{\max}(1+d_i) &= \sum_j e_{ij}(1+d_j) = \sum_j e^{\varepsilon_{ij}}(1+d_j) \\ &= \sum_j (1+\varepsilon_{ij}+o(\varepsilon))(1+d_j) \\ &= \sum_j \left(1+\varepsilon_{ij}+d_j+d_j\varepsilon_{ij}\right)+o(\varepsilon) \\ &= n+\sum_j \underbrace{d_j\varepsilon_{ij}}_{o(\varepsilon)}+o(\varepsilon) = n+o(\varepsilon) \end{aligned}$$

and, therefore, taking into account that on the basis of (A.12) $\lambda_{\max} = n+o(\varepsilon)$, the value of d_i can be approximated as

$$d_i = \frac{n+o(\varepsilon)}{n+o(\varepsilon)} - 1 = o(\varepsilon). \qquad (A.13)$$

Thus, the terms $d_j\varepsilon_{ij}^2$ of expression (A.12) are of order $o(\varepsilon^3)$ and we obtain

$$\lambda_{\max} = n+\frac{1}{2n}\sum_{ij}\varepsilon_{ij}^2+o(\varepsilon^3). \qquad (A.14)$$

The value of the consistency index will be

$$\mathrm{CI} = \frac{\lambda_{\max}-n}{n-1} = \frac{1}{2n(n-1)}\sum_{ij}\varepsilon_{ij}^2+o(\varepsilon^3) \qquad (A.15)$$

and, as the value of the geometric consistency index is given by

$$\frac{1}{(n-1)(n-2)}\sum_{ij}\varepsilon_{ij}^2,$$

we have

$$\mathrm{GCI} = \frac{2n}{n-2}\mathrm{CI}+o(\varepsilon^3). \qquad \square$$

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ORIGINAL PAPER

A heuristic rating estimation algorithm for the pairwise comparisons method

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Abstract The pairwise comparisons method is a powerful tool used for establishing the relative order between different concepts in situations in which it is difficult (or sometimes even impossible) to provide explicit rating. Appropriate ratings are determined by solving eigenvalue problem for the pairwise comparisons matrix. This study presents a new iterative heuristic rating estimation algorithm that tries to deal with the situation when exact estimations for some concepts (stimulus) C_K are a priori known and fixed, whilst the estimates for the others (unknown concepts C_U) need to be computed. The relationship between the local estimation error, understood as the average absolute error $E(c)$ over all direct estimates for the concept $c \in C_U$ and the pairwise comparisons matrix inconsistency index is shown. The problem of convergence of subsequent intermediate results is discussed and the convergence conditions are given.

Keywords Decision analysis · Pairwise comparisons · Iterative algorithms · Data inconsistency

1 Introduction

The *pairwise comparisons* (*PC*) *method* was introduced in its early form by Fechner (1966), then it was popularized and developed by Thurstone (1994). The introduction of hierarchical structures by Saaty (2008) was another important contribution to the *PC* method, providing the methodology and practical ways to deal with the large amounts of criterion parameters. Initially the *PC* method was used in the scientific study of psychometrics and psychophysic (Thurstone 1994), however, it then came to be used in other areas of applications, such as complex decision theory (Saaty 2008),

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economics (Peterson and Brown 1998), voting systems (Tideman 1987) and others. Since the data, which are input to the *PC* method are the result of human judgment, it is very easy for inaccuracy to occur. Hence, the input data set is frequently ambiguous and does not allow users to draw firm conclusions. There are several indexes of the data inconsistency (Bozóki and Rapcsak 2008), including the best known *Saaty's eigenvector method* (Saaty 1980), *Least Squares Method*, *Chi Squares Method*, *Koczkodaj's distance based inconsistency index* (Koczkodaj 1993), and others. Using these indexes, the reliability of the data can be assessed, hence, the confidence in the result can be evaluated. The answer to the question of how much input data must be consistent to ensure the result reliability is the subject of empirical research. For instance, according to *Saaty's* recommendation every occurrence of the consistency ratio greater than or equal to 0.1 should be the subject of re-examination of the pairwise judgments until the inconsistency becomes less than or equal to the desired value (Bozóki and Rapcsak 2008; Triantaphyllou et al. 1990). The main criticism of this approach relates to its separation from the data and lack of localizing the most problematic matrix elements (Bozóki and Rapcsak 2008; Koczkodaj 1993; Triantaphyllou et al. 1990). In contrast, *Koczkodaj's* inconsistency index has a meaningful interpretation and provides information about the inconsistency location, but it does not provide an exact answer to the question of how good the average data sample is. Inconsistency identified as too high, must be reduced to an acceptable level (ideally to zero). Since the ratio coefficients, which are the input to the *PC* method, frequently represent experts' judgements, thus a natural way of inconsistency reduction is to call the expert panel once again and ask the professionals gathered to agree on the opinion (Gomes 1993). Because such a solution is usually time-consuming and expensive, heuristic algorithms of inconsistency reduction have been proposed (Koczkodaj and Orłowski 1999; Koczkodaj and Szarek 2010; Gomes 1993; Xu and Wei 1999; Bozóki 2008; Temesi 2006; Cao et al. 2008). The result of these algorithms is a new set of data, which preserves most of the decision maker's original judgment structure and significantly reduces the data inconsistency.

The proposed innovative solution approaches the problem differently. It does not attempt to minimize inconsistency in the data, but rather proposes a way of using the data, which takes into account their inconsistency. Hence, knowing the exact values for a few concepts[1] and some inconsistent set of ratios between them, the data analyst is able to estimate values of all other concepts with errors depending on the degree of data inconsistency. The presented approach is comparable to the inconsistency reduction methods mentioned above, since the set of concepts for which the estimates are known can be easily transformed into a consistent set of data (as addressed in Sect. 8).

The first and second sections of the article focus on the presentation of the necessary facts and definitions concerning the pairwise comparisons methods. Section three formulates formally the problem considered in the work. The fourth and fifth, sections provide the *Heuristic Rating Estimation* (*HRE*) algorithm together with a numerical example demonstrating the algorithm application in practice. The sixth section shows the relationship between errors of estimations obtained by using the *HRE* algorithm and the data inconsistency. The next, seventh section addresses the case of which

[1] Thurstone (1994) originally called them stimulus.

the subsequent estimation sets converge to some limit. Finally, the last two sections contain the closing discussion and summary.

2 A pairwise comparisons method

A crucial part of the *PC* method is $M = (m_{ij}) \wedge m_{i,j} \in \mathbb{R}_{+} \wedge i, j \in \{1, \ldots, n\}$ a *PC* matrix that expresses some quantitative relation R over the finite set of concepts $C \stackrel{df}{=} \{c_i \in \mathscr{C} \wedge i \in \{1, \ldots, n\}\}$ where $\mathscr{C}$ is a non empty universe of concepts and $R(c_i, c_j) = m_{ij}$, $R(c_j, c_i) = m_{ji}$. Traditionally, these concepts are interpreted as subjective stimuli (Thurstone 1994), whilst the values m_{ij} and m_{ji} are considered as the relative importance indicators (stimulus intensities), so that according to the best knowledge of an expert the significance of c_i equals $m_{ij}c_j$.

Definition 1 A matrix M is said to be reciprocal if $\forall i, j \in \{1, \ldots, n\} : m_{ij} = \frac{1}{m_{ji}}$, and M is said to be consistent if $\forall i, j, k \in \{1, \ldots, n\} : m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$.

Since the knowledge stored in the *PC* matrix usually comes from different professionals in the field of relation R, it often results in inaccuracy. In such a case, reasoning using the data gathered in M may give ambiguous results. This observation gave rise to the research on the concept of data consistency. In the ideal case, M is consistent, and there is no doubt as regards the value assigned to the concept c_j if the value assigned to c_i and m_{ji} is known. Unfortunately, in practice the knowledge in M is inconsistent, and professionals using the *PC* method have to deal with this incoherence. Thus, it is important to answer the question of how inconsistent the knowledge in the *PC* matrix is. There are a number of inconsistency indexes, including *Eigenvecor Method* (Saaty 2008), *Least Squares Method*, *Chi Squares Method* (Bozóki and Rapcsak 2008), *Koczkodaj's distance based inconsistency index* (Koczkodaj 1993) and others. For the purpose of this article, the Koczkodaj's distance based inconsistency index has been adopted since it is the only localizing index amongst the above mentioned indexes.

Definition 2 Koczkodaj's distance based inconsistency index $\mathscr{K}$ of $n \times n$ and $(n > 2)$ reciprocal matrix M is equal to

$$\mathscr{K}(M) = max\left\{min\left\{\left|1 - \frac{m_{ij}}{m_{ik}m_{kj}}\right|, \left|1 - \frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\}\right\} \tag{1}$$

where $i, j, k = 1, \ldots, n$ and $i \neq j \wedge j \neq k \wedge i \neq k$.

Intuitively speaking, since in an "ideal" matrix $\forall i, j, k \in \{1, \ldots, n\} : m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$ the *Koczkodaj's* index localizes the worse triad (by the Euclidean distance) from this ideal.

PC matrices may be over-complete, i.e. there is more than one estimation describing one and the same relation between v_i and v_j, but they can be also incomplete, i.e. not all values m_{ji} are defined. While the first situation can be addressed in many ways, for instance the estimates related to the same pair of concepts can be averaged, the other is not desirable. It indicates that the model is lacking of knowledge, although the missing estimates can be somehow compensated (for example, by employing the properties of reciprocity, consistency or transitivity) (Koczkodaj et al. 1999). Hence, for further consideration, it will be assumed that the $n \times n$ *PC matrix* M is complete in the sense specified below.

Definition 3 A matrix M $(n \times n)$ is said to be complete if $\forall i, j = 1, \ldots, n : m_{ij}$ is defined.

Due to the further consideration it is useful to define a graph structure over the set C and matrix M.

Definition 4 Let a pairwise comparisons graph $G = (C, E, M)$ be a weighted directed graph over the matrix M, where

- C, is a set of vertices
- $E \subseteq C \times C$ so that $(v_i, v_j) = e \in E \Leftrightarrow m_{ji}$ exists, is a set of edges
- $M : E \rightarrow \mathbb{R}$ so that $M(e) \overset{df}{=} m_{ji} \wedge e = (v_i, v_j)$ is a function of experts' assessments

Wherever it does not raise doubts, instead of $M(e) \wedge e = (u, v)$ the function M will be written with two arguments i.e. $M(u, v)$.

3 Problem formulation

Since the concepts are linked to each other by a quantitative relation R, then assuming that the exact values of some concepts are known, the values of the others should be proposed. Thus, let $\mu : C \nrightarrow \mathbb{R}_+$ be a partial function that assigns to some concepts from $C \subset \mathscr{C}$ positive values from $\mathbb{R}_+$. Hence, the concepts for which the actual value μ is known are denoted by $C_K \subset C$ and called known concepts, whilst concepts for which μ need to be determined are denoted $C_U = C \backslash C_K$ and called unknown concepts. The relation between different concepts in terms of the function μ is represented in the form of the *PC* matrix M, so that $m_{ji}\mu(v_i) = \mu(v_j)$. Since m_{ji} usually aims to express how much greater or smaller v_i is than v_j with respect to μ by convention it is assumed that elements of the $n \times n$ *PC matrix* are positive real numbers, i.e. $m_{ji} \in \mathbb{R}_+$ where $i, j \in 1, \ldots, n$.

The presented method aims to provide an iterative heuristic estimation algorithm that for all $v \in C_U$ proposes the appropriate value of $\mu(v)$. In this approach $m_{ji}\mu(v_i)$ is treated as a sample of $\mu(v_j)$, hence the expected value of $\mu(v_j)$ is the arithmetic mean of values $m_{ji}\mu(v_i)$. The algorithm is iterative and sets the new expected values based on the ones previously determined. It stops either when it reaches a fixed number of iterations or (if convergent) when calculations reach the desired accuracy. Although during the course of the algorithm the new values for $\mu(v)$ and $v \in C_U$ are calculated, M remains unchanged and serves as reference data.

4 Heuristic rating estimation algorithm

The principle of operation of the rating estimation algorithm (Listing 1) is to iteratively assign the value $\mu(u)$ to every unknown vertex $u \in C_U$ by calculating the mean of

```
1  HRE($G$, $\epsilon$, $level$)
2    $L \leftarrow \{v : (u, v) \in E \wedge u \in C_K\} \cap C_U$
3    $l \leftarrow 0$
4    $tmp \leftarrow \emptyset$
5    $Est \leftarrow \emptyset$
6    while $stop(\mu, \mu_{old}, \epsilon, l, level)$ do
7      $\mu_{old} \leftarrow \mu$
8      while $L \neq \emptyset$ do
9        $u \leftarrow poll(L)$
10       $T \leftarrow \{e = (v, u) \in E \mid \mu(v)$ is defined$\}$
11       $tmp(u) \leftarrow \frac{1}{\#T} \sum_{(v,u) \in T} \mu(v) \cdot M(v, u)$
12     end while
13     foreach $u \in Dom(tmp)$
14       $\mu(u) \leftarrow tmp(u)$
15       foreach $v \in \{v : (u, v) \in E\} \cap C_U$
16         $L \leftarrow L \cup \{v\}$
17     $Est \leftarrow Est \cup \{\{\mu(u) : u \in C_U\}\}$
18     $l \leftarrow l + 1$
19     $tmp \leftarrow \emptyset$
20   end while
21   return chose_optimal_est($Est$)
```

Listing 1: Heuristic ratio estimation algorithm

its samples, and then choosing from among all the calculated estimations the one which is optimal. The idea of the procedure comes from the BFS algorithm (Cormen et al. 2009), which, layer by layer, traverses the graph of interest. In the presented approach, the estimates for the next layer are computed on the base of the previous layer, assuming that vertex repetition in different steps is allowed. The algorithm stops when either the appropriate layer is reached or if the algorithm converges for the given G, then it stops when the distance in the sense of the chosen metrics (see Eq. 47) between the subsequent estimates for elements in C_U is smaller than some desired ϵ. If the algorithm does not converge (when G is fixed) then the number of steps that need to be made is explicitly set at the beginning of the estimation procedure. In the case of standard complete *PC matrices* the graph is a directed clique, where every single vertex is connected to each other. In such a case it appears that even in the first step all the vertices are visited, whilst in the second step the computed estimations may take into account all possible values[2] gathered in M except those describing ratios between elements in C_K. Thus, when the algorithm is not convergent, a few iterations of the *HRE* procedure seem to be a reasonable choice.

The main principle of the algorithm (Listing 1) seems to be quite straightforward. It starts from assigning followers (in sense E) of all elements of C_K to the set L (Listing: 1, line: 2). For each u, for which $\mu(u)$ is unknown, all its predecessors in E are scanned. If $\mu(v)$ is already known then v becomes an element taken into account during the mean computation (Listing: 1, line: 10). Then, within the two loops *while* the current

[2] Following the reciprocity principle all the M values in the form $M(c_i, c_j)$ where $c_i \in C_U \wedge c_j \in C_K$ can be used to create expressions $\frac{\mu(c_j)}{M(c_i,c_j)}$ are treated as just one specimen of $\mu(c_i)$.

level is traversed (the outer loop Listing: 1, lines: 6–20), and appropriate estimates are computed (the inner loop Listing: 1, lines: 8–12). The predicate *stop* becomes false when either the auxiliary variable l reaches the assumed number of levels or it holds that $l > 1$ and $\rho(x_l, x_{l+1}) < \epsilon$, where $x_l = (\mu_{old}(c_{u_1}), \ldots, \mu_{old}(c_{u_k}))$ and $x_{l+1} = (\mu(c_{u_1}), \ldots, \mu(c_{u_k}))$ where the concepts $c_{u_i} \in C_U$ and ρ is one of the metrics defined in (Eq. 47). Otherwise it is true, and the outer loop continues traversing G. As long as L is not empty the inner loop proceeds as follows: removes one element from the set L and assigns it to the variable u (Listing: 1, line: 9), forms the auxiliary set T containing input edges for u so that their beginnings are predecessors of u in E for which μ is already known, and for all $u \in L$ adds to the auxiliary mapping $tmp : C_U \to \mathbb{R}_+$ a pair $(u, \mu(v)M(v, u))$ (Listing: 1, line: 11). When all the elements of L are processed then the inner loop ends. Next the outer loop rewrites the auxiliary mapping tmp to the result map μ (Listing: 1, lines: 13–14). Since for all the concepts that were previously in L the function μ is known, the next level is devoted to μ calculation for their followers in L, hence the set L is filled back (Listing: 1, lines: 15–16). At the end of the outer loop the step counter is incremented and the auxiliary mapping tmp is emptied. When the outer loop completes its operation, the set Est contains a sequence of subsequent sets of estimates for concepts from C_U. Then, at the end of the procedure (Listing: 1, line: 21) an optimal set of estimates needs to be chosen. For the purposes of this work it was assumed that the optimal set of estimates is one for which the average of the absolute mean estimation errors is minimal (Eq. 4). Let us define the absolute mean error $e(u)$ as an error indicator for some concept u and mapping μ as follows:

$$e_\mu(u) = \frac{1}{n}\sum_{i=0}^{n} |\mu(u) - \mu(v_i) \cdot M(v_i, u)| \tag{2}$$

where $u \in C_U$, $(v_i, u) \in E$ and $\mu(v_i)$ is already defined. Then the average error for all unknown concepts with respect to μ is defined as:

$$e_\mu(C_U) = \frac{1}{\#C_U}\sum_{c \in C_U} e_\mu(c) \tag{3}$$

hence

$$\text{chose\_optimal\_est}(Est) = \min_{\mu \in Est} \left\{ e_\mu(C_U) \right\} \tag{4}$$

In every iteration of the algorithm every weight $M(v, u)$ (Listing: 1, line: 11) is considered exactly once. Hence, the running time of the outer loop is $O(r \cdot |E|) = O(r \cdot n^2)$ where r is the number of iterations, and n is the size of C. Similarly, the calculation of the average error (Eq. 3) requires the consideration of every $M(v, u)$ exactly once, hence computing formulae (Eq. 4) also requires at most $r \cdot n^2$ steps. Thus, the overall running time of the *HRE* algorithm is also $O(r \cdot n^2)$.

5 Numerical example

Let us illustrate the algorithm defined above by a simple numerical example. Some state agency supports five innovative projects $u_1, \ldots, u_5$. After a while the two of them u_4

and u_5 come to an end and their actual cost becomes known $\mu(u_4) = 6,\ \mu(u_5) = 2$. As both projects exceeded the initial budget, the agency wants to reestimate the expected costs of the other projects using the already acquired knowledge. For this purpose the agency organizes a panel of experts under which a *PC* matrix M is formed (Eq. 5) reflecting the predicted cost relations between all the five projects.

$$\mathsf{M} = \begin{bmatrix} 1 & \frac{3}{5} & \frac{3}{4} & \frac{1}{2} & \frac{4}{3} \\ \frac{5}{4} & 1 & \frac{5}{4} & \frac{11}{12} & \frac{5}{2} \\ \frac{6}{5} & \frac{4}{5} & 1 & \frac{1}{2} & 2 \\ \frac{3}{2} & \frac{6}{5} & \frac{6}{5} & 1 & 3 \\ \frac{2}{3} & \frac{3}{5} & \frac{4}{7} & \frac{1}{3} & 1 \end{bmatrix} \tag{5}$$

Since experts were assigned to the pairs (u_i, u_j) randomly and did not know each other's estimates, the matrix M is neither reciprocal nor consistent. Based on this data the agency must calculate adequate estimates of the projects u_1, u_2 and u_3 and take appropriate decisions regarding the future of these projects. The reasoning presented below shows how to calculate the estimated cost of the projects using a *HRE* procedure (Listing: 1).

The matrix M is used for the generation graph G, the set of known concepts C_K is formed by the projects u_4 and u_5, and finally the unknown concept set is $C_U = \{u_1, u_2, u_3\}$. Since all concepts are reachable in the first step, and during the second step the μ mapping is computed for every $c \in C_U$ using all other concepts in C, then for the practical demonstration of the algorithm the number of levels traversed is limited to 2. In fact, for the matrix M as given (Eq. 5) the subsequent estimations calculated by *HRE* converge. Thus, instead of limiting the number of steps, an appropriately small ϵ can be chosen. This case will be discussed later, after the convergence criterion for *HRE* is defined.

Let us assume that the first vertex considered during the first step of the algorithm is u_1. Then, according to the presented procedure $\mu(u_1)$ is computed as follows:

$$\mu(u_1) = \frac{1}{2} \cdot (\mu(u_4) \cdot \mathsf{M}(\mathsf{u}_4, \mathsf{u}_1) + \mu(u_5) \cdot \mathsf{M}(\mathsf{u}_5, \mathsf{u}_1)) \tag{6}$$

hence,

$$\mu(u_1) = \frac{1}{2} \cdot \left(6 \cdot \frac{1}{2} \cdot +2 \cdot \frac{4}{3}\right) = \frac{17}{6} \approx 2.83 \tag{7}$$

and further, in the same way:

$$\mu(u_2) = \frac{1}{2} \cdot \left(6 \cdot \frac{11}{12} + 2 \cdot \frac{5}{2}\right) = \frac{21}{4} = 5.25 \tag{8}$$

$$\mu(u_3) = \frac{1}{2} \cdot \left(6 \cdot \frac{1}{2} + 2 \cdot 2\right) = \frac{7}{2} = 3.5 \tag{9}$$

The first turn of the outer loop (traversing the first level of G) brings estimations for all vertices in C_U. Thus, although the estimation process can be terminated, the agency concludes that it is better to take into account more data (for instance, due to the excluding of accidental errors of some experts), then decides to perform the second iteration of the algorithm. The new subsequent values $\mu(u_1)$, $\mu(u_2)$ and $\mu(u_3)$ are calculated below.

$$\mu(u_1) = \frac{1}{4} \cdot \left(\frac{21}{4} \cdot \frac{3}{5} + \frac{7}{2} \cdot \frac{3}{4} + 6 \cdot \frac{1}{2} \cdot \frac{1}{4} + 2 \cdot \frac{4}{3} \right) = \frac{1373}{480} \approx 2.86 \tag{10}$$

$$\mu(u_2) = \frac{1}{4} \cdot \left(\frac{17}{6} \cdot \frac{5}{4} + \frac{7}{2} \cdot \frac{5}{4} + 6 \cdot \frac{11}{12} + 2 \cdot \frac{5}{2} \right) = \frac{221}{48} \approx 4.6 \tag{11}$$

$$\mu(u_3) = \frac{1}{4} \cdot \left(\frac{17}{6} \cdot \frac{6}{5} + \frac{21}{4} \cdot \frac{4}{5} + 6 \cdot \frac{1}{2} + 2 \cdot 2 \right) = \frac{73}{20} \approx 3.65 \tag{12}$$

Recognizing the achieved result as optimal, the agency finishes the algorithm. It is worth noting that in the matrix M (Eq. 5) the two estimates do not come from the experts and have been introduced just for the matrix completeness. Namely, since the values $\mu(v_4)$ and $\mu(v_5)$ were previously known, the values $m_{4,5}$ and $m_{5,4}$ were calculated as ratios ${}^{\mu(v_4)}/_{\mu(v_5)}$ and ${}^{\mu(v_5)}/_{\mu(v_4)}$ respectively.

6 Data consistency and estimation error

For any algorithm, the result of which being inaccurate, a natural questions concern the accuracy of the resulting solution, and what should be done to improve the accuracy. The presented procedure is based on the fundamental sample mean estimation equation (Walpole 2012), where for the purpose of the algorithm the product $\mu(u_i) \cdot M(u_i, u_j)$ is treated as a sample, whilst $\mu(u_j)$ denotes the expected value inferred from samples. In this case the natural measure of the algorithm output accuracy is an estimation error understood as the distance between sample and the mean. Of course, the smaller the error, the more accurate the result. According to the popular adage "garbage in, garbage out" it is expected that even the best algorithm is not able to provide good output if the input data are bad. Hence, it might be expected that the estimation errors of *HRE* depend on data consistency, and are smaller in correlation with data inconsistency. The reasoning below supports this assertion.

Theorem 1 *For a complete PC matrix M, and PC graph G over M it holds that*

$$\mathscr{K}(M) \to 0 \Rightarrow e_{\mu}(u) \to 0 \tag{13}$$

where $\mathscr{K}(M)$ is Koczkodaj's distance based inconsistency index for M, $e(u)$ is the mean absolute estimation error for $u \in C_U$, and μ is any estimation provided by the HRE procedure.

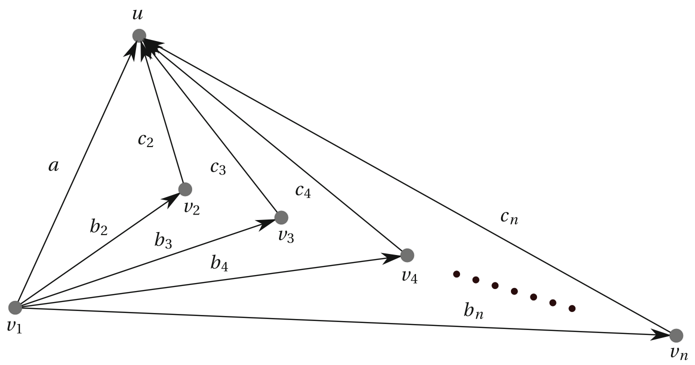

Fig. 1 Triads of vertices with u

Proof Let us consider some element $u \in C$ representing an initially unknown concept i.e. $u \in C_U$. Let $v_1, \ldots, v_n$ be its predecessors in G so that $(v_i, u) \in E \wedge i = 1, \ldots, n$, for which the value $\mu(v_i)$ is known. Since the matrix M is complete there must exist edges $(v_1, v_2), \ldots, (v_1, v_n) \in E$ so that $b_2 = M(v_1, v_2), b_3 = M(v_1, v_3), \ldots, b_n = M(v_1, v_n)$. Moreover, let us denote edges between v_i's and u as $a = M(v_1, u)$, and $c_2 = M(v_2, u), \ldots, c_n = M(v_n, u)$ (see Fig. 1). Following the Eq. 1, Koczkodaj's distance inconsistency index $\mathscr{K}(M)$, in short $\mathscr{K}$, means that the maximal local inconsistence for some maximal triad of three vertices $t_1, t_2, t_3 \in C$ is $\mathscr{K}$. Thus, in the case of triads composed of the concepts v_1, v_i, u it must hold that:

$$\mathscr{K} \geq min\left\{\left|1 - \frac{a}{b_i c_i}\right|, \left|1 - \frac{b_i c_i}{a}\right|\right\} \tag{14}$$

for all $i = 2, \ldots, n$. This implies that one of the following two statements is true:

$$a \leq b_i c_i \wedge \mathscr{K} \geq 1 - \frac{a}{b_i c_i} \tag{15}$$

$$b_i c_i \leq a \wedge \mathscr{K} \geq 1 - \frac{b_i c_i}{a} \tag{16}$$

Let us denote $\alpha \stackrel{df}{=} 1 - \mathscr{K}$ then the statements above can be written in the form:

$$a \leq b_i c_i \wedge \frac{1}{\alpha} \cdot a \geq b_i c_i \tag{17}$$

$$b_i c_i \leq a \wedge b_i c_i \geq \alpha a \tag{18}$$

Combining these two expressions (Eqs. 17 and 18) we obtain:

$$a \leq b_i c_i \leq \frac{1}{\alpha} \cdot a \vee \alpha a \leq b_i c_i \leq a \tag{19}$$

Let us denote $\beta_1 \overset{df}{=} a$ and $\beta_i \overset{df}{=} b_i \cdot c_i$, thus

$$a \leq \beta_i \leq \frac{1}{\alpha} \cdot a \vee \alpha a \leq \beta_i \leq a \tag{20}$$

Since $\alpha \leq 1$ (see Eqs. 15 and 16) thus the statement (Eq. 20) implies:

$$\alpha a \leq \beta_i \leq \frac{1}{\alpha} \cdot a \tag{21}$$

and of course

$$\alpha a \mu(v_1) \leq \beta_i \mu(v_1) \leq \frac{1}{\alpha} \cdot a \mu(v_1) \tag{22}$$

During the first step of the algorithm the μ is defined only for known concepts $v \in C_K$. Thus, all the concepts $v_1, \ldots, v_n$ are in C_K since only such elements are taken into account when calculating the value of $\mu(u)$. It is assumed that the value of the ratio $M(v_i, v_j) = m_{ji}$ for two a priori known concepts $v_i, v_j \in C_K$, corresponds to the actual fraction ${}^{\mu(v_j)}/{}_{\mu(v_i)}$, thus it holds that $\mu(v_1) \cdot b_i = \mu(v_i)$. Therefore the update equation for $\mu(u)$ (Listing: 1, line: 11) can be written in the form:

$$\mu(u) = \frac{1}{n} (\beta_1 + \cdots + \beta_2) \mu(v_1) \tag{23}$$

Since every component β_i is bounded (Eq. 21) their mean must also be within the same bounds, which leads to the conclusion that:

$$\alpha \cdot a \cdot \mu(v_1) \leq \mu(u) \leq \frac{1}{\alpha} \cdot a \cdot \mu(v_1) \tag{24}$$

The absolute estimation error for some u with respect to μ at the end of the first step of the algorithm (Listing: 1) can be written and bounded from above as follows:

$$e_1(u) = \frac{1}{n} \sum_{i=0}^{n} |\mu(u) - \beta_i \mu(v_1)| \leq \max_{j=1,\ldots,n} \left|\mu(u) - \beta_j \cdot \mu(v_1)\right| = |\mu(u) - \beta_k \cdot \mu(v_1)| \tag{25}$$

where $k \in \{1, \ldots, n\}$. Because both components of the absolute difference on the right side of the expression 25 have the same lower and upper bounds (Eqs. 22 and 24), then the maximal possible distance between them is limited by the difference between their upper and lower bounds. Thus,

$$|\mu(u) - \beta_k \cdot \mu(v_1)| \leq \frac{1}{\alpha} a \mu(v_1) - \alpha a \mu(v_1) = a \mu(v_1) \left(\frac{1}{\alpha} - \alpha \right) \tag{26}$$

then the absolute mean error for the purpose of traversing the first level of G is upper bounded by:

$$e_1(u) \leq a\mu(v_1)\left(\frac{1}{\alpha} - \alpha\right) \tag{27}$$

The second level is a bit more complicated. Under the conditions of the algorithm there is $u \in C_U$, and M is complete. Thus, there exists at least one known concept $v \in C_K$ which precedes u in E i.e. $(v, u) \in E$. Let us put $v_1 = v$. This means that during the first step of the algorithm either v_i (for $i = 2, \ldots, n$) was in C_K thus obviously $\mu(v_1) \cdot b_i = \mu(v_i)$ or $\mu(v_i)$ can be bounded using Eq. 24 (note that in order to use Eq. 24 a needs to be replaced by b_i). This leads to the following inequality:

$$\alpha \cdot b_i \cdot \mu(v_1) \leq \mu(v_i) \leq \frac{1}{\alpha} \cdot b_i \cdot \mu(v_1) \tag{28}$$

thus,

$$\alpha \cdot \beta_i \cdot \mu(v_1) \leq c_i \cdot \mu(v_i) \leq \frac{1}{\alpha} \cdot \beta_i \cdot \mu(v_1) \tag{29}$$

Since the Eq. 21 is valid for each triad, hence

$$\alpha^2 \cdot a \cdot \mu(v_1) \leq c_i \cdot \mu(v_i) \leq \frac{1}{\alpha^2} \cdot a \cdot \mu(v_1) \tag{30}$$

and then,

$$\alpha^2 \cdot a \cdot \mu(v_1) \leq \frac{(c_1 \cdot \mu(v_1) + \cdots + c_n \cdot \mu(v_n))}{n} \leq \frac{1}{\alpha^2} \cdot a \cdot \mu(v_1) \tag{31}$$

which provides the estimation for $\mu(u)$ for the purpose of the second step of the algorithm:

$$\alpha^2 \cdot a \cdot \mu(v_1) \leq \mu(u) \leq \frac{1}{\alpha^2} \cdot a \cdot \mu(v_1) \tag{32}$$

Once again, the absolute error with respect to μ at the end of the second step can be written and upper bounded as follows:

$$e_2(u) = \frac{1}{n}\sum_{i=0}^{n} |\mu(u) - \mu(v_i) \cdot c_i| \leq \max_{i=1,\ldots,n} |\mu(u) - \mu(v_i) \cdot c_i| = |\mu(u) - \mu(v_k) \cdot c_k| \tag{33}$$

where $k \in \{1, \ldots, n\}$. Then, similarly as in the first step, both components have the same upper and lower bounds (Eqs. 30 and 32), thus the distance between them must not be greater than the distance between these limits. So,

$$e_2(u) \leq |\mu(u) - \mu(v_k) \cdot c_k| \leq \frac{1}{\alpha^2} a\mu(v_1) - \alpha^2 a\mu(v_1) = a\mu(v_1)\left(\frac{1}{\alpha^2} - \alpha^2\right) \tag{34}$$

which means that the absolute mean error in the second step of the algorithm is bounded from above as follows:

$$e_2(u) \leq a\mu(v_1) \cdot \left(\frac{1}{\alpha^2} - \alpha^2 \right) \tag{35}$$

Let us consider the r'th step of the algorithm. Similarly as in the second step we consider $u \in C_U \wedge v_1 \in C_K$ where $(v_1, u) \in E$ (See Fig. 1). Let us assume by induction that every v_i for $i = 2, \ldots, n$ is bounded as follows:

$$\alpha^{r-1} \cdot b_i \cdot \mu(v_1) \leq \mu(v_i) \leq \frac{1}{\alpha^{r-1}} \cdot b_i \cdot \mu(v_1) \tag{36}$$

(compare with Eq. 28). Then by repeating the same reasoning as for the second step (Eqs. 29–35) we come to the conclusion that:

$$e_r(u) \leq a\mu(v_1) \cdot \left(\frac{1}{\alpha^r} - \alpha^r \right) \tag{37}$$

Due to the principle of induction the above inequality holds for eve $r \in \mathbb{N}_+$.

Since every $e_r(u)$ is a sum of absolute values then it cannot be negative i.e.

$$0 \leq e_r(u) \tag{38}$$

Moreover, the definition of α implies

$$\mathscr{K} \rightarrow 0 \Rightarrow \alpha \rightarrow 1 \tag{39}$$

Thus, due to the $({}^1\!/\!_{\alpha^r} - \alpha^r)$ component on the right side of the inequality 37, when $\alpha \rightarrow 1$ then the whole right side of Eq. 37 also approaches to 0. Therefore, it holds that for every step $r > 0$ it is true that:

$$\mathscr{K} \rightarrow 0 \Rightarrow e_r(u) \rightarrow 0 \tag{40}$$

The above assertion, in the light of the arbitrary choice of $u \in C_U$, satisfies the thesis of the theorem. $\square$

7 Convergence of solution

One of the immediate questions to come up is about the optimal number of iterations. The answer is not obvious, since the result of the algorithm depends on the input data. In particular, it is easy to construct simple graphs over the inconsistent PC matrix M in cases where every further step of the algorithm significantly increases the absolute error of estimation. For instance, the graph G such that $G = (C_K \cup C_U, E, M)$ where $C_K = v_1, C_U = v_2, v_3$ and $\{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_2)\} \subseteq E$ where the values $M(v_1, v_2),\ M(v_1, v_3),\ M(v_2, v_3),\ M(v_3, v_2)$ or all are less than or greater than 1. In the first case the subsequent estimates $\mu(v_2)$ and $\mu(v_3)$ tend to 0, and reversely, if all the values are greater than 1, then $\mu(v_2)$ and $\mu(v_3)$ tends to ∞. In the general case, the inequality 37 (Th. 1) suggests that with every subsequent step errors may increase. This leads to the conclusion that the optimal strategy is "the fewer steps the better". Therefore, if the algorithm is not convergent, traversing at most one or two levels seems like a good idea.

If there are some edges with weights below 1 and some edges with weights above 1, the behavior of the algorithm is not so obvious. During the conducted experiments, it turned out that very often the subsequent iterations produce series of estimations convergent to some fixed positive results. To explain this phenomenon let us write the algorithm in the form of an appropriate system of equations, describing the on-step calculations (for the second step and the following ones). For this purpose, let us assume that the first iteration of the procedure was performed, hence the values $\mu(v)$ are assigned to all the concepts $v \in C$. For simplicity, let us assume that $C_U = \{c_1, \ldots, c_k\}$, $C_K = \{c_{k+1}, \ldots, c_n\}$ and denote b_i for all $i = 1, \ldots, k$ as

$$b_i = \frac{1}{n-1} M(c_{k+1}, c_i)\mu(c_{k+1}) + \cdots + \frac{1}{n-1} M(c_n, c_i)\mu(c_n) \tag{41}$$

Thus, during the second and subsequent iterations the algorithm calculates the new estimation value $\mu(c_i)$ for each unknown concepts $c_i \in C_U$ according to one of the following equations:

$$\begin{aligned}
\mu(c_1) &= \frac{1}{n-1} M(c_2, c_1)\mu(c_2) + \cdots + \frac{1}{n-1} M(c_k, c_1)\mu(c_k) + b_1 \\
\mu(c_2) &= \frac{1}{n-1} M(c_1, c_2)\mu(c_2) + \frac{1}{n-1} M(c_3, c_2)\mu(c_3) + \cdots + \frac{1}{n-1} M(c_k, c_2)\mu(c_k) + b_2 \\
&\cdots \\
\mu(c_k) &= \frac{1}{n-1} M(c_1, c_k)\mu(c_1) + \cdots + \frac{1}{n-1} M(c_{k-1}, c_k)\mu(c_{k-1}) + b_k
\end{aligned} \tag{42}$$

Let us denote:

$$a_{ij} = \frac{1}{n-1} M(c_j, c_i) \wedge i \neq j \quad \text{and} \quad a_{ii} = M(c_i, c_i) = 1 \quad \text{and} \quad \mu(c_i) = x_i \tag{43}$$

Then, the equation system takes the form:

$$\begin{array}{llllll}
x_1 & -(1-a_{11})x_1 & -a_{12}x_2 & -\ldots & -a_{1k}x_k & = b_1 \\
x_2 & -a_{21}x_1 & -(1-a_{22})x_2 & -\ldots & -a_{2k}x_k & = b_2 \\
\ldots\ldots & & \ldots & \ldots & \ldots & \ldots \\
x_k & -a_{k1}x_1 & -a_{k2}x_2 & -\ldots & -(1-a_{kk})x_k & = b_k
\end{array} \tag{44}$$

Let us define the operator $T : \mathbb{R}^k \to \mathbb{R}^k$ (see Bronstein et al. 2005) as follows:

$$Tx = \left(x_1 - \sum_{r=1}^{k} a_{1r}x_r + b_1, \ldots, x_k - \sum_{r=1}^{k} a_{kr}x_r + b_k \right)^T \tag{45}$$

The considered equation system can be written in the form of a fixed point problem in the metric space $\mathbb{R}^k$:

$$x = Tx \tag{46}$$

Let us assume one of the following metrics:

$$\rho(x, y) = \sqrt{\sum_{i=1}^{k} |\xi_i - \eta_i|^2}, \quad \rho(x, y) = \max_{1 \leq i \leq n} |\xi_i - \eta_i|, \quad \rho(x, y) = \sum_{i=1}^{k} |\xi_i - \eta_i| \tag{47}$$

and $x = (\xi_1, \ldots, \xi_k)$, $y = (\eta_1, \ldots, \eta_k)$. It holds that if one of the following conditions Q_1, Q_2 or Q_3 is less than 1,

$$Q_1 = \sqrt{\sum_{i,j=1}^{k} |a_{ij}|^2}, \quad Q_2 = \max_{1 \leq i \leq n} \sum_{j=1}^{k} |a_{ij}|, \quad Q_3 = \max_{1 \leq j \leq n} \sum_{i=1}^{k} |a_{ij}| \tag{48}$$

then T turns out to be a contracting operator (Bronstein et al. 2005). According to the Banach fixed point theorem, there exists only one such point. Hence, in such a case there is only one set of values $\mu(c_1), \ldots, \mu(c_k)$ which is the limit of the sequence of *HRE* estimations.

Due to the fraction $1/n-1$ the smaller a_{ij} is the larger C is and the smaller $M(c_j, c_i)$ is. Moreover, the conditions (Eq. 48) can be more easily met with fewer concepts in C_U (that is because the summations embedded in the conditions (Eq. 48) include fewer elements). In other words, the estimation procedure has a high chance to be convergent if

1. The set C_U is relatively small (C_K is relatively large),
2. The estimated values $\mu(v)$ are similar.

Both of these conditions are quite intuitive and, in practice, are likely to be satisfied. The first of them reflects the natural desire to provide the experts with rather more than the lower number of known, reference concepts. The second corresponds to the common-sense observation that all the considered concepts should be similar to each other, because then, it is easy to compare them. In other words, the expert estimates are more reliable when the compared projects are more similar.

The convergence of the algorithm implies the convergence of the estimation error. Unfortunately, the limit towards which the estimation error tends is not necessarily the smallest possible error value. Hence, in order to minimize the error, the user needs to choose the best estimation from all the estimations generated during the course of the algorithm (Listing: 1, line: 21).

In the case of the previously considered example (Sect. 5) all the three conditions Q_1, Q_2 and Q_3 are below 1. Thus, the algorithm is convergent and the computed limits for $\mu(v_1)$, $\mu(v_2)$ and $\mu(v_3)$ are 2.758, 4.578 and 3.493, respectively. The absolute average estimation errors converge to $e_\mu(v_1) = 0.12$, $e_\mu(v_2) = 0.631$ and $e_\mu(v_3) = 0.338$, and are minimal with respect to the mean of errors $e_\mu(v_1)+e_\mu(v_2)+e_\mu(v_3)/3$.

8 Discussion

The idea underlying the *HRE* algorithm is the assumption that experts work independently and try to do their job as best as they can. Thus, they may be wrong in a random manner, and if so, it makes sense to accept their estimates as a part of samples, and the expected value of the sample (the arithmetic mean) as the alleged value of the estimated μ for the unknown concept. The absolute average estimation error may indicate how good such an estimation is. Depending on the data, the algorithm may or may not converge. If the algorithm does not converge (subsequent estimates are getting bigger) or converges to 0, selecting from among the first few estimates the one with the smallest error seems to be the best choice. If the algorithm converges to a non 0 value, it is very often useful to generate subsequent estimates until the current estimate does not differ from the limit (fixed point) with some ϵ, then to choose a set of estimations (among those generated) for which the average estimation error is minimal. Of course, this is not the only possible strategy. For instance, if some subset of C_U is particularly important, only errors for its elements may be taken into account while determining the optimal set of estimations.

However, the problem considered in the work relies on determining values of μ for elements from $C_U = \{c_1, \ldots, c_k\}$ on the basis of concepts $C_K = \{c_{k+1}, \ldots, c_n\}$ using the matrix M, it can be reformulated as computing a consistent approximation of a matrix M where certain elements are fixed. Indeed, at the end of the presented algorithm all the concepts $c \in C$ have assigned some values $\mu(c)$. Thus, defining $m'_{ij} = \frac{\mu(c_i)}{\mu(c_j)}$ allows the construction of a new matrix $M' = [m'_{ij}]$, which is a consistent approximation of M where $m_{ij} = m'_{ij}$ for $c_i, c_j \in C_K$. In the case of the previously considered example, assuming the values of $\mu(v_1)$, $\mu(v_2)$ and $\mu(v_3)$ corresponding to the smallest average absolute mean error $e_\mu(C_U)$, the matrix M' equals:

$$\mathsf{M}' = \begin{bmatrix} 1 & 0.602 & 0.789 & 0.46 & 1.379 \\ 1.66 & 1 & 1.311 & 0.763 & 2.289 \\ 1.266 & 0.763 & 1 & 0.582 & 1.746 \\ 2.175 & 1.311 & 1.718 & 1 & 3 \\ 0.725 & 0.437 & 0.572 & 0.333 & 1 \end{bmatrix} \tag{49}$$

In particular it holds that $m_{45} = m'_{45} = 3$ and $m_{54} = m'_{54} = \frac{1}{3}$.

Although the presented considerations in (Sect. 6 and 7) assume that the matrix M is complete (see Def. 3), the *HRE* algorithm seems to work without this assumption. For incomplete matrices, the value $\mu(u)$ for $u \in C_U$ can be determined as long as a path exists in the pairwise comparisons graph G over M (Def. 4) between u and some element $v \in C_K$. Hence, there is a chance that the *HRE* algorithm may support different pairwise comparisons techniques also when the input data are incomplete. In particular, it might be useful for the *AHP* (Saaty 1977) approach. The properties of the presented algorithm for the incomplete sets of input data will be the subject of future research.

9 Summary

The *HRE* algorithm presented here for computing estimations of initially unknown concepts C_U using information about known concepts C_K and the matrix M proposes a new approach to the pairwise comparisons method. It defines the intuitive algorithm of using the pairwise comparisons matrix M for determining the most probable values of unknown concepts (original stimulus) on the basis of the known concepts. The presented procedure iteratively generates sets of estimations, then chooses the set which has the smallest average absolute mean estimation error. According to the proven theorem, the size of the estimation error depends on $\mathscr{K}$*—Koczkodaj's distance based inconsistency index* shows that the lower the inconsistency index, the lower the estimation errors. The theorem can be particularly useful when the number of iterations of the *HRE* algorithm is small. In such cases, it may in practice be used to estimate the size of the estimation errors.

For some input data sets, the subsequent estimations produced by the *HRE* algorithm converges. If this happens, the estimation errors also converge to some limit, thus the number of estimation sets produced by the *HRE* algorithm does not need to be limited to a few elements. In such a case the sets of estimations can be generated until the limit towards which the *HRE* results converge will not be close enough. Such a situation has also been addressed in the paper. The given conditions of convergence have an intuitive explanation and in many practical situations are likely to be met.

The *HRE* algorithm is suitable for any matrix with positive elements, i.e., even in those situations where the applicability of the classical eigenvector method can be limited (finding the largest absolute eigenvalue of a nonreciprocal matrix may be difficult). The presented algorithm remains open for much more data than can be stored in a single pairwise comparisons matrix. Due to the graph representation of the problem, multiple values defining ratios between the same pairs of concepts can be easily encompassed within the algorithm as multiple arcs between the same pairs of vertices.

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Notes on the existence of a solution in the pairwise comparisons method using the heuristic rating estimation approach

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Abstract Pairwise comparisons (PC) is a well-known method for modeling the subjective preferences of a decision maker. The method is very often used in the models of voting systems, social choice theory, decision techniques (such as *AHP - Analytic Hierarchy Process*) or multi-agent AI systems. In this approach, a set of paired comparisons is transformed into one overall ranking of alternatives. Very often, only the results of individual comparisons are given, whilst the weights (indicators of significance) of the alternatives need to be computed. According to Heuristic Rating Estimation *(HRE),* the new approach discussed in the article, besides the results of comparisons, the weights of some alternatives can also be a priori known. Although *HRE* uses a similar method to the popular *AHP* technique to compute the weights of individual alternatives, the solution obtained is not always positive and real. This article tries to answer the question of when such a correct solution exists. Hence, the sufficient condition for the existence of a positive and real solution in the *HRE* approach is formulated and proven. The influence of inconsistency in the paired comparisons set for the existence of a solution is also discussed.

Keywords Decision support systems · Pairwise comparisons · AHP · Heuristic rating estimation · Data inconsistency

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1 Introduction

The ability to compare things is one of the most useful human skills [8]. When comparing the quality of products in a grocery store, the taste of foods in a restaurant or the fuel prices at a gas station, people are able to make the best choices. The problem starts when the list of possible options is too long and none of the options is clearly better than others. Probably anyone who has visited an electronics store felt slightly uncomfortable looking at several TVs, at the same time wondering which one is the best. In such a case, the pairwise comparisons *(PC)* method [23] may help. According to the method, instead of comparing all the alternatives (hereinafter referred to as concepts) at once, it is better to compare them in pairs. Then, knowing the results of each paired comparison, the final priority values for the considered concept can be calculated. The process of aggregating the results of individual comparisons into the common ranking list of compared concepts hereinafter will be referred to as the priority deriving procedure.

The first written evidence of the use of paired comparisons dates back to the thirteenth century and *RamonLlull's* binary election systems [11]. According to *Llull*, during several voting rounds, every set of two candidates is compared in a pair and the winner is the one who wins by a majority in the greatest number of binary[1] paired comparisons. The method was later repeatedly reinvented. *Condorcet* [12] and *Borda* [14] proposed it in the second half of the eighteen century in their voting systems. *Thurstone* uses the generalized[2] pairwise comparisons in experimental psychology [44]. *Llull's* basic system was then reinvented (with some minor modifications) by *Copeland* in the context of welfare economics [11, 13]. The pairwise comparisons method is a cornerstone of *AHP (Analytic Hierarchy Process)* - a multi-criteria decision technique [38]. According to *AHP*, each concept is compared with each other with respect to different criteria. In this way, each concept receives some ranking value associated with the given criterion. Then, the criteria are compared with each other. Hence, every criterion also gets a priority value. The final priority value assigned to the concept is the weighted sum of its criterion's specific priorities multiplied by priorities of the criteria [40].

Comparing alternatives in pairs is also widely used in other than *AHP* multi-criteria decision making methods such as *ELECTRE* [16, 19], *PROMETHEE* [7] or *MACBETH* [2]. In this study, however, pairwise comparisons have two equally important meanings: ordinal and cardinal. This makes the presented approach similar to *AHP* rather than *ELECTRE* or *PROMETHEE*. Thus, in this approach, unlike in some models known from the social choice theory [34, 43] the result of the comparison is a real number representing the relative value (strength) of the preference. In this sense the *PC* method as proposed by *Thurstone* [44], and then developed by *Saaty* [38] (hereinafter referred to as the *PC* method) seems to be closer to the generalized *Arrow's* model proposed by *Sen* [34, 42] than the earlier works. On the other hand in the *PC* method an expert (or a group of experts) is obligated to provide a matrix containing the results of the comparisons of any two alternatives. This makes it similar to the paired-comparisons voting rules such as the *Kemeny-Young* method or *Simpson-Kramer Min-Max* rule [33]. Although the *PC* method usually does not appear in the debate on the social choice theory it can be useful in this context [41].

[1]The result of a single paired comparison was binary: 0 or 1. Each element of the pair could be either a winner or a loser

[2]In contrast to the binary paired comparisons, the result of a generalized paired comparison was a number determining the ratio between the relative intensity of two stimuli

Despite their long history, paired comparisons are still an inspiration and a challenge to researchers. Examples of their exploration are the approaches based on using *rough sets* [22], fuzzy PC relation [35], incomplete *PC* relation [5, 18, 26], reduction of data inconsistency [28], non-numerical rankings [24], the social choice theory [33], additive *PC* [27] and weight effectiveness [3, 4].

A recent contribution to the *PC* method includes the *Heuristic Rating Estimation (HRE)* approach [30, 31] that allows the user to explicitly define a reference set of concepts, for which the utilities (the ranking values) are known a priori. In *HRE*, the relative value of a single non-reference concept is determined as the weighted average of all the other concepts. Such a proposition leads to the formulation of a linear equation system whose solution, a vector of weights, determines the desired ordering of concepts. The vector need to be strictly positive and real. The presented article is the first one which tries to provide the answer to the question when this vector is positive and real. The resulting outcome (Section 4) is an intuitive and easy to check criterion ensuring the existence of an admissible solution. Although the presented criterion is sufficient (but not necessary), it may be useful for a wide class of problems for which the reference concepts are roughly a bit more than half of all the objects (see Section 4, Remark 3).

Basic information about the *PC* method, the *M-matrix* theory and the *HRE* method can be found in Sections 1, 2.3 and 3 correspondingly. The main results of the work including an existence condition (Theorem 2) and three additional Remarks on its properties are presented in Section 4. A brief summary is provided in Section 5.

2 Preliminaries

2.1 The pairwise comparisons method

The pairwise comparisons method is very often used as a technique that allows an expert (or a group of experts) to synthesize individual pairwise judgments into one, common ranking. The subject of the rankings can be any tangible or intangible entities (anything that experts can assess), hereinafter referred to as concepts or alternatives. Let $C \overset{df}{=} \{c_1, \ldots, c_n\}$ be a finite set of concepts to be judged and/or analyzed, and $\{m_{1,1}, \ldots, m_{1,n}, m_{2,1}, \ldots, m_{2,n}, m_{3,1}, \ldots, m_{n,n}\}$ be the set of expert judgments about each pair of concepts $c_i, c_j \in C$. The judgments (preferences) of experts are represented in the form of real, positive numbers. Thus, assigning a particular value v to m_{ij}, expresses an opinion[3] that c_i is v times more important than c_j. A set of judgments can be conveniently represented as a *PC* matrix $M = (m_{ij})$. Because a comparison of a given concept to itself may not indicate a predominance of any of the two compared elements (since they are identical), the diagonal of M contains all ones.

Let us define the function that assigns the value of importance (also referred to as priority, preference or rank) to every $c \in C$.

Definition 1 The ranking function for C (the ranking of C) is a function $\mu : C \to \mathbb{R}_+$ that assigns a positive value from $\mathbb{R}_+$ to every compared concept.

[3] Sometimes, to help experts to express their verbal opinions in the form of numbers, different measurement scales are used. For example, in *AHP* the judgment values must lie between $1/9$ and 9, and each of the values $1/9, 1/8, \ldots, 8, 9$ has its own well-defined textual representation [38]

The μ function is usually defined as a vector of weights:

$$\mu \overset{df}{=} [\mu(c_1), \ldots, \mu(c_n)]^T \tag{1}$$

The values m_{ij} and m_{ji} represent subjective expert judgments as to the relative importance, utility or quality indicators of the concepts c_i and c_j. Thus, according to the best knowledge of experts, it should hold that $\mu(c_i) = m_{ij}\mu(c_j)$. This observation allows us to define the two properties of the matrix M: *reciprocity* and *consistency*.

Definition 2 A matrix M is said to be reciprocal if for every i, j such that $1 \leq i, j \leq n$ it holds that $m_{ij} = 1/m_{ji}$, and M is said to be consistent if for every i, j, k where $1 \leq i, j, k \leq n$ it holds that $m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$.

Although the matrix M may be neither reciprocal not consistent [21], still, in most cases is it assumed that reciprocity is satisfied. Unfortunately, usually M is not consistent. Since the data in the *PC* matrix represents the subjective opinions of the experts, they might be inconsistent. Hence, there may be a triad m_{ij}, m_{jk}, m_{ki} of entries in M for which $m_{ik} \cdot m_{kj} \neq m_{ij}$. This leads to a situation in which the relative importance of c_i with respect to c_j can be determined either as $m_{ik} \cdot m_{kj}$ or m_{ij} and both ways lead to two different results. In other words, either $\mu(c_i) = m_{ik}\mu(c_k) = m_{ik}\left(m_{kj}\mu(c_j)\right)$ or $\mu(c_i) = m_{ij}\mu(c_j)$, where $m_{ik}\left(m_{kj}\mu(c_j)\right) \neq m_{ij}\mu(c_j)$. In such a situation it seems natural to adopt the weighted mean of priorities of all the other concepts as a desired value of $\mu(c_i)$, i.e.,

$$\mu(c_i) = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} m_{ij}\mu(c_j) \tag{2}$$

AHP - one of the most popular decision making methods [38] is in line with the above postulate. It advocates users to adopt the principal eigenvector of M as the priority vector, rescaled so that the sum of all its entries is 1, i.e.

$$\mu_{ev} = \left[\frac{\mu_{max}(c_1)}{s_{ev}}, \ldots, \frac{\mu_{max}(c_n)}{s_{ev}}\right]^T \text{ and } s_{ev} = \sum_{i=1}^{n} \mu_{max}(c_i) \tag{3}$$

where μ_{ev} is the ranking function, μ_{max} is the principal eigenvector of M. Hence, it holds that:

$$\mu_{max}(c_i) = \frac{1}{\lambda_{max} - 1} \sum_{j=1, j\neq i}^{n} m_{ij}\mu_{max}(c_j) \tag{4}$$

where λ_{max} is a principal eigenvalue of M. Due to the *Peron-Frobenius* theory, when M is positive, such a real and positive λ_{max} exists [38]. In particular, if M is consistent then λ_{max} equals n, hence (4) is a weighted arithmetic mean as postulated in (2).

Let us see how the pairwise comparisons method works in practice by providing the following simple example in which four candidates $c_1, \ldots, c_4$ apply for the position of chancellor of some university. In the adopted election scheme, the university senate shall discuss the submitted applications and then proceed to vote. Then, taking into account the outcome of voting, the Rector shall select a candidate for the position of chancellor[4].

For the purpose of the example, let us assume that during the vote senators evaluate each of the six pairs $(c_1, c_2), (c_1, c_2), \ldots, (c_3, c_4)$ by assigning 1,2 or 3 either to the first or to

[4]The presented election scheme is quite popular in Poland. See for example (in Polish) Statute of AGH UST (in Polish), art. 19, par. 2.8, http://regent2.uci.agh.edu.pl/statut/statut-agh.pdf

the second candidate within the pair. By assigning 1 (to any out of the two in a pair) they indicate that both candidates are *equally preferred*. Assigning 2 to a candidate will mean that he/she is *more preferred* than the opponent in a pair and, finally, assigning 3 to the given candidate will mean that he/she is *much more preferred* than his/her opponent in the pair. To express intermediate judgments, voters are allowed to use intermediate values. For example, in order to express the opinion that c_i is *slightly more preferred* than c_j a voter may assign 1.5 to c_i. Voter assignments easily translate to the entries of the *PC* matrix. Whenever, considering the pair (c_i, c_j), the voter v_r assigns x to the concept c_i, the value m_{ij} is set to x, and correspondingly, m_{ji} is set to $1/x$.

Judgments expressed during the vote can be stored in the set of *PC* matrices $M^{(1)}, \ldots, M^{(q)}$ where every matrix $M^{(r)} = (m_{ij}^{(r)})$ corresponds to the opinion of one out of the q voters. The resulting matrices $M^{(1)}, \ldots, M^{(q)}$ can be aggregated into one *PC* matrix $\widehat{M} = (\widehat{m}_{ij})$ with the help of a geometric mean[5] [1], where:

$$\widehat{m}_{ij} = \left(\prod_{r=1}^{q} m_{ij}^{(r)} \right)^{1/q} \tag{5}$$

According to (3) the final ranking $\widehat{\mu}_{ev}$ is obtained as the rescaled principal eigenvector of $\widehat{M}$.

For the sake of the simplicity of calculations, let us assume that the voting was held by the senate committee consisting of three persons s_1, s_2 and s_3. Their votes were written down in the form of the following three matrices[6]:

$$M^{(1)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \mathbf{\frac{5}{2}} \\ \mathbf{2} & 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \frac{1}{2} & 1 & \mathbf{2} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}, \quad M^{(2)} = \begin{pmatrix} 1 & \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \frac{1}{2} & 1 & \frac{1}{2} & \mathbf{2} \\ \frac{1}{2} & \mathbf{2} & 1 & \mathbf{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \quad M^{(3)} = \begin{pmatrix} 1 & \mathbf{3} & \frac{1}{2} & \mathbf{2} \\ \frac{1}{3} & 1 & \frac{1}{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & 1 & \mathbf{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \tag{6}$$

Thus, according to the (5) the matrix of aggregated results is prepared as follows:

$$\widehat{M} = \begin{pmatrix} 1. & 1.44 & 0.794 & 2.47 \\ 0.693 & 1. & 0.794 & 2.29 \\ 1.26 & 1.26 & 1. & 2.29 \\ 0.405 & 0.437 & 0.437 & 1. \end{pmatrix} \tag{7}$$

The appropriately rescaled eigenvector of $\widehat{M}$ provides the desired ranking vector $\widehat{\mu}_{ev}$ (see 3).

$$\widehat{\mu}_{ev} = [\, 0.304 \;\; 0.248 \;\; 0.324 \;\; 0.123 \,]^T \tag{8}$$

The winner is the third candidate, whose application gets the highest rank $\widehat{\mu}_{ev}(c_3) = 0.324$. Thus, the senators recommend c_3 for the position of chancellor.

2.2 Matrix inconsistency

When the matrix M is inconsistent, it is difficult to unambiguously determine the relative importance of one concept with respect to the other. In particular, it may turn out that

[5] There are also other ways of aggregating the results in the multiple expert pairwise comparisons method [15, 17, 20], but the use of a geometric mean of judgments appears to be the most popular

[6] The values indicated by voters are written in bold

$m_{ij}m_{jk} \neq m_{ik}$ although both m_{ik} and $m_{ij}m_{jk}$ are equally well suited to determine the relative importance of c_i with reference to c_k. Thus, even if we compute the ranking value $\mu(c_i)$, the question arises regarding the extent to which it reflects the expert's actual opinion [39]? This question prompted researchers to define inconsistency (consistency) indices as methods for measuring the inconsistency of M.

There is a number of different consistency/inconsistency indices [6, 10, 23, 32]. Despite a few attempts of axiomatization [10, 29], there is no single commonly accepted definition of an inconsistency index. However, probably all known indices equal 0 for a fully consistent matrix M, and grow (or at least do not decrease) along with the increase in disturbances of triads (m_{ij}, m_{jk}, m_{ki}) (see Definition 2). Therefore, it is widely accepted that the lower the inconsistency index, the more consistent the *PC* matrix, and hence, the more reliable and trustworthy the results.

In his seminal work [38] Saaty defined the consistency index *(CI)* with the help of λ_{max} (the principal eigenvalue of M).

Definition 3 Given a $n \times n$ *PC* matrix M, *Saaty's CI* is defined as:

$$CI(M) = \frac{\lambda_{max} - n}{n - 1} \tag{9}$$

Indeed, since it holds that $\lambda_{max} = n$ for the fully consistent M, then for such a matrix $CI(M) = 0$. Similarly, the more products in the form $m_{ij}m_{jk}m_{ki}$ that differ from 1, the higher[7] the $CI(M)$. In most cases, $CI(M) < 0.1$ is considered as an acceptable level of inconsistency[8]. When the inconsistency is too high, the result of the ranking is regarded as inconclusive.

For the purpose of the rest of the article, the more restrictive *Koczkodaj's inconsistency index* is adopted [25]. Let us denote:

$$\kappa_{i,j,k} \overset{df}{=} \min\left\{\left|1 - \frac{m_{ij}}{m_{ik}m_{kj}}\right|, \left|1 - \frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\} \tag{10}$$

Definition 4 Koczkodaj's inconsistency index $\mathscr{K}$ of $n \times n$ and $(n > 2)$ reciprocal matrix M is equal to

$$\mathscr{K}(M) = \max_{1 \leq i,j,k \leq n} \left\{\kappa_{i,j,k}\right\} \tag{11}$$

where $i \neq j$, $j \neq k$ and $i \neq k$.

It is easy to see that Koczkodaj's inconsistency index also equals 0 when M is consistent. Similarly, the increase in disturbances of triads (m_{ij}, m_{jk}, m_{ki}) ultimately leads to the increase of $\mathscr{K}(M)$. It is assumed that the acceptable threshold of inconsistency, for most practical applications, is $\mathcal{K}(M) < 1/3$ [28]. A more complete overview of different indices, including a comparison of these two, can be found in the literature [6, 9].

In the context of the considered example (Section 2.1), the high inconsistency of matrices $M^{(1)}, M^{(2)}, M^{(3)}$ or $\widehat{M}$ may induce the Rector to make a decision contrary to the

[7] Some authors argue that the increase is too slow [29]

[8] The exact procedure for determining the acceptable value of inconsistency can be found the article [38]

recommendation of the senate committee. Conversely, the low inconsistency of these matrices is an argument for proceeding in accordance with the indication of the committee.

2.3 M-matrices

The analysis of the HRE method presented in the article requires knowledge of the concept of the *M-matrix* [37]. In order to introduce the notion of the *M-matrix* and its properties, let us denote $\mathcal{M}_{\mathbb{R}}(n)$ - the set of $n \times n$ matrices over $\mathbb{R}$, $\mathcal{M}_Z(n)$ - the set of all $A = (a_{ij}) \in \mathcal{M}_{\mathbb{R}}(n)$ with $a_{ij} \leq 0$ if $i \neq j$ and $1 \leq i, j \leq n$. Moreover, for every matrix $A \in \mathcal{M}_{\mathbb{R}}(n)$ and vector $b \in \mathbb{R}^n$ the notation $A \geq 0$ and $b \geq 0$ will mean that each entry of A and b is non-negative and neither A nor b equals 0. The spectral radius of A is defined as $\rho(A) = \max\{|\lambda| : \det(\lambda I - A) = 0\}$.

Definition 5 An $n \times n$ matrix that can be expressed in the form $A = sI - B$ where $B = [b_{ij}]$ with $b_{ij} \geq 0$ for $1 \leq i, j \leq n$, and $s \geq \rho(B)$, the maximum of the absolute value of the eigenvalues of B (i.e., $\rho(B) = \max_i |\lambda_i|$, where λ_i is an eigenvalue of B), is called an *M-matrix*.

In practice, solving many problems in the biological sciences and in the social sciences can be reduced to problems involving *M-matrices* [36]. For this reason, *M-matrices* have been of interest to researchers for a long time and many of their properties are known. Following the work of Plemmons [36] some of them are recalled below in the form of Theorem 1.

Theorem 1 (M-matrix properties) *For every $A \in \mathcal{M}_Z(n)$ the following conditions are equivalent:*

1. *A is inverse positive. That is, A^{-1} exists and $A^{-1} \geq 0$*
2. *A is semi-positive. That is, there exists vector $x > 0$ with $Ax > 0$*
3. *There exists a positive diagonal matrix D such that AD has all positive row sums.*
4. *A is a non-singular M-matrix*

Note that if A is non-singular then A^{-1} is also non-singular. Thus, the solution of $A\mu = b$ is $A^{-1}b$. Moreover for $b > 0$ and A - *M-matrix*, due to the theorem above $A^{-1} \geq 0$, the vector μ also must be strictly positive, i.e., $\mu = A^{-1}b > 0$.

3 Heuristic rating estimation approach

In the eigenvalue based approach [38], the ranking function μ for all the concepts $c \in C$ is initially unknown. Hence, every $\mu(c)$ needs to be determined by the priority deriving procedure. In real life, however, it may turn out that for some concepts the priority values are known. Sometimes decision makers have extra knowledge about the group of elements $C_K \subseteq C$ that allows them to determine $\mu(c)$ for all $c \in C_K$ in advance.

For example, let c_1, c_2 and c_3 be goods that company X intends to place on the market, whilst c_4 and c_5 have been available for some time in stores. In order to choose the most profitable and promising product out of $c_1, \ldots, c_3$, company X wants to calculate the function μ for c_1, c_2 and c_3. Due to some similarities between $c_1, \ldots, c_3$ and the pair c_4, c_5, company X wants to include them in the ranking, treating them as a reference. Of course, it makes no sense to ask experts about how profitable c_4 and c_5 are. The values $\mu(c_4)$ and $\mu(c_5)$ can be easily determined based on sales reports.

The situation as outlined in this simple example leads to the *Heuristic Rating Estimation* method *(HRE)* [30, 31]. The main heuristic of the *HRE* method assumes that the set of concepts C is composed of the unknown concepts $C_U = \{c_1, \ldots, c_k\}$ and the known (reference) concepts $C_K = \{c_{k+1}, \ldots, c_n\}$. Of course, only the values μ_j for $c \in C_U$ need to be estimated, whilst the values $\mu(c_i)$ for $c_i \in C_K$ are considered to be known. The idea behind the adopted heuristic (2), the same as for the eigenvalue based priority deriving method (3, 4) with the fully consistent *PC* matrix, is that for every unknown $c_j \in C_U$ the value $\mu(c_j)$ should be estimated as the arithmetic mean of all the other values $\mu(c_i)$ multiplied by the factor m_{ji}. Thus, the values $\mu(c_i)$ for each unknown concept $c_j \in C_U$ are calculated according to the following formulas:

$$\begin{array}{lcl} \mu(c_1) & = & \frac{1}{n-1}(m_{2,1}\mu(c_2) + \ldots\ldots\ldots\ldots\ldots + m_{n,1}\mu(c_n)) \\ \mu(c_2) & = & \frac{1}{n-1}(m_{1,2}\mu(c_1) + m_{3,2}\mu(c_3) + \ldots + m_{n,2}\mu(c_n)) \\ \ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu(c_k) & = & \frac{1}{n-1}\big(m_{1,k}\mu(c_1) + \ldots\ldots + m_{k-1,k}\mu(c_{k-1}) + \\ & & + m_{k+1,k}\mu(c_{k+1}) + \ldots\ldots + m_{n,k}\mu(c_n)\big) \end{array} \tag{12}$$

Since the values $\mu(c_{k+1}), \ldots, \mu(c_n)$ are known and constant ($c_{k+1}, \ldots, c_n$ are the reference concepts), they can be grouped together. Let us denote:

$$b_j = \frac{1}{n-1} m_{k+1,j}\mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{n,j}\mu(c_n) \tag{13}$$

Thus (12) could be written as the linear equation system $A\mu = b$ where the matrix A is:

$$A = \begin{bmatrix} 1 & \cdots & -\frac{1}{n-1}m_{1,k} \\ -\frac{1}{n-1}m_{2,1} & \cdots & -\frac{1}{n-1}m_{2,k} \\ \vdots & \ddots & \vdots \\ -\frac{1}{n-1}m_{k,1} & \cdots & 1 \end{bmatrix}, \tag{14}$$

and the vectors b and μ are:

$$b = \begin{bmatrix} \frac{1}{n-1}\sum_{i=k+1}^{n} m_{1,i}\mu(c_i) \\ \frac{1}{n-1}\sum_{i=k+1}^{n} m_{2,i}\mu(c_i) \\ \vdots \\ \frac{1}{n-1}\sum_{i=k+1}^{n} m_{k,i}\mu(c_i) \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu(c_1) \\ \mu(c_2) \\ \vdots \\ \mu(c_2) \end{bmatrix} \tag{15}$$

It is worth noting that $b > 0$, since every b_i for $i = 1, \ldots, k$ is the sum of strictly positive components.

Let us consider the following numerical example. In some local election there are 7 seats to be filled in a district council. Elections are conducted in accordance with the rule "7 best wins", hence only 7 people with the best election results are chosen to become members of the district council. Each party (election committee) may nominate any number of candidates. However, due to the cost of the election campaign, it is important that the nominated candidates actually have a chance of entering the council. On the other hand, based on the results of the previous election, it is known that the result of more than 2000 votes per candidate guaranteed a place in the council. Therefore, the parties are faced with the difficult task of identifying candidates who have a real chance of gathering at least 2000 votes.

One of the political parties participating in the elections plans to support at most five persons. As, during the inner-party meeting, seven candidates $c_1, \ldots, c_7$ have been put forward (including three current members c_5, c_6 and c_7 of the council), the party leadership has to decide whom to support. For this purpose, the party has hired a group of experts, whose task is to assess the chance of each candidate by comparing their election chances in pairs. During the meeting, the experts have prepared[9] the *PC* matrix $M = (m_{ij})$ such that every m_{ij} corresponds to the relative popularity (attractiveness to voters) of c_i candidate with respect to c_j.

$$M = \begin{pmatrix} 1 & 2 & 4.5 & 1.5 & 0.75 & 1.2 & 0.9 \\ \frac{1}{2} & 1 & 2 & 0.7 & 0.35 & 0.5 & 0.4 \\ 0.22 & \frac{1}{2} & 1 & 0.4 & 0.2 & 0.3 & 0.2 \\ 0.67 & 1.43 & 2.5 & 1 & 0.4 & 0.7 & 0.5 \\ 1.33 & 2.86 & 5. & 2.5 & 1 & \frac{3042}{2511} & \frac{3042}{3220} \\ 0.833 & 2. & 3.33 & 1.43 & \frac{2511}{3042} & 1 & \frac{2511}{3220} \\ 1.11 & 2.5 & 5. & 2. & \frac{3220}{3042} & \frac{3220}{2511} & 1 \end{pmatrix} \tag{16}$$

Since the current popularity of the party is similar to that during the previous elections, and it is known that previously c_5, c_6 and c_7 received $\mu(c_5) = 3042$, $\mu(c_6) = 2511$ and $\mu(c_7) = 3220$ votes correspondingly, then experts do not evaluate the pairs (c_5, c_6), (c_5, c_7) and (c_6, c_7). Instead, for each $i, j = 5, 6, 7$ the value $\mu(c_i)/\mu(c_j)$ as $m_{i,j}$ has been adopted.

To estimate the expected number of votes for other candidates the *HRE* method is used, where $C_U = \{c_1, \ldots, c_4\}$ and $C_K = \{c_5, c_6, c_7\}$. The matrix A and the vector b formed from the matrix M are as follows:

$$A = \begin{pmatrix} 1 & -0.33 & -0.75 & -0.25 \\ -0.083 & 1 & -0.33 & -0.12 \\ -0.037 & -0.083 & 1 & -0.067 \\ -0.11 & -0.24 & -0.42 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1400 \\ 600 \\ 330 \\ 760 \end{pmatrix} \tag{17}$$

The ranking vector μ as a solution of the equation $A\mu = b$ is as follows:

$$\mu = \begin{pmatrix} 2661.5 & 1226.48 & 643.048 & 1619.76 \end{pmatrix}^T \tag{18}$$

Hence, according to the experts, only c_1 (with approximately $\mu(c_1) = 2661$ votes) may count on the support of more than 2000 voters. Based on the results of the ranking, the party leadership decide to nominate the three current members of the district council $c_{5,} c_6$ and c_7 who can count on the support of 3042,2511 and 3220 votes correspondingly, and one new person, c_1, who expects to gain about 2661 votes.

4 Inconsistency based condition for the existence of a solution

To receive the ranking estimates in the *HRE* approach it is enough to solve the linear equation system $A\mu = b$. Therefore, on one hand, it is easy to find a solution by using almost

[9]For the purpose of the example, there is no need to specify how the experts obtained the matrix M. One of several possible methods [20] involving geometric averaging results provided by each expert has been presented in the previous example (Section 2.1)

any mathematical software including *Excel*®. On the other hand, a solution may not always exist, as the calculated μ may not always be positive and real. The following reasoning is an attempt to find an inconsistency related criterion that helps to decide on the existence of a solution in the *HRE* approach.

Let us note that the entries of $M = (m_{ij})$ are always positive as they represent the comparative opinions of experts. Thus, it holds that $M > 0$. For the same reason, the matrix A (14), formed on the basis of M, has positive entries only on the diagonal, i.e., $A \in \mathcal{M}_Z(n)$ (see Section 2.3). Therefore, proving that A satisfies any of the conditions of Theorem 1, implies that A is an *M-matrix*.

The sufficient condition for A to be an *M-matrix* is formulated with the help of the inconsistency index $\mathcal{K}(M)$ (Definition 4). The paired rankings for which the inconsistency index is too high are considered as unreliable [38]. Using an inconsistency index simplifies the evaluation of $A\mu = b$ and enables linking the reliability of expert assessments with the solution existence problem.

Theorem 2 (On the existence of a solution) *The linear equation system $A\mu = b$ introduced in the HRE approach has exactly one strictly positive solution if*

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \tag{19}$$

where $n = |C_U \cup C_K|$ is the number of all the estimated concepts, whilst $r = |C_K|$ - is the number of known concepts and $0 < r \leq n - 2$.

Proof Following Definition 4, the value of *Koczkodaj's inconsistency index* $\mathscr{K}(M)$, in short $\mathscr{K}$, means that the maximal inconsistency for some triad m_{pq}, m_{qr} and m_{pr} is $\mathscr{K}$. Thus, in the case of an arbitrarily chosen triad m_{ik}, m_{kj}, m_{ij} it must hold that:

$$\mathscr{K} \geq \kappa_{i,j,k} = \min\left\{\left|1 - \frac{m_{ij}}{m_{ik}m_{kj}}\right|, \left|1 - \frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\} \tag{20}$$

This means that either: $m_{ij} \leq m_{ik}m_{kj}$ implies that $\mathscr{K} \geq 1 - \frac{m_{ij}}{m_{ik}m_{kj}}$, or $m_{ik}m_{kj} \leq m_{ij}$ implies that $\mathscr{K} \geq 1 - \frac{m_{ik}m_{kj}}{m_{ij}}$. Denoting

$$\alpha \overset{df}{=} 1 - \mathscr{K} \tag{21}$$

we obtain that $m_{ij} \leq m_{ik}m_{kj}$ implies $m_{ij} \geq \alpha \cdot m_{ik}m_{kj}$, and $m_{ik}m_{kj} \leq m_{ij}$ implies $\frac{1}{\alpha} \cdot m_{ik}m_{kj} \geq m_{ij}$. It is easy to see that $0 \leq \mathcal{K} < 1$, thus $0 < \alpha \leq 1$. Thus, both these assertions lead to the common conclusion:

$$\alpha \cdot m_{ik}m_{kj} \leq m_{ij} \leq \frac{1}{\alpha} m_{ik}m_{kj} \tag{22}$$

for every i, j, k such that $1 \leq i, j, k \leq n$. This mutual relationship between the entries of M can be written as the parametric equation $m_{ij} = t \cdot m_{ik}m_{kj}$ where $\alpha \leq t \leq \frac{1}{\alpha}$. Using this equation, the matrix A (see 14) can be written as:

$$A = \begin{bmatrix} t_{1,1}m_{1,k}m_{k,1} & \cdots & -\frac{t_{1,k-1}m_{1,k}m_{k,k-1}}{n-1} & -\frac{m_{1,k}}{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{t_{k-1,1}m_{k-1,k}m_{k,1}}{n-1} & \ddots & t_{k-1,k-1}m_{k-1,k}m_{k,k-1} & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1}m_{k,1}}{n-1} & \cdots & -\frac{t_{k,k-1}m_{k,k-1}}{n-1} & 1 \end{bmatrix} \tag{23}$$

where $\alpha \leq t_{ij} \leq \frac{1}{\alpha}$, for i, j such that $1 \leq i, j \leq k-1$ (please note that the last column remained unchanged). Hence, finally the matrix A can be written as the matrix product $A = BC$ where:

$$B = \begin{bmatrix} t_{1,1}m_{1,k} & \cdots & -\frac{t_{1,k-1}m_{k-1,k}}{n-1} & -\frac{m_{1,k}}{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{t_{k-1,1}m_{k-1,k}}{n-1} & \vdots & t_{k-1,k-1}m_{k-1,k} & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1}}{n-1} & \cdots & -\frac{t_{k,k-1}}{n-1} & 1 \end{bmatrix} \tag{24}$$

and

$$C = \begin{bmatrix} m_{k,1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & m_{k,k-1} & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{25}$$

Since both t_{ij} and m_{ij} are strictly positive, it holds that $B \in \mathcal{M}_Z(n)$. Therefore, due to the third condition of Theorem 1 with $D \stackrel{df}{=} I$, B is a non-singular *M-matrix* if (and only if) the sums of the rows of $(n-1)B$ are positive. In other words, B is *an M-matrix* if all of the following inequalities (26) are true:

$$\begin{array}{rl} m_{1,k}(n-1)t_{1,1} - m_{1,k}(t_{1,2} + t_{1,3} + \ldots + t_{1,k-1} + 1) & > 0 \\ m_{2,k}(n-1)t_{2,2} - m_{2,k}(t_{2,1} + t_{2,3} + \ldots + t_{2,k-1} + 1) & > 0 \\ \vdots & \\ (n-1) - (t_{k,1} + t_{k,2} + \ldots\ldots\ldots\ldots\ldots\ldots + t_{k,k-1}) & > 0 \end{array} \tag{26}$$

Due to the constraints introduced by the inconsistency $\mathcal{K}(M)$, the minimal and the maximal value of every t_{ij} is α and $\frac{1}{\alpha}$ correspondingly. Thus, the inequalities (26) are true if the following two inequalities are satisfied[10]:

$$(n-1)\alpha > (\underbrace{\frac{1}{\alpha} + \ldots + \frac{1}{\alpha}}_{n-r-2} + 1) \quad \text{and} \quad (n-1) > (\underbrace{\frac{1}{\alpha} + \ldots + \frac{1}{\alpha}}_{n-r-1}) \tag{27}$$

where $r = n - k$ is the number of elements in C_K. In other words, B is *an M-matrix* if the following two conditions are met:

$$f(\alpha) > 0, \quad \text{where} \quad f(\alpha) \stackrel{df}{=} (n-1)\alpha^2 - \alpha - (n-r-2) \tag{28}$$

and

$$g(\alpha) > 0, \quad \text{where} \quad g(\alpha) \stackrel{df}{=} (n-1)\alpha - (n-r-1) \tag{29}$$

[10] Let us denote $p_i(t) \stackrel{df}{=} m_{i,k}(n-1)t$ and $q_i(t_1, \ldots, t_{k-1}) \stackrel{df}{=} m_{i,k}(t_1 + t_2 + \ldots + t_{k-1} + 1)$ for $0 < i < k$. Since every $\alpha < t, t_i < \frac{1}{\alpha}$, and $0 < \alpha < 1$, all but the last inequalities of (26) have a form $p_i(t) - q_i(t_1, \ldots, t_{k-1}) > 0$. It is easy to see that $p_i(t)$ reaches the minimum in the interval $\alpha < t < \frac{1}{\alpha}$ for $t = \alpha$, and similarly, $q_i(t_1, \ldots, t_{k-1})$ is maximal for every t_i such that $\alpha < t_i < \frac{1}{\alpha}$ when $t_1 = \ldots = t_{k-1} = \frac{1}{\alpha}$. Thus, the function $p_i(t) - q_i(t_1, \ldots, t_{k-1})$ reaches its minimum for $t = \alpha$ and $t_1 = \ldots = t_{k-1} = \frac{1}{\alpha}$. Since every $m_{i,k} > 0$, thus to decide the truth of all but the last inequalities of (26) it is enough to examine $\frac{1}{m_{i,k}}\left(p_i(\alpha) - q_i(\frac{1}{\alpha}, \ldots, \frac{1}{\alpha})\right) > 0$, i.e., $(n-1)\alpha - \left((k-2)\frac{1}{\alpha} + 1\right) > 0$. The same applies to the last inequality of (26)

Table 1 The upper bounds for $\mathcal{K}(M)$ for which there is a guarantee that A is an *M-matrix*

$0 \le \mathcal{K}(M) <$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
$n = 3$	0.5	-	-	-	-
$n = 4$	0.232	0.666	-	-	-
$n = 5$	0.156	0.359	0.75	-	-
$n = 6$	0.118	0.259	0.441	0.8	-
$n = 7$	0.095	0.204	0.333	0.5	0.833

By solving $f(\alpha) = 0$ and choosing the larger root[11] we obtain that:

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \tag{30}$$

whilst the right, linear, inequality $g(\alpha) > 0$ leads to

$$\mathcal{K}(M) < 1 - \frac{(n-r-1)}{(n-1)} \tag{31}$$

In order to decide which of these criteria are more restrictive and which should therefore be chosen, the following two cases need to be considered:

(a) $r = n - 2$
(b) $0 < r \le n - 3$

When $r = n - 2$, it is easy to see that $f(\alpha) = \alpha g(\alpha)$. Thus, both functions $f(\alpha)$ and $g(\alpha)$ take the 0 value for the same values of argument α. Hence, both criteria are equivalent.

If $0 < r \le n-3$, it is easy to prove (see Appendix A) that the first condition (30) is more restrictive than the second one, i.e., whenever (30) holds, the inequality (31) is also true. In other words, to provide a guarantee that B is an *M-matrix,* it is enough to consider the more restrictive condition (30).

The fact that B is *an M-matrix* implies that there is an inverse matrix $B^{-1} \ge 0$ (Theorem 1). Hence, due to the form of the matrix C, it is easy to see that the inverse matrix C^{-1} exists, thus A^{-1} exists and $A^{-1} = C^{-1}B^{-1} \ge 0$. Thus, due to the first condition of Theorem 1, A is *an M-matrix*, which means that the equation $A\mu = b$ has a unique strictly positive solution. This conclusion completes the proof of the theorem. □

Of course, the theorem proven above does not address the case $r = n-1$. This is because $r = n-1$ implies A is a scalar, hence solving $A\mu = b$ is trivial. When M is fully consistent, i.e., $\mathcal{K}(M) = 0$ and $\alpha = 1$, it is easy to see that both conditions (27) are satisfied. Thus, in such a case A is an *M-matrix*, and therefore $A\mu = b$ always has a strictly positive solution. Several upper bounds for $\mathcal{K}(M)$ related to the parameters n and r arising from the above theorem are gathered in Table 1. In a broader range, the relationship between n, r and $\mathcal{K}(M)$ is shown in (Fig. 1).

Remark 1 Let us note that for any combination of $r, n \in \mathbb{N}_+$ where $0 < r \le n - 2$, the right side of (30) is greater than 0. In other words, for a sufficiently low inconsistency, the equation $A\mu = b$ always has a feasible solution.

[11]The smaller root $\frac{1-\sqrt{1+4(n-1)(n-r-2)}}{2(n-1)} \le 0$ for any $n = 3, 4 \ldots$ and $0 < r \le n-2$, so it does not need to be taken into account

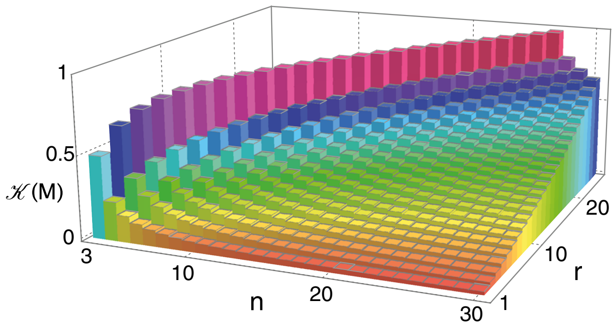

Fig. 1 Limit values of $\mathcal{K}(M)$ below which there is a guarantee that the *HRE* method has a solution

To prove this (see 30) it is enough to show that for $n = 3, 4, \ldots$ it holds that:

$$\frac{\left(1 + \sqrt{1 + 4(n-1)(n-r-2)}\right)}{2(n-1)} < 1 \tag{32}$$

Since $\sqrt{1 + 4(n-1)(n-r-2)} \le \sqrt{1 + 4(n-1)(n-3)}$, it is enough to show

$$\frac{\left(1 + \sqrt{1 + 4(n-1)(n-3)}\right)}{2(n-1)} < 1 \tag{33}$$

Thus,

$$\sqrt{1 + 4(n-1)(n-3)} < 2n - 3 \tag{34}$$

and

$$4(n-1)(n-3) < (2n-3)^2 - 1 \tag{35}$$

which is equivalent to

$$4(n-1)(n-3) < 4(n-1)(n-2) \tag{36}$$

Thus, for every $n > 1$ the above equation reduces to:

$$n - 3 < n - 2 \tag{37}$$

The last inequality is always satisfied, which proves that (32) is true for $n \ge 3$.

Remark 2 Another interesting observation is that the proof of Theorem 2 takes into account only those entries of the matrix M that form the matrix A. Hence, there is no need to analyze the inconsistency for the whole matrix M. Instead, it is enough to analyze $\widetilde{M}$ - the matrix obtained from M by removing rows and columns corresponding to the elements from the set of known concepts C_K. It also holds[12] that $\mathscr{K}(\widetilde{M}) \le \mathscr{K}(M)$. Thus, it may turn out that the inconsistency of $\widetilde{M}$ meets the condition (30), whilst the inconsistency of M is too high.

[12] By definition of the *Koczkodaj inconsistency index,* $\mathscr{K}(M)$ is the maximum of $T_M = \{\kappa_{i,j,r}$ such that $1 \le i, j, r \le n\}$. Similarly, $\mathscr{K}(\widetilde{M})$ is the maximum of $T_{\widetilde{M}} = \{\kappa_{i,j,r}$ such that $1 \le i, j, r \le k\}$, where k is the number of elements in C_U. Since $k < n$ thus, also $T_{\widetilde{M}} \subseteq T_M$. This implies that $\max T_{\widetilde{M}} \le \max T_M$, which leads to the observation that $\mathscr{K}(\widetilde{M}) \le \mathscr{K}(M)$

Table 2 The values of r that guarantee the existence of a solution in the HRE approach providing that $\mathcal{K}(M) < 1/3$

n	4	5	6	7	8	9	10	11	12
$r \geq$	2	2	3	3	4	5	5	6	6

Assuming that $C_U = \{c_1, \ldots, c_k\}$, the matrix $\widetilde{M}$ is as follows:

$$\widetilde{M} = \begin{bmatrix} 1 & \cdots & m_{1,k} \\ \vdots & \vdots & \vdots \\ m_{k-1,1} & \ddots & m_{k-1,k} \\ m_{k,1} & \cdots & 1 \end{bmatrix} \tag{38}$$

It might be noticed that, assuming $\alpha \overset{df}{=} 1 - \mathscr{K}(\widetilde{M})$ in (21), the proof of Theorem 2 does not change. Hence, instead of exploring the inconsistency of M it is sufficient to examine the inconsistency of the reduced matrix $\widetilde{M}$. Thereby, the upper bounds given in the Table 1 can be applied to $\mathscr{K}(\widetilde{M})$ instead of $\mathscr{K}(M)$.

Remark 3 For most practical applications, *Koczkodaj's* inconsistency lower than $1/3$ is recommended as acceptable [28]. Assuming $\mathcal{K}(M) = 1/3$ the condition (19) can be written as:

$$\frac{1}{3} < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \tag{39}$$

which is equivalent to

$$r > h(n), \quad \text{where} \quad h(n) \overset{df}{=} n - 2 - \frac{(4(n-1)-3)^2 - 9}{36(n-1)} \tag{40}$$

Hence, for the given $n \geq 3$ and $\mathcal{K}(M) < 1/3$, it is easy to compute how many known concepts guarantee the existence of a solution (Table 2).

In particular, in the example of the district council elections in Section 3, the inconsistency of the *PC* matrix (16) is $\mathcal{K}(M) \approx 0.308 < 1/3$. Since three out of the seven considered candidates have known ranking values, thus, according to the criterion (19), the solution must exist see Table 2. Moreover, it holds that:

$$\lim_{n\to\infty} \frac{h(n)}{n} = \frac{5}{9} \approx 0.5556 \tag{41}$$

Thus, whenever *Koczkodaj's* inconsistency index $\mathcal{K}(M)$ is lower than $1/3$ (i.e., inconsistency is considered as acceptable) the solution always exists if the known concepts are at least 55.56% of all the ranked concepts.

5 Summary

The reliability of the results achieved in the *PC* models are inseparably linked to the degree of inconsistency of the input data [38]. The lower the inconsistency the better and more reliable the results might be expected to be. Therefore, most practical applications of the *PC* method method seek to construct the *PC* matrix with the smallest possible inconsistency. The theorem proven in this article is in line with the tendency to seek *PC* solutions with low inconsistency. It shows that for an appropriately small inconsistency $\mathcal{K}(M)$ the recently proposed *HRE* method always has an admissible solution. Moreover, given that the inconsistency is acceptable, i.e., $\mathcal{K}(M) < 1/3$, 55.56 % or more of the known concepts guarantees the existence of a solution. This observation makes the provided criterion especially useful in situations where there are many known concepts.

The *HRE* approach is a relatively new estimation method of the relative order of concepts when a non-empty reference subset of concepts exists. The properties of this method are not yet fully understood. Thus, it requires further studies. The presented considerations are accompanied by two numerical examples demonstrating how the *PC* method and *HRE* can be used in different situations. Due to the possibility of applying the *HRE* method to solve various decision problems, it may be of interest to a wide range of researchers and practitioners.

Appendix A: Remark about the Restrictiveness of the Criteria

Assuming that $n - 3 \geq r > 0$ (i.e., $n - 2 > r > 0$) the criterion given as (30) is said to be more restrictive than the criterion given as (31), when the right side of (30) is smaller than the right side of (31), i.e., when:

$$1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} < 1 - \frac{(n-r-1)}{(n-1)} \tag{42}$$

The above inequality is true if and only if

$$\frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} > \frac{(n-r-1)}{(n-1)} \tag{43}$$

Let us denote $m \overset{df}{=} n - 1$, then (43) is equivalent to:

$$\frac{1 + \sqrt{1 + 4m(m-r-1)}}{2m} > \frac{(m-r)}{m} \tag{44}$$

Since $m > 0$, thus the above expression is equivalent to:

$$1 + \sqrt{1 + 4m(m-r-1)} > 2(m-r) \tag{45}$$

and consequently

$$\sqrt{1 + 4m(m-r-1)} > 2\left(m - r - \frac{1}{2}\right) \tag{46}$$

The above inequality holds if and only if

$$1 + 4m(m-r-1) > 4\left(m - \left(r + \frac{1}{2}\right)\right)^2 \tag{47}$$

thus,

$$1 + 4m^2 - 4mr - 4m > 4m^2 - 8m\left(r + \frac{1}{2}\right) + 4\left(r + \frac{1}{2}\right)^2 \tag{48}$$

and consequently

$$1 + 4m^2 - 4mr - 4m > 4m^2 - 8mr - 4m + 4r^2 + 4r + 1 \quad (49)$$

The above expression is equivalent to:

$$mr > (r + 1)r \quad (50)$$

which corresponds to

$$m - 1 > r \quad (51)$$

In the light of the assumptions that $n - 2 > r$ and the fact that $m = n - 1$ the inequality (51) is always met. Since, all the expressions (42) - (51) are equivalent the truth of (51) means that the criterion (30) is more restrictive than the criterion (31).

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A critical analysis of the eigenvalue method used to derive priorities in AHP

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Abstract

A lot of research has been devoted to the critical analysis of the Analytic Hierarchy Process (AHP), from various perspectives. However, as far as we know, no one has addressed a fundamental problem, discussed in this paper, concerning the meaning of the priority vector derived from the principal eigenvalue method used in AHP. The role of AHP's consistency ratio is also analysed.

Keywords: Decision analysis; Analytic Hierarchy Process; Eigenvalue method; Condition of order preservation

1. Introduction and objective of the analysis

Since Saaty (1977, 1980) introduced the *Analytic Hierarchy Process* (AHP), many applications in real-world decision-making have been reported (cf. Zahedi, 1986; Golden et al., 1989; Shim, 1989; Vargas, 1990; Saaty, 2000; Forman and Gass, 2001; Golden and Wasil, 2003; Vaidya and Kumar, 2006). In parallel, AHP has often been criticised in the literature, from several perspectives (see, for example, Watson and Freeling, 1982, 1983; Belton and Gear, 1983, 1985; French, 1988; Holder, 1990; Dyer, 1990a,b; Barzilai and Golany, 1994; Salo and Hämäläinen, 1997). A debate about the main criticisms of AHP can be found in Belton and Stewart (2002) and Smith and von Winterfeldt (2004). Saaty has frequently contested these critics (see, for example, Saaty et al., 1983; Saaty and Vargas, 1984; Saaty, 1990, 1997; Saaty and Hu, 1998) and, in essence, has not modified his original method (see Saaty, 2005). Independently of our agreement with some of those criticisms, the analysis of which is beyond the scope of this paper, we believe that the elicitation of pairwise comparison judgements and the possibility of expressing them verbally are cornerstones of the popularity of AHP.

There is, however, a key problem that, as far as we know, has never before been addressed in the literature. It concerns the meaning of the priority vector derived from the principal eigenvalue method used in AHP. The "AHP uses a principal eigenvalue

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method (EM) to derive priority vectors" (Saaty and Hu, 1998, p. 121). Following Saaty, the priority vector has two meanings: "The first is a numerical ranking of the alternatives that indicates an order of preference among them. The other is that the ordering should also reflect intensity or cardinal preference as indicated by the ratios of the numerical values (...)" (Saaty, 2003, p. 86). This second meaning requires, in our view, that these ratios preserve, whenever possible, the order of the respective preference intensities, which is not always the case for AHP priority vectors. Indeed, the ratios of AHP priority values can violate this order albeit the ratios of alternative priority values, derived from the same pairwise comparisons, preserve it. From our decision-aid perspective, this is a basic drawback of AHP. Consider the following condition:

Condition of Order Preservation (COP): For all alternatives x_1, x_2, x_3, x_4 such that x_1 dominates[2] x_2 and x_3 dominates x_4, if the evaluator's judgements indicate the extent to which x_1 dominates x_2 is greater that the extent to which x_3 dominates x_4, then the vector of priorities w should be such that, not only $w(x_1) > w(x_2)$ and $w(x_3) > w(x_4)$ (preservation of order of preference) but also that $w(x_1)/w(x_2) > w(x_3)/w(x_4)$ (preservation of order of intensity of preference).

For instance, if x_1 strongly dominates x_2 and x_3 moderately dominates x_4, it is from our view fundamental that, whenever possible, the vector of priorities w be such that $w(x_1)/w(x_2) > w(x_3)/w(x_4)$; indeed, these judgements indicate that the intensity of preference of x_1 over x_2 is higher than the intensity of preference of x_3 over x_4.

We will prove with simple examples that the AHP priority vector does not necessarily satisfy the COP, even though it is possible to respect this condition. In such cases, alternative priority values that satisfy COP can easily be found by a mathematical program including COP constrains. The particular program that we used is not important in the scope of this paper, since our intention is not at all to suggest an alternative procedure to AHP.

Note that a numerical scale that satisfies the COP does not always exist. In our constructive perspective, it is essential to detect these situations and discuss them with the evaluator before proposing any priority scale. A complementary objective of this paper is to analyse if the consistency ratio used in AHP can reveal such situations.

The rest of this paper is organised in the following manner: in Section 2, we review the principal eigenvalue method used in AHP to derive priority vectors; in Sections 3 and 4, we present some examples in which it would be possible to satisfy the COP, however, the AHP priority vectors violate it; in Section 5, we show that the AHP consistency ratio is not suitable for detecting the existence (or the non-existence) of a numerical scale satisfying the COP; a brief conclusion is presented in Section 6.

2. Overview of the principal eigenvalue method (EM)

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of elements and $\wp$ "a property or criterion that they have in common" (Saaty, 1996, p. 24) – for example, X could be a set of cars and $\wp$ their comfort. How can we help a person J quantify the relative priority (or importance) that the elements of X have for her, in terms of $\wp$?

The EM used in AHP to derive priorities for the elements of X requires that a number – denoted w_{ij} – be assigned to each pair of elements (x_i, x_j) representing, in the opinion of J, the ratio of the priority of the dominant element (x_i) relative to the priority of the dominated element (x_j) (Saaty, 1997). J is invited to compare the elements pairwise and can express her judgements in two different ways:

- either numerically, by giving a real number between 1 (inclusive) and 10 (exclusive) (Saaty, 1989) – for example, if x_i is a Chevrolet and x_j a Lada and if J judges the Chevrolet to be six times more comfortable than the Lada, than $w_{ij} = 6$;
- or verbally, by choosing one of the following expressions: equal importance, moderate dominance, strong dominance, very strong dominance, extreme dominance, or an intermediate judgement between two consecutive expressions; each verbal pairwise comparison elicited is then automatically converted into a number w_{ij} as exhibited in Table 1 – for example, if x_i is a Peugeot and x_j an Opel and if J judges the Peugeot to be moderately more comfortable than the Opel, then $w_{ij} = 3$.

During the elicitation process, a positive reciprocal matrix, in which each element $x_1, x_2, \ldots, x_n$ of X

[2] In this paper, "dominance" is used in the sense of "strict preference".

Table 1
Converting "verbal judgements" into "numbers"

Verbal expressions[a]	Corresponding numbers
Equal	1
Equal to moderate	2
Moderate	3
Moderate to strong	4
Strong	5
Strong to very strong	6
Very strong	7
Very strong to extreme	8
Extreme	9

[a] In Saaty (1996, 2005) the verbal expressions "equal to moderate", "moderate to strong", "strong to very strong" and "very strong to extreme" are replaced by "weak", "moderate plus", "strong plus" and "very, very strong", respectively.

is assigned one line and one column, can be filled by placing the corresponding number at the intersection of the line of x_i with the column of x_j

$$\begin{cases} w_{ij} & \text{if } x_i \text{ dominates } x_j, \\ 1/w_{ij} & \text{if } x_j \text{ dominates } x_i, \\ 1 & \text{if } x_i \text{ does not dominate } x_j \\ & \text{and } x_j \text{ does not dominate } x_i. \end{cases}$$

For example, assuming that for all $i,j \in \{1,2,\ldots,n\}$ x_i dominates x_j if and only if $i<j$, the format of the positive reciprocal matrix will be

$$\mathbf{W} = \begin{pmatrix} 1 & w_{12} & \cdots & w_{1n} \\ 1/w_{12} & 1 & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1/w_{1n} & 1/w_{2n} & \cdots & 1 \end{pmatrix}.$$

In order to assign a "priority" (or a "weight") to each element x_i – a numerical value that we will denote $w(x_i)$ – the principal eigenvalue $\lambda_{\max}$ of matrix $\mathbf{W}$ and its normalised eigenvector are calculated: the components of this vector are the $w(x_i)$. This procedure has a very interesting property: if the judgements of J are such that $w_{ij} \cdot w_{jk} = w_{ik}$ for all $i<j<k$ (cardinal consistency condition), the derived $w(x_i)$ are such that $w_{ij} = w(x_i)/w(x_j)$ for all $i<j$.

However, cardinal consistency is seldom observed in practice. Therefore, AHP makes use of a "consistency test" that prevents priorities from being accepted if the inconsistency level is high. In order to measure the deviation of matrix $\mathbf{W}$ from "consistency", a consistency index CI is defined as $\lambda_{\max} - n/(n-1)$ and a random index RI (of order n) is calculated as the average of the CI of many thousands reciprocal matrices (of order n) randomly generated from the scale 1 to 9, with reciprocals forced. The values of RI for matrices of size $1,2,\ldots,10$ can be found in Saaty (2005, p. 374). The ratio of CI to RI for the same order matrix is called the consistency ratio CR. According to Saaty (1980, p. 21), "a consistency ratio of 0.10 or less is considered acceptable". That is, an inconsistency is stated to be a matter of concern if CR exceeds 0.1, in which case the pairwise comparisons should be re-examined.

If the elements are to be compared according to several $\wp$, the AHP proposes that a hierarchy be built with the general goal on top, the elements at the bottom and the $\wp$ at intermediate levels. The procedure described above is then repeatedly applied bottom-up: to calculate a vector of priorities for the elements with respect to each $\wp$ situated at the bottom intermediate level; to calculate a vector of weights for each cluster of $\wp$ at the different levels. All this judgemental information is then synthesised from bottom to top by successive additive aggregations, in order to derive a vector of overall priorities for the elements.

3. Examples in which the COP is violated by the priority vector derived from the EM

We present in this section two examples proving that the COP may be violated by the priority vector given by the EM for each one of them, although scales exist that do respect it. Example 1 involves verbal judgements and Example 2 involves numerical judgements.

Example 1 (*Case of verbal judgements*). Let $X = \{x_1,x_2,x_3,x_4,x_5\}$ be a set of alternatives between which the following pairwise comparisons were formulated by a person J:

$\{x_1,x_2\}$: x_1 dominates x_2, equal to moderate dominance.
$\{x_1,x_3\}$: x_1 dominates x_3, moderate dominance.
$\{x_1,x_4\}$: x_1 dominates x_4, strong dominance.
$\{x_1,x_5\}$: x_1 dominates x_5, extreme dominance.
$\{x_2,x_3\}$: x_2 dominates x_3, equal to moderate dominance.
$\{x_2,x_4\}$: x_2 dominates x_4, moderate to strong dominance.
$\{x_2,x_5\}$: x_2 dominates x_5, extreme dominance.

$\{x_3, x_4\}$: x_3 dominates x_4, equal to moderate dominance.
$\{x_3, x_5\}$: x_3 dominates x_5, very strong to extreme dominance.
$\{x_4, x_5\}$: x_4 dominates x_5, very strong dominance.

From Table 1, the corresponding positive reciprocal matrix is

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 9 \\ 1/2 & 1 & 2 & 4 & 9 \\ 1/3 & 1/2 & 1 & 2 & 8 \\ 1/5 & 1/4 & 1/2 & 1 & 7 \\ 1/9 & 1/9 & 1/8 & 1/7 & 1 \end{pmatrix}$$

for which the normalised eigenvector corresponding to its principal eigenvalue is

$$\begin{pmatrix} 0.426 \\ 0.281 \\ 0.165 \\ 0.101 \\ 0.027 \end{pmatrix}.$$

Consequently, given the judgements of J, the priorities obtained through the EM are

$$w(x_1) = 0.426,$$
$$w(x_2) = 0.281,$$
$$w(x_3) = 0.165,$$
$$w(x_4) = 0.101,$$
$$w(x_5) = 0.027.$$

Then, in particular, $w(x_1)/w(x_4) \approx 4.218$ and $w(x_4)/w(x_5) \approx 3.741$, that is, $w(x_1)/w(x_4) > w(x_4)/w(x_5)$. Given that J judged that x_4 very strongly dominates x_5 and x_1 strongly dominates x_4, the priority vector given by the EM violates the COP. Yet, for example, the scale w^*

$$w^*(x_1) = 0.385,$$
$$w^*(x_2) = 0.275,$$
$$w^*(x_3) = 0.195,$$
$$w^*(x_4) = 0.125,$$
$$w^*(x_5) = 0.020,$$

respects the COP, as shown in Table 2. Let us also point out that the value of the consistency ratio for the judgements in Example 1 is 0.05, significantly smaller than the 0.10 threshold; therefore, in AHP's perspective the judgements need not be revised.

Example 2 (*Case of numerical judgements*). Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives between which the following pairwise comparisons were formulated by a person J:

$\{x_1, x_2\}$: x_1 dominates x_2 2.5 times.
$\{x_1, x_3\}$: x_1 dominates x_3 4 times.
$\{x_1, x_4\}$: x_1 dominates x_4 9.5 times.
$\{x_2, x_3\}$: x_2 dominates x_3 3 times.
$\{x_2, x_4\}$: x_2 dominates x_4 6.5 times.
$\{x_3, x_4\}$: x_3 dominates x_4 5 times.

The corresponding positive reciprocal matrix is

$$\begin{pmatrix} 1 & 2.5 & 4 & 9.5 \\ 1/2.5 & 1 & 3 & 6.5 \\ 1/4 & 1/3 & 1 & 5 \\ 1/9.5 & 1/6.5 & 1/5 & 1 \end{pmatrix}$$

for which the normalised eigenvector corresponding to its maximal eigenvalue is

$$\begin{pmatrix} 0.533 \\ 0.287 \\ 0.139 \\ 0.041 \end{pmatrix}.$$

Table 2
Example 1 – values of the ratios $w^*(x_i)/w^*(x_j)$

Possible verbal judgements	(x_i, x_j) pair(s) and respective $w^*(x_i)/w^*(x_j)$ ratios
Equal to moderate	(x_1, x_2): 1.40 (x_2, x_3): 1.41 (x_3, x_4): 1.56
Moderate	(x_1, x_3): 1.97
Moderate to strong	(x_2, x_4): 2.20
Strong	(x_1, x_4): 3.08
Strong to very strong	$\emptyset$
Very strong	(x_4, x_5): 6.25
Very strong to extreme	(x_3, x_5): 9.75
Extreme	(x_2, x_5): 13.75 (x_1, x_5): 19.25

Table 3
Example 2 – values of w_{ij} and $w(x_i)/w(x_j)$

	w_{ij}	$w(x_i)/w(x_j)$
$\{x_1, x_4\}$	9.5	13
$\{x_2, x_4\}$	6.5	7
$\{x_3, x_4\}$	5	3.39
$\{x_1, x_3\}$	4	3.83
$\{x_2, x_3\}$	3	2.06
$\{x_1, x_2\}$	2.5	1.86

Consequently, given the judgements of J, the priorities obtained through the EM are

$$w(x_1) = 0.533,$$
$$w(x_2) = 0.287,$$
$$w(x_3) = 0.139,$$
$$w(x_4) = 0.041.$$

For all $i,j \in \{1,2,3,4\}$ such that $i<j$, Table 3 presents the numerical value w_{ij} given by J when she judged how many times x_i dominates x_j, together with the respective value of the ratio $w(x_i)/w(x_j)$.

It is not surprising that the values of $w(x_i)/w(x_j)$ are not the same as the numerical judgements w_{ij} (because the latter are not cardinally consistent) but it is surprising to verify that their order is not preserved by the ratios. Indeed, $w_{34} > w_{13}$ but $w(x_3)/w(x_4) < w(x_1)/w(x_3)$. This proves that, again, the priority vector given by the EM violates the COP. Yet, for example, the scale w^*:

$$w^*(x_1) = 0.48,$$
$$w^*(x_2) = 0.32,$$
$$w^*(x_3) = 0.16,$$
$$w^*(x_4) = 0.04,$$

respects the COP. Indeed,

$$\frac{w^*(x_1)}{w^*(x_4)} = 12 > \frac{w^*(x_2)}{w^*(x_4)} = 8 > \frac{w^*(x_3)}{w^*(x_4)} = 4 > \frac{w^*(x_1)}{w^*(x_3)} = 3 > \frac{w^*(x_2)}{w^*(x_3)} = 2 > \frac{w^*(x_1)}{w^*(x_2)} = 1.5.$$

Moreover, the value of the consistency ratio for the judgements in Example 2 is 0.05, significantly smaller than the 0.10 threshold; therefore in AHP's perspective the judgements need not be revised.

4. Analysis of one of Saaty's examples

Example 3. In this section, we analyse the violation of the COP in one of the examples presented in Saaty (1977, pp. 254–256) and Saaty (1980, pp. 40–41) to empirically validate the EM. We refer to the example of pairwise comparisons of the GNP of several countries, in which, for a given matrix of verbal judgements, the priorities given by the AHP are remarkably close to the normalised GNP values. The countries are (Saaty's notation) "US, USSR, China, France, UK, Japan and W. Germany" and the matrix of judgements presented is

$$\begin{array}{c} \\ \text{US} \\ \text{USSR} \\ \text{China} \\ \text{France} \\ \text{UK} \\ \text{Japan} \\ \text{W. Germany} \end{array} \begin{pmatrix} \text{US} & \text{USSR} & \text{China} & \text{France} & \text{UK} & \text{Japan} & \text{W. Germany} \\ 1 & 4 & 9 & 6 & 6 & 5 & 5 \\ 1/4 & 1 & 7 & 5 & 5 & 3 & 4 \\ 1/9 & 1/7 & 1 & 1/5 & 1/5 & 1/7 & 1/5 \\ 1/6 & 1/5 & 5 & 1 & 1 & 1/3 & 1/3 \\ 1/6 & 1/5 & 5 & 1 & 1 & 1/3 & 1/3 \\ 1/5 & 1/3 & 7 & 3 & 3 & 1 & 2 \\ 1/5 & 1/4 & 5 & 3 & 3 & 1/2 & 1 \end{pmatrix}$$

The corresponding priorities are

$$w(\text{US}) = 0.427,$$
$$w(\text{USSR}) = 0.230,$$
$$w(\text{China}) = 0.021,$$
$$w(\text{France}) = 0.052,$$
$$w(\text{UK}) = 0.052,$$
$$w(\text{Japan}) = 0.123,$$
$$w(\text{W. Germany}) = 0.094.$$

These are the priorities appearing in Saaty (1980), which are a little different from those in Saaty (1977): 0.429, 0.231, 0.021, 0.053, 0.053, 0.119, and 0.095, respectively. Nevertheless, in both of these priority vectors the same five violations of the COP can be observed. We will analyse two of these hereafter.

(1) According to the matrix of judgements, US dominates USSR (4 times) more than Japan dominates France (3 times). But, $w(\text{US})/w(\text{USSR}) \approx 1.857$ and $w(\text{Japan})/w(\text{France}) \approx 2.365$, that is, $w(\text{US})/w(\text{USSR}) < w(\text{Japan})/w(\text{France})$.
(2) According to the matrix of judgements, Japan dominates China (7 times) more than US dominates UK (6 times). But, $w(\text{Japan})/w(\text{China}) \approx 5.857$ and $w(\text{US})/w(\text{UK}) \approx 8.212$, that is, $w(\text{Japan})/w(\text{China}) < w(\text{US})/w(\text{UK})$.

In spite of this, it is possible to avoid all of the violations of the COP, as for example with the following priority vector of priorities w^* (see Table 4):

Table 4
Verification of the COP

Possible verbal judgements	(x_i, x_j) pair(s) and respective $w^*(x_i)/w^*(x_j)$ ratios
Equal to moderate	(Japan, W. Germany): 1.23
Moderate	(W. Germany, France): 1.38 (W. Germany, UK): 1.38 (Japan, France): 1.70 (Japan, UK): 1.70 (USSR, Japan): 1.85
Moderate to strong	(US, USSR): 1.91 (USSR, W. Germany): 2.28
Strong	(USSR, France): 3.14 (USSR, UK): 3.14 (US, Japan): 3.54 (UK, China): 3.63 (France, China): 3.63 (US, W. Germany): 4.36
Strong to very strong	(US, France): 6.00 (US, UK): 6.00
Very strong	(Japan, China): 6.16 (USSR, China): 11.42
Very strong to extreme	$\varnothing$
Extreme	(US, China): 21.79

$w^*(\text{US}) = 0.414,$
$w^*(\text{USSR}) = 0.217,$
$w^*(\text{China}) = 0.019,$
$w^*(\text{France}) = 0.069,$
$w^*(\text{UK}) = 0.069,$
$w^*(\text{Japan}) = 0.117,$
$w^*(\text{W. Germany}) = 0.095.$

Let us also point out that the value of the consistency ratio for the judgements of this example is 0.08.

5. Discussion about the consistency ratio (CR)

Example 4. In this section, we present an example in which it is impossible to find a numerical scale satisfying the COP and analyse the value of the CR. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives between which the following pairwise comparison judgements were formulated by a person J:

$\{x_1, x_2\}$: x_1 dominates x_2, equal to moderate dominance.
$\{x_1, x_3\}$: x_1 dominates x_3, strong dominance.
$\{x_1, x_4\}$: x_1 dominates x_4, very strong dominance.
$\{x_1, x_5\}$: x_1 dominates x_5, extreme dominance.
$\{x_2, x_3\}$: x_2 dominates x_3, equal to moderate dominance.
$\{x_2, x_4\}$: x_2 dominates x_4, moderate dominance.
$\{x_2, x_5\}$: x_2 dominates x_5, very strong dominance.
$\{x_3, x_4\}$: x_3 dominates x_4, moderate dominance.
$\{x_3, x_5\}$: x_3 dominates x_5, strong dominance.
$\{x_4, x_5\}$: x_4 dominates x_5, equal to moderate dominance.

For this set of judgements, it is impossible to satisfy the COP. Indeed, one should simultaneously have:

(1) $w(x_1)/w(x_3) > w(x_2)/w(x_4)$, because, according to J's judgements, x_1 dominates x_3 (strong dominance) more than x_2 dominates x_4 (moderate dominance), and
(2) $w(x_3)/w(x_4) > w(x_1)/w(x_2)$, because, according to J's judgements, x_3 dominates x_4 (moderate dominance) more than x_1 dominates x_2 (equal to moderate dominance).

This is impossible because the product, member to member, of these two inequalities gives $w(x_1)/w(x_4) > w(x_1)/w(x_4)$.

In our view, this shows that we are in face of a real case of judgemental inconsistency because, contrary to Examples 1–3, the set of judgements in the present example is incompatible with a numerical representation that guarantees order preservation. And yet, the value of the CR corresponding to these judgements is very small (0.03), which means, in the AHP's perspective, that these judgements would not necessitate to be revised. Moreover, 0.03 is smaller than the values of the consistency ratios for Examples 1–3 (0.05, 0.05 and 0.08) in which, as shown in Sections 3 and 4, scales exist that satisfy the COP, unlike to the present example in which an inconsistency problem undoubtedly exists. This shows that the CR used in AHP is not suitable for detecting the existence (or the non-existence) of a numerical scale satisfying the COP.

6. Conclusion

In this article, we have addressed the foundations of AHP, by analysing the eigenvalue method (EM) used to derive a priority vector. Our main conclusion is that, although the EM is very elegant from a mathematical viewpoint, the priority vector derived from it can violate a condition of order preservation that, in our opinion, is fundamental in decision aiding – an activity in which it is essential to respect values and judgements. In light of that, and independently of all other criticisms presented in the literature, we consider that the EM has a serious fundamental weakness that makes the use of AHP as a decision support tool very problematic. As Saaty (2005, p. 346) points out, "the purpose of decision-making is to help people make decisions according to their own understanding", and "...methods offered to help make better decisions should be closer to being descriptive and considerably transparent".

Finally, it is worthwhile to note that the criticism of the EM, presented in this paper, is also valid for any other method that has been (or may be) conceived to derive a vector of priorities from a pairwise comparison matrix on the basis of a mathematical technique that does not integrate what we call the COP, or does not automatically guarantee its satisfaction.

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Consistency Measures for Pairwise Comparison Matrices

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ABSTRACT

We propose new measures of consistency of additive and multiplicative pairwise comparison matrices. These measures, the *relative consistency and relative error*, are easy to compute and have clear and simple algebraic and geometric meaning, interpretation and properties. The correspondence between these measures in the additive and multiplicative cases reflects the same correspondence which underpins the algebraic structure of the problem and relates naturally to the corresponding optimization models and axiom systems. The *relative consistency and relative error* are related to one another by the theorem of Pythagoras through the decomposition of comparison matrices into their consistent and error components. One of the conclusions of our analysis is that inconsistency is not a sufficient reason for revision of judgements. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: decision analysis; AHP; consistency; error measurement

1. INTRODUCTION

The analytic hierarchy process (AHP) is a widely used multicriteria decision analysis methodology (Saaty, 1980). In previous work on the mathematical foundations of the AHP (Barzilai *et al*., 1987; Barzilai *et al*., 1992; Barzilai and Golany, 1990, 1994; Barzilai, 1996, 1997) we studied the problem of deriving weight vectors from AHP pairwise comparison matrices. In this paper we study the related problem of measuring consistency of additive and multiplicative pairwise comparison matrices.

Typically, measures of consistency of pairwise comparison matrices are heuristics which are unrelated to the underlying mathematical structure of the problem, are defined in non-standard and non-intuitive ways, possess properties which are not fully understood and have no clear algebraic and geometric interpretation.

Saaty's (1980, p. 21) consistency index is defined in terms of the principal eigenvalue $\lambda_{\max}$ of A. Unfortunately, there is no sense in which $\lambda_{\max}(A_1) < \lambda_{\max}(A_2)$ corresponds to A_1 being more consistent than A_2 when both matrices are inconsistent, although this is exactly the relation such measures are supposed to convey. Furthermore, there is no explanation in the AHP literature of the meaning of comparisons involving the consistency index when comparing consistency of matrices of different dimensions, i.e. an explanation of the meaning of the relation $(\lambda_1 - n_1)/(n_1 - 1) < (\lambda_2 - n_2)/(n_2 - 1)$. Similar questions hold for the role of randomization in the construction of Saaty's consistency ratio and the meaning of the AHP 10% cut-off rule.

Golden and Wang (1989) consider Saaty's consistency ratio and the 10% cut-off rule 'somewhat arbitrary' and propose a measure based on the geometric mean. Their paper gives no reason for the use of the geometric mean and it is difficult to see what the properties of their measure are or how it relates to the structure of the problem. The papers by Islei and Lockett (1988) and Liang and Sheng (1990) are also relevant to our subject.

This paper proposes new measures of consistency, the *relative consistency and relative error* of pairwise comparison matrices. Section 2 provides the framework for our study by outlining the mathematical structure of the problem. Section 3 deals with optimization models and derived consistency measures. The *relative error* is defined and its basic properties are explored in Section 4. Section 5 deals with the decomposition of pairwise comparison matrices and the definition of the *relative consistency* measure. The results are extended to the multiplicative case in Section 6 and to hierarchies in Section 9. Numerical examples are given in Section 7 and Section 8 deals with the AHP measures. Conclusions from the analysis on the issue of revising judgements are drawn in Section 10 and the results are summarized in Section 11.

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2. STRUCTURE AND NOTATION

2.1. Conventions

Throughout this paper, all matrices are $n \times n$, vectors are n-dimensional and vector and matrix operations apply componentwise.

2.2. Definitions

1. $A = (a_{ij})$ is a pairwise multiplicative matrix if $0 < a_{ij} = 1/a_{ji}$.
2. $w = (w_k)$ is a *multiplicative weight vector* if $w_k > 0$ and $\Pi_{k=1}^n w_k = 1$.
3. $A = (a_{ij})$ is a *multiplicative consistent matrix* if $a_{ij} = w_i/w_j$ for some multiplicative weight vector w.
4. $A^\times$, $w^\times$ and $C^\times$ are the sets of all pairwise multiplicative matrices, multiplicative weight vectors and multiplicative consistent matrices respectively.
5. $f^\times$ is the set of all mappings from $A^\times$ to $w^\times$.
6. $A = (a_{ij})$ is a *pairwise additive matrix* if $a_{ij} = -a_{ji}$.
7. $w = (w_k)$ is an *additive weight vector* if $\Sigma_{k=1}^n w_k = 0$.
8. $A = (a_{ij})$ is an *additive consistent matrix* if $a_{ij} = w_i - w_j$ for some additive weight vector w.
9. A^+, w^+ and C^+ are the sets of all pairwise additive matrices, additive weight vectors and additive consistent matrices respectively.
10. f^+ is the set of all mappings from A^+ to w^+.

2.3. Structure

Proofs of the following observations are elementary and are omitted.

1. $A^\times$, $w^\times$ and $C^\times$ are all groups under componentwise multiplication; $C^\times$ is isomorphic to $w^\times$ and is a subgroup of $A^\times$.
2. A^+, w^+ and C^+ are all groups under componentwise addition; C^+ is isomorphic to w^+ and is a subgroup of A^+.
3. $A^\times$, $w^\times$ and $C^\times$ are isomorphic to A^+, w^+ and C^+ respectively. (The logarithmic function with any fixed basis applied componentwise is an isomorphism and the corresponding exponential function is its inverse.)

The normalization of weight vectors is necessary to ensure uniqueness. The normalization in the additive case is natural as the data are given in terms of differences, the fundamental nature of the problem is additive and the zero matrix and vector are the units of the groups A^+, C^+ and w^+. (See the discussion in Barzilai *et al.* (1987)). If w is a weight vector satisfying $\Sigma_{k=1}^n w_k = 0$, then $w' = w + \alpha/n$ satisfies $\Sigma_{k=1}^n w'_k = \alpha$. In particular, $w' = w + 1/n$ satisfies the common normalization $\Sigma_{k=1}^n w'_k = 1$.

3. OPTIMIZATION MODELS AND ERROR MEASUREMENT

We have established (Barzilai *et al.*, 1987; Barzilai and Golany, 1990, 1994; Barzilai, 1996, 1997) that the only acceptable solution for the additive problem of deriving weights from additive pairwise comparison matrices is the arithmetic mean mapping.

$$w_i = \frac{1}{n} \sum_{j=1}^{n} a_{ij}$$

The corresponding solution of the multiplicative problem is the geometric mean

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}$$

and the logarithmic/exponential isomorphisms of Section 2.3 link the two means:

$$\ln\left[\left(\prod_{j=1}^{n} e^{a_{ij}}\right)^{1/n}\right] = \frac{1}{n} \sum_{j=1}^{n} a_{ij}$$

We have also shown (Barzilai et al., 1992) that the arithmetic mean is the solution of the optimization problem

$$\begin{aligned} \min_{w} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} [a_{ij} - (w_i - w_j)]^2 \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i = 0 \end{aligned}$$

naturally associated with the additive problem of specifying $f \in f^+$ and that the geometric mean is the solution of the optimization problem

$$\begin{aligned} \min_{w} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} [\log a_{ij} - \log(w_i/w_j)]^2 \\ \text{s.t.} \quad & \prod_{i=1}^{n} w_i = 1, \quad w_i > 0, \quad i = 1, \ldots, n \end{aligned}$$

The group structures provided within the framework of Section 2 justify the association between the two optimization problems by means of the logarithmic isomorphism relating $A^\times$ and A^+.

The extremal values of the objective functions of the optimization problems provide measures of consistency in a natural way. In the additive case this measure is given by $\Sigma_{i,j=1}^{n} e_{ij}^2$, where $e_{ij} = a_{ij} - (w_i - w_j)$ are the error terms. (See Section 7 for numerical examples.) For the arithmetic mean solution the error terms satisfy

$$e_{ij} = a_{ij} - \frac{1}{n}\sum_{k=1}^{n}(a_{ik} - a_{jk}) = \frac{1}{n}\sum_{k=1}^{n}(a_{ij} + a_{jk} + a_{ki})$$

Since consistency means $a_{ij} + a_{jk} = a_{ik}$ or $a_{ij} + a_{jk} + a_{ki} = 0$, e_{ij} is the average inconsistency over all triplets with fixed i and j. Similarly, in the multiplicative case we have

$$e_{ij} = \left(\prod_{k=1}^{n} a_{ij}a_{jk}a_{ki}\right)^{1/n}.$$

The eigenvector solution is not applicable to the additive problem. Elegant as it is, the spectral analysis of positive matrices is not relevant to our decision analysis problem. While Saaty (1980, pp. 53, 223 and 234; 1983, pp. 248–250) does refer to difference and interval scales and to the arithmetic mean, the eigenvector solution limits the applicability of the AHP to positive matrices and hence to the multiplicative case. For example, Saaty (1983) proposes

$$\frac{1}{n^2}\sum_{i,j=1}^{n} a_{ij}$$

as a measure of inconsistency for additive matrices. There is no natural relationship between this measure and the eigenvalue-based multiplicative (AHP) consistency measure and this measure is meaningless since its value is zero for every matrix in A^+.

4. RELATIVE ERROR MEASURE

We first consider the additive case. Let $A = (a_{ij}) \in A^+$ and

$$w_i = \frac{1}{n}\sum_{j=1}^{n} a_{ij}$$

$$C = (c_{ij}) = (w_i - w_j)$$

$$E = (e_{ij}) = (a_{ij} - c_{ij})$$

Then $A = C + E$ is a decomposition of A into its consistent and error components (for reasons that will become clear later in this section, the error component will be called totally inconsistent). In addition, $m(A) = \Sigma_{ij} e_{ij}^2$ is a measure of the error or amount of inconsistency of A with at least the following properties.

1. $m(A)$ is a continuous function of A.
2. $m(A) = 0$ if and only if A is consistent.
3. $m(-A) = m(A)$
4. $m(A_1) > m(A_2)$ if and only if the related error terms satisfy $\Sigma_{ij} e_{ij,1}^2 > \Sigma_{ij} e_{ij,2}^2$.
5. m may be naturally extended to the multiplicative case by defining $m_1(A) = m(\log A)$ for $A \in A^\times$.

Property 1 is a reasonable requirement of any measure as it is difficult to interpret discontinuities in this context. Property 2 enables us to distinguish between consistent and inconsistent matrices but is insufficient to determine the degree of inconsistency of inconsistent matrices. Property 3 is related to the fundamental theory-of-measurement issue of independence of scale inversion (Barzilai, 1996, 1997). Some form of Property 4 must be satisfied by any meaningful measure and Property 5 is needed to link the parallel structures of the multiplicative and additive problems. Using the measure $m(A)$ as our starting point, we can improve on it by noting its deficiences.

- $m_1(A)$ depends on the logarithm base we use.
- A statement of the type 'A is less consistent than B if $m(A) > m(B)$' does not appear to be meaningful when A and B are of different dimensions.
- A cut-off rule of the type 'A is close enough to being consistent if $m(A) \leq \alpha$' for some fixed positive constant α, independent of n, does not appear to be meaningful.

A natural way to construct a measure that preserves the properties of $m(A)$ and addresses its deficiencies is to consider the *relative error* of A defined by $\mathrm{RE}(A) = 0$ if $A = 0$ and for $A \neq 0$ by

$$\mathrm{RE}(A) = \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2}$$

Note that $\mathrm{RE}(A) = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} e_{ij}^2 / \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} a_{ij}^2 = \Sigma_{i=1}^{n-1}\Sigma_{j=i+1}^{n} e_{ij}^2 / \Sigma_{i=1}^{n-1}\Sigma_{j=i+1}^{n} a_{ij}^2$ since A and E are antisymmetric. We will see that this measure and a related measure of *relative consistency* (formally defined in Section 5) are mappings onto the interval [0, 1] regardless of n, the size of A. Using these measures, we may compare consistency of matrices of different dimensions and justify the use of cut-off rules accepting A as sufficiently consistent if, for example, its relative error satisfies $\mathrm{RE}(A) \leq 0.10$ or, equivalently, if its relative consistency satisfies $\mathrm{RC}(A) \geq 0.90$. In addition, the extension of these measures to the multiplicative case fits the algebraic structure of the problem and is independent of logarithm bases. The following theorem establishes the range of $\mathrm{RE}(A)$.

Theorem 1
The relative error of any $A \in A^+$ satisfies $0 \leq \mathrm{RE}(A) \leq 1$.

Proof
$\mathrm{RE}(A)$ is zero when A is consistent and cannot be negative since it is the ratio of sums of squares. To prove that $\mathrm{RE}(A) \leq 1$, we need to show that $\Sigma_{ij} e_{ij}^2 \leq \Sigma_{ij} a_{ij}^2$. Since the error component minimizes the sum of squares of errors, we have

$$\sum_{ij} e_{ij}^2 = \sum_{ij} [a_{ij} - (w_i^* - w_j^*)]^2 \leq \sum_{ij} [a_{ij} - (w_i - w_j)]^2$$

where w^* is the arithmetic mean which minimizes the error and w is any weight vector. Substituting $w = 0$, we see that $\mathrm{RE}(A)$ is bounded from above by unity. □

A matrix A will be called totally inconsistent if its relative error or inconsistency is maximal, i.e. $\mathrm{RE}(A) = 1$. We will show in the next section that any matrix $A \in A^+$ can be uniquely decomposed into consistent and totally inconsistent components: $A = C + E$ with $\mathrm{RE}(C) = 0$ and $\mathrm{RE}(E) = 1$. It follows that each inconsistent $A \in A^+$ can be mapped into a totally inconsistent matrix through this decomposition.

An example of a totally inconsistent matrix is

$$X = \begin{pmatrix} 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 \end{pmatrix}$$

We will prove in Section 5 that $A \in A^+$ is totally inconsistent if and only if its row sums are all zero, but it is easy to be see by a direct computation that $\mathrm{RE}(X) = 1$. Reading the matrix X by rows, we see that it corresponds to ranking five objects A, B, C, D, E as follows:

D, E, A, B, C relative to A in row 1,
E, A, B, C, D relative to B in rwo 2,
A, B, C, D, E relative to C in row 3,
B, C, D, E, A relative to D in row 4,
C, D, E, A, B relative to E in row 5.

5. RELATIVE CONSISTENCY AND ORTHOGONAL DECOMPOSITION

Our derivation so far has been based on the classical idea of studying relative errors in an intuitive manner. We now prove a stronger result (Theorem 3 which implies Theorem 1) based on the algebraic properties of the decomposition defined in the previous section. Let $A = C + E$ be the decomposition of $A = (a_{ij}) \in A^+$ into its consistent and inconsistent components, i.e.

$$w_i = \frac{1}{n} \sum_{j=1}^{n} a_{ij}$$

$$C = (c_{ij}) = (w_i - w_j)$$

$$E = (e_{ij}) = (a_{ij} - c_{ij}).$$

We define the relative consistency of $A \neq 0$ as

$$\mathrm{RC}(A) = \frac{\sum_{ij} c_{ij}^2}{\sum_{ij} a_{ij}^2}$$

and $\mathrm{RC}(0) = 1$. The main result of this section is based on the classical *projection theorem* which we restate without proof (Gantmacher, 1959, Section 9.4; Luenberger, 1969, p. 51).

Theorem 2
Let A be an arbitrary vector in a Euclidean space R and let S be a subspace. Then A can be represented uniquely in the form $A = C + E$, where $C \in S$ and $E \perp S$. Furthermore, C and only C satisfies $\|A - C\| \leq \|A - X\|$ for all $X \in S$, where $\|A\|$ is the Euclidean norm of A.

We may rewrite any $n \times n$ matrix as an n^2-dimensional vector by ordering its elements by rows. In this n^2-dimensional Euclidean space the vectors corresponding to the consistent matrices in C^+ form a subspace. Combining the projection theorem with the minimization problem which characterizes the arithmetic mean, we see that the decomposition of A into its consistent and inconsistent components is an orthogonal decomposition, i.e. $C \perp E$ or $\Sigma_{ij} c_{ij} e_{ij} = 0$. Using this equation with $a_{ij} = c_{ij} + e_{ij}$, we obtain

$$\sum_{ij} a_{ij}^2 = \sum_{ij} c_{ij}^2 + 2\sum_{ij} c_{ij} e_{ij} + \sum_{ij} e_{ij}^2$$

and therefore

$$\sum_{ij} a_{ij}^2 = \sum_{ij} c_{ij}^2 + \sum_{ij} e_{ij}^2$$

(which is a restatement of the theorem of Pythagoras—see Gantmacher (1959, p. 244)). For $A \neq 0$ we then have

$$\frac{\sum_{ij} c_{ij}^2}{\sum_{ij} a_{ij}^2} + \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2} = 1$$

or

$$\mathrm{RC}(A) + \mathrm{RE}(A) = 1$$

which (regardless of the size n of the matrices) can be restated as $\cos^2(\alpha) + \sin^2(\alpha) = 1$, where α is the angle between the vectors A^* and C^* generated by rewriting the elements of the matrices A and C as n^2-dimensional vectors. (The analogy with measuring statistical coefficients of determination is clear.) Recalling that we defined $\mathrm{RC}(0) = 1$ and $\mathrm{RE}(0) = 0$, we have proved the following.

Theorem 3
For any $A \in A^+$, $\mathrm{RC}(A) + \mathrm{RE}(A) = 1$.

The following theorem provides an alternative characterization of totally inconsistent matrices.

Theorem 4
$0 \neq E \in A^+$ is totally inconsistent if and only if the row sums of E are all zero, i.e. $\Sigma_j e_{ij} = 0$ for all i.

Proof
If $w_i = (\Sigma_j e_{ij})/n = 0$, then $c_{ij} = w_i - w_j = 0$ and $e_{ij} = a_{ij}$ so that $\Sigma_{ij} e_{ij}^2 = \Sigma_{ij} a_{ij}^2$, implying $\mathrm{RE}(E) = 1$ and E is totally inconsistent.

Conversely, if E is totally inconsistent, then $\mathrm{RE}(E) = 1$ and, by Theorem 3, $\mathrm{RC}(E) = 0$, implying $\Sigma_{ij} c_{ij}^2 = 0$ and therefore $w_i = w_j$ for all i, j and $w_i = 0$ for all i since $\Sigma_i w_i = 0$. Finally, since $\Sigma_j e_{ij} = nw_i$ and $w_i = 0$, $\Sigma_j e_{ij} = 0$. □

The geometric interpretation of the projection theorem makes it clear that a non-zero matrix is totally inconsistent if and only if it is orthogonal to the subspace of all consistent matrices. The following theorem is the formal statement of this result.

Theorem 5
$0 \neq E \in A^+$ is totally inconsistent if and only if $E \perp C^+$, i.e. $E \perp C$ for all $C \in C^+$.

Proof
By Theorem 4, if $E = (e_{ij})$ is totally inconsistent, then $\Sigma_j e_{ij} = 0$ for all i and, since E is antisymmetric, $\Sigma_i e_{ij} = 0$ for all j as well. If C is a consistent matrix, then $C = (c_{ij}) = (w_i - w_j)$ for some weight vector w. To show that E is orthogonal to C, we calculate $\Sigma_{ij} (w_i - w_j) e_{ij} = \Sigma_{ij} w_i e_{ij} - \Sigma_{ij} w_j e_{ij} = 0$ since $\Sigma_{ij} w_i e_{ij} = \Sigma_i w_i \Sigma_j e_{ij} = 0$ and $\Sigma_{ij} w_j e_{ij} = \Sigma_j w_j \Sigma_i e_{ij} = 0$.

Conversely, if E is orthogonal to the subspace C^+, then $E \perp C$ for some $0 \neq C \in C^+$. Defining $A = C + E$, the projection theorem implies that $A = C + E$ is precisely the decomposition of A into its consistent and totally inconsistent components and therefore E is totally inconsistent. □

The following theorem is a corollary of the uniqueness—by the projection theorem—of the projection of A onto the subspace C^+.

Theorem 6
The decomposition of a consistent matrix $C \in C^+$ is given by $C = C + 0$. The decomposition of a totally inconsistent matrix E is given by $E = 0 + E$.

6. MULTIPLICATIVE CASE

The logarithmic isomorphism relating $A^\times$ and A^+ enables us to extend our results to the multiplicative case. Given $A \in A^\times$, define $L = (l_{ij})$ by $l_{ij} = \log_2 a_{ij}$ and define $\mathrm{RC}(A) = \mathrm{RC}(L)$ and $\mathrm{RE}(A) = \mathrm{RE}(L)$. This convenient notation is justified since $\mathrm{RC}(A)$ and $\mathrm{RE}(A)$ are independent of the logarithm base (any fixed base other than two may be used) and A cannot belong to both A^+ and $A^\times$. (In computing terms we may write a programme named RC which accepts an additive or multiplicative matrix A as its input and computes the relative consistency of this matrix. The programme first checks whether A is multiplicative by testing if $a_{11}=1$, in which case it replaces each a_{ij} with $\log_2 a_{ij}$. It then proceeds to compute the relative consistency of an additive matrix—either the input matrix or the one obtained by the above transformation from a multiplicative input matrix.)

The decomposition of $A=(a_{ij})\in A^{\times}$ into its consistent and inconsistent (or error) components is given by $A=C\times E$, where

$$w_i=\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}$$

$$C=(c_{ij})=(w_i\div w_j)$$

$$E=(e_{ij})=(a_{ij}\div c_{ij})$$

In analogy with the additive case, $A\in A^{\times}$ will be called totally inconsistent if its relative error or inconsistency is maximal, i.e. $\mathrm{RE}(A)=1$. The counterparts of Theorems 3 and 4 are now stated for the multiplicative case without proof.

Theorem 7
For any $A\in A^{\times}$, $\mathrm{RC}(A)+\mathrm{RE}(A)=1$.

Theorem 8
$1\neq E\in A^{\times}$ is totally inconsistent if and only if the row products of E are all ones, i.e. $\Pi_j e_{ij}=1$ for all i.

An example of a totally inconsistent multiplicative matrix is

$$Y=\begin{bmatrix} 1 & 2 & 4 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 & 4 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \\ 4 & \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ 2 & 4 & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

Note that $Y=(y_{ij})=(2^{x_{ij}})$, where $X=(x_{ij})$ is the totally inconsistent additive matrix defined in Section 4, $\mathrm{RE}(Y)=1$ and $\mathrm{RC}(Y)=0$.

The multiplicative decomposition is unique and is given for $C\in C^{\times}$ by $C=C\times 1$, while the decomposition of a multiplicative totally inconsistent matrix E is given by $E=1\times E$, where '1' is the symbol for the unit matrix with ones in all positions.

7. NUMERICAL EXAMPLES

Define the multiplicative matrix M and the additive matrix A as

$$M=\begin{pmatrix} 1 & 2 & \frac{1}{16} \\ \frac{1}{2} & 1 & 128 \\ 16 & \frac{1}{128} & 1 \end{pmatrix}$$

$$A=\begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

To compute the relative consistency of M, we first need to compute $A=(a_{ij})=(\log_2 m_{ij})$ as above. Note that any positive number rather than two can serve as the logarithm base since $\mathrm{RC}(kA)=\mathrm{RC}(A)$. Since A is antisymmetric, $\Sigma_{ij}a_{ij}^2=2(1^2+4^2+7^2)=132$. The row arithmetic mean vector for A is given by $w=(-1,\ 2,\ -1)$. Next we compute the consistent component of A.

$$C_A=(c_{ij})=(w_i-w_j)=\begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix}$$

We can now compute $\Sigma_{ij}c_{ij}^2=2(3^2+0^2+3^2)=36$ and

$$\mathrm{RC}(M)=\mathrm{RC}(A)=\frac{\sum_{ij} c_{ij}^2}{\sum_{ij} a_{ij}^2}=\frac{36}{132}=\frac{3}{11}=0.2727$$

indicating that the level of consistency of M and A is very low. The error component of A is computed by $E_A=A-C_A$ and the decomposition of A is given by

$$A=\begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}=C_A+E_A$$

$$=\begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 4 & -4 \\ -4 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix}$$

Note that in accordance with Theorem 4 the row sums of E_A are all zero. To demonstrate Theorem 3, note that $\Sigma_{ij} e_{ij}^2 = 2(3 \times 4^2) = 96$,

$$\mathrm{RE}(M) = \mathrm{RE}(A) = \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2} = \frac{96}{132} = \frac{8}{11} = 0.7273$$

indicating a high level of inconsistency, and $\mathrm{RC}(M) + \mathrm{RE}(M) = \mathrm{RC}(A) + \mathrm{RE}(A) = 1$. Finally, the decomposition of M can be computed from that of A:

$$M = \begin{pmatrix} 1 & 2 & \frac{1}{16} \\ \frac{1}{2} & 1 & 128 \\ 16 & \frac{1}{128} & 1 \end{pmatrix} = C_M \times E_M$$

$$= \begin{pmatrix} 1 & \frac{1}{8} & 1 \\ 8 & 1 & 8 \\ 1 & \frac{1}{8} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 16 & \frac{1}{16} \\ \frac{1}{16} & 1 & 16 \\ 16 & \frac{1}{16} & 1 \end{pmatrix}$$

8. AHP MEASURES

As mentioned above, in the additive case, Saaty (1983) proposes $(\Sigma_{ij} a_{ij})/n^2$ as a measure of consistency. This is not a meaningful measure since for any $A \in A^+$ the condition $a_{ij} = -a_{ji}$ implies $\Sigma_{ij} a_{ij} = 0$. Note also that the denominator of any reasonable variant of this measure should be $n(n-1)$ to reflect the constraint that the diagonal of matrices in the set A^+ is always zero, while the numerator should be an appropriate function of the errors, such as their absolute values, squares, etc.

In the multiplicative case, Saaty (1980, p. 21) defines the consistency index of A by

$$\mathrm{CI}(A) = \frac{\lambda_{\max} - n}{n - 1}$$

where $\lambda_{\max}$ is the principal eigenvalue of A and n is its size. He then uses an average random index $\mathrm{ARI}(n)$ (the average consistency index of a sample of randomly generated reciprocal matrices of size n) to define the consistency ratio of A by

$$\mathrm{CR}(A) = \frac{\mathrm{CI}(A)}{\mathrm{ARI}(n)}$$

Both $\mathrm{CI}(A)$ and $\mathrm{CR}(A)$ are heuristics with poorly understood properties and justification. AHP consistency analysis tends to centre on properties of the CI measure, although the CR measure is the one used almost exclusively in practice and the role of the randomization process used to derive the CR measure is obscure. The critical issue concerning these measures is that the statement that 'the closer $\lambda_{\max}$ is to n to the more consistent is the result' (Saaty, 1980, p. 21) is not justified anywhere in the AHP literature. Stated differently, it is clear, but of little value in measuring consistency, that $\mathrm{CI}(A)$ is zero if and only if A is consistent, but it is not clear at all in what sense $\mathrm{CI}(A_1) < \mathrm{CI}(A_2)$ corresponds to A_1 being more consistent that A_2 other than in circularly stating that the consistency index of A_1 is smaller than that of A_2.

9. CONSISTENCY OF A HIERARCHY

Saaty (1980, Section 4-5) proposes a measure of consistency for an entire hierarchy as follows.

> What we do is to multiply the index of consistency obtained from a pairwise comparison matrix by the priority of the property with respect to which the comparison is made and add all the results for the entire hierarchy. This is then compared with the corresponding index obtained by taking randomly generated indices, weighting them by the priorities and adding. The ratio should be in the neighborhood of 0.10 in order not to cause concern for improvements with the actual operation and in the judgments.

No basis is given for applying the operations of addition and multiplication of weights and consistency indexes of individual matrices for the purpose of this computation and the properties of such hierarchy consistency measures are unknown. Furthermore, similar aggregation rules for the computation of AHP weights have already been shown to be invalid (Barzilai and Golany, 1994). Note that an AHP hierarchy is a collection of multiplicative comparison matrices related to one another by their position in the hierarchy. Although the position of a given matrix in the hierarchy affects its contribution to the *overall weights*, the position in the hierarchy does not affect the contribution to the *overall inconsistency* of the hierarchy. In other words, two hierarchies with the same collection of matrices in different positions display the same level of inconsistency. Note also the discussion in the next section on the independence between consistency and acceptability of derived or projected weights.

Let A_i, $i \in S$, be a collection of either additive or multiplicative matrices. The natural way of measuring the relative consistency of this collection of matrices is to compute for each matrix A_i its consistent component C_i and produce the corresponding n_i^2-dimensional vectors A_i^* and C_i^*. (Note that it is sufficient to take the upper diagonal of these matrices as we do in the example below.) These vectors are then concatenated to produce the vectors $A^{**} = (A_1^*, A_2^*, A_3^*, \ldots)$ and $C^{**} = (C_1^*, C_2^*, C_3^*, \ldots)$. Finally, the relative consistency of the hierarchy RC(H) is given by

$$\mathrm{RC}(H) = \frac{\sum_i (c_i^{**})^2}{\sum_i (a_i^{**})^2}$$

Note that the vectors A_i^* and C_i^* may be concatenated in any order provided that the same order applies to both A^{**} and C^{**}. In the additive case the error vector may now be defined as $E^{**} = A^{**} - C^{**}$. The results of Sections 5 and 6 carry over and are summarized without proof in the following theorem which has an obvious multiplicative version.

Theorem 9
For an additive collection of matrices H, the vectors A^{**}, C^{**} and E^{**} satisfy $A^{**} = C^{**} + E^{**}$ and $C^{**} \perp E^{**}$. In addition, $\mathrm{RC}(H) + \mathrm{RE}(H) = 1$.

As an example, consider the following collection of two matrices:

$$A_1 = \begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{pmatrix} = C_{A_1} + E_{A_1}$$

$$= \begin{pmatrix} 0 & -3 & 0 \\ 3 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 & -4 \\ -4 & 0 & 4 \\ 4 & -4 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & -1 & 10 \\ 1 & 0 & -4 \\ 10 & 4 & 0 \end{pmatrix} = C_{A_2} + E_{A_2}$$

$$= \begin{pmatrix} 0 & 4 & 5 \\ -4 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -5 & 5 \\ 5 & 0 & -5 \\ -5 & 5 & 0 \end{pmatrix}$$

We have $A^{**} = (1, -4, 7, -1, 10, -4)$, $C^{**} = (-3, 0, 3, 4, 5, 1)$ and $E^{**} = (4, -4, 4, -5, 5, -5)$. The inner product of the vectors C^{**} and E^{**} is zero, verifying that they are indeed orthogonal, $\Sigma_i (a_i^{**})^2 = 183$, $\Sigma_i (c_i^{**})^2 = 60$, $\Sigma_i (e_i^{**})^2 = 123$, $\mathrm{RC}(H) = 60/183$ and $\mathrm{RE}(H) = 123/183$ with $\mathrm{RC}(H) + \mathrm{RE}(H) = 1$.

10. CONSISTENCY AND REVISION OF JUDGEMENTS

Section 3-5 in Saaty (1980) entitled 'Revising Judgments', opens with the following question.

> Assume that the consistency index is sufficiently large to warrant judgmental revision. When should it be made?

The underlying presumption here is that the projected weights should be accepted if the comparison matrix is sufficiently consistent and rejected otherwise. If the weights are rejected, the matrix is to be revised through an iterative procedure which converges to a consistent matrix.

The revision process leaves it unclear whether weights generated through this process are expected to be as close to the originally derived weights or as far away from these weights and on what basis. Another presumption that seems to underpin this proposal is that a convergent procedure is preferable to one that terminates in one step (i.e. replacing the input judgements with (w_i/w_j) which is consistent and does not involve a distortion of the decision maker's input judgements). To illustrate the difficulties with this proposal, consider the following matrices:

$$S_0 = \begin{pmatrix} 1 & 2.299 & 9.226 & 2.497 & 1.351 & 2.306 \\ 0.435 & 1 & 4.012 & 1.086 & 0.588 & 1.003 \\ 0.108 & 0.249 & 1 & 0.271 & 0.146 & 0.250 \\ 0.401 & 0.921 & 3.695 & 1 & 0.541 & 0.924 \\ 0.740 & 1.702 & 6.828 & 1.848 & 1 & 1.707 \\ 0.434 & 0.997 & 4.000 & 1.082 & 0.586 & 1 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 11.497 & 1.845 & 2.497 & 6.756 & 0.461 \\ 0.087 & 1 & 20.062 & 0.217 & 0.588 & 5.015 \\ 0.542 & 0.050 & 1 & 1.353 & 0.029 & 0.250 \\ 0.401 & 4.605 & 0.739 & 1 & 2.706 & 0.185 \\ 0.148 & 1.702 & 34.139 & 0.370 & 1 & 8.534 \\ 2.168 & 0.199 & 4.000 & 5.412 & 0.117 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 57.485 & 0.369 & 2.497 & 33.781 & 0.092 \\ 0.017 & 1 & 100.309 & 0.043 & 0.588 & 25.075 \\ 2.710 & 0.010 & 1 & 6.765 & 0.006 & 0.250 \\ 0.401 & 23.026 & 0.148 & 1 & 13.531 & 0.037 \\ 0.030 & 1.702 & 170.694 & 0.074 & 1 & 42.671 \\ 10.840 & 0.040 & 4.000 & 27.062 & 0.023 & 1 \end{pmatrix}$$

The matrix S_0 is consistent. For consistent matrices the (normalized) geometric mean and eigenvector are identical and these are given in this case by $gm = ev = (0.321,\ 0.140,\ 0.035,\ 0.128,\ 0.237,\ 0.139)$. In fact, to obtain the numbers in S_0, we computed in double precision (ev_i/ev_j), where the vector ev is the eigenvector solution for the School Example (Saaty, 1980, Table 1-2, p. 26).

The matrix S_1 is very inconsistent. Its principal eigenvalue, consistency index and consistency ratio are 12.40, 1.28 and 1.03 respectively. Its relative consistency and relative error are given by $\mathrm{RC}(S_1) = 0.36$ and $\mathrm{RE}(S_1) = 0.64$. For S_1 the geometric mean and eigenvector are again identical and are given as before by $gm = ev = (0.321,\ 0.140,\ 0.035,\ 0.128,\ 0.237,\ 0.139)$. The matrix S_1 was constructed as $S_1 = S_0 \times E_S$, where

$$E_S = \begin{pmatrix} 1 & 5 & 0.2 & 1 & 5 & 0.2 \\ 0.2 & 1 & 5 & 0.2 & 1 & 5 \\ 5 & 0.2 & 1 & 5 & 0.2 & 1 \\ 1 & 5 & 0.2 & 1 & 5 & 0.2 \\ 0.2 & 1 & 5 & 0.2 & 1 & 5 \\ 5 & 0.2 & 1 & 5 & 0.2 & 1 \end{pmatrix}$$

E_S is totally inconsistent since its row products are all ones. Recalling that S_0 is consistent, we see that $S_1 = S_0 \times E_S$ is the orthogonal decomposition of S_1, implying that S_1 and S_0 have the same consistent components and therefore the same geometric mean solution. The eigenvector solution is unchanged as well because the row sums of E_S are constant.

The matrix S_2 is more inconsistent that S_1. Its principal eigenvalue, consistency index and consistency ratio are 52.08, 9.22 and 7.43 respectively, $\mathrm{RC}(S_2) = 0.12$ and $\mathrm{RE}(S_2) = 0.88$. For S_2 the geometric mean and eigenvector are again identical and unchanged from the values above. The matrix S_2 was constructed as $S_2 = S_0 \times E_S^2$, where the squares in E_S^2 are computed componentwise.

In general, define $S_k = S_0 \times E_S^k$, where the powers in E_S^k are computed componentwise. The geometric mean and eigenvector of S_k for any real $k \geq 0$ are given by $gm = ev = (0.321,\ 0.140,\ 0.035,\ 0.128,\ 0.237,\ 0.139)$, while the inconsistency of S_k increases as k does ($\lambda \to \infty$ and $\mathrm{RC}(S_k) \to 0$). Since the projected weights of S_k for any $k \geq 0$ are constant, they are independent of its level of consistency. Clearly, the properties of this set of matrices have nothing to do with whether the weights are derived using the eigenvector or geometric mean and what consistency measures are used. The projected weights for these matrices are derived directly from the decision maker's undistorted input and are a true representation of the decision maker's preferences whether or not the decision maker is consistent. Regardless of the value of k, the decision analyst should remove whatever level of inconsistency present in the decision maker's input without injecting into it distorted or revised judgements—which is exactly what we do when we compute the projected weights.

Decomposing the input comparison matrix A into its components $A = C \times E$, we see that our aim is to reduce inconsistency by driving E as close as possible to '1'—the multiplicative group identity matrix. While revising E may appear desirable, distorting C in the process is not justified. However, if C is not to be distorted, no revision process is needed to determine the decision maker's preferences. We close this section by noting the last paragraph of Section 3-5 in Saaty (1980), which states the following.

> We caution against excessive use of this process of forcing the values of judgments to improve consistency. It distorts the answer. One would rather have naturally improved judgments arising from experience.

Forcing the values of judgements to improve consistency distorts the answer regardless of the level of consistency of A. In our opinion the projected weights should be presented to the decision maker as feedback from the analysis. If the decision maker can confirm that the matrix C is indeed an acceptable reflection of his/her preferences, no revision of judgements is necessary regardless of his/her level of consistency. Otherwise, a revision of the judgements is justified.

11. CONCLUSIONS

Based on our analysis of the mathematical foundations of the AHP, we constructed simple measures of consistency of additive and multiplicative pairwise comparison matrices. These measures, the *relative consistency* and *relative error*, are extended in a natural way to hierarchies or arbitrary collections of comparison matrices. They are derived from and fit the algebraic structure of the problem, are easy to compute and have clear and simple algebraic and geometric meaning, interpretation and properties. The correspondence between these measures in the additive and multiplicative cases reflects the same correspondence which underpins the algebraic structure of the problem and relates naturally to the corresponding optimization models and axiom systems.

The *relative consistency* and *relative error* are related to one another by the theorem of Pythagoras through the decomposition of comparison matrices into their consistent and error components. Since these consistency measures map comparison matrices of any size onto the common scale [0, 1], they make it possible to compare consistency of comparison matrices and collections of comparison matrices of diverse dimensions.

It is important to re-emphasize that for the *relative consistency* measure the fundamental comparison '$\mathrm{RC}(A_1) > \mathrm{RC}(A_2)$ if and only if A_1 is more consistent than A_2' and cut-off rules of the type 'A is sufficiently consistent if $\mathrm{RC}(A) \geq 0.90$' are meaningful and easy to understand and interpret. Finally, the insight gained from our analysis leads us to conclude that, by itself, inconsistency is not a sufficient reason to require the decision maker to revise his/her judgements.

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ORIGINAL PAPER

Studying a set of properties of inconsistency indices for pairwise comparisons

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Abstract Pairwise comparisons between alternatives are a well-established tool to decompose decision problems into smaller and more easily tractable sub-problems. However, due to our limited rationality, the subjective preferences expressed by decision makers over pairs of alternatives can hardly ever be consistent. Therefore, several inconsistency indices have been proposed in the literature to quantify the extent of the deviation from complete consistency. Only recently, a set of properties has been proposed to define a family of functions representing inconsistency indices. The scope of this paper is twofold. Firstly, it expands the set of properties by adding and justifying a new one. Secondly, it continues the study of inconsistency indices to check whether or not they satisfy the above mentioned properties. Out of the four indices considered in this paper, in their present form, two fail to satisfy some properties. An adjusted version of one index is proposed so that it fulfills them.

Keywords Pairwise comparisons · Consistency · Inconsistency indices · Analytic hierarchy process

1 Introduction

In decision making problems it is often common practice to use pairwise comparisons between alternatives as a basis to assign scores to the same alternatives. Pairwise comparisons allow the decision maker to decompose the problem of assigning scores to alternatives into smaller problems, where only two alternatives are considered at a time.

Another reason for using pairwise comparisons is that their use allows an estimation of the inconsistency of the preferences of a decision maker. In the literature, consistency of preferences is commonly related with the rationality of a decision maker and his ability in discriminating between alternatives (Irwin 1958). Consider, for sake of illustration, three

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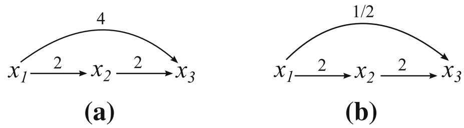


Fig. 1 Example of consistent and inconsistent triads of pairwise comparisons on x_1, x_2, x_3. **a** Consistent triad of pairwise comparisons and **b** inconsistent triad of pairwise comparisons

stones (alternatives) x_1, x_2, x_3. If, for instance, x_1 is reputed twice as heavy as x_2, and x_2 twice as heavy as x_3, then it is reasonable to assume that x_1 should be four times as heavy as x_3. This situation is called consistent, as the pairwise comparisons of the decision maker respect a principle of transitivity/rationality, and is depicted in Fig. 1a. An example of inconsistent pairwise comparisons is illustrated in Fig. 1b.

There is a meeting of minds on accepting preferences which are not consistent, but not too inconsistent either. In this paper, with the term *inconsistency* we mean a deviation from the condition of full consistency. In the theory of the AHP, Saaty (1993, 2013) required pairwise comparisons to be near consistent, i.e. not too inconsistent. As recalled by Gass (2005), Luce and Raiffa (1957) shared the same opinion in accepting inconsistencies and wrote "No matter how intransitivities arise, we must recognize that they exist, and we can take a little comfort in the thought that they are an anathema to most of what constitutes theory in the behavioral sciences today". On a similar note, Fishburn (1999) wrote that "Transitivity is obviously a great practical convenience and a nice thing to have for mathematical purposes, but long ago this author ceased to understand why it should be a cornerstone of normative decision theory".

It is in this context—where consistency is an auspicable but hardly ever achievable condition—that it becomes crucial to quantify inconsistency. Such quantification is indeed possible, since it is natural to envision that the notion of inconsistency is a matter or degree. Consequently, a wealth of inconsistency indices has been proposed in the literature; for instance the Consistency Index (Saaty 2013), the Harmonic Consistency Index (Stein and Mizzi 2007), the Geometric Consistency Index (Aguarón and Moreno-Jiménez 2003), the statistical index by Lin et al. (2013), and the index by Kułakowski (2015), just to cite few.

It is worth noting that the study of inconsistency of preferences is not limited to the single mathematical methods employing pairwise comparisons, as for instance the AHP. It is the case to remark that the study of inconsistency is immune from many of the criticisms moved against specific mathematical methods employing them. For instance, one of the critical points of the Analytic Hierarchy Process (AHP) is the rank reversal, which was discovered by Belton and Gear (1983) and recently surveyed by Maleki and Zahir (2013). Similarly, already Watson and Freeling (1982, 1983) questioned the interpretation of the weights in the AHP and their use in the aggregation of different priority vectors. In part, also the criticisms by Dyer (1990a, b) were triggered by the interpretation of the weights. Nevertheless, even though the above mentioned criticisms are to be taken into account, they are connected with the aggregation and interpretation of priority vectors proposed for the AHP, and therefore they will not affect the subject matter of inconsistency evaluation. Further support to the use of pairwise comparison matrices and their interpretation comes from the fact that pairwise comparison matrices as defined in this paper are group isomorphic (Cavallo and D'Apuzzo 2009)—and thus structurally identical—to the probabilistic preference relations studied by Luce and Suppes (1965). Such a strict connection between these two representations of valued preferences does not only make them mutually supportive, but increases the relevance of studying one of them—as it is going to be done in this paper—since abstract results are then extendible to the other one.

The use of the notion of inconsistency has gone beyond its mere quantification. One prominent use of inconsistency indices is that of localizing the inconsistency and detect what comparisons are the most contradictory (Ergu et al. 2011) and guide the decision maker when he tries to obtain sufficiently consistent preferences (Pereira and Costa 2015). This process was also advocated by Fishburn (1968) in a discussion on decision theory: "If the individual's preferences appear to violate a "rational" preference assumption, the theory suggests that he reexamine and revise one or more preference judgments to eliminate the inconsistency.". Another use of inconsistency indices regards pairwise comparison matrices with missing entries. In these situations, inconsistency indices have been used as objective functions to be minimized to find the most plausible values of the missing comparisons with respect to the elicited ones (Koczkodaj et al. 1999; Lamata and Pelaez 2002; Shiraishi et al. 1999; Chen et al. 2015). All this can be seen as evidence on the role played by inconsistency indices in the decision process, and consequently on the importance of having realiable indices.

Inconsistency of preferences has been studied empirically (Bozóki et al. 2013), and existing studies on inconsistency indices compared them numerically (Brunelli et al. 2013a) and showed that some indices are very different and therefore can lead to very different evaluations of the inconsistency of preferences. Conversely, it was proven that some of them are in fact proportional to each other (Brunelli et al. 2013b). Recently, Brunelli and Fedrizzi (2015a) and Koczkodaj and Szwarc (2014) proposed two formal approaches. Brunelli and Fedrizzi (2015a) proposed five properties in the form of axioms to formalize the concept of inconsistency index and then tested on some well-known indices.

In the pursuit of a formal treatment of inconsistency quantification, this paper presents some developments concerning the aforementioned set of properties. Firstly, in Sect. 3, a new property, of invariance under inversion of preferences, is introduced and its role is discussed. Secondly, Sect. 4 contains further results on the satisfaction of the properties by some known inconsistency indices. More specifically, we shall study four indices and discover that, in its present form, two do not fully satisfy the set of properties. An adjustment of one index is then proposed so that it satisfies them. Finally, Sect. 5 offers a concise discussion on the role of inconsistency quantification and on the results obtained in this paper.

2 Pairwise comparison matrices and inconsistency indices

Given a set $X = \{x_1, \dots, x_n\}$ of n alternatives, a *pairwise comparison matrix* is a positive square matrix $\mathbf{A} = (a_{ij})_{n \times n}$ such that $a_{ij}a_{ji} = 1$, where $a_{ij} > 0$ is the subjective assessment of the relative importance of the ith alternative with respect to the jth one. A pairwise comparison matrix can be seen as a convenient mathematical structure into which valued pairwise comparisons between alternatives are collected. Its general and its simplified (thanks to $a_{ij}a_{ji} = 1$) forms are the following,

$$\mathbf{A} = (a_{ij})_{n \times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \dots & 1 \end{pmatrix}.$$

The rest of the paper will follow the usual interpretation of entries a_{ij} in terms of ratios between quantities expressible on a ratio scale with a zero element. The classical example is that of x_1 and x_2 being stones and a_{ij} being the numerical estimation of the ratio between

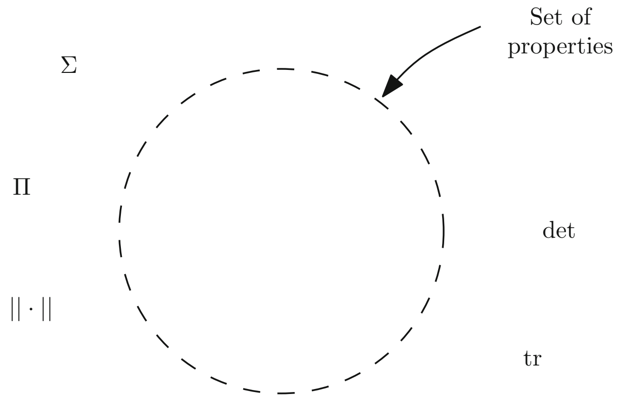


Fig. 2 The set of properties can be used to define a family of functions which can be used to estimate inconsistency, and discards functions which do not make sense if used as inconsistency indices, e.g. the trace and the determinant of **A**

their weights. Note that this approach considers entries $a_{ij} > 0$ taking values from an unbounded scale and complies with the formal treatment given by Herman and Koczkodaj (1996), Koczkodaj and Szwarc (2014). Furthermore, with this interpretation, a pairwise comparison matrix is *consistent* if and only if

$$a_{ik} = a_{ij}a_{jk} \quad \forall i, j, k, \tag{1}$$

which means that each direct comparison a_{ik} is exactly backed up by all indirect comparisons $a_{ij}a_{jk}\ \forall j$. For notational convenience, the set of all pairwise comparison matrices is defined as

$$\mathcal{A} = \left\{\mathbf{A} = (a_{ij})_{n\times n} | a_{ij} > 0, a_{ij}a_{ji} = 1 \quad \forall i,\ j, \quad n > 2\right\}.$$

The set of all *consistent* pairwise comparison matrices $\mathcal{A}^* \subset \mathcal{A}$ is defined accordingly,

$$\mathcal{A}^* = \{\mathbf{A} = (a_{ij})_{n\times n} | \mathbf{A} \in \mathcal{A}, a_{ik} = a_{ij}a_{jk} \quad \forall i, j, k\}.$$

An inconsistency index is a function $I : \mathcal{A} \to \mathbb{R}$ which evaluates the intensity of deviation of a pairwise comparison matrix **A** from its consistent form (1). In other words, the value $I(\mathbf{A})$ is an estimation of how much irrational the preferences collected in **A** are. Up to now, various inconsistency indices have been introduced heuristically, and an open question relates to what set of properties should be used to characterize them. That is, all the reasonable properties for a function I to fairly capture inconsistency could be used for various purposes; for example to check the validity of already proposed indices (Brunelli and Fedrizzi 2015a), devise new ones, and derive further properties (Brunelli and Fedrizzi 2015b). Figure 2 offers a snapshot of the meaning of the set of properties.

Brunelli and Fedrizzi (2015a) proposed five properties to characterize inconsistency indices. Since these properties were already justified and defined in the original work, they are here only briefly recalled. Note that they were organized in the form of an axiomatic systems, meaning that the soundness of single properties implies the soundness of the entire set of properties, i.e. the "logical intersection" of the properties.

P1 There exists a unique $\nu \in \mathbb{R}$ representing the situation of full consistency, i.e.

$$\exists! \nu \in \mathbb{R} \text{ such that } I(\mathbf{A}) = \nu \Leftrightarrow \mathbf{A} \in \mathcal{A}^*.$$

P2 Changing the order of the alternatives does not affect the inconsistency of preferences. That is,

$$I(\mathbf{PAP}^T) = I(\mathbf{A}),$$

for any permutation matrix $\mathbf{P}$.

P3 If preferences in $\mathbf{A}$ are intensified, then the inconsistency cannot decrease. More formally, since the power is the only meaningful function to intensify preferences, we defined $\mathbf{A}(b) = \left(a_{ij}^b\right)_{n\times n}$. Then, the property is as follows,

$$I(\mathbf{A}(b)) \geq I(\mathbf{A}) \quad \forall \mathbf{A} \in \mathcal{A}, \quad b \geq 1.$$

P4 Given a consistent pairwise comparison matrix and considering an arbitrary non-diagonal element a_{pq} (and its reciprocal a_{qp}) such that $a_{pq} \neq 1$, then, as we push its value far from its original one, the inconsistency of the matrix should not decrease. More formally, given a consistent matrix $\mathbf{A} \in \mathcal{A}^*$, let $\mathbf{A}_{pq}(\delta)$ be the inconsistent matrix obtained from $\mathbf{A}$ by replacing the entry a_{pq} with a_{pq}^δ, where $\delta \neq 1$. Necessarily, a_{qp} must be replaced by a_{qp}^δ in order to preserve reciprocity. Let $\mathbf{A}_{pq}(\delta')$ be the inconsistent matrix obtained from $\mathbf{A}$ by replacing entries a_{pq} and a_{qp} with $a_{pq}^{\delta'}$ and $a_{qp}^{\delta'}$ respectively. The property can then be formulated as

$$\begin{aligned} \delta' > \delta > 1 &\Rightarrow I(\mathbf{A}_{pq}(\delta')) \geq I(\mathbf{A}_{pq}(\delta)) \\ \delta' < \delta < 1 &\Rightarrow I(\mathbf{A}_{pq}(\delta')) \geq I(\mathbf{A}_{pq}(\delta)), \end{aligned} \tag{2}$$

for all $\delta \neq 1$, $p, q = 1, \ldots, n$, and $\mathbf{A} \in \mathcal{A}^*$.

P5 Function I is continuous with respect to the entries of $\mathbf{A}$.

3 A new property of invariance under inversion of preferences

Preferences expressed in the form of a pairwise comparison matrix $\mathbf{A}$ can be inverted by taking its transpose $\mathbf{A}^T$. For instance, if $a_{ij} = 2$ in $\mathbf{A}$ is inverted into $a_{ij} = 1/2$ we have that the intensity of preference is the same, but the direction is inverted. Clearly, by inverting all the preferences we change their polarity, but leave their structure unchanged. Thus, it is reasonable to expect a structural property of preferences—as inconsistency is—to be invariant under inversion. This can be formalized in the following property of invariance under inversion of preferences (P6).

Property 6 (P6) *An inconsistency index satisfies P6, if and only if* $I(\mathbf{A}) = I(\mathbf{A}^T)\ \forall \mathbf{A} \in \mathcal{A}$.

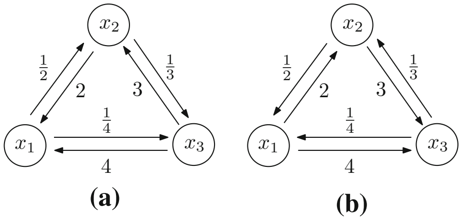
The previous justification of this property can be transposed into an example. Consider the following matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/3 \\ 4 & 3 & 1 \end{pmatrix} \quad \mathbf{A}^T = \begin{pmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 3 \\ 1/4 & 1/3 & 1 \end{pmatrix} \tag{3}$$

One can equivalently express the structure of the preferences by means of directed weighted graphs with nodes x_i and values of the edges a_{ij}. Figure 3 represents these graphs for $\mathbf{A}$ and $\mathbf{A}^T$, respectively.

The two graphs are identical, with the only exception of the directions of the arrows. Now, if we do not impose P6, we might end up with inconsistency indices which consider the violation of the condition of consistency in the direction $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ more or less (but

Fig. 3 Graphs of $\mathbf{A}$ and $\mathbf{A}^T$. **a** Graph of $\mathbf{A}$ and **b** Graph of $\mathbf{A}^T$



not equally) important than the violation in the direction $1 \to 3 \to 2 \to 1$, although both directions equivalently reflect the same structure of preferences.

The justification of P6 comes from theoretical intuition, but in some decision processes both a matrix $\mathbf{A}$ ad its transpose $\mathbf{A}^T$ can actually appear. Examples are applications in group decision making where pairwise comparison matrices are used in surveys on customers' needs and users' satisfactions, as for instance done by Nikou and Mezei (2013), Nikou et al. (2015). In these contexts there are as many pairwise comparison matrices as responding customers (usually a large number) and therefore it is not completely unlikely to find both preferences represented by $\mathbf{A}$ and $\mathbf{A}^T$, especially when there are few alternatives and intensities of preference are weak. i.e. values of entries are close to 1.

Note that, in general, an inversion of preferences cannot be obtained by row-column permutations. For example, given the matrices $\mathbf{A}$ and $\mathbf{A}^T$ in (3), there does *not* exist a permutation matrix $\mathbf{P}$ such that $\mathbf{PAP}^T = \mathbf{A}^T$.

One natural question is whether or not this new property, P6, is implied by a conjoint application of the others (independence) and if, when added to the set P1–P5, does not make it contradictory (logical consistency). The following theorem claims the independence and the logical consistency of the properties P1–P6.

Theorem 1 *Properties P1–P6 are independent and form a logically consistent axiomatic system.*

Proof See "Appendix". □

In light of the previously offered justification and Theorem 1, one concludes that P6 is another interesting property of inconsistency indices, and is independent from P1–P5.

4 Extending the analysis of the satisfaction of the axioms

Previous research (Brunelli and Fedrizzi 2015a; Cavallo and D'Apuzzo 2012) has made the effort of proving whether or not some known inconsistency indices satisfy the set of properties. This section continues the investigation on the satisfaction of the set of properties by testing four indices proposed in the literature and used in real-world decision making problems. For each index we shall recall the definition and highlight its relevance in both theory and practice.

4.1 Index K by Koczkodaj

The following index, K, was introduced by Koczkodaj (1993) and extended by Duszak and Koczkodaj (1994).

Definition 1 (*Index K* (Duszak and Koczkodaj 1994)) Given a pairwise comparison matrix **A**, the index K is

$$K(\mathbf{A}) = \max\left\{\min\left\{\left|1 - \frac{a_{ik}}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}}\right|\right\} : 1 \le i < j < k \le n\right\}. \quad (4)$$

This index has been used to estimate missing entries of incomplete pairwise comparisons (Koczkodaj et al. 1999) and in real-world applications in problems such as the evaluation of research institutions in Poland (Koczkodaj et al. 2014) and medical diagnosis (Kakiashvili et al. 2012). It was also compared to Saaty's Consistency Index (Bozóki and Rapcsák 2008) and on occasions even claimed superior to it (Koczkodaj and Szwarc 2014). Given its theoretical and practical relevance, it is therefore important to check what properties it satisfies. Here we show that index K satisfies the six properties P1–P6.

Proposition 1 *Index K satisfies the properties P1–P6.*

Proof It is straightforward, and thus omitted, to show that properties P1, P2, P5, and P6 are satisfied. For P3 we need to show that the local inconsistency for the generic transitivity (i, j, k),

$$\min\left\{\left|1 - \frac{a_{ik}^b}{a_{ij}^b a_{jk}^b}\right|, \left|1 - \frac{a_{ij}^b a_{jk}^b}{a_{ik}^b}\right|\right\}, \quad (5)$$

is non-decreasing for $b \ge 1$. We can do it by proving that $\frac{\partial K}{\partial b} \ge 0\ \forall b > 1$. With $x^b := \frac{a_{ik}^b}{a_{ij}^b a_{jk}^b}$, we study the two quantities

$$\mathrm{I} = |1 - x^b| \quad \mathrm{II} = |1 - x^{-b}|.$$

If the triple (i, j, k) is consistent, then $x = 1$ and P3 is satisfied. If the triple (i, j, k) is not consistent, then $x \ne 1$ and positive, and the derivatives of I and II in b are:

$$\frac{\partial \mathrm{I}}{\partial b} = -x^b \log(x) \mathrm{sgn}\left(1 - x^b\right)$$
$$\frac{\partial \mathrm{II}}{\partial b} = x^{-b} \log(x) \mathrm{sgn}\left(1 - x^{-b}\right).$$

Given $b \ge 1$, if $x \ne 1$, then $\frac{\partial \mathrm{I}}{\partial b}$ and $\frac{\partial \mathrm{II}}{\partial b}$ are positive, which proves that (5) is a non-decreasing function for $b \ge 1$. It follows that also K is a non-decreasing function of $b \ge 1$.

To prove the satisfaction of P4 we start considering

$$\min\left\{\left|1 - \frac{a_{ik}^\delta}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}^\delta}\right|\right\} \quad (6)$$

with $a_{ik} = a_{ij}a_{jk}$. By setting $y = a_{ik} = a_{ij}a_{jk}$ we can rewrite it as

$$\min\left\{\left|1 - y^{\delta-1}\right|, \left|1 - y^{1-\delta}\right|\right\}$$

and show that it is a non-decreasing function for $b \ge 1$ and a non-increasing function for $0 < b \le 1$. We then need to study the following quantities:

$$\mathrm{I} = |1 - y^{\delta-1}| \quad \mathrm{II} = |1 - y^{1-\delta}|$$

and their derivatives in δ

$$\frac{\partial \mathrm{I}}{\partial \delta} = -y^{\delta-1} \log(y) \mathrm{sgn}\left(1 - y^{\delta-1}\right) \quad \frac{\partial \mathrm{II}}{\partial \delta} = y^{1-\delta} \log(y) \mathrm{sgn}\left(1 - y^{1-\delta}\right).$$

By studying their sign we can derive that

$$0<\delta<1 \Rightarrow \frac{\partial \mathrm{I}}{\partial \delta}, \frac{\partial \mathrm{II}}{\partial \delta} \leq 0 \Rightarrow \frac{\partial K}{\partial \delta}<0$$
$$\delta>1 \Rightarrow \frac{\partial \mathrm{I}}{\partial \delta}, \frac{\partial \mathrm{II}}{\partial \delta} \geq 0 \Rightarrow \frac{\partial K}{\partial \delta}>0.$$

Similarly, P4 can be proven also in the case when the exponent δ is at the denominator of $\frac{a_{ik}}{a_{ij}a_{jk}}$ in (6). □

4.2 Index AI by Salo and Hämäläinen

Salo and Hämäläinen (1995, 1997) proposed their inconsistency index, AI, which stands for ambiguity index. Their inconsistency index has been implemented in the online decision making platform Web-HIPRE (Mustajoki and Hämäläinen 2000) and has been used, for instance, in the analysis of a real-world governmental decision on energy production alternatives (Salo and Hämäläinen 1995) and in traffic planning (Hämäläinen and Pöyhönen 1996).

Definition 2 Given a pairwise comparison matrix $\mathbf{A}$ and an auxiliary matrix $\mathbf{R}=(r_{ij})_{n\times n}$ with $r_{ij}=\{a_{ik}a_{kj}|k=1,\ldots,n\}$, then the index AI is

$$AI(\mathbf{A})=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\max(r_{ij})-\min(r_{ij})}{(1+\max(r_{ij}))(1+\min(r_{ij}))}. \tag{7}$$

The interpretation of AI is original and different from those of other indices. Consider that r_{ij} is *not* a real number but, instead, the set of possible values of a_{ij} as could be deduced from indirect comparisons $a_{ik}a_{kj}$ $\forall k$.
For example, given the matrix

$$\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 & 1/2 \\ 1/2 & 1 & 4 & 1/3 \\ 1/3 & 1/4 & 1 & 2 \\ 2 & 3 & 1/2 & 1 \end{pmatrix},$$

we have

$$r_{14}=\{a_{11}a_{14},a_{12}a_{24},a_{13}a_{34},a_{14}a_{44}\}=\left\{\frac{1}{2},\frac{2}{3},6\right\},$$

from which we obtain $\max(r_{14})=6$ and $\min(r_{14})=1/2$.

It is possible to build an interval-valued matrix

$$\bar{\mathbf{A}}=\left(\bar{a}_{ij}\right)_{n\times n}=\left([\min(r_{ij}),\max(r_{ij})]\right)_{n\times n}$$

such that the 'true value' of the comparison between x_i and x_j shall lie in the interval $\bar{a}_{ij}$. The larger the intervals are, the more inconsistent the matrix, and in fact AI is a normalized sum of the lengths of the intervals $\bar{a}_{ij}$. The following shows that AI satisfies all properties except P3.

Proposition 2 *Index AI satisfies P1, P2 and P4–P6, but not P3.*

Proof We shall prove all the properties separately.

P1 Assuming $\nu = 0$, then we should prove $AI(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \in \mathcal{A}^*$.
($\Rightarrow$): As all the terms of the sum in (7) are non-negative, if $AI(\mathbf{A}) = 0$, then they must all be equal to zero. Such terms equal zero only when all the numerators equal zero, i.e. when $\max(r_{ij}) = \min(r_{ij})\ \forall i < j$, which implies that $\mathbf{A} \in \mathcal{A}^*$.
($\Leftarrow$): If $\mathbf{A} \in \mathcal{A}^*$, then all the elements r_{ij} are singletons and therefore $\max(r_{ij}) = \min(r_{ij})\ \forall i < j$, implying that the numerators in (7) equals zero and $AI(\mathbf{A}) = 0$.

P2 Straightforward.

P3 It is sufficient to consider the following matrix $\mathbf{A}$ and its derived $\mathbf{A}(2)$ and $\mathbf{A}(3)$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 2 \\ 1/8 & 1/2 & 1 \end{pmatrix} \quad \mathbf{A}(2) = \begin{pmatrix} 1 & 2^2 & 8^2 \\ 1/2^2 & 1 & 2^2 \\ 1/8^2 & 1/2^2 & 1 \end{pmatrix} \quad \mathbf{A}(3) = \begin{pmatrix} 1 & 2^3 & 8^3 \\ 1/2^3 & 1 & 2^3 \\ 1/8^3 & 1/2^3 & 1 \end{pmatrix} \tag{8}$$

and observe that $I(\mathbf{A}(2)) \approx 0.108$ and $I(\mathbf{A}(3)) \approx 0.068$. Hence $I(\mathbf{A}(2)) > I(\mathbf{A}(3))$ and P3 is not satisfied.

P4 Given $a_{12}, a_{23}, \ldots, a_{n-1\,n}$, a *consistent* pairwise comparison matrix of order n can be equivalently written as

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{12}a_{23} & \ldots & a_{12} \cdot \ldots \cdot a_{n-1\,n} \\ \frac{1}{a_{12}} & 1 & a_{23} & \ldots & a_{23} \cdot \ldots \cdot a_{n-1\,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{a_{12} \cdot \ldots \cdot a_{n-2\,n-1}} & \frac{1}{a_{23} \cdot \ldots \cdot a_{n-2\,n-1}} & \frac{1}{a_{34} \cdot \ldots \cdot a_{n-2\,n-1}} & \ldots & a_{n-1\,n} \\ \frac{1}{a_{12} \cdot \ldots \cdot a_{n-1\,n}} & \frac{1}{a_{23} \cdot \ldots \cdot a_{n-1\,n}} & \frac{1}{a_{34} \cdot \ldots \cdot a_{n-1\,n}} & \ldots & 1 \end{pmatrix} \in \mathcal{A}^* \tag{9}$$

or, more compactly, as $\mathbf{B} = (b_{ij})_{n \times n}$ where

$$b_{ij} = \begin{cases} \prod_{p=i}^{j-1} a_{p\,p+1}, & \forall i < j \\ 1, & \forall i = j \\ 1/\prod_{p=i}^{j-1} a_{p\,p+1}, & \forall i > j \end{cases}$$

Then, each element of the auxiliary matrix $\mathbf{R}$ is as follows

$$r_{ij} = \{b_{ik}b_{kj} | k = 1, \ldots, n\} \quad \forall i, j.$$

Now, to test P4, without loss of generality, we fix the pair $(1, n)$ and replace a_{1n} and a_{n1} with a_{1n}^{δ} and a_{n1}^{δ}, respectively. Consequently, b_{1n} and b_{n1} are replaced by b_{1n}^{δ} and b_{n1}^{δ}. Hence, for all $i < j$

$$r_{ij} = \begin{cases} \left\{b_{ij}\right\}, & \forall i, j \notin \{1, n\} \\ \left\{b_{ij}, b_{1n}^{\delta} \frac{b_{ij}}{b_{1n}}\right\}, & \text{otherwise.} \end{cases}$$

Considering the definition of AI we reckon that the terms associated with r_{ij} for $i, j \notin \{1, n\}$ equals zero. Therefore, we shall prove that all the other terms are non-decreasing functions of δ. We can rewrite

$$\left\{b_{ij}, b_{1n}^{\delta} \frac{b_{ij}}{b_{1n}}\right\} = \left\{b_{ij}, b_{ij} b_{1n}^{\delta-1}\right\}$$

and with $x := b_{ij},\ y := b_{1n},\ \mu = \delta - 1$, it boils down to prove that

$$\frac{\max\{x, xy^{\mu}\} - \min\{x, xy^{\mu}\}}{(1 + \max\{x, xy^{\mu}\})(1 + \min\{x, xy^{\mu}\})} \tag{10}$$

is a non-decreasing function of $\mu > 0$ when also $x, y > 0$. Now we should examine the two cases (i) $x < xy^{\mu}$ and (ii) $x > xy^{\mu}$. We start with $x < xy^{\mu}$ and, considering that

$$xy^{\mu} > x \Leftrightarrow y^{\mu} > 1 \Leftrightarrow y > 1$$

and that therefore, for the case $xz > x$, y^{μ} is always an increasing function of μ. Hence, we can substitute y^{μ} with $z > 1$ and (10) can be replaced by

$$\frac{\max\{x, xz\} - \min\{x, xz\}}{(1+\max\{x, xz\})(1+\min\{x, xz\})} \quad (x > 0, z > 1). \tag{11}$$

Considering that we are in the case with $xy > x$, we simplify (11), and obtain

$$\phi_{(i)} = \frac{xz - x}{(1+xz)(1+x)}. \tag{12}$$

So now we shall prove that $\frac{\partial \phi_{(i)}}{\partial z}$ is positive for all $x > 0, z > 1$.

$$\begin{aligned}
\frac{\partial \phi_{(i)}}{\partial z} &= \frac{x}{(1+x)(1+xz)} - \frac{x(xz-x)}{(1+x)(1+xz)^2} \\
&= \frac{x(1+xz) - x(xz-x)}{(1+x)(1+xz)^2} \\
&= \frac{x(1+x)}{(1+x)(1+xz)^2} \\
&= \frac{x}{(1+xz)^2}.
\end{aligned}$$

This last quantity is always positive for $x > 0$. A very similar result can be derived for the case (ii) $x > xy$ and thus AI satisfies P4.

It can be checked that AI also satisfies properties P5 and P6. □

Although in its present form index AI does not satisfy P3, the underlying idea is ingenious and it is sufficient to adjust it, i.e. discard the normalization at the denominator, to make it satisfy P3.

Proposition 3 *The inconsistency index*

$$AI^*(\mathbf{A}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\max(r_{ij}) - \min(r_{ij})\right)$$

satisfies properties P1–P6.

Proof It follows from the proof of Proposition 2 that AI^* satisfies P1, P2, P4, P5, and P6. To show that P3 is satisfied, it is sufficient to take the arguments of the sum $\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\max(r_{ij}) - \min(r_{ij})\right)$ and consider that they are all non-negative, since $\max(r_{ij}) \geq \min(r_{ij})\ \forall i, j$. Consequently, the terms $\left(\max(r_{ij})^b - \min(r_{ij})^b\right) \geq 0$ are monotone non-decreasing functions with respect to $b > 1$, and P3 is satisfied. □

Example 1 Consider the pairwise comparison matrix $\mathbf{A}$ in (8) and its associated $\mathbf{A}(b) = \left(a_{ij}^b\right)_{3\times 3}$. Figure 4 contains the plots of AI and AI^* for $\mathbf{A}(b)$ as functions of b and shows their different behaviors.

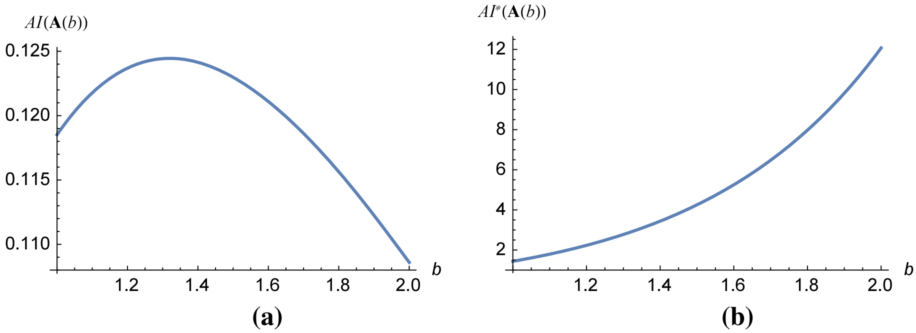


Fig. 4 Comparison between AI and AI^* with respect to P3. **a** Index AI can be decreasing w.r.t b, and even tend to 0, when $b \to \infty$ and **b** Index AI^* is monotone non-decreasing w.r.t. b

4.3 Index by Wu and Xu

Wu and Xu (2012) defined their inconsistency index using some properties of the Hadamard product of positive matrices.

Definition 3 (*Index by* Wu and Xu (2012)) The index defined by Wu and Xu is

$$CI_H(\mathbf{A}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} g_{ji} \, ,$$

where $g_{ij} = \left(\prod_{k=1}^{n} a_{ik} a_{kj} \right)^{\frac{1}{n}}$.

Note that the matrix $\mathbf{G} = \left(g_{ij}\right)_{n \times n} \in \mathcal{A}^*$ can be interpreted as a consistent approximation of $\mathbf{A}$. In the original paper CI_H was used in a mathematical model to manage consistency and consensus at once. Until now, no formal or numerical analysis has been made on CI_H and there is no information on its properties. However, with the following proposition we show that it satisfies P1–P6.

Proposition 4 *Index CI_H satisfies the properties P1–P6.*

Proof We shall show that P1 is satisfied, with $\nu = 1$. First we need to prove that $CI_H(\mathbf{A}) = 1 \Rightarrow \mathbf{A} \in \mathcal{A}^*$.

$$CI_H(\mathbf{A}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} g_{ji} = \frac{1}{n} + \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \underbrace{\left(a_{ij} g_{ji} + \frac{1}{a_{ij} g_{ji}} \right)}_{\psi(a_{ij}, g_{ji})}$$

Now it can be seen that each function $\psi(a_{ij}, g_{ji})$ attains its global minimum, equal to 2, when $a_{ij} g_{ji} = 1$, which is a restatement of the consistency condition. In this case, to receive the hint that $\nu = 1$, it is enough to simplify the sum,

$$CI_H(\mathbf{A}) = \frac{1}{n} + \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2 = \frac{1}{n} + \frac{1}{n^2} \cdot \frac{n(n-1)}{2} \cdot 2 = 1$$

Now in the other direction, $\mathbf{A} \in \mathcal{A}^* \Rightarrow CI_H(\mathbf{A}) = 1$, it suffices to expand $CI_H(\mathbf{A})$

$$CI_H(\mathbf{A}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} \left(\prod_{k=1}^{n} \frac{1}{a_{ik}a_{kj}} \right)^{1/n} \right).$$

Since consistency implies $a_{ik}a_{kj} = a_{ij}$ we have

$$CI_H(\mathbf{A}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} \frac{1}{a_{ij}} \right) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1 = 1.$$

It is simple, and thus omitted, to show that P2, P5 and P6 hold. To prove P3, we shall call $(ag)_{ij} = a_{ij}^b g_{ji}^b$. By expanding g_{ji},

$$(ag)_{ij} = a_{ij}^b \left(a_{j1}^b a_{1i}^b \cdot a_{j2}^b a_{2i}^b \cdot \ldots \cdot a_{jn}^b a_{ni}^b \right)^{1/n} = \underbrace{(a_{ij} g_{ji})}_{>0}{}^b.$$

Since $(ag)_{ij} = 1/(ag)_{ji}$, by summing $(ag)_{ij}$ and $(ag)_{ji}$ we obtain

$$(ag)_{ij} + (ag)_{ji} = \left(a_{ij} g_{ji}\right)^b + \frac{1}{\left(a_{ij} g_{ji}\right)^b} \quad \forall i, j,$$

which is an increasing function for $b > 0$. Since this holds for the general pair of indices $\{i, j\}$, the index satisfies P3.

To prove P4, assume, without loss of generality, that the element to be modified is a_{1n}. For sake of simplicity, we can modify it and its reciprocal by multiplying them by $\beta > 0$. All the $(ag)_{ij}$ with $i, j \notin \{1, n\}$ will be equal to 1. For the entries with one index i, j equal to either 1 or n we have

$$(ag)_{ij} = a_{ij} (\underbrace{a_{j1}a_{1i} \cdot a_{j2}a_{2i} \cdot \ldots \cdot a_{jn}a_{ni}}_{a_{ji}^n} \beta)^{1/n},$$

meaning that $g_{ji} = a_{ji}\beta^{1/n}$. As we know that $g_{ij} = 1/g_{ji}$, by summing $(ag)_{ij}$ and $(ag)_{ji}$ and simplifying, one obtains

$$\frac{1}{\beta^{1/n}} + \beta^{1/n},$$

which is a strictly convex function for $\beta > 0$ with minimum in $\beta = 1$. Similarly, for $(ag)_{1n}$ and $(ag)_{n1}$, it is

$$(ag)_{1n} + (ag)_{n1} = \frac{1}{\beta^{2/n}} + \beta^{2/n},$$

which shares the same property. □

4.4 Cosine Consistency Index and other indices

Many times it is not easy to prove whether an index satisfies some properties, but numerical tests and counterexamples can always be used to show that the index does not. This was the case with the Cosine Consistency Index.

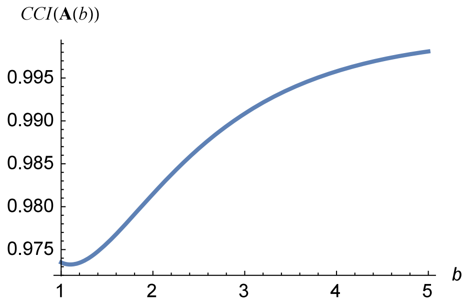


Fig. 5 For the matrix **A** in (13), an intensification of preferences decreases the inconsistency

Definition 4 (*Cosine Consistency Index* (Kou and Liu 2014)) The Cosine Consistency Index is

$$CCI(\mathbf{A}) = \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}\right)^2} \Big/ n,$$

where $b_{ij} = a_{ij} \Big/ \sqrt{\sum_{k=1}^{n} a_{kj}^2}$.

Note that $CCI(\mathbf{A}) \in [0, 1]$ and its interpretation is reversed, meaning that the greater its value the *less* inconsistent $\mathbf{A}$ is. It is simple, and it can also be found in the original paper, to show that CCI satisfies P1, P2, P5, and P6. For instance, in the case of P1, the proof comes directly from Eq. 6 and Theorem 3 in the paper by Kou and Liu (2014). However, the following counterexample suffices to show that CCI does *not* satisfy P3.

Example 2 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 1/3 & 1 & 1/2 \\ 1/7 & 2 & 1 \end{pmatrix} \tag{13}$$

and its associated $\mathbf{A}(b) = (a_{ij}^b)_{n\times n}$. The plot of $CCI(\mathbf{A}(b))$ is reported in Fig. 5 and shows that CCI does not satisfy P3.

Often, although in their present forms they do not satisfy P1–P6, ideas behind indices are valid and slight modifications are sufficient to make them satisfy a set of properties. One example is the index NI_n^σ proposed by Ramík and Korviny (2010) which was later studied (Brunelli 2011). Another concrete example is the Relative Error index by Barzilai (1998) which does not satisfy P4 and P5. In its original formulation such index is

$$RE(\mathbf{A}) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_{ij} - d_i + d_j\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_{ij}\right)^2},$$

where $p_{ij} = \log a_{ij}$ and $d_i = \frac{1}{n}\sum_{k=1}^{n} p_{ik}$, and where the denominator acts as a normalization factor. Here it can be proved that, if we discard the denominator, we obtain

$$RE^*(\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_{ij} - d_i + d_j\right)^2,$$

which, unlike RE, satisfies all the properties.

Proposition 5 *Index RE^* satisfies all the properties P1–P6.*

Proof It is easy to show, and therefore omitted, that P1, P2, P5 and P6 are satisfied. We shall prove P3 and P4 separately. To prove the satisfaction of P3 we consider $\mathbf{A}(b) = \left(a_{ij}^b\right)$ and note that applying the logarithmic transformation to its entries, we obtain $\left(b \log a_{ij}\right)_{n\times n} = \left(p_{ij} \cdot b\right)_{n\times n}$. Hence,

$$RE^*(\mathbf{A}(b)) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(b \cdot p_{ij} - \frac{1}{n}\sum_{k=1}^{n} b \cdot p_{ik} + \frac{1}{n}\sum_{k=1}^{n} b \cdot p_{jk}\right)^2 = b^2 \cdot RE^*(\mathbf{A}),$$

which implies that $RE^*(\mathbf{A}(b))$ is monotone non-decreasing for $b > 1$.

To show that P4 holds, let us consider the matrix $\mathbf{A} \in \mathcal{A}^*$, and its associated $\mathbf{P} = (p_{ij})_{n\times n} = (\log a_{ij})_{n\times n}$. P4 can equivalently be restated as the property that, if we take an entry p_{pq} and its reciprocal p_{qp} and substitute them with $p_{pq} + \xi$ and $p_{qp} - \xi$, respectively, then the inconsistency index RE^* is a quasi-convex function of ξ with minimum in $\xi = 0$. From the proof of Proposition 5 in (Brunelli and Fedrizzi 2015a) one recovers that, by introducing ξ, it is

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_{ij} - d_i + d_j\right)^2 = 4(n-2)\left(\frac{\xi}{n}\right)^2 + 2\left(\frac{n-2}{n}\xi\right)^2.$$

Thus one obtains,

$$RE^*(\mathbf{A}_{pq}(\xi)) = 4(n-2)\left(\frac{\xi}{n}\right)^2 + 2\left(\frac{n-2}{n}\xi\right)^2 = \frac{2(-2+n)\xi^2}{n} = \xi^2 \underbrace{\frac{2(n-2)}{n}}_{>0}$$

which is a decreasing function of ξ for $\xi < 0$ and an increasing function for $\xi > 0$. □

5 Discussion and conclusions

Choosing the most suitable inconsistency index is of considerable importance, yet formal studies had not been undertaken until very recently (Brunelli and Fedrizzi 2015a; Koczkodaj and Szwarc 2014). This is in contrast with the existence of long-standing studies on other aspects of pairwise comparisons. One of these is the choice of the method for deriving the priority vector, for which axiomatic studies have been proposed in the literature already in the Eighties (Cook and Kress 1988; Fichtner 1986) and in the Nineties (Barzilai 1997). Nevertheless, it has been shown by numerical studies (Ishizaka and Lusti 2006) that, excepts for some particular cases, such differences can be negligible and that therefore, in most of the cases, choosing one method or another does not really influence the final outcome.

In light of the recently proposed five properties for inconsistency indices, the contribution of this research is at least twofold:

- Firstly, it introduces and justifies a sixth property (P6) and shows that, together with the other five, it forms an an independent and logically consistent set of properties.
- Secondly, the paper further analyzes the satisfaction of the properties P1–P6. Four inconsistency indices have been considered from the literature and it was found that two of them fail to fully satisfy the set of properties P1–P6. A simple adjustment of one of these indices was proposed to make it fit P1–P6.

Table 1 Summary of propositions

✓ = property is satisfied, × = property is not satisfied, '—' = unknown. The original results presented in this research are separated from previous ones (Brunelli and Fedrizzi 2015a) by the dashed lines

	P1	P2	P3	P4	P5	P6
CI	✓	✓	✓	✓	✓	✓
GW	✓	✓	✗	—	✓	✓
GCI	✓	✓	✓	✓	✓	✓
RE	✓	✓	✓	✗	✗	✓
CI^*	✓	✓	✓	✓	✓	✓
HCI	✓	✓	✗	✓	✓	✓
NI_n^σ	✓	✓	—	✗	✓	✓
K (Definition 1)	✓	✓	✓	✓	✓	✓
AI (Definition 2)	✓	✓	✗	✓	✓	✓
CI_H (Definition 3)	✓	✓	✓	✓	✓	✓
CCI (Definition 4)	✓	✓	✗	—	✓	✓

Table 1 presents a summary of the findings of this research and shows how they expanded the original set of properties for inconsistency indices (Brunelli and Fedrizzi 2015a). It is remarkable that, in the form in which they were originally introduced in the literature, the majority of the indices satisfy only some of them. This seems to indicate that the definition of the properties and the analysis of their satisfaction is *not* a mere theoretical exercise.

The properties were here, and in previous research (Brunelli and Fedrizzi 2015a), justified. Nevertheless, clearly, this should not prevent anyone from criticizing and improving them: it is indeed desirable that a set of properties be openly discussed within a community. In this direction, if the system P1–P6 is considered too restrictive, it is worth noting that Theorem 1 implies that any subset of the properties P1–P6 also forms an independent and logically consistent set or properties (just a more relaxed one) which, indeed, can be used for the same purposes of P1–P6. In conclusion, it is the author's belief that a systematic study of inconsistency and inconsistency indices may bring new insights and more formal order into the evergreen topic of rational decision making. Furthermore, in the future, it should be possible to extend the set of properties to other types of numerical representations of preferences as, for instance, reciprocal preference relations (Tanino 1984) and skew-symmetric additive representations (Fishburn 1999).

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Appendix: Proof of Theorem 1

To prove *logical consistency*, it is sufficient to find an instance of $I : \mathcal{A} \rightarrow \mathbb{R}$ which satisfies all the properties P1–P6. One such instance is the following function

$$I^*(\mathbf{A}) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right) \tag{14}$$

To prove the *independence* of P1–P6, it is sufficient to find a function satisfying all properties except one, for all the properties. The examples of inconsistency indices proposed by Brunelli and Fedrizzi (2015a) to prove the independence of the system P1–P5 are invariant under transposition. If follows that P1–P5 are logically independent within the system P1–P6. It remains to show that P6 does *not* depend on P1–P5. Consider that, if $\mathbf{A}$ has one row, say H, whose non-diagonal elements are all greater than one, i.e. $a_{Hj} > 1\ \forall j \neq H$, then this property is shared by any matrix $\mathbf{PAP}^T$, where $\mathbf{P}$ is any permutation matrix, but not by its transpose $\mathbf{A}^T$. Taking into account the inconsistency index I^* in (14), and defining H as the row with the greatest non-diagonal element, then the function

$$I_{\neg 6}(\mathbf{A}) = I^*(\mathbf{A}) \cdot \underbrace{\left(1 + \max\left\{\min_{j\neq H}\{a_{Hj} - 1\}, 0\right\}\right)}_{M} \tag{15}$$

is invariant under row-column permutation but not under transposition. Hence, $I_{\neg 6}$ satisfies AP but not P6. To prove the independence of P6, it remains to show that (15) satisfies P1 and P3–P5. It is easy, and thus omitted, to show that P1, P3, and P5 are satisfied. To prove it for P4, we note that any $\mathbf{A} \in \mathcal{A}^*$ can be rewritten as

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{12}a_{23} & \dots & a_{12} \cdot \dots \cdot a_{n-1\,n} \\ \frac{1}{a_{12}} & 1 & a_{23} & \dots & a_{23} \cdot \dots \cdot a_{n-1\,n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{a_{12}\cdot\ldots\cdot a_{n-2\,n-1}} & \frac{1}{a_{23}\cdot\ldots\cdot a_{n-2\,n-1}} & \frac{1}{a_{34}\cdot\ldots\cdot a_{n-2\,n-1}} & \dots & a_{n-1\,n} \\ \frac{1}{a_{12}\cdot\ldots\cdot a_{n-1\,n}} & \frac{1}{a_{23}\cdot\ldots\cdot a_{n-1\,n}} & \frac{1}{a_{34}\cdot\ldots\cdot a_{n-1\,n}} & \dots & 1 \end{pmatrix} \in \mathcal{A}^* \tag{16}$$

Without loss of generality let us consider a_{1n} and its reciprocal a_{n1} and replace them with a_{1n}^{δ} and a_{n1}^{δ}, respectively. Then, by calling $\mathbf{A}_{1n}^{\delta}$ the new matrix and bearing in mind that $\mathbf{A} \in \mathcal{A}^*$, we have

$$\begin{aligned} I^*(\mathbf{A}_{1n}^{\delta}) &= \sum_{j=2}^{n-1}\left(\frac{a_{1n}^{\delta}}{a_{1j}a_{jn}} + \frac{a_{1j}a_{jn}}{a_{1n}^{\delta}} - 2\right) \\ &= (n-2)\left(\frac{(a_{12}\cdot\ldots\cdot a_{n-1\,n})^{\delta}}{a_{12}\cdot\ldots\cdot a_{n-1\,n}} + \frac{a_{12}\cdot\ldots\cdot a_{n-1\,n}}{(a_{12}\cdot\ldots\cdot a_{n-1\,n})^{\delta}} - 2\right) \end{aligned}$$

If $H \notin \{1, n\}$, then, in (15) M is constant and P4 holds in this case. Also if $H \in \{1, n\}$ and $\min_{j\neq H}\{a_{Hj}\} \neq a_{1n}$, then M is constant and P4 is satisfied. Finally, if $H = 1$ and $\min_{j\neq H}\{a_{Hj}\} = a_{1n}$, it is

$$I_{\neg 6}\left(\mathbf{A}_{1n}^{\delta}\right) = I^*\left(\mathbf{A}_{1n}^{\delta}\right) \cdot \left(1 + a_{1n}^{\delta} - 1\right) = I^*\left(\mathbf{A}_{1n}^{\delta}\right) \cdot a_{1n}^{\delta} \tag{17}$$

which can be reduced to

$$\begin{aligned} I_{\neg 6}(\mathbf{A}_{1n}^{\delta}) &= \overbrace{(n-2)\left(\frac{(a_{12}\cdot\ldots\cdot a_{n-1\,n})^{\delta}}{a_{12}\cdot\ldots\cdot a_{n-1\,n}} + \frac{a_{12}\cdot\ldots\cdot a_{n-1\,n}}{(a_{12}\cdot\ldots\cdot a_{n-1\,n})^{\delta}} - 2\right)}^{I^*(\mathbf{A}_{1n}^{\delta})} \\ &\quad \cdot \overbrace{(a_{12}\ldots a_{n-1\,n})^{\delta}}^{a_{1n}^{\delta}} \\ &= (n-2)\left(\frac{a_{1n}^{2\delta}}{a_{1n}} + a_{1n} - 2a_{1n}^{\delta}\right). \end{aligned}$$

Considering that, from $\mathbf{A} \in \mathcal{A}^*$ and $H = 1$, it follows that $a_{1n} \geq 1$ and the partial derivative in δ is

$$\frac{\partial I_{\neg 6}(\mathbf{A}_{1n}^{\delta})}{\partial \delta} = (n-2)\left(2a_{1n}^{2\delta-1}\log(a_{1n}) - 2a_{1n}^{\delta}\log(a_{1n})\right)$$
$$= \underbrace{(n-2)}_{>0}\underbrace{\left(2a_{1n}^{\delta-1}\right)}_{>0}\left(a_{1n}^{\delta} - a_{1n}\right)\underbrace{(\log a_{1n})}_{>0},$$

which is always non-negative for $\delta > 1$ and non-positive for $0 < \delta < 1$. Thus, P4 is satisfied and the properties P1–P6 are logically independent. □

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A General Unified Framework for Pairwise Comparison Matrices in Multicriterial Methods

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In a multicriteria decision making context, a pairwise comparison matrix $A = (a_{ij})$ is a helpful tool to determine the weighted ranking on a set X of alternatives or criteria. The entry a_{ij} of the matrix can assume different meanings: a_{ij} can be a preference ratio (multiplicative case) or a preference difference (additive case) or a_{ij} belongs to [0, 1] and measures the distance from the indifference that is expressed by 0.5 (fuzzy case). For the multiplicative case, a consistency index for the matrix A has been provided by T.L. Saaty in terms of maximum eigenvalue. We consider pairwise comparison matrices over an abelian linearly ordered group and, in this way, we provide a general framework including the mentioned cases. By introducing a more general notion of metric, we provide a consistency index that has a natural meaning and it is easy to compute in the additive and multiplicative cases; in the other cases, it can be computed easily starting from a suitable additive or multiplicative matrix.

1. INTRODUCTION

A crucial step in a decision making process is the determination of a weighted ranking on a set $X = \{x_1, x_2, \ldots, x_n\}$ of alternatives with respect to criteria or experts. A way to determine the weighted ranking is to start from a relation $\mathcal{A} : (x_i, x_j) \in X \times X \to a_{ij} = \mathcal{A}(x_i, x_j) \in G \subseteq R$ represented by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \tag{1.1}$$

that is called *pairwise comparison matrix* (PC matrix for short): a_{ij} expresses how much x_i is preferred to x_j and a condition of *reciprocity* is assumed in such way

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that the preference of x_i over x_j expressed by a_{ij} can be exactly read by means of the element a_{ji}. Under a suitable condition of *consistency*, X is totally ordered by $\mathcal{A}$ and there exists a vector $\underline{w}$, that perfectly represents the preferences over X. The reciprocity and consistency conditions depend on the different meaning given to the number a_{ij} as the following examples of PC matrices show.

1. **Multiplicative PC matrix**. $a_{ij} \in]0, +\infty[$ represents the preference ratio of x_i over x_j: $a_{ij} > 1$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < 1$ expresses the opposite preference and $a_{ij} = 1$ means that x_i and x_j are indifferent. Then, the condition of reciprocity is

   mr) $a_{ji} = \dfrac{1}{a_{ij}} \quad \forall\, i, j = 1, \ldots, n$ *(multiplicative reciprocity)*,

   so, $a_{ii} = 1$ for each $i = 1, 2, \ldots, n$. The consistency condition is given by

   mc) $a_{ik} = a_{ij} a_{jk} \quad \forall\, i, j, k = 1, \ldots, n$ *(multiplicative consistency)*.

   The matrix $A = (a_{ij})$ is consistent if and only if there is a positive vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $\frac{w_i}{w_j} = a_{ij}$.
2. **Additive PC matrix**. $a_{ij} \in]-\infty, +\infty[$ represents the difference of preference between x_i and x_j: $a_{ij} > 0$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < 0$ expresses the opposite preference and $a_{ij} = 0$ means that x_i and x_j are indifferent. Then, the condition of reciprocity is

   ar) $a_{ji} = -a_{ij} \quad \forall\, i, j = 1, \ldots, n$ *(additive reciprocity)*,

   thus, $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$. The consistency condition is given by

   ac) $a_{ik} = a_{ij} + a_{jk} \quad \forall\, i, j, k = 1, \ldots, n$ *(additive consistency)*.

   The matrix $A = (a_{ij})$ is consistent if and only if there is a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $w_i - w_j = a_{ij}$.
3. **Fuzzy PC matrix**. $a_{ij} \in [0, 1]$: $a_{ij} > 0.5$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < 0.5$ expresses the opposite preference and $a_{ij} = 0.5$ means that x_i and x_j are indifferent. Then, the condition of reciprocity is

   fr) $a_{ji} = 1 - a_{ij} \quad \forall\, i, j = 1, \ldots, n$ *(fuzzy reciprocity)*,

   thus, $a_{ii} = 0.5$ for all $i = 1, 2, \ldots, n$. The consistency condition is given by

   fc) $a_{ik} = a_{ij} + a_{jk} - 0.5 \quad \forall\, i, j, k = 1, \ldots, n$ *(fuzzy consistency)*.

   The matrix $A = (a_{ij})$ is consistent if and only if there is a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying the condition $w_i - w_j = a_{ij} - 0.5$.

The multiplicative PC matrices play a basic role in the Analytic Hierarchy Process, a procedure developed by Saaty at the end of the 70s,[1,2] and widely used by governments and companies[2–4] in fixing their strategies. Saaty indicates a scale translating the comparisons expressed in verbal terms into the preference ratios a_{ij}. By applying this scale, a_{ij} may only take value in $S^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$. The assumption of the Saaty scale restricts the decision maker's possibility to be consistent: indeed if the decision maker expresses the following preference ratios $a_{ij} = 5$ and $a_{jk} = 3$ then he will not be consistent because $a_{ij}a_{jk} = 15 > 9$. The assumption of any limited and closed set of values presents the same drawback for each one of the considered PC matrices. In particular, under the assumption that $a_{ij} \in [0, 1]$, the consistency property **fc** cannot be respected, for instance, by a decision maker who claims $a_{ij} = 0.9$ and $a_{jk} = 0.8$, because $a_{ij} + a_{jk} - 0.5 = 1.7 - 0.5 > 1$.

A measure of closeness to the consistency for a multiplicative PC matrix has been provided by Saaty[2,5] in terms of the principal eigenvalue $\lambda_{\max}$:

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1} \qquad (\textit{consistency index}),$$

and the right eigenvector $\underline{w}_{\lambda_{\max}} = (w_1, w_2, \ldots, w_n)$ associated to $\lambda_{\max}$ has been considered as weighting vector. Saaty[5] shows that the more CI is close to 0, the more the ratios $\frac{w_i}{w_j}$ are close to the preference ratios a_{ij}: so enough small values of CI would ensure a good representation of the preferences over X by means of $\underline{w}_{\lambda_{\max}}$.

To get a weighted ranking, other methods have also been considered by scholars; for example, weighted rankings are obtained by applying the arithmetic or geometric mean operators to the rows of the multiplicative PC matrix.[2,6,7]

The consistency index CI has been questioned because it is not easy to compute, has not a simple and geometric meaning[8,9] and, in some cases, seems to be unfair.[10] Also, the methods used to provide a weighted ranking have been questioned: indeed they may indicate rankings that do not agree with the expressed preference ratios a_{ij}.[11–15]

The aim of the present study is to define a general context in which different approaches to a PC matrix can be unified and provide a meaningful consistency index suitable for each type of matrix. The definitions of reciprocity and consistency in the multiplicative or additive case imply only an operation and its inverse (the multiplication and the division for a multiplicative PC matrix, the addition and the difference for an additive PC matrix): so in the study the set G, on which the relation $\mathcal{A}$ takes its values, is embodied only with a commutative group operation $\odot$ and a total order $\leq$ compatible with the operation; G is not necessary a real subset. The reciprocity and consistency conditions are expressed in terms of the group operation $\odot$ and a notion of distance $d_{\mathcal{G}}$, linked to the abelian linearly ordered group $\mathcal{G} = (G, \odot, \leq)$, is introduced (see Section 3). The assumption of *divisibility* for $\mathcal{G}$ allows to introduce the mean $m_{\odot}(a_1, \ldots, a_n)$ of n elements (see Section 2.1) and associate a *mean vector* $\underline{w}_{m_{\odot}}$ to a PC matrix $A = (a_{ij})$ (see Section 5). By using the mean operator $m_{\odot}$ and the distance $d_{\mathcal{G}}$, a consistency index $I_{\mathcal{G}}(A)$ for the matrix

A is also provided (see Section 6). $I_{\mathcal{G}}(A)$ is equal to the identity element of $\odot$ if and only if $A = (a_{ij})$ is consistent (see Section 6) and, in this case, the mean vector $\underline{w}_{m_\odot}$ provides weights $w_1, w_2, \ldots, w_n$ for the alternatives perfectly agreeing with the entries a_{ij}: indeed it results $w_i \div w_j = a_{ij} \quad \forall i, j = 1, 2, \ldots, n$, where $\div$ is the inverse of $\odot$. Moreover, for $n = 3$, in case of inconsistency the closeness of the elements $w_i \div w_j$ to the entries a_{ij} of the PC matrix can be expressed in terms of the consistency index $I_{\mathcal{G}}(A)$ (see Section 6.1). In this way, the study generalizes the multiplicative and the additive cases and finds, for these cases, a consistency index easy to compute and naturally grounded on a notion of distance. Moreover, if G is a real open interval, then the consistency index can be obtained by computing the consistency index of a suitable multiplicative or additive PC matrix.

In this approach, the definition of fuzzy consistency is modified in such way that the underlying operation is a group operation (see Proposition 4.2 and Remark 5.1) and the shown drawback, related to the possibility to build a consistent matrix, is removed.

2. ABELIAN LINEARLY ORDERED GROUPS

In this section, we recall some notions and properties related to abelian linearly ordered groups.

DEFINITION 2.1. *Let G be a nonempty set, $\odot : G \times G \to G$ a binary operation on G, $\leq$ a total weak order on G. Then $\mathcal{G} = (G, \odot, \leq)$ is an* abelian linearly ordered group, alo-group *for short, if and only if* $(G, \odot)$ *is an abelian group and*

$$a \leq b \Rightarrow a \odot c \leq b \odot c. \tag{2.1}$$

As an abelian group satisfies the cancellative law "$a \odot c = b \odot c \Leftrightarrow a = b$," Equation 2.1 is equivalent to the strict monotonicity of $\odot$ in each variable:

$$a < b \Leftrightarrow a \odot c < b \odot c. \tag{2.2}$$

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, we will indicate by:

- e the identity of $\mathcal{G}$,
- $x^{(-1)}$ the symmetric of $x \in G$ with respect to $\odot$,
- $\div$ the inverse operation of $\odot$ defined by $a \div b = a \odot b^{(-1)}$,
- $<$ the strict simple order defined by "$x < y \leftrightarrow x \leq y$ *and* $x \neq y$",
- $\geq$ and $>$ the opposite relations of $\leq$ and $<$, respectively.

Then

$$b^{(-1)} = e \div b, \quad (a \odot b)^{(-1)} = a^{(-1)} \odot b^{(-1)}, \quad (a \div b)^{(-1)} = b \div a; \tag{2.3}$$

moreover, assuming that G is no trivial, that is $G \neq \{e\}$, by Equation 2.2 we get

$$a < e \Leftrightarrow a^{(-1)} > e, \;\; a > e \Leftrightarrow e > a^{(-1)},$$

$$a \odot a > a \quad \forall a > e, \;\; a \odot a < a \quad \forall a < e. \tag{2.4}$$

If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then G is naturally equipped with the order topology induced by $\leq$ and $G \times G$ is equipped with the related product topology. We say that $\mathcal{G}$ is a *continuous* alo-group if and only if $\odot$ is continuous.

By definition, an alo-group $\mathcal{G}$ is a *lattice ordered group*,[16] that is there exists $a \vee b = max\{a, b\}$, for each pair $(a, b) \in G^2$. Nevertheless, by Equation 2.4, we get the following proposition.

PROPOSITION 2.1. *A nontrivial alo-group $\mathcal{G} = (G, \odot, \leq)$ has neither the greatest element nor the least element.*

Remark 2.1. By Proposition 2.1, neither the interval [0, 1] nor the Saaty set $S^* = \{1, 2, \ldots, 9, \frac{1}{2}, \frac{1}{3} \ldots, \frac{1}{9}\}$, embodied with the usual order $\leq$ on R, can be structured as linearly ordered group.

(*n*)-powers. Because of the associative property, the operation $\odot$ can be extended by induction to n-ary operation, $n > 2$, by setting

$$\bigodot_{i=1}^{n} x_i = \left(\bigodot_{i=1}^{n-1} x_i \right) \odot x_n. \tag{2.5}$$

Then, for a positive integer n, the *(n)- power* $x^{(n)}$ of $x \in G$ is defined by

$$\begin{cases} x^{(1)} = x \\ x^{(n)} = \bigodot_{i=1}^{n} x_i, \;\; x_i = x \forall i = 1, \ldots, n, \quad \text{for} \quad n \geq 2, \end{cases}$$

and verifies the following properties:

$$x^{(n)} \odot x^{(m)} = x^{(n+m)} = x^{(m)} \odot x^{(n)}, \quad (x^{(n)})^{(m)} = x^{(nm)} = (x^{(m)})^{(n)}, \tag{2.6}$$

$$x^{(n)} \odot y^{(n)} = (x \odot y)^{(n)}. \tag{2.7}$$

By the properties in Equations 2.2 and 2.4, we can get by induction

$$x < y \Leftrightarrow x^{(n)} < y^{(n)},$$

$$a^{(n)} > a \quad \forall a > e, \qquad a^{(n)} < a \quad \forall a < e. \tag{2.8}$$

We can extend the meaning of power $x^{(s)}$ to the case that s is a relative integer by setting

$$x^{(0)} = e \quad and \quad x^{(-n)} = \left(x^{(n)}\right)^{(-1)}. \tag{2.9}$$

By Equations 2.9 and 2.7, $x^{(n)} \odot x^{(-n)} = e = (x \odot x^{(-1)})^{(n)} = x^{(n)} \odot (x^{(-1)})^{(n)}$, so

$$x^{(-n)} = \left(x^{(-1)}\right)^{(n)}. \tag{2.10}$$

As a consequence, the properties in Equations 2.6 and 2.7 are satisfied for all integers m, n and, as particular case, we have

$$(a \div b)^{(n)} = \left(a \odot b^{(-1)}\right)^{(n)} = a^{(n)} \odot (b^{(n)})^{(-1)} = a^{(n)} \div b^{(n)}. \tag{2.11}$$

Isomorphism between alo-groups. An *isomorphism* between two alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ is a bijection $h : G \to G'$ that is both a lattice isomorphism and a group isomorphism, that is,

$$x < y \Leftrightarrow h(x) < h(y) \quad and \quad h(x \odot y) = h(x) \circ h(y). \tag{2.12}$$

Thus, $h(e) = e'$, where e' is the identity in $\mathcal{G}'$, and

$$h\left(x^{(-1)}\right) = (h(x))^{(-1)}. \tag{2.13}$$

By applying the inverse isomorphism $h^{-1} : G' \to G$, we get

$$h^{-1}(x' \circ y') = h^{-1}(x') \odot h^{-1}(y'), \quad h^{-1}\left({x'}^{(-1)}\right) = (h^{-1}(x'))^{(-1)}. \tag{2.14}$$

By the associativity of the operations $\odot$ and $\circ$, the equality in Equation 2.12 can be extended by induction to the n-operation $\bigodot_{i=1}^{n} x_i$, so that

$$h\left(\bigodot_{i=1}^{n} x_i\right) = \bigcirc_{i=1}^{n} h(x_i), \quad h\left(x^{(n)}\right) = h(x)^{(n)}. \tag{2.15}$$

2.1. Divisible Alo-Group, (n)-Roots and Mean Operator

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. By properties in Equation 2.8, for every positive integer n and every $a \in G$ there exists at most a solution $x \in G$ of the equation $x^{(n)} = a$. So, if there exists a solution b of the equation $x^{(n)} = a$, then this is the only one. Hence, we give the following definition:

DEFINITION 2.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. If $b^{(n)} = a$, then we say that b is the* (n)-root *of a and write $b = a^{(1/n)}$.*

DEFINITION 2.3. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, $\mathcal{G}$ is* divisible *if and only if for each positive integer n and each $a \in G$ there exists the (n)-root of a.*

PROPOSITION 2.2. *The (n)-root verifies the following properties:*

$$(a \odot b)^{(\frac{1}{n})} = a^{(\frac{1}{n})} \odot b^{(\frac{1}{n})}, \quad \left(a^{(-1)}\right)^{(\frac{1}{n})} = \left(a^{(\frac{1}{n})}\right)^{(-1)}, \tag{2.16}$$

$$a < b \Rightarrow a^{(1/n)} < b^{(1/n)}. \tag{2.17}$$

Proof. By Equation 2.7, $(a^{(\frac{1}{n})} \odot b^{(\frac{1}{n})})^{(n)} = (a^{(\frac{1}{n})})^n \odot (b^{(\frac{1}{n})})^n = a \odot b$ and so the first equality in Equation 2.16 is achieved. The second equality is also achieved, since $e = (a \odot a^{(-1)})^{(\frac{1}{n})} = a^{(\frac{1}{n})} \odot (a^{(-1)})^{(\frac{1}{n})}$. Finally, Equation 2.17 follows from Equation 2.8. ∎

DEFINITION 2.4 *Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible alo-group. Then, the $\odot$-* mean *$m_{\odot}(a_1, a_2, \ldots, a_n)$ of the elements $a_1, a_2, \ldots, a_n$ of G is defined by*

$$m_{\odot}(a_1, a_2, \ldots, a_n) = \begin{cases} a_1 & \text{for} \quad n = 1, \\ (\bigodot_{i=1}^{n} a_i)^{(1/n)} & \text{for} \quad n \geq 2. \end{cases}$$

In the sequel, for sake of simplicity, we say *mean* instead of $\odot$- *mean*.

PROPOSITION 2.3. *Let $h : G \to G'$ be an isomorphism between the alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$. Then, $\mathcal{G}$ is divisible if and only if $\mathcal{G}'$ is divisible. Moreover, under the assumption of divisibility:*

$$m_{\odot}(x_1, x_2, \ldots, x_n) = h^{-1}\big(m_{\circ}(h(x_1), h(x_2), \ldots, h(x_n))\big) \tag{2.18}$$

$$m_{\circ}(y_1, y_2, \ldots, y_n) = h\big(m_{\odot}(h^{-1}(y_1), h^{-1}(y_2), \ldots, h^{-1}(y_n))\big). \tag{2.19}$$

Proof. Let us set, for $x, x_i, a \in G$: $y = h(x)$, $y_i = h(x_i)$ and $b = h(a)$. By Equation 2.15, $x^{(n)} = a \Leftrightarrow y^{(n)} = b$, and so $\mathcal{G}$ is divisible if and only if $\mathcal{G}'$ is divisible.

Assume now that $\mathcal{G}$ and $\mathcal{G}'$ are divisible. Then, by Equation 2.15,

$$x^{(n)} = \bigodot_{i=1}^{n} x_i \Leftrightarrow h(x^{(n)}) = h\left(\bigodot_{i=1}^{n} x_i\right) \Leftrightarrow (h(x))^{(n)} = \bigcirc_{i=1}^{n} h(x_i).$$

Hence, $x = m_{\odot}(x_1, \ldots, x_n)$ if and only if $h(x) = m_{\circ}(h(x_1), \ldots, h(x_n))$ and Equation 2.18 is achieved. Equation 2.19 follows from Equation 2.18. ∎

3. $\mathcal{G}$-METRIC

Following Ref. 17, we give the following definition of norm:

DEFINITION 3.1. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the function:*

$$|| \cdot || : a \in G \rightarrow ||a|| = a \vee a^{(-1)} \in G \tag{3.1}$$

*is a $\mathcal{G}$-*norm*, or a* norm *on $\mathcal{G}$.*

PROPOSITION 3.1. *The $\mathcal{G}$-norm satisfies the properties:*

1. $||a|| = ||a^{(-1)}||$;
2. $a \leq ||a||$;
3. $||a|| \geq e$;
4. $||a|| = e \Leftrightarrow a = e$;
5. $||a^{(n)}|| = ||a||^{(n)}$;
6. $||a \odot b|| \leq ||a|| \odot ||b||$ (triangle inequality).

Proof. Items 1, 2, 3, 4 follow immediately from Definition 3.1. Item 5 follows by Equation 2.8 for which $a = x \vee x^{(-1)}$ if and only if $a^{(n)} = x^{(n)} \vee x^{(-1)^{(n)}}$. By Equation 2.1 and item 2, $a \odot b \leq ||a|| \odot ||b||$; so by item 1 and the second equality in Equation 2.3, the triangle inequality follows. ∎

DEFINITION 3.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d : (a, b) \in G^2 \rightarrow d(a, b) \in G$$

*is a $\mathcal{G}$-*metric *or $\mathcal{G}$-*distance *if and only if:*

1. $d(a, b) \geq e$;
2. $d(a, b) = e \Leftrightarrow a = b$;
3. $d(a, b) = d(b, a)$;
4. $d(a, b) \leq d(a, c) \odot d(b, c)$.

Proposition 3.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d_{\mathcal{G}} : (a, b) \in G^2 \to d_{\mathcal{G}}(a, b) = ||a \div b|| \in G \tag{3.2}$$

is a $\mathcal{G}$-distance.

Proof. The conditions 1, 2, and 3 are verified by $d_{\mathcal{G}}$ as consequence of the properties 3, 4, and 1 of the $\mathcal{G}$-norm and the equality $(a \div b)^{(-1)} = b \div a$. By applying the triangle inequality of the $\mathcal{G}$-norm, we get

$$||a \div b|| = ||a \odot c^{(-1)} \odot c \odot b^{(-1)}|| = ||(a \div c) \odot (c \div b)|| \leq ||a \div c|| \odot ||c \div b||;$$

thus, also the condition 4 is verified. ■

Proposition 3.3. *Let $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ be alo-groups and h: $G \to G'$ an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Then*

$$d_{\mathcal{G}'}(a', b') = h(d_{\mathcal{G}}(h^{-1}(a'), h^{-1}(b'))), \quad d_{\mathcal{G}}(a, b) = h^{-1}(d_{\mathcal{G}'}(h(a), h(b))). \tag{3.3}$$

Proof. By definition of $\mathcal{G}$-distance and properties in Equations 2.12 and 2.14:

$$h^{-1}(d_{\mathcal{G}'}(a', b')) = h^{-1}\big((a' \circ (b')^{(-1)}) \vee (b' \circ (a')^{(-1)})\big)$$

$$= \big(h^{-1}(a') \odot (h^{-1}(b'))^{(-1)}\big) \vee \big(h^{-1}(b') \odot (h^{-1}(a'))^{(-1)}\big) = d_{\mathcal{G}}\big(h^{-1}(a'), h^{-1}(b')\big)$$

and the first equality in Equation 3.3 is achieved. The second one is achieved in an analogous way. ■

4. CONTINUOUS ALO-GROUPS OVER A REAL INTERVAL

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R. If G is a proper interval of R then, by Proposition 2.1, it is an open interval.

Let Q be the set of the rational numbers, Q^+ the set of the positive rational numbers, $+$ the usual addition and $\cdot$ the usual multiplication on R. Then, we provide the following examples of real alo-groups.

Example 1. $\mathcal{R} = (R, +, \leq)$ and $\mathcal{Q} = (Q, +, \leq)$ are continuous alo-groups with: $e = 0$, $x^{(-1)} = -x$, $x^{(n)} = nx$, $x \div y = x - y$; the norm $||a|| = |a| = a \vee (-a)$ generates the usual distance over R (resp. Q):

$$|a - b| = (a - b) \vee (b - a).$$

$\mathcal{R}$ and $\mathcal{Q}$ are both divisible: the (n)-root $x^{(n)}$ of a is the solution of $nx = a$ that is usually indicated with the symbol a/n. The mean $m_+(a_1, a_2, \ldots, a_n)$ is the arithmetic mean: $\frac{\sum_i a_i}{n}$.

Example 2. $]0,+\infty[= (]0, +\infty[, \cdot, \leq)$ and $\mathcal{Q}^+ = (Q^+, \cdot, \leq)$ are continuous alo-groups with: $e = 1$, $x^{(-1)} = x^{-1} = 1/x$, $x^{(n)} = x^n$, $x \div y = \frac{x}{y}$ and $||a|| = a \vee a^{-1}$; so $d_{]0,+\infty[}(a, b)$ and $d_{\mathcal{Q}^+}(a, b)$ are both given by

$$\left|\left|\frac{a}{b}\right|\right| = \frac{a}{b} \vee \frac{b}{a} \in [1, +\infty[.$$

The alo-group **]0,+∞[** is divisible and the (n)-root of a is $x = \sqrt[n]{a}$. The mean $m_\cdot(a_1, \ldots, a_n)$ is the geometric mean: $\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}$.
The alo-group $\mathcal{Q}^+$ is not divisible: indeed $x^2 = 2$ has not solution in Q^+.

Let us consider the condition:

I) *G is a proper open interval of R and $\leq$ the total order on G inherited from the usual order on R.*

The following result of Aczél will be helpful to show that, under the condition **I**, a continuous real alo-group $\mathcal{G} = (G, \odot, \leq)$ can be built starting from the real alo-group $\mathcal{R}$ or the real alo-group **]0,+∞[**.

THEOREM 4.1. *Ref. 18. Under the assumption **I**, let $\odot$ be a binary operation over G. Then $\odot$ is a continuous, associative and cancellative operation if and only if there exists a continuous and strictly monotonic function $\phi : J \to G$ such that:*

$$x \odot y = \phi(\phi^{-1}(x) + \phi^{-1}(y)) \tag{4.1}$$

and J is R or one of real intervals $]-\infty, \gamma[$, $]-\infty, \gamma]$, $]\delta, +\infty[$, $[\delta, +\infty[$. The function ϕ in Equation 4.1 is unique up to a linear transformation of the variable (that is $\phi(x)$ may by replaced by $\phi(Cx)$, $C \neq 0$, but by no other function.)

COROLLARY 4.1. *Under the assumption **I**, let $\odot$ be a continuous, associative and cancellative operation over G. Then, $\odot$ is commutative and strictly increasing in each variable.*

Proof. *By Equation 4.1, commutativity of the addition and strict monotonicity of ϕ.* ■

THEOREM 4.2. *Under the assumption **I**, the following assertions are equivalent:*

1. *$\mathcal{G} = (G, \odot, \leq)$ is a continuous alo-group;*
2. *there exists a continuous and strictly increasing function $\phi : R \to G$ verifying the equality in Equation* 4.1;

3. *there exists a continuous and strictly increasing function* $\psi :]0, +\infty[\to G$ *verifying the equality*

$$x \odot y = \psi(\psi^{-1}(x) \cdot \psi^{-1}(y)). \tag{4.2}$$

Proof. $1 \Leftrightarrow 2$. By Theorem 4.1, Corollary 4.1, and Equation 2.2, $\mathcal{G}$ is a continuous alo-group if and only if there exists a continuous and strictly monotonic function $\phi : J \to G$ defined on a proper interval J of R and verifying the equality in Equation 4.1; this function can be chosen strictly increasing because it is unique up to a linear transformation of the variable. So, in order to prove the equivalence between item *1* and *2* it is enough to prove that the domain J of the function ϕ in Equation 4.1 coincides with R. To this purpose we observe that, by Equation 4.1,

$$x = x \odot e \Leftrightarrow \phi^{-1}(x) = \phi(x)^{-1} + \phi^{-1}(e) \Leftrightarrow \phi^{-1}(e) = 0,$$

thus $0 \in J$ and

$$x \odot x^{(-1)} = e \Leftrightarrow \phi^{-1}(x) + \phi^{-1}\left(x^{(-1)}\right) = \phi^{-1}(e) = 0 \Leftrightarrow \phi^{-1}\left(x^{(-1)}\right) = -\phi^{-1}(x);$$

so, if $a = \phi^{-1}(x) \in J$ then also $-a = \phi^{-1}(x^{(-1)}) \in J$. By Theorem 4.1, the equality $J = R$ follows.

$2 \Leftrightarrow 3$. Assume the assertion *2* is true. Then, by composing ϕ on the function $h : x \in]0, +\infty[\to log(x) \in R$, we get:

$$\psi : x \in]0, +\infty[\to \phi(\log(x)) \in G,$$

that is a bijection between $]0, +\infty[$ and G. Moreover $\psi^{-1}(y) = \exp(\phi^{-1}(y)$ and $\psi(\psi^{-1}(x) \cdot \psi^{-1}(y)) = \phi(\log(\exp(\phi^{-1}(x)) \cdot \exp(\phi^{-1}(y)))) = \phi(\phi^{-1}(x) + \phi^{-1}(y)) = x \odot y$. The implication $2 \Rightarrow 3$ is achieved. The reverse implication can be proved by an analogous reasoning. ■

COROLLARY 4.2. *Under the assumption* ***I****, a continuous alo-group* $\mathcal{G} = (G, \circ, \leq)$ *is isomorphic to* $\mathcal{R}$ *and to* $]0,+\infty[$ *and is divisible; moreover, if* ϕ *and* ψ *are the functions in items 2 and 3 of Theorem 4.2, then*

$$m_{\odot}(a_1, a_2, \ldots, a_n) = \phi\left(\frac{1}{n}\sum_{i=1}^{n}\phi^{-1}(a_i)\right) = \psi\left(\prod_{i=1}^{n}\psi^{-1}(a_i)\right)^{\frac{1}{n}},$$

$$d_{\mathcal{G}}(a, b) = \phi(d_{\mathcal{R}}(\phi^{-1}(a), \phi^{-1}(b))) = \psi(d_{]\mathbf{0},+\infty[}(\psi^{-1}(a), \psi^{-1}(b))).$$

Proof. The functions ϕ and ψ in items 2 and 3 of Theorem 4.2 are obviously isomorphisms between $\mathcal{R}$ and $\mathcal{G}$ and between **]0,+∞[** and $\mathcal{G}$, respectively; so, $\mathcal{G}$ is divisible by Proposition 2.3, and the equalities involving $m_{\odot}(a_1, a_2, \ldots, a_n)$ and $d_{\mathcal{G}}(a, b)$ follow by Propositions 2.3 and 3.3. ■

By applying Theorem 4.2, we provide, in the following propositions, two examples of continuous real alo-groups over a limited interval of R.

PROPOSITION 4.1. *Let* $\oplus :]-1,1[^2 \to]-1,1[$ *be the operation defined by*

$$x \oplus y = \frac{(1+x)(1+y) - (1-x)(1-y)}{(1+x)(1+y) + (1-x)(1-y)} \tag{4.3}$$

and $\leq$ *the order inherited by the usual order in* R. *Then* $\mathbf{]-1,1[} = (]-1,1[, \oplus, \leq)$ *is a continuous alo-group and it is* $e = 0$, $x^{(-1)} = -x$ *for each* $x \in]-1,1[$.

Proof. The function $g : t \in]0,+\infty[\to \frac{t-1}{t+1} \in]-1,1[$, is a bijection between $]0,+\infty[$ and $]-1,1[$, that is continuous and strictly increasing. For $a, b \in]0,+\infty[$ and $x = g(a),\ \ y = g(b)$, we get

$$g(a) \oplus g(b) = \frac{\left(1+\frac{a-1}{a+1}\right)\left(1+\frac{b-1}{b+1}\right) - \left(1-\frac{a-1}{a+1}\right)\left(1-\frac{b-1}{b+1}\right)}{\left(1+\frac{a-1}{a+1}\right)\left(1+\frac{b-1}{b+1}\right) + \left(1-\frac{a-1}{a+1}\right)\left(1-\frac{b-1}{b+1}\right)} = \frac{ab-1}{ab+1} = g(a \cdot b).$$

Thus, $x \oplus y = g(g^{-1}(x) \cdot g^{-1}(y))$, and Equation 4.2 in Theorem 4.2 is verified with $\psi = g$. Finally, it is easy to verify that $x \oplus 0 = x$ and $x \oplus (-x) = 0$. ∎

PROPOSITION 4.2. *Let* $\otimes :]0,1[^2 \to]0,1[$ *be the operation defined by*

$$x \otimes y = \frac{xy}{xy + (1-x)(1-y)}, \tag{4.4}$$

and $\leq$ *the order inherited by the usual order in* R. *Then* $\mathbf{]0,1[} = (]0,1[, \otimes, \leq)$ *is a continuous alo-group and it is* $e = 0,5$ *and* $x^{(-1)} = 1 - x$ *for each* $x \in]0,1[$.

Proof. The function

$$v : t \in]0,+\infty\Big[\to \frac{t}{t+1} \in \Big]0,1[, \tag{4.5}$$

is a bijection between $]0,+\infty[$ and $]0,1[$ that is continuous and strictly increasing. For $a, b \in]0,+\infty[$ and $x = v(a),\ \ y = v(b)$, we get:

$$v(a) \otimes v(b) = \frac{\frac{a}{a+1}\frac{b}{b+1}}{\frac{a}{a+1}\frac{b}{b+1} + \left(1-\frac{a}{a+1}\right)\left(1-\frac{b}{b+1}\right)} = \frac{ab}{ab+1} = v(a \cdot b).$$

Thus, $x \otimes y = v(v^{-1}(x) \cdot v^{-1}(y))$, and Equation 4.2 in Theorem 4.2 is verified with $\psi = v$. Finally, it is easy to verify that $x \otimes 0.5 = x$ and $x \otimes (1-x) = 0.5$. ∎

Let $\mathcal{R} = (R, +, \leq)$ and $]\mathbf{0},+\infty[= (]0, +\infty[, \cdot, \leq)$ be the alo-groups in Examples 1 and 2 and $]\mathbf{0}, \mathbf{1}[$ the alo-group in Proposition 4.2. Then, will call:

- $\mathcal{R}$ the *additive (real) alo-group*,
- $]\mathbf{0},+\infty[$ the *multiplicative (real) alo-group*,
- $]\mathbf{0}, \mathbf{1}[$ the *fuzzy (real) alo-group*.

Isomorphisms between $]\mathbf{0},+\infty[$ and $\mathcal{R}$ are

$$h : x \in]0, +\infty[\to \log x \in R, \quad h^{-1} : y \in R \to \exp(y) \in]0, +\infty[. \tag{4.6}$$

Isomorphisms between $]\mathbf{0},+\infty[$ and $]\mathbf{0}, \mathbf{1}[$ are the function v in Equation 4.5 and its inverse:

$$v^{-1} : y \in]0, 1\left[\to \frac{y}{1-y} \in \right] 0, +\infty[. \tag{4.7}$$

5. PAIRWISE COMPARISON MATRICES OVER A DIVISIBLE ALO-GROUP

In this section and in the next one, $\mathcal{G} = (G, \odot, \leq)$ denotes a divisible alo-group. A pairwise comparison system over $\mathcal{G}$ is a pair $(X, \mathcal{A})$ constituted by a set $X = \{x_1, \ldots, x_n\}$ and a relation $\mathcal{A} : (x_i, x_j) \in X^2 \to a_{ij} = \mathcal{A}(x_i, x_j) \in G$, represented by means of the PC matrix in Equation 1.1, with entries in G. In the context of an evaluation problem, the element a_{ij} can be interpreted as a measure on $\mathcal{G}$ of the preference of x_i over x_j: $a_{ij} > e$ implies that x_i is strictly preferred to x_j, whereas $a_{ij} < e$ expresses the opposite preference and $a_{ij} = e$ means that x_i and x_j are indifferent. Then $A = (a_{ij})$ is assumed to be *reciprocal* with respect to the operation $\odot$, that is,

$\mathbf{r}_{\odot}$) $\quad a_{ji} = a_{ij}^{(-1)} \quad \forall\, i, j = 1, \ldots, n$ *(reciprocity)*,

so $a_{ii} = e$ for each $i = 1, 2, \ldots, n$ and $a_{ij} \odot a_{ji} = e$ for $i, j \in \{1, 2, \ldots, n\}$.

In the sequel, $\mathcal{PC}_n(\mathcal{G})$ will denote the set of the reciprocal PC matrices of order $n \geq 3$ over $\mathcal{G}$. Then, a matrix of $\mathcal{PC}_n(]\mathbf{0}, +\infty[)$ is a *multiplicative* PC matrix, a matrix of $\mathcal{PC}_n(\mathcal{R})$ is an *additive* PC matrix. In this context, a *fuzzy* PC matrix is a matrix belonging to $\mathcal{PC}_n(]\mathbf{0}, \mathbf{1}[)$.

If $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ then we will denote by

- $\underline{a}_i$ the $i-th$ row of A: $\underline{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$;
- $\underline{a}^j$ the $j-th$ column of A: $\underline{a}^j = (a_{1j}, a_{2j}, \ldots, a_{nj})$;
- $m_{\odot}(\underline{a}_i)$ the mean $m_{\odot}(a_{i1}, a_{i2}, \ldots, a_{in})$;
- $\underline{w}_{m_{\odot}}(A)$ the mean vector $(m_{\odot}(\underline{a}_1), m_{\odot}(\underline{a}_2), \cdots, m_{\odot}(\underline{a}_n))$;
- ρ_{ijk} the element $a_{ik} \div (a_{ij} \odot a_{jk})$ of G.

Hence

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = ||\rho_{ijk}||. \tag{5.1}$$

Because of the assumption $\mathbf{r}_{\odot}$) the equality $a_{ik} = a_{ij} \odot a_{jk}$ does not depend on the considered order of the indexes i, j, k, that is,

$$a_{ik} = a_{ij} \odot a_{jk} \Leftrightarrow a_{ij} = a_{ik} \odot a_{kj} \Leftrightarrow a_{jk} = a_{ji} \odot a_{ik} \Leftrightarrow a_{ji} = a_{jk} \odot a_{ki} \Leftrightarrow \dots . \tag{5.2}$$

So the following definition is well done.

DEFINITION 5.1. *Let $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$. Then*

1. $A = (a_{ij})$ is consistent with respect to the $3 - subset$ $\{x_i, x_j, x_k\}$ of X *if and only if* $a_{ik} = a_{ij} \odot a_{jk}$;
2. $A = (a_{ij})$ is consistent *if and only if it is consistent with respect to each* $3 - subset$ $\{x_i, x_j, x_k\}$ *of X, that is*

$$\boldsymbol{c}_{\odot}) \quad a_{ik} = a_{ij} \odot a_{jk} \quad \forall i, j, k \qquad (consistency).$$

Remark 5.1. In our context, a fuzzy PC matrix is defined over $]\mathbf{0}, \mathbf{1}[$ (see Remark 2.1); then the condition of fuzzy consistency, by Definition 5.1, becomes:

$$\boldsymbol{c}_{\otimes}) \quad a_{ik} = \frac{a_{ij}a_{jk}}{a_{ij}a_{jk} + (1 - a_{ij})(1 - a_{jk})} \quad \forall i, j, k.$$

PROPOSITION 5.1. *The property of consistency is equivalent to each one of the following conditions:*

$$\boldsymbol{c'}_{\odot}) \quad a_{ik} \div a_{jk} = a_{ij} \quad \forall i, j, k;$$

$$\boldsymbol{c''}_{\odot}) \quad \rho_{ijk} = e \quad \forall i, j, k.$$

Proof. By Definition 5.1 and the meanings of $\div$ and ρ_{ijk}. ■

PROPOSITION 5.2. $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ *is consistent if and only if*

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = e \qquad \forall i, j, k. \tag{5.3}$$

Proof. By Equation 5.1 and Proposition 5.1. ■

Remark 5.2. Because of the equivalences in Equation 5.2 in checking the conditions $\boldsymbol{c}_{\odot}$), $\boldsymbol{c}'_{\odot}$), $\boldsymbol{c}''_{\odot}$) and Equation 5.2, we can limit ourselves to the case $i < j < k$.

DEFINITION 5.2. *A vector* $\underline{w} = (w_1, w_2, \cdots, w_n)$, $w_i \in G$, *is* consistent *with respect to* $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ *if and only if*

$$w_i \div w_j = a_{ij} \quad \forall i, j = 1, 2, \dots, n. \tag{5.4}$$

Remark 5.3. By Equation 5.4 and the equivalences ($w_i > w_j \Leftrightarrow w_i \div w_j > e$) and ($w_i = w_j \Leftrightarrow w_i \div w_j = e$), we get that $w_i > w_j \Leftrightarrow a_{ij} > e \quad and \quad w_i = w_j \Leftrightarrow a_{ij} = e$. Thus, the weights assigned to the alternatives by a consistent vector $\underline{w}$ agree with the preferences expressed by the entries a_{ij} of the PC matrix.

PROPOSITION 5.3. *$A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ is consistent if and only if there exists a consistent vector $\underline{w} = (w_1, w_2, \cdots, w_n)$, $w_i \in G$.*

Proof. Let $A = (a_{ij})$ be consistent. Then by $\mathbf{c'}_\odot$, $a_{ij} = a_{ik} \div a_{jk}$; so the equalities in Equation 5.4 are verified by $\underline{w} = \underline{a}^k$. Viceversa, if $\underline{w}$ is a consistent vector, then $a_{ij} \odot a_{jk} = (w_i \div w_j) \odot (w_j \div w_k) = w_i \odot w_j^{(-1)} \odot w_j \odot w_k^{(-1)} = w_i \odot w_k^{(-1)} = a_{ik}$. ■

PROPOSITION 5.4. *The following assertions related to $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ are equivalent:*

i) $A = (a_{ij})$ is consistent;
ii) each column $\underline{a}^k$ is a consistent vector;
iii) the mean vector $\underline{w}_{m_\odot}$ is a consistent vector.

Proof. $i) \Leftrightarrow ii)$ because of Proposition 5.1, condition $\mathbf{c'}_\odot$. $i) \Leftrightarrow iii)$. The implication $iii) \Rightarrow i)$ follows by Proposition 5.3. Under the assumption i) let us apply Equations 2.11 and 2.3 to get

$$(m_\odot(\underline{a}_i) \div m_\odot(\underline{a}_j))^{(n)} = m_\odot(\underline{a}_i)^{(n)} \div m_\odot(\underline{a}_j)^{(n)}$$

$$= (a_{i1} \odot a_{i2} \odot \cdots \odot a_{in}) \odot (a_{j1} \odot a_{j2} \odot \cdots \odot a_{jn})^{(-1)}$$

$$= (a_{i1} \odot a_{1j}) \odot (a_{i2} \odot a_{2j}) \odot \cdots \odot (a_{in} \odot a_{nj}) = a_{ij}^{(n)}.$$

So $\underline{w}_{m_\odot}$ verifies Equation 5.4 and the implication $i) \Rightarrow iii)$ is achieved. ■

PROPOSITION 5.5. *Let $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ be divisible alo-groups and $h : G \to G'$ an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Then*

$$H : A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G}) \to H(A) = A' = (h(a_{ij}))$$

is a bijection between $\mathcal{PC}_n(\mathcal{G})$ and $\mathcal{PC}_n(\mathcal{G}')$ that preserves the consistency, that is A is consistent if and only if A' is consistent.

Proof. H is an injection because h is an injective function. By applying h to the entries of the matrix $A = (a_{ij})$, we get the matrix $A' = (h(a_{ij}))$, that is reciprocal too, because of the equality in Equation 2.13: so $H(A) = (h(a_{ij})) \in \mathcal{PC}_n(\mathcal{G}')$. Moreover, by the equality in Equation 2.12, if $A = (a_{ij})$ is consistent, then the transformed $A' = H(A)$ is consistent too.

Viceversa, if $A' = (a'_{ij}) \in \mathcal{PC}_n(\mathcal{G}')$, by applying h^{-1} to the entries of A', we get the matrix $A = (h^{-1}(a'_{ij}))$ that belongs to $\mathcal{PC}_n(\mathcal{G})$ and, by Equation 2.14, is consistent if and only if A' is consistent too. ∎

Under the hypotheses of Proposition 5.5, we say that $A' = (h(a_{ij}))$ is the *transformed* of A by means of h, and $A = (h^{-1}(a'_{ij})) = H^{-1}(A')$ is the *transformed* of A' by means of h^{-1}. By Proposition 2.3, if $A' = (h(a_{ij}))$, then the mean vector $\underline{w}_{m_\odot}(A)$ is transformed, by means of h, in the mean vector $\underline{w}_{m_\circ}(A')$. Viceversa h^{-1} transforms the mean vector $\underline{w}_{m_\circ}(A')$ in the mean vector $\underline{w}_{m_\odot}(A)$.

6. A CONSISTENCY INDEX

Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible alo-group and $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$. By Definition 5.1, $A = (a_{ij})$ is *inconsistent* if and only if it is inconsistent in at least one $3 -$ *subset* $\{x_i, x_j, x_k\}$. The closeness to the consistency depends on the degree of consistency with respect to each $3 -$ *subset* $\{x_i, x_j, x_k\}$ and can be measured by an average of these degrees. So, in order to define a consistency index for $A = (a_{ij})$, we first consider the case that X has only three elements.

6.1. Consistency Index in the Case $n = 3$

Let X be the set $\{x_1, x_2, x_3\}$ and the relation $\mathcal{A}$ on X represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{PC}_3(\mathcal{G}). \tag{6.1}$$

By Proposition 5.2, $A = (a_{ij})$ and Remark 5.2 is inconsistent if and only if $d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) > e$. It is natural to say that the more A is inconsistent the more $d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23})$ is far from e. So, we give the following definition:

DEFINITION 6.1. *The* consistency index *of the matrix in Equation* 6.1 *is given by*

$$I_{\mathcal{G}}(A) = ||\rho_{123}|| = d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}). \tag{6.2}$$

As particular cases, we get

- if $A \in \mathcal{PC}_3(]\mathbf{0},+\infty[)$ then
$$I_{]\mathbf{0},+\infty[}(A) = \frac{a_{13}}{a_{12} \cdot a_{23}} \vee \frac{a_{12} \cdot a_{23}}{a_{13}} \in [1, +\infty[\tag{6.3}$$
and A is consistent if and only if $I_{]\mathbf{0},+\infty[}(A) = 1$;
- if $A \in \mathcal{PC}_3(\mathcal{R})$, then
$$\begin{aligned} I_{\mathcal{R}}(A) &= |a_{13} - a_{12} - a_{23}| \\ &= (a_{13} - a_{12} - a_{23}) \vee (a_{12} + a_{23} - a_{13}) \in [0, +\infty[\end{aligned} \tag{6.4}$$
and A is consistent if and only if $I_{\mathcal{R}}(A) = 0$;

- if $A \in \mathcal{PC}_3(]\mathbf{0,1}[)$, then

$$I_{]\mathbf{0,1}[}(A)$$

$$= \frac{a_{13}(1-a_{12}\otimes a_{23})}{a_{13}(1-a_{12}\otimes a_{23})+(1-a_{13})(a_{12}\otimes a_{23})} \vee \frac{(a_{12}\otimes a_{23})(1-a_{13})}{(a_{12}\otimes a_{23})(1-a_{13})+(1-a_{12}\otimes a_{23})a_{13}}$$

$$= \frac{a_{13}(1-a_{12})(1-a_{23})}{a_{13}(1-a_{12})(1-a_{23})+(1-a_{13})a_{12}a_{23}} \vee \frac{a_{12}a_{23}(1-a_{13})}{a_{12}a_{23}(1-a_{13})+(1-a_{12})(1-a_{23})a_{13}}; \qquad (6.5)$$

and A is consistent if and only if $I_{]\mathbf{0,1}[}(A) = 0.5$.

The following proposition shows that the more $I_{\mathcal{G}}(A)$ is close to e the more the mean vector $\underline{w}_{m_\odot}$ is close to be a consistent vector.

PROPOSITION 6.1. *Let $\underline{w}_{m_\odot} = (w_1, w_2, w_3)$ be the mean vector associated to the matrix in Equation* 6.1 *and $\rho = \rho_{123}$. Then*

$$d_{\mathcal{G}}(w_i \div w_j, a_{ij}) = ||\rho||^{\frac{1}{3}} \quad \forall i \neq j.$$

Proof. By definition of ρ

$$a_{13} = \rho \odot a_{12} \odot a_{23}, \quad a_{21} = \rho \odot a_{23} \odot a_{31} \quad \textit{and} \quad a_{32} = \rho \odot a_{12} \odot a_{31}.$$

By the above inequalities and the equality $a_{ii} = e$, we get

- $w_1 = (a_{11} \odot a_{12} \odot a_{13})^{(\frac{1}{3})} = ({a_{12}}^2 \odot \rho \odot a_{23})^{(\frac{1}{3})}$;
- $w_2 = (a_{21} \odot a_{22} \odot a_{23})^{(\frac{1}{3})} = (\rho \odot {a_{23}}^{(2)} \odot a_{31})^{(\frac{1}{3})}$;
- $w_3 = (a_{31} \odot a_{32} \odot a_{33})^{(\frac{1}{3})} = (\rho \odot {a_{31}}^{(2)} \odot a_{12})^{(\frac{1}{3})}$.

Thus:

1. $w_1 \div w_2 = (a_{12}^3 \odot \rho)^{(\frac{1}{3})} = a_{12} \odot \rho^{(\frac{1}{3})}$;
2. $w_2 \div w_3 = (\rho \odot {a_{23}}^{(2)} \odot a_{31} \odot a_{13} \odot a_{23})^{(\frac{1}{3})} = a_{23} \odot \rho^{(\frac{1}{3})}$;
3. $w_3 \div w_1 = (\rho \odot {a_{31}}^{(2)} \odot a_{12} \odot a_{21} \odot a_{31})^{(\frac{1}{3})} = a_{31} \odot \rho^{(\frac{1}{3})}$.

By item 1, we get: $(w_1 \div w_2) \div a_{12} = \rho^{(\frac{1}{3})}$ and $a_{12} \div (w_1 \div w_2) = (\rho^{(-1)})(\frac{1}{3})$, thus $d_{\mathcal{G}}(w_1 \div w_2, a_{12}) = ||\rho||^{(\frac{1}{3})}$.
By item 2, we get: $(w_2 \div w_3) \div a_{23} = \rho^{(\frac{1}{3})}$ and $a_{23} \div (w_2 \div w_3) = (\rho^{(-1)})(\frac{1}{3})$, thus $d_{\mathcal{G}}(w_2 \div w_3, a_{23}) = ||\rho||^{(\frac{1}{3})}$.
By item 3, $(w_3 \div w_1) \div a_{31} = \rho^{(\frac{1}{3})}$ and $a_{31} \div (w_3 \div w_1) = (\rho^{(-1)})(\frac{1}{3})$, thus $d_{\mathcal{G}}(w_3 \div w_1, a_{31}) = ||\rho||^{(\frac{1}{3})}$. ■

PROPOSITION 6.2. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij})) \in \mathcal{PC}_3(\mathcal{G}')$ the transformed of the matrix in Equation* 6.1*, by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}'}(A') = h(I_{\mathcal{G}}(A))$.*

Proof. By the equality $h(a_{12}) \circ h(a_{23}) = h(a_{12} \odot a_{23})$ and Proposition 3.3,

$$I_{\mathcal{G}'}(A') = d_{\mathcal{G}'}(h(a_{13}), h(a_{12}) \circ h(a_{23})) = h(d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23})) = h(I_{\mathcal{G}}(A)).$$

■

COROLLARY 6.1. *Under the assumption* ***I****, let* $\mathcal{G}$ *be a continuous alo-group and* $\phi : R \to G$ *and* $\psi :]0, +\infty[\to G$ *the functions in items 2 and 3 of Theorem 4.2. Then, the consistency index of the matrix in Equation* 6.1 *is*

$$I_{\mathcal{G}}(A = (a_{ij})) = \phi(I_{\mathcal{R}}(A' = (\phi^{-1}(a_{ij})))) = \psi(I_{]\mathbf{0},+\infty[}(A'' = (\psi^{-1}(a_{ij})))). \quad (6.6)$$

Proof. By Proposition 6.2 or Corollary 4.2. ■

COROLLARY 6.2. *Let* v *be the isomorphism in Equation 4.5 between* $]\mathbf{0}, +\infty[$ *and* $]\mathbf{0,1}[$*,* v^{-1} *the inverse isomorphism in Equation 4.7 and* $A' = (a'_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$ *the transformed of* $A = (a_{ij}) \in \mathcal{PC}_3(]\mathbf{0}, \mathbf{1}[)$ *by means of* v^{-1}*. Then*

$$A' = \left(\frac{a_{ij}}{1 - a_{ij}}\right) \quad \textit{and} \quad I_{]]\mathbf{0},\mathbf{1}[}(A) = v(I_{]\mathbf{0},+\infty[}(A')). \quad (6.7)$$

Proof. By Corollary 6.1. ■

For an example related to the equalities in Equation 6.7 see Example 8.1.

6.2. Consistency Index in the Case $n > 3$

Let $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$, $n > 3$. Then, we will denote by

- T the set of the $3 -$ *subset* $\{x_i, x_j, x_k\}$ of X;
- $n_T = \frac{n!}{3!(n-3)!}$ the cardinality of T.

Of course, n_T is also the cardinality of the set $T(A) = \{(a_{ij}, a_{jk}, a_{ik}), i < j < k\}$. For i, j, k, with $i < j < k$,

$$A_{ijk} = \begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix}$$

is a submatrix of A related to the $3 -$ *subset* $\{x_i, x_j, x_k\}$ and $I_{\mathcal{G}}(A_{ijk}) = ||\rho_{ijk}|| = d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk})$ is its consistency index. By item 2 of Definition 5.1 and Remark 5.2, a consistency index of A has to be expressed in terms of the consistency indices $I_{\mathcal{G}}(A_{ijk})$. Hence, we set

DEFINITION 6.2. *The consistency index of $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ is given by*

$$I_{\mathcal{G}}(A) = \left(\bigodot_{i<j<k} I_{\mathcal{G}}(A_{ijk}) \right)^{(1/n_T)} = \left(\bigodot_{i<j<k} d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) \right)^{(1/n_T)} . \tag{6.8}$$

PROPOSITION 6.3. *If $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ then $I_{\mathcal{G}}(A_{ijk}) \geq e$ and $A = (a_{ij})$ is consistent if and only if $I_{\mathcal{G}}(A) = e$.*

Proof. As $\odot$ is increasing with respect to each variable, the statement follows by the property 1 of a $\mathcal{G}$-distance (see Definition 3.2) and by Proposition 5.2. ■

As particular cases, we get

- if $A \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$, then $I_{]\mathbf{0},+\infty[}(A) = \left(\prod_{i<j<k} I_{]\mathbf{0},+\infty[}(A_{ijk}) \right)^{\frac{1}{n_T}} \geq 1$ and A is consistent if and only if $I_{]\mathbf{0},+\infty[}(A) = 1$;
- if $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{R})$, then $I_{\mathcal{R}}(A) = \frac{1}{n_T} \sum_{i<j<k} I_{\mathcal{R}}(A_{ijk}) \geq 0$, and A is consistent if and only if $I_{\mathcal{R}}(A) = 0$;
- if $A \in \mathcal{PC}_n(]\mathbf{0}, \mathbf{1}[)$, then $I_{]\mathbf{0},\mathbf{1}[}(A) = (\bigotimes_{i<j<k} I_{]\mathbf{0},\mathbf{1}[}(A_{ijk}))^{(\frac{1}{n_T})} \in [0.5, 1[$ and A is consistent if and only if $I_{]\mathbf{0},\mathbf{1}[}(A) = 0.5$.

PROPOSITION 6.4. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij})) \in \mathcal{PC}_n(\mathcal{G}')$ the transformed of $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}'}(A') = h(I_{\mathcal{G}}(A))$.*

Proof. By Propositions 6.2 and Proposition 2.3. ■

COROLLARY 6.3. *Under the assumption* ***I***, *let $\mathcal{G}$ be a continuous alo-group and $\phi : R \to G$ and $\psi :]0, +\infty[\to G$ the functions in items 2 and 3 of Theorem 4.2. Then, the consistency index of $A = (a_{ij}) \in \mathcal{PC}_n(\mathcal{G})$ verifies the equalities in Equation* 6.6.

COROLLARY 6.4. *Let $A' = (a'_{ij}) \in \mathcal{PC}_n(\mathcal{R})$ be the transformed of $A = (a_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$, by means of the isomorphism h between* **]0,+∞[** *and $\mathcal{R}$, given in Equation* 4.6. *Then, $A' = (log(a_{ij}))$, $A = (\exp(a'_{ij}))$ and*

$$I_{\mathcal{R}}(A') = log(I_{]\mathbf{0},+\infty[}(A)), \quad I_{]\mathbf{0},+\infty[}(A) = \exp(I_{\mathcal{R}}(A')).$$

COROLLARY 6.5. *Let $A' = (a'_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, +\infty[)$ be the transformed of $A = (a_{ij}) \in \mathcal{PC}_n(]\mathbf{0}, \mathbf{1}[)$, by means of the isomorphism v^{-1} in Equation* 4.7. *Then, the equalities in Equation 6.7 hold.*

For examples related to the above corollaries see Examples 8.2 and 8.3.

7. CONCLUSION AND FUTURE WORK

We have defined a general context in which different approaches to pairwise comparison matrices can be unified. We have also provided a meaningful consistency index suitable for each kind of matrix, naturally linked to a notion of distance and easy to compute in the additive and multiplicative case; in the other cases, this index is the transformed of the consistency index of a suitable multiplicative matrix or a suitable additive matrix.

Following the results in Refs. 13–15, 19 for the multiplicative case, our future work will be directed to investigate, in the new general context, the following problems related to a pairwise comparison matrix:

- to determine the conditions on a PC matrix inducing a qualitative ranking (*actual ranking*) on the set X;
- to individuate the conditions ensuring the existence of vectors representing the actual ranking at different levels.

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APPENDIX

In this section, we provide examples of computing consistency indices; multiplicative, additive, and fuzzy cases are considered. In Examples 8.1 and 8.2, we verify the relationship in Corollary 6.2 and in Corollary 6.4. In Example 8.3, we apply the Corollary 6.5.

Example 8.1. Let us consider the matrix

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.1 \\ 0.6 & 0.9 & 0.5 \end{pmatrix} \in \mathcal{PC}_3(]\mathbf{0}, \mathbf{1}[);$$

then, by Equation 6.5,

$$I_{]\mathbf{0},\mathbf{1}[}(A) = \frac{0.4 \cdot 0.7 \cdot 0.9}{0.4 \cdot 0.7 \cdot 0.9 + 0.3 \cdot 0.1 \cdot 0.6} \vee \frac{0.3 \cdot 0.1 \cdot 0.6}{0.4 \cdot 0.7 \cdot 0.9 + 0.3 \cdot 0.1 \cdot 0.6}$$

$$= 0.9\overline{3} \vee 0.0\overline{6} = 0.9\overline{3}.$$

By applying the isomorphism v^{-1} in Equation 4.7 to the entries of A, we get

$$A' = \begin{pmatrix} 1 & \frac{3}{7} & \frac{2}{3} \\ \frac{7}{3} & 1 & \frac{1}{9} \\ \frac{3}{2} & 9 & 1 \end{pmatrix} \in \mathcal{PC}_3(]\mathbf{0}, +\infty[)$$

which consistency index, by Equation 6.3, is $I_{]\mathbf{0},+\infty[}(A') = 14 \vee \frac{1}{14} = 14$.
Let v be the isomorphism in Equation 4.5, then in accordance with Corollary 6.2, $I_{]\mathbf{0},\mathbf{1}[}(A) = v(I_{]\mathbf{0},+\infty[}(A')) = v(14) = \frac{14}{15} = 0.9\overline{3}$.

Example 8.2. Let us consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{7} & \frac{1}{7} & \frac{1}{5} \\ 7 & 1 & \frac{1}{2} & \frac{1}{3} \\ 7 & 2 & 1 & \frac{1}{9} \\ 5 & 3 & 9 & 1 \end{pmatrix} \in \mathcal{PC}_4(]\mathbf{0}, +\infty[);$$

then

$$I_{]\mathbf{0},+\infty[}(A) = \sqrt[4]{I_{]\mathbf{0},+\infty[}(A_{234}) \cdot I_{]\mathbf{0},+\infty[}(A_{134}) \cdot I_{]\mathbf{0},+\infty[}(A_{124}) \cdot I_{]\mathbf{0},+\infty[}(A_{123})}$$

$$= \sqrt[4]{6 \cdot 12.6 \cdot 4.2 \cdot 2} = 5.02.$$

Let h be the isomorphism in Equation 4.6 between $\mathbf{]0,+\infty[}$ and $\mathcal{R}$. By applying h to the entries of A, we get

$$A' = \begin{pmatrix} 0 & -ln7 & -ln7 & -ln5 \\ ln7 & 0 & -ln2 & -ln3 \\ ln7 & ln2 & 0 & -ln9 \\ ln5 & ln3 & ln9 & 0 \end{pmatrix} \in \mathcal{PC}_4(\mathcal{R})$$

which consistency index is

$$I_{\mathcal{R}}(A') = \frac{I_{\mathcal{R}}(A'_{234}) + I_{\mathcal{R}}(A'_{134}) + I_{\mathcal{R}}(A'_{124}) + I_{\mathcal{R}}(A'_{123})}{4}$$

$$= \frac{1.7917 + 2.5336 + 1.4350 + 0.6931}{4} = 1.6134.$$

In accordance with Corollary 6.4, $I_{\mathcal{R}}(A') = log(I_{\mathbf{]0,+\infty[}}(A)) = \log(5.02) = 1.6134$.

Example 8.3. Let us consider the matrix

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.4 \\ 0.7 & 0.5 & 0.1 & 0.2 \\ 0.6 & 0.9 & 0.5 & 0.8 \\ 0.6 & 0.8 & 0.2 & 0.5 \end{pmatrix} \in \mathcal{PC}_4(\mathbf{]0,1[})$$

By applying the function v^{-1} in Equation 4.7 to the entries of A, we get

$$A' = \begin{pmatrix} 1 & \frac{3}{7} & \frac{2}{3} & \frac{2}{3} \\ \frac{7}{3} & 1 & \frac{1}{9} & \frac{1}{4} \\ \frac{3}{2} & 9 & 1 & 4 \\ \frac{3}{2} & 4 & \frac{1}{4} & 1 \end{pmatrix} \in \mathcal{PC}_4(\mathbf{]0,+\infty[})$$

which consistency index is

$$I_{\mathbf{]0,+\infty[}}(A') = \sqrt[4]{I_{\mathbf{]0,+\infty[}}(A'_{234}) \cdot I_{\mathbf{]0,+\infty[}}(A'_{134}) \cdot I_{\mathbf{]0,+\infty[}}(A'_{124}) \cdot I_{\mathbf{]0,+\infty[}}(A'_{123})}$$

$$= \sqrt[4]{\frac{16}{9} \cdot 4 \cdot \frac{56}{9} \cdot 14} = 4.9888.$$

Let v be the isomorphism in Equation 4.5, then, by Corollary 6.5,

$$I_{\mathbf{]0,1[}}(A) = v(I_{\mathbf{]0,+\infty[}}(A')) = \frac{4.9888}{5.9888} = 0.833.$$

Characterizations of Consistent Pairwise Comparison Matrices over Abelian Linearly Ordered Groups

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We consider the framework of pairwise comparison matrices over abelian linearly ordered groups. We introduce the notion of $\odot$-proportionality that allows us to provide new characterizations of the consistency, efficient algorithms for checking the consistency and for building a consistent matrix. Moreover, we provide a new consistency index. © 2010 Wiley Periodicals, Inc.

1. INTRODUCTION

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives or criteria. An useful tool to determine a weighted ranking on X is a *pairwise comparison matrix* (PCM)

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \tag{1}$$

where entry a_{ij} expresses how much the alternative x_i is preferred to alternative x_j. A condition of *reciprocity* is assumed for the matrix $A = (a_{ij})$ in such a way that the preference of x_i over x_j expressed by a_{ij} can be exactly read by using the element a_{ji}.

Under a suitable condition of *consistency* for $A = (a_{ij})$, X is totally ordered and the values a_{ij} can be expressed by means of the components w_i and w_j of a suitable vector $\underline{w}$, that is called *consistent vector* for the matrix $A = (a_{ij})$; then $\underline{w}$ provides the weights for the elements of X.

The shape of the reciprocity and consistency conditions depends on the different meaning given to the number a_{ij}, as the following well-known cases show.

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Multiplicative case: $a_{ij} \in]0, +\infty[$ is a preference ratio and the conditions of *multiplicative reciprocity* and *consistency* are given respectively by

$$a_{ji} = \frac{1}{a_{ij}} \quad \forall\, i, j = 1, \dots, n,$$

$$a_{ik} = a_{ij} a_{jk} \quad \forall\, i, j, k = 1, \dots, n.$$

A consistent vector is a positive vector $\underline{w} = (w_1, w_2, \dots, w_n)$ verifying the condition $\frac{w_i}{w_j} = a_{ij}$ and so perfectly representing the preferences over X.

Additive case: $a_{ij} \in] - \infty, +\infty[$ is a preference difference and the conditions of *additive reciprocity* and *consistency* are expressed as follows:

$$a_{ji} = -a_{ij} \quad \forall\, i, j = 1, \dots, n,$$

$$a_{ik} = a_{ij} + a_{jk} \quad \forall\, i, j, k = 1, \dots, n.$$

A consistent vector is a vector $\underline{w} = (w_1, w_2, \dots, w_n)$ verifying the condition $w_i - w_j = a_{ij}$.

Fuzzy case: $a_{ij} \in [0, 1]$ measures the distance from the indifference that is expressed by 0.5; the conditions of *fuzzy reciprocity* and *consistency* are

$$a_{ji} = 1 - a_{ij} \quad \forall\, i, j = 1, \dots, n,$$

$$a_{ik} = a_{ij} + a_{jk} - 0.5 \quad \forall\, i, j, k = 1, \dots, n.$$

A consistent vector is a vector $\underline{w} = (w_1, w_2, \dots, w_n)$ verifying the condition $w_i - w_j = a_{ij} - 0.5$.

The multiplicative *PCMs* play a basic role in the analytic hierarchy process, a procedure developed by T.L. Saaty at the end of the 1970s.[12–14] In Refs. 2–5, and 9, properties of multiplicative PCMs are analyzed in order to determine a qualitative ranking on the set of the alternatives and find vectors representing this ranking. Additive and fuzzy matrices are investigated for instance by Barzilai[1] and Herrera-Viedma et al.[11]

In the case of a multiplicative PCM, Saaty suggests that the comparisons expressed in verbal terms have to be translated into preference ratios a_{ij} taking value in $S^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$. Let us stress that the assumption of the Saaty scale restricts the decision maker's possibility to be consistent: indeed, if the decision maker (DM) expresses the following preference ratios $a_{ij} = 5$ and $a_{jk} = 3$, then he will not be consistent because $a_{ij} a_{jk} = 15 > 9$. Similarly, for the fuzzy case, the assumption that $a_{ij} \in [0, 1]$, restricts the possibility to realize the fuzzy consistency: indeed, if the DM claims $a_{ij} = 0.9$ and $a_{jk} = 0.8$, then he will not be consistent because $a_{ij} + a_{jk} - 0.5 = 1.7 - 0.5 > 1$.

In order to unify the several approaches to PCMs and remove the above drawbacks, in Ref. 6 we introduce PCMs whose entries belong to an abelian linearly ordered group (*alo-group*) $\mathcal{G} = (G, \odot, \leq)$. In this way, the reciprocity and consistency conditions are expressed in terms of the group operation $\odot$ and the drawbacks related to the consistency condition are removed; in fact the consistency condition is expressed by $a_{ik} = a_{ij} \odot a_{jk}$, where $a_{ij} \odot a_{jk}$ is an element of G, for each choice of $a_{ij}, a_{jk} \in G$. As a nontrivial alo-group $\mathcal{G} = (G, \odot, \leq)$ has neither the greatest element nor the least element (see Ref. 6), the Saaty set S* and the interval [0, 1], embodied with the usual order $\leq$ on R, cannot be structured as alo-groups. Moreover:

- the assumption of *divisibility* for $\mathcal{G}$ allows us to introduce the notion of mean $m_{\odot}(a_1, \ldots, a_n)$ of n elements and associate a *mean vector* $\underline{w}_{m_{\odot}}$ to a PCM;
- the introduction of a notion of distance $d_{\mathcal{G}}$, linked to the operation $\odot$ in a divisible alo-group $\mathcal{G}$, allows us to provide a measure of consistency for a PCM over $\mathcal{G}$: indeed the consistency index $I_{\mathcal{G}}(A)$ is defined as mean of the distances $d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk})$, with $i < j < k$.

In Ref. 7, we analyze some properties of a consistent PCM and provide algorithms, to check whether or not a matrix is consistent and to build a consistent matrix by means of $n - 1$ comparisons. Moreover, we provide a new consistency index linked to the index $I_{\mathcal{G}}(A)$ introduced in Ref. 6 but easier to compute.

In this paper, we extend the previous results and introduce the abelian group $\mathcal{G}^n = (G^n, \odot_{\times})$ and the notion of $\odot$-proportionality, that allows us to:

- provide new characterizations of a consistent PCM;
- introduce an equivalence relation on G^n and a bijection between the quotient set $G^n / \sim_{\odot}$ and the set of the consistent PCMs;
- provide more efficient algorithms to check the consistency;
- provide more efficient algorithms to build a consistent PCM.

The paper is organized as follows: Section 2 provides notations, definitions and some results useful in the next sections; starting from an alo-group $\mathcal{G} = (G, \odot, \leq)$, Section 3 introduces the abelian group $\mathcal{G}^n = (G^n, \odot_{\times})$ and the notion of $\odot$-proportional vectors; Section 4 provides characterizations of the consistency in terms of $\odot$-proportionality of rows or columns and algorithms to build consistent PCMs; furthermore, this section links the consistent PCMs to the related consistent vectors and provides the way to build consistent PCMs starting from a vector; Section 5 provides further characterizations that allows us to build consistent PCMs and to check the consistency in a more efficient way; Section 6 gives the consistency index for a PCM; Section 7 provides concluding remarks and directions for future work.

2. PRELIMINARIES

Let G be a nonempty set provided with a total weak order $\leq$ and a binary operation $\odot : G \times G \to G$. Then $\mathcal{G} = (G, \odot, \leq)$ is called *alo-group*, if and only if

$(G, \odot)$ is an abelian group and the the following implication holds:

$$a \leq b \Rightarrow a \odot c \leq b \odot c.$$

The above implication is equivalent to

$$a < b \Rightarrow a \odot c < b \odot c,$$

where $<$ is the strict simple order associated to $\leq$.
If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then G is naturally equipped with the order topology induced by $\leq$ and the abelian group $G \times G$ is equipped with the related product topology. $\mathcal{G}$ is called a *continuous* alo-group if and only if the function $\odot$ is continuous.
Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group, then we assume that e denotes the *identity* of $\mathcal{G}$, $x^{(-1)}$ the *symmetric* of $x \in G$ with respect to $\odot$, $\div$ the *inverse operation* of $\odot$ defined by

$$a \div b = a \odot b^{(-1)}.$$

PROPOSITION 1. *It results:*

i) $a = b \odot c$ *if and only if* $c = b^{(-1)} \odot a$;
ii) $a = b \odot c$ *if and only if* $a^{(-1)} = b^{(-1)} \odot c^{(-1)}$;
iii) $(a \div c)^{(-1)} = c \div a$.

Proof. Because of the associativity and the cancellation property of a commutative group operation, we have: $a = b \odot c \Leftrightarrow b^{-1} \odot a = b^{-1} \odot (b \odot c) \Leftrightarrow b^{-1} \odot a = c$. Moreover, $a = b \odot c \Leftrightarrow a^{(-1)} = (b \odot c)^{(-1)} \Leftrightarrow a^{(-1)} = b^{(-1)} \odot c^{(-1)}$. Finally, by applying (ii), we get $(a \div c)^{(-1)} = (a \odot c^{(-1)})^{(-1)} = a^{(-1)} \odot c = c \div a$. ■

For a positive integer n, the (n)-*power* $x^{(n)}$ of $x \in G$ is defined as follows:

$$x^{(1)} = x$$

$$x^{(n)} = \bigodot_{i=1}^{n-1} x_i \odot x_n = \bigodot_{i=1}^{n} x_i, \qquad x_i = x \quad i = 1, \ldots, n, \ n \geq 2.$$

If $b^{(n)} = a$, then we say that b is the (n)-*root* of a and write $b = a^{(1/n)}$.

$\mathcal{G}$ is *divisible* if and only if for each positive integer n and each $a \in G$ there exists the (n)-root of a.

Definition 1. *Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible alo-group. Then, the* $\odot$-mean *$m_\odot(a_1, a_2, \ldots, a_n)$ of the elements $a_1, a_2, \ldots, a_n$ of G is defined by*

$$m_\odot(a_1, a_2, \ldots, a_n) = \begin{cases} a_1 & n = 1, \\ \left(\bigodot_{i=1}^n a_i\right)^{(1/n)} & n \geq 2. \end{cases}$$

Definition 2. *An* isomorphism *between two alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ is a bijection $h : G \to G'$ that is both a lattice isomorphism and a group isomorphism, that is,*

$$x < y \Leftrightarrow h(x) < h(y)$$

$$h(x \odot y) = h(x) \circ h(y).$$

Proposition 2.[6] *Let $h : G \to G'$ be an isomorphism between the alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$. Then $\mathcal{G}$ is divisible if and only if $\mathcal{G}'$ is divisible and, under the assumption of divisibility*

$$m_\circ(y_1, y_2, \ldots, y_n) = h\big(m_\odot(h^{-1}(y_1), h^{-1}(y_2), \ldots, h^{-1}(y_n))\big).$$

2.1. The Notion of Distance

The *norm* of an element $a \in G$ is defined by setting

$$||a|| = a \vee a^{(-1)}. \tag{2}$$

Let us consider the operation

$$d_{\mathcal{G}} : (a, b) \in G^2 \to ||a \div b|| \in G. \tag{3}$$

In Ref. 6, we prove that $d_{\mathcal{G}}$ verifies the conditions:

1. $d_{\mathcal{G}}(a, b) \geq e$;
2. $d_{\mathcal{G}}(a, b) = e \Leftrightarrow a = b$;
3. $d_{\mathcal{G}}(a, b) = d_{\mathcal{G}}(b, a)$;
4. $d_{\mathcal{G}}(a, b) \leq d_{\mathcal{G}}(a, c) \odot d_{\mathcal{G}}(b, c)$,

so, we provide the following definition.

Definition 3. *The operation $d_{\mathcal{G}}$ in (3) is a $\mathcal{G}$*-metric *or $\mathcal{G}$*-distance.

Proposition 3.[6] *Let $h : G \to G'$ be an isomorphism between the alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$. Then,*

$$d_{\mathcal{G}'}(a', b') = h(d_{\mathcal{G}}(h^{-1}(a'), h^{-1}(b'))).$$

2.2. Real Alo-groups

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R

Under the above assumption for G and $\leq$, examples of real divisible and continuous alo-groups are the following:

Multiplicative alo-group. $]0, +\infty[$ $= (]0, +\infty[, \cdot, \leq)$, where $\cdot$ is the usual multiplication on R. Then, $e = 1$, $x^{(-1)} = 1/x$, $x^{(n)} = x^n$ and $x \div y = \frac{x}{y}$. So

$$d_{]0,+\infty[}(a, b) = \frac{a}{b} \vee \frac{b}{a}$$

and

$$m_{\cdot}(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$

is the geometric mean.

Additive alo-group. $\mathcal{R} = (R, +, \leq)$, where $+$ is the usual addition on R. Then, $e = 0$, $x^{(-1)} = -x$, $x^{(n)} = nx$, $x \div y = x - y$. So

$$d_{\mathcal{R}}(a, b) = |a - b| = (a - b) \vee (b - a)$$

and

$$m_{+}(a_1, \ldots, a_n) = \frac{\sum_i a_i}{n}$$

is the arithmetic mean.

Fuzzy alo-group. $]0,1[$ $= (]0, 1[, \otimes, \leq)$, where $\otimes :]0, 1[^2 \to]0, 1[$ is the operation defined by

$$x \otimes y = \frac{xy}{xy + (1 - x)(1 - y)}.$$

Then, $e = 0.5$, $x^{(-1)} = 1 - x$, $x \div y = \frac{x(1-y)}{x(1-y)+(1-x)y}$. So

$$d_{]0,1[}(a, b) = \frac{a(1 - b)}{a(1 - b) + (1 - a)b} \vee \frac{b(1 - a)}{b(1 - a) + (1 - b)a}.$$

We will compute the fuzzy mean $m_{\otimes}(a_1, \ldots, a_n)$ in the sequel of this paper.

Remark 1. Our choice of the operation structuring the ordered interval $]0, 1[$ as an alo-group wants to obey the requests: 0,5 is the identity element and $1 - x$ is the symmetric of x. In this way, the condition of reciprocity for a PCM over a fuzzy alo-group is given again by $a_{ji} = 1 - a_{ij}$, as defined in Section 1.

By setting $G =]0, 1[$ and

$$\psi : t \in]0, +\infty[\to \frac{t}{t+1} \in]0, 1[, \tag{4}$$

that is a continuous and strictly increasing function between $]0, +\infty[$ and $]0, 1[$, we get

$$x \otimes y = \psi(\psi^{-1}(x) \cdot \psi^{-1}(y)).$$

Thus $(G, \otimes, \leq)$ is a continuous alo-group as a consequence of the following result of Ref. 6:

THEOREM 1. *Let G be a proper open interval of R and $\leq$ the total order on G inherited from the usual order on R, then the following assertions are equivalent:*

1. *$\mathcal{G} = (G, \odot, \leq)$ is a continuous alo-group;*
2. *there exists a continuous and strictly increasing function $\psi :]0, +\infty[\to G$ verifying the equality*

$$x \odot y = \psi(\psi^{-1}(x) \cdot \psi^{-1}(y)).$$

Moreover, $\otimes$ verifies our requests about the identity and the symmetric of an element of the group.

Remark 2. The operation $\otimes$ is the restriction to $]0, 1[^2$ of the uninorm (see Ref. 10):

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\}; \\ \frac{xy}{xy + (1-x)(1-y)}, & \text{otherwise.} \end{cases}$$

The multiplicative, the additive and the fuzzy alo-groups are isomorphic; in fact the bijection

$$h : x \in]0, +\infty[\to \log x \in R$$

is an isomorphism between $\mathbf{]0,+\infty[}$ and $\mathcal{R}$ and ψ in (4) is an isomorphism between $\mathbf{]0,+\infty[}$ and $\mathbf{]0,1[}$. So, by Proposition 2, the mean $m_{\otimes}(a_1, \ldots, a_n)$ related to the fuzzy alo-group can be computed, by means of the function in (4), as follows:

$$m_{\otimes}(a_1, \ldots, a_n) = \psi\left(\left(\prod_{i=1}^{n} \psi^{-1}(a_i)\right)^{\frac{1}{n}}\right).$$

2.3. PCMs over an Alo-group

Let $X = \{x_1, x_2, \ldots x_n\}$ a set of alternatives, $A = (a_{ij})$ in (1) the related PCM. We claim that $A = (a_{ij})$ is a PCM over the alo-group $\mathcal{G}$ if and only $a_{ij} \in G,\ \forall i, j \in \{1, \ldots, n\}$. We will denote by

1. $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ the rows of A;
2. $\underline{a}^1, \underline{a}^2, \ldots, \underline{a}^n$ the columns of A.

If $\mathcal{G} = (G, \odot, \leq)$ is a divisible alo-group, then we consider the *mean vector* $\underline{w}_{m_\odot}(A)$, associated to A, that is defined as follows:

$$\underline{w}_{m_\odot}(A) = (m_\odot(\underline{a}_1), m_\odot(\underline{a}_1), \ldots, m_\odot(\underline{a}_n)). \tag{5}$$

From now on, we assume that $A = (a_{ij})$ is *reciprocal* with respect to $\odot$, that is,

$$a_{ji} = a_{ij}^{(-1)} \quad \forall\, i, j = 1, \ldots, n, \tag{6}$$

so

$$a_{ii} = e \quad \text{and} \quad a_{ij} \odot a_{ji} = e \ \ \forall i, j \in \{1, 2, \ldots, n\}. \tag{7}$$

DEFINITION 4.[6] *$A = (a_{ij})$ is a* consistent *PCM with respect to $\odot$, if and only if*

$$a_{ik} = a_{ij} \odot a_{jk} \quad \forall i, j, k.$$

In Ref. 6, we provide the following characterization.

PROPOSITION 4. *$A = (a_{ij})$ is a consistent PCM with respect to $\odot$, if and only if:*

$$a_{ik} = a_{ij} \odot a_{jk} \quad \forall\, i, j, k : i < j < k. \tag{8}$$

2.4. A Consistency Index for a PCM

Let T be the set $\{(a_{ij}, a_{jk}, a_{ik}), i < j < k\}$ and $n_T = |T|$. From Proposition 2.3., we derive the following.

PROPOSITION 5. *$A = (a_{ij})$ is a* consistent *matrix with respect to $\odot$, if and only if*

$$d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) = e \quad \forall\, i, j, k : i < j < k.$$

So, by Proposition 5, $A = (a_{ij})$ is inconsistent if and only if $d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) > e$ for some triple $(a_{ij}, a_{jk}, a_{ik}) \in T$. Thus, in Ref. 6, we provide the following definition of consistency index.

DEFINITION 5. *The* consistency index *of A is given by*

$$I_{\mathcal{G}}(A) = \begin{cases} d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) & n = 3, \\ (\bigodot_T d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}))^{(\frac{1}{n_T})} & n > 3. \end{cases}$$

with

$$n_T = \frac{n(n-2)(n-1)}{6}.$$

PROPOSITION 6. *$I_{\mathcal{G}}(A) \geq e$ and A is consistent if and only if $I_{\mathcal{G}}(A) = e$.*

3. THE ABELIAN GROUP $\mathcal{G}^N = (G^N, \odot_\times)$ AND THE NOTION OF $\odot$-PROPORTIONALITY

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group and

$$G^n = \underbrace{G \times G \times \ldots \times G}_{n}, \tag{9}$$

the cartesian product where elements are the vectors $\underline{w} = (w_1, w_2, \ldots w_n)$, $w_i \in G$, $\forall i = 1, \ldots, n$.
We consider G^n embodied with the binary operation

$$\odot_\times : (\underline{w}, \underline{v}) \rightarrow \underline{w} \odot_\times \underline{v} = (w_1 \odot v_1, w_2 \odot v_2, \ldots w_n \odot v_n). \tag{10}$$

The operation $\odot_\times$ is associative and commutative, has identity element equal to $\underline{e} = (e, e, \ldots, e)$, and, for each $\underline{w} = (w_1, w_2, \ldots w_n) \in G^n$, $\underline{w}^{(-1)} = (w_1^{(-1)}, w_2^{(-1)}, \ldots, w_n^{(-1)})$ is the symmetric element of $\underline{w}$. Thus, $\mathcal{G}^n = (G^n, \odot_\times)$ is an abelian group.

Let $\div_\times$ be the inverse operation of $\odot_\times$, defined as follows: $\underline{w} \div_\times \underline{v} = \underline{w} \odot_\times \underline{v}^{(-1)}$. Then

$$\underline{w} \div_{\times} \underline{v} = (w_1 \div v_1, w_2 \div v_2, \dots w_n \div v_n). \tag{11}$$

In the following, for simplicity of notation, we will use $\odot$ for $\odot_{\times}$ and $\div$ for $\div_{\times}$.

PROPOSITION 7. *It results:*

i) $\underline{w} = \underline{t} \odot \underline{v}$ *if and only if* $\underline{v} = \underline{t}^{(-1)} \odot \underline{w}$*;*
ii) $\underline{w} = \underline{t} \odot \underline{v}$ *if and only if* $\underline{w}^{(-1)} = \underline{t}^{(-1)} \odot \underline{v}^{(-1)}$*;*
iii) $(\underline{w} \div \underline{v})^{(-1)} = \underline{v} \div \underline{w}$*.*

Proof. From Proposition 1. ■

DEFINITION 6. *Let* $c \in G$ *and* $\underline{w} = (w_1, w_2, \dots w_n) \in G^n$*. Then the* $\odot$*-composition of* c *and* $\underline{w}$ *is the vector* $\underline{c} \odot \underline{w}$*, where* $\underline{c} = (c, c, \dots, c) \in G^n$*; that is,*

$$c \odot \underline{w} = (c \odot w_1, c \odot w_2, \dots, c \odot w_n).$$

Then, $c \div \underline{w}$ denotes the composition $c \odot \underline{w}^{(-1)}$; so

$$c \div \underline{w} = (c \div w_1, \dots, c \div w_n) = (c \odot w_1^{(-1)}, \dots, c \odot w_n^{(-1)}).$$

Similarly, $\underline{w} \div c$ denotes the vector

$$\underline{w} \div c = (w_1 \div c, \dots, w_n \div c) = (w_1 \odot c^{(-1)}, \dots, w_n \odot c^{(-1)}).$$

From Proposition 7, we derive

$$\underline{w} = c \odot \underline{v} \Leftrightarrow \underline{v} = c^{(-1)} \odot \underline{w}. \tag{12}$$

Hence we provide the following definition.

DEFINITION 7. *The vectors* $\underline{w}$ *and* $\underline{v}$ *are* $\odot$*-proportional if and only if there exists* $c \in G$ *such that* $\underline{w} = c \odot \underline{v}$*.*

PROPOSITION 8. *The vectors* $\underline{w}$ *and* $\underline{v}$ *are* $\odot$*-proportional if and only the vectors* $\underline{w}^{(-1)}$ *and* $\underline{v}^{(-1)}$ *are also* $\odot$*-proportional.*

Proof. From Proposition 7, $\underline{w} = c \odot \underline{v}$ if and only if $\underline{w}^{(-1)} = c^{(-1)} \odot \underline{v}^{(-1)}$. ■

PROPOSITION 9. *The vectors* $\underline{w}$ *and* $\underline{v}$ *are* $\odot$*-proportional if and only if*

$$w_i \div w_j = v_i \div v_j \; \forall i, j = 1, \ldots n. \tag{13}$$

Proof. Let $\underline{w}$ and $\underline{v}$ be $\odot$-proportional vectors. Then there exists $c \in G$ such that $w_i = c \odot v_i \forall i = 1, \ldots, n$. Thus, $w_i \div w_j = (c \odot v_i) \odot (c \odot v_j)^{-1} = (c \odot v_i) \odot (c^{-1} \odot v_j^{-1}) = v_i \div v_j$ for each $i, j = 1, \ldots, n$.
Let (13) be verified, then there exists $c \in G$ such that $w_j \div v_j = c \; \forall j = 1, \ldots, n$. So $\underline{w} = c \odot \underline{v}$. ∎

Example 1. Let us consider the multiplicative alo-group **]0, +∞[**.
The vectors $\underline{w} = (1, 2, 3)$ and $\underline{v} = (\frac{1}{2}, 1, \frac{3}{2})$ are $\cdot$-proportional because $\underline{v} = \frac{1}{2} \cdot \underline{w}$.
Then

$$\frac{w_1}{w_2} = \frac{v_1}{v_2} = \frac{1}{2}, \quad \frac{w_1}{w_3} = \frac{v_1}{v_3} = \frac{1}{3}, \quad \frac{w_2}{w_3} = \frac{v_2}{v_3} = \frac{2}{3}.$$

Example 2. Let us consider the additive alo-group $\mathcal{R}$.
The vectors $\underline{w} = (-2, 0, 1)$ and $\underline{v} = (0, 2, 3)$ are $+$-proportional because $\underline{v} = 2 + \underline{w}$.
Then

$$w_1 - w_2 = v_1 - v_2 = -2, \quad w_1 - w_3 = v_1 - v_3 = -3,$$
$$w_2 - w_3 = v_2 - v_3 = -1.$$

Example 3. Let us consider the fuzzy alo-group **]0,1[**.
The vectors $\underline{w} = (0.5, 0.6, 0.4)$ and $\underline{v} = (0.4, 0.5, 0.3077)$ are $\otimes$-proportional because $\underline{v} = 0.4 \otimes \underline{w}$.
Then

$$w_1 \otimes w_2^{(-1)} = v_1 \otimes v_2^{(-1)} = 0.4, \quad w_2 \otimes w_1^{(-1)} = v_2 \otimes v_1^{(-1)} = 0.6,$$
$$w_1 \otimes w_3^{(-1)} = v_1 \otimes v_3^{(-1)} = 0.6, \quad w_3 \otimes w_1^{(-1)} = v_3 \otimes v_1^{(-1)} = 0.4,$$
$$w_2 \otimes w_3^{(-1)} = v_2 \otimes v_3^{(-1)} = 0.6923, \quad w_3 \otimes w_2^{(-1)} = v_3^{(-1)} \otimes v_2^{(-1)} = 0.3077.$$

PROPOSITION 10. *Given $\underline{w} = (w_1, w_2, \ldots, w_n)$ and $\underline{v} = (v_1, v_2, \ldots, v_n)$ in G^n, the relation $\sim_{\odot}$, defined as*

$$\underline{w} \sim_{\odot} \underline{v} \Leftrightarrow \exists c \in G : \underline{w} = c \odot \underline{v}$$

is an equivalence relation.

Proof. As $\underline{w} = e \odot \underline{w}$ then $\underline{w} \sim_{\odot} \underline{w}$. So $\sim_{\odot}$ is a reflexive relation.
Because of the equivalence (12), $\sim_{\odot}$ is a symmetric relation.
If $\underline{w} = c_1 \odot \underline{v}$ and $\underline{v} = c_2 \odot \underline{u}$, then, by associativity property, $\underline{w} = (c_1 \odot c_2) \odot \underline{u}$.
Thus $(\underline{w} \sim_{\odot} \underline{v},\ \ \underline{v} \sim_{\odot} \underline{u}) \Rightarrow \underline{w} \sim_{\odot} \underline{u}$ and $\sim_{\odot}$ is a transitive relation. ■

We denote by $[\underline{w}]$ the $\odot$-equivalence class of $\underline{w}$, that is,

$$[\underline{w}] = \{\underline{v} \in G^n : \underline{w} \sim_{\odot} \underline{v}\}$$

and with $G^n / \sim_{\odot}$ the quotient set of G^n by $\sim_{\odot}$.

4. CONSISTENT PCMS: FIRST CHARACTERIZATIONS

In this section, $\mathcal{G} = (G, \odot, \leq)$ is a divisible alo-group and $A = (a_{ij})$ in (1) is a PCM over $\mathcal{G}$. By assumption, A is *reciprocal* with respect to $\odot$, so by (6) we get

$$\underline{a}^i = \underline{a}_i^{(-1)} \quad \forall\, i = 1, \ldots, n. \tag{14}$$

By means of the following proposition we reformulate the condition of consistency given in Definition 4.

PROPOSITION . *$A = (a_{ij})$ is a consistent PCM with respect to $\odot$, if and only if*

$$\underline{a}_i = a_{ij} \odot \underline{a}_j \quad \forall i, j = 1, 2, \ldots, n, \tag{15}$$

or, equivalently

$$\underline{a}^i = a_{ij}^{(-1)} \odot \underline{a}^j \quad \forall i, j = 1, 2, \ldots, n. \tag{16}$$

Proof. The equalities (15) follow from Definitions 4 and 6; the equalities (16) follow from the equalities (14), (15) and item (ii) of Proposition 7. ■

By the reciprocity property, the consistency conditions (15) can be written as follows:

$$\underline{a}_i = a_{ji}^{(-1)} \odot \underline{a}_j \quad \forall i, j = 1, 2, \ldots, n. \tag{17}$$

The conditions (16) and (17) characterize the consistent PCMs and allows us to build a consistent PCM starting from a column or a row, respectively.

By applying (17), we provide Algorithm 1 for building a consistent PCM starting from $n-1$ comparisons of the row $\underline{a}_{i^*}$; this algorithm generalizes and improves the algorithm proposed in Ref. 7.

Algorithm 1: BUILD A CONSISTENT PCM FROM THE SET $\{a_{i^*i} : i \neq i^*\}$

$a_{i^*i^*} = e$
for $i = 1 \ldots n$ **do**
  if $i \neq i^*$ **then**
    $\underline{a}_i = a_{i^*i}^{(-1)} \odot \underline{a}_{i^*}$
  end if
end for

Example 4. Let us consider the multiplicative alo-group $]\mathbf{0}, +\infty[$.
Let $\{x_1, x_2, x_3\}$ be a set of alternatives. We suppose that the DM expresses the following preference ratios: $a_{12} = 2$ and $a_{13} = 3$. Thus, $i^* = 1$ and the rows of the related consistent PCM are the following:

$$\underline{a}_1 = (1, 2, 3),$$

and

$$\underline{a}_2 = \frac{1}{2} \cdot \underline{a}_1 = \left(\frac{1}{2}, 1, \frac{3}{2}\right),$$

$$\underline{a}_3 = \frac{1}{3} \cdot \underline{a}_1 = \left(\frac{1}{3}, \frac{2}{3}, 1\right).$$

Example 5. Let us consider the additive alo-group $\mathcal{R}$.
Let $\{x_1, x_2, x_3\}$ be a set of alternatives. We suppose that the DM expresses the following preference differences: $a_{21} = 2$ and $a_{23} = 3$. Thus, $i^* = 2$ and the rows of the related consistent PCM are the following:

$$\underline{a}_2 = (2, 0, 3),$$

and

$$\underline{a}_1 = -2 + \underline{a}_2 = (0, -2, 1)$$

$$\underline{a}_3 = -3 + \underline{a}_2 = (-1, -3, 0).$$

Example 6. Let us consider the fuzzy alo-group **]0,1[**.
Let $\{x_1, x_2, x_3\}$ be a set of alternatives. We suppose that the DM expresses the following fuzzy preferences: $a_{12} = 0.4$ and $a_{13} = 0.6$. Thus, $i^* = 1$ and the rows of the related consistent PCM are the following:

$$\underline{a}_1 = (0.5, 0.4, 0.6),$$

and

$$\underline{a}_2 = 0.6 \otimes \underline{a}_1 = (0.6, 0.5, 0.69),$$

$$\underline{a}_3 = 0.4 \otimes \underline{a}_1 = (0.4, 0.31, 0.5).$$

The following theorem gives a characterization of a consistent PCM in terms of $\odot$-proportional vectors. This characterization allows us to check in an easy way the property of consistency.

THEOREM 2. *The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij})$ is a consistent PCM;*
2. *for every choice of $i, j = 1, 2, \ldots, n$ the rows $\underline{a}_i$ and $\underline{a}_j$ are $\odot$-proportional vectors; so for a fixed index i^*, $[\underline{a}_i] = [\underline{a}_{i^*}]$ for $i = 1, 2, \ldots, n$;*
3. *for every choice of $i, j = 1, 2, \ldots, n$ the columns $\underline{a}^i$ and $\underline{a}^j$ are $\odot$-proportional vectors; so for a fixed index i^*, $[\underline{a}^i] = [\underline{a}^{i^*}]$ for $i = 1, 2, \ldots, n$.*

Proof. 1. $\Leftrightarrow$ 2. If $A = (a_{ij})$ is a consistent PCM the item 2 follows immediately from the condition of consistency as formulated in (17). Alternatively, if item 2 is verified, then, for every choice of $i, j = 1, 2, \ldots, n$, there exists an element $c_{ij} \in G$ such that $\underline{a}_i = c_{ij} \odot \underline{a}_j$, so that $a_{ik} = c_{ij} \odot a_{jk}$ for each $k = 1, 2, \ldots, n$. By choosing $k = j$, we get $a_{ij} = c_{ij} \odot e = c_{ij}$; thus the condition (15) is verified and $A = (a_{ij})$ is a consistent PCM.
2 $\Leftrightarrow$ 3. From Proposition 8 or reasoning as above starting from equality (16). ■

Remark 3. We stress that, because of the assumption of reciprocity, the $\odot$-proportionality of the rows (resp. of the columns) is a necessary and sufficient condition for the consistency. Moreover, the condition of reciprocity allows us to determine the constants of $\odot$-proportionality by means of one of the equalities (15), (16) or (17).

Example 7. Let

$$A = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{3}{4} \\ 2 & \frac{4}{3} & 1 \end{pmatrix}$$

be a PCM over the multiplicative alo-group $]0,+\infty[$. A is consistent because $\underline{a}^2, \underline{a}^3 \in [\underline{a}^1]$. Indeed, $\underline{a}^2 = \frac{2}{3} \odot \underline{a}^1$ and $\underline{a}^3 = \frac{1}{2} \odot \underline{a}^1$.
The matrix

$$B = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{1}{4} \\ 2 & 4 & 1 \end{pmatrix}$$

over $]0,+\infty[$ is not consistent because $\frac{a_{21}}{a_{31}} = \frac{3}{4}$ is not equal to $\frac{a_{22}}{a_{32}} = \frac{1}{4}$, so for the column vectors $\underline{a}^1$ and $\underline{a}^2$ the condition (13) in Proposition 9 is not verified and $\underline{a}^2 \notin [\underline{a}^1]$.

COROLLARY 1. *The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij})$ is a consistent PCM;*
2. *for every choice of $i < n$, the rows $\underline{a}_i$ and $\underline{a}_{i+1}$ are $\odot$-proportional vectors,*
3. *for every choice of $i < n$, the columns $\underline{a}^i$ and $\underline{a}^{i+1}$ are $\odot$-proportional vectors.*

Proof. From Theorem 2 and the transitivity of the relation $\sim_{\odot}$. ∎

Remark 4. As we stressed in Remark 3 the constants of $\odot$-proportionality between two rows (resp. columns) are computed by means of (17) (resp. (16)), so

$$\underline{a}_{i+1} = a_{i\,i+1}^{(-1)} \odot \underline{a}_i \quad \forall i < n;$$

$$\underline{a}^{i+1} = a_{i+1\,i}^{(-1)} \odot \underline{a}^i \quad \forall i < n.$$

4.1. Consistent Vectors and Consistent Matrices

DEFINITION 8.[6] *A vector $\underline{w} = (w_1, \ldots, w_n)$, with $w_i \in G$, is a* consistent *vector for $A = (a_{ij})$ if and only if*

$$w_i \div w_j = a_{ij} \ \forall\ i, j = 1, 2, \dots, n.$$

PROPOSITION 12.[6] *$A = (a_{ij})$ is a consistent PCM if and only if there exists a consistent vector $\underline{w} = (w_1, w_2, \dots, w_n)$, $w_i \in G$.*

PROPOSITION 13. *Let $A = (a_{ij})$ be a consistent PCM and $\underline{w}$ a consistent vector for $A = (a_{ij})$. A vector $\underline{v}$ is consistent for $A = (a_{ij})$ if and only if $\underline{v} \in [\underline{w}]$.*

Proof. From Definition 8 and Proposition 9. ■

We stress that the condition of consistency in Definition 4 can be written as

$$a_{ik} \div a_{jk} = a_{ij} \quad \forall i, j, k,$$

thus, in a consistent PCM, each column is a consistent vector. In Ref. 6 the following result is provided.

PROPOSITION 14. *The following assertions related to $A = (a_{ij})$ are equivalent:*

i) $A = (a_{ij})$ is a consistent PCM;
ii) each column $\underline{a}^k$ is a consistent vector;
iii) the mean vector $\underline{w}_{m_\odot} = (m_\odot(\underline{a}_1), \dots, m_\odot(\underline{a}_n))$ is a consistent vector.

4.2. Bijection between $G^n/\sim_\odot$ and the Set of Consistent PCMs

THEOREM 3 . *Let $\mathcal{CM}$ be the set of the consistent PCM, then the relation:*

$$F : [\underline{v}] \in G^n/\sim_\odot \rightarrow A_v = \begin{pmatrix} v_1 \div v_1 & v_1 \div v_2 & \dots & v_1 \div v_n \\ v_2 \div v_1 & v_2 \div v_2 & \dots & v_2 \div v_n \\ \dots & \dots & \dots & \dots \\ v_n \div v_1 & v_n \div v_2 & \dots & v_n \div v_n \end{pmatrix} \in \mathcal{CM}$$

is a bijective function.

Proof. The function F is well defined because if $\underline{w} \in [\underline{v}]$ then, by Proposition 9, $w_i \div w_j = v_i \div v_j$ $\forall i, j = 1, \dots, n$ and so $A_{\underline{v}} = A_{\underline{w}}$.

Moreover,

- if $[\underline{v}] \neq [\underline{w}]$ then, by Proposition 9, $\exists i, j : w_i \div w_j \neq v_i \div v_j$, as a consequence $A_{\underline{v}} \neq A_{\underline{w}}$; so F is injective;
- if A is a consistent PCM, then, by Proposition 12, there exists a consistent vector $\underline{v}$; by Definition 8 and Proposition 13, $F([\underline{v}]) = A$. Thus F is surjective.

■

Theorem 3 allows us to build a consistent PCM starting form a vector; so we provide Algorithm 2.

Algorithm 2: BUILD A CONSISTENT PCM FROM A VECTOR $\underline{v} = (v_1, \ldots v_n)$

---

for $i = 1 \ldots n$ **do**
  $\underline{a}_i = v_i \div \underline{v}$
end for

Similarly, we can build the PCM for columns, by setting $\underline{a}^i = \underline{v} \div v_i$.

Example 8. Let us consider the multiplicative alo-group $\mathbf{]0, +\infty[}$.
Let us assume $\underline{v} = (2, 3, 4)$. Let us set

$$a_1 = v_1/\underline{v} = 2/(2, 3, 4) = \left(1, \frac{2}{3}, \frac{1}{2}\right)$$

$$a_2 = v_2/\underline{v} = 3/(2, 3, 4) = \left(\frac{3}{2}, 1, \frac{3}{4}\right)$$

$$a_3 = v_3/\underline{v} = 4/(2, 3, 4) = \left(2, \frac{4}{3}, 1\right).$$

So $\underline{v}$ and each vector in $[\underline{v}]$ generates the consistent PCM

$$A = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{3}{4} \\ 2 & \frac{4}{3} & 1 \end{pmatrix}.$$

Example 9. Let us consider the additive alo-group $\mathcal{R}$.
The vector $\underline{v} = (-2, 1, 2)$ and each vector in $[\underline{v}]$ generates the consistent PCM

$$A = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & -1 \\ 4 & 1 & 0 \end{pmatrix}.$$

Example 10. Let us consider the fuzzy alo-group **]0,1[**.
The vector $\underline{v} = (0.3, 0.4, 0.7)$ and each vector in $[\underline{v}]$ generates the consistent PCM

$$A = \begin{pmatrix} 0.5 & 0.39 & 0.155 \\ 0.61 & 0.5 & 0.22 \\ 0.845 & 0.78 & 0.5 \end{pmatrix}.$$

In order to give a simple way to check the consistency of a PCM we provide the following proposition.

PROPOSITION 15. *$A = (a_{ij})$ is a consistent PCM if and only if $A = F[\underline{a}^i]$ for some $i = 1, 2, \ldots, n$.*

Proof. From Theorem 3, Definition 8 and Proposition 14. ■

For instance, the matrix A in Example 7 is consistent as $A = F(\underline{a}^1)$.

5. FURTHER CHARACTERIZATIONS OF CONSISTENT PCMS

In Section 4, we have provided characterizations for building a consistent PCM, starting from comparisons contained in a row (resp. a column) of the built PCM or from a vector that represents a consistent vector for the built PCM, and for checking the consistency by checking each entry of the given PCM.

In this section, we provide new characterizations of the consistency that allows us to build a consistent PCM from a different set of comparisons and to check the consistency, by checking only a minimum number of entries of the given PCM.

5.1. Useful Characterizations for Building a Consistent PCM

PROPOSITION 16. *The following assertions are equivalent:*

1. *A is a consistent PCM with respect to $\odot$;*
2. $a_{ik} = a_{i\,i+1} \odot a_{i+1\,k} \quad \forall i, k : i < k$;
3. $a_{ik} = a_{i\,i+1} \odot a_{i+1\,i+2} \odot \ldots \odot a_{k-1\,k} \quad \forall i, k : i < k$.

Proof. $1 \Rightarrow 2$. It is straightforward because of Proposition 4.

$2 \Rightarrow 3$. By 2:

$$a_{ik} = a_{i\,i+1} \odot a_{i+1\,k}$$

$$a_{i+1\,k} = a_{i+1\,i+2} \odot a_{i+2\,k}$$

$$\vdots$$

$$a_{k-2\,k} = a_{k-2\,k-1} \odot a_{k-1\,k}$$

Thus, by associativity of $\odot$, 3 is achieved.

$3 \Rightarrow 1$. By Proposition 4, it is enough to prove that $3 \Rightarrow (8)$. Let $i < j < k$. By 3, we have that

$$a_{ik} = a_{i\,i+1} \odot \ldots a_{j-1\,j} \odot a_{j\,j+1} \ldots \odot a_{k-1\,k};$$

so, by associativity of $\odot$ and applying again 3, we have that

$$a_{ik} = (a_{i\,i+1} \odot a_{i+1\,i+2} \odot \ldots \odot a_{j-1\,j})$$

$$\odot(a_{j\,j+1} \odot a_{j+1\,j+2} \odot \ldots \odot a_{k-1\,k}) = a_{ij} \odot a_{jk}.$$

■

Proposition 16 generalizes to PCMs defined on alo-group a result provided in Ref. 8 for the fuzzy case.

By Proposition 16, the following corollary follows.

COROLLARY 2. *$A = (a_{ij})$ is a consistent PCM with respect to $\odot$, if and only if:*

$$d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k}) = e \quad \forall\, i, k : i < k.$$

We provide Algorithm 3 to build a consistent PCM starting from $a_{12}, a_{23}, \ldots a_{n-1\,n}$.

Algorithm 3: BUILDING A CONSISTENT PCM STARTING FROM $a_{12}, a_{23}, \ldots a_{n-1\,n}$

```
for i = 1, ... n − 2 do
  a_{i i+2} = a_{i i+1} ⊙ a_{i+1 i+2}
  temp = a_{i i+2}
  for k = i + 3 ... n do
    a_{i k} = temp ⊙ a_{k−1 k}
    temp = a_{i k}
  end for
end for
```

for $i = 2, \ldots n$ **do**
  for $j = 1, \ldots i - 1$ **do**
    $a_{i\ j} = a_{j\ i}^{(-1)}$
  end for
end for
for $i = 1, \ldots, n$ **do**
  $a_{ii} = e$
end for

Example 11. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives. We suppose that the DM expresses the following preference ratios (multiplicative case): $a_{12} = 2$, $a_{23} = 2$, $a_{34} = \frac{5}{4}$ and $a_{45} = \frac{6}{5}$. By means of Algorithm 3, we obtain

$$a_{13} = a_{12} \cdot a_{23} = 2 \cdot 2 = 4,$$

$$a_{14} = a_{12} \cdot a_{23} \cdot a_{34} = a_{13} \cdot a_{34} = 5,$$

$$a_{15} = a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{45} = a_{14} \cdot a_{45} = 6,$$

$$a_{24} = a_{23} \cdot a_{34} = \frac{5}{2},$$

$$a_{25} = a_{23} \cdot a_{34} \cdot a_{45} = a_{24} \cdot a_{45} = 3,$$

$$a_{35} = a_{34} \cdot a_{45} = \frac{3}{2}.$$

Computing the symmetric elements, we have

$$a_{21} = \frac{1}{2},$$

$$a_{31} = \frac{1}{4},\ a_{32} = \frac{1}{2},$$

$$a_{41} = \frac{1}{5},\ a_{42} = \frac{2}{5},\ a_{43} = \frac{4}{5},$$

$$a_{51} = \frac{1}{6},\ a_{52} = \frac{1}{3},\ a_{53} = \frac{2}{3},\ a_{54} = \frac{5}{6}.$$

Finally,

$$a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1.$$

Thus,

$$A = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ \frac{1}{2} & 1 & 2 & \frac{5}{2} & 3 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{5}{4} & \frac{3}{2} \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} & 1 & \frac{6}{5} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{5}{6} & 1 \end{pmatrix}.$$

Example 12. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives. We suppose that the DM expresses the following preferences (fuzzy case): $a_{12} = 0.6, a_{23} = 0.609, a_{34} = 0.632$ and $a_{45} = 0.692$. By means of Algorithm 3, we obtain

$$A = \begin{pmatrix} 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.4 & 0.5 & 0.609 & 0.727 & 0.857 \\ 0.3 & 0.391 & 0.5 & 0.632 & 0.794 \\ 0.2 & 0.273 & 0.368 & 0.5 & 0.692 \\ 0.1 & 0.143 & 0.206 & 0.308 & 0.5 \end{pmatrix}.$$

5.2. Useful Characterizations for Checking the Consistency

In the following, if $\underline{w} = (w_1, \dots, w_n) \in G^n$, then $\underline{w}[l : m]$, with $1 \le l \le m \le n$, will denote the vector $(w_l, \dots, w_m) \in G^{m-l+1}$.

PROPOSITION 17. *The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij})$ is a consistent PCM;*
2. *the sub-rows $\underline{a}_i[i+2 : n]$ and $\underline{a}_{i+1}[i+2 : n]$ are $\odot$-proportional vectors for $i = 1 \dots n-2$, that is*

$$a_{i+1\,k} = a_{i\,i+1}^{(-1)} \odot a_{ik} \quad \forall i = 1, \dots n-2 \;\forall k = i+2, \dots n;$$

3. *the sub-columns $\underline{a}^i[1 : i-1]$ and $\underline{a}^{i+1}[1 : i-1]$ are $\odot$-proportional vectors for $i = 2, \dots n-1$, that is,*

$$a_{k\,i} = a_{i\,i+1}^{(-1)} \odot a_{k\,i+1} \quad \forall i = 2, \dots n-1 \;\forall k = 1, \dots i-1.$$

Proof. By assumption of reciprocity for $A = (a_{ij})$, Corollary 1 and Remark 4. ■

In order to check whether or not a PCM is consistent, we use item *2* of Proposition 17 and we provide Algorithm 4. In Algorithm 4, we assume that

- i is the index of the rows of A, it is initialized to $i = 1$;
- k is the index of the columns of A, it is initialized to $k = i + 2$;
- n is the order of A;
- *ConsistentMatrix* is a boolean variable and the algorithm returns $ConsistentMatrix = true$ if and only if the matrix is consistent. It is initialized to $true$, but $ConsistentMatrix = false$ is immediately returned when an inconsistent triple $(a_{ik}, a_{i\,i+1}, a_{i+1\,k})$ occurs.

Similar algorithm can be provided if we use item *3* of Proposition 17.

Algorithm 4: CHECKING CONSISTENCY

$i = 1$;
ConsistentMatrix = true;
while $i \leq n - 2$ and ConsistentMatrix **do**
  $k = i + 2$;
  while $k \leq n$ and ConsistentMatrix **do**
    if $a_{i+1\,k} \neq a_{i\,i+1}^{(-1)} \odot a_{ik}$ **then**
      ConsistentMatrix = false;
    end if
    $k = k + 1$;
  end while
  $i = i + 1$;
end while
return ConsistentMatrix;

Example 13. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives and

$$A = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ \frac{1}{2} & 1 & 2 & \frac{5}{3} & 3 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{5}{4} & \frac{3}{2} \\ \frac{1}{5} & \frac{3}{5} & \frac{4}{5} & 1 & \frac{6}{5} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{5}{6} & 1 \end{pmatrix}$$

the related multiplicative PCM. In order to check whenever A is consistent we have to check the following equalities:

$$\underline{a}_2[3:5] = a_{12}^{-1} \cdot \underline{a}_1[3:5] = \frac{1}{2} \cdot \underline{a}_1[3:5]$$

$$\underline{a}_3[4:5] = a_{23}^{-1} \cdot \underline{a}_2[4:5] = \frac{1}{2} \cdot \underline{a}_2[4:5]$$

and

$$\underline{a}_4[5:5] = a_{34}^{-1} \cdot \underline{a}_3[5:5] = \frac{4}{5} \cdot \underline{a}_3[5:5].$$

We stress that, for instance, $a_{24} \neq \frac{1}{2} \cdot a_{14}$; so the PCM is not consistent.

6. A NEW CONSISTENCY INDEX

In Section 2.4, we have introduced a consistency index $I_{\mathcal{G}}(A)$. At the light of Corollary 2, it is reasonable to define a new consistency index, considering only the distances $d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k})$, with $i < k-1$; of course, if $i = k-1$ then $d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k}) = e$. Let T^* be the set $\{(a_{i\,i+1}, a_{i+1\,k}, a_{ik}), i < k-1\}$ and $n_{T^*} = |T^*|$, then we consider the following index:

$$I_{\mathcal{G}}^*(A) = \begin{cases} d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) & n = 3, \\ \left(\bigodot_{T^*} d_{\mathcal{G}}(a_{ik}, a_{i\,i+1} \odot a_{i+1\,k}) \right)^{\left(\frac{1}{n_{T^*}}\right)} & n > 3. \end{cases}$$

with

$$n_{T^*} = \frac{(n-2)(n-1)}{2}.$$

From Corollary 2, we have the following.

PROPOSITION 18. *$I_{\mathcal{G}}^*(A) \geq e$ and A is consistent if and only if $I_{\mathcal{G}}^*(A) = e$.*

As for $n > 3$ it results $n_{T^*} < n_T$, the index $I_{\mathcal{G}}^*(A)$ is more easy to compute than the consistency index $I_{\mathcal{G}}(A)$, thus, we provide the following definition.

DEFINITION 9. *A* consistency index *of A is given by $I_{\mathcal{G}}^*(A)$.*

PROPOSITION 19. *Let $\mathcal{G}' = (G', \circ, \leq)$ be a divisible alo-group isomorphic to $\mathcal{G}$ and $A' = (h(a_{ij}))$ the transformed of $A = (a_{ij})$ by means of the isomorphism $h : G \to G'$. Then $I_{\mathcal{G}}^*(A) = h^{-1}(I_{\mathcal{G}'}^*(A'))$.*

Example 14. Let us consider

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.4 \\ 0.7 & 0.5 & 0.1 & 0.2 \\ 0.6 & 0.9 & 0.5 & 0.8 \\ 0.6 & 0.8 & 0.2 & 0.5 \end{pmatrix}$$

that is a PCM over the fuzzy alo-group **]0,1[**. By applying the function ψ^{-1}, with ψ in 4, to the entries of A, we get the matrix

$$A' = \begin{pmatrix} 1 & \frac{3}{7} & \frac{2}{3} & \frac{2}{3} \\ \frac{7}{3} & 1 & \frac{1}{9} & \frac{1}{4} \\ \frac{3}{2} & 9 & 1 & 4 \\ \frac{3}{2} & 4 & \frac{1}{4} & 1 \end{pmatrix}.$$

A' is a PCM over the multiplicative alo-group $\mathbf{]0,+\infty[}$ and its consistency index is

$$I^*_{\mathbf{]0,+\infty[}}(A') = \sqrt[3]{I^*_{\mathbf{]0,+\infty[}}(A'_{123}) \cdot I^*_{\mathbf{]0,+\infty[}}(A'_{124}) \cdot I^*_{\mathbf{]0,+\infty[}}(A'_{234})}$$
$$= \sqrt[3]{14 \cdot \frac{56}{9} \cdot \frac{16}{9}} = 5.37.$$

Applying Proposition 19, we can compute the consistency index of A by means of the isomorphism ψ in 4:

$$I^*_{\mathbf{]0,1[}}(A) = \psi(I^*_{\mathbf{]0,+\infty[}}(A')) = \frac{5.37}{6.37} = 0.84.$$

7. CONCLUSION AND FUTURE WORK

We consider PCMs over an alo-group $\mathcal{G} = (G, \odot)$; in this framework, the several approaches to PCMs are unified and the drawbacks linked to the possibility of the DM to be consistent are removed. Moreover,

- we introduce the abelian group $\mathcal{G}^n = (G^n, \odot_\times)$ and the notion of $\odot$-proportionality;
- we provide new characterizations of a consistent PCM;
- we introduce an equivalence relation $\sim_\odot$ on G^n and a bijection between the quotient set $G^n/\sim_\odot$ and the set of the consistent PCMs;
- we provide efficient algorithms to check the consistency and to build a consistent PCM;
- we define a new consistency index.

Following the results in Refs. 2, 4, 5, and 9 for the multiplicative case, our future work will be directed to investigate, in the general context of the PCMs over alo-groups, conditions that allow us to obtain an evaluation vector $\underline{w} = (w_1, \ldots, w_n) \in G^n$, and able to represent the actual ranking of the alternatives at different levels.

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A Comparison of Two Methods for Determining the Weights of Belonging to Fuzzy Sets[1]

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Abstract. Saaty has solved a basic problem in fuzzy set theory using an eigenvector method to determine the weights of belonging of each member to the set. In this paper, a weighted least-square method is utilized to obtain the weights. This method has the advantage that it involves the solution of a set of simultaneous linear algebraic equations and is thus conceptually easier to understand than the eigenvector method. Examples are given for estimating the relative wealth of nations and the relative amount of foreign trade of nations. Numerical solutions are obtained using both the eigenvector method and the weighted least-square method, and the results are compared.

Key Words. Fuzzy sets, eigenvectors, weighted least squares, relative weight matrix.

1. Introduction

A basic problem in the theory of fuzzy sets (Ref. 1) is the determination of the degree of belonging of each member to the set. Saaty (Refs. 2 and 3) has shown that this problem can be reduced to a matrix eigenvalue problem. In Ref. 4, an imbedding method was applied to obtain the largest eigenvalue and eigenvector of the matrix. In this paper, a weighted least-square method is utilized, and the results are compared with the eigenvector method.

Let $w_i > 0$, $i = 1, 2, \ldots, n$, be the degrees of belonging of the n members. Forming the matrix of relative weights whose ijth element is w_i/w_j, Saaty observed that the vector $(w_1, w_2, \ldots, w_n)^T$ is an eigenvector

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corresponding to the largest eigenvalue (The Perron–Frobenius root). All the other eigenvalues are zero.

The idea is to estimate the matrix of relative weights and then obtain an estimate of the vector $(w_1, w_2, \ldots, w_n)^T$ as an eigenvector corresponding to the largest eigenvalue of the relative weight matrix. To compare a set of n objects in pairs according to their relative weights, Saaty denotes the objects by $A_1, \ldots, A_n$ and their weights by $w_1, \ldots, w_n$. The pairwise comparisons are represented by the matrix

$$A = \begin{array}{cc} \begin{matrix} A_1 & \quad A_2 & \quad \cdots & \quad A_n \end{matrix} & \\ \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & & & \\ w_n/w_1 & w_n/w_2 & & w_n/w_n \end{bmatrix} & \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} \end{array} \tag{1}$$

This matrix, called a reciprocal matrix, has positive entries everywhere and satisfies the reciprocal property

$$a_{ji} = 1/a_{ij}.$$

Multiplying this matrix by the vector $w = (w_1, \ldots, w_n)^T$, we have

$$Aw = nw, \tag{2}$$

or

$$(A - nI)w = 0. \tag{3}$$

This is a system of homogeneous linear equations which has a non-trivial solution iff the determinant of $(A - nI)$ vanishes, that is, n is an eigenvalue of A. The matrix A is also consistent; that is,

$$a_{jk} = a_{ik}/a_{ij}.$$

In the general case, the precise values of w_i/w_j are not known and must be estimated. Since the eigenvalues are perturbed by a small perturbation of the coefficients. Eq. (2) becomes

$$A'w' = \lambda_{\max} w', \tag{4}$$

where $\lambda_{\max}$ is the largest eigenvalue of A'. To simplify the notation, Eq. (4) is expressed in the form

$$Aw = \lambda_{\max} w, \tag{5}$$

where A is Saaty's matrix of pairwise comparisons. The eigenvector associated with the largest eigenvalue is the desired vector of weights.

Numerical methods for obtaining the largest eigenvalue and associated eigenvector were discussed in Ref. 4. In this paper, the equations for a weighted least-square method are derived. Numerical results are given for several examples and compared with the eigenvector results.

2. Weighted Least-Square Method

Consider the elements a_{ij} of Saaty's matrix A in Eq. (5). It is desired to determine the weights w_i, such that, given a_{ij},

$$a_{ij} \approx w_i / w_j. \tag{6}$$

The weights can be obtained by solving the constrained optimization problem

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2, \tag{7}$$

$$\sum_{i=1}^{n} w_i = 1, \tag{8}$$

$$\text{minimize } S. \tag{9}$$

An additional constraint is that $w_i > 0$. However, it is conjectured that the above problem can be solved such that $w_i > 0$ without this constraint. The least-square solution for

$$S_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - w_i / w_j)^2, \tag{10}$$

while more desirable than for the weighted least squares given by Eq. (7), is much more difficult to solve numerically.

In order to minimize S, form the sum

$$S' = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2 + 2\lambda \sum_{i=1}^{n} w_i, \tag{11}$$

where λ is the Lagrange multiplier. Differentiating Eq. (11) with respect to w_m, the following set of equations is obtained:

$$\sum_{i=1}^{n} (a_{im} w_m - w_i) a_{im} - \sum_{j=1}^{n} (a_{mj} - w_m) + \lambda = 0, \qquad m = 1, 2, \ldots, n. \tag{12}$$

Equations (12) and (8) form a set of $n+1$ inhomogeneous linear equations with $n+1$ unknowns. For example, for $n=2$, the equations are

$$(1+a_{21}^2)w_1-(a_{12}+a_{21})w_2+\lambda=0, \tag{13}$$

$$-(a_{12}+a_{21})w_1+(1+a_{12}^2)w_2+\lambda=0, \tag{14}$$

$$w_1+w_2=1. \tag{15}$$

Given the coefficients a_{ij}, Eqs. (13)–(15) can be solved for w_1, w_2, λ using a standard FORTRAN subroutine for solving simultaneous linear equations. In this simple case, however, an analytical solution is possible:

$$w_1=[(1+a_{12}^2)+a_{12}+a_{21}]/[(1+a_{12})^2+(1+a_{21})^2], \tag{16}$$

$$w_2=[(1+a_{21}^2)+a_{12}+a_{21}]/[(1+a_{12})^2+(1+a_{21})^2]. \tag{17}$$

Equations (16) and (17) show that, since the a_{ij}'s>0, then the w_i's>0.

In general, Eqs. (12) and (8) can be expressed in the matrix form

$$Bw=m, \tag{18}$$

where

$$w=(w_1, w_2, \ldots, w_n, \lambda)^T, \tag{19}$$

$$m=(0, 0, \ldots, 0, 1)^T, \tag{20}$$

$$B=(n+1)\times(n+1) \text{ matrix with elements } b_{ij}, \tag{21}$$

$$b_{ii}=(n-1)+\sum_{j\neq i}^{n} a_{ji}^2, \qquad i, j=1, \ldots, n, \tag{22}$$

$$b_{ij}=-a_{ij}-a_{ji}, \qquad i, j=1, \ldots, n, \tag{23}$$

$$b_{k,n+1}=b_{n+1,k}=1, \qquad k=1, \ldots, n, \tag{24}$$

$$b_{n+1,n+1}=0. \tag{25}$$

3. Wealth-of-Nations Matrix

Numerical results were obtained for Saaty's wealth-of-nations matrix given in Ref. 2 and repeated in Table 1. Saaty made estimates of the relative wealth of nations and showed that the eigenvector corresponding to the matrix agreed closely with the GNP. The power method for obtaining the eigenvector was utilized in Ref. 4 and compared to Saaty's results. Table 2 compares these results with the weighted least squares results. It is seen that the sums S and S_1, defined by Eqs. (7) and (10), are less for the weighted

Table 1. Wealth-of-nations matrix.

Country	US	USSR	China	France	UK	Japan	W. Germany
US	1	4	9	6	6	5	5
USSR	1/4	1	7	5	5	3	4
China	1/9	1/7	1	1/5	1/5	1/7	1/5
France	1/6	1/5	5	1	1	1/3	1/3
UK	1/6	1/5	5	1	1	1/3	1/3
Japan	1/5	1/3	7	3	3	1	2
W. Germany	1/5	1/4	5	3	3	1/2	1

least-square method than for the power method. The sums S_1 were computed for comparison, even though the minimizations were made with respect to the sums S.

4. Taiwan Trade Matrices

One of the authors was a student in the College of Chinese Culture in Taiwan. Using Saaty's scales (Ref. 2), she estimated the relative strengths of belonging of the US, Japan, S. America, and Europe to the fuzzy set of important trading partners with Taiwan. This was done with regard to exports, imports, and total trade (Ref. 5). Both methods for determining the

Table 2. Comparison of numerical results for wealth-of-nations matrix.

Country	Saaty's eigenvector ($\lambda_{max} = 7.61$)	Power method eigenvector ($\lambda_{max} = 7.60772$)	Weighted least-square method
US	0.429	0.427115	0.486711
USSR	0.231	0.230293	0.175001
China	0.021	0.0208384	0.0299184
France	0.053	0.0523856	0.0593444
UK	0.053	0.0523856	0.0593444
Japan	0.119	0.122719	0.10434
W. Germany	0.095	0.0942627	0.0853411
$S = \sum_i \sum_j (a_{ij}w_j - w_i)^2$		$S = 0.458232$	$S = 0.288071$
$S_1 = \sum_i \sum_j (a_{ij} - w_i/w_j)^2$		$S_1 = 187.898$	$S_1 = 124.499$

Table 3. Taiwan trade matrices.

	Country	US	Japan	S. America	Europe
Exports	US	1	3	9	5
	Japan	1/3	1	9	1/2
	S. America	1/9	1/5	1	1/2
	Europe	1/5	3	3	1
Imports	US	1	1	9	3
	Japan	1	1	7	3
	S. America	1/9	1/9	1	1/7
	Europe	1/3	1/2	7	1
Total trade	US	1	3	9	3
	Japan	1/4	1	7	3
	S. America	1/9	1/7	1	1/5
	Europe	1/5	1/2	5	1

relative weights were used, and comparisons were made with published trade data (Ref. 6) for the year 1975. Both methods yielded good agreement with those data. Table 3 gives the Taiwan trade matrices, and Table 4 gives a comparison of the numerical results. The sums S and S_1 are again lower for the weighted least-square method than for the power method in all cases, except for the imports sum S_1. In all cases, the dominant weight tends to be larger for the weighted least-square method.

5. Discussion

The numerical results tend to show that either the eigenvector or the weighted least-square method can be used to obtain the weights. For the examples used in this paper, the eigenvector method appeared to give answers closer to the expected values. However, the sum S and S_1 were generally smaller for the weighted least-square method. Also, the weighted least-square method, which involves the solution of a set of simultaneous linear algebraic equations, is conceptually easier to understand than the eigenvector method. Using the eigenvector method, it can be proved that the weights w_i are all greater than zero (Ref. 2). While we do not know whether such a theorem exists for the weighted least-square method, the numerical results given here indicate that the w_i's obtained by this method are also greater than zero and are comparable to those obtainable by the eigenvector method.

Table 4. Comparison of numerical results for Taiwan trade matrices.

Country	Fraction of exports	Power method eigenvector ($\lambda_{max} = 4.93$)	Weighted least-square method
US	0.525	0.540	0.641
Japan	0.204	0.193	0.160
S. America	0.038	0.052	0.047
Europe	0.233	0.215	0.152
		$S = 0.526$	$S = 0.277$
		$S_1 = 41.23$	$S_1 = 59.56$

Country	Fraction of imports	Power method eigenvector ($\lambda_{max} = 4.14$)	Weighted least-square method
US	0.394	0.399	0.414
Japan	0.418	0.382	0.396
S. America	0.014	0.036	0.042
Europe	0.174	0.183	0.148
		$S = 0.081$	$S = 0.040$
		$S_1 = 22.75$	$S_1 = 18.71$

Country	Fraction of total trade	Power method eigenvector ($\lambda_{max} = 4.09$)	Weighted least-square method
US	0.452	0.533	0.575
Japan	0.323	0.279	0.232
S. America	0.024	0.041	0.049
Europe	0.200	0.147	0.144
		$S = 0.179$	$S = 0.126$
		$S_1 = 19.94$	$S_1 = 19.92$

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AKADEMIA GÓRNICZO-HUTNICZA IM. STANISŁAWA STASZICA W KRAKOWIE

WYDZIAŁ ELEKTROTECHNIKI, AUTOMATYKI,
INFORMATYKI I INŻYNIERII BIOMEDYCZNEJ

KATEDRA INFORMATYKI STOSOWANEJ

Praca dyplomowa inżynierska

Biblioteka w językach R/Java wspierająca metodę porównywania parami
R/Java library supporting the pairwise comparisons method

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Kraków, 2017

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1. Wprowadzenie

1.1. Metoda porównań parami

Ludzie od wieków podejmują decyzje. Niektóre z nich są proste i przychodzą z łatwością, inne jednak, te bardziej skomplikowane, wymagają głębszej analizy. Jednym z przykładów jest handel wymienny (barter), który opiera się na zamianie określonych towarów. Skąd jednak mieć pewność, że przedmioty mają podobną wartość lub w jaki sposób oszacować koszt określonego towaru? Z takimi problemami musieli mierzyć się nasi pradziadowie. Na szczęście rozwój matematyki przyniósł nam ciekawe narzędzie, metodę porównań parami (ang. The Pairwise Comparisons (PC) method), która w prosty sposób pomaga w udzieleniu odpowiedzi na powyższe pytania. Pierwszym udokumentowanym przypadkiem użycia metody jest binarny system elekcyjny opisany przez Rajmunda Llula (kat. Ramon Llull) już w XIII wieku. Znaczny rozwój metody przypada jednak na wiek XIX i XX i wiąże się z działalnością Fechnera [1] i Thrustone [2]. Przełomowym momentem stało się wprowadzenie w 1980 roku *The Analytic Hierarchy Process (AHP)* [3], dokonane przez Saaty'ego, które pozwoliło na porównywanie coraz to bardziej skomplikowanych obiektów, poprzez tworzenie z nich rozbudowanej hierarchii. Kolejnym krokiem naprzód stało się stworzenie przez doktora Kułakowskiego *Heuristic Rating Estimation (HRE)* [4], które pokazuje nowy kierunek metody oraz nieodkryte dotąd możliwości.

Metoda PC opiera się na założeniu, że zamiast porównywać wszystkie alternatywy od razu, lepiej jest porównać je parami, a następnie zebrać wszystkie wyniki razem [5]. Takie podejście znacznie ułatwia wybór najlepszej alternatywy lub obiektu oraz daje bardziej wiarygodne rezultaty. Stwarza to również możliwość zebrania wyników od wielu osób, np. ekspertów w danej dziedzinie, połączenia ich ocen, porównania poszczególnych wartości i w końcu udzielenia odpowiedzi na pytanie: Która alternatywa (według określonego kryterium) jest naprawdę najlepsza?

1.2. Cele pracy

Celem poniższej pracy jest przedstawienie biblioteki *PariwiseComparisons*, która powstała w ramach pracy inżynierskiej. Biblioteka implementuje 49 funkcji związanych z metodą porównań parami, a jej funkcjonalność pokrywa się z istniejącym już pakietem napisanym w Wolfram Language i dostępnym w Internecie [6].

Biblioteka została stworzona z myślą, aby ułatwić tworzenie aplikacji wykorzystujących metodę porównań parami oraz zachęcić do poznawania jej możliwości. Dzięki tej pracy, programista, który rozwija taką aplikację, nie będzie musiał skupiać się na szczegółach implementacji i poświęcać wielu godzin na zgłębianie obliczeń matematycznych w danym języku. Celem tworzenia biblioteki było zaimplementowanie funkcji zarówno tych najbardziej podstawowych, np. związanych z tworzeniem macierzy porównań parami, jak również tych bardziej skomplikowanych, potrafiących obliczać całe rankingi. Dlatego pakiet może stać się pomocny nie tylko dla osób zajmujących się na co dzień metodą porównań parami, ale także dla tych, którzy chcą skorzystać tylko z jednej funkcjonalności metody, bez zagłębiania się w szczegóły.

Z powodu dużej popularności języka Java i mnogości aplikacji, które powstają w tym języku, powstała również druga wersja biblioteki, skompresowana do pliku Jar (pariwiseComparisons.jar), gotowa do użycia w projektach pisanych w tym Javie.

1.3. Zawartość i struktura pracy

Na kolejnych kartach niniejszej pracy prezentuję elementy metody porównań parami oraz funkcje z biblioteki PariwiseComPairsons, które implementują poszczególne funkcjonalności. W treści zawarte są również podpowiedzi w jaki sposób użyć pakietu w języku R oraz jak łatwo wywoływać funkcje z poziomu maszyny wirtualnej Javy.

Szczególną uwagę należy zwrócić na specyficzną strukturę pracy. Główna część, prezentująca metodę porównań parami, została podzielony na kilka mniejszych rozdziałów (2-5). W każdym z nich pierwszą część stanowią rozważania teoretyczne, bazujące na pracy naukowców i ich odkryciach. Druga część to prezentacja funkcji biblioteki PairwiseComPairsons, ze szczegółowym omówieniem najważniejszych z nich. Mam nadzieję, że taka budowa tej pracy pozwoli czytelnikowi poznać podstawowe zagadnienia związane z metodą porównań parami oraz zapoznać się z konkretnymi funkcjami przeznaczonymi do poszczególnych zagadnień.

2. Ranking AHP

2.1. Macierz PC

Kiedy chcemy podjąć prostą decyzję, sytuacja często sprowadza się do porównania dwóch wartości, np. gdy naszym zadaniem jest wybór cięższego owocu, sprawa jest oczywista. Wystarczy zważyć owoce lub oszacować ich wagę, a następnie wybrać cięższy. Wiele decyzji, z którymi spotykamy się na co dzień, jest jednak o wiele bardziej skomplikowanych. Może tak być w przypadku, gdy naszego kryterium nie można łatwo zmierzyć (np. użyteczność lub atrakcyjność), porównywanych obiektów jest dużo i znacznie się od siebie różnią lub są złożone, a kryterium wyboru zależy od kilku czynników. W każdej z tych sytuacji możemy posłużyć się metodą porównań parami, a konkretnie rankingiem AHP.

W celu utworzenia rankingu na samym początku należy rozpoznać problem, to znaczy zdefiniować decyzję, którą chcemy podjąć oraz znaleźć wszystkie opcje (alternatywy) brane pod uwagę. Następnie tworzymy kwadratową macierz *(PC matrix)*, w której nagłówkami kolumn i wierszy są zdefiniowane przez nas alternatywy.

Kolejnym krokiem jest uzupełnienie macierzy poprzez wykonanie odpowiedniej ilości porównań. Każdą alternatywę porównujemy z wszystkimi pozostałymi, tak więc dla n alternatyw otrzymujemy $\frac{n(n-1)}{2}$ porównań. Wyniki wpisujemy do macierzy, na przecięciu odpowiadających opcji.

Istnieje kilka skal, które możemy wykorzystać w czasie porównań, najbardziej popularną jest jednak zaproponowana przez Saaty'ege *(Satty's scale)* [7], którą również ja posłużę się w przykładach. Skala Saaty'ego przyjmuje następujące wartości:

$$\{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9\}$$

Ogólnie można powiedzieć, że im wyższa wartość przydzielona dla danej alternatywy, tym lepiej wypada ona w stosunku do drugiej opcji, a więc wartość $\frac{1}{9}$ można tłumaczyć jako *ekstrelamnie nie preferuję*, $\frac{1}{3}$ jako *nie preferuję*, 1 jako *tak samo ważna jak druga*, a 5 jako *silnie preferuję*, itd. Oczywiście nie porównujemy ze sobą tych samych alternatyw, lecz od razu wpisujemy do macierzy wartość 1.
Etap porównywania alternatyw i wyboru numerycznej wartości tego porównania jest najsłabszą stroną metody porównań parami, gdyż zależy od czynnika ludzkiego i narażony jest na błędy oraz niespójności. Problemy z tego wynikające opisuję w rozdziale (4).

Po wykonaniu wszystkich porównań i wpisaniu wartości otrzymujemy kompletną macierz PC, którą formalnie zapisujemy jako:

$$M = (m_{ij}) \wedge m_{i,j} \in R_+ \wedge i,j \in \{1,\ldots,n\}$$

Natomiast zbiór alternatyw oznaczamy jako

$$X = \{x_1,\ldots,x_n\}$$

Macierz dla n alternatyw wygląda więc następująco:

$$M = \begin{pmatrix} 1 & m_{12} & \ldots & m_{1n} \\ m_{21} & 1 & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \ldots & 1 \end{pmatrix}$$

Warto zauważyć, że wartości wykonanych przez nas porównań wpisujemy nad przekątną (tworząc *Upper Triangular Matrix*, natomiast pod nią automatycznie wprowadzamy wartości odwrotne. Nielogiczne byłoby ponowne porównywanie tych samych alternatyw, z odwróconą jedynie kolejnością. Ogólnie można powiedzieć, że zachodzi warunek $a_{ji} = \frac{1}{a_{ij}}$. Na przekątnej zaś zawsze znajdą się same jedynki.

Przyklad 2.1.1. *Chcemy podjąć decyzję, gdzie pojechać na wakacje. Rozważamy trzy opcje: Gdańsk, Zakopane lub Barcelona. W takim przypadku nasz zbiór alternatyw wygląda bardzo prosto:* $X = \{Gdansk, Zakopane, Barcelona\}$*. Następnie dokonujemy porównań parami, z których wynika, że bardziej preferujemy wyjazd do Gdańska niż do Zakopanego w stosunku* $3/1$*, parę Gdańsk - Barcelona oceniamy na* $1/2$*. Natomiast z dwójki Zakopane i Barcelona wybieramy Barcelonę i przypisujemy jej wartość* 6 *(silnie preferowana). W takim przypadku macierz porównań parowych wygląda następująco:*

$$M = \begin{pmatrix} 1 & 3 & \frac{1}{2} \\ \frac{1}{3} & 1 & \frac{1}{6} \\ 2 & 6 & 1 \end{pmatrix}$$

2.2. Wektor wag

Kolejnym krokiem do podjęcia decyzji jest utworzenie wektora wag (nazywany również *wektorem priorytetów*). Służy do tego funkcja, która każdej alternatywie ze zbioru X przyporządkowuje dodatnią liczbę rzeczywistą. Do wyliczenia wektora wykorzystuje macierz PC.

$$w = [w(c_1),\ldots,w(c_n)]^T$$

Wysoka wartość w wektorze wag oznacza, że dana alternatywa jest mocno preferowana. Bez trudu dochodzimy więc do wniosku, że najlepszą opcją w danym problemie jest ta, dla której przypisano najwyższą wartość w wektorze wag.

Dość naturalnie pojawia się pytanie, w jaki sposób działa funkcja wyznaczająca wektor wag. Istnieje kilka sposobów wyliczania tych wartości, poniżej przedstawię dwa najbardziej popularne, stosunkowo proste i dające rzetelne rezultaty:

1. Na podstawie macierzy PC wyznaczony zostaje wektor własny korespondujący z najwyższą wartością własną tej macierzy.

2. Dla każdego wiersza macierzy PC wyliczona zostaje średnia geometryczna wartości w danym wierszu.

Warto zaznaczyć, że otrzymany wektor wag należy przeskalować, w taki sposób, aby suma wartości w nim zawartych wynosiła 1.

Przyklad 2.2.1. *Na podstawie wyznaczonej w poprzenim przykładzie macierzy PC, korzystając z metody opartej na wektorach własnych, wyznaczamy wektor wag:*

$$w = \begin{bmatrix} 0.4423259 & 0.1474420 & 0.8846517 \end{bmatrix}^T$$

Po przeskalowaniu otrzymujemy kompletny wektor wag:

$$w = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.6 \end{bmatrix}$$

Z wyznaczonego wektora wag możemy wyciągnąć wnioski, że najlepszą alternatywą na wakacyjny wyjazd jest Barcelona, a najmniej preferowaną Zakopane. W ten prosty sposób udało nam się udzielić odpowiedzi na zadane pytanie.

2.3. AHP

Zdarzają się jednak sytuacje, w których problem jest bardziej skomplikowany, a alternatyw nie można zapisać w jednej macierzy porównań parowych. Wtedy z pomocą przychodzi wprowadzone przez Saaty'ego *The Analytic Hierarchy Process (AHP)* [3].

Podstawą AHP jest budowa hierarchii, która obrazuje proces powstawania decyzji. Prezentuje ona kolejne poziomy, od najbardziej ogólnego (decyzja), do szczegółowych, które mówią jakie porównania należy wykonać, aby rozwiązać problem. Najprostszą hierarchię mogliśmy zbudować już w poprzednim przykładzie, wyglądałaby ona następująco:

Dokąd pojechać na wakacje?

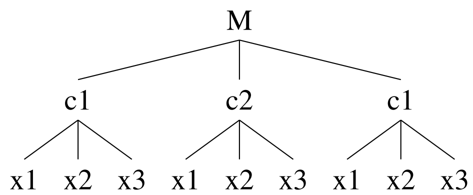
Gdańsk Zakopane Barcelona

Nie mówiliśmy wtedy jednak o AHP, gdyż wybór opierał się tylko na jednej macierzy PC.

AHP wprowadza kolejne pojęcie - kryterium. Kryterium to cecha posiadana przez każdą alternatywę i która wywiera wpływ na podejmowane decyzje. W celu stworzenia AHP i rozwiązania złożonego problemu należy zdefiniować zbiór kryteriów C.

$$C = \{c_1, \ldots, c_m\}$$

Następnym krokiem jest określenie kryteriów dla każdej alternatywy. Otrzymujemy wtedy kompletne dane, które pozwalają nam zbudować następującą hierarchię AHP:

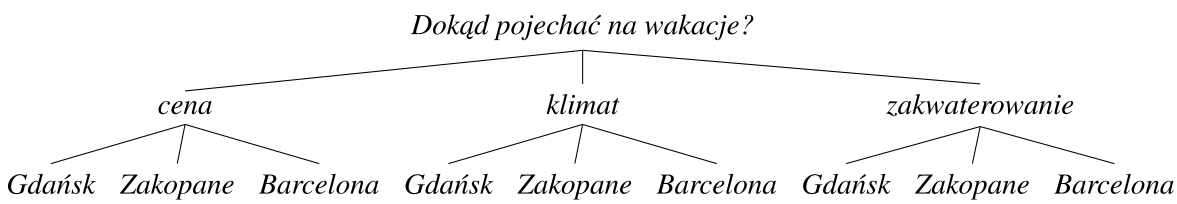


Przyklad 2.3.1. *Ponownie naszym zadaniem jest wybór miejsca, w które chcemy pojechać na wakacje. Oprócz zbioru alternatyw* $X = \{Gdansk, Zakopane, Barcelona\}$*, tym razem podajemy także kryteria, które są dla nas ważne w czasie podejmowania wyboru:* $C = \{cena, klimat, zakwaterowanie\}$.
Dla każdej z alternatyw określamy jej szczegóły:

Tabela 2.1. Szczegóły każdej alternatywy

	Alternatywy		
Kryteria	Gdańsk	Zakopane	Barcelona
cena	800zł	1300zł	2400zł
klimat	górski	bałtycki	śródmorski
zakwaterowanie	namiot	pensjonat	hotel

Na podstawie powyższych danych budujemy drzewo AHP:



Po utworzeniu hierarchii AHP możemy przystąpić do budowy macierzy PC. Tym razem nie będzie to jedna tablica, lecz kilka, które powiążemy ze sobą. Dla każdego poddrzewa składającego się z jednego wierzchołka-rodzica i kilku wierzchołków-dzieci konstruujemy osobną macierz PC oraz wykonujemy odpowiednie porównania. Następnie wyznaczamy przeskalowany wektor wag dla każdej macierzy. Jedyną różnicą w porównaniu z pierwszym przykładem jest sposób wyniku. Aby otrzymać odpowiedni wektor wynikowy, którego element sumują się do 1, należy zsumować wektory powstałe z macierzy PC na tym samym poziomie (najgłębszy poziom struktury), przemnażając je przez odpowiadające im wartości wag z poziomu wyższego. Tworzymy w ten sposób kombinację liniową wektorów priorytetów [8]. Dla n alternatyw wektor wag obliczamy według wzoru:

$$w = \hat{w}_1 w^{(c1)} + \hat{w}_2 w^{(c2)} + \cdots + \hat{w}_n w^{(c3)},$$

gdzie
$\hat{w}_1, \hat{w}_2, \ldots$ to kolejne elementy wektora wag kryteriów,
$w^{(c1)}, w^{(c2)}, \ldots$ to kolejne wektory wag obliczone dla poszczególnych kryteriów

Może zdarzyć się sytuacja, w której rozważany problem jest jeszcze bardziej rozbudowany. Wtedy powyższe kroki należy powtórzyć na każdym poziomie zagłębienia hierarchii AHP.

Przyklad 2.3.2. *W naszym przykładzie należy zbudować cztery macierze PC: jedna zdefiniuje priorytety kryteriów, trzy pozostałe określą preferencje w ramach poszczególnych kryteriów. Dla każdej macierzy wykonujemy odpowiednie porównania, a wyniki wpisujemy do macierzy:*

$$\hat{M} = \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ 2 & 1 & 4 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{pmatrix}$$

$$M_{cena} = \begin{pmatrix} 1 & 3 & 7 \\ \frac{1}{3} & 1 & 4 \\ \frac{1}{7} & \frac{1}{4} & 1 \end{pmatrix} \qquad M_{klimat} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 2 & 2 & 1 \end{pmatrix} \qquad M_{zakwaterowanie} = \begin{pmatrix} 1 & 2 & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ 3 & 4 & 1 \end{pmatrix}$$

Wyznaczamy wektor wag dla każdej macierzy

$$\hat{w} = \begin{bmatrix} 0.32 \\ 0.56 \\ 0.12 \end{bmatrix} \qquad w_c = \begin{bmatrix} 0.66 \\ 0.26 \\ 0.08 \end{bmatrix} \qquad w_k = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix} \qquad w_z = \begin{bmatrix} 0.24 \\ 0.14 \\ 0.62 \end{bmatrix}$$

Warto zwrócić uwagę, że wektor oznaczony jako $\hat{w}$ jest poziom wyżej od pozostałych i przez jego wartości będziemy przemnażać pozostałe wektory.

Ostateczny wynik:

$$w = 0.32w_c + 0.56w_k + 0.12mu_z = \begin{bmatrix} 0.379 \\ 0.240 \\ 0.381 \end{bmatrix}$$

Otrzymaliśmy rezultat i rozwiązanie zdefiniowanego problemu. Najlepszym miejscem na wakacyjny wyjazd okazała się Barcelona. Osiągnęła jednak minimalną przewagę nad Gdańskiem.

2.4. Biblioteka PairwiseComparisons - AHP

Przedstawię kilka zasad tworzenia zmiennych w językach R i Java, które można przekazywać do funkcji z biblioteki PairwiseComparisons.

Informacje dotyczące języka R:

1. Aby utworzyć wektor wystarczy wywołać funckję *c*, przekazując jej dowolną ilość wartości.
2. Do transpozycji wektora służy funkcja *t*.
3. Istnieje kilka funkcji, które tworzą macierz, najważniejsze z nich to:
   - *matrix* – jako argumenty przekazujemy wektor z danymi, liczbę wierszy i liczbę kolumn.
   - *rbind* – jako argumenty przekazujemy wektory z danymi. Każdy z wektorów utworzy jeden wiersz.
   - *cbind* – jako argumenty przekazujemy wektory z danymi. Każdy z wektorów utworzy jedną kolumną.

   Dodatkowe opcje każdej z funkcji można doczytać w dokumentacji. Przykładowe wywołania:
   matrix(c(1,2,0.5,1),2,2); *rbind(c(1,2),c(0.5,1))*; *cbind(c(1,0.5),c(2,1))*;

Informacje dotyczące języka Java:

1. Aby przekazać macierz do funkcji z omawianej biblioteki, wystarczy utworzyć i uzupełnić danymi tablicę dwuwymiarową typu *double*.
2. Aby przekazać wektor do funkcji, należy utworzyć i zainicjalizować tablicę typu *double*.

Przykładowe utworzenie odpowiednich zmiennych:
double[2][2] matrix = {{1,0.5},{2,1}};
double[2] vector = {0.4,0.6};

Dla niektórych funkcji powstały również wersje symboliczne, które mają dokładnie takie same argumenty, zwracają tę samą wartość i działają minimalnie szybciej. W wyjątkowych sytuacjach może jednak się zdarzyć, że wersja symboliczna nie zwróci prawidłowego rezultatu.

Funkcje z biblioteki PairwiseComparisons pomocne w powyższych obliczeniach:

Największa wartość własna macierzy

Nazwa

principalEigenValue

Opis

Oblicza największą wartość własną z macierzy.

Argumenty

matrix – PC matrix

Zwracana wartość

Największa wartość własna macierzy

Dodatkowe informacje

W bibliotece dostępna jest również symboliczna wersja tej funkcji *principalEigenValueSym*

Wektor własny macierzy

Nazwa

principalEigenVector

Opis

Oblicza wektor własny macierzy korespondujący z jej największą wartością własną (patrz *principalEigenValue*).

Argumenty

matrix – PC matrix

Zwracana wartość

Wektor własny macierzy

Dodatkowe informacje

W bibliotece dostępna jest również symboliczna wersja tej funkcji *principalEigenVectorSym*

Ranking na podstawie wartości własnych

Nazwa

eigenValueRank

Opis

Oblicza ranking macierzy na podstawie metody wartości własnych. Oblicza wektor własny, a następnie przekształca go w taki sposób, że suma elementów wynosi 1.

Argumenty

matrix – PC matrix

Zwracana wartość

Przeskalowany wektor własny macierzy

Dodatkowe informacje

W bibliotece dostępna jest również symboliczna wersja tej funkcji *eigenValueRankSym*

Wektor średnich geometrycznych

Nazwa

geometricRank

Opis

Oblicza wektor, którego elementy są średnimi geometrycznymi każdego wiersza przekazanej macierzy. Wektor ten posłuży do stworzenia rankingu opartego o metodę średnich geometrycznych.

Argumenty

matrix – PC matrix

Zwracana wartość

Wektor średnich geometrycznych

Ranking na podstawie średnich geometrycznych

Nazwa

geometricRescaledRank

Opis

Oblicza ranking macierzy na podstawie metody średnich geometrycznych. Oblicza wektor, którego elementy są średnimi geometrycznymi każdego wiersza przekazanej macierzy, a następnie skaluje ten wektor w taki sposób, że suma elementów wynosi 1.

Argumenty

matrix – PC matrix

Zwracana wartość

Przeskalowany wektor średnich geometrycznych macierzy

Ranking AHP na podstawie wartości własnych

Nazwa

ahp

Opis

Oblicza wielokryteriowy ranking AHP na podstawie metody bazującej na wartościach własnych macierzy. Wykorzystuje macierz kryteriów i macierze alternatyw. Jest to podstawowy, trójpoziomowy ranking AHP. Ilość przekazanych wartości do funkcji zależy od użytkownika i wynosi $n + 1$, gdzie n to ilość kryteriów w rozważanym problemie.

Argumenty

M – macierz kryteriów o wymiarach *nxn*

... – lista n macierzy PC, z których każda dotyczy jednego kryterium.

Zwracana wartość

Ranking AHP

3. Heuristic Rating Estimation

3.1. Wstęp do HRE

Istnieją również inne decyzje i problemy do rozwiązania. To sytuacje, w których pewne wartości są niepodważalne, z góry określone lub narzucone przez kogoś. Nie chcemy nimi manipulować ani dyskutować z ich wiarygodnością. Zależy nam natomiast, aby do tej relacji wprowadzić nowe obiekty, których wartość (w szerokim tego słowa znaczeniu) określimy w tej samej skali. Jako przykład możemy wyobrazić sobie targ z owocami, na którym sadownicy wymieniają się zbiorami bez użycia pieniędzy. Jedyną ustaloną i tradycyjną już wymianą jest sprzedawanie dwóch gruszek w zamian za trzy jabłka. Trzymając się tego wyznacznika, chcemy określić *ceny* innych owoców. Jak tego dokonać?

W takiej sytuacji z pomocą przychodzi nam wprowadzone przez doktora Kułakowskiego *Heuristic Rating Estimation (HRE)*. W mojej pracy ogólnie omówię to zagadnienie, zainteresowanych szczegółami odsyłam do źródła [4] [9].

3.2. Zbiór alternatyw

Pierwszym ważnym elementem HRE jest zbiór alternatyw. W tej metodzie nie będzie on już tylko zbiorem cech lub obiektów, których wartości poszukujemy. HRE zakłada, że pewne wartości są z góry określone. Nie chcemy ich zmieniać, w szczególności manipulować ich wzajemną relacją. Waga tych obiektów pozostanie niezmienna w czasie obliczeń. Drugą część zbioru tworzą jednak alternatywy, których wartości poszukujemy. Interesuje nas ich stosunek względem siebie i względem wag, które znamy. Jeśli więc poprzez C_K oznaczymy alternatywy, których wartości są znane od początku (*known concepts*), a poprzez C_U opcje, których wartości chcemy przybliżyć (*unknown concepts*), to zbiór alternatyw określamy jako sumę wszystkich elementów i zapisujemy jako:

$$C = C_K \cup C_U$$

3.3. Macierz PC w metodzie HRE

Drugim krokiem, podobnie jak w AHP, jest stworzenie macierzy porównań parowych, która posłuży do dalszych obliczeń. Przykładowa macierz PC wygląda następująco:

$$M = \begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{21} & 1 & m_{22} & m_{24} \\ m_{31} & m_{32} & 1 & \frac{2}{3} \\ m_{41} & m_{42} & \frac{3}{2} & 1 \end{pmatrix}$$

Przedstawiona macierz obrazuje sytuację, w której znamy wartości trzeciej i czwartej alternatywy, łatwo więc obliczamy ich stosunek, który wpisujemy bezpośrednio do macierzy. Z tak przygotowanej macierzy generujemy porównania parowe, które należy wykonać, a wyniki wpisujemy w odpowiednie miejsca. Porównań, w zależności od sytuacji, możemy dokonać sami lub poprosić o nie ekspertów w danej dziedzinie. Oczywiście pomijamy te, które są już znane, a więc w naszym przypadku nie będziemy szacować stosunku alternatywy trzeciej do czwartej.

3.4. Metody HRE

Kolejnym i zarazem najważniejszym krokiem omawianego problemu są obliczenia, które należy wykonać na uzupełnionej macierzy PC. Z racji na ich stopień zaawansowania oraz fakt, że bardzo czytelnie i szczegółowo zostały przedstawione w źródle, nie będę w pełni ich przytaczał. Pokażę tylko kilka etapów składających się na tę metodę i do których odpowiednie funkcje zostały zaimplementowane w bibliotece.

Głównym zagadnieniem jest sprowadzenie problemu to równania postaci

$$Aw = b,$$

gdzie:
A to macierz $r \times r$, gdzie r to ilość elementów poszukiwanych, oznaczamy $|C_U|$,
b to wektor wartości wyliczonych na podstawie znanych alternatyw,
w to poszukiwany wektor wag postaci

$$w = \begin{bmatrix} w(c_1) \\ \vdots \\ w(c_U) \end{bmatrix}$$

Po wyliczeniu przedstawionego równania, wektor w uzupełniamy o wartości ze zbioru C_K i otrzymujemy ostateczny rezultat. Należy pamiętać, że suma elementów w wyliczonym wektorze w jest różna od 1, gdyż wartości znane nie zmieniły się, alternatywy wyliczone zaś są również podane w odniesieniu do nich. Jeśli zależy nam, aby suma elementów wyniosła 1, możemy przeskalować wektor, dzieląc każdą wartość przez sumę wszystkich elementów.

Przyklad 3.4.1. *Prowadzimy handel wymienny owoców. Wiemy, że gruszka jest* 1.5 *razy cenniejsza od jabłka. W zbiorach mamy jeszcze brzoskwienie, truskawki i maliny. Chcemy oszacować wartości wszystkich owoców. Sporządzamy więc zbiory alternatyw znanych, nieznanych i wszystkich razem.*

$$C_K = [jabko, gruszka], \qquad C_U = [malina, brzoskwinia, truskawka].$$
$$C = [jabko, gruszka, malina, brzoskwinia, truskawka]$$

Budujemy macierz PC. Przeprowadzamy odpowiednie porównania parami, a wyniki zapisujemy do macierzy.:

$$M = \begin{pmatrix} 1 & \frac{2}{3} & 10 & 4 & 7 \\ \frac{3}{2} & 1 & 15 & 3 & 5 \\ \frac{1}{10} & \frac{1}{15} & 1 & \frac{1}{3} & \frac{1}{2}\frac{1}{4} \\ \frac{1}{3} & 3 & 1 & 2 & \\ \frac{1}{7} & \frac{1}{5} & 2 & \frac{1}{2} & 1 \end{pmatrix}$$

Korzystając z funkcji HRE możemy od razu wyliczyć wektor wag, aby zaprezentować jednak sposób działania metody, prześledźmy kolejne etapy powstawania wyniku.
Wykorzystujemy algorytmy do obliczenia macierzy A, wektora b, a następnie w:

$$Ab = w$$

$$A = \begin{pmatrix} 1 & -\frac{1}{12} & -\frac{1}{8} \\ -\frac{3}{4} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{8} & 1 \end{pmatrix} \qquad b = \begin{bmatrix} 0.1 \\ 0.375 \\ 0.2214286 \end{bmatrix}$$

Po rozwiązaniu otrzymujemy:

$$w = \begin{bmatrix} 0.22 \\ 0.75 \\ 0.42 \end{bmatrix}.$$

Dodajemy wartości znane i otrzymujemy ostateczny rezultat:

$$\hat{w} = \begin{bmatrix} 2 \\ 3 \\ 0.2150966 \\ 0.7475316 \\ 0.4224183 \end{bmatrix}.$$

Ostatnią rzeczą, na którą należy zwrócić uwagę jest kwestia wyboru metod, które służą do obliczeń. W Heuristic Rating Estimation, podobnie jak w AHP możemy wybrać sposób liczenia oparty na wartościach własnych macierzy lub średnich geometrycznych. Obie drogi dają bardzo zbliżone rezultaty. W przykładzie posłużyłem się metodą z wykorzystaniem wartości własnych.

3.5. Biblioteka PairwiseComparisons - HRE

Funkcje z biblioteki PairwiseComparisons pomocne w powyższych obliczeniach:

Macierz A metody HRE

Nazwa
HREmatrix

Opis
W oparciu o wartości własne macierzy, oblicza macierz, która razem z wektorem b (patrz *HREconstantTermVector*) utworzy układ równań liniowych postaci $Aw = b$.

Argumenty
matrix – PC matrix
knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość
Macierz A, służąca do utworzenia równania $Aw = b$.

Wektor b znanych wartości metody HRE (wykorzystuje wartości własne)

Nazwa

HREconstantTermVector

Opis

Na podstawie znanych alternatyw, w oparcu o wartości własne macierzy, oblicza wektor b, który razem z macierzą A (patrz *HREmatrix*) utworzy układ równań liniowych postaci $Aw = b$.

Argumenty

matrix – PC matrix

knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wektor b, służący do utworzenia równania $Aw = b$.

Ranking nieznanych wartości HRE (wykorzystuje wartości własne)

Nazwa

HREpartialRank

Opis

W oparcu o wartości własne macierzy oblicza nieznane alternatywy.

Argumenty

matrix – PC matrix

knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wartości nieznanych alternatyw

Pełny ranking HRE (wykorzystuje wartości własne)

Nazwa

HREfullRank

Opis

W oparciu o wartości własne macierzy oblicza nieznane alternatywy i dodaje je do wektora znanych wartości.

Argumenty

matrix – PC matrix

knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wektor wag znanych i nieznaych alternatyw.

Przeskalowany ranking HRE (wykorzystuje wartości własne)

Nazwa

HRErescaledRank

Opis

Oblicza pełny ranking HRE (patrz *HREfullRank*), a następnie skaluje wynikowy wektor wag w taki sposób, aby suma elementów wyniosiła 1.

Argumenty

matrix – PC matrix
knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Przeskalowany wektor wag znanych i nieznanych alternatyw.

Macierz A metody HRE (wykorzystuje średnie geometryczne)

Nazwa

HREgeomMatrix

Opis

W oparciu o średnie geometryczne wierszy macierzy, oblicza macierz, która razem z wektorem b (patrz *HREgeomConstantTermVector*) utworzy układ równań liniowych postaci $Aw = b$.

Argumenty

matrix – PC matrix
knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Macierz A, służąca do utworzenia równania $Aw = b$.

Wektor b znanych wartości metody HRE (wykorzystuje średnie geometryczne)

Nazwa

HREgeomConstantTermVector

Opis

Na podstawie znanych alternatyw, w oparciu o średnie geometryczne, oblicza wektor b, który razem z macierzą A (patrz *HREgeomMatrix*) utworzy układ równań liniowych postaci $Aw = b$.

Argumenty

matrix – PC matrix
knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wektor b, służący do utworzenia równania $Aw = b$.

Pośredni ranking nieznanych wartości HRE

Nazwa

HREgeomIntermediateRank

Opis

W oparciu o średnie geometryczne oblicza podstawę, która posłuży do obliczenia wartości nieznanych alternatyw. Współczynniki te zostaną przemnożone przez 10 (patrz *HREgeomPartialRank*)

Argumenty

matrix – PC matrix

knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wektor pośrednich wartości nieznanych alternatyw

Ranking nieznanych wartości HRE (wykorzystuje średnie geometryczne)

Nazwa

HREgeomPartialRank

Opis

W oparciu o średnie geometryczne oblicza nieznane alternatywy.

Argumenty

matrix – PC matrix

knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wartości nieznanych alternatyw

Pełny ranking HRE (wykorzystuje średnie geometryczne)

Nazwa

HREgeomFullRank

Opis

W oparciu o średnie geometryczne oblicza nieznane alternatywy i dodaje je do wektora znanych wartości.

Argumenty

matrix – PC matrix

knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Wektor wag znanych i nieznaych alternatyw.

Przeskalowany ranking HRE (wykorzystuje średnie geometryczne)

Nazwa

HREgeomRescaledRank

Opis

Oblicza pełny ranking HRE (patrz *HREgeomFullRank*), a następnie skaluje wynikowy wektor wag w taki sposób, aby suma elementów wyniosiła 1.

Argumenty

matrix – PC matrix
knownVector – wektor znanych alternatyw, pozostałe oznaczone jako 0

Zwracana wartość

Przeskalowany wektor wag znanych i nieznanych alternatyw.

4. Niespójność

4.1. Problemy związane z metodami porównań parami

Głównym problemem metody porównań parami jest niespójność danych. To najczęstszy zarzut, jaki można usłyszeć ze strony krytyków. Być może to także jeden z powodów, dla których AHP i HRE nie zdobyły jeszcze tak dużej popularności. Mimo, iż metoda opiera się na obliczeniach matematycznych i jest potwierdzona dowodami oraz twierdzeniami, zawiera jednak jeden *słabszy* element - czynnik ludzki. Człowiek współtworzy przecież obliczenia tej metody poprzez dostarczanie wyników porównań, bez których pozostałe składniki nie mają sensu. Dlaczego tak trudno jest dostarczyć spójne dane wejściowe i czy eksperci, którzy proszeni są o dokonanie porównań nie mogliby się tego po prostu nauczyć?

Oczywiście ludzie, którzy znają zasady działania metody potrafią tak dobrać wartości macierzy PC, aby była ona spójna. Jeśli jednak widzimy tylko sparowane alternatywy i mamy przypisać im, zgodnie z naszymi odczuciami, preferencje, okazuje się to już o wiele trudniejsze. Czasem zdarzają się sytuacje, w których ciężko jest przypisać konkretne liczby do stosunku, jaki posiadamy do przedstawionych alternatyw lub określić *stopień preferowania*. Skale ocen spośród których wybieramy ocenę jest umowna, a konkretne wartości mogą zostać różnie, subiektywnie odczytane przez poszczególne osoby.

Zastanówmy się kiedy macierz jest niespójna. Aby łatwiej zrozumieć na czym polega problem, przedstawię dwa proste przykłady. Pierwszy z nich okaże się niespójny nawet bez zwracania uwagi na konkretne wartości.

Przyklad 4.1.1. *Sporządzamy ranking zabawek, aby wybrać ulubiony przedmiot dziecka. Przedstawiamy mu piłkę i rower, a dziecko wybiera piłkę. Następnie pokazujemy rower i hulajnogę, wybór pada na rower. Ostatnie porównanie to hulajnoga i piłka, tym razem dziecko wskazuje na hulajnogę.*

Nie trzeba znać się na matematyce, aby szybko zorientować się, że ranking w tej sytuacji nie ma sensu, ponieważ w teorii, jeśli obiekt A jest lepszy od B, zaś B lepszy od C, to oczekujemy, że obiekt A jest zdecydowanie bardziej preferowany niż C. W praktyce czasami okazuje się inaczej. W tej sytuacji dane są niespójne.

Przyklad 4.1.2. *Porównujemy obiekt A z obiektem B i przypisujemy rezultat 2. Następnie zestawiamy B i C, tutaj również wybieramy wartość 2. A więc, mówiąc potocznie, obiekt A jest dwa razy lepszy od B, który z kolei 2 razy lepszy od C. Jakiego rezultatu oczekujemy więc w porównaniu obiektów A i C?*

Po chwili namysłu dochodzimy do wniosku, że obiekt A w stosunku do obiektu C powinien przyjąć wartość 4. Właśnie wtedy nasze dane będą całkowicie spójne.

Drugi z przedstawionych przykładów prowadzi nas do wniosku, na którym opiera się teoria spójności danych w metodzie porównań parami:

$$m_{ik} = m_{ij}m_{jk} \qquad\qquad \forall_{i,j,k} \tag{4.1}$$

W praktyce okazuje się, że bardzo rzadko otrzymujemy idealnie spójną macierz, dlatego do metod porównań parami wprowadzony został współczynnik niespójności. Informuje on o stopniu niespójności danych i na jego podstawie możemy zdecydować, czy warto wykonywać obliczenia na danej macierzy PC, czy może należy poprosić o ponowne wykonanie porównań. Na przestrzeni lat powstało wiele sposobów obliczania współczynnika niespójności, w mojej pracy przedstawię dwa z nich.

4.2. Współczynnik Saaty'ego

Aby wyznaczyć współczynnik Saaty'ego należy ponownie wykorzystać maksymalną wartość własną macierzy. To właśnie w oparciu o ten parametr Saaty przedstawił swoje rozważania [7]. Wykorzystał fakt, że największa wartość własna każdej macierzy jest równa jej wymiarowi wtedy i tylko wtedy, kiedy dana macierz jest spójna. Na tej podstawie zaproponował współczynnik niespójności (ang. *Consistency Index*). Dla macierzy o wymiarze *n* wyraża się wzorem:

$$CI(A) = \frac{\lambda_{max} - n}{n - 1}$$

W najprostszej wersji obliczania współczynnika niespójności możemy w tym miejscu zakończyć nasze rozważania. Przyjmuje się, że jeżeli wyznaczona wartość CI jest mniejsza niż 0.1, to macierz jest spójna, w przeciwnym wypadku należy poprawić wartości porównań.

Przyklad 4.2.1.

$$M = \begin{pmatrix} 1 & 2 & 8 \\ \frac{1}{2} & 1 & \frac{3}{4} \\ \frac{1}{8} & \frac{4}{3} & 1 \end{pmatrix}$$

Wyznaczamy największą wartość własną: $\lambda_{max} = 3.319518$*, a następnie współczynnik:*

$$CI(M) = \frac{3.319518 - 3}{3 - 1} = 0.159759.$$

Otrzymany rezultat informuje nas, że macierz M *nie jest spójna.*

Nieco bardziej dokładny sposób obliczania niespójności zaproponowany przez Saaty'ego zestawia wartość CI z współczynnikiem zależnym od wymiaru macierzy. Pozwala to na bardziej szczegółowe określenie wielkości niespójności danych. W tym przypadku należy wykorzystać tabelę (4.1) i wyznaczyć współczynnik nazywany *Consistency Ratio* (CR) według wzoru:

$$CR(A) = \frac{CI(A)}{RI_n}$$

Tabela 4.1. Wartości RI_n

n	3	4	5	6	7
RI_n	0.5247	0.8816	1.1086	1.2479	1.3417

W przypadku naszej macierzy M wynosi on: $\frac{0.159759}{0.5247} = 0.3044768$. W tej metodzie również przyjmuje się, że warunkiem spójności jest spełnienie nierówności: $CR \leq 0.1$, więc klasyfikujemy macierz M jako niespójną.

4.3. Metoda odległościowa - Koczkodaj

Jedną z głównych wad współczynnika Saaty'ego jest fakt, że wartości własne są wielkościami charakteryzującymi całą macierz, nie pozwalają więc określić, które elementy powodują wystąpienie niespójności. Rozwiązaniem tego problemu jest wprowadzona przez Koczkodaja [10] metoda odległościowa, którą krótko przedstawię. Nieco bardziej rozbudowany opis, napisany przyjaznym językiem, można znaleźć w [11].
Zacznijmy od rozważenia macierzy o wymiarach 3×3:

$$A = \begin{pmatrix} 1 & a & b \\ \frac{1}{a} & 1 & c \\ \frac{1}{b} & \frac{1}{c} & 1 \end{pmatrix}$$

Odwołując się do 4.1 możemy wnioskować, że $b = ac$. Nasza macierz będzie spójna, jeśli spełniony zostanie ten warunek.

W celu zmierzenia ewentualnej niespójności, ponownie wykorzystując 4.1, możemy stworzyć trzy wektory, w których jedna wartość zostanie wyliczona jako kombinacja dwóch pozostałych. Otrzymujemy więc wektory: $(\frac{b}{c}, b, c)$, (a, ac, c) i $(a, b, \frac{b}{a})$. Następnie sprawdzamy odległość każdego z tych wektorów od danego w przykładzie wektora (abc). Wybieramy ten, którego wartość odległości jest najniższa. Uzyskany rezultat to współczynnik niespójności. Zapis formalny przedstawionego algorytmu wygląda następująco:

$$CM(a,b,c) = min\{\frac{1}{a}|a - \frac{b}{c}|, \frac{1}{b}|b - ac|, \frac{1}{c}|c - \frac{b}{a}|\}$$

Teraz możemy przejść do macierzy o większych wymiarach. Okazuje się, że wystarczy wyszukać wszystkie trójki liczb, które powinny być od siebie zależne. Trójki te zostały nazwane *triadami*. Aby wyznaczyć współczynnik niespójności macierzy wystarczy wybrać triad, dla którego wyliczona wartość jest największa:

$$CM(A) = max\{\{|1 - \frac{b}{ac}|, |1 - \frac{ac}{b}|\} \qquad \forall_{triady \quad (a,b,c) \quad macierzy \quad A}$$

Przyjmuje się, że macierz jest spójna, jeżeli $CM(A) \leq \frac{1}{3}$.

Warto zauważyć, że metoda odległościowa pozwana nie tylko zbadać niespójność, ale także wskazuje miejsce, które ma na nią największy wpływ. Dlatego można poprawić wprowadzone do macierzy PC wartości w jednym, konkretnym miejscu i przez to zmniejszyć współczynnik niespójności.

4.4. Biblioteka PairwiseComparisons - niespójność

Funkcje z biblioteki PairwiseComparisons pomocne w powyższych obliczeniach:

Współczynnik Saaty'ego

Nazwa

saatyIdx

Opis

Oblicza współczynnik niespójności macierzy w podstawowej wersji zaproponowanej przez Saaty'ego.

Argumenty

matrix – PC matrix

Zwracana wartość

Wartość współczynnika Saaty'ego.

Dodatkowe informacje

W bibliotece dostępna jest również symboliczna wersja tej funkcji.*saatyIdxSym*

Współcznnik niespójności Koczkodaja dla triady

Nazwa

koczkodajTriadIdx

Opis

Oblicza współczynnik niespójności dla triady metodą Koczkodaja

Argumenty

triad – wektor trzech liczb

Zwracana wartość

Współczynnik Koczkodaja

Najbardziej niespójny triad macierzy

Nazwa

koczkodajTheWorstTriad

Opis

Znajduje triad, którego wartość współczynnika niespójności Koczkodaja jest największa.

Argumenty

matrix – PC matrix

Zwracana wartość

Triad o największym współczynniku niespójności.

Najbardziej niespójne triady macierzy

Nazwa

koczkodajTheWorstTriads

Opis

Znajduje triady, których wartości współczynnika niespójności Koczkodaja są największe.

Argumenty

matrix – PC matrix

n – ilość poczukiwanych triad

Zwracana wartość

Triady o największym współczynniku niespójności.

Współczynnik niespójności Koczkodaja

Nazwa

koczkodajIdx

Opis

Oblicza współczynnik niespójności macierzy w podstawowej wersji zaproponowanej przez Koczkodaja.

Argumenty

matrix – PC matrix

Zwracana wartość

Wartość współczynnika Koczkodaja.

Spójny triad

Nazwa

koczkodajConsistentTriad

Opis

Na podstawie przekazanej trójki liczb, znajduje triad , którego wartość współczynnika niespójności jest najmniejsza.

Argumenty

triad – wektor trzech liczb

Zwracana wartość

Triad o najmniejszym współczynniku niespójności.

Poprawiona macierz

Nazwa

koczkodajImprovedMatrixStep

Opis

Znajduje miejsce (triad), w którym macierz jest najbardziej niespójna, a następnie poprawia wartości w tych miejsach, w taki sposób, aby współczynnik niespójności zmniejszył nię

Argumenty

matrix – PC matrix

Zwracana wartość

Macierz o mniejszym współczynniku niespójności.

5. Pozostałe metody

W ciągu wielu lat rozwijania metody porównań parami i badań prowadzonych w tej dziedzinie, powstało wiele funkcji, które nie są bezpośrednio częścią PC, przyczyniają się jednak do weryfikacji prawidłowości działania metody, upraszczają pracę z macierzami, również tymi niekompletnymi, czy pozwalają sprawdzić otrzymane wyniki. W tym rozdziale przedstawię kilka z takich metod.

5.1. Łączenie rankingów

Kiedy chcemy rozwiązać jakiś określony problem wykorzystując metodę porównań parami, możemy sami dokonać porównań lub poprosić o nie ekspertów w danej dziedzinie. To daje nam nadzieję, że wyniki będą obiektywne i odzwierciedlające rzeczywistość. Po otrzymaniu od nich macierzy PC, chcemy połączyć te tabele i utworzyć z nich jedną, która będzie odzwierciedlać preferencje wszystkich ekspertów. Drugą alternatywą jest obliczenie wektora wag dla każdej z otrzymanych macierzy, a następnie połączenie ich w jeden, sumaryczny wektor.

Aby tego dokonać możemy posłużyć się metodami *Aggregating individual judgments and priorities (AIJ i AIP)*, które wprowadzili Forman i Peniwati [12]. Proponują oni użycie funkcji, które wyliczają średnie arytmetyczne i geometryczne, zarówno dla macierzy, jak i dla wektorów.

Przyklad 5.1.1.

$$M_1 = \begin{pmatrix} 1 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & \frac{1}{3} & 4 \\ 3 & 1 & 9 \\ \frac{1}{4} & \frac{1}{9} & 1 \end{pmatrix}$$

Obliczamy macierz PC, która odzwierciedla obie macierze:

– *z wykorzystaniem średniej arytmetycznej:*

$$M_{art} = \begin{pmatrix} 1 & \frac{7}{6} & 4 \\ 1\frac{3}{4} & 1 & 5\frac{1}{2} \\ \frac{1}{4} & \frac{11}{36} & 1 \end{pmatrix}$$

– *z wykorzystaniem średniej geometrycznej:*

$$M_{geom} = \begin{pmatrix} 1 & 0.8164966 & 4 \\ 1.224745 & 1 & 4.242641 \\ 0.25 & 0.2357023 & 1 \end{pmatrix}$$

5.2. Wektor wag a macierz PC

Przyjrzyjmy się jeszcze raz przedstawionemu już wcześniej warunkowi spójności macierzy: $m_{ik} = m_{ij}m_{jk}$. Wyprowadzenie tego wzoru jest proste i opiera się na spostrzeżeniu, że stosunek dwóch elementów wektora wag powinien być równy odpowiadającej im wartości w macierzy PC. Przykładowo, jeśli alternatywa druga ma wartość 0.4, a alternatywa trzecia 0.2, to spodziewamy się, że wartość porównania drugiej i trzeciej alternatywy wynosi $\frac{0.4}{0.2} = 2$. Szczegółową wersję spójności macierzy można więc przedstawić następująco:

$$a_{ij}a_{jk} = \frac{w_i}{w_j}\frac{w_j}{w_k} = \frac{w_i}{w_k} = a_{ik}. \tag{5.1}$$

Właśnie na tym spostrzeżeniu bazują Bana e Costa i Vansnick [13] w swoich badaniach, wprowadzając dwa sposoby sprawdzania spójności i wykrywania miejsc, w których macierz nie jest spójna: *(first and second Condition of Order Preservation - COP)*. Pierwszy sposób (COP1) sprawdza, czy wybór lepszej alternatywy w każdej parze przekłada się na większą wartość danej alternatywy w wektorze wag. Drugi sposób (COP2) jest bardziej dokładny i wprost weryfikuje warunek 5.1 dla każdej pary alternatyw.

5.3. Odległość między wektorami

Ciekawym algorytmem jest obliczanie odległości między wektorami. Sposób ten przedstawił Maurice Kendall [14]. Opiera się on na idei sortowania bąbelkowego. W czasie sortowania tego rodzaju, kolejno porównujemy sąsiadujące ze sobą elementy i jeśli nie są we właściwej kolejności, to zamieniamy je miejscami. Czynność powtarzamy aż do momentu, gdy cała lista elementów jest posortowana. Kendall zaproponował algorytm zliczający ilość *zamian miejscami*, które należałoby wykonać w danym wektorze, by stał się identyczny do drugiego wektora.

5.4. Inne funkcje usprawniające pracę z macierzami PC

W bibliotece PairwiseComparisons znalazły się także inne funkcje, które bezpośrednio nie dotyczą metody porównań parami, ułatwiają jednak prace z macierzami, w szczególności z macierzami PC. Przykładem może być funkcja *recreatePCMatrix*, która przyjmuje macierz z uzupełnionymi wartościami tylko powyżej przekątnej, a pozostałe elementy uzupełnia automatycznie.

5.5. Biblioteka PairwiseComparisons - pozostałe metody

Funkcje z biblioteki PairwiseComparisons pomocne w powyższych obliczeniach:

Agregacja macierzy/wektorów (średnia arytmetyczna)

Nazwa

AIJadd

Opis

Oblicza macierz lub wektor, którego elementy są średnią arytmetyczną przekazanych macierzy lub wektorów.

Argumenty

... – lista macierzy lub wektorów tych samych wymiarów

Zwracana wartość

Średnia macierz/wektor

Agregacja macierzy/wektorów (średnia geometryczna)

Nazwa

AIJgeom

Opis

Oblicza macierz lub wektor, którego elementy są średnią geometryczną przekazanych macierzy lub wektorów.

Argumenty

... – lista macierzy lub wektorów tych samych wymiarów

Zwracana wartość

Średnia macierz/wektor

Lista COP1

Nazwa

cop1ViolationList

Opis

Wyznacza listę indeksów macierzy, które nie spełniają pierwszego warunku *Condition of Order Preservation (COP1)*

Argumenty

matrix – macierz PC

resultList – wektor wag macierzy

Zwracana wartość

Lista indeksów niespełniających $COP1$

COP1

Nazwa

cop1Check

Opis

Sprawdza czy każda para indeksów macierzy spełnia pierwszy warunek *Condition of Order Preservation (COP1)*

Argumenty

matrix – macierz PC
resultList – wektor wag macierzy

Zwracana wartość

true jeśli $COP1$ jest spełniony, w przeciwnym razie *false*

Lista COP2

Nazwa

cop2ViolationList

Opis

Wyznacza listę indeksów macierzy, które nie spełniają drugiego warunku *Condition of Order Preservation (COP2)*

Argumenty

matrix – macierz PC
resultList – wektor wag macierzy

Zwracana wartość

Lista indeksów niespełniających $COP2$

COP2

Nazwa

cop2Check

Opis

Sprawdza czy każda para indeksów macierzy spełnia drugi warunek *Condition of Order Preservation (COP2)*

Argumenty

matrix – macierz PC
resultList – wektor wag macierzy

Zwracana wartość

true jeśli $COP2$ jest spełniony, w przeciwnym razie *false*

Rozbieżność rankingu

Nazwa

errorMatrix

Opis

Oblicza rozbieżność dla każdego elementu macierzy i tworzy macierz E zawierającą elementy obliczane według wzoru $e_{ij} = m_{ij}\frac{r_i}{r_j}$. Jeśli macierz M nie zawiera elementów rozbieżnych (warunek *COP2* jest spełniony), każdy element e_{ij} wynosi 1.

Argumenty

matrix – macierz PC

resultList – wektor wag macierzy

Zwracana wartość

Macierz błędów $E = [e_{ij}]$

Lokalna rozbieżność rankingu

Nazwa

localDiscrepancyMatrix

Opis

Oblicza wielkość rozbieżności (patrz *errorMatrix*) dla każdego elementu.

Argumenty

matrix – macierz PC

resultList – wektor wag macierzy

Zwracana wartość

Macierz z lokalnymi rozbieżnościami

Globalna rozbieżność rankingu

Nazwa

globalDiscrepancy

Opis

Znajduje największą lokalną rozbieżność (patrz *localDiscrepancyMatrix*).

Argumenty

matrix – macierz PC

resultList – wektor wag macierzy

Zwracana wartość

Maksymalna wartość lokalnej rozbieżności

Odległość między wektorami

Nazwa

kendallTauDistance

Opis

Oblicza odległość Kendall Tau (sortowanie bąbelkowe) pomiędzy dwoma wektorami.

Argumenty

list1 - pierwszy wektor do porównania

list2 - drugi wektor do porównania

Zwracana wartość

Liczba *zamian miejscami*, które należy wkonać, aby kolejność elementów w wektorach była identyczna

Znormalizowana odległość między wektorami

Nazwa

normalizedKendallTauDistance

Opis

Oblicza odległość Kendall Tau (sortowanie bąbelkowe) pomiędzy dwoma wektorami (patrz *kendallTauDistance* i dzieli je przez liczbę wszystkich możliwych *zamian miejscami*.

Argumenty

list1 - pierwszy wektor do porównania

list2 - drugi wektor do porównania

Zwracana wartość

Stosunek liczba *zamian miejscami*, które należy wkonać, aby kolejność elementów w wektorach była identyczna do liczby wszystkich możliwych *zamian miejscami*

Usuń wiersze

Nazwa

deleteRows

Opis

Usuwa wybrane wiersze z macierzy

Argumenty

matrix – macierz PC

listOfRows – wektor indeksów wierszy, które należy usunąć

Zwracana wartość

Macierz po usunięciu wskazanych wierszy

Usuń kolumny

Nazwa

deleteColumns

Opis

Usuwa wybrane kolumny z macierzy

Argumenty

matrix – macierz PC

listOfColumns – wektor indeksów kolumn, które należy usunąć

Zwracana wartość

Macierz po usunięciu wskazanych kolumn

Usuń wiersze i kolumny

Nazwa

deleteRowsAndColumns

Opis

Usuwa wybrane wiersze i kolumny z macierzy

Argumenty

matrix – macierz PC

listOfRowsAndColumns – wektor indeksów wierszy i kolumn, które należy usunąć

Zwracana wartość

Macierz po usunięciu wskazanych wierszy i kolumn

Zwróć element macierzy

Nazwa

getMatrixEntry

Opis

Zwraca $[r, c]$ element z macierzy.

Argumenty

matrix – macierz PC

r – numer rzędu

c – numer kolumny

Zwracana wartość

Wskazany element macierzy

Utwórz kompletną macierz PC

Nazwa

recreatePCMatrix

Opis

Na podstawie macierzy z uzupełnionymi wartościami nad przekątną, tworzy kompletną macierz PC

Argumenty

matrix – macierz PC

Zwracana wartość

Kompletna macierz PC

Puste elementy macierzy

Nazwa

harkerMatrixPlaceHolderCount

Opis

Sprawdza ile elementów w rzędzie ma wartość 0.

Argumenty

matrix – macierz PC

row – numer rzędu, którego elementy są sprawdzane

Zwracana wartość

Ilość pustych elementów w wierszu

Napraw macierz

Nazwa

harkerMatrix

Opis

Tworzy macierz gotową do użycia w metodach porównań parami. Niewłaściwe elementy zastępowane są wartością 0, a na przekątnej ustawiana jest wartość 1.

Argumenty

matrix – macierz z możliwymi błędnymi wartościami

Zwracana wartość

Macierz PC gotowa do użycia w metodach porównań parami.

Utwórz spójną macierz

Nazwa

consistentMatrixFromRank

Opis

Na podstawie rankingu wag W tworzy spójną macierz PC, której elemnty to $m_{ij} = \frac{w_i}{w_j}$.

Argumenty

rankList – wektor wag macierzy

Zwracana wartość

Spójna macierz PC

Sortuj ranking

Nazwa

rankOrder

Opis

Sortuje malejąco wartości wektora wag od najwyższej do najniższej

Argumenty

rankList – wektor wag macierzy

Zwracana wartość

Posortowany wektor wag macierzy

6. Podsumowanie i wnioski

Cel pracy, którym było stworzenie biblioteki PairwiseComparisons, został osiągnięty. Powstała biblioteka, zgodnie z założeniami, pokrywa funkcjonalność pakietu [6] i służy do wykonywania obliczeń matematycznych, na których opiera się metoda porównań parami. Biblioteka pozwala obliczać gotowe rankingi poprzez wywołanie tylko jednej funkcji, jak również udostępnia wiele metod, które umożliwiają stopniowe budowanie rankingów.

Pakiet napisany został w języku R, który przeznaczony jest do obliczeń matematycznych i wykorzystywany głównie w tym celu. Jego składnia i sposób definiowania własnych funkcji jest intuicyjny, a sposób działania szybki. Powstała również biblioteka napisana w języku Java umożliwiająca korzystanie z funkcji z poziomu maszyny wirtualnej Javy. Wykorzystuje bibliotekę *RCaller* [15], która pozwala na wywoływanie kodu źródłowego napisanego w języku R - przeznaczonego przecież właśnie do obliczeń. Rozwiązanie to pozwoliło w prosty sposób połączyć oba języki. Dzięki niemu nie musiałem wprost implementować szczegółów metody porównań parami w języku Java. Wykorzystanie dobrodziejstw R w pakiecie Javy okazało się dobrym wyborem i uświadomiło mi, że w ramach pisania kodu źródłowego w jednym języku, warto czasem pewne specyficzne fragmenty *odesłać* do innego, aby ułatwić implementację i usprawnić pracę całego rozwiązania.

Biblioteka może posłużyć osobom, które znają już metodę porównań parami i pracują z nią na co dzień. Powinna przyspieszyć ich działania i pozwolić skupić się na rozwijaniu metody, a nie wyszukiwaniu sposobów na zaimplementowanie podstawowych funkcjonalności w języku Java. Pakiet może zostać również wykorzystany do wprowadzenia nowych osób w zagadnienie metody porównań parami poprzez zaprezentowanie prostych przykładów jej działania. Kiedy opowiada się o PC, jednym z elementów powodującym zniechęcenie są obliczenia matematyczne, których implementacja może wydawać się skomplikowane. Powstała biblioteka pozwoli odwrócić od niej uwagę i pokazać, że można korzystać z metody w prosty sposób. Mam nadzieję, że wpłynie to pozytywnie na odbiór metody, pozwoli zwiększyć jej popularność i zainteresować nią nowe osoby.

Metoda porównań parami ciągle się rozwija i z pewnością w przyszłości będzie poszerzana o nowe funkcjonalności, a także wykorzystywana w coraz to nowych dziedzinach. Nowe rozwiązania mogą zostać wprowadzane do biblioteki, której kolejne wersje będą odpowiadać aktualnemu stopniowi rozwoju metody. Czekam więc na kolejne odkrycia w tej dziedzinie, aby zaimplementować je w językach R i Java, a następnie dostarczyć badaczom w celu usprawnienia ich pracy.

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Generalization of a new definition of consistency for pairwise comparisons

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1. Consistency

The generalization of a new definition of consistency for pairwise comparisons (see [3]) is proposed. Making comparative judgments of intangible criteria (e.g. the degree of an environmental hazard or pollution factors) involves not only imprecise or inexact knowledge but also inconsistency in our own judgments. The improvement of knowledge elicitation by controlling the inconsistency of experts' judgments is not only desirable but absolutely necessary.

Checking consistency in the pairwise comparison method could be compared to checking that the divisor is not equal to 0. It simply does not make sense to divide anything by 0 and all proposed (heuristic) solutions to pairwise comparison models are based on an assumption that the given reciprocal matrix is consistent (see [5]). Can we not assume that the reciprocal matrix must be consistent? Requesting all the judgments to be consistent is not the answer. We know that most judgements are subjective and nearly always contain some type of bias.

The definition of consistency of a pairwise comparison matrix **A**, based on eigenvalues, was introduced by Saaty [5]. His consistency definition is given by the following formula,

$$cf = \lambda_A - \text{order}(A)/(\text{order}(A) - 1)\,\lambda_{\text{random}},$$

where λ_A is the largest eigenvalue of the reciprocal matrix **A** and λ_{random} is the largest eigenvalue of randomly generated reciprocal matrix of the same order as matrix **A** (see [5]).

However, the above formula leads to some theoretical problems (see [6] and [3]). A new definition of consistency (see [3]) is based on one triad (A_i, A_j, A_k) of the comparisons matrix **A**. In this case, the pairwise comparisons matrix reduces to the following 3×3 basic reciprocal matrix **A3**,

$$\mathbf{A3}(A_i, A_j, A_k) = \begin{vmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{vmatrix},$$

where a expresses an expert's relative preference of criterion A_i, over A_j, b expresses preference of criterion A_i, over A_k, and c is a relative preference of

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stimulus A_j, over stimulus A_k. Matrix $\mathbf{A3}(A_i, A_j, A_k)$ is consistent if and only if $b = ac$.

The new definition of consistency of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k$ is based on the following intuition: it is a measure of deviation from the nearest basic consistent reciprocal matrix. The interpretation of the consistency measure becomes more apparent when we reduce a basic reciprocal matrix to a vector of three coordinates $[a, b, c]$. We know that $b = ac$ holds for each consistent reciprocal matrix. Therefore, we can always produce three consistent reciprocal matrices (therefore three vectors) by computing one coordinate from the combination of the remaining two coordinates. These three vectors are: $[b/c, b, c]$, $[a, ac, c]$, and $[a, b, b/a]$. The inconsistency measure will be defined as the relative distance to the nearest consistent reciprocal matrix represented by one of these three vectors for a given metric. In the case of Euclidean (or Chebysheff) metric we have the following definition (see [3]):

Definition 1. Consistency measure $\mathbf{CM}(A_i, A_j, A_k)$ of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k)$ with the following ratios of comparisons $A_i/A_j = a$, $A_i/A_k = b$, $A_j/A_k = c$, where $a, b, c > 0$, is equal to

$$\begin{aligned}&\mathbf{CM}(A_i, A_j, A_k)\\ &\quad = \min\{|a - b/c|/a, |b - ac|/b, |c - b/a|/c\}.\end{aligned}$$

Note that the consistency measure is not a metric. It is a global matrix characteristic, which could be compared (in spirit) to the entropy of a probabilistic sample space. The definition of $\mathbf{CM}$ can be extended to the reciprocal matrices of any order. To achieve this we must first prove that Definition 1 does not depend on the order of criteria.

Lemma 2. *If $\mathbf{A3}(A_i, A_j, A_k)$ is a basic reciprocal matrix then, for each permutation $\tau : \mathbb{N}^3 \to \mathbb{N}^3$, $\mathbf{CM}(A_i, A_j, A_k) = \mathbf{CM}(A_{\tau(i)}, A_{\tau(j)}, A_{\tau(k)})$ where $\mathbb{N}$ denotes the set of all positive integers.*

Proof. Assume the following ratios, $A_i/A_j = a$, $A_i/A_k = b$, $A_j/A_k = c$ and $a, b, c > 0$; then:

$$\begin{aligned}&\mathbf{CM}(A_i, A_j, A_k) = \mathbf{CM}(a, b, c)\\ &\quad = \min\{|a - b/c|/a, |b - ac|/b, |c - b/a|/c\}\\ &\quad = \min\{|ac - b|/ac, |b - ac|/b, |b - ac|/ac\}\\ &\quad = \min\{|1 - b/ac|, |1 - ac/b|\}.\end{aligned}$$

For each permutation of indexes i, j, k we have:

$$\begin{aligned}&\mathbf{CM}(A_i, A_k, A_j) = \mathbf{CM}(b, a, 1/c)\\ &\quad = \min\{|1 - ac/b|, |1 - b/ac|\},\\ &\mathbf{CM}(A_k, A_i, A_j) = \mathbf{CM}(1/b, 1/c, a)\\ &\quad = \min\{|1 - b/ac|, |1 - ac/b|\},\\ &\mathbf{CM}(A_k, A_j, A_i) = \mathbf{CM}(1/c, 1/b, 1/a)\\ &\quad = \min\{|1 - ac/b|, |1 - b/ac|\},\\ &\mathbf{CM}(A_j, A_i, A_k) = \mathbf{CM}(1/a, c, b)\\ &\quad = \min\{|1 - ac/b|, |1 - b/ac|\},\\ &\mathbf{CM}(A_j, A_k, A_i) = \mathbf{CM}(c, 1/a, 1/b)\\ &\quad = \min\{|1 - b/ac|, |1 - ac/b|\}.\end{aligned}$$

The above equations complete the proof. □

As an immediate consequence of the above lemma we have obtained the following, equivalent formula for calculating the consistency measure of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k)$.

Definition 3. Consistency measure $\mathbf{CM}(A_i, A_j, A_k)$ of a basic reciprocal matrix $\mathbf{A3}(A_i, A_j, A_k)$ with ratios of comparisons $A_i/A_j = a$, $A_i/A_k = b$, $A_j/A_k = c$, $a, b, c > 0$ is equal to

$$\begin{aligned}&\mathbf{CM}(A_i, A_j, A_k)\\ &\quad = \mathbf{CM}(a, b, c) = \min\{|1 - b/ac|, |1 - ac/b|\}.\end{aligned}$$

We can proceed now to the general case of any $n \times n$ reciprocal matrix.

Lemma 4. *The number of all possible triads of the $n \times n$ comparison matrix is equal to $n(n-1)(n-2)/3!$.*

Proof. To have inconsistent judgments we must have at least three criteria to be compared. Consequently we may assume that all indexes i, j, k must be pairwise different. According to Lemma 2, we may calculate inconsistencies only for triads with indexes holding the property $1 \leqslant i < j < k \leqslant n$. It is known that the number of such indexes equals the number of all three element subsets of the set $\{1, 2, \ldots, n\}$. By

Newton's formula it is equal $n!/(n-3)!3!$, which is $n(n-1)(n-2)/3!$. □

Both of the above lemmas give the following definition of the consistency measure of any $n \times n$ pairwise comparison matrix **A**.

Definition 5. Consistency measure $\mathbf{CM}(\mathbf{A})$ of an n by n $(n > 2)$ reciprocal matrix **A** is given by the following formula:

$$\mathbf{CM}(\mathbf{A}) = \max\{\mathbf{CM}(A_i, A_j, A_k) \mid 1 \leqslant i < j < k \leqslant n)\}.$$

Definition 5 gives the opportunity to design an algorithm for reducing the inconsistency of the expert's judgments. It should be seen as a technique for data validation in the knowledge acquisition process. The consistency measure of a comparison matrix is the measure of the validity of knowledge. To"improve" the quality of the knowledge, experts, with the help of computer software, compute the consistency of their judgments. The program displays the triad with the largest inconsistency, so the experts have a possibility to revise their preferences. The important point is that the system does not force the experts to change their judgment. Instead, the computer program flags the most critical spot in the set of judgments.

2. Applications

The authors worked in conjuction with the mining rehabilitation experts from the Provincial Ministry of Northern Development and Mines. An expert system is implemented for rehabilitation problems connected with abandoned mines in Ontario. The system will assist in semistructered decision situations. The main goal of this system is to provide management with the most comprehensive and most updated information necessary to make consistent decisions. Four major objectives of the system are:

- to develop a practical tool for priority setting and decision making procedures in coping with abandoned mine issues in Ontario,
- to ensure the most effective use of public funds allocated for mine rehabilitation work,
- to protect public interest for Ontario residents in public safety, public health, environmental concerns, social concerns and economic concerns,
- to find the way to link, in many cases contradictory, technical and socio-economic factors included in the decision making process.

In building our expert system, the process of knowledge acquisition is based on the pairwise comparisons method. The consistency measure described in this paper is the basic tool in the knowledge validation process (see [1] and [2] for details).

The authors are also involved in another project for a university library. The purpose is to help librarians select the most appropriate CD-ROM collection. The system will prioritize the CD-ROM titles according to the library policies and the librarians' preferences.

3. Conclusions

Weights, reflecting the relative importance of the objectives concerned are a valuable piece of information. There exist various ways of formulating priorities: trade-off methods, ratings, rankings, verbal statements, pairwise comparisons. Voogd (see [7]) reports that the ranking method is preferable by the majority of interviewed experts. There is no tool, however, for knowledge validation in this case. Under some circumstances it might be attractive to use a method that enables decision makers to express their priorities in a more refined way. Therefore, the pairwise comparisons method is proposed with the consistency measure as a knowledge elicitation technique. This approach, which we call a consistency-driven knowledge acquisition, improves the problem understanding and enhances quality of the knowledge acquired for the design of an expert system. The generalized definition is simplified for a more efficient computability and extended to a reciprocal matrix of any order.

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Theory and Methodology

Aggregating individual judgments and priorities with the Analytic Hierarchy Process

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Abstract

The Analytic Hierarchy Process (AHP) is often used in group settings where group members either engage in discussion to achieve a consensus or express their own preferences. Individual judgments can be aggregated in different ways. Two of the methods that have been found to be most useful are the aggregation of individual judgments (AIJ) and the aggregation of individual priorities (AIP). We propose that the choice of method depends on whether the group is assumed to act together as a unit or as separate individuals and explain why AIJ is appropriate for the former while AIP is appropriate for the latter. We also address the relationships between the choice of method, the applicability of the Pareto principle, and the use of arithmetic or geometric means in aggregation. Finally, we discuss Ramanathan and Ganesh's method to derive priorities for individual decision-makers that can be used when aggregate group preferences of individuals whose judgments are not all equally weighted. We conclude that while this method can be useful, it is applicable only in special circumstances.

Keywords: Analytic Hierarchy Process; Aggregating individual judgments; Aggregating individual priorities; Geometric mean

1. Introduction

The Analytic Hierarchy Process (AHP) of Saaty (1980) is one of the most popular and powerful techniques for decision making in use today. AHP is generally used to derive priorities based on sets of pairwise comparisons. The AHP is built on a human being's intrinsic ability to structure his perceptions or his ideas hierarchically, compare pairs of similar things against a given criterion or a common property, and judge the intensity of the importance of one thing over the other. The AHP then synthesizes all the judgments, using the framework given by the hierarchy, to obtain the overall priority of the elements. There are several possible ways to aggregate information when more than one (perhaps many) individuals participate in a decision process,

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including: (1) aggregating the individual judgments for each set of pairwise comparisons into an 'aggregate hierarchy'; (2) synthesizing each of the individual's hierarchies and aggregating the resulting priorities; and (3) aggregating the individual's derived priorities in each node in the hierarchy. In any case, the relative importance of the decision-makers may either be assumed to be equal, or else incorporated in the aggregation process. We will focus on the first two of these methods here and refer to them as aggregating individual judgments (AIJ) and aggregating individual priorities (AIP). While technically the AHP can deal with the third method, it is less meaningful and not commonly used.

Three fundamental questions need to be addressed interdependently to meaningfully obtain group preference from individual information with the AHP. First, whether the group is assumed to be a synergistic unit or simply a collection of individuals. The response to this question determines whether to use the AIJ or AIP method. Second, what mathematical procedure should be used to aggregate individual judgments? The answer is dependent on the response to the first question. Third, if the individuals are not weighted equally, how to obtain their weights and how to incorporate them in the aggregation process.

Both the geometric mean and the arithmetic mean are appropriate procedures for ratio scales. Ramanathan and Ganesh (1994) observe that employing AIJ violates the Pareto principle of social choice theory. Insisting that the principle should apply, they have proposed that a weighted AIP be used instead. We view AIJ and AIP as two circumstances that are philosophically different. We consider when the Pareto principle is relevant and under what circumstances AIJ or AIP should be used. We also provide reasons for using the geometric rather than the arithmetic mean.

2. The group: A new 'individual' or a collection of independent individuals

There are two basic ways to aggregate individual preferences into a group preference, depending on whether the group wants to act together as a unit or as separate individuals. An example of the former is a group of department heads meeting to set corporate policy. An example of the latter is a group consisting of representative constituencies with stakes in welfare reform, such as taxpayers, those on welfare, politicians, etc. One needs to decide which situation is applicable in order to determine the proper procedure for AIP. Ramanathan and Ganesh do not distinguish between these situations, considering only the latter.

3. Aggregating individual judgments (AIJ)

When individuals are willing to, or must relinquish their own preferences (values, objectives) for the good of the organization, they act in concert and pool their judgments in such a way that the group becomes a new 'individual' and behaves like one. There is a synergistic aggregation of individual judgments. Individual identities are lost with every stage of aggregation and a synthesis of the hierarchy produces the group's priorities. Because we are not concerned with individual priorities, and because each individual may not even make judgments for every cluster of the hierarchy, there is no synthesis for each individual – individual priorities are irrelevant or non-existent. Thus, the Pareto principle, which Ramanathan and Ganesh claim is violated, is irrelevant. Furthermore, since the group becomes a new 'individual' and behaves like one, the reciprocity requirement for the judgments must be satisfied and the geometric mean rather than an arithmetic mean must be used as will be shown below. Note that while individual identities are lost when synthesizing the hierarchy, they are maintained for each cluster of elements where an individual provides judgments. Inconsistencies in an individual's set of judgments can be examined and the group can ask an individual to consider revising one or more judgments if the inconsistency is deemed to be too high. The group could also decide to exclude an individual's judgments from the geometric average for a cluster, based on the inconsistency ratio.

4. Aggregating individual priorities

When individuals are each acting in his or her own right, with different value systems, we are concerned about each individual's resulting alternative priorities. An aggregation of each individual's resulting priorities can be computed using either a geometric or arithmetic mean. Neither method will violate the Pareto principle (as will be shown below). Ramanathan and Ganesh erroneously conclude that the arithmetic mean must be used.

5. The Pareto principle and AHP

The Pareto (unanimity, agreement) principle essentially says that given two alternatives A and B, if each member of a group of individuals prefers A to B, then the group must prefer A to B. The principle has been formulated and applied in the social sciences in the AIP context described above. The AIJ approach on the other hand, is a synergistic aggregation of individual judgments when individuals are willing to, or must out of necessity, relinquish their own preferences for the good of the organization. This new perception of a group of individuals was not possible with ordinal measurement methods prior to the AHP. With AIJ the individuals first work together to agree on a common hierarchy before they can work on aggregating their judgments. The agreement on a common hierarchy is the first step in 'merging' the different individuals into a new 'individual' representing the group. The next 'merging' process occurs at the judgment level. Even at this level, the 'merging process' occurs step by step from the most general at the higher level to the more specific at the lower level of the hierarchy. In other words, after agreeing to the general structure of the hierarchy, the group 'merges' further by agreeing on the relative importance of the criteria. Once this process is done, the previous individual judgments with respect to the relative importance of the criteria become irrelevant, the same way their original hierarchies do as soon as a common hierarchy is agreed upon. Consequently there is no synthesis for each individual and the Pareto principle is inapplicable.

6. Geometric mean or arithmetic mean with AIJ and AIP

In general one must decide in advance whether to represent a group by aggregating their individual judgments (AIJ) or by aggregating their individual (final) priorities (AIP), but not both. Treating the group as a new 'individual' with AIJ requires satisfaction of the reciprocity condition for the judgments. Aczel and Saaty (1983), and in a more general paper Aczel and Roberts (1989), have shown that when aggregating the judgments of n individuals where the reciprocal property is assumed even for a single n-tuple, only the geometric mean satisfies the Pareto principle (unanimity condition) and the homogeneity condition (if all individuals judge a ratio t times as large as another ratio, then the synthesized judgment should also be t times as large). Thus, for AIJ, the geometric mean *must* be used. As for AIP, either an arithmetic or geometric mean can be used to aggregate the individuals' priorities. Most people were taught, and have grown up to feel comfortable with the arithmetic mean, or what is commonly referred to as the mean or average. In general, if one wishes to take an 'average' of measurements possessing only interval scale meaning, an arithmetic average must be used since it is meaningless to multiply interval scale numbers. However, for ratio scale measurements (as we always have with AHP) both arithmetic and geometric averages are meaningful. The aggregation of individual priorities will satisfy the Pareto principle with *either* an arithmetic or geometric average:

If $a_i \geqslant b_i$, $i = 1, 2, \ldots, n$ then $\sum_{i=1}^{n} a_i/n \geqslant \sum_{i=1}^{n} b_i/n$ for an arithmetic mean, and

$$\sqrt[n]{\prod_{i=1}^{n} a_i} \geqslant \sqrt[n]{\prod_{i=1}^{n} b_i} \quad \text{for a geometric mean}$$

provided $a_i \geqslant 0$ and $b_i \geqslant 0$, $i = 1, 2, \ldots, n$.

James and James (1968) provide definitions of arithmetic and geometric means that furnish additional insight into the choice of using an arithmetic or geometric mean for the AIP case:

> The arithmetic average of two numbers is the middle term in an arithmetic progression of three terms including the two given numbers.

Thus, for example, the arithmetic mean of 1 and 9 is 5 since the arithmetic progression from 1 to 9 occurs in two equal *intervals* of 4.

> The geometric average of two numbers is the middle term in a geometric progression of three terms including the two given numbers.

Thus, for example, the geometric mean of 1 and 9 is 3, since the geometric progression from 1 to 9 occurs in two equal *ratios* of 3.

While either an arithmetic or geometric mean can be used for AIP, the geometric mean is more consistent with the meaning of both judgments and priorities in AHP. In particular, judgments in AHP represent ratios of how many times more important (preferable) one factor is than another. Synthesized alternative priorities in AHP are *ratio* scale measures and have meaning such that the ratio of two alternatives' priorities represents how many times more preferable one alternative is than the other.

7. Weighted arithmetic and geometric means

When calculating the geometric average of the judgments (AIJ) or either the arithmetic or geometric average of priorities (AIP) we often assume that the individuals are of equal importance. If, however, group members are not equally important, we can form a weighted geometric mean or weighted arithmetic mean as follows:

Weighted geometric mean of judgments (AIJ):

$$J_g(k,l) = \prod_{i=1}^{n} J_i(k,l)^{w_i},$$

where: $J_g(k,l)$ refers to the group judgement of the relative importance of factors k and l, $J_i(k,l)$ refers to individual i's judgment of the relative importance of factors k and l, w_i is the weight of individual i; $\sum_{i=1}^{n} w_i = 1$; and n the number of decision-makers.

Weighted (Un-normalized) geometric mean of priorities (AIP):

$$P_g(A_j) = \prod_{i=1}^{n} P_i(A_j)^{w_i},$$

where $P_g(A_j)$ refers to the group priority of alternative j, $P_i(A_j)$ to individual i's priority of alternative j, w_i is the weight of individual i; $\sum_{i=1}^{n} w_i = 1$; and n the number of decision-makers.

Weighted arithmetic mean of priorities (AIP):

$$P_g(A_j) = \sum_{i=1}^{n} w_i P_i(A_j),$$

where $P_g(A_j)$ refers to the group priority of alternative j, $P_i(A_j)$ to individual i's priority of alternative j, w_i is the weight of individual i; $\sum_{i=1}^{n} w_i = 1$; and n is the number of decision-makers.

The question arises as to how to compute the w_i's. Saaty (1994) suggests forming a hierarchy of factors such as expertise, experience, previous performance, persuasive abilities, effort on the problem, etc. to determine the priorities of the decision-makers. But who is to provide judgments for this hierarchy? If it *cannot* be agreed that one person (a supra decision-maker) will provide the judgments, it is possible to ask the same decision-makers who provided judgments for the original hierarchy to provide judgments for this hierarchy as well. If so, we have a meta-problem of how to weight their individual judgments or priorities in the aggregation process to determine the weights for the decision-makers to apply to the aggregation of the original hierarchy. One possibility is to assume equal weights. Ramanathan and Ganesh provide another method, which they call the eigenvector method of weight derivation. They reason that, if $\overline{w} = (w_1, w_2, \ldots, w_n)^t$ is the 'true' (but unknown) weight priority vector for the individual's weights, and if the individual weight priority vectors derived from the judgments from each of the individuals are arranged in a matrix: $\overline{M} = (\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_n)$, then we can aggregate to find the priorities of the n individuals, $\bar{x}$, where $\bar{x} = \overline{M} * \bar{w}$. Then Ramanathan and Ganesh reason that $\bar{x} = \bar{w}$, resulting in the eigenvector equation: $\bar{w} = \overline{M} * \bar{w}$. We observe that this method is attractive but reasonable *only if* the weights for obtaining priorities of the decision-makers are assumed to be the same as the weights to be used to aggregate the decision-makers' judgments/priorities for obtaining the alternative priorities in the original hierarchy. In general, this need not be the case.

8. Summary and conclusions

When several individuals provide judgments with the Analytic Hierarchy Process, one may AIJ or AIP. The choice of method depends on whether the group is assumed to act together as a unit or as separate individuals. In the former case, the geometric average of individual judgments (AIJ) satisfies the reciprocity requirement, implying a synergistic aggregation of individual preferences in such a way that the group becomes a new 'individual' and behaves like one. Individual identities are lost with every stage of aggregation and the Pareto principle is irrelevant. When group members act as individuals (AIP), one may take either a geometric mean (representing an average ratio) or an arithmetic mean (representing an average interval) of their resulting priorities. While the Pareto principle will not be violated in either case, the geometric mean is more consistent with the meaning of both judgments and priorities in AHP.

If the group members are not considered to be of equal importance, a weighted geometric mean can be used with AIJ or weighted geometric or arithmetic mean with AIP. A separate hierarchy can be constructed to derive priorities of the decision-makers. There is great flexibility in determining who makes the judgments for this hierarchy. When the original group members themselves make these judgments, Ramanathan and Ganesh's eigenvector method can be used *provided* the relative importance of the decision-makers in aggregating to obtain decision-maker priorities are assumed to be the same as the priority of the decision-makers in aggregating the priorities of the hierarchy of the original problem.

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The quality of priority ratios estimation in relation to a selected prioritization procedure and consistency measure for a Pairwise Comparison Matrix

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Abstract: An overview of current debates and contemporary research devoted to the modeling of decision making processes and their facilitation directs attention to the Analytic Hierarchy Process (AHP). At the core of the AHP are various prioritization procedures (PPs) and consistency measures (CMs) for a Pairwise Comparison Matrix (PCM) which, in a sense, reflects preferences of decision makers. Certainly, when judgments about these preferences are perfectly consistent (cardinally transitive), all PPs coincide and the quality of the priority ratios (PRs) estimation is exemplary. However, human judgments are very rarely consistent, thus the quality of PRs estimation may significantly vary. The scale of these variations depends on the applied PP and utilized CM for a PCM. This is why it is important to find out which PPs and which CMs for a PCM lead directly to an improvement of the PRs estimation accuracy. The main goal of this research is realized through the properly designed, coded and executed seminal and sophisticated simulation algorithms in *Wolfram Mathematica 8.0.* These research results convince that the embedded in the AHP and commonly applied, both genuine PP and CM for PCM may significantly deteriorate the quality of PRs estimation; however, solutions proposed in this paper can significantly improve the methodology.

***Keywords**: pairwise comparisons, priority ratios, consistency, AHP, Monte Carlo simulations*

Introduction

It is agreed that the world is a complex system of interacting elements. It is obvious that human minds have not yet evolved to the point where they can clearly perceive relationships of this global system and solve crucial issues associated with them. In order to deal with complex and fuzzy social, economic, and political issues, people must be supported and guided on their way to order priorities, to agree that one goal out-weighs another from a perspective of certain established criterion, to make tradeoffs in order to be able to serve the greatest common interest (Caballero, Romero & Ruiz 2016; García-Melón et al. 2016).

Obviously, intuition cannot be trusted, although many commonly do so, attempting to devise solutions for complex problems which demand reliable answers. Overwhelming scientific evidence indicates that the unaided human brain is simply not capable of simultaneous analysis of many different competing factors and then synthesizing the results for the purpose of rational decision. It is presumably the principal reason why scientists continuously deal with explanations and modeling of decisional problems in a way to make them widely comprehendible. That is why many supportive methodologies have been elaborated in order to make the decision making process easier, more credible and sometimes even possible. Indeed, numerous psychological experiments (Martin 1973),

including the well-known Miller study (Miller 1956) put forth the notion that humans are not capable of dealing accurately with more than about seven (±2) things at a time (the human brain is limited in its short term memory capacity, its discrimination ability and its bandwidth of perception).

Principles of the analytic thinking process

Humans learn about anything by two means. The first involves examining and studying some phenomenon from the perspective of its various properties, and then synthesizing findings and drawing conclusions. The second entails studying some phenomenon in relation to other similar phenomena and relating them by making comparisons (Saaty 2008). The latter method leads directly to the essence of the matter i.e. judgments regarding the phenomenon. Judgments can be relative and absolute. An absolute judgment is the relation between a single stimulus and some information held in short or long term memory. A relative judgment, on the other hand, can be defined as the identification of some relation between two stimuli both present to the observer (Blumenthal 1977). It is said that humans can make much better relative judgments than absolute ones (Saaty 2000). It is probably so because humans have better ability to discriminate between the members of a pair, than compare one thing against some recollection from long term memory.

For detailed knowledge, the mind structures complex reality into its constituent parts, and these in turn into their elements. The number of parts usually ranges between five and nine. By breaking down reality into homogenous clusters and subdividing those into smaller ones, humans can integrate large amounts of information into the structure of a problem and form a more comprehensive picture of the whole system. Abstractly, this process entails the decomposition of a system into a hierarchy which is a model of a complex reality. Thus, a hierarchy constitutes a structure of multiple levels where the first level is the objective followed successively by levels of factors, criteria, sub-criteria, and so on down to a bottom level of alternatives. The goal of this hierarchy is to evaluate the influence of higher level elements on those of a lower level or alternatively the contribution of elements in the lower level to the importance or fulfillment of the elements in the level above. In this context the latter elements serve as criteria and are called properties.

Generally, a hierarchy can be functional or structural. The latter closely relates to the way a human brain analyzes complexity by breaking down the objects perceived by the senses into clusters and sub-clusters, and so on. Thus, in structural hierarchies, complex systems are structured into their constituent parts in descending order according to their structural properties. In contrast, in functional hierarchies complex systems are decomposed into their constituent parts in accordance to their essential relationships.

A large number of hierarchies in application are available in the literature (Saaty 1993). Supposedly, the hierarchical classification is the most powerful method applied by the human mind during intellectual reasoning and ordering of information and/or observations. Thus, we may agree that an efficient and effective multiple criteria decision making process should encompass the following steps:

– transpose the problem into a hierarchy;
– derive judgments that reflect ideas and feelings or emotions;
– represent these judgments with meaningful numbers values;
– apply those number values for computing priorities for the elements in the hierarchy;

– synthesize the results in order to establish an overall outcome.

There is a multiple criteria decision making support methodology which meets the prescription developed above. It is called the Analytic Hierarchy Process (AHP) and was developed at the Wharton School of Business by Thomas Saaty (1977). Although it is a very popular and widely implemented theory of choice, it is also controversial, thus very often validated and valuated from the perspective of its methodology. From that perspective, most recent papers, such as Grzybowski (2016); Kazibudzki (2016a); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); Aguarón, Escobar & Moreno-Jiménez (2014); Lin, Kou & Ergu (2013); Brunelli, Canal & Fedrizzi (2013), unfold new research areas in this matter which should be thoroughly examined and provoke questions which should be answered, that is:

1) Is the principal right eigenvector (REV), as the prioritization procedure (PP), necessary and sufficient for the AHP?

2) Is the reciprocity of the Pairwise Comparison Matrix (PCM) a reasonable condition leading to the improvement of the priority ratios estimation quality?

3) Are PCM consistency measures, commonly applied and embedded in the AHP, really conducive to the improvement of the priority ratios estimation quality?

Principles of the Analytic Hierarchy Process

Preliminaries

The AHP seems to be the most widely used multiple criteria decision making approach in the world today. Probably, the most recent list of application oriented papers can be found in Grzybowski (2016). Actual applications in which the AHP results were accepted and used by competent decision makers can be found in: Saaty (2008); Ishizaka & Labib (2011); Ho (2008); Vaidya & Kumar (2006); Bhushan & Ria (2004); or Saaty & Vargas (2006). However, regardless of AHP popularity, the genuine methodology is also undeniably the most validated, developed and perfected contemporary methodology, see for example: Kazibudzki (2016b); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); or Aguarón, Escobar & Moreno-Jiménez (2014).

The AHP allows decision makers to set priorities and make choices on the basis of their objectives, knowledge and experience in a way that is consistent with their intuitive thought process. AHP has substantial theoretical and empirical support encompassing the study of human judgmental process by cognitive psychologists. It uses the hierarchical structure of the decision problem, pairwise relative comparisons of the elements in the hierarchy, and a series of redundant judgments. This approach reduces errors and provides a measure of the consistency of judgments. The process permits accurate priorities to be derived from verbal judgments even though the words themselves may not be very precise. Thus, it is possible to use words for comparing qualitative factors and then to derive ratio scale priorities that can be combined with quantitative factors.

To make a proposed solution possible i.e. derive ratio scale priorities on the basis of verbal judgments, a scale is utilized to evaluate the preferences for each pair of items. Apparently, the most popular is Saaty's numerical scale which comprises of the integers from one (equivalent to the verbal judgment - 'equally preferred') to nine (equivalent to the verbal judgment - 'extremely preferred'), and their reciprocals. However, in conventional AHP applications it may be desirable to utilize other scales also i.e. a geometric and/or numerical scale. The former usually consists of the numbers computed in accordance with the formula $2^{n/2}$ where n comprises of the integers from minus eight to eight. The latter involves arbitrary integers from one to n and their reciprocals.

The first step in using AHP is to develop a hierarchy by breaking a problem down into its primary components. The basic AHP model includes the goal (a statement of the overall objective), criteria (the factors that should be considered in reaching the ultimate decision) and alternatives (the feasible alternatives that are available to achieve said ultimate goal). Although the most common and basic AHP structure consists of a goal-criteria-alternatives sequence (Fig.1). AHP can easily support more complex hierarchies. A variety of basic hierarchical structures include:
– goal, criteria, sub-criteria, scenarios, alternatives;
– goal, players, criteria, sub-criteria, alternatives;
– goal, criteria, levels of intensities, many alternatives.

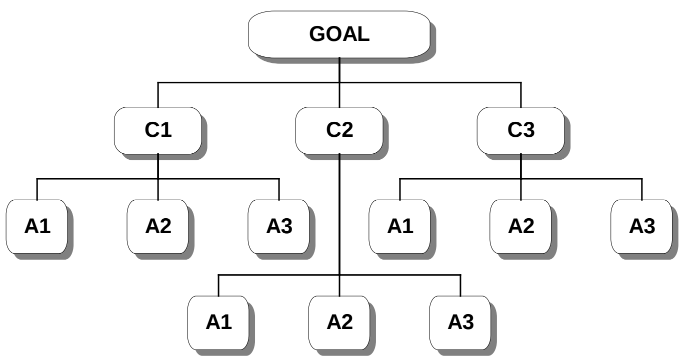


Fig. 1 - Example of a fundamental three level hierarchy encompassing three criteria and three alternatives under each criterion

Mathematics behind the Analytic Hierarchy Process

The conventional procedure of priority ranking in AHP is grounded on the well-defined mathematical structure of consistent matrices and their associated right-eigenvector's ability to generate true or approximate weights.

The German mathematician, Oscar Perron, proved in 1907 that, if $\boldsymbol{A}=(a_{ij})$, $a_{ij}>0$, where i, $j=1,\ldots,n$, then $\boldsymbol{A}$ has a simple positive eigenvalue $\lambda_{\max}$ called the principal eigenvalue of $\boldsymbol{A}$ and $\lambda_{\max}>|\lambda_k|$ for the remaining eigenvalues of $\boldsymbol{A}$. Furthermore, the principal eigenvector $\boldsymbol{w}=[w_1,\ldots,w_n]^T$ that is a solution of $\boldsymbol{A}\boldsymbol{w}=\lambda_{\max}\boldsymbol{w}$ has $w_i>0$, $i=1,\ldots,n$. Thus, the conventional concept of AHP can be presented as follows:

$$\begin{bmatrix} w1/w1 & w1/w2 & w1/w3 & \dots & w1/wn \\ w2/w1 & w2/w2 & w2/w3 & \dots & w2/wn \\ w3/w1 & w3/w2 & w3/w3 & \dots & w3/wn \\ \vdots & \vdots & \vdots & & \vdots \\ wn/w1 & wn/w2 & wn/w3 & \dots & wn/wn \end{bmatrix} \times \begin{bmatrix} w1 \\ w2 \\ w3 \\ \vdots \\ wn \end{bmatrix} = \begin{bmatrix} nw1 \\ nw2 \\ nw3 \\ \vdots \\ nwn \end{bmatrix} \quad (1)$$

If the relative weights of a set of activities are known, they can be expressed as a Pairwise Comparison Matrix (PCM) as shown above $\boldsymbol{A}(w)$. Now, knowing $\boldsymbol{A}(w)$ but not $\boldsymbol{w}$ (vector of priority ratios), Perron's theorem can be applied to solve this problem for $\boldsymbol{w}$. The solution leads to n unique values for λ, with an associated vector $\boldsymbol{w}$ for each of the n values.

PCMs in the AHP reflect relative weights of considered activities (criteria, scenarios, players, alternatives, etc.), so the matrix $\boldsymbol{A}(w)$ has a special form. Each subsequent row of that matrix is a constant multiple of its first row. In this case a matrix $\boldsymbol{A}(w)$ has only one non-zero eigenvalue, and since the sum of the eigenvalues of a positive matrix is equal to the sum of its diagonal elements, the only non-zero eigenvalue in such a case equals the size of the matrix and can be denoted as $\lambda_{\max} = n$.

The norm of the vector $\boldsymbol{w}$ can be written as $\|\boldsymbol{w}\|=\boldsymbol{e}^{\mathrm{T}}\boldsymbol{w}$ where: $e=[1, 1,\dots, 1]^{\mathrm{T}}$ and $\boldsymbol{w}$ can be normalized by dividing it by its norm. For uniqueness, $\boldsymbol{w}$ is referred to in its normalized form.

Theorem 1: A positive n by n matrix has the ratio form $\boldsymbol{A}(w)=(w_i / w_j)$, $i, j=1,\dots, n$, if, and only if, it is consistent.

Theorem 2: The matrix of ratios $\boldsymbol{A}(w)=(w_i / w_j)$ is consistent if and only if n is its principal eigenvalue and $\boldsymbol{Aw}=n\boldsymbol{w}$. Further, $\boldsymbol{w}>0$ is unique to within a multiplicative constant.

Definition 1: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ij}=1/w_{ji}$ for all i, $j=1,\dots, n$ then the matrix $\boldsymbol{A}(w)$ is called *reciprocal*.

Definition 2: The matrix $\boldsymbol{A}(w)$ is called *ordinal transitive* if the following conditions hold:

(A) if for any $i=1,\dots, n$, an element a_{ij} is not less than an element a_{ik} then $a_{ij} \ge a_{ik}$ for $i=1,\dots, n$, and

(B) if for any $i=1,\dots, n$, an element a_{ji} is not less than an element a_{ki} then $a_{ji} \ge a_{ki}$ for $i=1,\dots, n$.

Definition 3: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ik}w_{kj}=w_{ij}$ for all $i, j, k=1,\dots, n$, and the matrix is *reciprocal*, then it is called *consistent* or *cardinal transitive*.

Certainly, in real life situations when AHP is utilized, there is not an $\boldsymbol{A}(w)$ which would reflect weights given by the vector of priority ratios. As was stated earlier, the human mind is not a reliable measurement device. Assignments such as, 'Compare – applying a given ratio scale – your feelings concerning alternative 1 versus alternative 2', do not produce accurate outcomes. Thus, $\boldsymbol{A}(w)$ is not established but only its estimate $\boldsymbol{A}(x)$ containing intuitive judgments, more or less close to $\boldsymbol{A}(w)$ in accordance with experience, skills, specific knowledge, personal taste and even temporary mood or overall disposition. In such case, consistency property does not hold and the relation between elements of $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$ can be expressed as follows:

$$x_{ij} = e_{ij} w_{ij} \tag{2}$$

where e_{ij} is a perturbation factor fluctuating near unity. In the statistical approach e_{ij} reflects a realization of a random variable with a given probability distribution.

It has been shown that for any matrix, small perturbations in the entries imply similar perturbations in the eigenvalues, that is why in order to estimate the true priority vector ***w***, conventional AHP utilizes Perron's theorem. The solution of the matrix equation $\boldsymbol{A}\boldsymbol{w}=\lambda_{max}\boldsymbol{w}$, gives us ***w*** as the Right Principal Eigenvector (REV) associated with λ_{max}.

In practice the REV solution is obtained by raising the matrix $\boldsymbol{A}(x)$ to a sufficiently large power, then the rows of $\boldsymbol{A}(x)$ are summed and the resulting vector is normalized in order to receive ***w***. This concept can be also delivered in the form of the following formula:

$$w = \lim_{k\to\infty}\left(\frac{A^k \times e}{e^T \times A^k \times e}\right) \tag{3}$$

where: $e=[1, 1,\ldots, 1]^T$.

Description of the first problem

It has been promoted that the REV prioritization procedure (PP) is necessary and sufficient to uniquely establish the ratio scale rank order inherent in inconsistent pairwise comparison judgments (Saaty & Hu 1998). However, there are alternative PPs devised to cope with this problem. Many of them are optimization based and seek a vector w, as a solution of the minimization problem given by the formula:

$$\min \boldsymbol{D}(\boldsymbol{A}(x), \boldsymbol{A}(w)) \tag{4}$$

subject to some assigned constraints such as, for example, positive coefficients and normalization condition. Because the distance function $\boldsymbol{D}$ measures an interval between matrices $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$, different ways of its definition lead to various prioritization concepts and prioritization results. As an example, Choo et al. (2004) describes and compares eighteen estimation procedures for ranking purposes although some authors suggest there are only fifteen that are different. Furthermore, since the publication of the above article, a few additional procedures have been introduced to the literature, see for example: Grzybowski (2012).

Certainly, when the PCM is consistent, all known procedures coincide. However, in real life situations, as was discussed earlier, human judgments produce inconsistent PCMs. The inconsistency is a natural consequence of human brain dynamics described earlier and also a consequence of the questioning methodology, mistaken entering of judgment values, and scaling procedure (i.e. rounding errors). It seems crucial to emphasize here that usually even perfectly consistent PCMs, only because of rounding errors are not error-free. It can be illustrated on the basis of the following hypothetic example.

The genuine priority vector: $\boldsymbol{w}=[7/20, 1/4, 1/4, 3/20]$ is considered and derived from it, $\boldsymbol{A}(w)$ which can be presented as follows:

$$\boldsymbol{A}(w)=\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Now it is considered $\boldsymbol{A}(x)$ produced by a hypothetic decision maker (DM), whose judgments are perfectly consistent. Even if it is assumed that the selected DM is very trustworthy and can express judgments very precisely, DM is still somehow limited by the necessity of expressing judgments on a scale (the example utilizes Saaty's scale). As such, the DM will produce the PCM ($\boldsymbol{A}(x)$) which is not error-free because the entries must be in this case rounded to the closest values of Saaty's scale. Since $\boldsymbol{A}(x)$ must be reciprocal (the fundamental requirement of the AHP) the PCM appears as follows:

$$\boldsymbol{A}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

It may be noticed that the above PCM is perfectly consistent, so this construct seems to be exemplary. However, the hypothetic DM, despite best intentions, is burdened with inescapable estimation errors. In the above situation the priority vector (PV) derived from $\boldsymbol{A}(x)$ by any PP, provides the following priority ratios (PRs): $\boldsymbol{x}$=[2/7, 2/7, 2/7, 1/7] which are not equal to those considered exemplary: $\boldsymbol{w}$=[7/20, 1/4, 1/4, 3/20]. Obviously, the deviation between those PVs can also be expressed by their Mean Absolute Error (MAE), for instance, established by the following formula:

$$MAE(w,x)=\frac{1}{n}\sum_{i=1}^{n}\left|w_i-x_i\right| \tag{5}$$

where n is the number of elements within the particular PV. Noticeably, in the above example, MAE equals 1/28.

From that perspective, Saaty & Hu's (1998) declaration articulating that the REV *is the only valid PP for deriving the PV from a PCM, particularly when the PCM is inconsistent* seems at least questionable. However, they provide an example of a situation where variability in ranks does not occur for each individual judgment matrix, it occurs in the overall ranking of the final alternatives due to the application of different PPs and the multi-criteria process itself. They argue that only the REV possesses a sound mathematical background directly dealing with the question of inconsistency. Furthermore, as they state, only the REV captures the rank order inherent in the inconsistent data in a unique manner. It appears to be time to verify the credibility of these statements utilizing the Monte Carlo simulations.

For that purpose, apart from the REV, four different PPs have been arbitrarily selected ranked as the best within AHP methodology (Kazibudzki & Grzybowski 2013; Lin 2007; Choo & Wedley 2004) – Table 1.

Table 1 – Formulae for the prioritization procedures

The Prioritization Procedure	Formula for the Prioritization Procedure
Logarithmic Utility Approach – LUA –	$w_{(LUA)} = \min \sum_{i=1}^{n} \ln^2 \left(\sum_{j=1}^{n} \frac{a_{ij} w_j}{n w_i} \right)$
Sum of Squared Relative Differences Method – SRDM	$w_{(SRDM)} = \min \sum_{i=1}^{n} \left(\frac{1}{n w_i} \sum_{j=1}^{n} a_{ij} w_j - 1 \right)^2$
Logarithmic Least Squares Method – LLSM –	$w_{(LLSM)} = \min \sum_{i=1}^{n} \sum_{j=1}^{n} \ln^2 \left(a_{ij} \frac{w_j}{w_i} \right)$
Simple Normalized Column Sum – SNCS –	$w_{i(SNCS)} = \frac{1}{n} \sum_{j=1}^{n} \left(a_{ij} \Big/ \sum_{k=1}^{n} a_{kj} \right)$

The first problem study

The objective of this chapter is to verify the above statement i.e. *the REV is the only valid method for deriving the PV from a PCM, particularly when the matrix is inconsistent*.

Taking into account the exemplary study of Saaty & Hu (1998), it seems that the best way to analyze the problem is to examine whether different PPs are really inferior in the estimation of true PVs whose intent is accurate estimation. From that perspective, only computer simulations can illuminate the question, for it is possible to elaborate an algorithm which enables simulation of different kinds of errors which may occur during the process of judgment, and enables assessment which one from the selected PPs delivers better estimates (from a given perspective) of the genuine PV.

Thus, the following simulation algorithm was constructed. Assuming that the decisional problem can be presented in the form of a three level hierarchy (goal, criteria and alternatives – see Figure 1). In order to emulate the problem presented in Saaty & Hu (1998), the hypothetical hierarchy is also designed as a four criteria and four alternatives structure i.e. n=4 and m=4. In agreement with these assumptions, it is possible to elaborate and execute the simulation algorithm **SA|1|** comprising of the following steps:

Step 1. Randomly generate a priority vector $\boldsymbol{k}=[k_1,\ldots, k_n]^T$ of assigned size [n x 1] for criteria and related perfect PCM(k)=$\boldsymbol{K}(k)$

Step 2. Randomly generate exactly n priority vectors $\boldsymbol{a_n}=[a_{n,1},\ldots, a_{n,m}]$ of assigned size [m x 1] for alternatives under each criterion and related perfect PCMs(a)=$\boldsymbol{A_n}(a)$

Step 3. Compute a total priority vector $\boldsymbol{w}$ of the size [m x 1] applying the following procedure: $w_x=k_1a_{1,x} + k_2a_{2,x} +\ldots+ k_na_{n,x}$

Step 4. Randomly choose a number e from the assigned interval [α; β] on the basis of assigned probability distribution π

Step 5. Apply separately *Step 5A* and *Step 5B*:

<u>*Step 5A*</u> *– the case of PCM forced reciprocity implementation;*

replace all elements a_{ij} for $i<j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with ea_{ij}, and all elements k_{ij} for $i<j$ of $\boldsymbol{K}(k)$ with ek_{ij}

<u>*Step 5B*</u> *– the case of arbitrary PCM acceptance;*

replace all elements a_{ij} for $i\neq j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with ea_{ij}, and all elements k_{ij} for $i\neq j$ of $\boldsymbol{K}(k)$ with ek_{ij}

Step 6. Apply separately *Step 6A* and *Step 6B*:

<u>*Step 6A*</u> *– when Step 5A is performed;*

round all values of elements a_{ij} for $i<j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$, and all values of elements k_{ij} for $i<j$ of $\boldsymbol{K}(k)$ to the closest values from a considered scale, then replace all elements a_{ij} for $i>j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with $1/a_{ij}$, and all elements k_{ij} for $i>j$ of $\boldsymbol{K}(k)$ with $1/k_{ij}$

<u>*Step 6B*</u> *– when Step 5B is performed;*

round all values of elements a_{ij} for $i\neq j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$, and all values of elements k_{ij} for $i\neq j$ of $\boldsymbol{K}(k)$ to the closest values from a considered scale

Step 7. On the basis of all perturbed $\boldsymbol{A}_{\mathbf{n}}(a)$ denoted as $\boldsymbol{A}_{\mathbf{n}}(a)^*$ and perturbed $\boldsymbol{K}(k)$ denoted as $\boldsymbol{K}(k)^*$ compute their respective priorities vectors $\boldsymbol{a}_{\mathbf{n}}^*$ and $\boldsymbol{k}^*$ with application of assigned estimation procedure (EP), i.e.: REV, LUA, SRDM, LLSM, and SNCS.

Step 8. Compute a total priority vectors $\boldsymbol{w}^*$(EP) of the size [m x *1*] applying the following procedure: $w^*_x=k^*_1a^*_{1,x}+k^*_2a^*_{2,x}+\ldots+k^*_na^*_{n,x}$

Step 9. Calculate *Spearman rank correlation coefficients* – $\mathrm{SRC}_{\gamma,\chi}(\boldsymbol{w^*}(\mathrm{EP}),\boldsymbol{w})$ between all $\boldsymbol{w^*}$(EP) and $\boldsymbol{w}$, as well designated estimation precision characteristics, i.e.: mean relative errors:

$$RE_{\gamma,\chi}\left(w^*(EP),w\right)=\frac{1}{m}\sum_{i=1}^{m}\frac{\left|w_i-w_i^*(EP)\right|}{w_i} \tag{6}$$

along with mean relative ratios:

$$RR_{\gamma,\chi}\left(w^*(EP),w\right)=\frac{1}{m}\sum_{i=1}^{m}\frac{w_i^*(EP)}{w_i} \tag{7}$$

Step 10. Repeat Steps 4 to 9, χ times, where χ denotes a size of the sample

Step 11. Repeat Steps 1 to 9, γ times, where γ denotes a number of considered AHP models

Step 12. Return arithmetic average values of all $\mathrm{SRC}_{\gamma,\chi}(\boldsymbol{w^*}(\mathrm{EP}),\boldsymbol{w})$, $\mathrm{RE}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, and $\mathrm{RR}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ computed during all runs in Steps: 10 and 11, i.e.:

$$MSRC\left(w^*(EP),w\right)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}SRC_i\left(w^*(EP),w\right) \tag{8}$$

$$MRE\left(w^*(EP),w\right)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}RE_i\left(w^*(EP),w\right) \tag{9}$$

$$MRR\left(w^*(EP),w\right)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}RR_i\left(w^*(EP),w\right) \tag{10}$$

where: $\mathrm{MSRC}(\boldsymbol{w^*}(\mathrm{EP}),\boldsymbol{w})$, $\mathrm{MRE}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ and $\mathrm{MRR}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ denotes: *mean Spearman rank correlation coefficient*, *average mean relative error* and *average mean relative ratio*, respectively.

In the first experiment, the probability distribution π attributed in Step 4 to the perturbation factor *e* is selected arbitrarily to be the *gamma* or *uniform* distribution. These are two of the distribution types most frequently considered in literature for various implementation purposes (Grzybowski 2016). Usually recommended are such types as: *gamma*, *log-normal*, *truncated normal*, or *uniform*. Apart from these most popular π, one

can find applications of the Couchy, Laplace, or either *triangle* and *beta* probability distributions (see e.g. Dijkstra 2013).

The first simulation scenario also assumes that the perturbation factor e will be drawn from the interval $e\in[0.01; 1.99]$. Noticeably, in each case hereafter, the parameters of different probability distributions applied are set in such a way that the expected value of e in each particular simulation scenario $EV(e)=1$. It seems a very reasonable assumption, because although human judgments are not accurate, they undeniably aim perfect ones.

Furthermore, the number of alternatives and criteria in a single AHP model will be assigned randomly. By 'randomly' – without any other explicit specification – hereafter defined as a process operating under uniform distribution. All simulation scenarios also assume application of the rounding procedure which always operates according to the *geometric* scale described earlier in this paper.

Finally, the first scenario also takes into account the obligatory assumption in conventional AHP applications i.e. the PCM reciprocity condition. In such cases, only judgments from the upper triangle of a given PCM are taken into account and those from the lower triangle are replaced by the inverses of the former.

The outcomes i.e. mean characteristics for 30,000 cases (χ=15 and γ=2000) of the first simulation scenario are presented in Table 2. It may be noticed from Table 2, that the REV can be undeniably classified as the worst PP from the perspective of PRs derived from ranks established on the basis of three different prioritization quality measures i.e. MRE, MSRC, and MRR. The best two PPs from the viewpoint of this classification are LLSM, known also as Geometric Mean Procedure (GM), and LUA. Certainly, the first scenario experiment was designed only to contrast the results presented by Saaty & Hu (1998). It is the intention to establish wider and more fundamental relationships among the presented PPs.

Table 2 – Mean performance measures of arbitrarily selected PPs for 30,000 cases

Scenario Details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean Rank
Geometric Scale	*gamma* distribution	*FR*-PCM	LLSM	0.438438	1	0.682300	2	1.21242	1	1.3(3)
			REV	0.452614	5	**0.668380**	5	1.22051	4	4.6(6)
			LUA	0.447349	2	0.673067	3	1.21792	2	2.3(3)
			SRDM	0.448759	3	0.671380	4	1.21870	3	3.3(3)
			SNCS	0.450734	4	0.692453	1	1.24398	5	3.3(3)
	uniform distribution	*FR*-PCM	LLSM	0.288608	1	0.804860	2	1.12813	1	1.3(3)
			REV	0.302346	4	**0.792580**	5	1.13530	4	4.3(3)
			LUA	0.298401	2	0.795767	3	1.13350	2	2.3(3)
			SRDM	0.299400	3	0.794820	4	1.13400	3	3.3(3)
			SNCS	0.303463	5	0.808333	1	1.15450	5	3.6(6)

Note: *FR*-PCM denotes *forced reciprocity* applied to PCM during simulations

The second simulation scenario was designed to encompass new assumptions not yet taken into account in the literature. First of all, taken into consideration were results obtained not only on the basis of reciprocal PCM, but also the simulation outcomes of nonreciprocal PCM. Secondly, it was decided to implement into simulations new intervals for random errors and apply their new probability distribution. As is known, many

simulation analyses presented in literature assume very non symmetric intervals for a perturbation factor (considering its influence on the particular element of PCM). For example consider the interval for perturbation factor applied in the first simulation scenario i.e. $e \in [0.01;1.99]$. Under this assumption, it becomes apparent that if some entry of PCM is modified *in plus* by the perturbation factor from that particular interval, it is multiplied maximal by the number 1.99, so if the original entry is 3, the modified value will be around 6. However, if some entry of PCM is modified *in minus* by the perturbation factor from that particular interval, it may result that some entry will be multiplied by the number 0.01, so in fact the entry will be divided by 100. Thus, in the situation where the original entry is 9, the modified value will be 0.09, which can be rounded to 1/9 on the Saaty's scale. It may be noticed that this modification practically reverses the preference of DM from e.g. extremely preferred A over B, to extremely preferred B over A (applying the Saaty scale).

It is obvious that this very common assumption is imposed by another very crucial and logical assumption which states that the expected value of e in every particular simulation scenario should equal one i.e. $EV(e)=1$. It is quite easy to fulfill that requirement on the basis of an asymmetric interval for the perturbation factor (from the perspective of its influence on a particular element of PCM). However, it is rather a challenge to have this assumption implemented with a symmetric interval for the perturbation factor. That is why commonly applied simulation scenarios minimize the range for the perturbation factor in order to achieve at least the delusion of symmetry for $e \in [0.5;1.5]$. Nevertheless, that objective has been attained with the present research, yet to be achieved by other researchers. Presently it seems reasonable to apply symmetric intervals to simulations for the perturbation factor because they better reflect true life situations. Thus, different kinds of probability distributions (PDs) were experimented with and it was discovered that Fisher-Snedecor PD possesses the feature that can be useful in the present analysis. It occurs that for $n_1=14$ and $n_2=40$ degrees of freedom for one thousand randomly generated numbers on the basis of this PD, their mean equals 1.03617, so it is very close to unity, and these numbers fluctuate from 0.174526 to 5.57826. So, with these assumptions, we have $e \in [0.174526; 5.57826]$, which gives a very symmetric distribution for the perturbation factor, and $EV(e) \approx 1$. The results of prioritization quality for different selected PPs and assumed prioritization quality measures i.e. MSRC, MRE, and MRR obtained on the basis of described earlier simulation scenario, are presented in Table 3.

As can be noticed, the REV again is not the dominant PP from the perspective of all simulation scenarios under prescribed frameworks (it takes third place in the total classification order). Certainly, apparent differences in the PV estimation quality in relation to the selected PP are noticeable for nonreciprocal PCMs.

Then, the LUA and SRDM or LLSM dominate over the rest of the selected PPs, especially from the perspective of rank correlations which are the crucial issue from the viewpoint of rank preservation phenomena. These issues will be treated in the section entitled '*Breakthroughs and milestones of this research*'.

Table 3 – Mean performance measures of arbitrarily selected five different ranking procedures for various uniformly drawn 100,000 AHP models – 1,000 hypothetic decisional problems perturbed 100 times each (*)

Scenario Details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean Rank
Geometric Scale	*n*, *m*∈{3, 4…, 7}	FRPCM	LLSM	0.123288	4	0.916281	1	1.04646	3	2.6(6)
			REV	0.123030	1	0.915056	5	1.04546	1	2.3(3)
			LUA	0.123044	3	0.915489	2	1.04699	4	3
			SRDM	0.123038	2	0.915476	3	1.04567	2	2.3(3)
			SNCS	0.132926	5	0.915228	4	1.05865	5	4.6(6)
		APCM	LLSM	0.100511	1	0.930242	4	1.02953	4	3
			REV	0.101523	4	0.930164	5	1.02938	3	4
			LUA	0.100658	2	0.930965	2	1.02926	2	**2**
			SRDM	0.101310	3	0.930510	3	1.02925	1	**2.3(3)**
			SNCS	0.108689	5	0.931026	1	1.04315	5	3.6(6)
	n, *m*∈{8, 9…, 12}	FRPCM	LLSM	0.079748	4	0.931396	1	1.03319	4	3
			REV	0.079110	1	0.928266	5	1.03116	1	2.3(3)
			LUA	0.079321	3	0.928817	2	1.03173	3	2.6(6)
			SRDM	0.079286	2	0.928769	4	1.03166	2	2.6(6)
			SNCS	0.086223	5	0.928799	3	1.03935	5	4.3(3)
		APCM	LLSM	0.063936	4	0.943393	3	1.02252	4	3.6(6)
			REV	0.062735	3	0.942399	5	1.02070	1	3
			LUA	0.061757	1	0.944593	1	1.02109	3	**1.6(6)**
			SRDM	0.061852	2	0.944314	2	1.02105	2	**2**
			SNCS	0.068981	5	0.942764	4	1.02879	5	4.6(6)
Saaty's scale	*n*, *m*∈{3, 4…, 7}	FRPCM	LLSM	0.143650	4	0.911381	1	1.06578	4	3
			REV	0.142967	1	0.911151	4	1.06498	1	2
			LUA	0.143069	3	0.911347	2	1.06520	3	2.6(6)
			SRDM	0.143054	2	0.911320	3	1.06517	2	2.3(3)
			SNCS	0.155694	5	0.910735	5	1.07850	5	5
		APCM	LLSM	0.116095	1	0.927455	1	1.04681	3	**1.6(6)**
			REV	0.116994	4	0.926955	4	1.04705	4	4
			LUA	0.116337	2	0.927129	3	1.04657	1	**2**
			SRDM	0.116962	3	0.926532	5	1.04658	2	3.3(3)
			SNCS	0.127154	5	0.927397	2	1.06051	5	4
	n, *m*∈{8, 9…, 12}	FRPCM	LLSM	0.100279	4	0.917231	1	1.04856	4	3
			REV	0.098084	1	0.915833	4	1.04630	1	2
			LUA	0.098648	3	0.916245	2	1.04695	3	2.6(6)
			SRDM	0.098569	2	0.916193	3	1.04687	2	2.3(3)
			SNCS	0.106674	5	0.915633	5	1.05424	5	5
		APCM	LLSM	0.078464	4	0.938192	3	1.03563	4	3.6(6)
			REV	0.077002	3	0.937837	4	1.03422	1	2.6(6)
			LUA	0.076762	1	0.939669	1	1.03469	3	**1.6(6)**
			SRDM	0.076789	2	0.939415	2	1.03464	2	**2**
			SNCS	0.084307	5	0.937796	5	1.04125	5	5
Average Mean Rank			**LLSM**	**2.958**	***REV***	***2.792***	**LUA**	**2.292**	***SRDM***	***2.417***
									SNCS	**4.542**
Order			**4**		**3**		**1**		**2**	**5**

Note: (*) AHP models drawn randomly (uniformly) for assigned set of criteria and alternatives. The scenario assumes application of both: perturbation factor drawn with F-Snedecor probability for n_1=14 and n_2=40

degrees of freedom, and rounding errors associated with a given scale (geometric or Saaty's). It assumes calculation of performance measures either for reciprocal PCMs (FRPCM) or nonreciprocal PCMs (APCM).

Description of the second problem

In the previous two sections of this research, it was determined that the quality of PV estimation depends on the selected PP. This section will focus on the other facet of the problem i.e. how the quality of PV estimation depends on the type of PCM Consistency Measure (PCM-CM) engaged in the prioritization process. The difference between the meaning of consistency of a given PCM and the particular PCM-CM is intentionally stressed at this point. Indeed, there are several PCM-CMs provided in the literature called consistency indices (CIs), nevertheless the scientific meaning of PCM consistency is given by the definition (Definition 3).

The most popular and certainly less intuitive is the PCM-CM proposed by Saaty. He proposed his PCM-CM on the basis of his PP which involves eigenvectors and eigenvalues calculations. Thus, the indication of the fact that for the consistent PCM its $\lambda_{max}= n$, for the purpose of PCM consistency measurement, Saaty proposes his CI be determined by the following formula:

$$CI_{REV} = \frac{\lambda_{\max} - n}{n - 1} \tag{11}$$

where n indicates the number of alternatives within the particular PCM. The significant disadvantage of this PCM-CM is the fact it can operate exclusively with reciprocal PCMs. In the case of nonreciprocal PCMs, this measure is useless (its values are meaningless) which in consequence seriously diminishes the value of the whole Saaty approach (Linares et al. 2014).

However, as mentioned earlier, there are a number of additional PCM-CMs. Some of them, as in the case of CI_{REV}, originate from the PPs devised for the purpose of the PV estimation process. Their distinct feature is the fact that all of them can operate equally efficiently in conditions where reciprocal and nonreciprocal PCMs are accepted. A number of them, selected on the basis of their popularity (but not only) and up-to-date nature (Kazibudzki 2016b) are presented in Table 4.

Noticeably, there are few propositions of PCM-CMs which are not connected with any PP and are devised on the basis of the PCM consistency definition (Definition 3). Koczkodaj's (1993) idea is the first to be considered. His PCM-CM is grounded on his concept of triad consistency. The notion of a triad:

Statement 1: For any three distinguished decision alternatives A_1, A_2, and A_3, there are three meaningful priority ratios i.e. α, β, and χ, which have their different locations in a particular $\boldsymbol{A}(w)=[w_{ij}]_{nxn}$

Definition 4: If $\alpha=w_{ik}$, $\chi=w_{kj}$, $\beta=w_{ij}$ for some different $i\leq n$, $j\leq n$, and $k\leq n$, then the tuple (α, β, χ) is called a *triad.*

Definition 5: If the matrix $\boldsymbol{A}(w)=[w_{ij}]_{nxn}$ is consistent, then $\alpha\chi=\beta$ for all triads.

Table 4 – Formulae for the PCM-CMs related to their PPs

Symbol of the PP	Formula for the PCM-CM
LUA	$CI_{LUA}=\frac{1}{n}\sqrt{\min\sum_{i=1}^{n}\ln^2\left(\sum_{j=1}^{n}\frac{a_{ij}w_j}{nw_i}\right)}$
SRDM	$CI_{SRDM}=\sqrt{\frac{1}{n}\min\sum_{i=1}^{n}\left(\frac{1}{nw_i}\sum_{j=1}^{n}a_{ij}w_j-1\right)^2}$
LLSM	$CI_{LLSM}=\frac{2}{(n-1)(n-2)}\sum_{i<j}\log^2\left(\frac{a_{ij}w_j}{w_i}\right)$

In consequence, either of the equations 1–β/αχ=0 and 1–αχ/β=0 have to be true. Taking the above into consideration, Koczkodaj proposed his measure for triad inconsistency by the following formula:

$$TI(\alpha,\beta,\chi)=\min\left[\left|1-\frac{\beta}{\alpha\chi}\right|,\left|1-\frac{\alpha\chi}{\beta}\right|\right] \tag{12}$$

Following his idea, he then proposed the following CM of any reciprocal PCM:

$$K(TI)=\max[TI(\alpha,\beta,\chi)] \tag{13}$$

where the maximum value of TI(α,β,χ) is taken from the set of all possible triads in the upper triangle of a given PCM.

On the basis of Koczkodaj's idea of triad inconsistency, Grzybowski (2016) presented his PCM consistency measure determined by the following formula:

$$A(TI)=\frac{1}{N}\sum_{i=1}^{N}[TI_i(\alpha,\beta,\chi)] \tag{14}$$

Finally, following the idea, that ln(αχ/β) = minus ln(β/αχ), Kazibudzki (2016a) redefined triad inconsistency and proposed:

– two formulae for its measurement -

$$LTI_1(\alpha,\beta,\chi)=|\ln(\alpha\chi/\beta)| \tag{15}$$

$$LTI_2(\alpha,\beta,\chi)=\ln^2(\alpha\chi/\beta) \tag{16}$$

– and one meaningful formula for PCM-CM -

$$A(LTI_i)=\frac{1}{N}\sum_{j=1}^{N}[LTI_{ij}(\alpha,\beta,\chi)] \tag{17}$$

which can be calculated on the basis of triads from the upper triangle of the given PCM when it is reciprocal, or all triads within the given PCM when it is nonreciprocal.

The second problem study

This section begins with the fundamental question which should encourage all researchers who deal with the problem of PR estimation quality to seek appropriate PCM consistency measurement. The question asks:
Does a growth of the PCM consistency directly lead to the betterment of the priority vector estimation quality?

Apparently, the answer to this question seems to be affirmative. Commonly, this is the reason why one strives to keep the consistency of the PCM at the highest possible level. However, *is it a good idea to use universally recognized PCM-CMs for this purpose?* To answer this question a preliminary analysis of the example provided and examined in the section entitled '*Description of the first problem*' can be initiated.

Thus, the genuine PV is reconsidered, ***w***=[7/20, 1/4, 1/4, 3/20] and ***A***(w) derived from that PV can be presented as follows:

$$\boldsymbol{A}(w)=\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Now considering two PCMs i.e. ***R***(x) and ***A***(x) produced by a hypothetical DM, whose judgments are rounded to Saaty's scale – DM is very trustworthy and is able to express judgments very precisely. In the first scenario, entries of ***A***(w) are rounded to Saaty's scale and the entries are made reciprocal (a principal condition for a PCM in the AHP) producing:

$$\boldsymbol{R}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

In the second scenario only entries of ***A***(w) are rounded to Saaty's scale (nonreciprocal case)producing:

$$\boldsymbol{A}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

It should be noted that ***R***(x) is perfectly consistent and ***A***(x) is not. Tables 5 and 6 present selected values of the PPs related PCM-CMs (that is CI_{REV}, CI_{LUA}, and CI_{LLSM}) for ***R***(x) and ***A***(x) together with PVs derived from ***R***(x) and ***A***(x); Mean Absolute Errors (MAEs) [Formula (18)], among ***w****(PP) and the genuine ***w*** for the case; Spearman Rank Correlation Coefficients (SRCs) among ***w****(PP) and the genuine ***w*** for the case.

$$MAE\left(w^*(PP),w\right)=\frac{1}{n}\sum_{i=1}^{n}\left|w_i-w_i^*(PP)\right| \tag{18}$$

Table 5 – Values of the PCM-CMs for ***R***(x) and proposed characteristics of PVs estimates (*) quality in relation to the genuine PV for the case

PP	Estimates	Performance measures		
		CI(*PP*)	MAE	SRC
REV	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
LUA	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
LLSM	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966

(*) derived from ***R***(x) with application of a particular PP

Table 6 – Values of the PCM-CMs for ***A***(x) and proposed characteristics of PVs estimates (*) quality in relation to the genuine PV for the case

PP	Estimates	Performance measures		
		CI(*PP*)	MAE	SRC
REV	$[0.309401, 0.267949, 0.267949, 0.154701]^T$	– 0.0893164	0.0202995	1
LUA	$[0.306135, 0.268645, 0.268645, 0.156576]^T$	0.0344483	0.0219326	1
LLSM	$[0.314288, 0.264284, 0.264284, 0.157144]^T$	0.0400378	0.0178559	1

(*) derived from ***A***(x) with application of a particular PP

Surprisingly, a very interesting phenomenon can be noted on the basis of information provided in Tables 5 and 6. The nonreciprocal version of the analyzed PCM contains non-zero values for the selected PCM-CMs. In cases similar to this example, the value of Saaty's PCM-CM always becomes negative which makes it inexplicable and in consequence useless under such circumstances (as already mentioned earlier). The other two measures are positive and higher than zero which indicates that the particular PCM is not consistent. On the basis of the same indicators in the case of the reciprocal version of the analyzed PCM, its perfect consistency is apparent because all selected PCM-CMs in this case are equal to zero. However, the estimation precision measures (MAE and SRC) i.e. characteristics of the particular PV estimation quality, indicate something quite opposite. Surprisingly, apparent are smaller values of MAEs and perfect correlation of ranks between estimated and genuine PV for nonreciprocal version of the analyzed PCM. Certainly, this conclusion concerns all analyzed PPs and it is very true in the situation when the particular PCM is apparently less consistent (on the basis of selected exemplary PCM-CMs).

It has been suggested that these discoveries inevitably lead to the conclusion that the time has just come to revise the common yet erroneous approach to the PCM consistency measurement which can be described as … *the lower PCM-CM, the better PR estimation quality*.

Therefore, it becomes apparent that there are actually three significantly different consistency notions: (1) the consistency of PCM stated by Definition 3, and reflected by a value of the specific CM which in its way denotes a deviation of the analyzed PCM from its fully consistent counterpart; (2) the consistency of DM i.e. their reliability from the viewpoint of their expertise, measured by a comparison of DM judgments reflected by the particular PCM with judgments made more or less randomly; and (3) the PCM consistency stated by Definition 3 and reflected by a value of the specific CM which denotes the

particular PCM applicability for PRs derivation in the way that minimizes estimation errors.

The third notion is of particular interest from the perspective of the Multiple Criteria Decision Making (MCDM) quality. The key concept of the issue was first presented by Grzybowski (2016) and enhanced by Kazibudzki (2016a). It was decided to examine the phenomenon described therein and further develop it to improve the quality of MCDM. The simulation framework for this purpose was adopted from Kazibudzki (2016a) as the only way to examine said phenomena through computer simulations. The simulation algorithm **SA|2|** thus comprises of the following phases:

Phase 1 Generate randomly a priority vector $\boldsymbol{w}=[w_1,\ldots, w_n]^T$ of assigned size $[n \times 1]$ and related perfect PCM(w)=$\boldsymbol{K}(w)$

Phase 2 Select randomly an element w_{xy} for $x<y$ of $\boldsymbol{K}(w)$ and replace it with $w_{xy}e_B$ where e_B is a relatively significant error, randomly drawn (*uniform* distribution) from the interval $e_B\in[2;4]$. Errors of that magnitude are basically considered as "significant", see e.g.: Grzybowski (2012), Dijkastra (2013), Lee (2007).

Phase 3 For each other element w_{ij}, $i<j\leq n$ select randomly a value e_{ij} for the relatively small error in accordance with the given probability distribution π (applied in equal proportions as: *gamma*, *log-normal*, *truncated normal*, and *uniform* distribution) and replace the element w_{ij} with the element w_{ij} e_{ij} where e_{ij} is randomly drawn (*uniform* distribution) from the interval $e_{ij}\in[0{,}5;1{,}5]$

Phase 4 Round all values of $w_{ij}\, e_{ij}$ for $i<j$ of $\boldsymbol{K}(w)$ to the nearest value of a considered scale

Phase 5 Replace all elements w_{ij} for $i>j$ of $\boldsymbol{K}(w)$ with $1/w_{ij}$

Phase 6 After all replacements are done, return the value of the examined index as well as the estimate of the vector $\boldsymbol{w}$ denoted as $\boldsymbol{w}^*$(PP) with application of assigned prioritization procedure (PP). Then return the mean absolute error MAE between $\boldsymbol{w}$ and $\boldsymbol{w}^*$(PP). Remember values computed in this phase as one record.

Phase 7 Repeat *Phases* from 2 to 6 N_n times.

Phase 8 Repeat *Phases* from 1 to 7 N_m times.

Phase 9 Return *all* records to one database file.

Once again, all parameters of the applied PDs – *gamma*, *log-normal*, *truncated normal*, and *uniform* – in the above simulation framework are set as previously in such a way that the expected value EV(e_{ij})=1.

The simulation begins from n=4, because simulations for n=3 are not interesting due to direct interrelation of considered PCM consistency measures (Bozóki & Rapcsak 2008, Dijkstra 2013). For the sake of objectivity, the simulation data is gathered in the following way: all values of selected consistency measures are split into 15 separate sets designated by the quantiles Q of order p from 1/15 to 14/15. The 15 intervals are defined as: the first is from 0 to the quantile of order 1/15 i.e. VRCM$_1$=[0, $Q_{1/15}$), where VRCM represents *a Value Range of the Selected PCM Consistency Measure*; the second denotes VRCM$_2$=[$Q_{1/15}$, $Q_{2/15}$), and so on… to the last one which starts from the quantile of order 14/15 and goes on to infinity i.e. VRCM$_{15}$=[$Q_{14/15}$, ∞). The following variables are examined: Mean VRCM$_n$, average MAE within VRCM$_n$ between $\boldsymbol{w}$ and $\boldsymbol{w}^*$(PP), MAE quantiles of the following orders, 0.05, 0.1, 0.5, 0.9, 0.95, and relations between all of them. In the preliminary simulation program, it was decided that PP=LLSM. The application of the rounding procedure was also assumed which in this preliminary research operates according to Saaty's scale.

Lastly, the scenario takes into account the compulsory assumption in conventional AHP applications i.e. the PCM reciprocity condition. The results are based on N_n=20, and N_m=500, i.e. 10,000 cases.

In the case of a good PCM-CM, one could assume that MAE quantiles of any order should monotonically grow concurrently with the growth of the selected PCM-CM e.g. VRCM index. The same relation should occur for Mean VRCM_n and average MAE for VRCM_n. The results of the proposed simulation framework, or any other similar simulation scenario which would contradict such a pertinent relationship would unequivocally lead to the conclusion that the examined PCM-CM does not serve its purpose.

An examination from that point of view is in order, the performance of six PCM-CMs selected from among very common or recently proposed (Fig.2): Saaty CI_{REV} – (Plot A), together with Crawford & Williams CI_{LLSM} – (Plot B), and Koczkodaj $K(TI)$ – (Plot C), together with Grzybowski $A(TI)$ – (Plot D), as well as Kazibudzki $A(LTI_1)$ and CI_{LUA} – (Plots E-F).

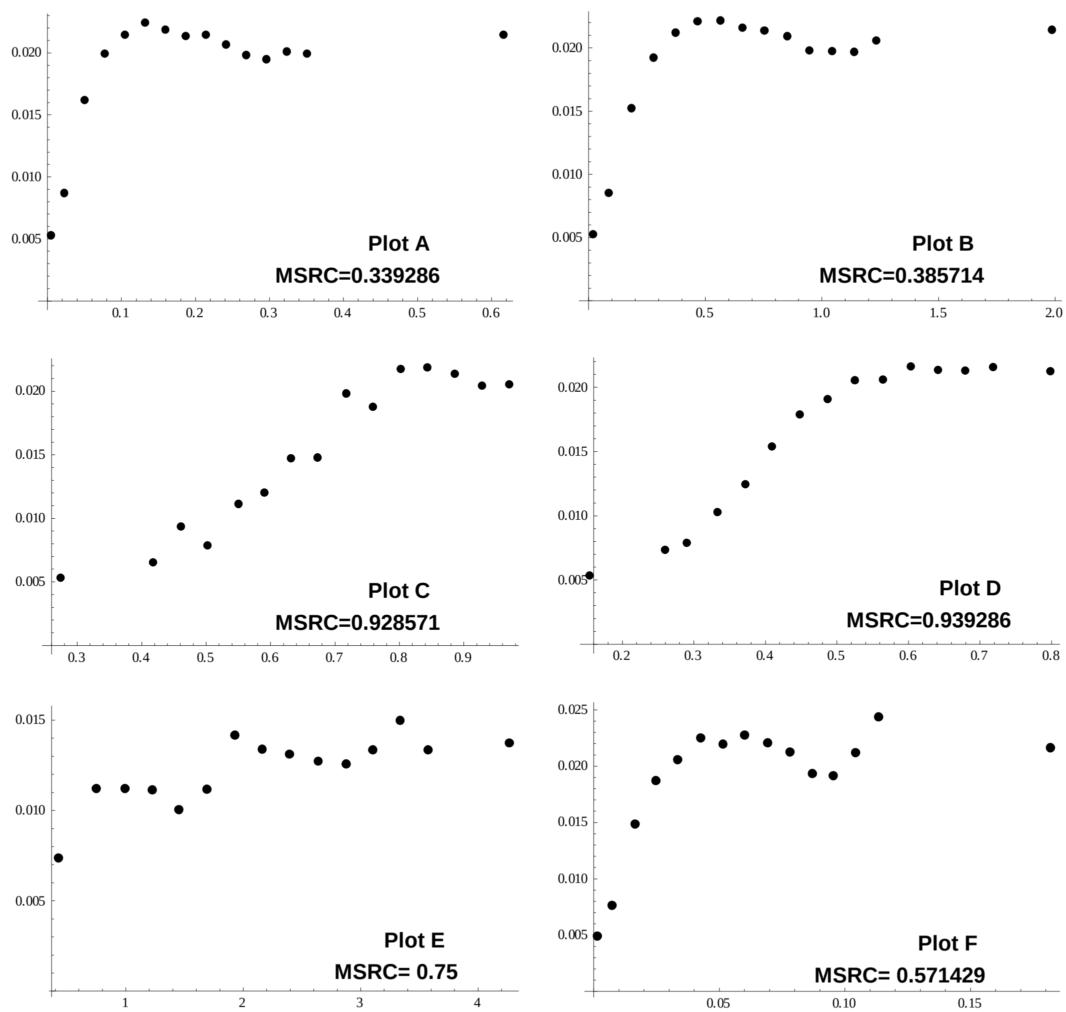


Fig. 2 – Performance of selected PCM-CMs – The plots present the relation between a mean value of a given PCM-CM within a given interval (VRCM_n) and quantiles of order 0.05 of MAEs distribution concerning estimated and genuine PV for the case. The results are generated with application of LLSM as the PP. Plots are based on 10,000 random reciprocal PCMs for n=4. The relation strength MSRC denotes Mean Spearman Rank Correlation Coefficient between analyzed variables.

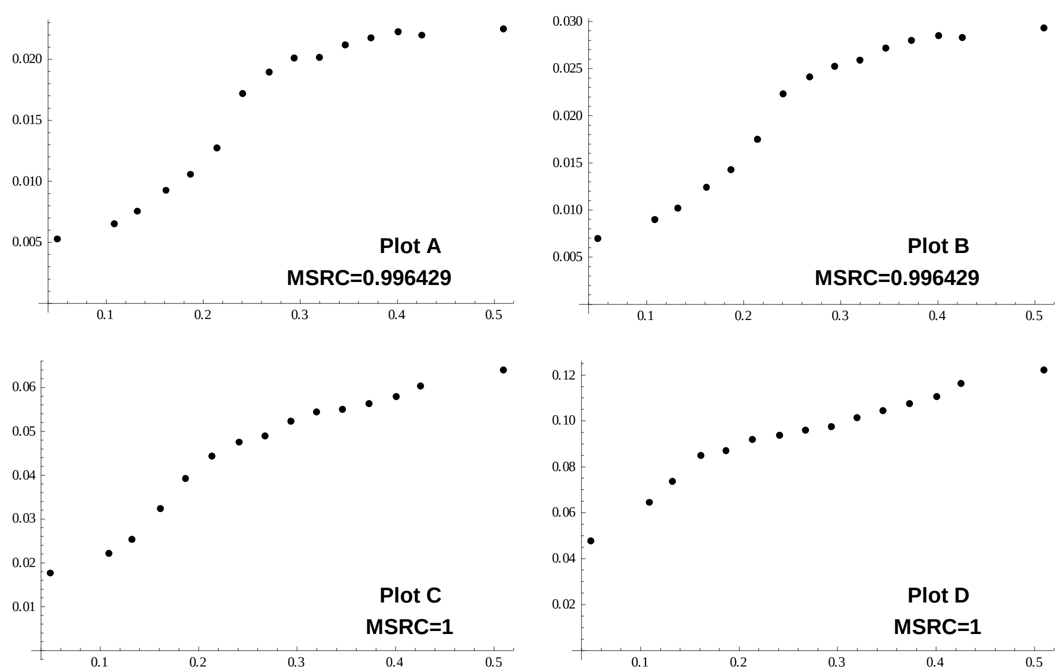
Noticeably, when the quality of PV in MCDM process of AHP is taken into consideration, the presented relations indicate that the analyzed performance of selected PCM-CMs vary more or less from the target. Indeed, the relations indicate that most of the analyzed indices may even misinform DMs about their judgment applicability for the construct of the PV which best converge with the ideal one i.e. obtained from a fully consistent PCM. As seen similarly in the example provided earlier in this paper (Tab. 5 and 6), taking the particular index as the measure of PCM consistency, one can expect both i.e. the betterment of PRs estimation quality (increase of the estimation accuracy) together with the increase of the particular CI (decrease of PCM consistency); and inversely, the deterioration of PRs estimation quality (decrease of the estimation accuracy) together with the descent of the particular CI (improvement of PCM consistency).

Noticeably, the analyzed PCM-CMs are not selected without a reason as they are commonly applied and/or suggested as good solutions in the process of PV estimation on the basis of inconsistent PCMs (for discussion see also Grzybowski 2016). This was the motivation to search for a PCM-CM which relation to PV estimation errors, reflected by SRC, would be very close or equal to 1 (the most desirable situation).

Thus, a seminal solution is proposed in this matter. On the basis of triad inconsistency measure $LTI_2(\alpha,\beta,\chi)=\ln^2(\alpha\chi/\beta)$ introduced by Kazibudzki (2016a), the following PCM-CM is submitted:

$$CM(LTI_2)=\frac{MEAN[LTI_2(\alpha,\beta,\chi)]}{1+MAX[LTI_2(\alpha,\beta,\chi)]} \tag{19}$$

The proposed PCM-CM is denoted as *the Triads Squared Logarithm Corrected Mean* and an examination of its performance on the basis of simulation algorithm **SA|2|** proposed earlier in this paper was carried out.



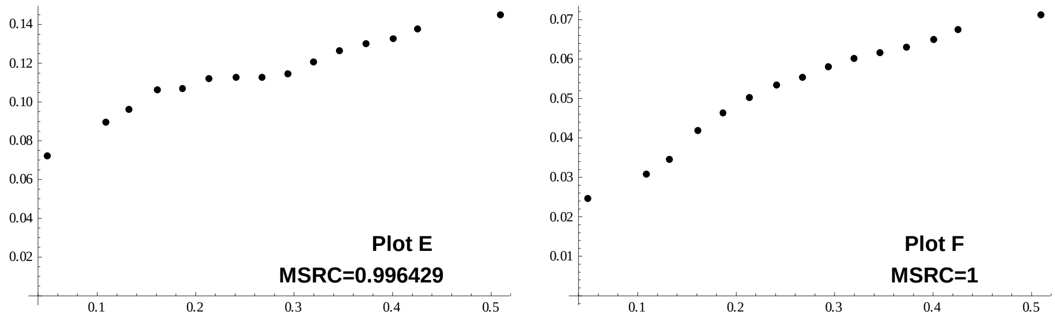


Fig. 3 – Performance of the new PCM-CM: CM(LTI_2) The plots present a relation between a mean value of CM(LTI_2) within a given interval ($VRCM_n$) and quantiles of order 0.05, 0.1, 0.5, 0.9, 0.95 of MAEs distribution as well their average values for estimated and genuine PVs. The results are generated with application of LLSM as the PP. Plots are based on 10,000 random reciprocal PCMs for *n*=4.

As can be noticed, the proposed CM(LTI_2) significantly outperforms the other PCM-CMs analyzed earlier in this paper. It is undeniably a seminal revelation that unquestionably opens a new chapter in MCDM on the basis of AHP – especially because CM(LTI_2) is suitable for both reciprocal and nonreciprocal PCM.

Breakthroughs and milestones of the research

As was said in 1990 by the creator of AHP: … *there is a well-known principle in mathematics that is widely practiced, but seldom enunciated with sufficient forcefulness to impress its importance. A necessary condition that a procedure for solving a problem be a good one is that if it produces desired results, and we perturb the variables of the problem in some small sense, it gives us results that are 'close' to the original ones. (…) An extension of this philosophy in problems where order relations between the variables are important is that on small perturbations of the variables, the procedure produces close, order preserving results* (Saaty 1990, p. 18).

The quality of PR estimation in relation to the selected PP

With said notion in mind, an effort was undertaken to verify the statement of followers of the REV, boldly spreading the idea that so long as inconsistency is accepted, the REV is the paramount theoretical basis for deriving a scale and no other concepts qualify.

It is a fact that in order to support some theory, one must verify it through many experiments to validate its reliability. On the other hand one needs only one example showing it does not work in order to abolish its credibility. Thus, numerous examples were provided indicating that the REV concedes with other devised PP to determine ranking of alternatives. However, although data obtained during simulation experiments are unequivocal, they support the above notion only generally. That is why scientific verification of their meaning is carried out on the basis of the statistical hypothesis testing theory (SHTT).

If $MSRC_{PP}$ and $MSRC_{REV}$ respectively are denoted as mean SRC of selected PP and mean SRC of the REV, their difference significance can be tested using "t" statistics defined by the following formula:

$$t = R\sqrt{\frac{n-2}{1-R^2}} \tag{20}$$

where R is the difference between particular MSRCs.

This statistic has a distribution of *t–student* with n minus 2 degrees of freedom df, where n equals the size of the sample. The following hypothesis was tested:

$$H_0\text{: } \mathrm{MSRC}_{PP} - \mathrm{MSRC}_{REV} = 0$$

versus

$$H_1\text{: } \mathrm{MSRC}_{PP} - \mathrm{MSRC}_{REV} > 0$$

In order to conform to the example presented by Saaty & Hu (1998), the data gathered in Table 2 was considered. The simulation framework of that case is df=29,998. Thus, for assumed levels of significance α=0.01, α=0.02 or α=0.03, the critical values of *t–student* statistics equal consecutively $t_{0.01}$ = 2.326472, $t_{0.02}$ = 2.053838, or $t_{0.03}$ = 1.880865.

In the situation when a level of tested *t–student* statistics is higher than its critical value for the assumed level of significance, the hypothesis H_0, must be rejected in favor of alternative hypothesis H_1. In the opposite situation, there are no foundations to reject H_0. The selected statistics and their values for the problem evaluation are presented in Table 7.

Clearly, the results of the simulation scenario, designed in accordance with the framework presented in Saaty & Hu (1998), indicate two PPs which on the basis of SHTT always perform better than the REV, regardless of the preselected PD. It should be emphasized that the performance of selected PPs is examined here from the perspective of rank preservation phenomena which is reflected in our research by the MSRC between genuine and perturbed PV. It should be evident that the above conclusions, unlike any other before, are the effect of sound statistical reasoning (rigorous significance level) based on the seminal approach toward AHP methodology evaluation grounded on precisely planned and performed simulation study.

Table 7 – MSRC values and principal statistics for the performance test of the REV versus other selected PPs

Scenario details			Procedure	MSRC	*R*	R^2	*t–value*	*α–level**
Geometric Scale	*gamma* distribution	*FR*-PCM	LLSM	0.682300	0.01392	0.00019	2.411167969	0.01
			REV	0.668380	><	><	><	><
			LUA	0.673067	0.00469	0.00002	0.811794069	–
			SRDM	0.671380	0.00300	0.00001	0.519600260	–
			SNCS	0.692453	0.02407	0.00058	4.170635557	0.01
	uniform distribution	*FR*-PCM	LLSM	0.804860	0.01228	0.00015	2.127047876	0.02
			REV	0.792580	><	><	><	><
			LUA	0.795767	0.00319	0.00001	0.551988995	–
			SRDM	0.794820	0.00224	0.00001	0.387967421	–
			SNCS	0.808333	0.01575	0.00025	2.728747286	0.01

Note: (*) the closest significance level providing the ground to reject a tested hypothesis

In order to develop the concept further it was decided to expand the simulation program. The results of this endeavor are presented in Table 3. They should be considered as surprising, especially when one realizes that the PP embedded in the AHP merely takes third place in the overall performance ranking. The ranking takes into account not only

MSRC, but MRE and MRR also, the latter never taken into consideration in previous simulation research. The MRR will now be examined to expand its concept and highlight its novelty.

Let's consider a vector $\boldsymbol{k}$ of values to be estimated, $\boldsymbol{k}$=[3, 3, 3, 3], and three of its estimates, $\boldsymbol{k_1}$=[2, 4, 2, 4], $\boldsymbol{k_2}$=[2, 2, 2, 2], $\boldsymbol{k_3}$=[4, 4, 4, 4]. It may be noted that the MREs of all the estimates (given by formula (6)) are the same and equal 1/3. However, MRRs of the estimates (given by formula (7)) are not the same and equal respectively, $\mathrm{MRR}_1(k,k_1)=1$, $\mathrm{MRR}_2(k,k_2)=2/3$, $\mathrm{MRR}_3(k,k_3)=4/3$. Obviously, the goal of estimation is both i.e. to minimize MREs and maintain the MRRs close to unity. This prerequisite is of great importance when one deals with PVs i.e. vectors normalized to unity, as in the case of AHP. Certainly, one can encounter the following three estimation scenarios.

Scenario 1 — Consider a vector $\boldsymbol{w}$ of genuine PRs trying to estimate $\boldsymbol{w}$=[0.25, 0.25, 0.25, 0.25], and its estimate $\boldsymbol{w_1}$=[0.01, 0.49, 0.05, 0.45]. This scenario gives a rather high MRE of 0.88, which indicates the mean 88% volatility of estimated PRs in relation to their primary values, and MRR=1.

Scenarios 2–3 — Consider a vector $\boldsymbol{p}$ of genuine PRs trying to estimate $\boldsymbol{p}$=[0.1, 0.2, 0.3, 0.4], and its two estimates $\boldsymbol{p_1}$=[0.15, 0.3, 0.25, 0.3], and $\boldsymbol{p_2}$=[0.05, 0.1, 0.35, 0.5]. This situation entails a moderate MRE of 0.35425 for both estimates, and two MRRs i.e. $\mathrm{MRR}_1(p,p_1)=1.145$, and $\mathrm{RR}_2(p,p_2)=0.85425$, for the second and third scenario respectively.

Obviously, during the PRs estimation process, it is desirable to avoid situations exemplified by the first and second scenario. Noticeably, they both have something in common. Apart from estimation discrepancies they lead to rank reversal of the initial priorities (emphasis added).

Turning back to Table 3, having in mind the imposed simulation scenario, F-Snedecor PD mean value of a perturbation factor EV(e)=1.03617, we can conclude as follows:

1) the applied measures (MRE, MSRC, MRR) reflecting the quality of PR estimation process within the simulation framework are always better for nonreciprocal PCMs in relation to their reciprocal equivalents;
2) the applied measures of the quality of PR estimation within the simulation framework indicate better estimation results for a relatively higher number of alternatives;
3) both MRE and MRR values indicate that the quality of PR estimation within the simulation framework is better when geometric scale is implemented instead of Saaty's scale for preferences expression of DMs (MRR is then more often less than 1.03617 which indicates less risk of rank reversal);
4) and last but not least, the REV procedure IS NOT a dominating procedure during PR estimation in the simulated framework of the AHP.

The quality of PR estimation in relation to the CM of the PCM

Thus far the alterability of prioritization quality in consequence of the application of preselected PP, preference scale and reciprocal or nonreciprocal PCM in the AHP has been dealt with. This chapter endeavors to focus and conclude the findings concerning the

alterability of prioritization quality in relation to the applied method of the PCM consistency measurement.

Figure 2 demonstrates the basic relation between the distribution of estimation MAEs and values of selected PCM-CMs when LLSM is applied as the PP. The objective was to realize that those measures are not a good indicator of the quality of PR estimation, although the quality of PR estimation should be the core of PCM consistency measurement. Thus, a seminal solution for this problem was introduced i.e. the novel PCM-CM - CM(LTI_2) and depicted its performance in relation to the quality of PR estimation (Fig. 3). As noted, its performance is much better than the PCM-CMs presented earlier (Fig. 2), independently of the MAEs distribution characteristics applied. Below (Tables 8–9), detailed characteristic data is presented for CM(LTI_2) for LLSM and LUA as the PPs, and Saaty's scale as the preferred applied scale.

Table 8 – Performance of the CM(LTI_2) index Statistical characteristics of the MAEs distribution in relation to various $VRCM_i$ for i=1,…,15 of CM(LTI_2) values. The results were generated for n=4 on the basis of **SA|2|** as the simulation algorithm and are based on 10,000 perturbed random reciprocal PCMs. The scenario assumed LLSM as the PP.

i	$VRCM_i$ for CM(LTI_2)	Mean CM(LTI_2) in $VRCM_i$	p–quantiles of MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LLSM)					Average MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0.00, 0.0934)	0.049049	0.0052533	0.0070072	0.0177090	0.0478280	0.0722714	0.0246076
2	[0.0934, 0.12)	0.108368	0.0065427	0.0089647	0.0221881	0.0646356	0.0896474	0.0307832
3	[0.120, 0.147)	0.131977	0.0075604	0.0101952	0.0254009	0.0735568	0.0962640	0.0346824
4	[0.147, 0.173)	0.161289	0.0092812	0.0124076	0.0323969	0.0848569	0.1062240	0.0418424
5	[0.173, 0.200)	0.186567	0.0106050	0.0142825	0.0392350	0.0872419	0.1070400	0.0463689
6	[0.200, 0.227)	0.213651	0.0127312	0.0174795	0.0443425	0.0921171	0.1121160	0.0503101
7	[0.227, 0.253)	0.240868	0.0171655	0.0223103	0.0474780	0.0939184	0.1129280	0.0534051
8	[0.253, 0.280)	0.267645	0.0189530	0.0241065	0.0489027	0.0959089	0.1126270	0.0554222
9	[0.280, 0.307)	0.293803	0.0200809	0.0252443	0.0523480	0.0975035	0.1147230	0.0580895
10	[0.307, 0.333)	0.319702	0.0201740	0.0259357	0.0544712	0.1014610	0.1208420	0.0601639
11	[0.333, 0.360)	0.345876	0.0211796	0.0271488	0.0550490	0.1043660	0.1267140	0.0615576
12	[0.360, 0.387)	0.372744	0.0217402	0.0279791	0.0563253	0.1076280	0.1302670	0.0630527
13	[0.387, 0.413)	0.400500	0.0222736	0.0284786	0.0579657	0.1105020	0.1326590	0.0649738
14	[0.413, 0.440)	0.425325	0.0219914	0.0282637	0.0603546	0.1163910	0.1378310	0.0674297
15	[0.440, ∞)	0.509413	0.0224786	0.0293611	0.0639097	0.1220180	0.1448450	0.0711265

Noted, all statistical characteristics of the MAEs distribution in relation to various $VRCM_i$ for i=1,…,15 of CM(LTI_2) values monotonically grow in both cases. This phenomenon ascertains that the proposed measure of the quality of PR estimation in relation to PCM-CM outperforms other commonly known or recently introduced means of PCM consistency control which were examined during this research. The paramount position of the CM(LTI_2) is additionally strengthened by the fact that its performance improves significantly for higher numbers of alternatives without regard to which PP is employed.

It should be noted that all characteristics presented herein are of great importance in MCDM, because one has to consider the potential of rejecting a "good" PCM, and vice versa i.e. the possibility of acceptance a "bad" PCM, as in the classic SHTT. However, for first time in the course of the AHP development history, the possibility of selecting the level of trustworthiness and basing decisions on statistical facts has been demonstrated. For

instance, considering some hypothetic PCM for n=4, with its CM(LTI_2)≈0.319702 (Tab. 8), one can expect with 95% certainty that MAE should not exceed the value of 0.1208420.

Table 9 – Performance of the CM(LTI_2) index Statistical characteristics of the MAEs distribution in relation to various VRCM$_i$ for i=1,…,15 of CM(LTI_2) values. The results were generated for n=4 on the basis of **SA|2|** as the simulation algorithm and are based on 10,000 perturbed random reciprocal PCMs. The scenario assumed LUA as the PP.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LUA)					Average MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0.00, 0.0921)	0.0483805	0.0051862	0.0070132	0.0176693	0.0485092	0.0721248	0.0246818
2	[0.0921, 0.119)	0.107336	0.0065804	0.0087362	0.0223436	0.0668610	0.0901757	0.0310714
3	[0.119, 0.145)	0.130827	0.00728515	0.0096983	0.0248230	0.0756282	0.0986153	0.0345387
4	[0.145, 0.172)	0.159831	0.0097014	0.0126492	0.0318836	0.0839675	0.1048160	0.0417584
5	[0.172, 0.199)	0.185763	0.0108996	0.0147705	0.0390121	0.0867685	0.1087370	0.0464742
6	[0.199, 0.226)	0.212789	0.0127518	0.0171452	0.0444749	0.0906489	0.1110220	0.0502253
7	[0.226, 0.252)	0.239711	0.0168641	0.0221950	0.0483727	0.0943307	0.1121290	0.0538373
8	[0.252, 0.279)	0.266664	0.0191223	0.0243966	0.0499312	0.0963741	0.1128810	0.0561933
9	[0.279, 0.306)	0.292923	0.0210745	0.0265733	0.0536876	0.0971178	0.1136750	0.0590709
10	[0.306, 0.332)	0.318738	0.0222280	0.0280330	0.0570706	0.1018000	0.1224680	0.0622836
11	[0.332, 0.359)	0.344798	0.0229873	0.0290093	0.0582741	0.1054570	0.1267530	0.0640174
12	[0.359, 0.386)	0.371865	0.0237677	0.0299580	0.0592460	0.1080910	0.1309080	0.0652984
13	[0.386, 0.412)	0.399489	0.0243569	0.0309199	0.0612529	0.1118710	0.1350460	0.0678424
14	[0.412, 0.439)	0.424271	0.0245079	0.0311770	0.0630208	0.1197740	0.1443030	0.0707793
15	[0.439, ∞)	0.507848	0.0240355	0.0310822	0.0660800	0.1264270	0.1500270	0.0737878

At the same time, one can expect with 95% certainty that it will be higher than 0.0201740. Whether one decide to accept such a PCM or reject it, obviously depends on the quality requirements of PR estimation and the attitude regarding these errors. Indeed, the outcome of the research finally creates the potential for true consistency control in an unprecedented way i.e. directly related to the PR estimation quality.

Consider the following PV as $\boldsymbol{w}$=[0.345, 0.335, 0.32] of DM preferences for alternatives, A_1, A_2, A_3, respectively. Taking into consideration earlier assumed level of CM(LTI_2)≈0.319702, the order of alternatives ranks i.e. A_1=1, A_2=2, A_3=3, can be very deceptive, and is rather meaningless. Indeed, in such a situation one can expect with 95% certainty that MAE>0.0201740 which makes one aware that the true rank order of examined preferences may look otherwise, due to estimation errors related to DM inconsistency e.g. $\boldsymbol{w}$=[(0.345–0.04), (0.335+0.01), (0.32+0.03)]=[0.305, 0.345, 0.35], which designates a different order for alternatives ranks, A_1=3, A_2=2, A_3=1.

On the other hand, consider PV as $\boldsymbol{w}$=[0.6, 0.35, 0.05] of DM preferences for alternatives: A_1, A_2, A_3, consecutively, as previously. Again, assuming CM(LTI_2)≈0.319702, it can be anticipated with 95% certainty that MAE<0.1208420 which insures confidence in the order of alternatives ranks.

In order to conserve the length of the paper, but at the same time enable similar analyses concerning different numbers of alternatives the exemplary generalized (results are averaged for geometric scale and Saaty's scale applied fifty-fifty) characteristics of CM(LTI_2) performance for n>4, and for selected PP in appendices to this article are provided (Tables: A1–A2).

Concluding, this simulation framework a performance of different PCM-CMs in relation to implementation of the most popular PPs, preference scales, and number of alternatives were compared. The research findings can be stated as follows:

1) it is possible to significantly improve the quality of PR estimation when CM(LTI_2) is applied as the PCM-CM;
2) LLSM and LUA as the PP, differ insignificantly from the perspective of CM(LTI_2) performance, the same concerns other examined PP;
3) when the number of alternatives grows, the performance of examined PCM-CMs improves.

Conclusions and further research areas

The objective of the article was to generate answers to the following questions: *Is the REV as the PP necessary and sufficient for the AHP? Is the reciprocity of PCMs a reasonable condition leading to the betterment of the PRs estimation quality? Are PCM-CMs, commonly applied and embedded in the AHP, really conducive to the improvement of the PRs estimation quality?*

The thorough and seminal investigation which significantly upgrades the AHP methodology provides the following answers to these questions:

1) the REV as the PP is not necessary and sufficient for the AHP. Moreover, the research reveals two PP which outperform the REV;
2) the reciprocity of PCM in the AHP is the artificial condition and directly leads to deterioration of the PR estimation quality.
3) the commonly applied PCM-CMs embedded in the AHP, mislead and in consequence often directly lead to deterioration of the PR estimation quality.

Proposed: resign from known PCM-CMs embedded in the AHP in favor of CM(LTI_2) that can operate both types of PCM i.e. reciprocal and nonreciprocal, withhold the PCM reciprocity requirement from the AHP and consider the replacement of the REV as the PP within the AHP in favor of LUA or LLSM.

Certainly, there is a need for further research in the field. Firstly, one should examine the performance of CM(LTI_2) when nonreciprocal PCM are applied. Secondly, one may study its performance from the perspective of relative estimation errors, and last but not least, one could evaluate its performance from the perspective of the entire hierarchy as opposed to a single PCM.

To recapitulate; in conjunction with other contemporary and seminal research papers e.g. Grzybowski (2016); Kazibudzki (2016a, 2016b); García-Melón et al. (2016); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); Aguarón, Escobar & Moreno-Jiménez (2014); Lin, Kou & Ergu (2013); Brunelli, Canal & Fedrizzi (2013), the results of this scientific research enriches the state of knowledge about the true value of the AHP which is widely recognized as an applicable MCDM support system. Hopefully, the results of this freshly finished authentic examination will improve the quality of human's prospective choices.

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Appendices

Table A1 – Performance of CM(LTI_2) index under the action of LLSM as the PP. Statistical characteristics of the MAEs distribution in relation to various levels of CM(LTI_2) within a given VRCM$_i$ for i=1,…,15. The results are based on 10,000 perturbed random reciprocal PCMs (geometric and Saaty's scales applied fifty-fifty), and were generated on the basis of **SA|2|** as the simulation algorithm. The table contains results for $n\in\{5, 6, 7, 8, 9\}$, presented consecutively.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0899)	0.057912	0.0039186	0.0049954	0.0109799	0.0221753	0.0274887	0.0127898
2	[0.0899, 0.107)	0.099124	0.0056158	0.0073876	0.0157136	0.0324243	0.0398139	0.0183201
3	[0.107, 0.124)	0.116088	0.0063140	0.0079525	0.0184299	0.0389673	0.0490687	0.0214159
4	[0.124, 0.142)	0.133907	0.0075132	0.0102233	0.0230429	0.0443459	0.0539668	0.0258898
5	[0.142, 0.159)	0.151127	0.0099921	0.0129535	0.0261046	0.0486044	0.0581258	0.0290851
6	[0.159, 0.176)	0.167911	0.0113191	0.0142543	0.0289546	0.0558904	0.0682777	0.0328936
7	[0.176, 0.193)	0.184671	0.0125612	0.0158052	0.0320054	0.0594491	0.0730399	0.0357402
8	[0.193, 0.211)	0.201896	0.0136853	0.0171375	0.0339101	0.0640703	0.0789391	0.0380755
9	[0.211, 0.228)	0.219329	0.0142803	0.0178080	0.0361548	0.0711273	0.0839402	0.0408705
10	[0.228, 0.245)	0.236371	0.0150518	0.0185369	0.0380656	0.0762136	0.0919801	0.0435024
11	[0.245, 0.262)	0.253302	0.0161087	0.0208189	0.0405464	0.0789105	0.0929572	0.0462684
12	[0.262, 0.280)	0.270523	0.0160427	0.0205586	0.0431223	0.0821647	0.0965329	0.0482168
13	[0.280, 0.297)	0.288211	0.0165698	0.0209757	0.0457022	0.0865715	0.100490	0.0504072
14	[0.297, 0.314)	0.305099	0.0177870	0.0226112	0.0455671	0.0859316	0.100544	0.0507868
15	[0.314, ∞)	0.357080	0.0186614	0.0241816	0.0493007	0.0932224	0.107664	0.0547348
I	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0901)	0.0618775	0.0036042	0.0044511	0.0090509	0.0175099	0.0212066	0.0102634
2	[0.0901, 0.102)	0.096694	0.00584585	0.0072472	0.0147877	0.0255157	0.0297767	0.0158427
3	[0.102, 0.115)	0.109186	0.0071783	0.0088408	0.0167774	0.0304119	0.0360603	0.0186253
4	[0.115, 0.127)	0.121228	0.00831565	0.0102091	0.0192100	0.0349865	0.0420209	0.0214601
5	[0.127, 0.139)	0.133028	0.0088771	0.0109435	0.0208206	0.0393504	0.0481357	0.0236802
6	[0.139, 0.151)	0.144977	0.0097898	0.0118208	0.0225534	0.0439163	0.0538868	0.0259512
7	[0.151, 0.163)	0.156874	0.0101678	0.0126009	0.0248914	0.0500528	0.0613696	0.0288113
8	[0.163, 0.176)	0.169306	0.0113233	0.0138144	0.0274455	0.0552421	0.0656847	0.0317599
9	[0.176, 0.188)	0.181783	0.0120341	0.0147276	0.0297646	0.0587297	0.0700824	0.0339487
10	[0.188, 0.200)	0.193745	0.0124796	0.0157621	0.0317564	0.0613300	0.0720410	0.0356610
11	[0.200, 0.212)	0.205758	0.0137977	0.0167981	0.0329443	0.0622977	0.0721443	0.0368687
12	[0.212, 0.225)	0.218204	0.0140878	0.0175574	0.0347152	0.0652521	0.0774105	0.0386492
13	[0.225, 0.237)	0.230723	0.0140705	0.0177333	0.0369638	0.0672822	0.0764555	0.0402684

14	[0.237, 0.249)	0.242818	0.0146810	0.0186397	0.0381558	0.0692375	0.0786225	0.0413928
15	[0.249, ∞)	0.279499	0.0168309	0.0207854	0.0401272	0.0721349	0.0829652	0.0439267
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.07975)	0.061626	0.00329141	0.0040063	0.0079292	0.0153184	0.017980	0.0089902
2	[0.07975, 0.09)	0.085354	0.00558449	0.0066781	0.0124836	0.0217916	0.0254877	0.0136394
3	[0.09, 0.10)	0.095128	0.00622084	0.0074651	0.0136346	0.0241580	0.0288301	0.0151410
4	[0.10, 0.11)	0.105046	0.00677146	0.0081844	0.0150571	0.0277432	0.0343089	0.0170348
5	[0.11, 0.12)	0.114884	0.00728075	0.0089529	0.0164708	0.0329745	0.0408782	0.0192642
6	[0.12, 0.13)	0.124902	0.00792417	0.0097471	0.0185168	0.0378364	0.0464765	0.0217170
7	[0.13, 0.14)	0.134949	0.00851189	0.0104389	0.0202075	0.0415434	0.0507830	0.0236614
8	[0.14, 0.15)	0.144883	0.00952136	0.0115606	0.0224145	0.0446314	0.0531641	0.0257116
9	[0.15, 0.161)	0.155416	0.0101888	0.0121602	0.0241178	0.0465694	0.0553538	0.0275101
10	[0.161, 0.171)	0.165845	0.0110535	0.0132394	0.0261677	0.0499157	0.0583309	0.0293786
11	[0.171, 0.181)	0.175874	0.0116123	0.0139639	0.0273006	0.0515428	0.0596329	0.0304575
12	[0.181, 0.191)	0.185981	0.0121824	0.0150547	0.0293308	0.0532065	0.0613544	0.0320030
13	[0.191, 0.201)	0.195819	0.0122294	0.0152015	0.0299135	0.0553010	0.0642011	0.0330142
14	[0.201, 0.211)	0.205937	0.0132402	0.0164008	0.0321805	0.0552598	0.0636846	0.0343310
15	[0.211, ∞)	0.235348	0.0147413	0.0179580	0.0321805	0.0586515	0.0682411	0.0363445
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.06861)	0.056493	0.0029930	0.0036359	0.0071723	0.0129253	0.0151100	0.0079193
2	[0.06861, 0.078)	0.073616	0.0047668	0.0056647	0.0098201	0.0165545	0.0197820	0.0107659
3	[0.078, 0.087)	0.082558	0.0051148	0.00615425	0.0108293	0.0189720	0.0232764	0.0121106
4	[0.087, 0.095)	0.090957	0.0054815	0.0067644	0.0120701	0.0230822	0.0289636	0.0139455
5	[0.095, 0.104)	0.0995085	0.0062208	0.0074045	0.0134360	0.0267488	0.0338094	0.0157422
6	[0.104, 0.113)	0.108507	0.0065308	0.0079148	0.0148310	0.0307300	0.0379708	0.0175495
7	[0.113, 0.122)	0.117503	0.0073636	0.0087983	0.0166204	0.0342287	0.0402093	0.0192815
8	[0.122, 0.131)	0.126447	0.0077367	0.00920785	0.0182781	0.0366835	0.0432579	0.0209778
9	[0.131, 0.140)	0.135467	0.0081883	0.0099817	0.0200944	0.0391024	0.0463982	0.0227669
10	[0.140, 0.149)	0.144406	0.00893715	0.0109052	0.0215995	0.0404294	0.0465999	0.0240918
11	[0.149, 0.158)	0.153395	0.0096365	0.0118788	0.0228543	0.0420224	0.0488816	0.0252208
12	[0.158, 0.167)	0.162379	0.0105213	0.0128739	0.0250637	0.0441591	0.0509963	0.0270496
13	[0.167, 0.176)	0.171319	0.0109917	0.0133182	0.0253654	0.0446033	0.0525163	0.0275815
14	[0.176, 0.185)	0.180246	0.0120041	0.0144395	0.0266159	0.0464516	0.0529339	0.0289197
15	[0.185, ∞)	0.205854	0.0127740	0.0155662	0.0283804	0.0479564	0.0549310	0.0304352
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.059795)	0.051197	0.0026372	0.0031677	0.0061278	0.0107141	0.0122502	0.0066588
2	[0.05979, 0.068)	0.064092	0.0040166	0.0047872	0.0079722	0.0133870	0.0158901	0.0087678
3	[0.068, 0.076)	0.072019	0.0044127	0.0052033	0.0089180	0.0154595	0.0189767	0.0099818
4	[0.076, 0.085)	0.080495	0.0046625	0.0055923	0.0098746	0.0183837	0.0234965	0.0114040
5	[0.085, 0.093)	0.089047	0.0052813	0.0062378	0.0110158	0.0221043	0.0279174	0.0129826
6	[0.093, 0.101)	0.097017	0.0056575	0.0067669	0.0124636	0.0261396	0.0326188	0.0147615
7	[0.101, 0.109)	0.105051	0.0061505	0.0074036	0.0138920	0.0290254	0.0358010	0.0164984
8	[0.109, 0.118)	0.113488	0.0066692	0.0079686	0.0153474	0.0308319	0.0365922	0.0177312
9	[0.118, 0.126)	0.122009	0.0073133	0.0087907	0.0171076	0.0330852	0.0388189	0.0193438
10	[0.126, 0.134)	0.129857	0.0076181	0.0092912	0.0186416	0.0343317	0.0401595	0.0204982
11	[0.134, 0.142)	0.137970	0.0083801	0.0102174	0.0199779	0.0355818	0.0416818	0.0216939
12	[0.142, 0.151)	0.146298	0.0091112	0.0107807	0.0212040	0.0376109	0.0430078	0.0229256
13	[0.151, 0.159)	0.154883	0.0097330	0.0118942	0.0219635	0.0378245	0.0435528	0.0237785
14	[0.159, 0.167)	0.162793	0.0102563	0.0125995	0.0228089	0.0390630	0.0443591	0.0244409
15	[0.167, ∞)	0.184864	0.0115601	0.0138072	0.0242891	0.0403012	0.046879	0.0259996

Table A2 – Performance of CM(LTI_2) index under the action of LUA as the PP. Statistical characteristics of the MAEs distribution in relation to various levels of CM(LTI_2) within a given VRCM$_i$ for i=1,…,15. The results are based on 10,000 perturbed random reciprocal PCMs (geometric and Saaty's scales applied fifty-fifty), and were generated on the basis of **SA|2|** as the simulation algorithm. The table contains results for $n\in\{5, 6, 7, 8, 9\}$, presented consecutively.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.08867)	0.057344	0.0040500	0.0051112	0.0109956	0.0223145	0.0281738	0.0129222
2	[0.08867, 0.106)	0.097631	0.0051370	0.0066813	0.0149743	0.0314682	0.0402326	0.0179091
3	[0.106, 0.123)	0.115154	0.0062273	0.0080033	0.0178583	0.0404214	0.0493824	0.0215433
4	[0.123, 0.140)	0.132429	0.0077722	0.0103867	0.0235191	0.0443250	0.0533440	0.0260091
5	[0.140, 0.158)	0.149841	0.0100660	0.0130848	0.0264063	0.0492151	0.0598150	0.0293785
6	[0.158, 0.175)	0.166943	0.0122130	0.0152940	0.0305507	0.0567287	0.0669818	0.0339792
7	[0.175, 0.192)	0.183544	0.0134146	0.0168104	0.0341529	0.0622582	0.0730947	0.0376190
8	[0.192, 0.209)	0.200556	0.0144681	0.0180775	0.0371079	0.0664060	0.0801886	0.0405209
9	[0.209, 0.227)	0.217798	0.0152484	0.0195489	0.0387389	0.0726297	0.0886177	0.0432569
10	[0.227, 0.244)	0.235136	0.0161576	0.0201625	0.0403835	0.0771441	0.0945720	0.0454089
11	[0.244, 0.261)	0.252143	0.0164634	0.0205743	0.0428687	0.0812496	0.0997771	0.0479053
12	[0.261, 0.278)	0.269128	0.0174125	0.0217309	0.0445472	0.0844031	0.1015070	0.0498806
13	[0.278, 0.296)	0.286560	0.0184856	0.0235664	0.0474587	0.0907092	0.1046180	0.0527022
14	[0.296, 0.313)	0.304366	0.0176996	0.0228077	0.0479535	0.0900992	0.1047390	0.0532388
15	[0.313, ∞)	0.354236	0.0192203	0.0244908	0.0503011	0.0929620	0.1098880	0.0556579
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.09033)	0.063185	0.0035880	0.0044267	0.0089867	0.0179120	0.0222610	0.0103740
2	[0.09033, 0.103)	0.097359	0.0063423	0.0078999	0.0155573	0.0273932	0.0319008	0.0168600
3	[0.103, 0.115)	0.109410	0.00722965	0.0091969	0.0177403	0.0325171	0.0386338	0.0196173
4	[0.115, 0.127)	0.121343	0.0084960	0.0107027	0.0210117	0.0381138	0.0455231	0.0233632
5	[0.127, 0.139)	0.133108	0.0094898	0.0116495	0.0229152	0.0420076	0.0524375	0.0257261
6	[0.139, 0.152)	0.145538	0.0108994	0.0132421	0.0253036	0.0481306	0.0607595	0.0287714
7	[0.152, 0.164)	0.157728	0.0114276	0.0139558	0.0271455	0.0539605	0.0656762	0.0310679
8	[0.164, 0.176)	0.169681	0.0121504	0.0150640	0.0292961	0.0575169	0.0691811	0.0336092
9	[0.176, 0.189)	0.182266	0.0128272	0.0159433	0.0313538	0.0612481	0.0726740	0.0356428
10	[0.189, 0.201)	0.194801	0.0138039	0.0173398	0.0328835	0.0629099	0.0736509	0.0370798
11	[0.201, 0.213)	0.206815	0.0140270	0.0173321	0.0347352	0.0651391	0.0772600	0.0387081
12	[0.213, 0.225)	0.218752	0.0145777	0.0184396	0.0366233	0.0669927	0.0785993	0.0400494
13	[0.225, 0.238)	0.231199	0.0152711	0.0185568	0.0383407	0.0700653	0.0825784	0.0420203
14	[0.238, 0.250)	0.243637	0.0158614	0.0199098	0.0378087	0.0715807	0.0841889	0.0422926
15	[0.250, ∞)	0.280974	0.0171267	0.0210891	0.0412814	0.0738420	0.0859174	0.0450109
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.07955)	0.061285	0.0032254	0.0039749	0.0080564	0.0157394	0.0187853	0.0091669
2	[0.07955, 0.09)	0.085237	0.0055756	0.00703635	0.0133354	0.0239340	0.0276372	0.0145973
3	[0.09, 0.10)	0.095217	0.0067076	0.00822896	0.0151476	0.0267136	0.0313365	0.0168334
4	[0.10, 0.11)	0.105055	0.0071842	0.0087319	0.0167347	0.0300692	0.0372910	0.0187971
5	[0.110, 0.120)	0.114948	0.0078851	0.0096590	0.0184274	0.0350956	0.0451092	0.0211768
6	[0.120, 0.130)	0.124976	0.00841585	0.0104132	0.0202074	0.0399982	0.0487099	0.0233008
7	[0.130, 0.140)	0.134961	0.0094352	0.0114042	0.0217320	0.0442261	0.0533524	0.0258727
8	[0.140, 0.150)	0.144948	0.0097585	0.0120015	0.0234560	0.0471241	0.0565387	0.0269760
9	[0.150, 0.160)	0.154896	0.0105670	0.0127710	0.0253768	0.0489510	0.0573372	0.0286380
10	[0.160, 0.171)	0.165237	0.0113250	0.0138414	0.0270817	0.0499840	0.0598609	0.0301303
11	[0.171, 0.181)	0.175843	0.0120019	0.0146386	0.0287811	0.0534214	0.0617005	0.0317433
12	[0.181, 0.191)	0.185759	0.0126595	0.0153651	0.0298787	0.0548531	0.0642158	0.0329712
13	[0.191, 0.201)	0.195725	0.0128283	0.0154763	0.0313703	0.0562006	0.0639533	0.0339822
14	[0.201, 0.211)	0.205744	0.0139552	0.0170698	0.0326421	0.0578949	0.0674790	0.0354460
15	[0.211, ∞)	0.235674	0.0149229	0.0180402	0.0341664	0.0601557	0.0694405	0.0371047
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0688)	0.056836	0.0030787	0.0036205	0.0073926	0.0145693	0.0176789	0.0084662
2	[0.0688, 0.078)	0.073666	0.0051105	0.0061309	0.0109223	0.0189602	0.0222010	0.0121584
3	[0.078, 0.087)	0.082561	0.0055045	0.0066116	0.0121236	0.0213335	0.0264824	0.0136680

4	[0.087, 0.096)	0.091556	0.0059645	0.0072229	0.0133996	0.0253087	0.0332026	0.0154277
5	[0.096, 0.105)	0.100528	0.0063149	0.0077633	0.0145624	0.0296739	0.0375550	0.0170392
6	[0.105, 0.114)	0.109496	0.0069818	0.0085509	0.0161950	0.0334936	0.0413081	0.0189646
7	[0.114, 0.123)	0.118520	0.0075613	0.0090468	0.0175431	0.0357544	0.0434245	0.0204242
8	[0.123, 0.132)	0.127416	0.0079922	0.0097597	0.0192472	0.0395417	0.0473757	0.0223206
9	[0.132, 0.141)	0.136453	0.0086852	0.0105264	0.0209858	0.0407715	0.0482480	0.0238363
10	[0.141, 0.150)	0.145395	0.0097858	0.0118143	0.0228944	0.0419213	0.0491251	0.0254135
11	[0.150, 0.159)	0.154468	0.0100095	0.0125284	0.0241689	0.0438311	0.0507983	0.0265045
12	[0.159, 0.168)	0.163355	0.0106847	0.0132036	0.0259776	0.0454986	0.0526910	0.0279660
13	[0.168, 0.177)	0.172363	0.0111935	0.0138710	0.0271393	0.0471614	0.0543407	0.0291270
14	[0.177, 0.186)	0.181389	0.0124251	0.0149686	0.0273290	0.0474666	0.0540064	0.0294350
15	[0.186, ∞)	0.206400	0.0133439	0.0162347	0.0294684	0.0496426	0.0567018	0.0315851
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.05999)	0.051571	0.0027899	0.0033529	0.0066145	0.0129846	0.0152304	0.00757575
2	[0.05999, 0.068)	0.064108	0.00425595	0.0049956	0.0089923	0.0154707	0.0180262	0.0099654
3	[0.068, 0.076)	0.0720425	0.0045459	0.0053806	0.0096431	0.0170663	0.0215704	0.0110540
4	[0.076, 0.085)	0.080557	0.00484755	0.0059128	0.0107496	0.0200923	0.0264333	0.0124749
5	[0.085, 0.093)	0.089092	0.0053555	0.0064602	0.0118465	0.0241501	0.0307256	0.0140711
6	[0.093, 0.101)	0.096974	0.0059146	0.0070858	0.0130955	0.0270450	0.0340523	0.0154649
7	[0.101, 0.109)	0.105011	0.0064092	0.0077271	0.0143894	0.0307524	0.0372307	0.0172088
8	[0.109, 0.118)	0.113412	0.0070543	0.0083974	0.0158975	0.0323549	0.0386885	0.0184995
9	[0.118, 0.126)	0.121966	0.0076984	0.0092627	0.0180989	0.0344896	0.0405523	0.0204213
10	[0.126, 0.134)	0.129881	0.0081201	0.0097966	0.0194020	0.0364334	0.0431290	0.0216106
11	[0.134, 0.142)	0.137905	0.0088861	0.0106243	0.0206471	0.0371653	0.0437476	0.0225820
12	[0.142, 0.151)	0.146345	0.0094580	0.0114928	0.0216677	0.0380785	0.0443759	0.0235540
13	[0.151, 0.159)	0.154771	0.0101348	0.0124254	0.0231652	0.0396143	0.0456328	0.0248599
14	[0.159, 0.167)	0.162795	0.0108284	0.0129963	0.0234617	0.0396838	0.0458155	0.0250440
15	[0.167, ∞)	0.184832	0.0122591	0.0145491	0.0255388	0.0426019	0.0490175	0.0273476

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NEW OPTIMIZATION-BASED METHOD FOR ESTIMATING PRIORITY WEIGHTS

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Abstract. The estimation of priority vectors from pairwise comparison matrices is a core of the Analytic Hierarchy Process. Perhaps the most popular approach for deriving the priority weights is the right eigenvalue method (EM). Despite its popularity, various shortcomings of the EM have been described in literature. In this paper a new method for deriving priority vectors is proposed. This method makes use of the idea underlying the EM but in difference to the latter, the new one is optimization-based. Important features of this new technique are studied via computer simulations and illustrated by some numerical examples.

Keywords: *AHP, eigenvalue method, priority vector, constrained optimization*

Introduction

In the AHP, pairwise comparisons of various alternatives are performed by the decision-maker (DM) and then the pairwise comparison matrix (PCM) is built. The elements of the matrix represent the DM judgments about the values of the priority ratios. Priority weights - assigned to each alternative and/or criterion - measure their relative importance. The weights form a so-called priority vector (PV). Generating PVs from the PCMs is the core of the AHP. In the early 1980s Saaty [1] suggested the right eigenvalue prioritization method (EM), that became the most popular method for deriving priority weights. During the last decades several other prioritization methods have been proposed in literature. However, each known method has its advantages and disadvantages. One can find in literature a number of papers devoted to comparative studies of various prioritization methods [2-8]. Despite its popularity, the EM also has its share of criticism, see [4, 6, 9-11]. Some of its shortcomings will be addressed later on in more details.

This study proposes a new approach for deriving PVs. The basic concept is to combine Saaty's idea with some optimization procedures. As a result, we obtain estimation procedures which do not suffer from the above-mentioned EM's drawbacks and, moreover, provide us with naturally meaningful indicators of inconsistency of the PCMs.

In the following section we state the prioritization problem formally. Next, in Section 2, the main prioritization approaches are briefly described. In Section 3 we introduce our proposals for deriving PVs. Section 4 presents the results of the simulation studies of the performance of the introduced methods as well as some numerical examples illustrating the advantages resulting from our approach. In Section 5 a problem connected with the COP is addressed.

1. Formal statement of the prioritization problem

A problem of deriving priority weights from PCM is to estimate a PV $\mathbf{w} = (w_1,\ldots,w_n)$ on the base of the matrix $\mathbf{A} = [a_{ij}]_{\mathrm{nxn}}$. Usually, the priority weights $w_i, i = 1,\ldots,n$ are chosen to be positive and normalized to unity: $\sum_i^n w_i = 1$. The elements a_{ij} of the matrix $\mathbf{A}$ are the DM judgments about the priority ratios w_i/w_j, $i,j = 1,\ldots,n$. The judgments are usually expressed in linguistic terms and then transformed into an appropriate numeric scale. A given PCM is said to be *reciprocal* (RPCM) if $a_{ij} = 1/a_{ji}$. PCM is called *consistent* if it is reciprocal and its elements satisfy the condition: $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k = 1,\ldots,n$. It is proved that a necessary and sufficient condition for a positive matrix $\mathbf{A}$ to be consistent is an existence of a unique PV $\mathbf{w}$ satisfying $a_{ij} = w_i/w_j$ for $i,j = 1,\ldots,n$. PCM is said to be (ordinally) *transitive* if the following condition holds: (A) if for any $l = 1,\ldots,n$ element a_{lj} is not less than a_{lk} then $a_{ij} \geq a_{ik}$ for $i = 1,\ldots,n$ and (B) if for any $l = 1,\ldots,n$ element a_{jl} is not less than a_{kl} then $a_{ji} \geq a_{ki}$ for $i = 1,\ldots,n$.

It is obvious that in reality it cannot be expected that the elements of PCM give exact priority ratios. The evaluations of the ratios may depend on personal taste, experience, changes in one's knowledge, and may vary in time. One cannot also neglect rounding errors which can be quite big if we use discrete numeric scale, especially if it offers only a few values for consideration. Therefore in reality the PCM is typically inconsistent. Then the relation between the PCM elements and the priority weights can be expressed in the form

$$a_{ij} = \varepsilon_{ij} \frac{w_i}{w_j} \tag{1}$$

where ε_{ij} is a perturbation factor which is expected to be near 1 [2, 12, 13]. In the statistical approach and in various simulation studies the perturbation factor is interpreted as realization of a random variable.

In conventional AHP, the elements of PCM are collected only for the upper triangle of the matrix $\mathbf{A}$, and the remaining elements are computed as the inverse of the corresponding symmetric elements in the upper triangle that ensures the reciprocity of the PCM. This method of data collection artificially forces some consistency of judgments which is not always natural, see e.g. [2, 6, 10, 11]. Some authors argue that enforcing this kind of consistency on the input data creates unnecessary dependency among observations and loses additional information contained perhaps in the elements of the lower triangle of **A** which may lead to poorer estimates of the priorities.

2. Prioritization methods discussed in literature

Choo et al. [2] discussed and compared 18 methods which may be used for deriving priority weights. These methods are derived from different concepts of the estimation quality criteria and under different assumptions about the perturbation factor structure. Except for the EM, most prioritization methods are optimization-based. Such methods may assume a different criterion function and, consequently, result in different prioritization estimates. Among them the most popular is the logarithmic least squares method (also known as geometric mean method - GM) [3, 4, 9]. Imposing the normalization condition $\sum_{i=1}^{n} w_i = 1$, the weights in GM can be estimated from the following formula:

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n} \Bigg/ \sum_{i=1}^{n} \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n} \tag{2}$$

As it was already pointed out, the most commonly used prioritization method is the EM. This method is not optimization-based. Let us describe the idea underlying EM in more detail. In a perfect judgment case, where there are no perturbations ($\varepsilon_{ij} = 1$) we have

$$\mathbf{A}\mathbf{w} = n\mathbf{w} \tag{3}$$

Thus in this case the PV **w** can be calculated by solving the eigenvector equation (3). It turns out that for a consistent matrix **A** the number n is the principal eigenvalue of **A**, i.e. the largest solution of the characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. It is also the only nonzero eigenvalue in this case. The relation (3) plays the main role in Saaty's approach. In the case where the matrix **A** is perturbed, the Saaty proposal is to use the normalized right eigenvector associated with the largest eigenvalue as an estimate of the true priority vector. Hence, to obtain the estimate we need to solve the general eigenvector equation

$$\mathrm{Aw} = \lambda_{\max}\mathrm{w} \tag{4}$$

where $\lambda_{\max}$ is the principal eigenvalue. For an arbitrary positive *reciprocal* matrix **A** the value $\lambda_{\max}$ is always real, unique and not smaller than n.

Apart from deriving priority vectors, another very important problem is how to measure the degree of inconsistence of the PCM. It is obvious that the significant violations of the consistency makes the inference useless (or at least questionable).

In practice the only widely accepted rule of inconsistency measuring is due to Saaty and is closely related to EM. According to this concept the index CI(n) is given in the following form:

$$\mathrm{CI}(n) = \frac{\lambda_{\max} - n}{n-1} \tag{5}$$

The justification of the formula (5) as well as a more detailed description of the approach to consistency measuring can be found in various articles, see e.g. [1, 6, 12]. However, the index can hardly be interpreted in any intuitive way and is more and more often criticised and new indices which can be used for this purpose are proposed in literature, see [6, 14]. What is more, the index CI(n) is useful only for RPCM and even then it can be very misleading, see e.g. [6] and references therein.

Although the EM has attracted much attention, especially in practical application, it has also been criticized in literature for several different reasons. In particular, some authors have pointed out that Saaty's procedure does not optimize any performance criterion. Thus it cannot be interpreted in statistical or optimization fashion and it is difficult to compare resulting PVs with the ones obtained with the help of other methods, see e.g. [3, 4, 9]. Moreover, unlike many optimization models, EM does not allow DM to introduce any additional constraints for the priority vector which, according to decision-maker opinions, should be satisfied by the weights [6].

Another drawback of this method is that it should be used only for the so-called reciprocal PCMs, and as such it has a limited range of applications. In real-world problems the reciprocity condition is artificially enforced by Saaty's method, and many authors argue that it would be much better if the DM made all comparisons because then the PCM would contain more information about the unknown priority vector, see e.g. [3, 6, 10, 11]. In the sequel a new prioritization method which does not suffer from these drawbacks is introduced. The performance of the new method will be compared with the performance of the most frequently recommend and used in AHP practice methods- the EM and GM, see [4, 8].

3. Proposal of a new prioritization method

In this proposal an idea of deriving the weights based on the relation (3) is adopted. However, in a new approach proposed in [5, 6], instead of solving the eigenvalue equation (4) one looks for a vector **w** which best *approximates* the relation (3), i.e. a vector for which

$$\mathbf{A}\mathbf{w} \cong n\mathbf{w} \tag{6}$$

Consequently, as an estimate of the DM priorities, a vector **w** is used, which is a solution to the following optimization problem:

$$\min D(\mathbf{Aw}, n\mathbf{w}) \tag{7}$$

subject to $\sum_{i=1}^{n} w_i = 1, w_i > 0$, i=1,…,n. The function D in (7) measures a distance between the vectors appearing on the right- and left-hand side of the relation (6).

Let $\mathbf{d} = (d_1,\ldots,d_n)^{\mathrm{T}} = \mathbf{Aw}-n\mathbf{w}$. There are various distance measures that can be used in () to derive the priorities. Some of them were discussed in [6] and [7]. Here we consider one of the most widely accepted distance measures, which appears in various optimization models. It is the sum of squared deviations $\sum_{i=1}^{n} d_i^2$. This measure leads to a prioritization method, which will be called in the sequel *least squared deviation approximation* (LSDA). To obtain the PV in this approach one needs to solve the following quadratic programming problem

$$\min SSD(\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\mathbf{Bw} \tag{8}$$

subject to

C1. $\sum_{j=1}^{n} w_j = 1,$ and C2. $w_i > 0,\ i = 1,\ldots,n$

in the above model $\mathbf{B} = [(\mathbf{A}-n\mathbf{I})^{\mathrm{T}}(\mathbf{A}-n\mathbf{I})]$ with $\mathbf{I}$ being an identity matrix of order n.

It is easy to see that if the matrix $\mathbf{A}$ is inconsistent then the matrix $\mathbf{B}$ is nonsingular. In such a case we can find a closed-form formula for a vector $\mathbf{w}_{LS}$ minimizing (8) and satisfying the constraint C1. Using well-known solutions for minimizing a quadratic form under linear constraints, the following formula can be derived

$$\mathbf{w}_{LS} = \mathbf{B}^{-1}\mathbf{e}/(\mathbf{e}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{e}) \tag{9}$$

where $\mathbf{e} = (1,\ldots,1)^{\mathrm{T}}$ is a column vector with all elements equal to 1. Obviously, the vector given by (9) does not have to satisfy the positivity constraint C2. However, it is rarely the case if the PCM is both transitive and reciprocal. We have generated randomly 10 000 such matrices (each of random order drawn from the interval [4, 12]) and found only 16 examples when the vectors given by (9) have some negative coefficients. It is only 0.16% of all generated PCMs. The analogous percentage computed for PCMs which were transitive and nonreciprocal is greater and amounts to 12.7% while in the case of reciprocal and nontransitive PCMs the percentage equals 59.7% Obviously, in all such cases, we can use computer software to solve the minimization problem (8) numerically. One may notice that nowadays very efficient computer software is available for minimizing the quadratic form (8) under linear constraints of the type C1 and C2.

The existence of the closed-form formula (9) that allows to derive priorities in many practical problems is a very appealing future of the proposed method. It also has another important feature - easily interpretable consistency index. Let the minimum of (8) subject to constraints C1 and C2 be denoted as MSSD. Its value can be considered as an inconsistency measure. In a consistent case MSSD equals 0. For inconsistent PCMs the index takes positive values. It is easy to note that the sum of the coefficients of the vector standing on the right-hand side of the relation (6) is equal to n. Thus, if we divide MSSD by n, and take the square root of the result, then we obtain an index of inconsistency $\mathrm{II}_{\mathrm{LS}} = \sqrt{\mathrm{MSSD}/n}$. The value of $\mathrm{II}_{\mathrm{LS}}$ can be compared with 1, the total sum of priority weights. Roughly speaking, the value tells us what part of the total mass of weights has to be changed (added or subtracted dependently on the coefficient) to achieve equality in the formula (6).

Unlike the EM, the proposed prioritization method and resulting inconsistency index can be applied to any type of PCM and thus provide us with a tool for dealing with nonreciprocal PCMs. It is well known that in the case of nonreciprocal PCMs the index CI may take negative values. In such a case CI has no interpretation. In the next section we present an example of such a situation.

4. Methods comparison via computer simulation - results and examples

In this section we present some examples illustrating the usefulness of the proposed method as well as results of simulation comparison of the LSDA with EM and GM, the most popular and recommended methods, see [4, 8]. To compare the accuracy of the estimates obtained by the considered methods we simulate various situations related to various sources of the inconsistency of PCM. In the simulations we assume, similarly as in e.g. [4, 8, 13], that we know the true PV. Next we generate an inconsistent PCM related to the known PV. The inconsistence is a result of various types of errors and/or perturbation factors. In literature various sources of inconsistence of the PCMs were named. The sources can be divided into two groups: errors resulting from the nature of human judgments and errors resulting from the technical realization of the comparison procedure. The errors from the second group are mainly the rounding errors and errors resulting from the forced reciprocity. The presence of the rounding errors is connected with the numerical ratio scale, whose values should be used by the DM to express its judgments, see e.g. [11, 12]. In conventional AHP the most popular is Saaty numerical scale which consists of the integers 1 to 9 and their reciprocals. Possible problems resulting from the rounding errors illustrate the following example.

Example 1. Let the true PV be as follows: $\mathbf{w} = (0.691, 0.173, 0.126, 0.010)^{\mathrm{T}}$. Then the approximated PCM (in the sequel denoted as APCM) containing the numbers from the Saaty scale which are the closest to the elements of the true ratio matrix is:

$$\begin{bmatrix} 1 & 4 & 5 & 9 \\ 1/4 & 1 & 1 & 9 \\ 1/6 & 1/2 & 1 & 9 \\ 1/9 & 1/9 & 1/9 & 1 \end{bmatrix}$$

It is worth noticing, that due to the rounding errors only, a nonreciprocal PCM is obtained. Indeed, for example the ratio w_1/w_3 equals 5.484 and the closest value taken from the ratio scale is 5, but the ratio w_3/w_1 equals 0.182 and the closest value from the scale is 1/6. Now, let us estimate the PV using the EM, GM and LSDA. We compare the results with the true PV. In these comparisons we use performance measures known from literature, see e.g. [2, 4, 8, 13]: the Pearson correlation coefficient r between the estimated and true vectors, the Spearman rank correlation coefficient ρ, mean absolute deviation (MAD) and root-mean-square-deviation (RMS). The two latter measures are given by the following formulae:

$$\mathrm{MAD}(\mathbf{w},\mathbf{w}^*) = \frac{1}{n}\sum_{i=1}^{n}\left|w_i - w_i^*\right|$$

$$\mathrm{RMS}(\mathbf{w},\mathbf{w}^*) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(w_i - w_i^*)^2}$$

The results of the comparison are presented in Table 1. One can see that in this case the new method performs slightly better than the two others.

Table 1

Comparison of EM, GM and LSDA. Performance for rounding errors - Example 1

method	estimate	r	ρ	MAD	RMS
EM	$(0.6067, 0.1976, 0.1615, 0.0342)^T$	0.9982	1	0.0422	0.0489
GM	$(0.6158, 0.1955, 0.1582, 0.0305)^T$	0.9980	1	0.0408	0.0472
LSDA	$(0.6158, 0.1954, 0.1581, 0.0307)^T$	0.9985	1	0.0376	0.0436

Now let us address the problem of forced reciprocity. According to the conventional AHP setup, the elements of the lower triangle of the PCM should be computed as reciprocals of the appropriate elements from the upper triangle. Thus, in our example, the elements a_{31} and a_{32} should be changed to 1/5 and 1, respectively. Results obtained for such a PCM are presented in Table 2.

Table 2

Comparison of EM, GM and LSDA. Performance for rounding errors and forced reciprocity - Example 1

method	estimate	r	ρ	MAD	RMS
EM	$(0.5955, 0.1894, 0.1825, 0.0326)^T$	0.9951	1	0.0478	0.0572
GM	$(0.6143, 0.1843, 0.1767, 0.0247)^T$	0.9935	1	0.0519	0.0619
LSDA	$(0.6144, 0.1841, 0.1763, 0.0252)^T$	0.9961	1	0.0383	0.0468

We see that the performance of all considered methods are now poorer than in the nonreciprocal case. It can also be observed in Tables 1 and 2 that the LSDA outperforms the other two methods with respect to all criteria. One may say that it is just an example and in other situations the ranking of the methods may be different. It appears, however, that the phenomena and relations observed in Example 1 are typical. In our studies we simulate a thousand such problems. In each case the number of alternatives n is drawn from the set $\{4,\ldots,12\}$ and then a random PV is generated for which we compute all performance characteristics described in Example 1. Table 3 presents the average results for PCMs with the rounding errors only (APCM) and for PCMs with rounding errors and forced reciprocity (FR-APCM).

Table 3

Comparison of EM, GM and LSDA. Performance for PCMs with rounding errors - average results for 1000 random PV

	APCM				FR-APCM			
method	r	ρ	MAD	RMS	r	ρ	MAD	RMS
EM	0.979	0.997	0.0120	0.0090	0.980	0.996	0.0136	0.0100
GM	0.978	0.996	0.0129	0.0095	0.980	0.995	0.0146	0.0106
LSDA	0.976	0.997	0.0114	0.0086	0.980	0.996	0.0123	0.0091

Now let us consider the errors resulting from the nature of human judgments. They are often treated as realization of random variables and are commonly represented in the form (1), see e.g. [2, 6, 14]. Probability distributions (p.d.) of the perturbation factor ε_{ij} mainly involve gamma, log-normal and uniform ones [6, 13]. Following [4, 8] we consider three comparison frameworks that will be denoted CF1, CF2, and CF3. In CF1 the PCMs contain many small errors. In our simulations small error ε_{ij}, see (1), has p.d. that is uniform on the interval [0.8, 1.2]. In the second comparison framework, the PCMs contain many small and one large error, in CF3 the PCMs contain many large errors. Large errors ε_{ij} are generated according to the p.d. having uniform distribution on the interval [0.2, 1.8].

Although we adopt the comparison approach described in [4, 8] we propose some changes. First, the simulations described in [4, 8] are based only on *one* priority vector. Moreover, the vector is not normalized and thus the observed average errors cannot be compared with errors corresponding to other vectors having different dimensions and priority values. To make the results more representative in our simulation we use 30 random normalized vectors having random dimensions. The dimension of each random vector is drawn from the set $\{4,\ldots,12\}$. For each vector we generate 250 randomly disturbed PCMs with the perturbation factor having the p.d. dependent on the comparison framework, as described above. Similarly as in Example 1, we consider two types of approximation: with and without forced reciprocity. Next the performance measures are computed.

Another difference is that in our studies we also take into account the rounding errors. Therefore the randomly disturbed ratios are rounded to the closest values from Saaty's scale. Such simulation framework seems to be more realistic, because in the AHP procedure DM is always expected to express his/her opinions in a given scale. Table 4 presents the average results obtained in these studies.

Table 4

Comparison of EM, GM and LSDA under simulation frameworks CF1, CF2 and CF3

CF1 - average results for 30 random PVs								
	APCM				FR-APCM			
method	r	ρ	MAD	RMS	r	ρ	MAD	RMS
EM	0.972	0.932	0.0264	0.0203	0.993	0.973	0.0372	0.0284
GM	0.966	0.922	0.0292	0.0225	0.991	0.973	0.0370	0.0283
LSDA	0.971	0.930	0.0266	0.0204	0.993	0.973	0.0369	0.0283
CF2 - average results for 30 random PVs								
EM	0.990	0.951	0.0225	0.0166	0.988	0.944	0.0243	0.0178
GM	0.986	0.954	0.0239	0.0187	0.986	0.945	0.0259	0.0187
LSDA	0.989	0.951	0.0214	0.0157	0.988	0.944	0.0224	0.0165
CF3 - average results for 30 random PVs								
EM	0.977	0.945	0.0239	0.0180	0.948	0.923	0.0337	0.0248
GM	0.971	0.938	0.0266	0.0200	0.949	0.927	0.0341	0.0249
LSDA	0.977	0.944	0.0237	0.0179	0.948	0.923	0.0325	0.0243

All the presented results confirm that the methods EM, GM, and LSDA perform very similarly. However the LSDA seems to be the best method with respect to the values of average errors, both MAD and RMS. Taking into account these criteria, it demonstrates the best performance under 5 out of 6 simulation frameworks.

In Table 4, another interesting observation can be made. All considered methods perform significantly better in the case of the nonreciprocal matrices. This fact indicates the importance of making use of the nonreciprocal PCMs in AHP. In such a case, however, the Saaty’s inconsistency index may be uninterpretable even in the case of slightly inconsistent PCMs resulting e.g. from the rounding errors. The problem is illustrated by the example below.

Example 2. Let the PCM be as follows:

$$\begin{bmatrix} 1 & 2 & 5 & 9 \\ 1/2 & 1 & 3 & 9 \\ 1/6 & 1/4 & 1 & 3 \\ 1/9 & 1/9 & 1/3 & 1 \end{bmatrix}$$

We see that $a31 \neq 1/a13$ and $a32 \neq 1/a23$. However the PCM can hardly be considered as seriously inconsistent because, as we have already seen such perturbations can be observed even in the case of perfect judgments and may appear simply due to rounding errors. Indeed, if for example the true ratios are w_1/w_3 = 5.49, and w_2/w_3 = 3.45, then the APCM will have values of the related elements as in our example. But, in this case, the Saaty inconsistency index CI equals –0.017 and, as negative, is uninterpretable, and that is why nonreciprocal PCMs are forbidden in conventional AHP. As we can see from the above simulation results, forced reciprocity often leads to the loss of quality of the resulting estimates. To avoid this drawback, one may use LSDA and the related index of inconsistency. In this case the index $\mathrm{II}_{\mathrm{LS}}$ is equal to 0.002, and its small value indicates good consistency of the PCM. Roughly speaking, only less than 0.2% of the total mass of weights should be modified (added to or subtracted from the coefficients of the vector) to achieve the equality in (6). Indeed, based on such intuition, one may say that the inconsistency is not significant. However, in real world applications of the AHP, the DM needs a more precise and better justified threshold value which separates the acceptable judgment PCMs, and PCMs that should be rejected as "randomly generated". To obtain such thresholds, following Saaty's approach, let us study the empirical distribution of the values of $\mathrm{II}_{\mathrm{LS}}$ computed for random PCMs. This random index will be denoted by RII. Table 5 presents the statistical characteristics of empirical distributions obtained for RII generated for reciprocal PCM.

Table 5

Statistical characteristics of empirical distributions of RII(*n*) obtained for random reciprocal PCMs. For each number of alternatives *n* results are based on 10 000 random reciprocal PCMs

Empirical distribution characteristics		number of alternatives n						
		$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
Mean	(MRII)	0.239	0.282	0.307	0.320	0.325	0.321	0.323
p-Quantiles	$p = .01$	0.008	0.018	0.041	0.055	0.100	0.107	0.150
	$p = .05$	0.015	0.032	0.074	0.119	0.148	0.166	0.203
	$p = .10$	0.024	0.053	0.106	0.151	0.187	0.206	0.226
	$p = .15$	0.030	0.081	0.137	0.186	0.211	0.228	0.241

However, the reciprocity is usually forced artificially, whilst the most traditional definition to characterize the consistency of PCMs is to use transitivity (see e.g. [10] and references in there) and intransitivity is prohibited by most theories. Thus in Table 6 results related to random PCMs with forced transitivity are also presented.

Adopting Saaty's approach for inference based on the new index, one may treat a given PCM as consistent enough if $\mathrm{II}_{\mathrm{LS}}(n) < t_\alpha(n)$, with $t_\alpha(n) = \alpha$ MRII(n), for some prescribed consistency level α. On the other hand, one may prefer to adopt the conventional statistical approach, and select a proper quantile of the empirical distribution of the corresponding RII. The order p of the quantile should reflect the attitude of the DM towards the probability of accepting random PCM as a consistent one and may be interpreted as the required consistency level.

Table 6

Statistical characteristics of empirical distributions of RII(*n*) obtained for random transitive PCMs. For each number of alternatives *n* results are based on 10 000 random transitive PCMs

Empirical distribution characteristics		number of alternatives n						
		$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
Mean	(MRII)	0.055	0.040	0.029	0.026	0.022	0.018	0.016
p-Quantiles	$p = .01$	.0007	.0006	.0006	.0005	.0003	.0002	.0002
	$p = .05$	.0046	.0029	.0024	.0021	.0017	.0015	.0012
	$p = .10$	.0086	.0060	.0046	.0041	.0034	.0031	.0027
	$p = .15$	.0135	.0088	.0067	.0061	.0050	.0043	.0040

For instance, let us consider Example 2. The PCM is transitive. It is well known that very often DMs, based on their experience, assume that their judgments *should* be transitive and the transitivity is usually "forced" *by the DM personal beliefs*. Thus one may assume that it was filled in as such. Consequently, we should compare the index of inconsistency with the quantiles given in Table 6. If we assume the required consistency level equal to $p = 0.05$, then the matrix from this example can be considered as the consistent one because ICS for this matrix equals 0.002 and the value is significantly less than 0.0046 - the quantile of order $p = 0.05$ related to empirical distribution of RII(4). A more thorough discussion of the problem can be found in [6].

Final remarks

The approach proposed in this paper provides DM not only with a relatively simple prioritization technique, but also with intuitive and easily computable inconsistency indices. The simulation results described in Section 4 show that the new method, the LSDA, demonstrates similar or even better performance than EM or GM. However the advantage of the LSDA is that it can be used for both reciprocal and nonreciprocal PCMs because the related inconsistency index is naturally meaningful for all types of PCMs. As a optimization based technique the LSDA also allows the DM to implement various other conditions which, according to his/her opinions, should be satisfied by the weights. Such additional constraints based on a prior knowledge or resulting from the DM requirements can be easily included in the LSDA but not in the EM. An additional advantage of the LSDA is the simple closed-form formula (9) which can be often used for computing the PV.

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ALTERNATIVE MODES OF QUESTIONING IN THE ANALYTIC HIERARCHY PROCESS

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Abstract—The standard mode of questioning in the Analytic Hierarchy Process (AHP) requires the decision maker to complete a sequence of positive reciprocal matrices by answering $n(n-1)/2$ questions for each matrix, each entry being an approximation to the ratio of the weights of the n items being compared. This paper presents two extensions of the eigenvector approach of the AHP which allows the decision maker to say "I don't know" or "I'm not sure" to some of the questions being asked, and to approximate nonlinear functions of the ratios of the weights. In this way, the questioning process can be substantially shortened and better representations of the responses to certain stimuli may be derived.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP) is a decision-aiding method which has received increasing attention in the literature and in application since its development by Saaty [1]. The reader is referred to the recent review paper by Zahedi [2] for a listing of the current literature on this subject. The basis of the AHP is the completion of an $n \times n$ matrix $A = (a_{ij})$ at each level of the decision hierarchy. This matrix A is of the form $a_{ij} = 1/a_{ji}$, $a_{ij} > 0$; i.e. A is a *positive, reciprocal matrix*. The basic theory, as developed by Saaty [1, 3], is based on the fact that a_{ij} is an approximation to the relative weights (w_i/w_j) of the n alternatives under consideration; the value assigned to a_{ij} is typically in the interval $[1/9, 9]$. Given the $n(n-1)/2$ approximations to these weights which the decision maker supplies when completing the matrix A, the weights $\mathbf{w} = (w_i)$ are found by solving the following eigenvector problem:

$$A\mathbf{w} = \lambda_{\max}\mathbf{w}, \tag{1}$$

where $\lambda_{\max}$ is the Perron root or principal eigenvalue of A. A complete discussion of the reasons for using equation (1) to derive the weights $\mathbf{w}$ can be found in Ref. [4].

This paper presents two extensions of the method described above for the elicitation and computation of weights from a set of pairwise comparisons. First, the completion of $n(n-1)/2$ comparisons at each level of the hierarchy can become an onerous task if n is large. Thus, one would like to find a method in which the decision maker could complete less than $n(n-1)/2$ comparisons but still answer enough comparisons in order to derive a meaningful measure of the alternatives' relative weights. Also, it is often the case that a decision maker, when faced with a particular comparison between alternatives i and j, would rather not answer this comparison directly or may simply not yet have a good understanding of his or her preferences for those two alternatives. The first case arises when the elicitation of a_{ij} calls for the decision maker to publically state a tradeoff between two sensitive criteria; e.g. mortality risk vs cost when comparing a set of measures to lessen hazardous materials risks. It may be easier for the decision maker to skip this question and have the judgment being made indirectly through the other responses he or she has provided. The ability to skip certain direct questions may make the decision maker more willing to participate in a structured decision analysis exercise. The final reason for considering the completion of less than $n(n-1)/2$ judgments stems from the fact that the decision maker may not have formed a strong opinion on a particular question and rather than forcing this individual to make an often wild guess or to have the entire process slowed due to one question, one can simply skip that comparison. The next section will describe a method based upon a theory of *nonnegative, quasi-reciprocal matrices* which can be employed to deal with incomplete comparisons.

The second extension considered in this paper is the ability to deal with nonlinear relative preferences. The standard AHP model assumes that a_{ij} is an approximation to w_i/w_j. However, there is substantial evidence in the psychology literature [5, 6] that individuals often respond in a nonlinear fashion to stimuli; e.g. a_{ij} is an approximation to $(w_i/w_j)^\alpha$, where α is a positive scalar. Using a recent result concerning a nonlinear extension of the Perron–Frobenius theorem, a method is developed to deal with nonlinear ratio estimations in the context of the AHP; this result is presented in Section 3.

Thus, this paper presents two methodological extensions of the AHP which should simplify the elicitation process and allow greater flexibility in the modeling of preference structures with nonlinear reciprocal judgments.

2. A METHOD FOR INCOMPLETE COMPARISONS

As was discussed in the previous section, there are three basic reasons why one would want to consider the completion of less than $n(n-1)/2$ pairwise comparisons in the context of the AHP:

—the time to complete all $n(n-1)/2$ comparisons;
—unwillingness to make a direct comparison between two alternatives;
—being unsure of some of the comparisons.

Harker [7] has presented a method to deal with incomplete comparisons in the context of an iterative scheme for the elicitation of the matrix A which is based upon the approximation of the missing elements of A with the data available from the completed comparisons. This approximation of a_{ij} is formed by taking the geometric mean of the intensity of all paths in the directed graph $D(A)$ associated with the partially completed matrix A which connect the alternatives i and j. This approximation scheme in some sense mimics what the decision makers would have to perform if he or she were forced to complete a given comparison.

In this section a more natural approach to dealing with the missing entries a_{ij} will be considered. Instead of approximating the missing entry a_{ij}, which is itself an approximation to the ratio w_i/w_j, let us simply set a_{ij} to be equal to w_i/w_j. In other words, let us complete the missing entries by setting them equal to the value which they seek to approximate. Of course, one does not know *a priori* the value w_i/w_j. The purpose of this section is to derive the necessary theory to deal with the situation in which some a_{ij}'s take on the functional form w_i/w_j instead of as numerical value.

In order to begin, let us reiterate a well-known result in linear algebra and graph theory (see, for example, Ref. [3, Theorem 7.1] for a proof).

Definition 1

A square matrix A is *irreducible* if it cannot be decomposed into the form

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix},$$

where A_1 and A_3 are square matrices and 0 is the zero matrix.

Theorem 1

An $n \times n$ matrix A is irreducible iff its directed graph $D(A)$ is strongly connected.

Therefore, a matrix A is irreducible iff there exists a path between every ordered pair of nodes in the graph of A. In the AHP context, this states that there must exist a direct or indirect comparison between every pair of alternatives under consideration. Given that one always has $a_{ji} = 1/a_{ij}$ when $a_{ij} > 0$ in the AHP, completing the top row of the matrix A will be sufficient to guarantee that A is irreducible.

The following definition will be necessary in what follows.

Definition 2

An $n \times n$ matrix A is called a *nonnegative, quasi-reciprocal matrix* if

$$a_{ij} \geqslant 0 \quad \text{and} \quad a_{ij} > 0 \qquad \text{implies } a_{ji} = 1/a_{ij}, \qquad \forall\, i, j = 1, 2, \ldots, n.$$

Note that all positive, reciprocal matrices are quasi-reciprocal but that the class of quasi-reciprocal matrices allows for zero entries.

Let us now assume that the decision maker has considered a set of n alternatives and has completed some subset of the $n(n-1)/2$ pairwise comparisons to form a matrix $C = (c_{ij})$. For those questions which the decision maker responded, one as $c_{ij} > 0$, $c_{ji} = 1/c_{ij}$. Let us assume that the completed questions form an irreducible matrix. From the above discussion on positive reciprocal matrices and the graph theoretic interpretation of the matrix, it is clear that the completion of the top row of questions is sufficient to make the matrix C irreducible. By definition one has $c_{ii} = 1$, $\forall i = 1, 2, \ldots, n$. For those questions which were not answered, let $c_{ij} = w_i/w_j$. For example, in comparing three alternatives one may have:

$$C = \begin{bmatrix} 1 & 2 & w_1/w_3 \\ 1/2 & 1 & 2 \\ w_3/w_1 & 1/2 & 1 \end{bmatrix}. \tag{2}$$

Computing $C\mathbf{w}$ one obtains the vector $(2w_1 + 2w_2, 1/2w_1 + w_2 + 2w_3, 1/2w_2 + 2w_3)$, which defines a new matrix A where $A\mathbf{w} = C\mathbf{w}$:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{bmatrix}. \tag{3}$$

Thus, the problem of computing the right principal eigenvector $\mathbf{w}$ for the matrix C which contains the functional relations becomes that of computing $\mathbf{w}$ for the nonnegative, quasi-reciprocal matrix A. Thus, the issue of dealing with incomplete pairwise comparisons becomes that of studying the properties of nonnegative, quasi-reciprocal matrices. Let us now formalize this issue.

Let $B = (b_{ij})$ be an $n \times n$ matrix formed from the partially completed matrix C as follows:

$$\begin{aligned} b_{ij} &= c_{ij} && \text{if } c_{ij} \text{ is a real number } > 0 \\ &= 0 && \text{otherwise} \\ b_{ii} &= m_i, && \text{the number of unanswered questions in row } i = 1, 2, \ldots, n. \end{aligned}$$

In our example, B is given by

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1/2 & 1 \end{bmatrix}. \tag{4}$$

and the matrix $(I + B)$ equals A in equation (3). By assumption we have that B will be an irreducible matrix. Defining $A = (I + B)$ to be the nonnegative, quasi-reciprocal matrix formed from the partial pairwise comparisons which will obviously also be irreducible, one has from the Perron–Frobenius theorem that $\lambda_{\max}$ will be a real positive and simple eigenvalue which is not exceeded in modulus by any other eigenvalue of A. Furthermore, the following results are known.

Theorem 2 [3, Theorem 8-5]

If B is a nonnegative irreducible matrix of order n we have $(I + B)^{n-1} > 0$; i.e. $A = (I + B)$ is a primitive matrix.

Theorem 3 [3, Theorem 7-13]

For a primitive matrix A

$$\lim_{k \to \infty} \frac{A^k \mathbf{e}}{\mathbf{e}\mathrm{T} A^k \mathbf{e}} = c\mathbf{w}, \tag{5}$$

where $\mathbf{e}$ is the unit vector, c is a constant and $\mathbf{w}$ is the principal eigenvector of A.

Therefore, the matrix B formed from the partial comparison information does lead to a primitive matrix A, and the same convergence result (5) as in the case of positive, reciprocal matrices holds. Equation (5) thus becomes the means by which one computes $\mathbf{w}$, just as in the current theory of the AHP. For example, the matrix B in equation (4) leads to the following matrix A:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1/2 & 1 \end{bmatrix}. \tag{6}$$

The limiting value of iterations (5) leads to the Perron eigenvector $\mathbf{w} = (4/7, 2/7, 1/7)$ and root $\lambda_{\max} = 3$.

Therefore, nonnegative, quasi-reciprocal matrices can be used in exactly the same manner as positive, reciprocal matrices. The only question remaining is the relationship between $\lambda_{\max}$ and n for this class of matrices. For positive, reciprocal matrices it is known that $\lambda_{\max} \geqslant n$ and that $\lambda_{\max} = n$ iff the matrix A is consistent; i.e. $a_{ij}a_{jk} = a_{ik}$, $\forall\, i,j,k$. The following theorem establishes this same result for quasi-reciprocal matrices.

Theorem 4

Let A be a nonnegative, irreducible, quasi-reciprocal matrix. Then the Perron root of A, $\lambda_{\max}$, is $\geqslant n$, the rank of A, and $\lambda_{\max} = n$ iff A is consistent; i.e. $a_{ij}a_{kj} = a_{ik}$, $\forall\, i, j, k$, with a_{ij}, a_{jk}, a_{ik} positive.

Proof. It is well-known that the trace of A, $\operatorname{tr}(A)$, equals the sum of the eigenvalues of A:

$$\operatorname{tr}(A) = n + \sum_i m_i = \sum_i \lambda_i, \tag{7}$$

where the summations are over $i = 1, 2, \ldots, n$. Given that A is irreducible, we know from the Perron–Frobenius theorem that $w_i > 0$, $\forall\, i = 1, 2, \ldots, n$, and hence one can divide the ith row of $A\mathbf{w} = \lambda_{\max}\mathbf{w}$ to form

$$\lambda_{\max} = \sum_j a_{ij} w_j / w_i \tag{8}$$

$$= (1 + m_i) + \sum_{j \neq i} a_{ij} w_j / w_i. \tag{9}$$

Summing equation (9) over $i = 1, 2, \ldots, n$, yields

$$n\lambda_{\max} = n + \sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} [a_{ij}(w_j/w_i) + a_{ji}(w_i/w_j)]. \tag{10}$$

Defining $\mu = (\lambda_{\max} - n)/(n - 1)$ and placing equation (10) into this definition, one obtains

$$\mu = -1 + [n(n-1)]^{-1}\left[\sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} a_{ij}(w_j/w_i) + a_{ji}(w_i/w_j)\right]. \tag{11}$$

Defining $a_{ij} = (w_i/w_j)(1 + \delta_{ij})$, $\delta_{ij} > -1$ if $a_{ij} > 0$, zero otherwise, and placing this into equation (11) yields, after some manipulation:

$$\mu = -1 + [n(n-1)]^{-1}\left[\sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} 2 + (\delta_{ij})^2/(1 + \delta_{ij})\right] \tag{12}$$

$$= -1 + n(n-1)[n(n-1)]^{-1} + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} (\delta_{ij})^2/(1 + \delta_{ij}) \tag{13}$$

$$= \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} (\delta_{ij})^2/(1 + \delta_{ij}). \tag{14}$$

Since $\delta_{ij} > -1$, one has $\mu \geqslant 0$ and $\mu = 0$ ($\lambda_{\max} = n$) iff $\delta_{ij} = 0$, $\forall\, i, j$, with nonzero a_{ij}. Thus, $\mu = 0$ iff A is consistent. Q.E.D.

Therefore, the theory developed by Saaty [1, 3] for positive reciprocal matrices follows completely when one considers incomplete pairwise comparisons in the context of quasi-reciprocal matrices.

In order to illustrate the workings of this partial comparison method, consider the distance to Philadelphia example from Ref. [3] which is reproduced in Table 1. Table 2 illustrates the matrix A when the minimum number of questions (five) are answered, and Table 3 presents the results of sequentially answering more questions. As one can see, the weights derived from partial questioning are fairly accurate at about 10–11 responses; i.e. one typically does not need to answer a large number of questions to get fairly accurate weightings.

3. NONLINEAR RESPONSES IN THE AHP

The second extension to the AHP which we shall consider in this paper deals with nonlinear responses to the pairwise comparisons. In the standard theory it is assumed that a_{ij} is an approximation to the ratio w_i/w_j. However, one could have situations in which a_{ij} is an approximation to some function of this ratio $f(w_i/w_j)$. For example, the function $f = (w_i/w_j)^{\alpha}$ has been widely studied in the psychology literature for various values of the power α with respect to differing stimuli; Table 4 lists several of these values. Saaty [3, Theorem 7-28] studied the use of the power function when it is assumed that the responses by the decision maker are *perfectly consistent*. In this section, the requirement of consistency will be dropped and a recent result concerning the Perron–Frobenius theorem will be used to study the general functional form $f(w_i/w_j)$.

Table 1. Distance from Philadelphia example [3, p. 42]

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	1	9	3	3	9
Chicago	1/8	1/9	1	1/6	1/5	2
San Francisco	1/3	1/3	6	1	1/3	6
London	1/3	1/3	5	3	1	6
Montreal	1/7	1/9	1/2	1/6	1/6	1
Distance (miles)	5729	7449	660	2732	3658	400
Relative distance	0.2777	0.3611	0.0320	0.1324	0.1773	0.0194

Table 2. Distance example with five questions answered

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	5	0	0	0	0
Chicago	1/8	0	5	0	0	0
San Francisco	1/3	0	0	5	0	0
London	1/3	0	0	0	5	0
Montreal	1/7	0	0	0	0	5
Eigenvector **w**	0.20265	0.60796	0.02533	0.06755	0.6755	0.02895
$\lambda_{\max}$	6.000					

Table 3. Results of the distance example with incomplete questioning

No. of questions	Eigenvector **w**						
	w_1	w_2	w_3	w_4	w_5	w_6	$\lambda_{\max}$
5	0.20265	0.60796	0.02553	0.06755	0.06755	0.02895	6.0000
6	0.23761	0.53684	0.04345	0.07498	0.07498	0.03214	6.0563
7	0.25471	0.47756	0.04052	0.11649	0.07751	0.03322	6.0954
8	0.26563	0.43977	0.03873	0.11097	0.11097	0.03392	6.1188
9	0.27265	0.42794	0.03848	0.11008	0.11008	0.04077	6.1215
10	0.26919	0.41975	0.03235	0.13368	0.10582	0.03921	6.1701
11	0.26793	0.41690	0.02985	0.12797	0.11860	0.03875	6.1832
12	0.26650	0.41041	0.03553	0.13374	0.12291	0.03092	6.2810
13	0.26331	0.40137	0.03437	0.10901	0.16290	0.02904	6.4223
14	0.26178	0.39729	0.03338	0.11613	0.16502	0.02640	6.4538
15	0.26185	0.39749	0.03343	0.11639	0.16424	0.02660	6.4536

If one assumes that a_{ij} is an approximation to $(w_i/w_j)^\alpha$ for any positive value of α, the eigenvector problem can be written as

$$A\mathbf{w}^\alpha = \lambda_{\max}\mathbf{w}^\alpha, \tag{15}$$

where $\mathbf{w}^\alpha = (w_1^\alpha, w_2^\alpha, \ldots, w_n^\alpha)$. Defining $\mathbf{v}$ to be equal to $\mathbf{w}^\alpha$ one can immediately see that equation (15) becomes our standard eigenvector problem in the AHP which can be dealt with in the standard fashion or by the incomplete comparison method derived in the previous section. Thus, equation (15) can be solved with any matrix A; *consistency is not a requirement for the power law to be applicable.* One need only convert from $\mathbf{v}$ to $\mathbf{w}$ by taking the αth root of each component of the vector $\mathbf{v}$.

The effect of employing a power different from 1 is illustrated in Table 5 which is based on the matrix from Saaty's [3, p. 19] optics example:

$$A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ & 1 & 4 & 6 \\ & & 1 & 4 \\ & & & 1 \end{bmatrix}.$$

As Table 5 illustrates, a value of $\alpha < 1$ tends to amplify the differences between the various alternatives, and $\alpha > 1$ tends to dampen these differences. Thus, one might wish to employ some power of the eigenvector in order to see greater differences or to satisfy a decision maker who tends to favor equality of weights rather than stark contrasts—the "even keel" mentality. There is a great amount of empirical research which should be performed in order to ascertain the appropriate values of α in various decision contexts; e.g. risk assessments, assessment of cost items, assessment of probabilities, etc. Through this line of research one may be able to "tune" the AHP to the various stimuli which a decision maker is likely to face.

One can generalize the functional forms to be considered in the nonlinear AHP by using recent results in functional analysis which deal with a nonlinear extension of the Perron–Frobenius theorem by Kohlberg [8] (which is based upon the earlier work by Kohlberg and Pratt [9]) and

Table 4. Values of power α for various stimuli [3, p. 189]

Stimuli	Power α
Loudness	0.3
Brightness	0.33–0.5
Length	1.1
Duration	1.15
Numerousness	1.34
Heaviness	1.45
Velocity	1.77
Electric shock	4.0

Table 5. Eigenvector values for various powers α

Power α	w_1	w_2	w_3	w_4
0.1	0.999937	0.000063	0.000000	0.000000
0.2	0.991985	0.007898	0.000115	0.000002
0.4	0.907923	0.081015	0.009761	0.001301
0.6	0.792956	0.158341	0.038626	0.010078
0.8	0.694322	0.207404	0.071992	0.026282
1.0	0.618669	0.235323	0.100934	0.045074
1.2	0.561702	0.251002	0.123972	0.063324
1.4	0.518344	0.259875	0.141963	0.079818
1.6	0.484705	0.264915	0.156077	0.094302
1.8	0.458070	0.267740	0.167292	0.106899
2.0	0.436572	0.269252	0.176338	0.117839
5.0	0.321306	0.264825	0.223581	0.190288
10.0	0.284770	0.258532	0.237548	0.219150

Rath [10]. In order to summarize these results, consider a continuous mapping $F: R_+^n \to R_+^n$ which satisfies

homogeneity of degree 1: $F(\eta \mathbf{x}) = \eta F(\mathbf{x})$

primitive: for some integer $l > 0$, $\mathbf{x} \geqslant \mathbf{y}$ implies $F^l(\mathbf{x}) > F^l(\mathbf{y})$, where F^l denotes the lth application of F.

Using these two concepts Kohlberg [8] has shown the following.

Theorem 5

Let $F: R^n_+ \to R^n_+$ be a continuous mapping which is homogeneous of degree 1 and primitive, then:

(a) there exists a vector $\mathbf{x}^0 > 0$ which is unique up to proportionality such that $F(\mathbf{x}^0) = \lambda_0 \mathbf{x}^0$ for some $\lambda_0 > 0$;

(b) $\lim_{k \to \infty} F^l(\mathbf{x})/\|F^l(\mathbf{x})\| = c\mathbf{x}^0, \forall\, \mathbf{x} \geqslant 0$;
i.e. the nonlinear map will converge for any starting point $\mathbf{x}$ and $\|\cdot\|$ is any norm in R^n.

Let us now consider the case where a_{ij} represents an approximation of an arbitrary function $f(w_i/w_j)$ instead of the special case of $(w_i/w_j)^\alpha$ which is discussed above. For example, $f(w_i/w_j)$ could take the form $\exp[\beta(w_i/w_j)]/\exp(\beta)$; i.e. the response of the decision maker is an exponential function of the weights on the alternatives, or in other words, the ratio of the alternatives' weights varies with the logarithm of a_{ij}:

$$w_i/w_j = 1 + (1/\beta)\ln a_{ij}.$$

Which exact functional forms one should employ will not be discussed in this paper; this issue is left for future research.

Given the type of pairwise function $f(w_i/w_j)$ described above, the ith component of the mapping $F: R^n_+ \to R^n_+$ will take the form

$$F_i(\mathbf{w}) = w_i \sum_j a_{ij} [f(w_i/w_j)]^{-1}. \tag{16}$$

In order to illustrate the meaning of this relationship in the context of the AHP, consider $F_i(\mathbf{w}) = \lambda_0 \mathbf{w}$. Remember that a_{ij} is defined to be an approximation to the function $f(w_i/w_j)$. If a_{ij} were exactly equal to $f(w_i/w_j)$, then the sum in equation (16) and hence λ_0 would equal n, the number of alternatives; i.e. perfect consistency is achieved. In the special case of $f(w_i/w_j) = (w_i/w_j)^\alpha$, the eigenvector problem $F(\mathbf{w}) = \lambda\mathbf{w}$ defined by equation (16) would simplify to equation (15). Thus, equation (16) is the natural representation for the use of general functional forms in the AHP.

If $f(\cdot)$ is a continuous, positive-valued function of degree 0, then the function $F(\mathbf{w})$ defined by equation (16) will be continuous and homogeneous of degree 1. The proof that $F(\mathbf{w})$ is primitive for certain classes of functions $f(\cdot)$ is very involved and depends upon the relative values of the a_{ij}s. It suffices to state that with general functional forms $f(w_i/w_j)$, one need only try the iterative scheme in Theorem 5 and if convergence is achieved, one has obtained the Perron vector for this nonlinear map. Further research is necessary to ascertain if there exist any functional forms beyond the power function $(w_i/w_j)^\alpha$ which are primitive and are empirically useful.

In order to illustrate the general nonlinear mapping, define

$$f(w_i/w_j) = \exp[\beta(w_i/w_j)]/\exp(\beta) \tag{17}$$

and consider two alternatives with $a_{11} = a_{22} = 1, a_{12} = 2, a_{21} = 0.4$. Note that a general functional form such as equation (17) need no longer obey the reciprocal property of the AHP and hence, the entire matrix must be completed. In fact, Saaty [3, Theorem 7-28] has shown that a power function $(w_i/w_j)^\alpha$ is the only form of $f(w_i/w_j)$ which retains the reciprocal property of the standard AHP. There exists some ancedotal evidence that decision makers may not always obey strict reciprocity. This fact points to the need for further research in understanding if nonreciprocal judgments are empirically meaningful and if so, which functional forms $f(w_i/w_j)$ best represent these nonreciprocal judgments. For the moment, let us assume that one can derive meaningful functions of the form $F_i(\mathbf{w})$, as in equation (16). By Theorem 5 one knows that an iterative scheme will converge if $F(\mathbf{w})$ is primitive. For example, Table 6 lists the results of the iterative scheme for the nonlinear map (17) for various values of β. As one can see, the primitivity of the map depends upon the relative values of the parameters (β) and that in this case large β tends to smooth the weights and small β tends to accent the differences between the alternatives.

In summary, there exists a method for dealing with situations in which a_{ij} is an approximation of some function of the weights $\mathbf{w}$. It is a very interesting research question to ascertain which functional relationship are usable and meaningful in the context of the AHP.

Table 6. Results of the nonlinear example

β	w_1	w_2
0.2	0.89094	0.10906
0.4	0.80983	0.19017
0.6	0.75088	0.24912
0.8	0.70797	0.29203
1.0	0.67620	0.32380
1.2	0.65213	0.34787
1.4	0.63346	0.36654
1.6	0.61866	0.38134
1.8	0.60668	0.39332
2.0	0.59682	0.40318
2.2	*did not converge*	*imprimitive map*

4. CONCLUSIONS

This paper has presented two extensions of the AHP methodology to deal with incomplete pairwise comparisons and nonlinear ratio scales. These two extensions should both speed up the elicitation process and provide the analyst with greater flexibility in the modeling of the decision maker's responses to the stimuli of comparing decision alternatives. However, several interesting research questions remain. First, what are the appropriate values of α in the power function approach and if this power law is not applicable, what other functional forms $f(w_i/w_j)$ can be employed in the AHP context? Also, the question as to how the techniques discussed in this paper can be extended to deal more efficiently and effectively with the overall hierarchical structure rather than with a single matrix also remains for future research.

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ALTERNATIVE MODES OF QUESTIONING IN THE ANALYTIC HIERARCHY PROCESS

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Abstract—The standard mode of questioning in the Analytic Hierarchy Process (AHP) requires the decision maker to complete a sequence of positive reciprocal matrices by answering $n(n-1)/2$ questions for each matrix, each entry being an approximation to the ratio of the weights of the n items being compared. This paper presents two extensions of the eigenvector approach of the AHP which allows the decision maker to say "I don't know" or "I'm not sure" to some of the questions being asked, and to approximate nonlinear functions of the ratios of the weights. In this way, the questioning process can be substantially shortened and better representations of the responses to certain stimuli may be derived.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP) is a decision-aiding method which has received increasing attention in the literature and in application since its development by Saaty [1]. The reader is referred to the recent review paper by Zahedi [2] for a listing of the current literature on this subject. The basis of the AHP is the completion of an $n \times n$ matrix $A = (a_{ij})$ at each level of the decision hierarchy. This matrix A is of the form $a_{ij} = 1/a_{ji}$, $a_{ij} > 0$; i.e. A is a *positive, reciprocal matrix*. The basic theory, as developed by Saaty [1, 3], is based on the fact that a_{ij} is an approximation to the relative weights (w_i/w_j) of the n alternatives under consideration; the value assigned to a_{ij} is typically in the interval $[1/9, 9]$. Given the $n(n-1)/2$ approximations to these weights which the decision maker supplies when completing the matrix A, the weights $\mathbf{w} = (w_i)$ are found by solving the following eigenvector problem:

$$A\mathbf{w} = \lambda_{\max}\mathbf{w}, \tag{1}$$

where $\lambda_{\max}$ is the Perron root or principal eigenvalue of A. A complete discussion of the reasons for using equation (1) to derive the weights $\mathbf{w}$ can be found in Ref. [4].

This paper presents two extensions of the method described above for the elicitation and computation of weights from a set of pairwise comparisons. First, the completion of $n(n-1)/2$ comparisons at each level of the hierarchy can become an onerous task if n is large. Thus, one would like to find a method in which the decision maker could complete less than $n(n-1)/2$ comparisons but still answer enough comparisons in order to derive a meaningful measure of the alternatives' relative weights. Also, it is often the case that a decision maker, when faced with a particular comparison between alternatives i and j, would rather not answer this comparison directly or may simply not yet have a good understanding of his or her preferences for those two alternatives. The first case arises when the elicitation of a_{ij} calls for the decision maker to publically state a tradeoff between two sensitive criteria; e.g. mortality risk vs cost when comparing a set of measures to lessen hazardous materials risks. It may be easier for the decision maker to skip this question and have the judgment being made indirectly through the other responses he or she has provided. The ability to skip certain direct questions may make the decision maker more willing to participate in a structured decision analysis exercise. The final reason for considering the completion of less than $n(n-1)/2$ judgments stems from the fact that the decision maker may not have formed a strong opinion on a particular question and rather than forcing this individual to make an often wild guess or to have the entire process slowed due to one question, one can simply skip that comparison. The next section will describe a method based upon a theory of *nonnegative, quasi-reciprocal matrices* which can be employed to deal with incomplete comparisons.

The second extension considered in this paper is the ability to deal with nonlinear relative preferences. The standard AHP model assumes that a_{ij} is an approximation to w_i/w_j. However, there is substantial evidence in the psychology literature [5, 6] that individuals often respond in a nonlinear fashion to stimuli; e.g. a_{ij} is an approximation to $(w_i/w_j)^\alpha$, where α is a positive scalar. Using a recent result concerning a nonlinear extension of the Perron–Frobenius theorem, a method is developed to deal with nonlinear ratio estimations in the context of the AHP; this result is presented in Section 3.

Thus, this paper presents two methodological extensions of the AHP which should simplify the elicitation process and allow greater flexibility in the modeling of preference structures with nonlinear reciprocal judgments.

2. A METHOD FOR INCOMPLETE COMPARISONS

As was discussed in the previous section, there are three basic reasons why one would want to consider the completion of less than $n(n-1)/2$ pairwise comparisons in the context of the AHP:

—the time to complete all $n(n-1)/2$ comparisons;
—unwillingness to make a direct comparison between two alternatives;
—being unsure of some of the comparisons.

Harker [7] has presented a method to deal with incomplete comparisons in the context of an iterative scheme for the elicitation of the matrix A which is based upon the approximation of the missing elements of A with the data available from the completed comparisons. This approximation of a_{ij} is formed by taking the geometric mean of the intensity of all paths in the directed graph $D(A)$ associated with the partially completed matrix A which connect the alternatives i and j. This approximation scheme in some sense mimics what the decision makers would have to perform if he or she were forced to complete a given comparison.

In this section a more natural approach to dealing with the missing entries a_{ij} will be considered. Instead of approximating the missing entry a_{ij}, which is itself an approximation to the ratio w_i/w_j, let us simply set a_{ij} to be equal to w_i/w_j. In other words, let us complete the missing entries by setting them equal to the value which they seek to approximate. Of course, one does not know *a priori* the value w_i/w_j. The purpose of this section is to derive the necessary theory to deal with the situation in which some a_{ij}'s take on the functional form w_i/w_j instead of as numerical value.

In order to begin, let us reiterate a well-known result in linear algebra and graph theory (see, for example, Ref. [3, Theorem 7.1] for a proof).

Definition 1

A square matrix A is *irreducible* if it cannot be decomposed into the form

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix},$$

where A_1 and A_3 are square matrices and 0 is the zero matrix.

Theorem 1

An $n \times n$ matrix A is irreducible iff its directed graph $D(A)$ is strongly connected.

Therefore, a matrix A is irreducible iff there exists a path between every ordered pair of nodes in the graph of A. In the AHP context, this states that there must exist a direct or indirect comparison between every pair of alternatives under consideration. Given that one always has $a_{ji} = 1/a_{ij}$ when $a_{ij} > 0$ in the AHP, completing the top row of the matrix A will be sufficient to guarantee that A is irreducible.

The following definition will be necessary in what follows.

Definition 2

An $n \times n$ matrix A is called a *nonnegative, quasi-reciprocal matrix* if

$$a_{ij} \geqslant 0 \quad \text{and} \quad a_{ij} > 0 \qquad \text{implies } a_{ji} = 1/a_{ij}, \qquad \forall\, i, j = 1, 2, \ldots, n.$$

Note that all positive, reciprocal matrices are quasi-reciprocal but that the class of quasi-reciprocal matrices allows for zero entries.

Let us now assume that the decision maker has considered a set of n alternatives and has completed some subset of the $n(n-1)/2$ pairwise comparisons to form a matrix $C = (c_{ij})$. For those questions which the decision maker responded, one as $c_{ij} > 0$, $c_{ji} = 1/c_{ij}$. Let us assume that the completed questions form an irreducible matrix. From the above discussion on positive reciprocal matrices and the graph theoretic interpretation of the matrix, it is clear that the completion of the top row of questions is sufficient to make the matrix C irreducible. By definition one has $c_{ii} = 1$, $\forall i = 1, 2, \ldots, n$. For those questions which were not answered, let $c_{ij} = w_i/w_j$. For example, in comparing three alternatives one may have:

$$C = \begin{bmatrix} 1 & 2 & w_1/w_3 \\ 1/2 & 1 & 2 \\ w_3/w_1 & 1/2 & 1 \end{bmatrix}. \tag{2}$$

Computing $C\mathbf{w}$ one obtains the vector $(2w_1 + 2w_2, 1/2w_1 + w_2 + 2w_3, 1/2w_2 + 2w_3)$, which defines a new matrix A where $A\mathbf{w} = C\mathbf{w}$:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{bmatrix}. \tag{3}$$

Thus, the problem of computing the right principal eigenvector $\mathbf{w}$ for the matrix C which contains the functional relations becomes that of computing $\mathbf{w}$ for the nonnegative, quasi-reciprocal matrix A. Thus, the issue of dealing with incomplete pairwise comparisons becomes that of studying the properties of nonnegative, quasi-reciprocal matrices. Let us now formalize this issue.

Let $B = (b_{ij})$ be an $n \times n$ matrix formed from the partially completed matrix C as follows:

$$\begin{aligned} b_{ij} &= c_{ij} \quad \text{if } c_{ij} \text{ is a real number } > 0 \\ &= 0 \quad \text{otherwise} \\ b_{ii} &= m_i, \quad \text{the number of unanswered questions in row } i = 1, 2, \ldots, n. \end{aligned}$$

In our example, B is given by

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1/2 & 1 \end{bmatrix}. \tag{4}$$

and the matrix $(I + B)$ equals A in equation (3). By assumption we have that B will be an irreducible matrix. Defining $A = (I + B)$ to be the nonnegative, quasi-reciprocal matrix formed from the partial pairwise comparisons which will obviously also be irreducible, one has from the Perron–Frobenius theorem that $\lambda_{\max}$ will be a real positive and simple eigenvalue which is not exceeded in modulus by any other eigenvalue of A. Furthermore, the following results are known.

Theorem 2 [3, Theorem 8-5]

If B is a nonnegative irreducible matrix of order n we have $(I + B)^{n-1} > 0$; i.e. $A = (I + B)$ is a primitive matrix.

Theorem 3 [3, Theorem 7-13]

For a primitive matrix A

$$\lim_{k \to \infty} \frac{A^k \mathbf{e}}{\mathbf{e}TA^k\mathbf{e}} = \mathbf{c}\mathbf{w}, \tag{5}$$

where $\mathbf{e}$ is the unit vector, c is a constant and $\mathbf{w}$ is the principal eigenvector of A.

Therefore, the matrix B formed from the partial comparison information does lead to a primitive matrix A, and the same convergence result (5) as in the case of positive, reciprocal matrices holds. Equation (5) thus becomes the means by which one computes $\mathbf{w}$, just as in the current theory of the AHP. For example, the matrix B in equation (4) leads to the following matrix A:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1/2 & 1 \end{bmatrix}. \tag{6}$$

The limiting value of iterations (5) leads to the Perron eigenvector $\mathbf{w} = (4/7, 2/7, 1/7)$ and root $\lambda_{\max} = 3$.

Therefore, nonnegative, quasi-reciprocal matrices can be used in exactly the same manner as positive, reciprocal matrices. The only question remaining is the relationship between $\lambda_{\max}$ and n for this class of matrices. For positive, reciprocal matrices it is known that $\lambda_{\max} \geqslant n$ and that $\lambda_{\max} = n$ iff the matrix A is consistent; i.e. $a_{ij}a_{jk} = a_{ik}$, $\forall\, i,j,k$. The following theorem establishes this same result for quasi-reciprocal matrices.

Theorem 4

Let A be a nonnegative, irreducible, quasi-reciprocal matrix. Then the Perron root of A, $\lambda_{\max}$, is $\geqslant n$, the rank of A, and $\lambda_{\max} = n$ iff A is consistent; i.e. $a_{ij}a_{kj} = a_{ik}$, $\forall\, i, j, k$, with a_{ij}, a_{jk}, a_{ik} positive.

Proof. It is well-known that the trace of A, $\operatorname{tr}(A)$, equals the sum of the eigenvalues of A:

$$\operatorname{tr}(A) = n + \sum_i m_i = \sum_i \lambda_i, \tag{7}$$

where the summations are over $i = 1, 2, \ldots, n$. Given that A is irreducible, we know from the Perron–Frobenius theorem that $w_i > 0$, $\forall\, i = 1, 2, \ldots, n$, and hence one can divide the ith row of $A\mathbf{w} = \lambda_{\max}\mathbf{w}$ to form

$$\lambda_{\max} = \sum_j a_{ij} w_j / w_i \tag{8}$$

$$= (1 + m_i) + \sum_{j \neq i} a_{ij} w_j / w_i. \tag{9}$$

Summing equation (9) over $i = 1, 2, \ldots, n$, yields

$$n\lambda_{\max} = n + \sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} [a_{ij}(w_j/w_i) + a_{ji}(w_i/w_j)]. \tag{10}$$

Defining $\mu = (\lambda_{\max} - n)/(n - 1)$ and placing equation (10) into this definition, one obtains

$$\mu = -1 + [n(n-1)]^{-1}\left[\sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} a_{ij}(w_j/w_i) + a_{ji}(w_i/w_j)\right]. \tag{11}$$

Defining $a_{ij} = (w_i/w_j)(1 + \delta_{ij})$, $\delta_{ij} > -1$ if $a_{ij} > 0$, zero otherwise, and placing this into equation (11) yields, after some manipulation:

$$\mu = -1 + [n(n-1)]^{-1}\left[\sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} 2 + (\delta_{ij})^2/(1 + \delta_{ij})\right] \tag{12}$$

$$= -1 + n(n-1)[n(n-1)]^{-1} + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} (\delta_{ij})^2/(1 + \delta_{ij}) \tag{13}$$

$$= \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} (\delta_{ij})^2/(1 + \delta_{ij}). \tag{14}$$

Since $\delta_{ij} > -1$, one has $\mu \geqslant 0$ and $\mu = 0$ ($\lambda_{\max} = n$) iff $\delta_{ij} = 0$, $\forall\, i, j$, with nonzero a_{ij}. Thus, $\mu = 0$ iff A is consistent. Q.E.D.

Therefore, the theory developed by Saaty [1, 3] for positive reciprocal matrices follows completely when one considers incomplete pairwise comparisons in the context of quasi-reciprocal matrices.

In order to illustrate the workings of this partial comparison method, consider the distance to Philadelphia example from Ref. [3] which is reproduced in Table 1. Table 2 illustrates the matrix A when the minimum number of questions (five) are answered, and Table 3 presents the results of sequentially answering more questions. As one can see, the weights derived from partial questioning are fairly accurate at about 10–11 responses; i.e. one typically does not need to answer a large number of questions to get fairly accurate weightings.

3. NONLINEAR RESPONSES IN THE AHP

The second extension to the AHP which we shall consider in this paper deals with nonlinear responses to the pairwise comparisons. In the standard theory it is assumed that a_{ij} is an approximation to the ratio w_i/w_j. However, one could have situations in which a_{ij} is an approximation to some function of this ratio $f(w_i/w_j)$. For example, the function $f = (w_i/w_j)^\alpha$ has been widely studied in the psychology literature for various values of the power α with respect to differing stimuli; Table 4 lists several of these values. Saaty [3, Theorem 7-28] studied the use of the power function when it is assumed that the responses by the decision maker are *perfectly consistent*. In this section, the requirement of consistency will be dropped and a recent result concerning the Perron–Frobenius theorem will be used to study the general functional form $f(w_i/w_j)$.

Table 1. Distance from Philadelphia example [3, p. 42]

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	1	9	3	3	9
Chicago	1/8	1/9	1	1/6	1/5	2
San Francisco	1/3	1/3	6	1	1/3	6
London	1/3	1/3	5	3	1	6
Montreal	1/7	1/9	1/2	1/6	1/6	1
Distance (miles)	5729	7449	660	2732	3658	400
Relative distance	0.2777	0.3611	0.0320	0.1324	0.1773	0.0194

Table 2. Distance example with five questions answered

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	5	0	0	0	0
Chicago	1/8	0	5	0	0	0
San Francisco	1/3	0	0	5	0	0
London	1/3	0	0	0	5	0
Montreal	1/7	0	0	0	0	5
Eigenvector $\mathbf{w}$	0.20265	0.60796	0.02533	0.06755	0.6755	0.02895
$\lambda_{\max}$	6.000					

Table 3. Results of the distance example with incomplete questioning

No. of questions	Eigenvector $\mathbf{w}$						
	w_1	w_2	w_3	w_4	w_5	w_6	$\lambda_{\max}$
5	0.20265	0.60796	0.02553	0.06755	0.06755	0.02895	6.0000
6	0.23761	0.53684	0.04345	0.07498	0.07498	0.03214	6.0563
7	0.25471	0.47756	0.04052	0.11649	0.07751	0.03322	6.0954
8	0.26563	0.43977	0.03873	0.11097	0.11097	0.03392	6.1188
9	0.27265	0.42794	0.03848	0.11008	0.11008	0.04077	6.1215
10	0.26919	0.41975	0.03235	0.13368	0.10582	0.03921	6.1701
11	0.26793	0.41690	0.02985	0.12797	0.11860	0.03875	6.1832
12	0.26650	0.41041	0.03553	0.13374	0.12291	0.03092	6.2810
13	0.26331	0.40137	0.03437	0.10901	0.16290	0.02904	6.4223
14	0.26178	0.39729	0.03338	0.11613	0.16502	0.02640	6.4538
15	0.26185	0.39749	0.03343	0.11639	0.16424	0.02660	6.4536

If one assumes that a_{ij} is an approximation to $(w_i/w_j)^\alpha$ for any positive value of α, the eigenvector problem can be written as

$$A\mathbf{w}^\alpha = \lambda_{\max}\mathbf{w}^\alpha, \tag{15}$$

where $\mathbf{w}^\alpha = (w_1^\alpha, w_2^\alpha, \ldots, w_n^\alpha)$. Defining $\mathbf{v}$ to be equal to $\mathbf{w}^\alpha$ one can immediately see that equation (15) becomes our standard eigenvector problem in the AHP which can be dealt with in the standard fashion or by the incomplete comparison method derived in the previous section. Thus, equation (15) can be solved with any matrix A; *consistency is not a requirement for the power law to be applicable.* One need only convert from $\mathbf{v}$ to $\mathbf{w}$ by taking the αth root of each component of the vector $\mathbf{v}$.

The effect of employing a power different from 1 is illustrated in Table 5 which is based on the matrix from Saaty's [3, p. 19] optics example:

$$A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ & 1 & 4 & 6 \\ & & 1 & 4 \\ & & & 1 \end{bmatrix}.$$

As Table 5 illustrates, a value of $\alpha < 1$ tends to amplify the differences between the various alternatives, and $\alpha > 1$ tends to dampen these differences. Thus, one might wish to employ some power of the eigenvector in order to see greater differences or to satisfy a decision maker who tends to favor equality of weights rather than stark contrasts—the "even keel" mentality. There is a great amount of empirical research which should be performed in order to ascertain the appropriate values of α in various decision contexts; e.g. risk assessments, assessment of cost items, assessment of probabilities, etc. Through this line of research one may be able to "tune" the AHP to the various stimuli which a decision maker is likely to face.

One can generalize the functional forms to be considered in the nonlinear AHP by using recent results in functional analysis which deal with a nonlinear extension of the Perron–Frobenius theorem by Kohlberg [8] (which is based upon the earlier work by Kohlberg and Pratt [9]) and Rath [10]. In order to summarize these results, consider a continuous mapping $F: R^n_+ \to R^n_+$ which satisfies

Table 4. Values of power α for various stimuli [3, p. 189]

Stimuli	Power α
Loudness	0.3
Brightness	0.33–0.5
Length	1.1
Duration	1.15
Numerousness	1.34
Heaviness	1.45
Velocity	1.77
Electric shock	4.0

Table 5. Eigenvector values for various powers α

Power α	w_1	w_2	w_3	w_4
0.1	0.999937	0.000063	0.000000	0.000000
0.2	0.991985	0.007898	0.000115	0.000002
0.4	0.907923	0.081015	0.009761	0.001301
0.6	0.792956	0.158341	0.038626	0.010078
0.8	0.694322	0.207404	0.071992	0.026282
1.0	0.618669	0.235323	0.100934	0.045074
1.2	0.561702	0.251002	0.123972	0.063324
1.4	0.518344	0.259875	0.141963	0.079818
1.6	0.484705	0.264915	0.156077	0.094302
1.8	0.458070	0.267740	0.167292	0.106899
2.0	0.436572	0.269252	0.176338	0.117839
5.0	0.321306	0.264825	0.223581	0.190288
10.0	0.284770	0.258532	0.237548	0.219150

homogeneity of degree 1: $F(\eta\mathbf{x}) = \eta F(\mathbf{x})$

primitive: for some integer $l > 0$, $\mathbf{x} \geqslant \mathbf{y}$ implies $F^l(\mathbf{x}) > F^l(\mathbf{y})$, where F^l denotes the lth application of F.

Using these two concepts Kohlberg [8] has shown the following.

Theorem 5

Let $F: R^n_+ \to R^n_+$ be a continuous mapping which is homogeneous of degree 1 and primitive, then:

(a) there exists a vector $\mathbf{x}^0 > 0$ which is unique up to proportionality such that $F(\mathbf{x}^0) = \lambda_0 \mathbf{x}^0$ for some $\lambda_0 > 0$;

(b) $\lim_{k \to \infty} F^l(\mathbf{x})/\|F^l(\mathbf{x})\| = c\mathbf{x}^0, \forall\, \mathbf{x} \geqslant 0$;
i.e. the nonlinear map will converge for any starting point $\mathbf{x}$ and $\|\cdot\|$ is any norm in R^n.

Let us now consider the case where a_{ij} represents an approximation of an arbitrary function $f(w_i/w_j)$ instead of the special case of $(w_i/w_j)^\alpha$ which is discussed above. For example, $f(w_i/w_j)$ could take the form $\exp[\beta(w_i/w_j)]/\exp(\beta)$; i.e. the response of the decision maker is an exponential function of the weights on the alternatives, or in other words, the ratio of the alternatives' weights varies with the logarithm of a_{ij}:

$$w_i/w_j = 1 + (1/\beta)\ln a_{ij}.$$

Which exact functional forms one should employ will not be discussed in this paper; this issue is left for future research.

Given the type of pairwise function $f(w_i/w_j)$ described above, the ith component of the mapping $F: R^n_+ \to R^n_+$ will take the form

$$F_i(\mathbf{w}) = w_i \sum_j a_{ij} [f(w_i/w_j)]^{-1}. \tag{16}$$

In order to illustrate the meaning of this relationship in the context of the AHP, consider $F_i(\mathbf{w}) = \lambda_0 \mathbf{w}$. Remember that a_{ij} is defined to be an approximation to the function $f(w_i/w_j)$. If a_{ij} were exactly equal to $f(w_i/w_j)$, then the sum in equation (16) and hence λ_0 would equal n, the number of alternatives; i.e. perfect consistency is achieved. In the special case of $f(w_i/w_j) = (w_i/w_j)^\alpha$, the eigenvector problem $F(\mathbf{w}) = \lambda \mathbf{w}$ defined by equation (16) would simplify to equation (15). Thus, equation (16) is the natural representation for the use of general functional forms in the AHP.

If $f(\cdot)$ is a continuous, positive-valued function of degree 0, then the function $F(\mathbf{w})$ defined by equation (16) will be continuous and homogeneous of degree 1. The proof that $F(\mathbf{w})$ is primitive for certain classes of functions $f(\cdot)$ is very involved and depends upon the relative values of the a_{ij}s. It suffices to state that with general functional forms $f(w_i/w_j)$, one need only try the iterative scheme in Theorem 5 and if convergence is achieved, one has obtained the Perron vector for this nonlinear map. Further research is necessary to ascertain if there exist any functional forms beyond the power function $(w_i/w_j)^\alpha$ which are primitive and are empirically useful.

In order to illustrate the general nonlinear mapping, define

$$f(w_i/w_j) = \exp[\beta(w_i/w_j)]/\exp(\beta) \tag{17}$$

and consider two alternatives with $a_{11} = a_{22} = 1, a_{12} = 2, a_{21} = 0.4$. Note that a general functional form such as equation (17) need no longer obey the reciprocal property of the AHP and hence, the entire matrix must be completed. In fact, Saaty [3, Theorem 7-28] has shown that a power function $(w_i/w_j)^\alpha$ is the only form of $f(w_i/w_j)$ which retains the reciprocal property of the standard AHP. There exists some ancedotal evidence that decision makers may not always obey strict reciprocity. This fact points to the need for further research in understanding if nonreciprocal judgments are empirically meaningful and if so, which functional forms $f(w_i/w_j)$ best represent these nonreciprocal judgments. For the moment, let us assume that one can derive meaningful functions of the form $F_i(\mathbf{w})$, as in equation (16). By Theorem 5 one knows that an iterative scheme will converge if $F(\mathbf{w})$ is primitive. For example, Table 6 lists the results of the iterative scheme for the nonlinear map (17) for various values of β. As one can see, the primitivity of the map depends upon the relative values of the parameters (β) and that in this case large β tends to smooth the weights and small β tends to accent the differences between the alternatives.

In summary, there exists a method for dealing with situations in which a_{ij} is an approximation of some function of the weights $\mathbf{w}$. It is a very interesting research question to ascertain which functional relationship are usable and meaningful in the context of the AHP.

Table 6. Results of the nonlinear example

β	w_1	w_2
0.2	0.89094	0.10906
0.4	0.80983	0.19017
0.6	0.75088	0.24912
0.8	0.70797	0.29203
1.0	0.67620	0.32380
1.2	0.65213	0.34787
1.4	0.63346	0.36654
1.6	0.61866	0.38134
1.8	0.60668	0.39332
2.0	0.59682	0.40318
2.2	*did not converge*	*imprimitive map*

4. CONCLUSIONS

This paper has presented two extensions of the AHP methodology to deal with incomplete pairwise comparisons and nonlinear ratio scales. These two extensions should both speed up the elicitation process and provide the analyst with greater flexibility in the modeling of the decision maker's responses to the stimuli of comparing decision alternatives. However, several interesting research questions remain. First, what are the appropriate values of α in the power function approach and if this power law is not applicable, what other functional forms $f(w_i/w_j)$ can be employed in the AHP context? Also, the question as to how the techniques discussed in this paper can be extended to deal more efficiently and effectively with the overall hierarchical structure rather than with a single matrix also remains for future research.

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Inconsistency indices for pairwise comparison matrices: a numerical study

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Abstract Evaluating the level of inconsistency of pairwise comparisons is often a crucial step in multi criteria decision analysis. Several inconsistency indices have been proposed in the literature to estimate the deviation of expert's judgments from a situation of full consistency. This paper surveys and analyzes ten indices from the numerical point of view. Specifically, we investigate degrees of agreement between them to check how similar they are. Results show a wide range of behaviors, ranging from very strong to very weak degrees of agreement.

Keywords Pairwise comparison matrices · Consistency · Analytic hierarchy process

1 Introduction

In the field of studies on the human behavior and decision processes, a large amount of papers investigate consistency in judgments and evaluations (for a recent review, see Karelaia and Hogarth 2008). The measurement of the quality of expertise of the decision makers has been a matter of much debate, especially in domains lacking external criteria which enable verification (for a discussion, see Shanteau et al. 2003). Consensus exists

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among some traits of experts' judgments including selectivity in information search/use (Phelps and Shanteau 1978; Shanteau 1989; Slovic 1969), discrimination among different stimuli (Weiss and Shanteau 2003), and intra-individual consistency (Weiss et al. 2006; Karelaia and Hogarth 2008). Unfortunately, there is evidence that many professional judgments do not fulfill these requirements and consequently their stated judgments are not consistent.

In decision making processes, when an expert has to grade alternatives subjectively, in order to form a rating, he/she is often asked to express his/her preferences by means of cardinal preference relations. In this framework, the decision maker is then asked to pairwise compare alternatives and, for each pair, to express his degree of preference of the first alternative over the second, or vice versa. This approach was formally introduced by Thurstone (1927) and its importance was highlighted by Koczkodaj (1993). Although there is not a unique way for representing subjective preferences under the form of pairwise comparisons over pairs, it is possible to define a consistency condition such that, if it holds, the decision maker is considered coherent and his/her judgments are not contradictory. Namely, a decision maker is consistent if his/her opinions respect some cardinal transitivity conditions of preferences on triplets of alternatives. Nevertheless, it is well known that in making paired comparisons, people do not have the intrinsic logical ability to always be consistent (Saaty 1994). Despite some contrary opinions, e.g. Linares (2009), consistency has always been regarded as a desirable property. Consequently, for each type of representation of preferences, there is a meeting of minds on the right definition of consistency, whereas there is a rather large number of proposals to evaluate the amount of inconsistency contained in the preferences. The firstly introduced and most popular index for measuring inconsistency is the Consistency Index, defined by Saaty (1977, 1980) in the Analytic Hierarchy Process (AHP). Originally introduced as a tool for estimating degrees of membership in fuzzy sets (Saaty 1974), the AHP is a method for decisions which draws heavily on the theory of preference relations. Within the framework of the AHP, consistency evaluation has probably been one of the most debated issues. Since there is no consensually accepted method for measuring inconsistency, every index proposed in the literature is itself a different definition of inconsistency degree. Since there is not a satisfactory comparative study in the literature, in this paper we numerically investigate the agreement between different indices in evaluating inconsistency of cardinal preference relations.

Note that the relevance of inconsistency indices goes beyond the quantification of inconsistency as they are also often used as methods to complete pairwise comparison matrices with missing elements.

This paper is outlined as follows. In Sect. 2 we introduce pairwise comparison matrices and the concepts underlying the idea of inconsistency, meant as a deviation from a situation of perfect consistency. In Sect. 3 we recall the definitions of the ten inconsistency indices that we are going to analyze numerically. Section 4 describes the numerical study and exposes the results. In Sect. 5, we comment and discuss the findings of Sect. 4.

2 Pairwise comparison matrices and their inconsistency

Given a non-empty finite set of alternatives $X = \{x_1, \dots, x_n\}$, a pairwise comparison matrix (*PCM*) is a positive real valued matrix $\mathbf{A} = (a_{ij})_{n\times n}$ with (i) $a_{ii} = 1$ $\forall i$ and (ii) $a_{ij}a_{ji} = 1$ $\forall i, j$. Despite this definition, in his papers on the AHP, Saaty (1977) claimed that, for psychological and behavioral reasons (Miller 1956), decision makers should use the bounded discrete scale $1/9, 1/8, \dots, 1/2, 1, 2, \dots, 8, 9$ instead of the set of all positive real numbers.

We are going to use this restrictive approach also because it allows us to generate random matrices. A *PCM* is said to be fully consistent if and only if the following transitivity condition holds

$$a_{ik} = a_{ij}a_{jk} \quad \forall i, j, k. \tag{1}$$

Moreover, if and only if $\mathbf{A}$ is consistent, then there exists a vector $\mathbf{w} = (w_1, \dots, w_n)$ such that

$$a_{ij} = \frac{w_i}{w_j} \quad \forall i, j. \tag{2}$$

If $\mathbf{A}$ is consistent, then vector $\mathbf{w}$ can be obtained by using the geometric mean method

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}} \quad \forall i. \tag{3}$$

It is clear that a definition of consistency has been given. However, the same cannot be said for inconsistency, which is roughly seen as a deviation from the condition of full consistency. A number of indices have been proposed in the literature to quantify the extent of this deviation and they will be presented in the next section. Let us just recall that, in general, an inconsistency index is a function

$$I : \mathcal{A} \to \mathbb{R}$$

where $\mathcal{A}$ is the set of all the *PCM*s, i.e. $\mathcal{A} = \{\mathbf{A} = (a_{ij})_{n\times n} | a_{ij} > 0,\ a_{ij}a_{ji} = 1\ \forall i, j,\ n > 2\}$. In words, an inconsistency index is a function which associates *PCM*s with real numbers, where the real numbers are quantifications of inconsistency. Consequently, different indices are not alternative ways for evaluating a well-defined quantity. Instead, different indices represent different definitions of the degree of inconsistency of pairwise comparison matrices $\mathbf{A} \in \mathcal{A}$.

3 Inconsistency indices

In this study we consider ten inconsistency indices, but there exist alternative approaches. For instance, Salo (1993) and Salo and Hämäläinen (1995) considered that ambiguity of judgments goes arm-in-arm with their inconsistency and therefore introduced, and then pushed forward, an ambiguity index which can also be used as an estimation of inconsistency. Osei-Bryson (2006) proposed an interpretable parametric optimization problem where the value of the objective function can be used as an inconsistency index. Unlike other inconsistency indices, the index proposed by Osei-Bryson (2006) allows the decision maker to establish a priori some acceptability thresholds for the inconsistencies associated with single entries of the *PCM*.

Although we shall keep the description of the indices self-contained—as their full illustration would be beyond the scope of this paper—we skip the details and invite the reader to refer to the original works. We only preliminarily note that some indices were originally called consistency indices whereas some others were called inconsistency indices. Here we maintain the original names but we clarify that, in spite of these terminological difficulties, the nature of the indices is the same, i.e. the greater the value of the index, the greater the inconsistency of the *PCM*.

3.1 CI and CR

According to the result that given a *PCM* $\mathbf{A}$, its maximum eigenvalue, $\lambda_{\max}$, is equal to n if and only if the matrix is consistent (and greater than n otherwise), Saaty (1977) proposed a consistency index

$$CI = \frac{\lambda_{\max} - n}{n-1}. \tag{4}$$

However, empirical analyses confirmed that the expected value of *CI* of a random matrix of size $n+1$ is, in average, greater than the expected value of *CI* of a random matrix of order n. Consequently, *CI* is not reliable in comparing matrices of different size. Therefore, it needs to be rescaled.

CR, which stands for *Consistency Ratio*, is the standardized version of *CI*. Given a matrix of order n, *CR* can be obtained dividing *CI* by a real number *RI* (*Random Index*) which is nothing else but the average *CI* obtained from a large enough set of randomly generated matrices of size n. Hence,

$$CR = \frac{CI}{RI}. \tag{5}$$

3.2 Index of determinants and c_3

This index (Pelàez and Lamata 2003) is mainly based on the following property of a *PCM* of order three. Expanding the determinant of a 3×3 real matrix one obtains

$$\det(\mathbf{A}) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2.$$

If the matrix is not consistent, then $\det(\mathbf{A}) > 0$, because $\frac{a}{b} + \frac{b}{a} - 2 > 0\ \forall a \neq b,\ a, b > 0$.

It is possible to generalize this result for matrices of order greater than three and define an inconsistency index as the arithmetic mean of the determinants of all the possible submatrices $\mathbf{T}_{ijk}$ of a given *PCM*, constructed in a way so that they respect the following formulation

$$\mathbf{T}_{ijk} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ a_{ji} & 1 & a_{jk} \\ a_{ki} & a_{kj} & 1 \end{pmatrix}, \quad \forall i < j < k.$$

The number of so constructed submatrices is $\binom{n}{3} = \frac{n!}{3!(n-3)!}$. The result is a standardized index and its value is the average inconsistency computed for all the submatrices $\mathbf{T}_{ijk}$ $(i < j < k)$

$$CI^* = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right) \Big/ \binom{n}{3}. \tag{6}$$

The coefficient c_3 of the characteristic polynomial of a *PCM* was also proposed to act as an inconsistency index by Shiraishi and Obata (2002) and Shiraishi et al. (1998, 1999). In fact, by definition, the characteristic polynomial has the following form

$$P_{\mathbf{A}}(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n,$$

with $c_1, \ldots, c_n$ that are real numbers and λ the unknown. Shiraishi et al. (1998) proved that, if $c_3 < 0$, then the matrix at issue cannot be fully consistent. In fact, this is evident if one reckons that—in light of the Perron-Frobenius theorem—the only possible formulation of the characteristic polynomial which yields $\lambda_{\max} = n$, is

$$P_{\mathbf{A}}(\lambda) = \lambda^{n-1}(\lambda - n). \tag{7}$$

Thus, the presence of c_3 is certainly a symptom of inconsistency. Moreover, they also proved that c_3 has the following analytic expression

$$c_3 = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}}\right). \tag{8}$$

Since it was showed that indices CI^* (6) and c_3 (8) are proportional (Brunelli et al. 2013), then they are equivalent, and, to avoid redundancy, in the following we will consider them as a unique index.

3.3 Squared differences index

The definition of this index (Chu et al. 1979) is based on characterization (2) and it assumes that each deviation from the desirable situation should be considered a symptom of inconsistency. Thus, the sum of the squares of the deviations $(a_{ij} - w_i/w_j)$ $\forall i \neq j$ is here considered a fair and global quantification of inconsistency

$$LS = \min_{w_1,\dots,w_n} \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \left(a_{ij} - \frac{w_i}{w_j}\right)^2 \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, \quad w_i > 0. \tag{9}$$

Index LS, which stands for least squares, is also easy to be standardized since the number of non-diagonal terms in the sum, as noted above, is $n(n-1)$. Let us note that the argument minimizing (9) is the priority vector $\mathbf{w}^* = (w_1^*, \dots, w_n^*)$ associated with the pairwise comparison matrix $\mathbf{W}^* = (w_i^*/w_j^*)_{n\times n}$ which minimizes the Frobenius norm $\|\mathbf{A} - \mathbf{W}\|_2$ with $\mathbf{W} = (w_i/w_j)_{n\times n}$. Despite the elegant formulation, optimization problem (9) is tough to be solved numerically, multiple solutions can exist and no exact method or analytic solution has been found to solve it. Bózoki (2008) built an equation system whose roots yield to the optimal components of $\mathbf{w}$. However, even this method, suffers of a huge computational complexity. In order to overcome this problem, Anholcer et al. (2011) proposed some simplifications which are based on some uncertain assumptions.

3.4 Geometric consistency index

This index had been implicitly introduced by Crawford and Williams (1985), then it was reexamined by Aguaròn and Moreno-Jimènez (2003). It considers the priority vector to be estimated by means of the geometric mean method (3). With the estimated weights it is possible to build a local estimator of inconsistency,

$$e_{ij} = a_{ij}\frac{w_j}{w_i}, \quad i, j = 1, \dots, n. \tag{10}$$

For consistent matrices the value of e_{ij} is equal to 1 because it is the result of a multiplication of an entry times its reciprocal. Therefore, since $a_{ij} = \frac{w_i}{w_j} \Rightarrow \ln e_{ij} = 0$, it is then possible to define a global inconsistency index, i.e. the Geometric Consistency Index (GCI), that is

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln^2 e_{ij}. \tag{11}$$

Furthermore, Brunelli et al. (2013) proved that GCI is linearly proportional to the following quantity

$$\rho = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} (\log_9 a_{ik} a_{kj} a_{ji})^2, \tag{12}$$

which means that such index can be expressed as a measure of deviation from either (1) or (2).

3.5 Harmonic consistency index

If and only if $\mathbf{A}$ is a consistent *PCM*, then its columns are proportional and $\text{rank}(\mathbf{A}) = 1$. Therefore, it is fair to suppose that the less proportional are the columns, the less consistent is the matrix. An index of inconsistency based on proportionality between columns was then proposed by Stein and Mizzi (2007) and based on this sufficient condition of consistency. Given a matrix $\mathbf{A}$, the authors proposed to construct an auxiliary vector $\mathbf{s} = (s_1, \ldots, s_n)$ with $s_j = \sum_{i=1}^n a_{ij} \ \forall j$. It was proven that $\sum_{j=1}^n s_j^{-1} = 1$ if and only if $\mathbf{A}$ is consistent, and smaller than 1 otherwise. The harmonic mean of the components of vector $\mathbf{s}$ is then the result of the following

$$HM = \frac{n}{\sum_{j=1}^n \frac{1}{s_j}}. \tag{13}$$

HM itself could be an index of inconsistency, but the authors, according to computational experiments, proposed a normalization in order to align its behavior with that of *CI*. The Harmonic Consistency Index is then

$$HCI = \frac{(HM - n)(n+1)}{n(n-1)}. \tag{14}$$

3.6 Cavallo-D'Apuzzo

Besides proposing a general framework based on abelian linearly ordered groups for some representations of cardinal preferences Cavallo and D'Apuzzo (2009, 2010) introduced an approach based on some new metrics and the following inconsistency index

$$I_{CD} = \prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^{n} \left(\max\left\{ \frac{a_{ik}}{a_{ij} a_{jk}}, \frac{a_{ij} a_{jk}}{a_{ik}} \right\} \right)^{\frac{1}{\binom{n}{3}}}. \tag{15}$$

Remarkably, thanks to the fact that the justification of this index has been grounded on group theory, the authors equivalently formulated it for other well-known types of preference relations.

3.7 Relative error

The relative error index was formulated by Barzilai (1998) and requires the construction of an auxiliary matrix $\mathbf{A}^+ = (a_{ij}^+)_{n \times n} = (\log_2 a_{ij})_{n \times n}$ which is skew symmetric and represents an 'additive' *PCM* and to derive a weight vector $\mathbf{w}^+ = (w_1^+, \ldots, w_n^+)$ with $w_i^+ = \frac{1}{n} \sum_{j=1}^n a_{ij}^+$. Having done this, the consistent part of $\mathbf{A}^+$ is obtained as $\mathbf{C} = (c_{ij})_{n \times n} = (w_i^+ - w_j^+)_{n \times n}$. Another matrix, $\mathbf{E} = (e_{ij})_{n \times n} = (a_{ij} - c_{ij})_{n \times n}$, is also obtained to represent the inconsistent part of $\mathbf{A}$, such that $\mathbf{C} + \mathbf{E} = \mathbf{A}$. At this point the relative error index is derived as

$$RE = \frac{\sum_{i=1}^n \sum_{j=1}^n e_{ij}^2}{\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2}. \tag{16}$$

3.8 Koczkodaj index

Koczkodaj (1993) proposed to estimate the inconsistency of a *PCM* of order three as

$$K_{ijk} = \min\left\{\frac{1}{a_{ij}}\left|a_{ij} - \frac{a_{ik}}{a_{jk}}\right|, \frac{1}{a_{ik}}|a_{ik} - a_{ij}a_{jk}|, \frac{1}{a_{jk}}\left|a_{jk} - \frac{a_{ik}}{a_{ij}}\right|\right\}. \tag{17}$$

This method was then generalized by Duszak and Koczkodaj (1994) for $n \geq 3$,

$$K = \max\{K_{ijk} | 1 \leq i < j < k \leq n\}. \tag{18}$$

Note that, for sake of simplicity, it was proven (Duszak and Koczkodaj 1994) that (17) collapses into the following

$$K_{ijk} = \min\left\{\left|1 - \frac{a_{ik}}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}}\right|\right\}. \tag{19}$$

3.9 Golden-Wang index

Golden and Wang (1989) assumed that the priority vector be computed thanks to the row geometric mean and then normalized such that its components sum up to one. They call the so obtained vector $\mathbf{g}^* = (g_1^*, \dots, g_n^*)$. The same operation is similarly repeated on the columns of $\mathbf{A}$. Namely, entries on the jth column are divided by a constant $k_j = \sum_{i=1}^n a_{ij}$ and the new matrix can then be called $\mathbf{A}^* = (a_{ij}^*)$. At this point, the index is as follows,

$$GW = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n \left|a_{ij}^* - g_i^*\right|. \tag{20}$$

3.10 Ramík-Korviny index

Ramík and Korviny (2010) and Ramík and Perzina (2010) presented an inconsistency index for *PCM*s whose entries are triangular fuzzy numbers. However, as they treat it as a more general case, their index can be adapted to work with *PCM*s with real valued entries. They proposed to use the geometric mean method to estimate the priority vector (3) and formulated the inconsistency index as follows,

$$NI_n^\sigma = \gamma_n^\sigma \max_{i,j}\left\{\left|a_{ij} - \frac{w_i}{w_j}\right|\right\}, \tag{21}$$

where

$$\gamma_n^\sigma = \begin{cases} \frac{1}{\max\{\sigma - \sigma^{\frac{2-2n}{n}}, \sigma^2((\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}})\}}, & \text{if } \sigma < (\frac{n}{2})^{\frac{n}{n-2}}, \\ \frac{1}{\max\{\sigma - \sigma^{\frac{2-2n}{n}}, \sigma^{\frac{2n-2}{n}} - \sigma\}}, & \text{if } \sigma \geq (\frac{n}{2})^{\frac{n}{n-2}}, \end{cases}$$

is a positive normalization factor. Let us incidentally note that this index has been criticized by Brunelli (2011) and an alternative metric was proposed.

4 Numerical study on indices' agreement

The main aim of this paper is to compare different inconsistency evaluation proposals. All the ten different indices described in the previous section aim to measure exactly the same notion, i.e. the deviation of a given *PCM* from consistency. As a consequence, some questions naturally arise: is there a good agreement between different indices? Do they classify pairwise comparison matrices in a similar way? Is their use interchangeable? Can the choice of different inconsistency indices affect the result of a decision process? In our opinion, it is important to give reliable answers to these questions, since many decision models proposed in the literature assume that consistency plays an important role in the decision process. In group decision making, for instance, the aggregation of the individual preferences can be performed taking into account their closeness to consistency. In this section we study the inconsistency indices from a statistical perspective through numerical simulations. We are aware of the difficulties in comparing indices defined in different frameworks. In particular, the use of bounded/unbounded scales leads to different consequences on consistency. As specified in Sect. 2, in our study we chose to use the Saaty's bounded scale due to its importance and popularity. By modifying, as described below, initial values of this scale, it is clearly possible to obtain different real numbers. We performed our study on *PCM*s of order 4, 6 and 8 since these dimensions fits well real world applications. For reasons of space, we report in this section only the results corresponding to *PCM*s of order 6. Then, in Sect. 5, we briefly discuss and compare the results obtained with *PCM*s of different order.

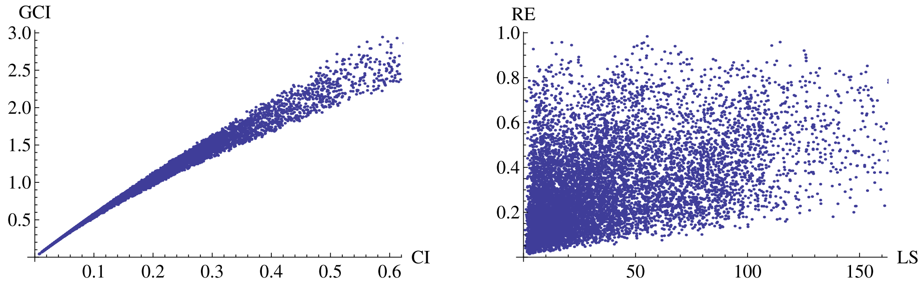
First, in Sect. 4.1, we present the scatter plots reporting the values of the indices taken two by two. Then, we focus on three statistical tools. In Sect. 4.2, we compute the Pearson's correlation coefficient for each pair of indices in order to highlight the linear correlation between them. In Sect. 4.3, by means of the Spearman's rank correlation coefficient, we study the comonotonicity between the indices. In Sect. 4.4, by means of Cohen's kappa coefficient, we study the agreement between indices in classifying pairwise comparison matrices according to their consistency.

We use two main different classes of *PCM*s: a set S_1 of randomly generated matrices and a set S_2 of consistent matrices perturbed by a random noise, like in Choo and Wedley (2004). With this choice, we study the consistency evaluation of the most general type of matrices as well as that of matrices possibly elicited by decision makers in real life problems. The numerical data set we use in Sects. 4.1, 4.2, 4.3 and 4.4 was constructed as follows and refers to sets S_1 and S_2.

- We generated the set $S_1 = \{\mathbf{A}_1, \dots, \mathbf{A}_N\}$ with $N = 10000$ *PCM*s of order 6 by randomly sampling the upper diagonal entries from Saaty's scale

$$\{1/9, 1/8, \dots, 1/2, 1, 2, \dots, 8, 9\}$$

  and consequently computed the lower diagonal entries according to reciprocity, $a_{ji} = 1/a_{ij}$.
- We generated the second set $S_2 = \{\mathbf{B}_1, \dots, \mathbf{B}_N\}$ with $N = 10000$ *PCM*s of order 6 obtained by means of a random perturbation on consistent *PCM*s. More precisely, the following procedure was repeated 10000 times: first, a consistent *PCM* $\mathbf{B} = (b_{ij})$ is constructed by setting $(b_{ij}) = (\frac{w_i}{w_j})$, where $(w_1, \dots, w_6)$ is a randomly generated vector with $w_i \in [1, 9]$, so that $b_{ij} \in [1/9, 9]$. Then, each consistent *PCM* is modified by means of a random perturbation on single elements above the diagonal $b_{ij} \to b_{ij}(1 + \beta)$, where β is a random variable with normal distribution, $\beta \sim N(0, \sigma)$, with $\sigma = 0.5$. The elements below the diagonal of the *PCM* are modified accordingly to preserve reciprocity, $b_{ji} = 1/b_{ij}$.



(a) Scatter plot CI – GCI (b) Scatter plot LS – RE

Fig. 1 Two scatter plots

- We computed the 10 inconsistency indices $I_1(\mathbf{A}_p), \ldots, I_{10}(\mathbf{A}_p)$, for every PCM in the random matrices set, $\mathbf{A}_p \in S_1$. Note that, for simplicity, we now denote the ten inconsistency indices by $I_1, \ldots, I_{10}$, whereas in Sect. 3 we used the original notation introduced by the various authors. The notation correspondence preserves the previous presentation order, namely, $I_1 = CI$, $I_2 = CI^*$, $I_3 = LS$, $I_4 = GCI$, $I_5 = HCI$, $I_6 = I_{CD}$, $I_7 = RE$, $I_8 = K$, $I_9 = GW$, $I_{10} = NI_n^\sigma$.
- Similarly, we considered set S_2 and for every PCM $\mathbf{B}_p \in S_2$, we computed the 10 inconsistency indices $I_1(\mathbf{B}_p), \ldots, I_{10}(\mathbf{B}_p)$.
- To summarize, our data set is composed of two 10000×10 tables containing the inconsistency values corresponding to the ten inconsistency indices for each set S_1 and S_2.

Finally, following Choo and Wedley (2004), we considered consistent matrices where only one entry was modified to make them inconsistent. The results corresponding to this last set of matrices are discussed in Sect. 4.5.

4.1 Graphical representations

By first considering set S_1, for each pair of indices $\{I_i, I_j\}$ we produced a scatter plot for the 10000 PCMs. Each one of the 10000 PCMs is represented by a point of the plot, the coordinates being the values of I_i and I_j respectively. The same study was performed on the matrices of set S_2. A cloud of dispersed points indicates a scarce relationship between the two indices, whereas a regular set of points closely disposed along a curve indicates a strong functional relationship. In Fig. 1(a), for example, indices $I_1 = CI$ and $I_4 = GCI$ are compared and the plot suggests a good agreement between the two indices. In Fig. 1(b), conversely, a poor agreement between indices $I_3 = LS$ and $I_7 = RE$ is evidenced. Both scatter plots refer to set S_2.

The scatter plots for all the comparisons among the 10 indices are compactly represented in Fig. 2 for the PCMs in set S_1 and in Fig. 3 for the PCMs in set S_2. In Figs. 2 and 3 the scatter plots of 500 PCMs are reported for a clearer graphical visualization. Note that the plots in the diagonal boxes are straight lines since identical values are involved. The plots evidence remarkably different behaviors of the indices and we will comment some relevant cases in the next section.

4.2 Pearson correlation coefficient

In order to study the linear correlation between pairs of inconsistency indices described in Sect. 3, we considered the Pearson Correlation Coefficient and proceeded as fol-

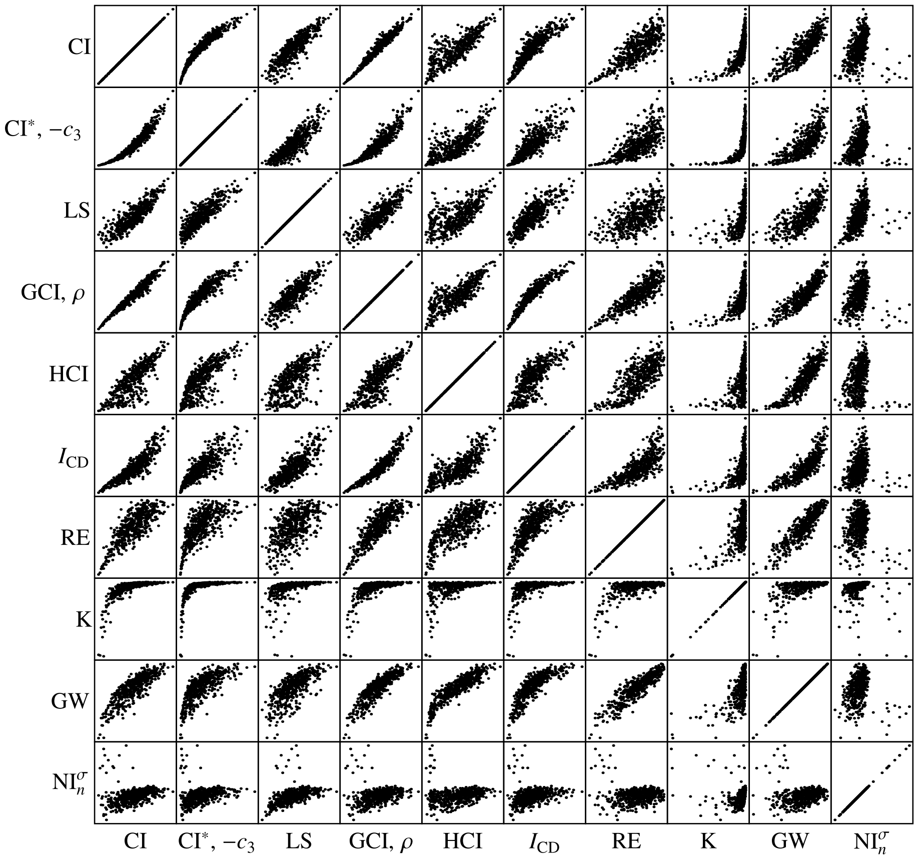


Fig. 2 Scatter plots of inconsistency indices for random matrices

lows. We first considered the set S_1 of random *PCM*s. We fixed two inconsistency indices, say I_i and I_j, and we considered the two associated sets of inconsistency values $\{I_i(\mathbf{A}_1), \dots, I_i(\mathbf{A}_N)\}$ and $\{I_j(\mathbf{A}_1), \dots, I_j(\mathbf{A}_N)\}$. We computed the corresponding Pearson Correlation Coefficient $r(i, j)$,

$$r(i,j) = \frac{\sum_{p=1}^{N}(I_i(\mathbf{A}_p) - \bar{I}_i)(I_j(\mathbf{A}_p) - \bar{I}_j)}{(N-1)s_i s_j}, \tag{22}$$

where $\bar{I}_i$ and s_i are the mean and the standard deviation of $I_i(\mathbf{A}_p)$, respectively. Analogously, $\bar{I}_j$ and s_j are the mean and the standard deviation of $I_j(\mathbf{A}_p)$. We did the same for all $\binom{10}{2}$ pairs of indices $\{I_i, I_j\}$, $i = 1, \dots, 9$; $j = i, \dots, 10$. We reported the $r(i, j)$ values in Table 1. Note that indices *CI** and $-c_3$ are reported in the same row/column due to their proportionality, as pointed out in Sect. 3. The same holds for indices *GCI* and ρ. We repeated the same study on the set S_2 and the results are reported in Table 2.

4.3 Spearman rank correlation coefficient

While the Pearson's correlation coefficient measures the *linear* correlation between pairs of observations, the Spearman index (Spearman 1904; Snedecor and Cochran 1980) measures

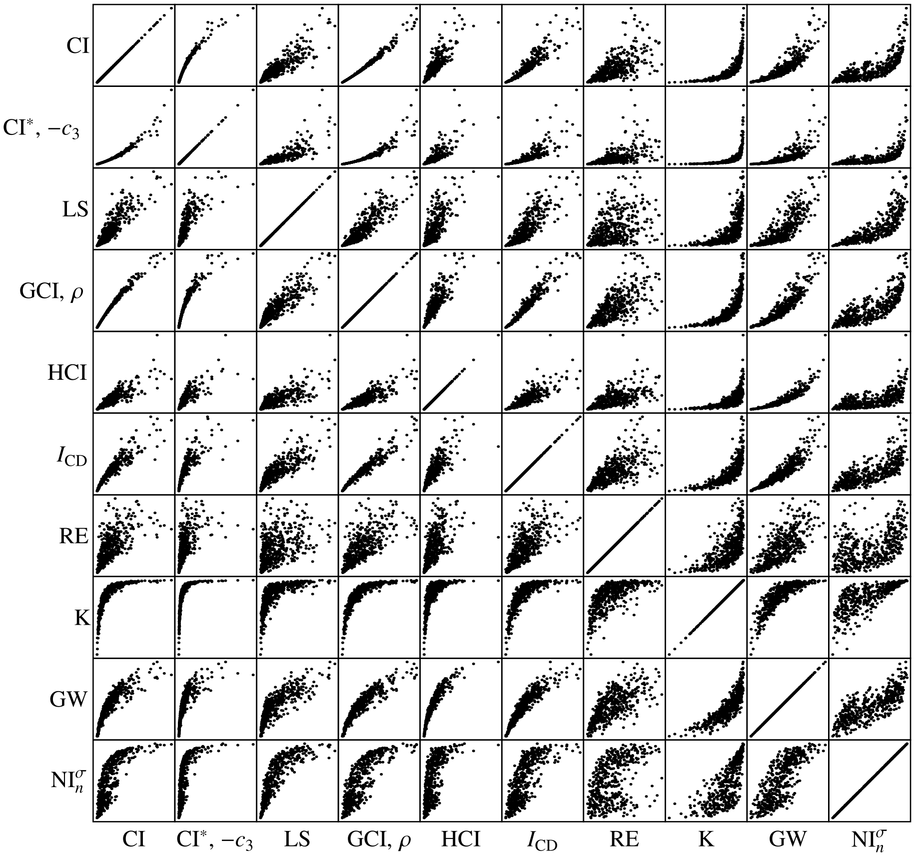


Fig. 3 Scatter plots of inconsistency indices for perturbed consistent matrices

to which extent the pairs of observations are comonotone. The Spearman index compares the way two indices rank the matrices of a fixed set of *PCM*s and reaches its maximum value 1 when two indices produce the same ranking, from the best matrix to the worst one. This means that the two indices are related by a monotone increasing function, no matter the linearity of the relation. Therefore, the Pearson Correlation Coefficient and the Spearman Rank Correlation Coefficient measure a different type of association.

Let $r(I_i(\mathbf{A}_p))$ be the rank of matrix $\mathbf{A}_p$ according to inconsistency index I_i and $r(I_j(\mathbf{A}_p))$ be the rank of matrix $\mathbf{A}_p$ according to inconsistency index I_j. For example, $r(I_i(\mathbf{A}_p)) = 1$ means that index I_i considers matrix $\mathbf{A}_p$ the most consistent one in the set. The Spearman index is defined as

$$\varrho(i, j) = 1 - \frac{6\sum_{p=1}^{N} d_p^2}{N(N^2-1)}, \tag{23}$$

where $d_p^2 = [r(I_i(\mathbf{A}_p)) - r(I_j(\mathbf{A}_p))]^2$.

Similarly to Sect. 4.2, we computed the Spearman Coefficient (23) for all $\binom{10}{2}$ pairs of indices $\{I_i, I_j\}$, $i = 1, \ldots, 9$; $j = i, \ldots, 10$. We report the $\varrho(i, j)$ values obtained for set S_1 in Table 3 and the values obtained for set S_2 in Table 4.

Table 1 Linear correlation computed on 10000 randomly generated *PCM*s of order 6

Index	CI	$CI^*, -c_3$	LS	GCI, ρ	HCI	I_{CD}	RE	K	GW	NI_n^σ
CI	1.	0.952	0.880	0.977	0.835	0.902	0.796	0.608	0.851	0.331
$CI^*, -c_3$	0.952	1.	0.868	0.921	0.787	0.837	0.699	0.513	0.759	0.342
LS	0.880	0.868	1.	0.870	0.707	0.843	0.590	0.474	0.733	0.496
GCI, ρ	0.977	0.921	0.870	1.	0.846	0.949	0.853	0.578	0.897	0.292
HCI	0.835	0.787	0.707	0.846	1.	0.803	0.730	0.470	0.896	0.210
I_{CD}	0.902	0.837	0.843	0.949	0.803	1.	0.778	0.447	0.833	0.276
RE	0.796	0.699	0.590	0.853	0.730	0.778	1.	0.552	0.890	0.132
K	0.608	0.513	0.474	0.578	0.470	0.447	0.552	1.	0.574	0.091
GW	0.851	0.759	0.733	0.897	0.896	0.833	0.890	0.574	1.	0.208
NI_n^σ	0.331	0.342	0.496	0.292	0.210	0.276	0.132	0.091	0.208	1

Table 2 Linear correlation computed on 10000 perturbed consistent *PCM*s of order 6

Index	CI	$CI^*, -c_3$	LS	GCI, ρ	HCI	I_{CD}	RE	K	GW	NI_n^σ
CI	1.	0.954	0.885	0.989	0.898	0.935	0.692	0.715	0.880	0.746
$CI^*, -c_3$	0.954	1.	0.807	0.909	0.838	0.839	0.602	0.571	0.756	0.625
LS	0.885	0.807	1.	0.894	0.789	0.863	0.475	0.676	0.824	0.819
GCI, ρ	0.989	0.909	0.894	1.	0.908	0.967	0.710	0.762	0.924	0.771
HCI	0.898	0.838	0.789	0.908	1.	0.888	0.670	0.675	0.924	0.656
I_{CD}	0.935	0.839	0.863	0.967	0.888	1.	0.680	0.727	0.925	0.719
RE	0.692	0.602	0.475	0.710	0.670	0.680	1.	0.625	0.710	0.480
K	0.715	0.571	0.676	0.762	0.675	0.727	0.625	1.	0.812	0.764
GW	0.880	0.756	0.824	0.924	0.924	0.925	0.710	0.812	1.	0.766
NI_n^σ	0.746	0.625	0.819	0.771	0.656	0.719	0.480	0.764	0.766	1.

4.4 Cohen's κ coefficient

The last statistical tool we used to evaluate the agreement between the inconsistency indices is the Cohen's kappa Coefficient κ (Cohen 1968). This coefficient has been designed to measure the strength of agreement in classifying processes by taking into account the agreement occurring by chance. We proceeded as follows. For each fixed inconsistency index I_i, we partitioned the set $\mathcal{A}$ of all *PCM*s in ten subsets of the same cardinality according to the inconsistency values $I_i(\mathbf{A}_p)$. As a result, index I_i classifies in the first subset the 'best' 10 % matrices, in the second subset the subsequent 10 % matrices and so on. We denote by $\{\mathcal{A}_i^1, \dots, \mathcal{A}_i^{10}\}$ the partition of $\mathcal{A}$ induced by I_i, $i = 1, \dots, 10$. From each partition, the thresholds values of I_i between the classes were derived. We estimated the threshold values for each inconsistency index by randomly sampling 30000 *PCM*s from $\mathcal{A}$.

We carried on the study on the agreement between the inconsistency indices based on the Cohen's kappa Coefficient similarly to the studies described in the previous subsections. First, we fixed one of the two sets of *PCM*s described above, say S_1. Then, for each inconsistency index I_i we classified the matrices in S_1 assigning them to the corresponding classes $\{\mathcal{A}_i^1, \dots, \mathcal{A}_i^{10}\}$ according to the threshold values obtained above. Obviously, this classifying process retraces the one used for obtaining the threshold values in the partition of $\mathcal{A}$, but we

Table 3 Spearman index computed on 10000 randomly generated *PCM*s of order 6

Index	CI	$CI^*, -c_3$	LS	GCI, ρ	HCI	I_{CD}	RE	K	GW	NI_n^σ
CI	1.	0.976	0.876	0.974	0.831	0.919	0.767	0.804	0.847	0.459
$CI^*, -c_3$	0.976	1.	0.857	0.938	0.795	0.850	0.727	0.876	0.806	0.460
LS	0.876	0.857	1.	0.860	0.703	0.834	0.577	0.736	0.748	0.583
GCI, ρ	0.974	0.938	0.860	1.	0.841	0.967	0.839	0.714	0.906	0.404
HCI	0.831	0.795	0.703	0.841	1.	0.807	0.714	0.608	0.900	0.321
I_{CD}	0.919	0.850	0.834	0.967	0.807	1.	0.811	0.620	0.886	0.370
RE	0.767	0.727	0.577	0.839	0.714	0.811	1.	0.506	0.884	0.263
K	0.804	0.876	0.736	0.714	0.608	0.620	0.506	1.	0.589	0.484
GW	0.847	0.806	0.748	0.906	0.900	0.886	0.884	0.589	1.	0.338
NI_n^σ	0.459	0.460	0.583	0.404	0.321	0.370	0.263	0.484	0.338	1

Table 4 Spearman index computed on 10000 perturbed consistent *PCM*s of order 6

Index	CI	$CI^*, -c_3$	LS	GCI, ρ	HCI	I_{CD}	RE	K	GW	NI_n^σ
CI	1.	0.999	0.904	0.999	0.926	0.973	0.755	0.960	0.949	0.837
$CI^*, -c_3$	0.999	1.	0.903	0.997	0.924	0.967	0.754	0.968	0.945	0.838
LS	0.904	0.903	1.	0.903	0.790	0.878	0.505	0.866	0.855	0.928
GCI, ρ	0.999	0.997	0.903	1.	0.928	0.981	0.754	0.951	0.955	0.830
HCI	0.926	0.924	0.790	0.928	1.	0.915	0.753	0.878	0.976	0.716
I_{CD}	0.973	0.967	0.878	0.981	0.915	1.	0.735	0.899	0.955	0.787
RE	0.755	0.754	0.505	0.754	0.753	0.735	1.	0.723	0.740	0.497
K	0.960	0.968	0.866	0.951	0.878	0.899	0.723	1.	0.883	0.821
GW	0.949	0.945	0.855	0.955	0.976	0.955	0.740	0.883	1.	0.777
NI_n^σ	0.837	0.838	0.928	0.830	0.716	0.787	0.497	0.821	0.777	1.

preferred to keep the two phases distinct for procedural correctness. Finally, for all $\binom{10}{2}$ pairs of indices $\{I_i, I_j\}$, $i = 1, \dots, 9$; $j = i, \dots, 10$ we computed the Cohen's kappa Coefficient obtaining the agreement between I_i and I_j in classifying the matrices of S_1. We denote, for clarity, the obtained values by $\kappa(i, j)$. When there are more than two classes expressed on an ordinal scale (in our case we have ten classes), disagreements may not all be equally important and a greater penalty can be applied if the classes are further apart. To account for these inequalities, Cohen (1968) introduced weights in the formulation of the agreement index, leading to the *weighted* kappa coefficient

$$\kappa(i, j) = \frac{p_{o(w)} - p_{e(w)}}{1 - p_{e(w)}}. \tag{24}$$

In (24), $p_{o(w)} = \sum_{s=1}^{10} \sum_{t=1}^{10} w_{st} p_{st}$ is the observed agreement proportion, $w_{st} = 1 - \frac{(s-t)^2}{9^2}$ are the quadratic weights, and p_{st} are the observed proportions, namely, the proportion of matrices that index I_i classifies in class $\mathcal{A}_s$ and index I_j classifies in class $\mathcal{A}_t$. Finally, $p_{e(w)} = \sum_{s=1}^{10} \sum_{t=1}^{10} w_{st} p_{s\cdot} p_{\cdot t}$ is the agreement proportion expected by chance, where $p_{s\cdot}$ and $p_{\cdot t}$ are the marginal values. We reported the values $\kappa(i, j)$ obtained for the *PCM*s of S_1 in Table 5. Then, we repeated the same study on the set S_2 and we reported the results in Table 6.

Table 5 Cohen's kappa computed on 10000 randomly generated *PCM*s of order 6

Index	CI	CI^*, $-c_3$	LS	GCI, ρ	HCI	I_{CD}	RE	K	GW	NI_n^σ
CI	1.	0.966	0.865	0.965	0.820	0.909	0.757	0.795	0.836	0.458
CI^*, $-c_3$	0.966	1.	0.846	0.928	0.783	0.839	0.717	0.868	0.795	0.459
LS	0.865	0.846	1.	0.850	0.692	0.824	0.567	0.726	0.738	0.580
GCI, ρ	0.965	0.928	0.850	1.	0.828	0.958	0.829	0.706	0.894	0.405
HCI	0.820	0.783	0.692	0.828	1.	0.796	0.703	0.599	0.891	0.321
I_{CD}	0.909	0.839	0.824	0.958	0.796	1.	0.801	0.611	0.876	0.368
RE	0.757	0.717	0.567	0.829	0.703	0.801	1.	0.498	0.873	0.262
K	0.795	0.868	0.726	0.706	0.599	0.611	0.498	1.	0.579	0.484
GW	0.836	0.795	0.738	0.894	0.891	0.876	0.873	0.579	1.	0.338
NI_n^σ	0.458	0.459	0.580	0.405	0.321	0.368	0.262	0.484	0.338	1

Table 6 Cohen's kappa computed on 10000 perturbed consistent *PCM*s of order 6

Index	CI	CI^*, $-c_3$	LS	GCI, ρ	HCI	I_{CD}	RE	K	GW	NI_n^σ
CI	1.	0.905	0.580	0.805	0.540	0.513	0.067	0.569	0.603	0.037
CI^*, $-c_3$	0.905	1.	0.553	0.667	0.573	0.354	0.080	0.720	0.567	0.048
LS	0.580	0.553	1.	0.481	0.576	0.332	0.078	0.483	0.709	0.067
GCI, ρ	0.805	0.667	0.481	1.	0.445	0.756	0.041	0.313	0.593	0.020
HCI	0.540	0.573	0.576	0.445	1.	0.267	0.137	0.519	0.753	0.088
I_{CD}	0.513	0.354	0.332	0.756	0.267	1.	0.023	0.142	0.418	0.010
RE	0.067	0.080	0.078	0.041	0.137	0.023	1.	0.159	0.104	0.391
K	0.569	0.720	0.483	0.313	0.519	0.142	0.159	1.	0.419	0.104
GW	0.603	0.567	0.709	0.593	0.753	0.418	0.104	0.419	1.	0.054
NI_n^σ	0.037	0.048	0.067	0.020	0.088	0.010	0.391	0.104	0.054	1.

4.5 Single entry modification

Following Choo and Wedley (2004), we considered the consistent matrix $\mathbf{A} = (\frac{w_i}{w_j})$ corresponding to the preference vector $\mathbf{w} = (1, 2.5, 4, 5.5, 7, 8.5)$. Then, we proceeded to generate the $(n(n-1))/2 = 15$ matrices $\mathbf{C}_1, \dots, \mathbf{C}_{15}$ by modifying a single entry of the matrix $\mathbf{A}$, $a_{ij} \to a_{ij}(1+0.9)$, $i = 1, \dots, 5$; $j = i+1, \dots, 6$ and putting accordingly $a_{ji} = 1/a_{ij}$. Note that this change corresponds to modifying the evaluation of a single comparison, namely the one between alternative x_i and x_j. For each one of the 15 matrices, we computed the value of the 10 inconsistency indices $I_1(\mathbf{C}_p), \dots, I_{10}(\mathbf{C}_p)$, $p = 1, \dots, 15$. It is interesting to note a different behavior of the indices with respect to the change of a single comparison. Five indices out of ten assign an inconsistency value which is independent from p, i.e. from the modified comparison. For example, it is $CI(\mathbf{C}_1) = CI(\mathbf{C}_2) = \cdots = CI(\mathbf{C}_{15})$ and the same holds for CI^*, GCI, I_{CD} and K. Conversely, indices LS, HCI, RE, GW and NI_n^σ assign different inconsistency values to the 15 *PCM*s. As an example, in Fig. 4 the comparison between indices CI and HCI evidences the constant value $CI(\mathbf{C}_p) \approx 0.0093$ for $p = 1, \dots, 15$ and the different values of HCI. We performed also on the matrices $\mathbf{C}_1, \dots, \mathbf{C}_{15}$ the study described in Sects. 4.1, 4.2, 4.3 and 4.4 for sets S_1 and S_2 but we omit, for reasons of space, to report in

Fig. 4 Scatter plot *CI–HCI*

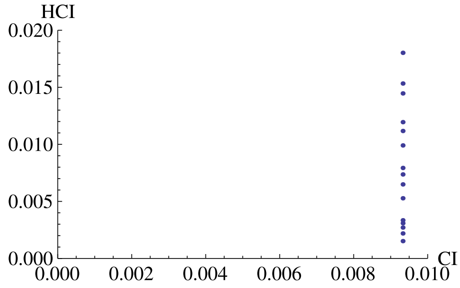


Table 7 Minimum mean value of the Spearman index

Dimension	mmv	Outlier
10	0.468	NI_n^σ
9	0.717	K
8	0.790	RE
7	0.840	

detail the corresponding results. However, we remain available to send the interested readers the tables and plots.

5 Discussion

A relevant finding emerges from the study described above. A majority of indices evidences a good internal agreement, whereas few indices perform as outliers, with poor agreement when compared with all the others. In particular, we highlight this behavior by referring to Table 3 and by operating as follows. First, we compute for each row of the table, namely for each inconsistency index, the mean value. Then, we consider the index with the minimum mean value as an 'outlier', we remove it from the table, thus obtaining a 9×9 table. We repeat the same operations on this table by identifying and removing the next outlier. We proceed the same way until for each remained inconsistency index the mean value of the Spearman index is greater than 0.8, which is considered a threshold for a good agreement. In Table 7 we report the minimum mean value of the Spearman index, say mmv, in the tables obtained as described above, for dimension 10, 9, 8 and 7. It can be checked that the indices iteratively classified as outliers are $I_{10} = NI_n^\sigma$, $I_8 = K$, and $I_7 = RE$.

Some comments are needed, for trying to interpret and justify this result. Indices NI_n^σ and K are defined using the max operator, thus focusing on a single piece of information evidencing a local maximal inconsistency. Conversely, the other indices synthesize several inconsistency contributions. Therefore, indices NI_n^σ and K could be conveniently used when it is crucial to identify the maximal violation of consistency rather than to evaluate the global amount of this violation. Nevertheless, the rather poor agreement between NI_n^σ and K evidenced in Sect. 4 remarks the diversity between the two indices.

Index $I_7 = RE$ was proposed by Barzilai (1998) and has the relevant property of being invariant w.r.t. transformation $f(a_{ij}) = a_{ij}^b$ (Fedrizzi and Brunelli 2009). This means that index RE takes into account only the mutual coherence of the judgments and it is independent from the size of a PCM's entries. Therefore, also a PCM which is very close to a consistent matrix, according to the euclidean metric, can have a large value of RE if the preferences are not coherent and, possibly, several cycles are contained. Formally, this property induces a discontinuity of *RE* in moving away from consistency. To our best knowledge, no other inconsistency index shares the same property. Moreover, Barzilai assumed that preferences can be quantified by means of every positive real number, $a_{ij} \in \mathbb{R}$. It is therefore not surprising that *RE* (16) behaves differently from the other inconsistency indices.

If we instead focus on the common characteristics of indices with a very high agreement, i.e. *GCI*, CI^*, I_{CD} and *CI*, we note that all of them, apart from *CI*, are more or less explicit averages of the single deviations of triples of pairwise comparisons from the condition of transitivity (1). Note also that, from the study performed in Sect. 4.5, all these four indices evidence the same constant behavior in changing a single comparison.

From the analysis performed on matrices of order 4 and 8 we observed that the results were coherent with those described in Sect. 4 for matrices of order 6. Nevertheless, we noted that, as the order of the matrices increases, there is a general weak decrease of the coefficients described in Sects. 4.1, 4.2, 4.3 and 4.4, namely Pearson's, Spearman's and Cohen's coefficients. In our opinion, this fact may indicate a divergent behavior of the considered inconsistency indices, thus evidencing that the problem of consistency evaluation becomes more critical as the number of involved alternatives increases. As previously specified in Sect. 4.5, we encourage the interested readers to contact us and ask for specific data which, for reasons of space, we did not include in this paper.

We conclude by stressing again that every inconsistency index is in fact a different *definition* of inconsistency degree and that there is still no recognized benchmark to measure the goodness of the various proposals.

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Introduction to the Analytic Hierarchy Process *

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Preface

Why would anyone feel urged to write another book on the Analytic Hierarchy Process (AHP), given those already written? I felt urged because the existing books on the AHP are conservative, too anchored to the original framework, and do not cover recent results, whereas lots of questions have been addressed in the last years. Apparently, the interest in the AHP has not faded in the last years, and we shall see that this view is also supported by other studies, as well as by the years of publication of many of the references used in this booklet.

Now, the next question one should ask himself when writing a tutorial should regard to whom the tutorial is for. With the premise that a decision scientist might find these pages too simplistic, in my intentions, the readership should include the following categories.

- *Practitioners* and *consultants* willing to apply, and *software developers* willing to implement, the AHP. Some collateral issues, for instance the incompleteness of judgments, are usually neglected in didactic expositions, but remain fundamental in practical implementations. On the software development side, at present, there is still not a modern and free software which covers all the aspects of the AHP presented in this booklet

- Recent advances in the theory have been disseminated in different journals and, as research requires, are narrow, technical, and often use heterogeneous notation and jargon. Therefore, I also hope that *students* who have been introduced to the AHP and want to have an updated exposition on, and references to, the state of the art can find these pages useful

- Even the *applied mathematicians* might find it interesting. The mathematics behind the method is simple, but some of its extensions have been a fertile ground for the application of non-trivial concepts stemming from abstract algebra and functional analysis, just to mention two areas of interest.

The following pages assume neither previous knowledge of the AHP, nor higher mathematical preparation than some working knowledge of calculus and linear algebra with eigenvector theory. A brief tutorial on eigenvalues and eigenvectors is provided in the appendix. Moreover, some sections are marked with the symbol * to indicate that they contain further discussions and references to research literature. The reader interested in the fundamentals might want to skip them.
Ideally, this booklet is also articulated to suit different levels of readership. I believe that the following three can serve as approximate guidelines:

- A *basic* exposition is given in Chapter 1 with the exclusion of the section marked with *. The reader can then proceed examining Section 2.1 until the end of §2.1.1, Section 2.2 until the end of §2.2.1, and Section 3.2 with the exclusion of the subsection marked with *. A basic understanding allows the reader to use the AHP only at a superficial level.

- A *complete* exposition of the AHP can be gained by reading this booklet in its entirety, with the exclusion of the sections marked with *. A complete understanding allows the reader to choose between different tools to perform different tasks.

- An *advanced* understanding of the method is like the complete, but with the addition of the sections marked with *. Compared to the complete understanding, in the advanced, the reader will familiarize with the most recent results and the ongoing discussions, and will be able to orient through the literature.

I shall also spell out that I will not refrain from giving a personal perspectives on some problems connected with the AHP, as the method has been a matter of heated debate since its inception.

I hereby wish to thank those who helped me. Among them, I am particularly gratetul to Michele Fedrizzi, who also taught me much of the material contained in this booklet. I am also grateful to Springer, especially in the person of Matthew Amboy. Furthermore, this project has been financed by the Academy of Finland.

It goes without saying that I assume the paternity of all imprecisions and mistakes and that the reader is welcome to contact me.

Espoo, Finland, November 2014 — Matteo Brunelli

1. Introduction and fundamentals

> Beauty started when people began to choose.
>
> Roberto Benigni

In a world whose complexity is rapidly growing, making the best decisions becomes an increasingly demanding task for managers of companies, governmental agencies and many other decision and policy makers. In recent years, this has gone arm-in-arm with the growth of what are now known as *decision analytics* methodologies. Namely, decision makers are more reluctant to make gut decisions based of feelings and hunches, and instead prefer to use analytic and quantitative tools, and base and analyze their decisions on a solid ground. Many methods stemming from applied mathematics and operations research have proved useful to help decision makers making informed decisions, and among these methods there are also those requiring, as inputs, subjective judgments from a decision maker or an expert. It is in this context that the Analytic Hierarchy Process (AHP) becomes a useful tool for analyzing decisions.

What is the AHP? Broadly speaking, the AHP is a theory and methodology for *relative measurement*. In relative measurement we are not interested in the exact measurement of some quantities, but rather on the *proportions* between them. Consider a pair of stones. In classical measurement we might be interested in knowing their exact weights and the pair of measurements $(2, 1)$ is not correct unless the weight of the first stone is 2kgs and the weight of the second is 1kg. Conversely, in relative measurement we confine our interest to the knowledge of how much heavier each object is compared to another. Hence, the pair of measurements $(2, 1)$ is correct as long as the weight of the first stone is double the weight of the second. It follows that, in this example, if we use relative measurement theory the pairs of measurements $(2/3, 1/3)$ $(4, 2)$, $(8, 4)$ are also correct for the two stones. Relative measurement theory suits particularly well problems where the best alternative has to be chosen. In fact, in many cases we are not really interested in the precise scores of the alternatives but it is sufficient to know their relative measurements to know which alternative is the best. Moreover, when attributes of alternatives are intangible, it is difficult to devise a measurement scale and using relative measurements simplifies the analysis. The ultimate scope of the AHP is that of using pairwise comparisons between alternatives as inputs, to produce a rating of alternatives, compatibly with the theory of relative measurement.

In what field of study do we stand when we talk of the AHP? It is the author's opinion that the AHP should be placed in the intersection between decision analysis and operations research. Keeney and Raiffa [76] gave the following definition of *decision* *analysis*:

> The theory of decision analysis is designed to help the individual make a choice among a set of prespecified alternatives.

Hence, as long as the AHP is used as a technology for aiding decisions, it seems that its study belongs to decision analysis.
On the other hand, to justify its connection with operations research, without going too far, we can refer to some definitions reported by Saaty, the main developer of the AHP, in one of the first graduate textbooks in operations research [100]. In his book, curious and thought-provoking definitions can be found: operations research was defined as "quantitative common sense" and, perhaps in the intent of underlining its limitations, as "the art of giving bad answers to problems to which otherwise worse answers are given". Such definitions are surely thought-provoking but they capture the essence of quantitative methods, which is that of helping make better decisions. Consulting the Merriam-Webster dictionary one can find the following definition of *operations research*:

> The application of scientific and especially mathematical methods to the study and analysis of problems involving complex systems.

Hence, it is straightforward to conclude that the study of the AHP belongs to operations research too. Within operations research, two different types of studies appeared. The classical operations research, more mathematically oriented, which studies the modeling and solution of structured problems can be called 'hard' operations research. Conversely, especially recently, the effort of applying the reasoning of operations research to problems which, by nature, are unstructured, has gone under the name of 'soft' operations research. Perhaps the fact that the AHP mostly deals with subjective judgments and intangible attributes gave the false idea that it did not belong to the tools or 'hard' operations research, but rather to its 'soft' side. However, in recent discussions [90] the role of the AHP has been revisited and now it seems clearer that it has been a matter of study for 'hard' operations research. The positioning of the AHP is depicted in Figure 1.1.

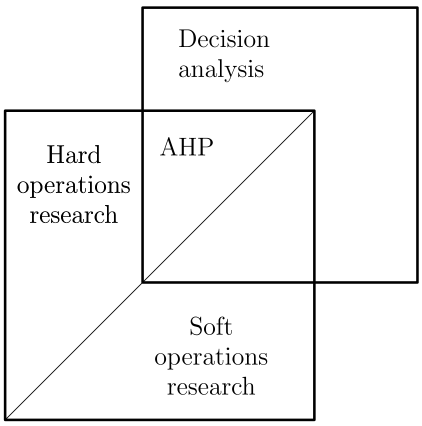


Figure 1.1.: The position of the AHP in the scientific debate.

Although the utility of the AHP is not limited to the following, it is safe to say that it has been especially advocated to be used with *intangible* criteria and alternatives, and thus used to solve *multi-criteria decision making* (MCDM) problems, which are choice problems where alternatives are evaluated with respect to multiple criteria. Tangible properties of alternatives, for example the weight of different stones or the salary of different employees, can be measured without ambiguity and subjectivity. Hence, the machinery of the AHP becomes unnecessary. Conversely, when the magnitude of some properties of alternatives, such as the dexterity of a sportsman or the aesthetic appeal of a bridge, cannot be easily grasped and measured we are in the domain of the intangibles, which is where the AHP gives its best.

The organization of this booklet is quite unorthodox and differs from the approach used in other expositions [19, 20, 74, 102, 103, 106]. Here, at the very beginning, the AHP is presented through a normative lens with lots of assumptions. That is, the AHP is introduced as a method which works in a rational world with full information. However, since this is clearly not the world we are living in, successively, by pointing out the limits of this normative approach, binding assumptions are relaxed and the AHP more fully explained. In this sense the reader should not be deceived: the exposition of the AHP contained in this first chapter is by no means complete, and it is even narrower than the one given originally by Saaty. But, as said, this little trick shall hopefully help to expose the AHP in a more natural and painless way.

In the following we shall use a standard notation where vectors are noted in boldface, e.g. $\mathbf{w} = (w_1, \ldots, w_n)^T$ and matrices (all square) in capital boldface, e.g. $\mathbf{A} = (a_{ij})_{n \times n}$. The set of real numbers is $\mathbb{R}$ and the set of positive real numbers is $\mathbb{R}_>$. We shall use open square brackets to indicate open intervals, e.g. $]0, 1[$.

1.1. Fundamentals

As already mentioned, in our framework, the AHP can be applied to a multitude of decision making problems involving a finite number of alternatives. Formally, in this setting, in a decision process there is one *goal* and a finite set of *alternatives*, $X = \{x_1, \ldots, x_n\}$, from which the decision maker, is usually asked to select the best one.
Explaining the AHP is like teaching a child how to tie the shoestrings: easier to show with an example than to explain with words. Hence, it is time to present a prototypical example which will accompany us for the rest of this section: a family has to decide which European city to visit during their holidays. Reasonably, the goal of the family is the highest satisfaction with their destination. Alternatives may be some cities, in our simple example

$$X = \{\underbrace{\text{Rome}}_{x_1}, \underbrace{\text{Barcelona}}_{x_2}, \underbrace{\text{Reykjavik}}_{x_3}\}, \tag{1.1}$$

and the structure of the problem represented in Figure 1.2.
Often, in decision processes, the decision maker is asked to assign a score to each alternative and then to choose the one with the maximum value. That is, given a set of

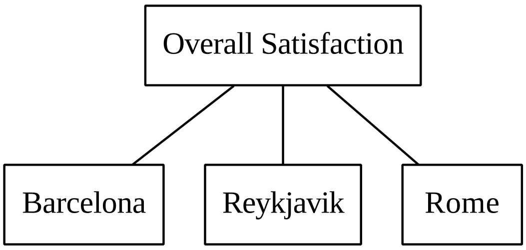


Figure 1.2.: Evaluating alternatives with respect to an overall goal.

alternatives, $X = \{x_1, \ldots, x_n\}$, the decision maker should provide a weight vector

$$\mathbf{w} = (w_1, \ldots, w_n)^T, \tag{1.2}$$

where w_i is a value which coherently estimates the score of alternative x_i. That is, the greater w_i, the better the ith alternative. Similarly to what happens for value theory [55], the rule is that alternative x_i is preferred to alternative x_j if and only if $w_i > w_j$. Weight vectors are nothing else but ratings, and their components w_i are called priorities, or weights, of the alternatives x_i. For example, $\mathbf{w} = (0.4, 0.2, 0.3, 0.1)^T$ implies $x_1 \succ x_3 \succ x_2 \succ x_4$ where $x_i \succ x_j$ means that alternative x_i is preferred to x_j. Possible ties are expressed as $x_i \sim x_j$.

Example 1. *Consider the example of the choice of the best site for holidays. If the vector* $\mathbf{w} = (0.3, 0.5, 0.2)^T$ *was associated with the set of alternatives*

$$X = \{\mathit{Rome}, \mathit{Barcelona}, \mathit{Reykjavik}\}$$

then we would have that

$$\mathit{Barcelona} \succ \mathit{Rome} \succ \mathit{Reykjavik}$$

because $w_2 > w_1 > w_3$.

Making decisions in this way seems easy, but it becomes a hard task when complexity increases. As we will see, complexity augments arm-in-arm with the number of alternatives and criteria.

From the priority vector to the pairwise comparison matrix

It is clear that a decision maker could run into troubles when asked to submit a rating in the form of a numerical vector for a large number of alternatives. Does not it often happen that we cannot decide among several alternatives? Even worse, do not we decide and eventually realize that it was not the best decision? This is a matter of fact and originates from our cognitive limits and the impossibility of effectively comparing several alternatives at the same time.
An effective way to overcome this problem is to use *pairwise comparisons*. The reason

for doing so, is that this allows the decision maker to consider two alternatives at a time. Thus, the strategy is that of decomposing the original problem into many smaller subproblems and deal with these latter ones. Formally, the pairwise comparisons are collected into a *pairwise comparison matrix*, $\mathbf{A} = (a_{ij})_{n\times n}$, structured as follows

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{1.3}$$

with $a_{ij} > 0$ expressing the degree of preference of x_i to x_j. More precisely, according to Saaty's theory, each entry is supposed to approximate the ratio between two weights

$$a_{ij} \approx \frac{w_i}{w_j} \quad \forall i, j. \tag{1.4}$$

This means that, if the entries exactly represent ratios between weights, then the matrix $\mathbf{A}$ can be expressed in the following form,

$$\mathbf{A} = (w_i/w_j)_{n\times n} = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix}. \tag{1.5}$$

Note that, as soon as we account for (1.4) and consider (1.5), a condition of multiplicative reciprocity $a_{ij} = 1/a_{ji}\ \forall i, j$ holds, and $\mathbf{A}$ can be simplified and rewritten,

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{pmatrix}. \tag{1.6}$$

In words, the simplified structure of pairwise comparison matrices in this form follows from the assumption that if, for example, x_1 is 2 times better than x_2, then we can deduce that x_2 is $1/2$ as good as x_1.
Let us now proceed with the example and imagine a pairwise comparison matrix for the set of cities X as defined previously, in §1.1. In this case, and only in this case, to facilitate the understanding, the labels x_1, x_2, x_3 are attached to the rows and columns of the matrix.

$$\mathbf{A} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 3 & 6 \\ 1/3 & 1 & 2 \\ 1/6 & 1/2 & 1 \end{pmatrix} \end{array} \tag{1.7}$$

From this matrix, in particular from entry a_{12}, one can figure out that x_1 (Rome) is considered three times better than x_2 (Barcelona). That is, $a_{12} = 3$ suggests us that

$w_1 = 3w_2$. Once a pairwise comparison matrix is completed, there are many methods to derive the priority vector $\mathbf{w}$. In the example it can be checked that the condition $a_{ij} = w_i/w_j \ \forall i, j$ is satisfied by, for instance, the following vector with its components summing up to one,

$$\mathbf{w} = \begin{pmatrix} 6/9 \\ 2/9 \\ 1/9 \end{pmatrix},$$

and thus Rome (x_1) is ranked the best. To summarize, whenever the number of alternatives is too large, pairwise comparing them is an effective way for obtaining a rating. Perhaps we have spent a bit more of our time but the rating of alternatives contained in $\mathbf{w}$ is now more robust than it would have been if it had been estimated directly, without using $\mathbf{A}$. We shall here ask the reader for a leap of faith and leave the issue of the weight determination open and discuss it later.

From the pairwise comparison matrix to the hierarchy

At this point, it is time to wonder why the pairwise comparison matrix $\mathbf{A}$ was filled in that particular way and what factors influenced the decision maker's judgments. Needless to say, such decision factors are few if the expert is choosing the type of bread to buy (mainly price and quality) whereas they are several when a member of a parliament has to vote a proposition (sake of the electors, own reputation, likelihood of reelection, and surely many others). First, we should start using the word criterion instead of factor and reckon that, if we can make decisions and account for multiple, and possibly conflicting criteria, we are in the realm of Multi Criteria Decision Making (MCDM) methods.

Formally, in the decision making process, the expert has to consider a set of *criteria* $C = \{c_1, \ldots, c_m\}$, which are characteristics making one alternative preferable to another with respect to a given goal. In the example, which regards the location for holidays, the set of criteria could be

$$C = \{\underbrace{\text{climate}}_{c_1}, \underbrace{\text{sightseeing}}_{c_2}, \underbrace{\text{environment}}_{c_3}\}, \tag{1.8}$$

At this point we need at least a graphical formalism to combine alternatives, criteria and goals and represent the structure of the problem in an intuitive way. In the AHP, a *hierarchy* serves this purpose and is compounded by:

- the goal
- the set of alternatives
- the set of criteria
- a relation on the goal, the criteria and the alternatives.

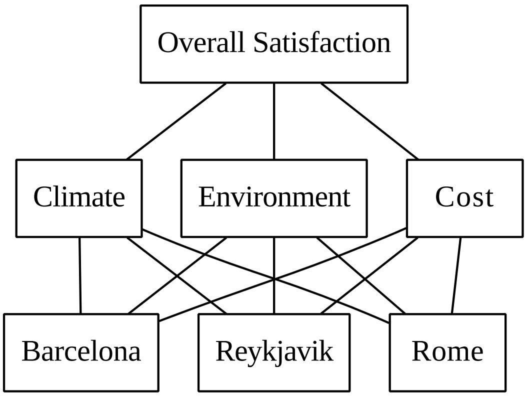


Figure 1.3.: Hierarchy for the European city selection problem. At the top level there is the goal, at the bottom there are the alternatives, and criteria are in intermediate levels. A line connecting two elements marks the existence of a relation of hierarchical dependence between them.

A graphical example of hierarchy for the decision on the European city is depicted in Figure 1.3. Note that in this booklet we shall not dwell on the hierarchy in more formal terms since it would be beyond its scope.

The main drawback of the pairwise comparison matrix $\mathbf{A}$ in (1.7) is that it compared alternatives without considering criteria. Simply, when filling it, the decision maker was only thinking about the overall satisfaction with the alternatives and did not make any separate reasoning about the criteria—cost, sightseeing and environment in the example—contributing to the global satisfaction.

Once again, complexity can be a problem and the solution is to decompose it. This is why, at this point, Saaty [101] suggested to build a different matrix for each criterion. Hence, in the following, a matrix $\mathbf{A}^{(k)}$ is the matrix of pairwise comparisons between alternatives according to criterion k. For example, using the convention c = climate, s = sightseeing, e = environment, entry a_{13} of matrix $\mathbf{A}^{(c)}$ below entails that the decision maker prefers Rome to Reykjavik if he compares these two cities *exclusively* under the climatic point of view. The following three matrices can be taken as examples of preferences expressed by a decision maker on the three cities according to the three different criteria.

$$\mathbf{A}^{(c)} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 1/4 & 1/4 & 1 \end{pmatrix} \qquad \mathbf{A}^{(s)} = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{pmatrix} \qquad \mathbf{A}^{(e)} = \begin{pmatrix} 1 & 1/2 & 1/8 \\ 2 & 1 & 1/4 \\ 8 & 4 & 1 \end{pmatrix}.$$

Then, we estimate (no worries for the moment about the method) their priority vectors

$$\mathbf{w}^{(c)} = \begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} \qquad \mathbf{w}^{(s)} = \begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} \qquad \mathbf{w}^{(e)} = \begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix}.$$

Now we have three vectors instead of one! Their interpretation is at least twofold: (i) as they are 3 vectors of dimension 3, one can imagine them as 3 points in the 3-dimensional Euclidean space; (ii) vectors are ratings and they can be contradictory: climate-wise Barcelona is preferred to Reykjavik, but, on the other hand, the opposite is true if the criterion is the environment.

It is reasonable to assume that the solution should be a compromise between vectors $\mathbf{w}^{(c)}, \mathbf{w}^{(s)}, \mathbf{w}^{(e)}$. However, the simple arithmetic mean is not the best way to aggregate the vectors because, most likely, criteria have different degrees of importance. For instance, an old and rich man may not care much about the cost and just demand a quiet and peaceful place for his holidays—in this hypothetical case the criterion 'environment' would be judged more important than 'cost'. Hence, we need another type of averaging function and the compromise that we are looking for is the *weighted arithmetic mean*, in this case a convex (linear) combination of vectors. Now, the question is how to find the weights to associate to different vectors. The only thing we know is that the weight associated to a vector should be proportional to the importance of the criterion associated with it. The proposed solution is to use the same technique used so far. First, we build a pairwise comparison matrix $\hat{\mathbf{A}} = (\hat{a}_{ij})_{n\times n}$ which compares the importance of criteria with respect to the achievement of the goal. In the example, the matrix could be

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/2 \\ 4 & 2 & 1 \end{pmatrix}.$$

Then, we derive a vector $\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)^T$ (again, no worries on how it is derived)

$$\hat{\mathbf{w}} = \begin{pmatrix} 1/7 \\ 2/7 \\ 4/7 \end{pmatrix}$$

whose components are the weights of criteria. According to this vector the decision maker—in our case the family—is mainly interested in the third criterion, i.e. the environment. We proceed with the linear combination of $\mathbf{w}^{(c)}$, $\mathbf{w}^{(s)}$ and $\mathbf{w}^{(e)}$.

$$\begin{aligned} \mathbf{w} &= \hat{w}_1\mathbf{w}^{(c)} + \hat{w}_2\mathbf{w}^{(s)} + \hat{w}_3\mathbf{w}^{(e)} \\ &= \frac{1}{7}\begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} + \frac{2}{7}\begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} + \frac{4}{7}\begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix} \\ &\approx \begin{pmatrix} 0.287 \\ 0.253 \\ 0.460 \end{pmatrix}. \end{aligned}$$

We have a final ranking and we can choose the best alternative, which is the one rated the highest, then x_3 which, in our example, is Reykjavik. Formally, the best alternative is any element of the set $\{x_i | w_i \geq w_j, \ \forall i, j\}$.

The role of the criteria weights can be stressed by a numerical example. Consider the priority vector for criteria $(1/7, 4/7, 2/7)^T$ instead of $(1/7, 2/7, 4/7)^T$. Then, the final priorities become $(0.43, 0.29, 0.28)^T$ and the best alternative is now x_1 (Rome).

Note that hierarchies can contain more levels of criteria. For example, for the selection of the best city for holidays, the criterion 'climate' could have been refined into subcriteria such as 'chance of rain', 'temperature', 'length of the daylight', each of which contributes to the concept of climate. For reasons of space we cannot provide a numerical example of a hierarchy with more criteria levels, but we invite the reader to consider the following exercise.

Problem 1. *Convince yourself that the AHP can work out the hierarchy in Figure 1.4. Note the complication that 'wheels' is a subcriterion of both 'mechanics' and 'aesthetics'.*

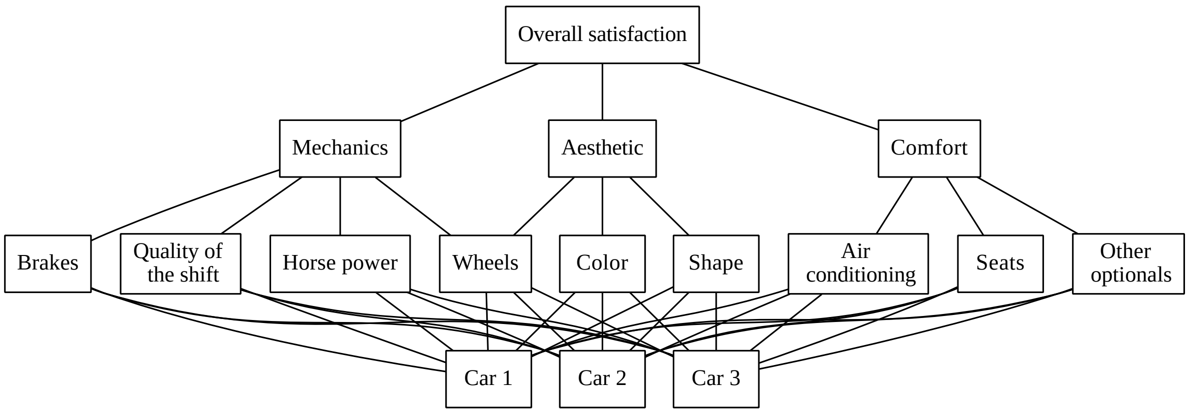


Figure 1.4.: Selection of an automobile.

Certainly, this is an ad hoc toy-example, but the reader might be interested to know that the AHP has actually been proposed for automobile selection [32].

Let us conclude this section by remarking that, by using naïve examples we have already seen the three basic steps of the AHP:

1. Problem structuring and definition of the hierarchy
2. Elicitation of pairwise comparisons
3. Derivation of priority vectors and their linear combinations.

Nevertheless, so far, we have considered an idyllic situation which used various assumptions and only in the next chapters we shall see how the AHP can be used as a more flexible model. Even so, what we know is already sufficient to understand the next section on some applications.

1.2. Applications

Our previous example was simple and aimed at understanding the principles behind the AHP and, needless to say, real-world applications have presented a much higher level of complexity. In this section we shall skim through some applications to show their vast range and hopefully whet the readers' appetite for the AHP, whose full potential has not been revealed yet. Nowadays, applications are so many that no survey can be comprehensive enough. However, albeit not recent, the surveys by Golden *et al.* [64], Zahedi [140], and Vargas [130] remain the best reference points.

City evaluation and planning

Saaty [104] proposed to use the AHP to rank a set of cities from the most to the least livable. In his study, he considered a set of cities in the United States. Indeed, the satisfaction of the final goal 'livability' can be decomposed into the satisfaction of some criteria, such as 'environment', 'services', 'security', and each of these criteria can itself be decomposed into subcriteria. For instance, the 'services' criterion might depend on subcriteria such as 'transportation facilities', 'health care', and so forth. Some cities are undoubtedly more livable than others[1]. Interestingly, in this application, the AHP questionnaire was given to six decision makers representing different demographic groups and light was shed on differences of preferences between them. The research concluded presenting some conjectures on the reasons behind these discrepancies.

Another innovative application was proposed by Saaty and Sagir [113]. By looking at metropolitan areas, the authors were able to classify most of the world cities into one category, out of seven, each representing an alternative model of developing a city. Some alternatives were: compact, 3-dimensional (New York City), flat (Riyadh). The AHP was used to systematically take into account good and bad points of each type of city by means of an AHP-based cost-benefit analysis.

Country ranking

Until the late Eighties, ranking of countries was based on their gross domestic product per capita, or at least that was the most significant measure. More recently, starting in the early Nineties, a more inclusive and composite measure accounting for multiple criteria called Human Development Index has been popularized by economists such as the Nobel laureate Amartya Sen [4]. Few know that, in 1987, the AHP was already proposed to rank countries taking into account multiple criteria [97]. Clearly, in this study, the alternatives were the countries themselves, and the criteria simply all those characteristics which could make one country better than another. Indeed, with a suitable choice of criteria, this use of the AHP can be seen as a primer in the multivariate ranking of countries.

[1] It is the author's of this booklet half joke to say that the choice of San Francisco as the most livable served as a sure validation of the AHP.

Mobile value services

With the widespread use of (smart) mobile devices, mobile services and applications are becoming more and more successful and part of end-users' everyday life, but why are some devices and services successful while others are not? It is indeed of great importance to identify and understand critical success factors driving the acceptance and adoption of mobile devices and different mobile services. Traditional models mainly consider a limited set of adoption factors, focusing on the perceived values of mobile services (usefulness, ease of use, cost). Nikou and Mezei [92] proposed to use the AHP to determine the most important decision criteria driving the customers' adoption of mobile devices and mobile services. The main attributes considered include payment mode, functionality, added value, perceived quality, cost, and performance. The results of this type of studies can be essential for various service providers (operators, mobile handset manufacturers) to design profitable applications that generate value for the end-users.

Organ transplant

It is a fact that there are more people needing human organ transplants than available organs, and that different allocations of organs can make the difference between death and life. Some combinatorial optimization problems have been proposed to match donors with organs in the best possible way, and to be fair, such algorithms account for the fact that some patients require an organ in a shorter time than others. In a study, Lin and Harris [84] proposed to use the AHP to decompose the four criteria 'urgency', 'efficiency', 'benefit', and 'equity' into subcriteria and eventually estimate their importance in the donors-organs matching process. Patients were treated as alternatives, but it is clear that their huge amount would have made the use of subjective judgments impossible. Fortunately, in this case, the pairwise comparison matrices at the alternative level were filled automatically since different criteria were quite easily quantifiable. For instance, if the life expectancies of two patients are 1 and 2 years, it can be automatically derived that under that criterion, the first patient is two times more 'urgent' than the second.

Chess prediction

The AHP has been used for forecasting too. In sports, athletes can be seen as alternatives and their characteristics as criteria, and the player rated the highest shall be regarded as the most likely to win. Here we refer to an application of the AHP for the prediction of winners in chess matches—The AHP was used to evaluate the outcome of the Chess World Championships [118] as well as of the matches between Fischer and Spassky in 1972 and Karpov and Korchnoi in 1978 [116]. A possible hierarchy for this problem is represented in Figure 1.5. It is interesting to see that the values of the weights $w_1, \ldots, w_n$ in this sort of problems about forecasting can be interpreted as *subjective probabilities* [136]. For example, in this case, w_1 and w_2 could be interpreted as the subjective probabilities of the victories of the two chess players.

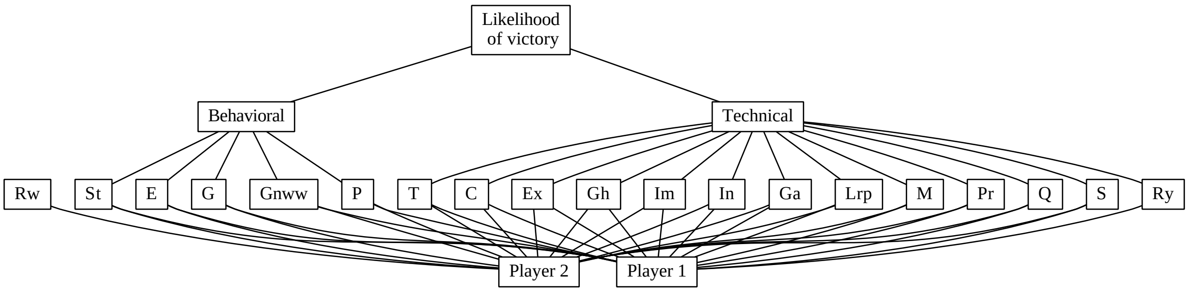


Figure 1.5.: A hierarchy for the chess competition problem. Abbreviations are as follows: Gamesmanship (G), Good Nerves and Will to Win (GN), Personality (P), Stamina (ST), Ego (E), Calculation (Q), Experience (EX), Good Health (GH), Imagination (IM), Intuition (IN), Game Aggressiveness (GA), Long Range Planning (LRP), Memory (M), Preparation (PR), Quickness (Q), Relative Youth (RY), Seconds (S), Technique (M). For a fuller description see the original paper [118].

Facility location

In Turku, a city in the South-West of Finland, the AHP has been used to find the best location the new ice hockey stadium (now called Turku Arena). Several criteria were used to evaluate different locations. Among the criteria one can find the accessibility of the arena, the possibility of having car parking, the quality of the soil on which the arena shall be built, and so forth. Carlsson and Walden [33] gave a frank political account of the decision process, which involved the local administration, and whose selected alternative was the third best ranked, and not the best.

1.3. Criticisms and open debates *

Accounts of successful applications and empirical studies [71], have brought evidence on the AHP as an appealing method for decision making. Notwithstanding, as Shakespeare put it "All that glitters is not gold".
Thus, since any fair exposition must take into account its drawbacks and open issues, we should spell it out: the AHP is *not* a flawless method. Like the driver of a race car knows the limits of the machine, users of the AHP too shall be aware of its limitations and possible misuses. In this section we shall dwell on three of them. Further matters and open debates will be recalled later when they are related to specific topics of interest of some sections of this booklet. Even so, let us now focus on the three of them which can be already understood at this stage of the exposition.

Rank reversal

The most spirited criticisms against the AHP have been based on the *rank reversal* phenomenon. Since the treatise of Von Neumann and Morgenstern [91] some axioms have been required to hold for decision analysis methodologies. One of these axioms requires that, if a new alternative is added to the original set of alternatives, then the order relation $\succ$ on the old set of alternatives should not change. Transposing this concept to our daily lives, if one has to select one meal and he prefers pasta to soup, when they offer him fish, this should *not* change his original preference of pasta to soup. Belton and Gear [14] proposed the following example to show that the AHP can suffer of rank reversal. Consider the matrices

$$\mathbf{A}^{(a)} = \begin{pmatrix} 1 & 1/9 & 1 \\ 9 & 1 & 9 \\ 1 & 1/9 & 1 \end{pmatrix} \quad \mathbf{A}^{(b)} = \begin{pmatrix} 1 & 9 & 9 \\ 1/9 & 1 & 1 \\ 1/9 & 1 & 1 \end{pmatrix} \quad \mathbf{A}^{(c)} = \begin{pmatrix} 1 & 8/9 & 8 \\ 9/8 & 1 & 9 \\ 1/8 & 1/9 & 1 \end{pmatrix}$$

which compare three alternatives with respect to three criteria, respectively—remember that a similar situation was proposed in the example of Figure 1.3. Assuming that the three criteria have equal weight, i.e. 1/3, it follows that the final priority vector is $(0.45, 0.47, 0.08)^T$, and thus the alternatives are ranked $x_2 \succ x_1 \succ x_3$. So far so good, but suppose now that a new alternative, x_4, is added to the initial set, and the new judgments are

$$\mathbf{A}^{(a)} = \begin{pmatrix} 1 & 1/9 & 1 & 1/9 \\ 9 & 1 & 9 & 1 \\ 1 & 1/9 & 1 & 1/9 \\ 9 & 1 & 9 & 1 \end{pmatrix} \quad \mathbf{A}^{(b)} = \begin{pmatrix} 1 & 9 & 9 & 9 \\ 1/9 & 1 & 1 & 1 \\ 1/9 & 1 & 1 & 1 \\ 1/9 & 1 & 1 & 1 \end{pmatrix} \quad \mathbf{A}^{(c)} = \begin{pmatrix} 1 & 8/9 & 8 & 8/9 \\ 9/8 & 1 & 9 & 1 \\ 1/8 & 1/9 & 1 & 1/9 \\ 9/8 & 1 & 9 & 1 \end{pmatrix}.$$

Note that the preferences on the original three alternatives have been unchanged. However, still considering the criteria to be equally important, the new priority vector becomes $(0.37, 0.29, 0.06, 0.29)^T$ and thus the new ranking is $x_1 \succ x_2 \sim x_4 \succ x_3$. Now x_1 is ranked the best! The gravity of this drawback is made evident if we consider the initial example of the European city, where considering one more city, say Stockholm, might have changed the original ranking of the other three, let alone more important real-world problems. In a scientific context, and especially in decision analysis where everything should be justifiable, the rank reversal has been pivotal in the debate on the theoretical soundness of the AHP. On the other hand, many scholars ignore that the rank reversal is avoided if priority vectors are aggregated taking their component-wise geometric mean, instead of a convex linear combination. Although opposed by some [131], the use of this technique to avoid the rank reversal has been proven mathematically [11]. For a review of the rank reversal the interested reader can see the original discussion [14, 15, 118], a survey [89], and an account of the AHP versus Multi Attribute Utility Theory debate [61].

The nature of the AHP

The discovery of the rank reversal has been the spark for further discussions. One of the most important relates with the nature of the AHP. In other words, on what fundamental theory is the AHP based on? As whispered before, the AHP has something in common with *value theory*. In both these theories, there is a set of alternatives which are eventually matched with real numbers such that $w_i \geq w_j \Leftrightarrow x_i \succeq x_j$. Ultimately, in value theory, there is a function $v : X \rightarrow \mathbb{R}$, where $v(x_i) = w_i$. It follows that, however complicated the function v is, rank reversal cannot happen, since alternatives are evaluated independently one from another and hence adding real or fictitious alternatives does not change the order of the existing ones. As seen before, this invariance is required by one axiom of value theory, which is violated by the AHP, because of the rank reversal. Thus, the existence of the rank reversal excludes that the AHP belongs to value theory. After Saaty's [105] attempt to axiomatize it and a debate initiated by Dyer [49], nowadays the AHP is considered to be grounded on *relative measurement theory* which can be seen as a theory where what matters are only the ratios between measurements of whatever entities under consideration. Hence, from a very high perspective, the AHP can be seen an a mathematical tool for relative measurement. The interested reader can refer to Saaty [104, 108] for an exposition of the AHP under this point of view and consider that, very recently, Bernasconi *et al.* [16] reinterpreted the AHP using the theory of psychological measurement.

Different scales

It does not take much to see that, in spite of the elegance of the relative measurement theory, a decision maker could have troubles to state that, under the climatic point of view, Barcelona is 4 times better than Reykjavik. In everyday life, people are more inclined to use linguistic expressions like "I *slightly* prefer pasta with salmon to pasta with cheese" or "I *strongly* prefer one banana to one apple". To help the decision maker, some linguistic expressions have been proposed and then linked to different values assignable to the entries a_{ij}. Hence, the decision maker can express opinions on pairs using linguistic terms, which are then associated to real numbers. In his original paper on the AHP, Saaty proposed an association between verbal judgments and values for pairwise comparisons. Other scales have been proposed and studied, among others, by Ji and Jiang [75] to which the reader can refer for a short overview. One of the foremost is the balanced scale proposed by Pöhjönen *et al.* [98]. The balanced scale, Saaty's scale, and their matching with verbal judgments are reported in Table 1.1.

Verbal description	Saaty's scale	Balanced scale
Indifference	1	1
—	2	1.22
Moderate preference	3	1.5
—	4	1.86
Strong preference	5	2.33
—	6	3
Very strong or demonstrated preference	7	4
—	8	5.67
Extreme preference	9	9

Table 1.1.: Two scales and their association with verbal judgments.

Which scale is better is still an open debate, but it is safe to say that, most likely, Saaty's scale is not optimal. It is a fact that it was introduced as a rule of thumbs, whereas other scales seem to have more supporting evidence. For instance, the balanced scale has been proposed on the basis of empirical experiments with people. Reasonably, this topic will require more research from the behavioral point of view than from the mathematical one.

One last remark is that, in spite of the open debate on the association between linguistic labels and numerical values, there is a meeting of minds on using bounded numerical scales, of which the most famous is the set of all integers up to 9 and their reciprocals,

$$\left\{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9\right\}.$$

The main reason for this choice is our limited ability of processing information, also corroborated by psychological studies according to which our capacity of reckoning alternatives is limited to 7 ± 2 of them [112]. Nevertheless, although in practice this discrete scale is employed, in the following, unless otherwise specified, we shall not restrict the discussion and adopt the more general $\mathbb{R}_{>}$. In support of this approach comes also the fact that, mathematically speaking, the algebra of the AHP, and more generally of relative measurement theory, builds on positive real numbers.

2. Priority vector and consistency

> The average man's judgment is so poor, he runs a risk every time he uses it.
>
> Ed Howe

It is important to reflect on the fact that in the previous chapter, almost unconsciously, a number of very restrictive assumptions were imposed. Let us summarize them within one sentence, where the assumptions are highlighted in italic.

A *single* decision maker is *perfectly rational* and can *precisely* express his preferences on *all pairs* of *independent* alternatives and criteria using *positive real numbers*.

Some of these assumptions had already been relaxed in Saaty's original works, and some others were relaxed later. In this and in the next chapter we shall present the ways in which these assumptions have been relaxed in the literature to provide the users of the AHP with a more flexible method. Everytime one assumption is relaxed, the previous box will be presented again and the assumption at stake emphasized in boldface. We are now ready to depart from a normative view on the AHP (how decisions should be made in a perfect world) to adopt a more descriptive view (how decisions are actually made).

2.1. Priority vector

We have seen that one pivotal step in the AHP is the derivation of a *priority vector* for each pairwise comparison matrix. Note that if each entry a_{ij} of the matrix is exactly the *ratio* between two weights w_i and w_j, then all the columns of $\mathbf{A}$ are proportional one another and consequently the weight vector is equal to any normalized column of $\mathbf{A}$ (see the matrices in Chapter 1). In this case the information contained in the matrix $\mathbf{A}$ can be perfectly synthesized in $\mathbf{w}$ and there is no loss of information.
However, we do not even bother dwelling on this case and technique to derive the weights, since it is hardly ever the case that a decision maker is so accurate and rational to give exactly the entries as ratios between weights. In this, and in the next section on consistency, we shall investigate how the AHP can cope with irrational pairwise

comparisons. Let us then represent again the box with the relaxed assumption now in boldface.

A *single* decision maker is ***perfectly rational*** and can *precisely* express his preferences on *all pairs* of *independent* alternatives and criteria using *positive real numbers*.

When the entries of the matrix $\mathbf{A}$ are not obtained exactly as ratios between weights, there does *not* exist a weight vector which perfectly synthesize the information in $\mathbf{A}$. Nonetheless, since the AHP cannot make it without the weight vectors, it is necessary to devise some smart ways of estimating a 'good' priority vector.
Several methods for eliciting the priority vector $\mathbf{w} = (w_1, \dots, w_n)^T$ have been proposed in the literature. Each method is just a rule for synthesizing pairwise comparisons into a rating, and mathematically is a function $\tau : \mathbb{R}^{n\times n}_{>} \rightarrow \mathbb{R}^{n}_{>}$. Clearly, different methods might lead to different priority vectors, except when the entries of the matrix are representable as ratios between weights, in which case all methods shall lead to the same vector $\mathbf{w}$. Needless to say, in the case of perfect rationality, the same vector $\mathbf{w}$ obtained with any method must be such that $(w_i/w_j)_{n\times n} = \mathbf{A}$.

2.1.1. Eigenvector method

The most popular method to estimate a priority vector is that proposed by Saaty himself, according to which the priority vector should be the principal eigenvector of $\mathbf{A}$. In linear algebra it is often called the Perron-Frobenius eigenvector, from the homonymic theorem [70]. The method stems from the following observation. Taking a matrix $\mathbf{A}$ whose entries are exactly obtained as ratios between weights and multiplying it by $\mathbf{w}$ one obtains

$$\mathbf{A}\mathbf{w} = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ \vdots \\ nw_n \end{pmatrix} = n\mathbf{w}.$$

From linear algebra, we know that a formulation of the kind $\mathbf{A}\mathbf{w} = n\mathbf{w}$ implies that n and $\mathbf{w}$ are an *eigenvalue* and an *eigenvector* of $\mathbf{A}$, respectively [1]. Moreover, by knowing that the other eigenvalue of $\mathbf{A}$ is 0, and has multiplicity $(n-1)$, then we know that n is the largest eigenvalue of $\mathbf{A}$. Hence, if the entries of $\mathbf{A}$ are ratios between weights, then the weight vector is the eigenvector of $\mathbf{A}$ associated with the eigenvalue n. Saaty proposed to extend this result to all pairwise comparison matrices by replacing n with the more generic maximum eigenvalue of $\mathbf{A}$. That is, vector $\mathbf{w}$ can be obtained from any pairwise comparison matrix $\mathbf{A}$ as the solution of the following equation system,

$$\begin{cases} \mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w} \\ \mathbf{w}^T\mathbf{1} = 1 \end{cases}$$

[1] A short overview of eigenvector theory in the AHP can be found in the Appendix.

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{A}$, and $\mathbf{1} = (1, \ldots, 1)^T$. Although this problem can easily be solved by mathematical software and also spreadsheets, its interpretation remains cumbersome for practitioners.

2.1.2. Geometric mean method

Another widely used method to estimate the priority vector is the *geometric mean method*, proposed by Crawford and Williams [43]. According to this method each component of $\mathbf{w}$ is obtained as the geometric mean of the elements on the respective row divided by a normalization term so that the components of $\mathbf{w}$ eventually add up to 1,

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}} \Bigg/ \underbrace{\sum_{i=1}^{n} \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}}}_{\text{normalization term}} . \tag{2.1}$$

Example 2. *Let us take into account the following matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 1/2 & 1/4 & 3 \\ 2 & 1 & 1/2 & 2 \\ 4 & 2 & 1 & 2 \\ 1/3 & 1/2 & 1/2 & 1 \end{pmatrix} \tag{2.2}$$

for which, by using (2.1), one obtains

$$\mathbf{w} \approx (0.119, 0.208, 0.454, 0.219)^T$$

Problem 2. *Prove that, if $a_{ij} = w_i/w_j \; \forall i, j$, then the geometric mean method (2.1) returns the vector* $\mathbf{w}$ *whose ratios between components are the elements of* $\mathbf{A}$.

By looking at (2.1) it is apparent that the geometric mean method is very appealing for practical applications since, in contrast to the eigenvector method, the weights can be expressed as analytic functions of the entries of the matrix. Furthermore, even the final weights of the whole hierarchy can be expressed as analytic expressions of the entries of all the matrices in the hierarchy. This is particularly important since it opens avenues to perform efficiently some sensitivity analysis. Moreover, on a more mathematical note, it is interesting to note that the vector $\mathbf{w}$ obtained with this method, can equivalently be obtained as the argument minimizing the following optimization problem

$$\begin{aligned} \underset{(w_1,\ldots,w_n)}{\text{minimize}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln a_{ij} + \ln w_j - \ln w_i \right)^2 \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i = 1, \quad w_i > 0 \, \forall i \end{aligned} \tag{2.3}$$

Problem 3. *Prove that the argument optimizing (2.3) is the same vector (up to multiplication by a suitable scalar) which could be obtained with the geometric mean method.*

This optimization problem has some interpretations, the following being quite straightforward. We know that, in the case of perfect rationality, $a_{ij} = w_i/w_j \, \forall i, j$. Indeed, it is fair to consider $\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - w_i/w_j\right)^2$ as a distance between $\mathbf{A}$ and the matrix $(w_i/w_j)_{n\times n}$ associated with the weight vector $\mathbf{w}$. Another metric can be found by using the natural logarithm ln, which is a monotone increasing function, thus obtaining $\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln a_{ij} - \ln\left(w_i/w_j\right)\right)^2$. The rest is done by observing that the logarithm of a quotient is the difference of the logarithms. Then the minimization problem (2.3) is introduced to find a suitable priority vector associated to a 'close' consistent approximation $(w_i/w_j)_{n\times n}$ of the matrix $\mathbf{A}$.

2.1.3. Other methods and discussion *

A large number of alternative methods to compute the priority vector have been proposed in the literature. Choo and Wedley [39] listed 18 different methods and proposed a numerical and comparative study. Lin [83] reconsidered and simplified their framework. Another comparative study was offered by Ishizaka and Lusti [73]. Instead, Cook and Kress [40] presented a more axiomatic analysis where some desirable properties were stated. From all these studies it appears that, besides the eigenvector and the geometric mean method, other two methods have gained some popularity.

- The so-called *least squares method* where the priority vector is the argument solving the following optimization problem

$$\begin{aligned} &\underset{(w_1,\dots,w_n)}{\text{minimize}} \quad && \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_i}{w_j}\right)^2 \\ &\text{subject to} && \sum_{i=1}^{n} w_i = 1, \quad w_i > 0 \, \forall i\,. \end{aligned} \tag{2.4}$$

  In spite of its elegance, this optimization problem can have local minimizers where the optimization algorithms get trapped. For a discussion on this method and its solutions the reader can refer to Bozóki [22].

- The other one is the *normalized columns method* which requires the normalization of all the columns of $\mathbf{A}$ so that the elements add up to 1 before the arithmetic means of the rows are taken and normalized to add up to 1 to yield the weights $w_1, \dots, w_n$. This is the simplest method but lacks solid theoretical foundation.

  Example 3. *Consider the pairwise comparison matrix (2.2) already used to illustrate the geometric mean method. Then, the matrix with normalized columns and the priority vector are the following, respectively,*

$$\begin{pmatrix} 3/22 & 1/8 & 1/9 & 3/8 \\ 6/22 & 2/8 & 2/9 & 2/8 \\ 12/22 & 4/8 & 4/9 & 2/8 \\ 1/22 & 1/8 & 2/9 & 1/8 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 21/163 \\ 42/163 \\ 84/163 \\ 16/163 \end{pmatrix}.$$

Nevertheless, in spite of the great variety of methods, it is safe to say that the eigenvector and the geometric mean method have been the most used and therefore it is convenient to confine further discussions to these two. Saaty and Vargas [117] claimed the superiority of the eigenvector method and concluded that:

> In fact it is the only method that should be used when the data are not entirely consistent in order to make the best choice of alternative.

Saaty and Hu [111] proposed a theorem claiming the necessity of the eigenvector method, and Saaty [107] also proposed ten reasons for not using other methods. Fichtner [53] proposed some axioms and showed that the eigenvector method is the only one satisfying them. Curiously, supporters of the geometric mean method have used similar arguments. For instance, Barzilai at al. [10] proposed another axiomatic framework and proved that the geometric mean method is the only one which satisfies his axioms. Seemingly, the existence of two axiomatic frameworks leading to different conclusions suggest that the choice of the method depends on what set of properties we want the method to satisfy. Supporters of the geometric mean method also gave precise statements on the use of this method and, to summarize one of his papers, Barzilai [8] wrote:

> We establish that *the geometric mean is the only method* for deriving weights from multiplicative pairwise comparisons which satisfies fundamental consistency requirements.

Bana e Costa and Vansnick [41] also moved a criticism against the eigenvector method based on what they called the *condition of order preservation* (COP). The COP states that, if x_i more strongly dominates x_j than x_k does with x_l, it means that $a_{ij} > a_{kl}$, and then it is natural to expect that the priority vector be such that $w_i/w_j > w_k/w_l$. Formally,

$$a_{ij} > a_{kl} \Rightarrow \frac{w_i}{w_j} > \frac{w_k}{w_l} \quad \forall i, j, k, l.$$

Bana e Costa and Vansnick showed some examples of cases where, given a pairwise comparison matrix $\mathbf{A}$, the eigenvector method does not return a priority vector satisfying the COP, although there exists a set of other vectors satisfying it.

On a similar note, a recent discovery related to what economists call Pareto efficiency. The reasonable idea behind this is suggested also by (2.3) and (2.4) and is that, having estimated the priority vector $\mathbf{w}$, the matrix $(w_i/w_j)_{n\times n}$ should be as near as possible to the original preferences in $\mathbf{A}$. Blanquero *et al.* [18] showed that, if $\mathbf{w}$ is estimated by the eigenvector method, in some cases there exists a vector $\mathbf{v} = (v_1, \ldots, v_n)^T \neq \mathbf{w}$ such that

$$\left| \frac{v_i}{v_j} - a_{ij} \right| \leq \left| \frac{w_i}{w_j} - a_{ij} \right| \quad \forall i, j.$$

The fact that $\mathbf{w} \neq \mathbf{v}$ implies that the inequality is strict for some i, j. To summarize, this means that there can be vectors which are closer than the eigenvector to the preferences expressed in $\mathbf{A}$. At the time of writing this manuscript, it seems that in some cases the differences between $\mathbf{v}$ and $\mathbf{w}$ can be relevant [23].

2.2. Consistency

A perfectly rational decision maker should be able to state his pairwise preferences exactly, i.e. $a_{ij} = w_i/w_j \ \forall i, j$. So, let us consider the ramifications of this condition on the entries of the pairwise comparison matrix $\mathbf{A}$. If we write $a_{ij}a_{jk}$ and apply the condition $a_{ij} = w_i/w_j \ \forall i, j$, then we can derive the following

$$a_{ij}a_{jk} = \frac{w_i}{w_j}\frac{w_j}{w_k} = \frac{w_i}{w_k} = a_{ik}.$$

Hence, we discovered that, if all the entries of the pairwise comparison matrix $\mathbf{A}$ satisfy the condition $a_{ij} = w_i/w_j \ \forall i, j$, then the following condition holds [2],

$$a_{ik} = a_{ij}a_{jk} \quad \forall i, j, k\,, \tag{2.5}$$

which means that each direct comparison a_{ik} is exactly confirmed by all indirect comparisons $a_{ij}a_{jk} \ \forall j$. Formally, a decision maker able to give perfectly consistent pairwise comparisons does not contradict himself. A matrix for which this transitivity condition holds is called *consistent*.

Example 4. *Consider the characteristic 'weight' of three stones x_1, x_2, x_3. If the decision maker says that x_1 is three times heavier than x_3 ($a_{13} = 3$), and then also says that x_1 is two times heavier than x_2 ($a_{12} = 2$), and x_2 is also two times heavier than x_3 ($a_{23} = 2$), then he contradicts himself, because he directly states that $a_{13} = 3$, but indirectly states that the value of a_{13} should be $a_{12}a_{23} = 2 \cdot 2 = 4$ and not 3.*

Evidently the whole reasoning can be translated into the language of pairwise comparison matrices.

Example 5. *Consider this other example with the two pairwise comparison matrices*

$$\begin{pmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1/2 \\ 1/2 & 1 & 2 \\ 2 & 1/2 & 1 \end{pmatrix}$$

for which we have the two diagrams in Figure 2.1 respectively.

$$\begin{tikzcd} x_1 \arrow[r, "2"] \arrow[rr, bend left, "4"] & x_2 \arrow[r, "2"] & x_3 \end{tikzcd}$$

(a) The matrix is consistent

$$\begin{tikzcd} x_1 \arrow[r, "2"] \arrow[rr, bend left, "1/2"] & x_2 \arrow[r, "2"] & x_3 \end{tikzcd}$$

(b) The matrix is inconsistent

Figure 2.1.: Examples of consistent and inconsistent transitivities.

[2] As we will see, the 'if' condition is in fact an 'if and only if'.

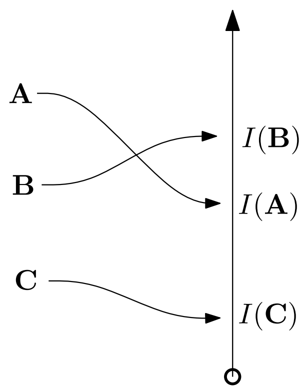


Figure 2.2.: An inconsistency index can be seen as a 'thermometer', which takes pairwise comparison matrices as inputs and evaluates how inconsistent the judgments are.

Being consistent is seldom possible because many factors can determine the emergency of inconsistencies. For instance, the decision maker might be asked to use integer numbers and their reciprocals; in this case if $a_{ij} = 3$ and $a_{jk} = 1/2$ it is impossible to find a consistent value for a_{ik}. Moreover, the number of independent transitivities (i, j, k) in a matrix of order n is equal to $\binom{n}{3}$, thus evidencing the difficulty of being fully consistent.

Example 6. *In a matrix of order 6, there are* $\binom{6}{3} = 20$ *independent transitivities; that is the number of possible assignments of values to* i, j, k *such that* $1 \leq i < j < k \leq 6$. *In a matrix of order 4, there are* $\binom{4}{3} = 4$ *transitivities. They are* $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$ *and* $(2, 3, 4)$.

In spite of the difficulty in being fully transitive, it is undeniable that consistency is a desirable property. In fact, an inconsistent matrix could be a symptom of the decision maker's incapacity or inexperience in the field. Additionally, it is possible to envision that violations of the condition of consistency (2.5) can be of different extent and gravity and imagine inconsistency as a gradual notion. Hence, on the ground that a matrix should deviate as less as possible from the condition of transitivity, a number of inconsistency indices have been proposed in the literature to quantify this deviation. Formally, an *inconsistency index* is a function mapping pairwise comparison matrices into the real line (see Figure 2.2 for an oversimplification).

There exist various inconsistency indices in the literature and this variety is in part justified by the fact that the condition of consistency can be formulated in many *equivalent* ways. Among them, it is the case to reckon the following four:

i) $a_{ik} = a_{ij}a_{jk} \; \forall i, j, k$,

ii) There exists a vector $(w_1, \ldots, w_n)^T$ such that $a_{ij} = w_i/w_j \; \forall i, j$,

iii) The columns of $\mathbf{A}$ are proportional, i.e. $\mathbf{A}$ has rank one,

iv) The pairwise comparison matrix $\mathbf{A}$ has its maximum eigenvalue, $\lambda_{\max}$, equal to n.

In this section we explore some inconsistency indices, each inspired by one of these equivalent consistency conditions.

2.2.1. Consistency index and consistency ratio

According to the result that given a pairwise comparison matrix $\mathbf{A}$, its maximum eigenvalue, $\lambda_{\max}$, is equal to n if and only if the matrix is consistent (and greater than n otherwise), Saaty [101] proposed the *Consistency Index*

$$CI(\mathbf{A}) = \frac{\lambda_{\max} - n}{n-1}. \tag{2.6}$$

However, numerical studies showed that the expected value of CI of a random matrix of size $n+1$ is, on average, greater than the expected value of CI of a random matrix of order n. Consequently, CI is not fair in comparing matrices of different orders and needs to be rescaled.
The *Consistency Ratio*, CR, is the rescaled version of CI. Given a matrix of order n, CR can be obtained dividing CI by a real number RI_n (*Random Index*) which is nothing else but an estimation of the average CI obtained from a large enough set of randomly generated matrices of size n. Hence,

$$CR(\mathbf{A}) = \frac{CI(\mathbf{A})}{RI_n} \tag{2.7}$$

Estimated values for RI_n are reported in Table 2.1. Note that the generation of random matrices requires the definition of a bounded scale where the entries take values, for instance the interval $[1/9, 9]$. According to Saaty [102], in practice one should accept matrices with values $CR \leq 0.1$ and reject values greater than 0.1. A value of $CR = 0.1$ means that the judgments are 10% as inconsistent as if they had been given randomly.

n	3	4	5	6	7	8	9	10
RI_n	0.5247	0.8816	1.1086	1.2479	1.3417	1.4057	1.4499	1.4854

Table 2.1.: Values of RI_n [3].

Example 7. *Consider the pairwise comparison matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 9 & 1 \\ 1/2 & 1 & 1/3 & 1/6 \\ 1/9 & 3 & 1 & 2 \\ 1 & 6 & 1/2 & 1 \end{pmatrix}. \tag{2.8}$$

It can be calculated that its maximum eigenvalue is $\lambda_{\max} = 5.28$*. Using the formula for* CI*, we obtain* $CI(\mathbf{A}) = 0.42667$*. Dividing it by* RI_4 *one obtains* $CR(\mathbf{A}) \approx 0.48$ *which is significantly greater than the threshold* 0.1*. In a decision problem it is common practice to ask the decision maker to revise his judgments until a value of* CR *smaller than* 0.1 *is reached.*

2.2.2. Index of determinants

The index of determinants was proposed by Peláez and Lamata [95] and comes from the following property of a matrix of order three. Expanding the determinant of a 3×3 real matrix one obtains

$$\det(\mathbf{A}) = \frac{a_{13}}{a_{12}a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - 2.$$

If $\mathbf{A}$ is not consistent, then $a_{13} \neq a_{12}a_{23}$ and $\det(\mathbf{A}) > 0$, because, in general, $\frac{a}{b} + \frac{b}{a} - 2 > 0 \;\; \forall\, a \neq b,\; a, b > 0$.
It is possible to generalize this result to matrices of order greater than three and define this inconsistency index as the average of the determinants of all the possible submatrices $\mathbf{T}_{ijk}$ of a given pairwise comparison matrix, constructed as follow,

$$\mathbf{T}_{ijk} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ a_{ji} & 1 & a_{jk} \\ a_{ki} & a_{kj} & 1 \end{pmatrix}, \quad \forall i < j < k.$$

The number of so constructed submatrices is $\binom{n}{3} = \frac{n!}{3!(n-3)!}$. The result is an index whose value is the average inconsistency computed for all the submatrices $\mathbf{T}_{ijk}$ $(i < j < k)$

$$CI^*(\mathbf{A}) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \underbrace{\left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right)}_{\det(\mathbf{T}_{ijk})} \Big/ \binom{n}{3}. \tag{2.9}$$

Example 8. *Consider the matrix* $\mathbf{A}$ *in (2.8). It is then possible to calculate the average of the determinants of all the submatrices* $\mathbf{T}_{ijk}$ *with* $i < j < k$.

$$\begin{aligned} CI^*(\mathbf{A}) =& \frac{\overbrace{\det\begin{pmatrix} 1 & 2 & 9 \\ 1/2 & 1 & 1/3 \\ 1/9 & 3 & 1 \end{pmatrix}}^{\mathbf{T}_{123}} + \cdots + \overbrace{\det\begin{pmatrix} 1 & 1/3 & 1/6 \\ 3 & 1 & 2 \\ 6 & 1/2 & 1 \end{pmatrix}}^{\mathbf{T}_{234}}}{4} \\ =&(11.5741 + 1.3333 + 16.0556 + 34.0278)/4 = 15.7477. \end{aligned}$$

Interestingly, CI^* is proportional to another inconsistency index called c_3 [27]. The coefficient c_3 of the characteristic polynomial of a pairwise comparison matrix was proposed to act as an inconsistency index by Shiraishi and Obata [124] and Shiraishi *et al.* [125, 126]. By definition, the characteristic polynomial[3] of a matrix $\mathbf{A}$ has the following form

$$P_{\mathbf{A}}(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n,$$

with $c_1, \ldots, c_n$ that are real numbers and λ the unknown. Shiraishi *et al.* [125] proved that, if $c_3 < 0$, then the matrix cannot be fully consistent. In fact, this is evident if one

[3]See appendix on eigenvalues and eigenvectors

reckons that—in light of the Perron-Frobenius theorem—the only possible formulation of the characteristic polynomial which yields $\lambda_{\max} = n$, is

$$P_{\mathbf{A}}(\lambda) = \lambda^{n-1}(\lambda - n). \tag{2.10}$$

Thus, the presence of $c_3 < 0$ contradicts this last formulation and is certainly a symptom of inconsistency. Moreover, Shiraishi *et al.* [125] also proved that c_3 has the following analytic expression

$$c_3 = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}} \right) \tag{2.11}$$

which highlights its proportionality with CI^*.

2.2.3. Geometric consistency index

This index was introduced by Crawford [42] and reexamined by Aguarón and Moreno-Jiménez [2]. It considers the priority vector to be estimated by the geometric mean method (2.1). With the so estimated weights it is possible to build a local quantification of inconsistency e_{ij} for each entry a_{ij} such that

$$e_{ij} = a_{ij}\frac{w_j}{w_i}, \quad i,j = 1,\ldots,n. \tag{2.12}$$

Obviously, for consistent matrices the value of e_{ij} is equal to 1 because it becomes the multiplication of an entry times its reciprocal. Note that,

$$a_{ij} = \frac{w_i}{w_j} \;\Rightarrow\; \ln e_{ij} = 0.$$

It is now possible to define an index of global inconsistency as the normalized sum of the local contributions to the inconsistency of $\mathbf{A}$. This index of global inconsistency, the Geometric Consistency Index (GCI), is the following:

$$GCI(\mathbf{A}) = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\ln e_{ij}\right)^2 . \tag{2.13}$$

Example 9. *We refer to the matrix* $\mathbf{A}$ *presented in (2.8). By using the geometric mean method, the priority vector is* $\mathbf{w} \approx (2.06, 0.408, 0.904, 1.316)^T$. *Next, it could be convenient to collect values* e_{ij} *obtained with (2.12) into the following auxiliary matrix*

$$\mathbf{E} = (e_{ij})_{n\times n} = \left(a_{ij}\frac{w_j}{w_i}\right)_{n\times n} = \begin{pmatrix} 1 & 0.3964 & 3.9482 & 0.6389 \\ 2.5227 & 1 & 0.7379 & 1.8612 \\ 0.2538 & 1.3554 & 1 & 2.9129 \\ 1.5651 & 1.8612 & 0.3432 & 1 \end{pmatrix}.$$

Finally, the last computation is achieved applying formula (2.13), which yields $GCI(\mathbf{A}) \approx 1.52$.

2.2.4. Harmonic consistency index

If and only if $\mathbf{A}$ is a consistent pairwise comparison matrix, then its columns are proportional and $\text{rank}(\mathbf{A}) = 1$. Therefore, it is fair to suppose that the less proportional are the columns, the less consistent is the matrix. An index of inconsistency loosely based on proportionality between columns was then proposed by Stein and Mizzi [127]. Given a matrix $\mathbf{A}$, they proposed to construct an auxiliary vector $\mathbf{s} = (s_1, \ldots, s_n)^T$ with $s_j = \sum_{i=1}^{n} a_{ij} \; \forall j$. It was proven that $\sum_{j=1}^{n} s_j^{-1} = 1$ if and only if $\mathbf{A}$ is consistent, and smaller than 1 otherwise. The harmonic mean of the components of vector $\mathbf{s}$ is then the result of the following

$$HM(\mathbf{s}) = \frac{n}{\sum_{j=1}^{n} \frac{1}{s_j}}. \tag{2.14}$$

The function HM itself could be an index of inconsistency, but the authors, according to computational experiments, proposed a normalization in order to align its behavior with that of CI. The Harmonic Consistency Index is then

$$HCI(\mathbf{A}) = \frac{(HM(\mathbf{s}) - n)(n+1)}{n(n-1)}. \tag{2.15}$$

Example 10. *Considering the matrix* $\mathbf{A}$ *in (2.8), then the vector* $\mathbf{s}$ *is*

$$\mathbf{s} = \left(\frac{47}{18}, 12, \frac{65}{6}, \frac{25}{6} \right)^T$$

whose harmonic mean is

$$HM(\mathbf{s}) = \frac{4}{\frac{1}{\frac{47}{18}} + \frac{1}{12} + \frac{1}{\frac{65}{6}} + \frac{1}{\frac{25}{6}}} = \frac{733200}{146387} = 5.00864$$

Now it is possible to derive the value of the harmonic consistency index by plugging the value $HM(\mathbf{s})$ *into (2.15) and obtain* $HCI(\mathbf{A}) \approx 0.42$.

2.2.5. Ambiguity index

Salo and Hämäläinen [121, 122] proposed an *ambiguity index* which can be used as an inconsistency index too. It requires the construction of an auxiliary interval-valued matrix

$$\bar{\mathbf{A}} = (\bar{a}_{ij})_{n\times n} = \begin{pmatrix} [a_{11}^L, a_{11}^R] & \ldots & [a_{1n}^L, a_{1n}^R] \\ \vdots & \ddots & \vdots \\ [a_{n1}^L, a_{n1}^R] & \ldots & [a_{nn}^L, a_{nn}^R] \end{pmatrix}$$

where $a_{ij}^L = \min\{a_{ik}a_{kj} | k = 1, \ldots, n\}$ and $a_{ij}^R = \max\{a_{ik}a_{kj} | k = 1, \ldots, n\}$. Namely, each interval's lower (upper) bound a_{ij}^L (a_{ij}^R) is the smallest (greatest) possible value assignable to the entry if it was estimated indirectly using a transitivity of the pairwise comparison matrix. Surely, if $\mathbf{A}$ is consistent, then all the intervals collapse into real numbers. From this insight, Salo and Hämäläinen deduced that the wider the intervals, the more inconsistent $\mathbf{A}$ should be. Hence, they presented their consistency measure,

$$CM(\mathbf{A}) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{a_{ij}^R - a_{ij}^L}{\left(1 + a_{ij}^R\right)\left(1 + a_{ij}^L\right)},$$

Example 11. *Salo and Hämäläinen proposed the following simple example [122]. Consider*

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 4 \\ 1/7 & 1 & 1/5 \\ 1/4 & 5 & 1 \end{pmatrix}$$

from which one can derive

$$\bar{\mathbf{A}} = \begin{pmatrix} 1 & [7,20] & [7/5,4] \\ [1/20,1/7] & 1 & [1/5,4/7] \\ [1/4,5/7] & [7/4,5] & 1 \end{pmatrix}. \tag{2.16}$$

It follows that

$$CM(\mathbf{A}) = \frac{2}{3 \cdot 2}\left(\frac{20-7}{(20+1)\cdot(7+1)} + \frac{4-\frac{7}{5}}{(4+1)\cdot(7+1)} + \frac{\frac{4}{5}-\frac{1}{5}}{(\frac{4}{7}+1)\cdot(\frac{1}{5}+1)}\right) = 0.16\,.$$

2.2.6. Other indices and discussion *

Many other inconsistency indices have been proposed. For instance, Koczkodaj [80] proposed an inconsistency index for matrices of order three which was later extended to matrices of greater order [48]. Golden and Wang formulated an index which considers a metric between the normalized columns of the matrix and the priority vector obtained either with the eigenvector or the geometric mean method [63]. Cavallo and D'Apuzzo proposed an interpretation of pairwise comparison matrices using group theory and introduced their own index [35, 36]. Barzilai, first transformed the entries of the matrix by means of a logarithmic function and then proposed another index [9]. Gass and Rapcsák [62] defined an index based on the singular value decomposition of matrices. Furthermore, consider that even the objective functions of the optimization problems of the logarithmic least squares (2.3) and the least squares (2.4) used in §2.1 to derive the priority vector can be considered inconsistency index. The interested reader can refer to a survey paper with numerical tests on various indices [26].

More broadly, and to include the most updated results, it is the case to remark that recently some questions have been answered.

- Some questions were open on the behavioral side of consistency. For instance, does the order in which the comparisons are asked affect the final inconsistency? Does inconsistency increase with the order of the matrix? These, and other questions, have been answered by means of empirical experiments with real decision makers [24].

- It used to be unclear whether different inconsistency indices were similar or choosing one or another really made a difference. Namely, the formulations of the indices are often so dissimilar that we cannot understand if they tend to give similar results. By means of numerical simulations it was discovered that some indices are very similar whereas some others can give very different results [26]. Curiously, some indices have even been proved to be proportional, and thus equivalent, in estimating inconsistency [27].

- Inconsistency indices have been introduced empirically and an inquisitive reader might not take their validity for granted. Clearly, functions like the product of all the entries of a matrix, $\Pi(\mathbf{A})$, or the trace of a matrix $\text{tr}(\mathbf{A})$ cannot capture the inconsistency of a matrix. Five axioms were proposed and considered necessary to characterize inconsistency indices and it was proved that, in fact, some inconsistency indices fail to satisfy some of them and can be suspected of ill-behavior in some situations [28]. Figure 2.3 is a snapshot of the axiomatic system and its role.

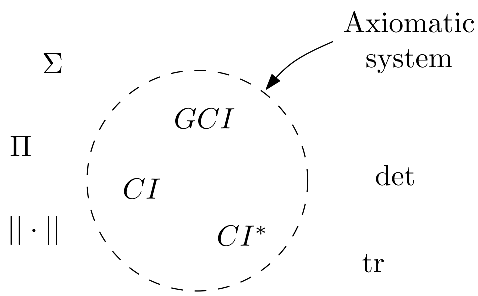


Figure 2.3.: The axiomatic system [28] defines a set of functions suitable to estimate inconsistency

In the literature it is often assumed that one inconsistency index together with a threshold value should be used to test whether the inconsistency of a matrix is tolerable or not. At least three points can be raised in connection with this standard procedure. Firstly, unlike for the Consistency Index, thresholds have been rarely presented for inconsistency indices. That is, indices were introduced *tout court*, without discussions on acceptance rules for sufficiently consistency matrices. Hence, presently, without thresholds the use of many indices is limited to stating if a matrix is more (or less) inconsistent that another.
Secondly, a wise proposal would be that of using two indices, perhaps quite dissimilar, to test if a matrix is not too inconsistent. New acceptance rules might be devised too; considering two indices, one could for instance accept matrices respecting the threshold value for both the first *and* the second inconsistency index.
Thirdly, it is simple to envision that a decision maker can hardly ever be completely consistent, and therefore, instead of requiring him to achieve a sufficiently low value for an inconsistency index, one might want to lower the bar and introduce less demanding conditions which can realistically be fully satisfied. This kind of reasoning has pushed some authors to research on weaker conditions of consistency. One natural way to force

transitivity, but in a weaker sense, would be that of doing without the degrees of preference and simply require that if a decision maker prefers x_i to x_j and x_j to x_k, then it should also prefer x_i to x_k. This condition, which is nothing else but a restatement of ordinal transitivity for binary relations, can be formalized as follows:

$$a_{ij} > 1 \text{ and } a_{jk} > 1 \Rightarrow a_{ik} > 1 \quad \forall i, j, k.$$

This condition can be strengthened into the more restrictive *weak consistency* condition:

$$a_{ij} > 1 \text{ and } a_{jk} > 1 \Rightarrow a_{ik} > \max\{a_{ij}, a_{jk}\} \quad \forall i, j, k.$$

A deeper analysis of weaker consistency conditions, and their implications on the stability of the ranking of alternatives, can be found in [12, 13].

3. Missing comparisons and group decisions

3.1. Missing comparisons

Having, and manipulating, a complete and consistent pairwise comparison matrix means dealing with rich and reliable information and therefore it represents the most desirable situation in a decision making problem with the AHP. However, sometimes, it is not possible for the decision maker to express all the pairwise comparisons and therefore, it is nowadays common practice to accept that some entries of a pairwise comparison matrix be missing. Let us now reprise the famous box and highlight the assumption that we are going to relax

A *single* decision maker is *perfectly rational* and can *precisely* express his preferences on ***all pairs*** of *independent* alternatives and criteria using *positive real numbers*.

In complex problems like those considered in §1.2, it may happen that the decision maker cannot complete a preference relation due to lack of time, the typology of problem, his incapacity in comparing two alternatives of different nature, and so forth [29]. Besides, sometimes, even if it was possible to obtain all the pairwise comparisons, doing so could be discouraged, since, due to information overload, the last ones could be given with less attention and care [34]. Certainly, it might be better to have few pairwise comparisons carefully given, than many, but given with scarce attention.
Hereafter, we shall call *incomplete pairwise comparison matrix* any pairwise comparison matrix with some missing entries. Considering all the diagonal elements of the matrix as given and the fact that, thanks to reciprocity, we only need to know a_{ij} to derive its reciprocal a_{ji}, then a pairwise comparison matrix of order n requires $n(n-1)/2$ *independent* comparisons.

Example 12. *Consider a decision problem with 6 alternatives evaluated with respect to 5 criteria. Then, the number of independent comparisons is*

$$5\frac{6(6-1)}{2}+\frac{5(5-1)}{2}=75+10=85.$$

Problem 4. *How many independent comparisons are required by the hierarchy in Figure 1.4 used in Problem 1?*

All in all, the range of reasons for leaving a matrix incomplete is wide and the real problem is how to derive a priority vector when there is not full information about the preferences on alternatives. In fact, we have seen that the eigenvector method and the geometric mean method were defined only for complete matrices. Several methods have been implemented to face this problem and, despite their diversities, considering $\mathbf{A}$ and $\dot{\mathbf{A}}$ to be a complete and an incomplete pairwise comparison matrix, respectively, the decision maker can proceed in one of the two following alternative ways:

- Complete the matrix by means of the information provided by the existing comparisons, ①. This operation is usually carried out following some principles of consistency, in the sense that the missing comparisons should be as coherent as possible with the existing ones. Having done this, it is possible to estimate the priority vector by means of one of the methods discussed earlier in §2.1 ②

- Estimate directly the priority vector by means of some modified algorithms which work even when some comparisons are missing, ③.

These two ways of proceeding are illustrated in the diagram below.

$$\begin{tikzcd}
\dot{\mathbf{A}} \arrow[r, "①"] \arrow[dr, "③"'] & \mathbf{A} \arrow[d, "②"] \\
& \mathbf{w}
\end{tikzcd} \tag{3.1}$$

The following two subsections describe an algorithm of the first kind as well as one of the second kind.

3.1.1. Optimization of the coefficient c_3

The name c_3 refers to the coefficient of λ^{n-3} in the characteristic polynomial of the matrix $\mathbf{A}$. Shiraishi *et al.* [126] observed that c_3 can be considered an inconsistency index for a pairwise comparison matrix. This was already discussed in §2.2.2 and the analytic formula of c_3 was given in (2.11). Then, in order to complete $\dot{\mathbf{A}}$ following a principle of consistency, the authors considered the m missing comparisons as variables $\alpha_1, \ldots, \alpha_m$ and proposed to maximize c_3 (reckon that the greater c_3 the smaller the inconsistency) as a function of these variables, thus obtaining the values of the missing comparisons by solving

$$\begin{aligned}
&\underset{(\alpha_1,\ldots,\alpha_m)}{\text{maximize}} && c_3 \\
&\text{subject to} && \alpha_1, \ldots, \alpha_m > 0\,.
\end{aligned} \tag{3.2}$$

Note that there always exists an optimum for (3.2), but when there are too many missing comparisons uniqueness is not guaranteed [124].

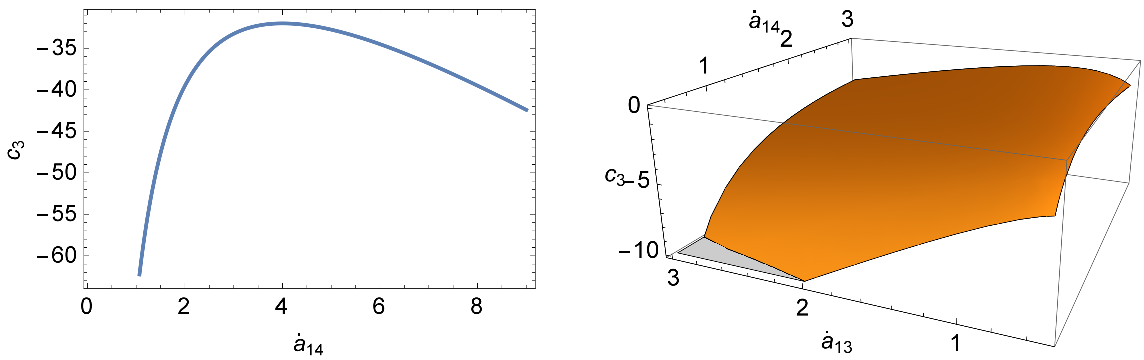
Example 13. *First, we present an incomplete pairwise comparison matrix* $\dot{\mathbf{A}}$

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 4 & \dot{a}_{14} \\ 1/2 & 1 & 1/3 & 1 \\ 1/4 & 3 & 1 & 2 \\ 1/\dot{a}_{14} & 1 & 1/2 & 1 \end{pmatrix}. \tag{3.3}$$

Its missing comparison can be estimated by solving (3.2) with $\alpha_1 = \dot{a}_{14}$. *The plot of* c_3 *as a function of* $\dot{a}_{14}$ *is depicted in Figure 3.1a. The optimal solution is* $\dot{a}_{14} = 4$. *If we further assume that also* $\dot{a}_{13}$ *is missing, then the new incomplete pairwise comparison matrix becomes*

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & 2 & \dot{a}_{13} & \dot{a}_{14} \\ 1/2 & 1 & 1/3 & 1 \\ 1/\dot{a}_{13} & 3 & 1 & 2 \\ 1/\dot{a}_{14} & 1 & 1/2 & 1 \end{pmatrix}. \tag{3.4}$$

and the plot of c_3 *as a function of both* $\dot{a}_{13}$ *and* $\dot{a}_{14}$ *is in Figure 3.1b. In this case, the optimal solution is* $\dot{a}_{13} \approx 0.763143$ *and* $\dot{a}_{14} \approx 1.74716$.



(a) Plot of c_3 as a function of $\dot{a}_{14}$ (b) Plot of c_3 as a function of $\dot{a}_{14}$ and $\dot{a}_{13}$

Figure 3.1.: Plots of c_3

Problem 5. *Consider the matrix* $\dot{\mathbf{A}}$ *in (3.3). Find a way to recover* $\dot{a}_{14} = 4$ *as the* analytic *solution of the nonlinear optimization problem (3.2).*

The optimization of the coefficient c_3 provides values for the missing entries, and, upon completion of the matrix, it becomes straightforward to derive a priority vector.

3.1.2. Revised geometric mean method

This method, proposed by Harker [68], is not explicitly based on the optimization of an objective function, but refers to the eigenvector approach by Saaty. Practically, it extends Saaty's approach to non-negative quasi-reciprocal matrices, in order to apply it to the case of incomplete preferences. Unlike the optimization of c_3 proposed in (3.2), this method does not reconstruct the matrix but instead finds a priority vector using less information. The algorithm requires to construct an auxiliary matrix $\mathbf{C} = (c_{ij})_{n\times n}$ as follows

$$c_{ij} = \begin{cases} 1+m_i, & \forall i = j \\ \dot{a}_{ij}, & \forall i \neq j \text{ and } \dot{a}_{ij} \text{ not missing} \\ 0, & \dot{a}_{ij} \text{ is missing} \end{cases}$$

where m_i is the number of missing comparisons on the ith row. Having done this, the priority vector can be estimated by means of the eigenvector method. The following case, proposed by Harker [68], provides a numerical toy example and more insight on the method.

Example 14. *Consider the following pairwise comparison matrix,*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & \dot{a}_{13} \\ 1/2 & 1 & 2 \\ \dot{a}_{31} & 1/2 & 1 \end{pmatrix},$$

and replace the missing comparison and its reciprocal entry with their theoretical values w_i/w_j so that the new matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & \frac{w_1}{w_3} \\ 1/2 & 1 & 2 \\ \frac{w_3}{w_1} & 1/2 & 1 \end{pmatrix}.$$

It is possible to observe what is obtainable through the operation $\mathbf{Bw}$

$$\mathbf{Bw} = \begin{pmatrix} 1 & 2 & \frac{w_1}{w_3} \\ 1/2 & 1 & 2 \\ \frac{w_3}{w_1} & 1/2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 2w_1 + 2w_2 \\ w_1/2 + w_2 + 2w_3 \\ w_2/2 + 2w_3 \end{pmatrix}.$$

We can reach the same result considering $\mathbf{Cw}$ with

$$\mathbf{C} = \begin{pmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{pmatrix}.$$

Finally, we can certainly state that

$$\mathbf{Bw} = \mathbf{Cw}. \tag{3.5}$$

We can proceed with the elicitation of weights, extending what was stated above. Therefore,

$$\mathbf{Bw} = \mathbf{Cw} = \lambda_{\max}\mathbf{w}. \tag{3.6}$$

Since $\mathbf{B}$ has some non-numerical entries, we can solve the eigenvector problem for $\mathbf{C}$. Needless to say, the result is $\mathbf{w} = (4, 2, 1)^T$.

Problem 6. *Formulate the auxiliary matrix $\mathbf{C}$ associated with $\mathbf{A}$ in (3.4).*

3.1.3. Other methods and discussion ⋆

There are other methods, of both types. For instance, Harker proposed the application of the concept of *connecting path.* A connecting path of length r between i and j is a product $a_{i\,i_1}a_{i_1i_2}\cdots a_{i_{r-2}i_{r-1}}a_{i_{r-1}j}$, where the special case with $r=2$ collapses to the consistency condition $a_{ik} = a_{ij}a_{jk}$. Harker proposed to compute the missing entries taking the geometric mean of all their connecting paths. Although sound, this proposal suffers of computational complexity when the size of the matrix is large enough, and is difficult to implement when several comparisons are missing.

It seems that one natural way to estimate missing comparisons is that of using some principles of consistency. For example, an inconsistency index can be optimized and the missing comparisons be used as variables. True, the foremost inconsistency index has been the CI which, fixed a value of n, is a positive affine transformation of the maximum eigenvalue $\lambda_{\max}$, which in turn is a root of a polynomial of degree n, hence impossible to be expressed analytically, except in few cases. In spite of this problem, using some convexity properties, Bozóki *et al.* [25] were able to formulate an optimization problem and a special algorithm to minimize $\lambda_{\max}$ keeping the missing comparisons as variables. It is indeed a very valuable proposal, but remains too cumbersome to be explained in this booklet.

On a more general level, a deeper discussion on missing comparisons goes back to the philosophy of the AHP and question how many comparisons the decision maker should provide. Is it carved in stone that the matrix has to be complete? Can, instead, some comparisons be missing? How many? This question has at least two possible answers, one algorithmic and one connected with common sense.

- From the *algorithmic* point of view, different methods for dealing with incomplete pairwise comparisons give different answers. Considering the revised geometric mean method presented in §3.1.2 one can observe that, in fact, this method works even when all the nondiagonal entries of $\dot{\mathbf{A}}$ are missing, in which case it returns a priority vector where all the weights are equal. More generally, it was also shown that one needs only $(n-1)$ independent comparisons to complete a matrix and make it consistent in a univocal way [69]. In fact, the knowledge of a set of comparisons, as for instance the set of entries right above the main diagonal, $\{a_{12}, a_{23}, \ldots, a_{n-1\,n}\}$, or the set of non-diagonal entries on, say, the first row, $\{a_{12}, a_{13}, \ldots, a_{1n}\}$ suffices to reconstruct the missing entries using the condition $a_{ik} = a_{ij}a_{jk}\ \forall i,j,k$.

Problem 7. *Consider the following matrix $\dot{\mathbf{A}}$ and reconstruct its missing entries using the consistency condition $a_{ik} = a_{ij}a_{jk}\ \ \forall i,j,k$.*

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 1/2 & 1 & \dot{a}_{23} & \dot{a}_{24} \\ 1/4 & 1/\dot{a}_{23} & 1 & \dot{a}_{34} \\ 1/3 & 1/\dot{a}_{24} & 1/\dot{a}_{34} & 1 \end{pmatrix}$$

- On the *common sense* side of the subject matter, one would surely refrain from estimating the priority vector from a totally incomplete matrix, and probably even question the convenience of reconstructing a (consistent) matrix from $(n-1)$ comparisons, since at that point the original $(n-1)$ comparisons would have been sufficient to estimate the priority vector directly. Moreover, by giving up a large number of comparisons one also gives up the possibility to estimate the inconsistency of preferences and thus to detect potential flaws in the decision maker's judgments. One last reason for not leaving too many comparisons missing is that evaluation errors can better compensate, and tend to cancel each other, when there are many comparisons then when there are few.

Another open question regards what comparisons should be elicited and what can be left missing. For example, knowing that the decision maker is willing to express his opinions on a subset of pairwise comparisons, but not all, then which ones should he express, and in what order? The quest for optimal completion rules and optimal completion paths has inspired some papers, as for instance those by Harker [67] and Fedrizzi and Giove [52].

There are various research papers on methods for dealing with incomplete preferences but very few investigated the relation between the number of missing comparisons and the stability of the obtained priority vector. One of these rare studies was by Carmone *et al.* [34] and it is safe to say that there is need and space for further investigation.

3.2. Group decisions

A further assumption was made regarding the number of decision makers: so far opinions have been given by a single decision maker. Even in the introductory exposition of the AHP given in Chapter 1, in the example of the European city, the family was considered as an unique entity and we did not account for possibly different opinions of family members. However, in many real-world contexts, decisions are made by groups of people, committees, boards, teams of experts, and so forth. Whenever there is a multitude of experts bringing diverse evidence on a problem, it is good practice to account for them.

> A ***single*** decision maker is *perfectly rational* and can *precisely* express his preferences on *all pairs* of *independent* alternatives and criteria using *positive real numbers*.

In his popular book *The Wisdom of the Crowd*, Surowiecki [128] argued that collective intelligence often outperforms individual one. An anecdote, originally by Galton [60], is reported in the introductory part of the book: at a county fair, individuals were asked to estimate the exact weight of an ox. Remarkably, by averaging the responses of the crowd members, they could obtain an estimate of the weight of the ox which was closer to the true weight then any of the individual judgments which were instead given by a

number of cattle experts. In other words, the collective wisdom of the crowd was more accurate than the estimates of true experts in the field. The problem of the ox was a problem of measurement, and we should not forget that the AHP is a theory of relative measurement.
The AHP for group decisions has been acclaimed by some researchers, for instance Dyer and Forman [50]. Peniwati [96] proposed some desirable properties, e.g. 'Scientific and mathematical generality', 'psychophysical applicability', and 'applicability to conflict resolution', for group decisions with MCDM methods, and according to her qualitative analysis, the AHP is a valuable decision methodology for group decisions.
The AHP can be used in many different ways as a group decision making method and it can be implemented in the so-called *Delphi method* [85]. In a nutshell, the Delphi method prescribes a number of meetings led by a moderator, where, after each meeting, the decision makers can revise their opinions. The role of the moderator is to make the opinions of different decision makers converge towards a consensual solution. Nonetheless, in spite of its seeming triviality, any short description of the Delphi method would be an oversimplification, and any lengthy one would go beyond the scope of this exposition. We shall therefore use some mathematical notation and focus on another way to make sense of the AHP in group decisions.

Suppose that m ($m \geq 2$) decision makers are involved in a decision and we want to take into account and synthesize their opinions, i.e. we want to *aggregate* them. This suggests that we ought to average different opinions. More specifically, starting from their pairwise comparison matrices

$$\underbrace{\left(a_{ij}^{(1)}\right)_{n\times n}}_{\mathbf{A}_1}, \ldots, \underbrace{\left(a_{ij}^{(m)}\right)_{n\times n}}_{\mathbf{A}_m}$$

we eventually want to obtain *one* representative group priority vector $\mathbf{w}^G = (w_1^G, \ldots, w_n^G)^T$. According to Forman and Peniwati [57] there are two methods to derive a vector $\mathbf{w}^G$ from a set of pairwise comparison matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ and they differ in where the aggregation takes place.

- *Aggregation of individual judgments* (AIJ): Matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ can be aggregated into a single pairwise comparison matrix, $\mathbf{A}^G = (a_{ij}^G)$, and then the priority vector be calculated from $\mathbf{A}^G$ with any of the methods described in §2.1. In this case the aggregation happens before the elicitation of the priorities

- *Aggregation of individual priorities* (AIP): Priority vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ can be derived from the original set of matrices. These vectors are then aggregated into $\mathbf{w}^G$. In this case, the aggregation happens after the derivation of the priority vectors.

The following scheme shall clarify the difference between AIJ and AIP. Also, it should be evident that, either way, going from a set of pairwise comparison matrices to a single priority vector entails a double process of aggregation.

$$
\begin{tikzcd}
\mathbf{A}_1,\dots,\mathbf{A}_m \arrow[r, "\S 2.1"] \arrow[d, "\mathrm{AIJ}"'] & \mathbf{w}_1,\dots,\mathbf{w}_m \arrow[r, "\mathrm{AIP}"] & \mathbf{w}^G \\
\mathbf{A}^G \arrow[d, "\S 2.1"] & & \\
\mathbf{w}^G & &
\end{tikzcd}
\tag{3.7}
$$

Obviously, the crucial point is that of finding a suitable aggregation function. For the aggregation of individual *judgments* (AIJ), the reader can check that a basic function like the arithmetic mean fails since the resulting matrix $\mathbf{A}^G$ would not be reciprocal. Aczel and Saaty [1] and Saaty and Alsina [110] proposed a set of reasonable properties for the aggregation of preferences and, by using functional analysis, proved that in this context the only meaningful and non-trivial aggregation method is the weighted geometric mean. Namely, entries of the group matrix $\mathbf{A}^G = (a_{ij}^G)_{n\times n}$ are obtained using the following parametric formula,

$$a_{ij}^G = \prod_{h=1}^{m} a_{ij}^{(h)^{\lambda_h}}$$

with $\lambda_h > 0 \ \forall h$ and $\lambda_1 + \cdots + \lambda_m = 1$. The most common interpretation of a given λ_h is that it should be proportional to the importance of the hth decision maker. When $\lambda_h = 1/m \ \forall h$ then all the decision makers have the same importance. Conversely, if $\lambda_h > \lambda_k$, then the relative importance of the hth decision maker is greater than that of the kth.

Example 15. *Consider the very simple case of two decision makers with preferences expressed as*

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 2 & 1/2 \\ 1/2 & 1 & 3 \\ 2 & 1/3 & 1 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 7 & 2 \\ 1/7 & 1 & 1/4 \\ 1/2 & 4 & 1 \end{pmatrix} \tag{3.8}$$

and suppose that the first decision maker should be twice as influential as the second. This suggests the use of $\lambda_1 = 2/3$ and $\lambda_2 = 1/3$. Hence, the group matrix computed with AIJ is

$$\mathbf{A}^G = \begin{pmatrix} 1 & 2^{\frac{2}{3}}7^{\frac{1}{3}} & (1/2)^{\frac{2}{3}}2^{\frac{1}{3}} \\ (1/2)^{\frac{2}{3}}1/7^{\frac{1}{3}} & 1 & 3^{\frac{2}{3}}(1/4)^{\frac{1}{3}} \\ 2^{\frac{2}{3}}(1/2)^{\frac{1}{3}} & (1/3)^{\frac{2}{3}}4^{\frac{1}{3}} & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 3.04 & 0.79 \\ 0.22 & 1 & 1.31 \\ 1.26 & 0.76 & 1 \end{pmatrix},$$

from which a group priority vector be easily derived.

If we turn our attention to the aggregation of *priorities* (AIP), two formulas are accepted, the weighted geometric mean and the weighted arithmetic mean,

$$w_i^G = \left(\prod_{h=1}^{m} w_i^{(h)^{\lambda_i}} \right), \qquad w_i^G = \left(\sum_{h=1}^{m} \lambda_i w_i^{(h)} \right).$$

These two formulas clearly lead to different priority vectors, but they are both accepted in the literature, perhaps with a slight preference for the geometric mean [17].

Example 16. *Consider the two matrices in (3.8) and their priority vectors*

$$\mathbf{w}_1 \approx (0.331313, 0.379259, 0.289428)^T$$
$$\mathbf{w}_2 \approx (0.602629, 0.082342, 0.315029)^T$$

obtained with the eigenvector method (but we could have used any other method). At this point the geometric mean aggregation can be applied component-wise to $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *to estimate the group priority vector*

$$\mathbf{w}^G = \begin{pmatrix} 0.331313^{\frac{2}{3}} 0.602629^{\frac{1}{3}} \\ 0.379259^{\frac{2}{3}} 0.082342^{\frac{1}{3}} \\ 0.289428^{\frac{2}{3}} 0.315029^{\frac{1}{3}} \end{pmatrix} \approx \begin{pmatrix} 0.404429 \\ 0.227945 \\ 0.297722 \end{pmatrix}.$$

It is noteworthy that, when the geometric mean method is used to derive the priorities and the geometric mean is used to aggregate judgments, the diagram (3.7) becomes commutative, as depicted in (3.9), and thus using AIP or AIJ makes no difference.

$$\begin{tikzcd} \mathbf{A}_1, \ldots, \mathbf{A}_m \arrow[r, "\S 2.1"] \arrow[d, "\text{AIJ}"'] & \mathbf{w}_1, \ldots, \mathbf{w}_m \arrow[d, "\text{AIP}"] \\ \mathbf{A}^G \arrow[r, "\S 2.1"] & \mathbf{w}^G \end{tikzcd} \tag{3.9}$$

An interesting question could then refer to how much difference there is between weight vectors $\mathbf{w}^G$ when the geometric mean method is not used, and perhaps the eigenvector method is employed. According to a recent study [17], in these cases, the differences between vectors are often negligible.

Compatibility index

Very often, it is desirable that a sufficient level of consensus is reached, before opinions of different experts are aggregated. Namely, in many procedures different decision makers are encouraged to discuss, clarify issues and make their opinions converge towards a consensual solution. Therefore, it is important to define an index of similarity between opinions of decision makers. One of these indices was defined by Saaty [109] and goes under the name of *compatibility index*. Recall that, given two matrices of the same order $\mathbf{A} = (a_{ij})_{n \times n}$ and $\mathbf{B} = (b_{ij})_{n \times n}$, their Hadamard product $\mathbf{A} \circ \mathbf{B}$ is defined as the entry-wise multiplication, i.e. $\mathbf{A} \circ \mathbf{B} = (a_{ij} \cdot b_{ij})_{n \times n}$. At this point the compatibility index of two pairwise comparison matrices of order n was defined in a matrix form as

$$\text{comp}(\mathbf{A}, \mathbf{B}) = \frac{1}{n^2} \mathbf{1}^T \left(\mathbf{A} \circ \mathbf{B}^T \right) \mathbf{1},$$

where $\mathbf{1} = (1, \ldots, 1)^T$. Note that it can be rewritten as

$$\text{comp}(\mathbf{A}, \mathbf{B}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} \cdot b_{ji}).$$

Example 17. *For illustrative simplicity, let us consider two matrices differing only for one entry and its reciprocal,*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 5 \\ 1/3 & 1/5 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}.$$

By using the definition of compatibility index we obtain

$$comp(\mathbf{A}, \mathbf{B}) = \frac{1}{n^2}(1,1,1) \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 5 \\ 1/3 & 1/5 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 1/2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Proceeding by solving the Hadamard product,

$$comp(\mathbf{A}, \mathbf{B}) = \frac{1}{n^2}(1,1,1) \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 5/2 \\ 1 & 2/5 & 1 \end{pmatrix}}_{\mathbf{A} \circ \mathbf{B}^T} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

At this point the remaining simplifies to taking the arithmetic mean of all the entries of $\mathbf{A} \circ \mathbf{B}^T$, *which returns* $comp(\mathbf{A}, \mathbf{B}) = 99/90 = 1.1$.

The example shows that the minimum value attained by the compatibility index is 1 and it represents perfect consensus. Additionally, it was proven by Saaty (see Theorem 1 in [109]) that there is a connection between this metric and the method of the eigenvector. Considering $\mathbf{W}$ the matrix constructed by using the priority vector of $\mathbf{A}$ obtained by using the eigenvector method, then $\text{comp}(\mathbf{A}, \mathbf{W}) = \lambda_{\max}/n$. In this sense, this quantification of consensus results appealing to those who prefer the eigenvector method. It goes without saying that many other metrics, e.g. matrix norms of $(\mathbf{A} - \mathbf{B})$, can be used to estimate the distances between preferences of experts.

3.2.1. Integrated methods *

More models have been built to deal with many relaxations at once. These mathematical models can be called integrated, in the sense that they incorporate different purposes in the same model. Many times, integrated models can be formulated in very simple forms[1] and here we can even make up one of them for the purpose of the exposition. In this case we are interested to derive the priority vector from a set of *incomplete* pairwise comparison matrices $\dot{\mathbf{A}}_1, \ldots, \dot{\mathbf{A}}_m$. Hence, *two* problems, the incompleteness of preferences and the multiplicity of decision makers, can be accommodated in *one* single

[1] However, it is the opinion of the author of this manuscript that recently, in several papers, an apparent effort has been made to complicate things which could have been left simple.

optimization problem as, for example, the following

$$\begin{aligned} &\underset{(w_1^G,\ldots,w_n^G)}{\text{minimize}} && \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{m} \delta_{ij}^{(h)} \left(\log \dot{a}_{ij}^{(h)} + \log w_j^G - \log w_i^G\right)^2 \\ &\text{subject to} && \sum_{i=1}^{n} w_i^G = 1, \end{aligned} \tag{3.10}$$

with

$$\delta_{ij}^{(h)} = \begin{cases} 1, & \text{if } \dot{a}_{ij}^{(h)} \text{ is given} \\ 0, & \text{if } \dot{a}_{ij}^{(h)} \text{ is missing} \end{cases}$$

Note that the constrained optimization problem (3.10) aims at finding the closest (using a logarithmic metric) possible matrix $(w_i^G/w_j^G)_{n\times n}$ to the preferences expressed by the decision makers and the variables $\delta_{ij}^{(h)} \in \{0,1\}$ make all the terms containing a missing comparison vanish.

Example 18. *Suppose that three decision makers, which could be the three members of the family of the initial example, express their preferences on three alternatives in the form of the following incomplete pairwise comparison matrices,*

$$\dot{\mathbf{A}}_1 = \begin{pmatrix} 1 & 2 & \dot{a}_{13}^{(1)} \\ 1/2 & 1 & 3 \\ \dot{a}_{31}^{(1)} & 1/3 & 1 \end{pmatrix} \quad \dot{\mathbf{A}}_2 = \begin{pmatrix} 1 & \dot{a}_{12}^{(2)} & 1/2 \\ \dot{a}_{21}^{(2)} & 1 & 3 \\ 2 & 1/3 & 1 \end{pmatrix} \quad \dot{\mathbf{A}}_3 = \begin{pmatrix} 1 & \dot{a}_{12}^{(3)} & \dot{a}_{13}^{(3)} \\ \dot{a}_{21}^{(3)} & 1 & 5 \\ \dot{a}_{31}^{(3)} & 1/5 & 1 \end{pmatrix}.$$

Then, solving the optimization problem (3.10), one obtains

$$\mathbf{w}^G = (0.312391, 0487379, 0.20023)^T.$$

Surely the reader can imagine more integrated models and the next section, on extensions of the AHP, will hopefully provide more food for thought also under this lens.

4. Extensions

In this chapter we shall proceed and analyze further extensions for pairwise comparison matrices. The common denominator of the following extensions is that they all involve the domain of representation of the pairwise comparisons a_{ij}, that is the set of possible values attained by a_{ij}.

4.1. Equivalent representations

So far we have expressed pairwise comparisons using the so-called *multiplicative* scale, i.e. the judgments have been expressed by means of positive real numbers, $a_{ij} > 0 \ \forall i, j$. The multiplicative scale is often taken from granted, but here we shall keep our minds open and observe that this should not be the case. Let us follow the tradition and highlight the assumption which will be relaxed in this section.

> A *single* decision maker is *perfectly rational* and can *precisely* express his preferences on *all pairs* of *independent* alternatives and criteria using ***positive real numbers***.

Alternative numerical representations have been proposed to model pairwise comparisons. The most popular and studied are the additive representation and the one based on reciprocal relations. In this section we shall discuss these two, see that concepts as reciprocity and consistency can be similarly replicated in these other two frameworks, and finally suggest that there is a deeper connection among these representations which can be formalized by using abstract algebra.

4.1.1. Additive pairwise comparison matrices

The so-called *additive* representation of preferences by means of additive pairwise comparison matrices was well-presented by Barzilai [9] and has been used in methods alternative, yet very similar, to the AHP such as the Simple Multi-Attribute Rating Technique (SMART) [86] and the Ratio Estimations in Magnitudes or deci-Bells to Rate Alternatives which are Non-Dominated Technique (REMBRANDT) [7, 93]. The domain of representation of preferences is the real line, indifference is represented by 0 and, if we call $\mathbf{P} = (p_{ij})_{n \times n}$ the *additive pairwise comparison matrix* containing the preferences in this form, then the condition of reciprocity becomes $p_{ij} + p_{ji} = 0 \ \forall i, j$, whence the name 'additive'. The condition of consistency becomes

$$p_{ik} = p_{ij} + p_{jk} \ \forall i, j, k. \tag{4.1}$$

If and only if a matrix is consistent, then there exists a priority vector $\mathbf{u} = (u_1, \ldots, u_n)^T$ such that $p_{ij} = u_i - u_j \; \forall i, j$. One natural question regards the relation between pairwise comparison matrices and their additive representations. Namely, is there a way to associate a pairwise comparison matrix to its additive version and vice versa? A minimum requirement is that this transformation could map consistent pairwise comparison matrices into their consistent counterparts. The answer is positive and any logarithmic function would make it. For instance, using the natural logarithm, given a pairwise comparison matrix $\mathbf{A}$, we can obtain its additive representation $\mathbf{P} = (p_{ij})_{n \times n}$ with $p_{ij} = \ln a_{ij}$. Conversely, to go back to the multiplicative representation one can use its inverse, the exponential transformation $a_{ij} = e^{p_{ij}}$.

Example 19. *Consider the consistent pairwise comparison matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 4 \\ 1/8 & 1/4 & 1 \end{pmatrix}. \tag{4.2}$$

Using the logarithm in base 2 *one obtains the following skew-symmetric matrix*

$$\mathbf{P} = \begin{pmatrix} \log_2 1 & \log_2 2 & \log_2 8 \\ \log_2 1/2 & \log_2 1 & \log_2 4 \\ \log_2 1/8 & \log_2 1/4 & \log_2 1 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix},$$

for which the additive consistency condition (4.1) holds, e.g. $1 + 2 = 3$. *Moreover, one can check that the priority vector associated with* $\mathbf{P}$ *is*

$$\mathbf{u} = (2, 1, -1)^T.$$

Problem 8. *Can you find a way to derive the vector* $\mathbf{u}$ *from a consistent additive pairwise comparison matrix* $\mathbf{P}$*?*

One convenient fact about this representation is that, fixed a value for n, the set of all additive pairwise comparison matrices $\mathbf{P}$ of order n is a subspace of the linear space $\mathbb{R}^{n \times n}$. Note that also the set of *consistent* additive pairwise comparison matrices is a subspace of $\mathbb{R}^{n \times n}$ [81]. Thus, the possible loss of results that we get from giving up working with positive matrices is here compensated by the gain of the suite of tools from linear algebra. To explain it with an example, remember that the inconsistency index CI for pairwise comparison matrices was based on some results on positive square matrices (the fact that $\lambda_{\max}$ of $\mathbf{A}$ is always a real number) and therefore they are not directly replicable for additive pairwise comparison matrices. On the other hand, additive pairwise comparison matrices $\mathbf{P}$ form linear spaces, which allows us to draw from linear algebra to obtain original results. The interested reader might want to see, for instance, the inconsistency index defined by Barzilai [9] as it relies on considerations stemming from linear algebra. The priority vector $\mathbf{u}$ has a different interpretation than $\mathbf{w}$. In $\mathbf{u}$ the information is captured by the *differences* $(u_i - u_j)$ between priorities and not their ratios. Unlike for the components of $\mathbf{w}$, the ratio between u_i and u_j has no meaning. Consider that some

components u_i can be negative too.
One last remark regards the apparent similarity of this approach to the one with pairwise comparison matrices. In a consistent pairwise comparison matrix each column is equal to any other column multiplied times a suitable scalar. In the additive approach each column is equal to any other plus a suitable scalar. The same reasoning affects also the priority vectors. Priority vectors $\mathbf{w}$ of consistent pairwise comparison matrices are unique up to multiplication, whereas vectors $\mathbf{u}$ are unique, but up to addition.

4.1.2. Reciprocal relations

Another prominent representation of preferences is based on reciprocal relations [44], often called *fuzzy preference relations* [69] in the fuzzy sets literature. The notion of reciprocal relation became popular in the framework of fuzzy sets, but it can be verified that its inception dates back, at least, to the study by Luce and Suppes [88] on probabilistic preference relations.

A *reciprocal relation* can be represented by a matrix $\mathbf{R} = (r_{ij})_{n\times n}$ with $r_{ij} \in]0,1[$ satisfying the reciprocity condition $r_{ij} + r_{ji} = 1$ and with the indifference represented by the value 0.5. The *consistency* condition for reciprocal relations is

$$\frac{r_{ik}}{r_{ki}} = \frac{r_{ij}}{r_{ji}}\frac{r_{jk}}{r_{kj}} \quad \forall i,j,k. \tag{4.3}$$

Most of the references to this condition refer to Tanino [129] but the very same condition was already used by Luce and Suppes [88] and Shimura [123]. Furthermore, to make it more homogeneous with respect to the conditions of consistency for pairwise comparison matrices and additive pairwise comparison matrices, Chiclana *et al.* [38] showed that (4.3) can be equivalently written as

$$r_{ik} = \frac{r_{ij}r_{jk}}{r_{ij}r_{jk} + (1-r_{ij})(1-r_{jk})} \quad \forall i,j,k. \tag{4.4}$$

If and only if this consistency condition is satisfied, then there exists a weight vector $\mathbf{w}$ such that $r_{ij} = w_i/(w_i + w_j)$. The problem of finding the weight vector arises also for reciprocal relations and even in this environment many methods have been proposed. Among them, the most straightforward is probably the following,

$$w_i = \left(\prod_{j=1}^{n} \frac{r_{ij}}{1-r_{ij}}\right)^{\frac{1}{n}},$$

which was proven [51] to be the counterpart of the geometric mean method for pairwise comparison matrices. Similarly to what was established for additive pairwise comparison matrices, pairwise comparison matrices can be transformed into reciprocal relations by means of the following function

$$r_{ij} = \frac{a_{ij}}{1+a_{ij}}, \tag{4.5}$$

and its inverse $a_{ij} = r_{ij}/r_{ji}$ can be used to transform reciprocal relations to pairwise comparison matrices.

Example 20. *Consider the consistent pairwise comparison matrix* $\mathbf{A}$ *in (4.2). Using the transformation (4.5) one obtains*

$$\mathbf{R} \approx \begin{pmatrix} 1/2 & 2/3 & 8/9 \\ 1/3 & 1/2 & 4/5 \\ 1/9 & 1/5 & 1/2 \end{pmatrix},$$

for which the consistency condition (4.4) holds (check!). Moreover, one can check that the priority vector associated with $\mathbf{R}$ *is* $\mathbf{w} = (8/13, 4/13, 1/13)^T$ *and corresponds to the vector that would have been obtained from* $\mathbf{A}$.

The reader should be aware that another type of consistency condition, called *additive* consistency [129], for reciprocal relations was proposed and later developed, but we shall not dwell on it in this booklet. The reader can refer to [51] for an overview of transformations between pairwise comparison matrices and reciprocal relations and a method to derive the priority vector from these latter, and to Xu [135] for a survey which elaborates on different representations of pairwise preferences. Table 4.1 draws a parallel and summarizes the different representations of preferences and their main characteristics.

	Multiplicative	Additive	Reciprocal
Domain of representation	$\mathbb{R}_>$	$\mathbb{R}$	$]0,1[$
Reciprocity condition	$a_{ij} = 1/a_{ji}$	$p_{ij} = -p_{ji}$	$r_{ij} = 1 - r_{ji}$
Value for indifference between alternatives	1	0	0.5
Consistency condition	$a_{ik} = a_{ij}a_{jk}$	$p_{ik} = p_{ij} + p_{jk}$	$r_{ik} = \frac{r_{ij}r_{jk}}{r_{ij}r_{jk}+(1-r_{ij})(1-r_{jk})}$
Weight vector characterization	$a_{ij} = \frac{w_i}{w_j}$	$p_{ij} = u_i - u_j$	$r_{ij} = \frac{w_i}{w_i+w_j}$

Table 4.1.: Representations of pairwise preferences and their properties.

The transformations between different representations are instead depicted in Figure 4.1.

These three representations of preferences have different origins, but, if we look backwards, their similarities were already visible years ago. Consider, for example, that the problem of inconsistency and intransitivity, which can occur in all three representations, was in fact considered (and treated similarly) in each of them. As Gass [61] noted, Fishburn [56], whose skew symmetric representation of preferences is the progenitor of additive pairwise comparison matrices, wrote:

> Transitivity is obviously a great practical convenience and a nice thing to have for mathematical purposes, but long ago this author ceased to understand why it should be a cornerstone of normative decision theory.

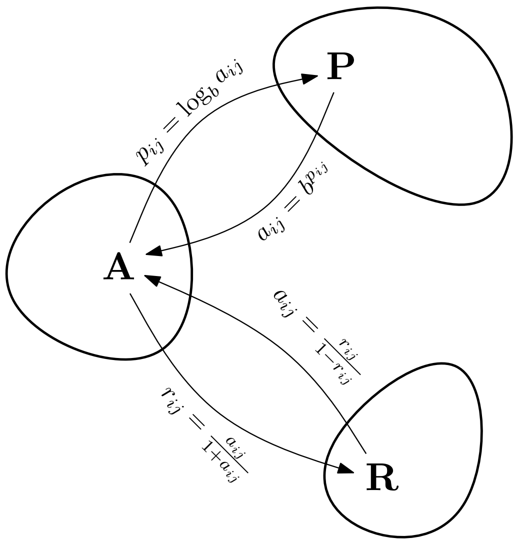


Figure 4.1.: Transformations between different representations of valued preferences.

Even Luce and Raiffa [87] whose work can be seen as an inception of reciprocal relations, wrote:

> No matter how intransitivities arise, we must recognize that they exist, and we can take a little comfort in the thought that they are an anathema to most of what constitutes theory in the behavioral sciences today.

The same view was also shared by Saaty who, already in his seminal paper on the AHP [101], wrote:

> As a realistic representation of the situation in preference comparisons, we wish to account for inconsistency in judgments because, despite their best efforts, people's feelings and preferences remain inconsistent and intransitive.

4.1.3. Group isomorphisms between equivalent representations *

It is apparent that these three representations of preferences are very similar and we can shift from one approach to another, but to what extent are they interchangeable? The non-trivial, yet elegant answer, is to the extent to which the domains of representations of preferences, together with their conditions of consistency are isomorphic groups. Recall that, in group theory, a *group* is a set S equipped with a binary operator $* : S \times S \to S$ such that

- the set S is closed under the operator $*$, i.e. $a * b \in S \ \ \forall a, b \in S$
- the operator $*$ is associative, i.e. $a * (b * c) = (a * b) * c \ \ \forall a, b, c \in S$
- there exists an identity element e such that $s * e = s \ \ \forall s \in S$
- for each $s \in S$ there exists an inverse element $s^{-1} \in S$ such that $s * s^{-1} = e$.

A group is represented by a pair $(S, *)$ where the first component is the set and the second is the operator. Two groups $(S, *)$ and $(Q, \odot)$ are *group isomorphic* if and only if there exists a bijection (group isomorphism) $f : S \to Q$ such that, for all $x, y \in S$, it is

$$f(x) * f(y) = f(x \odot y)\,.$$

Now, if we look at Table 4.1 we shall check that each domain of representation together with its consistency operation is a group where the identity element e is the value expressing indifference between alternatives and where the inverse element is determined by means of the reciprocity condition. Moreover, it can be checked that they are isomorphic groups, the isomorphisms being the functions in Figure 4.1.

Example 21. *The logarithm relates* $(\mathbb{R}_>, \cdot)$ *with* $(\mathbb{R}, +)$ *and is perhaps the most famous group isomorphism. In fact, from basics of calculus we know the rule*

$$\log(x) + \log(y) = \log(x \cdot y) \;\; \forall x, y > 0\,,$$

which exposes the relation between pairwise comparison matrices and additive pairwise comparison matrices.

The reader familiar with group theory must have understood the strength and the implication of group isomorphism which, in words, was described by Fraleigh in his textbook [58] as "the concept of two systems being structurally identical, that is, one being just like the other except for names". The existence of group isomorphisms between different representations of preference is not a mere theoretical exercise but a precious result as it helps to naturally extend concepts from one representation to another one. For a deep and theoretical analysis of the group isomorphisms between these representations of preferences the reader might find the papers by Cavallo and D'Apuzzo [35, 36] enlightening.

4.2. Interval AHP

In §2.2.5 the reader was already presented with a pairwise comparison matrix whose entries were intervals instead of real numbers. In that case the interval-valued matrix was functional in the definition of an inconsistency index, but it is natural to imagine that a decision maker could express his judgments by means of intervals. This is natural to cope with uncertainty and imprecision. In this and in the next section we shall dwell on representations of preferences when the decision maker cannot state them precisely and with absolute certainty and see what the literature has to offer.

A *single* decision maker is *perfectly rational* and can ***precisely*** express his preferences on *all pairs* of *independent* alternatives and criteria using *positive real numbers*.

This section shall introduce the principles behind what probably is the most widely known extension of the AHP with intervals. Salo and Hämäläinen considered interval judgments $\bar{a}_{ij} = [a_{ij}^L, a_{ij}^R]$ as bounds for the values of the 'true' weights, i.e. the interval-valued comparison $\bar{a}_{ij} = [a_{ij}^L, a_{ij}^R]$, entails that $a_{ij}^L \leq w_i/w_j \leq a_{ij}^R$. At this point, it is important to know what values different weights can attain, given the constraints imposed by the interval pairwise comparisons. What is, for instance, the maximum possible value of w_i given an interval-valued matrix $\bar{\mathbf{A}}$? To solve this problem, we first need to define the set of all normalized priority vectors with n components as

$$W_n = \left\{ (w_1, \ldots, w_n)^T \middle| \sum_{i=1}^{n} w_i = 1,\ w_i > 0\ \forall i \right\},$$

Such a set is depicted in Figure 4.2 for the case with $n = 3$. Furthermore, the set of

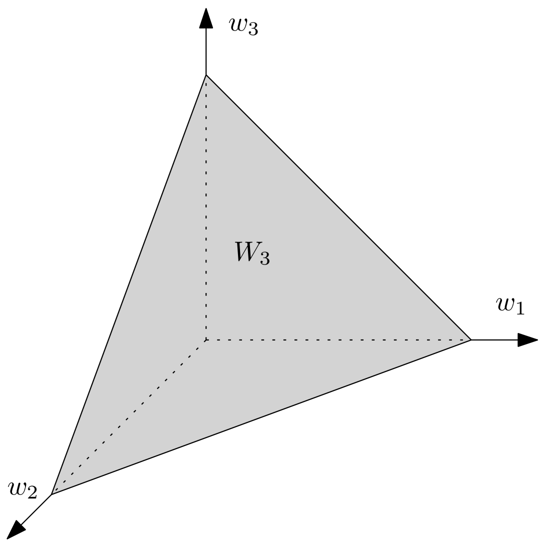


Figure 4.2.: Graphical representation of W_3

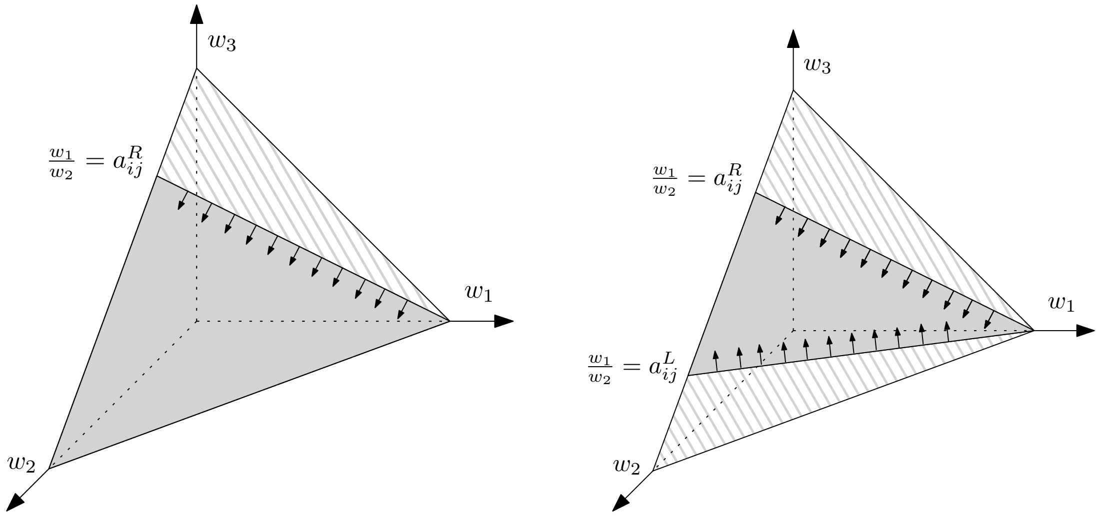
feasible weight vectors according to the interval-valued pairwise comparison matrix $\bar{\mathbf{A}}$ is

$$S_{\bar{\mathbf{A}}} = \left\{ (w_1, \ldots, w_n)^T \middle| a_{ij}^L \leq \frac{w_i}{w_j} \leq a_{ij}^R\ \forall i < j \right\}$$

As showed in Figure 4.3, adding the constraints characterizing $S_{\bar{\mathbf{A}}}$ to W obviously reduces the set of feasible solutions. It follows that the 'true' normalized weight vector must be an element of the set $W_n \cap S_{\bar{\mathbf{A}}}$, as pictured in Figure 4.4. Then it is possible to construct an interval-valued vector $\bar{\mathbf{w}} = (\bar{w}_1, \ldots, \bar{w}_n)^T$ with $\bar{w}_i = [w_i^L, w_i^R]$ where w_i^L and w_i^R are the smallest and the greatest possible values for w_i respectively. Hence, they can be computed as follows,

$$w_i^L = \underset{\mathbf{w} \in W_n \cap S_{\bar{\mathbf{A}}}}{\text{minimize}}\ w_i \qquad i = 1, \ldots, n, \tag{4.6}$$

$$w_i^R = \underset{\mathbf{w} \in W_n \cap S_{\bar{\mathbf{A}}}}{\text{maximize}}\ w_i \qquad i = 1, \ldots, n. \tag{4.7}$$



(a) Application of the constraint $w_1/w_2 \leq a_{ij}^R$ (b) Adding the constraint $w_1/w_2 \geq a_{ij}^L$

Figure 4.3.: The sequential application of the constraints reduces the region of feasible solutions

Example 22. *Considering the matrix*

$$\bar{\mathbf{A}} = \begin{pmatrix} 1 & [7, 20] & [7/5, 4] \\ [1/20, 1/7] & 1 & [1/5, 4/7] \\ [1/4, 5/7] & [7/4, 5] & 1 \end{pmatrix} \tag{4.8}$$

already used in (2.16). Then the weight w_1^R *is the optimal value of the following optimization problem.*

$$\begin{aligned} &\underset{(w_1,w_2,w_3)}{\textit{maximize}} && w_1 \\ &\textit{subject to} && \left.\begin{array}{l} 7 \leq w_1/w_2 \leq 20, \\ 7/5 \leq w_1/w_3 \leq 4, \\ 1/5 \leq w_2/w_3 \leq 4/7, \end{array}\right\} \Rightarrow (w_1, w_2, w_3)^T \in S_{\bar{\mathbf{A}}} \\ & && \left.\begin{array}{l} w_1 + w_2 + w_3 = 1, \\ w_1, w_2, w_3 > 0 \end{array}\right\} \Rightarrow (w_1, w_2, w_3)^T \in W_3 \end{aligned} \tag{4.9}$$

By proceeding in this way, we ask what the greatest possible value achievable by w_1 *is, when* $(w_1, w_2, w_3) \in W_n \cap S_{\bar{\mathbf{A}}}$. *The interval-valued priority vector derivable from* $\bar{\mathbf{A}}$ *in (2.16) is*

$$\bar{\mathbf{w}} = \begin{pmatrix} [0.54, 0.77] \\ [0.04, 0.10] \\ [0.18, 0.38] \end{pmatrix} \tag{4.10}$$

In the example, the vector (4.10) provided enough information and we knew that the best alternative was x_1 since its weight cannot be smaller than the weights of the

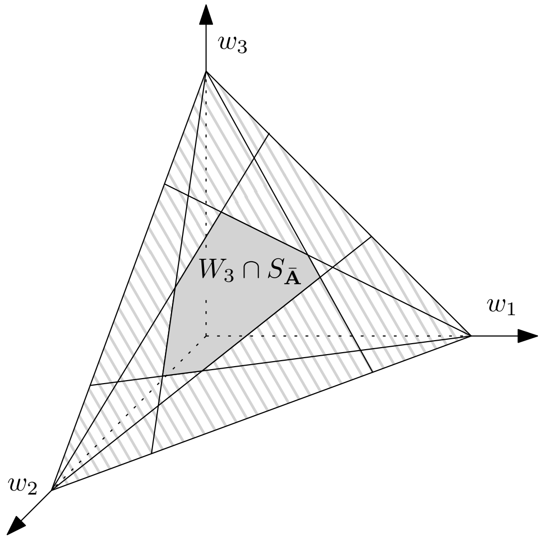


Figure 4.4.: The set $W_3 \cap S_{\bar{\mathbf{A}}}$.

other alternatives. However, in other cases, when intervals overlap, selection of the best alternative is non-trivial. To solve this problem one can use different strategies. Firstly, the decision maker can be asked to refine his judgments until the best alternative is clearly identified. Secondly, when this is not a viable solution, some methods for ranking intervals can be employed. Among such methods, there are the pairwise dominance [121] and the methods for ranking fuzzy quantities [133].
In this section we described a method for deriving weights which can be used on a single interval-valued pairwise comparison matrix and not on a whole hierarchy. The extension to the whole hierarchy is methodologically straightforward but quite lengthy to be explained, and therefore the reader can refer to the original contribution [121].

Euclidean center of $W_n \cap S_{\bar{\mathbf{A}}}$

The problem of ranking interval weights and their hierarchical composition can be solved by means of a shortcut, which is used to derive real valued weights from interval valued comparison matrices. The following is due to Arbel and Vargas [6]. Their solution is based on the fact that the set of constraints characterizing $W_n \cap S_{\bar{\mathbf{A}}}$ can be equivalently stated as a set of linear constraints since those containing ratios can be splitted into two linear constraints. Considering for instance the first constraint in (4.9), one can see that

$$\begin{aligned} 7 \leq w_1/w_2 \leq 20 \;\; &\Leftrightarrow \; 7w_2 \leq w_1 \text{ and } w_2 \leq 20w_1 \\ &\Leftrightarrow \; 7w_2 - w_1 \leq 0 \text{ and } w_2 - 20w_1 \leq 0. \end{aligned}$$

Hence, since $W_n \cap S_{\bar{\mathbf{A}}}$ is a bounded set defined by linear constraints it is a polytope. Arbel and Vargas proposed to take the real valued weights $w_1, \ldots, w_n$ as the coordinates of the Euclidean center of the polyhedron $W_n \cap S_{\bar{\mathbf{A}}}$. In words, the Euclidean center of a polytope is the center of the maximum radius ball which can be inscribed in the polytope and it can be found by solving a linear optimization problem [1]. Figure 4.5 reports a graphical example of the Euclidean center of a 2-dimensional polytope, i.e. a polygon.

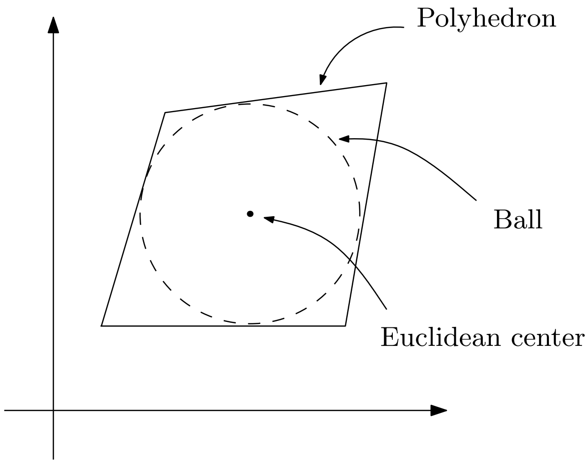


Figure 4.5.: Graphical example of the Euclidean center of a polygon.

Let us see how to write down a linear optimization problem to find the ball with the largest radius r in $W_n \cap S_{\bar{\mathbf{A}}}$. Since $W_n \cap S_{\bar{\mathbf{A}}}$ is a polytope, it can be defined by a set of inequalities $\mathbf{a}_i^T \mathbf{w} \leq b_i$. The problem is how to model the constraints with respect to the center of the ball. Consider a single constraint. A ball of radius r pointed in $\mathbf{w}$, $B(r, \mathbf{w})$, satisfies the ith inequality, if $\mathbf{a}_i^T \mathbf{y} \leq b_i \ \forall \mathbf{y} \in B(r, \mathbf{w})$. The trick is to write the inequality in such a way that we consider a point $\mathbf{y}^*$ which is the point in $B(r, \mathbf{w})$ with the greatest value when multiplied by $\mathbf{a}_i^T$, i.e. $\mathbf{a}_i^T \mathbf{y}^* \geq \mathbf{a}_i^T \mathbf{y} \ \forall \mathbf{y} \in B(r, \mathbf{w})$. This point is the point $\mathbf{y}^* = \mathbf{w} + r \frac{\mathbf{a}_i}{\|\mathbf{a}_i\|}$ (convince yourself graphically in 2-dimensions) and therefore the inequality can be written as

$$\mathbf{a}_i^T \underbrace{\left(\mathbf{w} + r \frac{\mathbf{a}_i}{\|\mathbf{a}_i\|} \right)}_{\mathbf{y}^*} \leq b_i \, .$$

Hence, the optimization problem becomes

$$\begin{aligned} \underset{r, \mathbf{w}}{\text{maximize}} \quad & r \\ \text{subject to} \quad & \mathbf{a}_i^T \left(\mathbf{w} + r \frac{\mathbf{a}_i}{\|\mathbf{a}_i\|} \right) \leq b_i \qquad i = 1 \ldots, N \end{aligned} \tag{4.11}$$

where N is the number of inequalities used to define P. The optimization problem (4.11) can be seen as max-min optimization problem too. In fact, the variable r, which is maximized, eventually is the distance between the center of the ball $\mathbf{w}$ and the closest (least distant) face of the polyhedron $W \cap S_{\bar{\mathbf{A}}}$.

[1] Note that in some other sources it is referred to as the Chebychev center. See, for instance, the book by Boyd and Vandenberghe [21].

Example 23. *For sake of homogeneity we still consider the matrix $\bar{\mathbf{A}}$ from (2.16). Then we have*

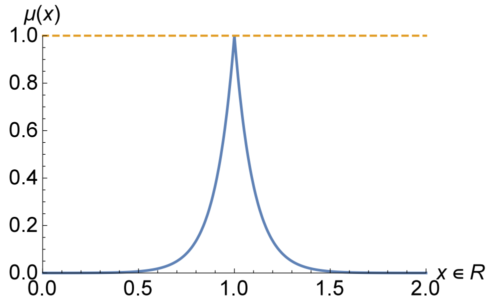
$$
\begin{array}{ll}
\underset{r,\mathbf{w}}{\text{maximize}} & r \\
\text{subject to} & \left.\begin{array}{l}
(-1,7,0)\left(\mathbf{w}+r\dfrac{(-1,7,0)^T}{5\sqrt{2}}\right) \leq 0, \\
(1,20,0)\left(\mathbf{w}+r\dfrac{(1,20,0)^T}{\sqrt{401}}\right) \leq 0, \\
(-1,0,7/5)\left(\mathbf{w}+r\dfrac{(-1,0,7/5)^T}{\sqrt{74}/5}\right) \leq 0, \\
(1,0,-4)\left(\mathbf{w}+r\dfrac{(1,0,-4)^T}{\sqrt{17}}\right) \leq 0, \\
(0,-1,1/5)\left(\mathbf{w}+r\dfrac{(0,-1,1/5)^T}{\sqrt{26}/5}\right) \leq 0, \\
(0,1,-4/7)\left(\mathbf{w}+r\dfrac{(0,1,-4/7)^T}{\sqrt{65}/7}\right) \leq 0,
\end{array}\right\} \Rightarrow (w_1,w_2,w_3)^T \in S_{\bar{\mathbf{A}}} \\
& \left.\begin{array}{l}
w_1+w_2+w_3=1, \\
w_1,w_2,w_3>0
\end{array}\right\} \Rightarrow (w_1,w_2,w_3)^T \in W_3
\end{array}
\tag{4.12}
$$

Note that the constraints defining W_3 are left unchanged: the constraint $w_1 + w_2 + w_3 = 1$ is an equality and therefore it must hold exactly, and the positivity constraints $w_1, w_2, w_3 > 0$ could even be deleted since they are made redundant by those defining $S_{\bar{\mathbf{A}}}$. The vector maximizing r in the optimization problem is $\mathbf{w} \approx (0.72, 0.07, 0.21)^T$.

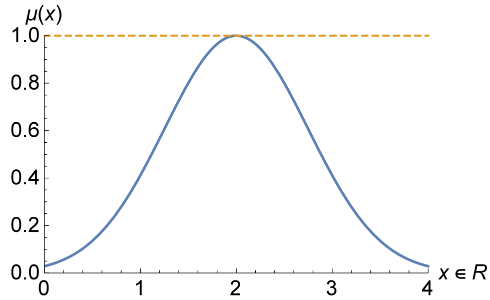
This approach to interval judgments, which considers intervals as implicitly defining bounds for weights, was initially proposed by Arbel [5]. Conversely, for a probabilistic approach to interval pairwise comparisons the reader can refer to Saaty and Vargas [119].

4.3. Fuzzy AHP

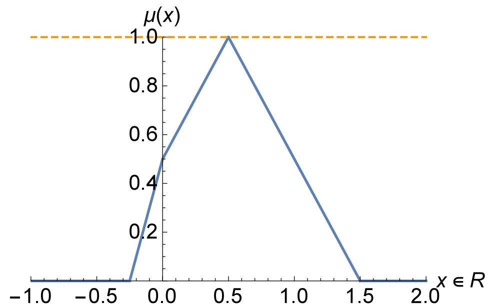
The *fuzzy AHP* is an even more popular methodology to account for uncertainty. In the fuzzy AHP entries of the pairwise comparison matrices are expressed in the form of fuzzy numbers. A function $\mu : \mathbb{R} \to [0,1]$ is a *fuzzy number* if and only if there exists an x_0 such that $\mu(x_0) = 1$ and all the upper level sets of μ are convex, i.e. the set $\{x \in \mathbb{R} | \mu(x) \geq \alpha\}$ is convex for all $0 < \alpha \leq 1$. Figure 4.6 reports some instances of fuzzy numbers. Also a real interval can be treated as a fuzzy number; considering the interval $[a,b] \subset \mathbb{R}$, then the value of its membership function is 1 for all $x \in [a,b]$ and 0 otherwise. The fuzzy AHP draws from the theory of fuzzy sets initiated by Zadeh [137] and described, for instance, in the excellent monographs by Klir and Yuan [79]



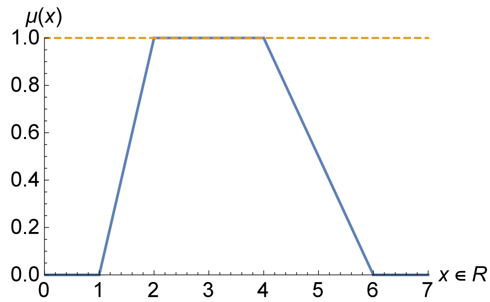
(a) Cuspidal fuzzy number



(b) Bell shaped fuzzy number



(c) Piecewise linear fuzzy number



(d) Trapezoidal fuzzy number

Figure 4.6.: Four examples of fuzzy numbers.

and Dubois and Prade [47]. Even so, to keep the description short and self-contained we shall here skip all the unnecessary details on fuzzy sets theory and go straight to the point.

4.3.1. Fuzzy AHP with triangular fuzzy numbers

One of the most used shapes of fuzzy numbers for modeling preferences, and more generally to represent uncertain quantities, is triangular. A *triangular fuzzy number* is defined by the following function

$$\mu(x) = \begin{cases} 0, & x \leq a^L \\ (x-a)/(b-a), & a^L \leq x \leq a^C \\ (c-x)/(c-b), & a^C \leq x \leq a^R \\ 0, & x \geq a^R \end{cases}$$

with $a^L \leq a^C \leq a^R$. Observe that there exists a one-to-one correspondence between triangular fuzzy numbers and triples $\tilde{a} = (a^L, a^C, a^R)$ with $a^L \leq a^C \leq a^R$. An example of triangular fuzzy number is reported in Figure 4.7. To many, the shape of a triangular fuzzy number might resemble a probability distribution, just with the normalization such that the area subtended by the curve is equal to one replaced by the condition $\sup_{x \in \mathbb{R}} \mu(x) = 1$. True, a fuzzy number can be seen as a distribution indicating the

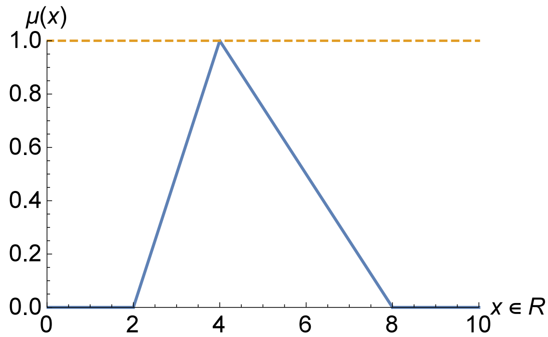


Figure 4.7.: Triangular fuzzy number $\tilde{a} = (2, 4, 8)$

likelihood of events, but within the framework of possibility theory [139], and not probability. According to Klir [77] the value $1 - \mu(x)$ can be interpreted as the degree of surprise to discover that x is the 'true' value of the variable under observation. Leaving aside the technicalities of this theory, for which the reader can be referred to the monograph by Klir [78], for practical purposes it is common to interpret the three values a^L, a^C, a^R characterizing a triangular fuzzy number as the smallest possible, the most likely, and the greatest possible values for the uncertain quantity under study. The use of triangular shapes for fuzzy numbers has been advocated by many, for instance Pedrycz [94], and a whole arithmetic has been developed to perform operations on fuzzy sets and fuzzy numbers in particular. In one of the first papers on fuzzy AHP, van Laarhoven and Pedrycz [82] defined the operations of addition ($\oplus$), multiplication ($\otimes$), logarithm ($\widetilde{\ln}$), inversion, and power as follows, respectively:

$$\begin{aligned} \tilde{a} \oplus \tilde{b} =& (a^L + b^L, a^C + b^C, a^R + b^R) \\ \tilde{a} \otimes \tilde{b} \approx& (a^L \cdot b^L, a^C \cdot b^C, a^R \cdot a^R) \\ \widetilde{\ln}(a_L, a_C, a_R) \approx& (\ln a_L, \ln a_C, \ln a_R) \\ \tilde{a}^{-1} \approx& \left(\frac{1}{a^R}, \frac{1}{a^C}, \frac{1}{a^L} \right) \\ e^{(a^L, a^C, a^R)} =& (e^{a^L}, e^{a^C}, e^{a^R}) \end{aligned} \tag{4.13}$$

The primal issue with a fuzzy pairwise comparison matrix is that of deriving the priority vector, and one straightforward approach could be that of using these operations on matrices with triangular fuzzy entries, i.e. $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{n \times n} = (a_{ij}^L, a_{ij}^C, a_{ij}^R)$, in the same way their corresponding operations were used with pairwise comparison matrices. Hereafter, we shall focus on the problem of finding a suitable priority vector for a fuzzy pairwise comparison matrix. To this scope, we should distinguish *a priori* between two types of methods:

- Methods to derive a vector of fuzzy weights.
- Methods to derive a vector of weights expressed as real numbers.

We shall here dwell a bit more on these two methodologies by explaining how they have been treated in the literature.

Obtaining fuzzy weights

One straightforward solution to this problem was recently suggested by Ramík and Korviny [99]. According to this method, the components of the priority vector are fuzzy numbers and can be estimated by an extension of the geometric mean method. Namely, the priority vector appears as $\tilde{\mathbf{w}} = (\tilde{w}_1, \ldots, \tilde{w}_n)^T$, where the components $\tilde{w}_i = (w_i^L, w_i^C, w_i^R)$ themselves are triangular fuzzy numbers. Following this method, the priority vector with triangular fuzzy components is estimated as the minimizer of the following constrained optimization problem.

$$
\begin{aligned}
\underset{(\tilde{w}_1,\ldots,\tilde{w}_n)}{\text{minimize}} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\big(\ln a_{ij}^L - \ln w_i^L + \ln w_j^L\big)^2 + \big(\ln a_{ij}^C - \ln w_i^C + \ln w_j^C\big)^2 + \\
& \big(\ln a_{ij}^R - \ln w_i^R + \ln w_j^R\big)^2\Big) \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i^C = 1, \\
& w_i^U \geq w_i^C \geq w_i^L > 0 \ \forall i.
\end{aligned} \tag{4.14}
$$

Ramík and Korviny proved (see Theorem 1 in their paper [99]) that the analytic solution of this optimization problem is

$$
w_k^L = c_{\min} \cdot \frac{\left(\prod_{j=1}^{n} a_{ij}^L\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^C\right)^{\frac{1}{n}}} \quad \forall k, \tag{4.15}
$$

$$
w_k^C = \frac{\left(\prod_{j=1}^{n} a_{ij}^C\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^C\right)^{\frac{1}{n}}} \quad \forall k, \tag{4.16}
$$

$$
w_k^R = c_{\max} \cdot \frac{\left(\prod_{j=1}^{n} a_{ij}^R\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^C\right)^{\frac{1}{n}}} \quad \forall k, \tag{4.17}
$$

where

$$
c_{\min} = \min_{i=1,\ldots,n}\left\{\frac{\left(\prod_{j=1}^{n} a_{ij}^C\right)^{\frac{1}{n}}}{\left(\prod_{j=1}^{n} a_{ij}^L\right)^{\frac{1}{n}}}\right\} \qquad c_{\max} = \max_{i=1,\ldots,n}\left\{\frac{\left(\prod_{j=1}^{n} a_{ij}^C\right)^{\frac{1}{n}}}{\left(\prod_{j=1}^{n} a_{ij}^U\right)^{\frac{1}{n}}}\right\}
$$

Example 24. *Consider the following matrix*

$$\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{3\times 3} = \begin{pmatrix} (1,1,1) & (1/2,2,3) & (1,1,2) \\ (1/3,1/2,2) & (1,1,1) & (1/3,2,4) \\ (1/2,1,1) & (1/4,2,3) & (1,1,1) \end{pmatrix} \tag{4.18}$$

Then, the weight vector obtained by using (4.15)–(4.17) is

$$\tilde{\mathbf{w}} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)^T = \begin{pmatrix} (0.412599, 0.412599, 0.412599) \\ (0.249914, 0.32748, 0.454124) \\ (0.259921, 0.259921, 0.32748) \end{pmatrix} \tag{4.19}$$

For a critical analysis of this method and a broader overview on the use of fuzzy sets in decision making, the interested reader can refer to the recent paper by Dubois [45].

One method was proposed by van Laarhoven and Pedrycz [82] themselves, but a lot has happened since then and their proposal has been refined a number of times. Here we should present one of the most recent refinement, which can be seen as a fuzzy extension of the geometric mean method in the optimization form that we encountered in (2.3) and seemingly resembles the optimization problem (4.14). Note that, again, the solution is itself a priority vector whose components are triangular fuzzy numbers and is here denoted as $\tilde{\mathbf{w}} = (\tilde{w}_1, \dots, \tilde{w}_n)^T$ with $\tilde{w}_i = (w_i^L, w_i^C, w_i^R)$.

$$\begin{aligned}
\underset{(\tilde{w}_1,\dots,\tilde{w}_n)}{\text{minimize}} \quad & \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \Big(\big(\ln a_{ij}^L - \ln w_i^L + \ln w_j^L\big)^2 + \big(\ln a_{ij}^C - \ln w_i^C + \ln w_j^C\big)^2 + \\
& \big(\ln a_{ij}^R - \ln w_i^R + \ln w_j^R\big)^2 \Big) \\
\text{subject to} \quad & w_i^L + \sum_{j=1, j\neq i}^{n} w_j^U \geq 1, \ \forall i \\
& w_i^U + \sum_{j=1, j\neq i}^{n} w_j^L \leq 1, \ \forall i \\
& \sum_{i=1}^{n} w_i^C = 1, \\
& \sum_{i=1}^{n} \big(w_i^L + w_i^R\big) = 2, \\
& w_i^U \geq w_i^C \geq w_i^L > 0 \ \forall i.
\end{aligned} \tag{4.20}$$

Since all these methods return a vector $\tilde{\mathbf{w}}$ whose components are fuzzy numbers, the question on how to select the best alternative remains. In fact, if there exists a universally accepted order on the set $\mathbb{R}$—that is, given two different real numbers we can always say which one is the greatest—the situation is more ambiguous in the case of fuzzy numbers. Consider, for instance, the weights in (4.19). If it is intuitive to say that $\tilde{w}_1$ is greater than $\tilde{w}_3$, then the situation between $\tilde{w}_1$ and $\tilde{w}_2$ is much more ambiguous. Which one should be considered greater, and which one be the best between x_1 and x_2? Although much research has been done on the topic, there is still not a meeting of minds on how to order fuzzy numbers. The interested reader can refer to Wang and Kerre [133] and Brunelli and Mezei [30] for an axiomatic and a numerical study of methods for ranking fuzzy numbers, respectively.

Obtaining a real-valued priority vector

From the literature, it seems that the problem of ranking fuzzy numbers and its ambiguity can be bypassed by using methods which recover real valued priority vectors. There are few doubts that the most popular method for deriving a real valued priority vector $\mathbf{w}$ for a pairwise comparison matrix with fuzzy entries $\tilde{\mathbf{A}}$ is the so called *extent analysis*, proposed by Chang [37]. The extent analysis can be described in five algorithmic steps.

1. For each row, calculate its sum $\tilde{s}_i = \tilde{a}_{i1} \oplus \cdots \oplus \tilde{a}_{in}$.
2. Normalize all the $\tilde{s}_i$'s in the following way: $\tilde{r}_i = \tilde{s}_i \otimes (\tilde{s}_1 \oplus \cdots \oplus \tilde{s}_n)^{-1}$.
3. Calculate the degree of possibility that $\tilde{r}_i$ be greater than $\tilde{r}_j$ as follows

$$\mathrm{Pos}\left(\tilde{r}_i \gtrsim \tilde{r}_j\right) = \begin{cases} 1, & \text{if } r_i^C \geq r_j^C \\ \frac{r_i^R - r_j^L}{(r_i^R - r_i^C) + (r_j^C - r_j^L)}, & \text{if } r_i^C < r_j^C \text{ and } r_j^L \leq r_i^U \\ 0, & \text{otherwise} \end{cases}$$

   The second case looks cumbersome but has a simple geometric interpretation: it is the value of the membership function for which the 'right leg' of $\tilde{r}_i$ and the 'left leg' of $\tilde{r}_j$ intersect. The concepts is illustrated in Figure 4.8.

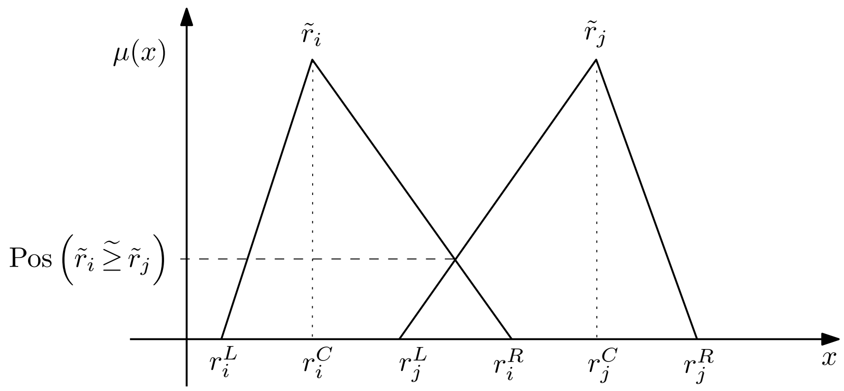


Figure 4.8.: Assessing the degree of possibility that the fuzzy number $\tilde{r}_i$ be greater than $\tilde{r}_j$.

4. Generalize the previous step by considering that

$$\mathrm{Pos}\left(\tilde{r}_i \gtrsim \tilde{r}_j | j = 1, \ldots, n\,, j \neq i\right) = \min_{j \in \{1,\ldots,n\}, j \neq i} \mathrm{Pos}\left(\tilde{r}_i \gtrsim \tilde{r}_j\right).$$

5. The real valued priority vector $\mathbf{w}$ is obtained by normalizing the values obtained in the previous steps:

$$w_i = \frac{\mathrm{Pos}\left(\tilde{r}_i \widetilde{\geq} \tilde{r}_j | j = 1, \ldots, n\,, j \neq i\right)}{\sum_{k=1}^{n} \mathrm{Pos}\left(\tilde{r}_i \widetilde{\geq} \tilde{r}_j | j = 1, \ldots, n\,, j \neq k\right)}.$$

Let us check the extent analysis method with a numerical example.

Example 25. *Consider the matrix $\tilde{\mathbf{A}}$ in (4.18) as the starting point. Then, the sums of the fuzzy numbers on the rows are calculated by means of the operation at step 1 and can be collected in the following vector,*

$$\begin{pmatrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \end{pmatrix} = \begin{pmatrix} (2.5, 4, 6) \\ (5/3, 3.5, 7) \\ (7/4, 4, 5) \end{pmatrix}.$$

To normalize the components of this vector, one calculates $(\tilde{s}_1 \oplus \cdots \oplus \tilde{s}_n) = \left(\frac{71}{12}, 10, 18\right)$ *where* $71/12 = 2.5 + 5/3 + 7/4$ *and uses it to obtain, as described in the step 2,*

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{pmatrix} = \begin{pmatrix} (0.138889, 0.4, 1.01408) \\ (0.0925926, 0.35, 1.1831) \\ (0.0972222, 0.25, 0.84507) \end{pmatrix}$$

Then we can construct the matrix of possibilities according to step 3

$$\mathbf{V} = \begin{pmatrix} - & 1 & 1 \\ 0.954305 & - & 1 \\ 0.824804 & 0.882695 & - \end{pmatrix} \tag{4.21}$$

where each nondiagonal entry is a value $\mathrm{Pos}\left(\tilde{r}_i \widetilde{\geq} \tilde{r}_j\right)$*. Now, considering the algorithmic steps 4 and 5 together we can obtain the following priority vector,*

$$\mathbf{w} = \begin{pmatrix} \frac{1}{1+0.954305+0.824804} \\ \frac{0.954305}{1+0.954305+0.824804} \\ \frac{0.824804}{1+0.954305+0.824804} \end{pmatrix} = \begin{pmatrix} 0.359828 \\ 0.343385 \\ 0.296787 \end{pmatrix}.$$

More on the extent analysis will follow in the next section. For the moment it is sufficient to observe that, although an algorithm for ranking fuzzy numbers has not been explicitly mentioned, it has nevertheless been implicitly used. The matrix $\mathbf{V}$ in (4.21) is, *de facto*, a representation of a fuzzy ordering relation [138] which does induce a ranking on the fuzzy numbers $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$. Hence, we can conclude that, even by using the extent analysis, the ambiguity inherent to the ranking of fuzzy numbers is not avoided, but rather swept under the carpet.

4.3.2. Is the fuzzy AHP valid? ⋆

The question posed in the title of this subsection is as provocative as still standing, and it has definitely been answered in a negative sense by many. Since the seminal papers by van Laarhoven and Pedrycz [82] and Buckley [31], the fuzzy AHP has attracted most of the criticisms directed to the AHP plus a good deal of original others due to the (mis)use of fuzzy sets.

The first criticism is that the operations for triangular fuzzy numbers commonly used in the fuzzy AHP, and here reported in (4.13), are only approximations of the correct operations. The correct operations are defined by means of the extension principle and, according to these latter, for instance, the product of two triangular fuzzy numbers is *not* a triangular fuzzy number, but something nonlinear. Part of the scientific community accepts the approximations (4.13) as a necessary compromise to mitigate computational complexity while the other part does not. The reader can refer to Dubois and Prade [46] and Klir and Yuan [79] for a correct definition of arithmetic operations with fuzzy sets and fuzzy numbers.

Saaty and Tran [115] criticized the fuzzy AHP by saying that the traditional and real valued AHP suffices to account for all the imprecision in human judgments. Ramík and Korviny [99] rebutted that the traditional AHP can be seen as a special case of the fuzzy AHP—and not as a different method—and therefore it is difficult to see how the fuzzy AHP, which is more general, could perform worse than the AHP, which is the less general of the two.

A delicate point in the fuzzy AHP is that of ranking the components of the priority vector, when these are fuzzy numbers. Although it is not a real drawback, the fact that different ranking methods for fuzzy numbers could give very different results [30] can be perceived as a lack or robustness of the method, especially because there is not a prime ranking method. On the other hand, also methods which derive a real valued priority vector such as the extent analysis are not immune to criticisms. For instance, the extent analysis was criticized by Wang *et al.* [134] as, among many criticisms, they pointed out that the final weights are surely useful to rank alternatives but they cannot be interpreted as weights in a multiplicative sense. It is sufficient to see that in the extent analysis some weights can be equal to zero. However, in the case of null weights this does not mean that one alternative is infinitely better than another.

Recently, Zhü [141] moved some criticisms to the AHP, but it seems that many of them are pretentious and unsupported and others stem from a very narrow view of the method, which differs in large part from the more open minded view offered in this booklet. For instance, one of the criticisms moved by Zhü is that there is not an unique inconsistency index for fuzzy pairwise comparison matrices. This, clearly, stems from a vision of the AHP (very much *à la* Saaty) where, as Zhü [141] admitted, the Consistency Index CI is considered as the only reasonable consistency index and all others considered inferior.

5. Conclusions

As reported by Saaty and Sodenkamp [114], in 2008 Saaty was awarded by the INFORMS for the inception and development of the AHP. Part of the motivation for the award was the following:

> The AHP has revolutionized how we resolve complex decision problems... the AHP has been applied worldwide to help decision makers in every conceivable decision context across both the public and private sectors, with literally thousands of reported applications.

Moreover, from a survey by Wallenius *et al.* [132] it seems that the AHP has been by far the most studied and applied MCDM method, at least judging by the number of publications. The reader should have noticed that only basic mathematical and technical knowledge is required to use the AHP. For instance, if we consider that the priority vector can be derived using the method of the normalized columns, mentioned in §2.1.3, and consistency can be estimated by using the harmonic consistency index, in §2.2.4, then one can use the AHP at a basic level by using only elementary operations! Nevertheless, in spite of this possible simplicity, it is difficult to find an aspect of the AHP, or of pairwise comparison matrices, which has not been object of heated debates. Many of these debates are still open and probably will be so for much longer. However, even if inconclusive, it would be a mistake to regard them as pointless, since they contributed to create awareness around the AHP. Still, for the same sake of awareness, in this last part we shall overview some aspects of the AHP which have not been considered in the exposition.

Analytic Network Process

The observant reader might have also noticed that one of the assumptions has not been relaxed yet. Let us do it now.

A *single* decision maker is *perfectly rational* and can *precisely* express his preferences on *all pairs* of ***independent*** alternatives and criteria using *positive real numbers*.

It is possible that, in some decisions, two criteria might affect each other. For instance, considering the selection of a resort for holidays, one can envision that the two criteria 'cost' and 'environment' are not independent since, probably, *ceteris paribus*, the resort in the best environment will also be more expensive. The best-known methodology for dealing with interdependencies between parts of the hierarchy is the *Analytic Network Process* (ANP), which can be seen as a generalization of the AHP. That is, the AHP is a special case of the ANP without dependencies. Although more general than the AHP, the ANP still lacks a fundamental discussion and an axiomatization. Also, a self contained exposition of the ANP would require the introduction of new concepts, some of which of difficult digestion for those who are not in the field. For these reason, and the fear of making a sloppy job and possibly not do justice to the method, we shall here not dwell on the ANP. The interested reader can refer to a dedicated book by Saaty and Vargas [120] or, for an easier and more superficial treatment, to the book by Ishizaka and Nemery [74]. Let us incidentally note that the term Analytic Network Process was not coined by Saaty but, instead, by Hämäläinen and Seppäläinen [66].

Alternative methods

The AHP is a decision analysis methodology, but it is not the only one. Although nowadays geographical distinctions are arguably meaningless, for historical reasons, in decision analysis there has been two schools, the American and the French [54]. Here we shall touch upon one method of each type.

- *Multi-Attribute Value Theory* (MAVT) belongs to the so-called American school of decision analysis [76] and assumes that alternatives are fully described by their attributes. Then, each attribute state is mapped into a real number, and finally the numerical expressions of the different attributes are aggregated into a unique representative value. We shall now change notation and consider x the alternative and x_i as the state of the ith attribute in the alternative x. Consider a car, represented by the following list of characteristics:

  $$x = (x_1, x_2, x_3) = (\text{blue}, 180, 3)$$

  where the attributes are 'color', 'max speed in km/h' and 'safety level', respectively. Consider X_i as the set of possible states of the ith attribute. Then, according to value theory, for each attribute, there is a function $u_i : X_i \to [0, 1]$. The greater the value, the greater the satisfaction of the attribute. Given the existence of these functions, we can suppose that the car represented by x be mapped into the following vector,

  $$\mathbf{x} = (u_1(\text{blue}), u_2(180), u_3(3)) = (0.5, 0.7, 0.6) \in [0, 1]^3.$$

  At this point, the values attained for the attributes, in this case three, are aggregated into a single value by means of a function $v : [0, 1]^3 \to [0, 1]$ and a single real number is used to synthesize the value of an alternative. This process is represented in Figure 5.1. Functions $u_1, \ldots, u_n, v$ are defined once for all, and therefore their application is automatic when new alternatives are considered. The selection rule is simple: the greater the value, the better the alternative, i.e. with

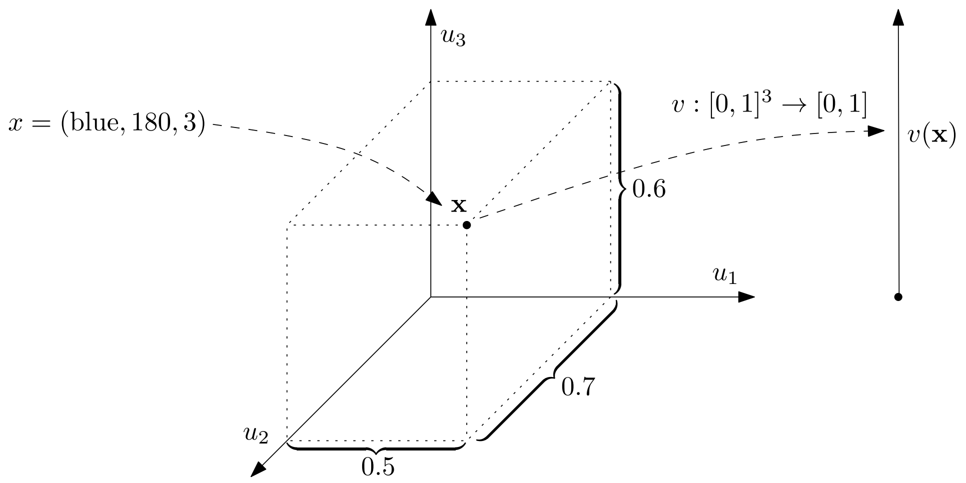


Figure 5.1.: A list of attribute states x is mapped into an $\mathbf{x} \in [0,1]^3$, which, in turn, is synthesized into $v(\mathbf{x})$.

$x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$, which are two alternatives described by m attributes,

$$x \succeq y \Leftrightarrow v(\underbrace{u_1(x_1), \dots, u_m(x_m)}_{\mathbf{x}}) \geq v(\underbrace{u_1(y_1), \dots, u_m(y_m)}_{\mathbf{y}})$$

A strength of value theory is its elegance and explanatory power on how decisions are made. Conversely, practical uses of this theory are limited by the difficulties in the estimation of the functions $u_1, \dots, u_n, v$.

- The acronym *ELECTRE* stands for *ELimination Et Choix Traduisant la REalité*, and it is used to denote a family of methods from the French school. Nowadays many variants of the original ELECTRE methods exist and are applied to problems of ranking and also sorting. These methods are based on pairwise comparisons between alternatives, and to each comparison degrees of concordance and discordance are attached. A number of parameters and a non-trivial algorithm are necessary for the implementation of these methods, whose interpretation, possibly due to the aforementioned reasons, is not as straightforward as the one of the AHP.

Software

It is difficult to say whether much software appeared thanks to the popularity of the AHP or the popularity of the AHP is due to the wealth of software. Perhaps both propositions are to some extent true and the popularity of the method and of its software have gone arm-in-arm and boosted each other.

The foremost software is called *Expert Choice* and was first developed by Saaty and Forman in 1983. Expert Choice adopts Saaty's approach, according to which the priority vector is calculated with the eigenvector method and CI is used to estimate the inconsistency of preferences. Expert Choice was described and discussed by Ishizaka

and Labib [72]. Expert Choice's natural evolution and generalization to the ANP is called *SuperDecisions*. The name of the software comes from the 'supermatrix', which is a special matrix used in the ANP.
A direct and recent competitor of Expert Choice is *MakeItRational*, which was described by Ishizaka and Nemery [74]. One of the characteristics of MakeItRational is its ease of use, together with a captivating interface.
The software listed so far is not free and the user has to pay for its use. Among the free available software there is *HIerarchical PREference analysis on the World Wide Web* (Web-HIPRE) software which is part of Decisionarium [65], an online platform offering software for decision-making. Web-HIPRE allows the use of both the original scale of Saaty and the balanced scale (see §1.3). Two inconsistency indices can be used in Web-HIPRE: Saaty's CI and the index CM by Salo and Hämäläinen [122]. Web-HIPRE was the first online platform for decision making with the AHP and has a module which supports group decision making.
A comparative study between three software for the AHP was proposed by French and Xu [59]. Although other software exist, at present there is not an updated *and* free software for the AHP. An auspicable characteristic of such a free software is that it include different inconsistency indices, prioritization method, and methods to deal with with incomplete pairwise comparison matrices.

Sensitivity analysis

A module which is included in most AHP software allows for sensitivity analysis. In mathematical modeling, sensitivity analysis studies how the output of a mathematical model reacts to variations in the inputs. In the introductory chapter we encountered a numerical case where three weight vectors rating alternatives with respect to three criteria were aggregated using the weighs of criteria as factors in a linear combination.

$$
\begin{aligned}
\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= \hat{w}_1 \mathbf{w}^{(c)} + \hat{w}_2 \mathbf{w}^{(s)} + \hat{w}_3 \mathbf{w}^{(e)} \\
&= \frac{1}{7} \begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} + \frac{2}{7} \begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix} \approx \begin{pmatrix} 0.287 \\ 0.253 \\ 0.460 \end{pmatrix}.
\end{aligned}
$$

Now, we can assume that we want to see what happens to the final ranking of alternatives if we allow the weight of the third criterion to take values in $[0,1]$ and rescale the weights of the other two criteria accordingly. In this case, the final rating can be expressed as follows and it becomes a function of $\hat{w}_3$,

$$
\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{3}(1-\hat{w}_3) \begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} + \frac{2}{3}(1-\hat{w}_3) \begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} + \hat{w}_3 \begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix}.
$$

For example the weight of the first alternative is

$$w_1 = \frac{74(1-\hat{w}_3)}{135} + \frac{\hat{w}_3}{11} = \frac{74}{135} - \frac{679}{1485}\hat{w}_3,$$

that is, an affine function of the weight of the third criterion $\hat{w}_3$. The same property of affinity holds also for w_2 and w_3 and, when the dimension of the problem allows it, sensitivity analysis lends itself nicely to graphical interpretations. In this case the graphical interpretation of w_1, w_2 and w_3 as functions of $\hat{w}_3$ is in Figure 5.2, which we should briefly comment.

The original weight assigned to $\hat{w}_3$ was $4/7 \approx 0.51$. From the picture we can see that, if

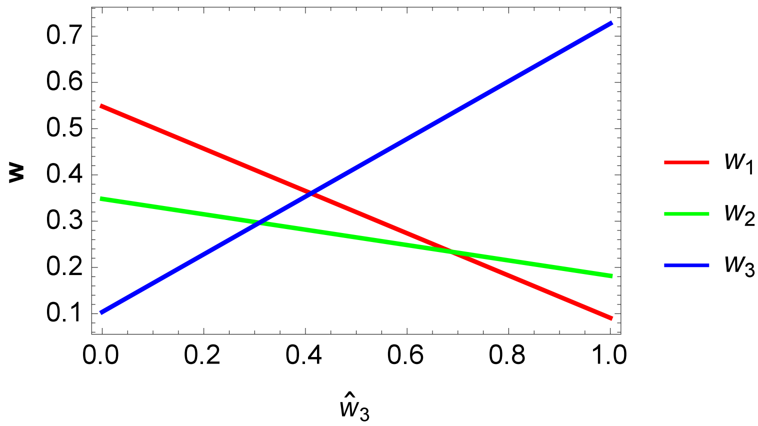


Figure 5.2.: Sensitivity analysis.

$\hat{w}_3 \geq 4/7$, then the solution is stable and alternative x_3 is always the best. Conversely, if the weight $\hat{w}_3$ is decreased, then, at some point, alternative x_1 will prevail. Sensitivity analysis is a precious tool for testing the robustness of solutions and their stability with respect to the inputs, in this case subjective judgments of experts. Moreover, here we have only presented the most popular way of performing sensitivity analysis, but it is easy to figure out that, by using the geometric mean method as the prioritization method, we can make the final ranking depend directly on entries of pairwise comparison matrices.

Future studies

The AHP is a fundamentally simple method which, in its simplest implementations, consists of three steps:

1. Problem structuring and definition of the hierarchy
2. Elicitation of pairwise comparisons
3. Derivation of priority vectors and their linear combinations.

In spite of its ease of interpretation, research has been going on for the last forty and more years and although many issues are still open, and perhaps are bound to be open for very long, nowadays it is safe to say that this technology has reached the maturity. We have seen in this booklet that a wide range of methods have been proposed to perform tasks within the AHP. Consider, for example, the wide range of methods for estimating the priority vector or the wealth of the inconsistency indices.

Unlike for some other areas of applied mathematics and mathematical modeling, in the case of the AHP, more often than not, new methods, indices, and extensions have been introduced heuristically and without results showing their originality and superiority. This practice generated an overabundance of material. In the future, it is auspicable that new numerical and axiomatic studies clarify and polish the state of the art, and when new methods are introduced, clear evidence on their originality and feasibility be provided.

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A. Eigenvalues and eigenvectors

The AHP is an important field of application of linear algebra, and especially of its theory regarding positive matrices. This appendix contains an introduction to eigenvalues and eigenvectors focused on their relevance for the AHP. At present, there are many ways to work out the AHP without getting dirty with eigenvalues and eigenvectors. Thus, in a certain sense knowing about them is superfluous. However, by knowing them the reader will figure out the connection between AHP and linear algebra and hopefully see the AHP from a higher observation point.

Definition 1 (Eigenvalues and eigenvectors)**.** *Consider an $n \times n$ square matrix $\mathbf{A}$ and an n-dimensional vector $\mathbf{w}$. Then, $\mathbf{w}$ and λ are an eigenvector and an eigenvalue of $\mathbf{A}$, respectively, if and only if*

$$\mathbf{A}\mathbf{w} = \lambda \mathbf{w}. \tag{A.1}$$

Example 26. *Consider the matrix and the vector as follows*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then, one reckons that $\mathbf{w}$ is an eigenvector of $\mathbf{A}$ for $\lambda = 2$. In fact

$$\begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

.

Note that, if $\mathbf{w}$ is an eigenvector of $\mathbf{A}$, then all vectors $\alpha\mathbf{w}$ for $\alpha \in \mathbb{R}$ are also eigenvectors of $\mathbf{A}$, we call this set of vectors the *eigenspace* of $\mathbf{A}$ associated to that eigenvector (or its respective eigenvalue). Now one natural question arises; how to find the eigenvalues and the eigenvectors of a given matrix. By considering the identity matrix $\mathbf{I}$ and the null vector $\mathbf{0} = (0, \dots, 0)^T$, we can rewrite (A.1),

$$\begin{aligned} \mathbf{A}\mathbf{w} &= \lambda \mathbf{w} \\ \mathbf{A}\mathbf{w} - \lambda \mathbf{w} &= \mathbf{0} \\ \mathbf{A}\mathbf{w} - \lambda \mathbf{I}\mathbf{w} &= \mathbf{0} \\ (\mathbf{A} - \lambda \mathbf{I})\,\mathbf{w} &= \mathbf{0} \end{aligned} \tag{A.2}$$

Now, from the basics of linear algebra we know that, if $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$, then there is only one solution to (A.2), which is the trivial solution $\mathbf{w} = (0, \dots, 0)^T$. We are instead interested in the case where other solutions exists, then to the case $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Hence, by changing notation $\rho_{\mathbf{A}}(\lambda) := \det(\mathbf{A} - \lambda\mathbf{I})$, we need to find the roots of $\rho_{\mathbf{A}}(\lambda)$. Such a polynomial is called the *characteristic polynomial* of $\mathbf{A}$.

Example 27. *Reprising the matrix of the previous example*

$$
\begin{aligned}
\rho_{\mathbf{A}}(\lambda) &= \det \begin{pmatrix} 1-\lambda & 2 \\ 1/2 & 1-\lambda \end{pmatrix} \\
&= (1-\lambda)(1-\lambda) - \frac{1}{2}2 \\
&= \lambda(\lambda - 2)
\end{aligned}
$$

and by imposing $\lambda(\lambda-2) = 0$ it follows that $\rho_{\mathbf{A}}(\lambda) = 0$ for $\lambda = 0, 2$. Now, considering for example the eigenvalue $\lambda = 2$ the associated eigenvector can be found by solving

$$
\begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
$$

from which we derive that $w_1 = 2w_2$ and that $\mathbf{w} = (2,1)^T$ is the eigenvector associated to $\lambda = 2$. Clearly, also all the eigenvectors of the eigenspace spanned by $\mathbf{w}$ are eigenvectors of $\lambda = 2$, e.g. $(1, 0.5)^T$.

Note that the eigenvalues can be ordered from the greatest to the smallest according to their absolute value. We call *maximum eigenvalue* the one with the greatest absolute value and we denote it as $\lambda_{\max}$. In Example 27, we have $\lambda_{\max} = 2$. Going back to the computational part, with the increasing size of a matrix, things get more complicated, especially when it comes to find the roots of the characteristic polynomial. However, the idea remains the same.

Example 28. *Consider the following matrix*

$$
\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 4 \\ 1/8 & 1/4 & 1 \end{pmatrix}.
$$

Then, by putting its characteristic polynomial equal to 0, and by skipping the elementary steps, one recovers

$$
\rho_{\mathbf{A}}(\lambda) = \lambda^2(3-\lambda) = 0 \tag{A.3}
$$

The eigenvalues are then $\lambda = 0, 3$. In this case we say that the algebraic multiplicity of $\lambda = 0$ is equal to 2. Roughly speaking, with algebraic multiplicity *we indicate the number of times that a solution appears in the equation. In this case the multiplicity 2 of $\lambda = 0$ is obvious if we rewrite (A.3) as follows,*

$$
\rho_{\mathbf{A}}(\lambda) = \lambda\lambda(3-\lambda) = 0 \tag{A.4}
$$

Note that in the previous example one eigenvalue was equal to n, and the other, with multiplicity $(n-1)$ was equal to 0. This is not a case, but a more general result.

Proposition 1. *Given a pairwise comparison matrix $\mathbf{A}$, if and only if $\mathbf{A}$ is consistent, then one eigenvalue, $\lambda_{\max}$ is equal to n and the other is equal to 0, with multiplicity $(n-1)$.*

Proceeding further, another question arises and regards the behavior of $\lambda_{\max}$ when $\mathbf{A}$ is not consistent. As $\lambda_{\max}$ cannot be equal to n, then what else can it be? Eigenvalues are roots of polynomials and it is natural to suspect that $\lambda_{\max}$ could be a complex number. Fortunately, this cannot happen for pairwise comparison matrices and we can restrict the search to real numbers. This is formalized in the following theorem.

Theorem 1 (Perron-Frobenius)**.** *Given a square matrix* $\mathbf{A}$*, if* $\mathbf{A}$ *is positive, i.e.* $a_{ij} > 0 \; \forall i, j$*, then its maximum eigenvalue is real,* $\lambda_{\max} \in \mathbb{R}$*.*

Example 29. *Consider the following matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 1/4 \\ 1/8 & 4 & 1 \end{pmatrix}$$

Using the rule of Sarrus we compute

$$\begin{aligned} \rho_{\mathbf{A}}(\lambda) &= (1-\lambda)^3 + \left(2 \cdot \frac{1}{4} \cdot \frac{1}{8}\right) + \left(8 \cdot \frac{1}{2} \cdot 4\right) - (1-\lambda) - (1-\lambda) - (1-\lambda) \\ &= (1-\lambda)^3 + \frac{2}{32} + \frac{32}{2} - 3(1-\lambda) \\ &= \frac{225}{16} + 3\lambda^2 - \lambda^3. \end{aligned}$$

By solving $\frac{225}{16} + 3\lambda^2 - \lambda^3 = 0$ *we find that* $\lambda_{\max} \approx 3.9167$*. The other two roots are conjugate complex and we are not interested in them. Such solution can be easily found by any mathematical software. Now, with this solution, we need to solve the equation system*

$$\begin{pmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 1/4 \\ 1/8 & 4 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 3.9167 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

To aid the process and avoid the problem of infinitely many solutions we add the condition $w_1 + w_2 + w_3 = 1$ *and solve*

$$\begin{cases} w_1 + 2w_2 + 8w_3 = 3.9167 w_1 \\ \frac{1}{2} w_1 + w_2 + \frac{1}{4} w_3 = 3.9167 w_2 \\ \frac{1}{8} w_1 + 4 w_2 + w_3 = 3.9167 w_3 \\ w_1 + w_2 + w_3 = 1 \end{cases}$$

from which we obtain

$$\mathbf{w} \approx (0.660761, 0.131112, 0.208127)^T.$$

Note that in the this last example $\mathbf{A}$ was inconsistent and $\lambda_{\max} > n$. The following proposition clarifies the range of possible values attained by $\lambda_{\max}$.

Proposition 2 (Saaty [101]). *Let* **A** *be a pairwise comparison matrix. Then* $\lambda_{\max} = n$ *if and only if* **A** *is consistent and strictly greater than* n *otherwise.*

Nowadays, all textbooks on linear algebra cover the theory of eigenvalues and eigenvectors. For a less didactic and more involving exposition of eigenvalues and eigenvectors with an eye on positive matrices the reader can refer to the book by Horn and Johnson [70].

B. Solutions

Solution to Problem 2: Consider that we assumed $a_{ij} = w_i/w_j \; \forall i,j$. Then we write w_i and w_j as the respective geometric means and see what happens if we account for the assumption.

$$a_{ij} = \frac{\left(\prod_{k=1}^{n} a_{ik}\right)^{1/n}}{\left(\prod_{k=1}^{n} a_{jk}\right)^{1/n}} = \left(\frac{a_{i1}a_{i2}\cdots a_{in}}{a_{j1}a_{j2}\cdots a_{jn}}\right)^{\frac{1}{n}}.$$

Since we assumed that $a_{ij} = w_i/w_j$ we can substitute these in the equation and rewrite it as

$$a_{ij} = \left(\frac{\frac{w_i}{w_1}\frac{w_i}{w_2}\cdots\frac{w_i}{w_n}}{\frac{w_j}{w_1}\frac{w_j}{w_2}\cdots\frac{w_j}{w_n}}\right)^{\frac{1}{n}} = \left(\frac{\frac{w_i^n}{w_1w_2\cdots w_n}}{\frac{w_j^n}{w_1w_2\cdots w_n}}\right)^{\frac{1}{n}} = \frac{w_i}{w_j}.$$

The original assumption is correctly recovered and therefore, when $a_{ij} = w_i/w_j \; \forall i,j$, the geometric mean method returns the correct vector.

Solution to Problem 3: The proof was provided by the Crawford and Williams [43]. See Theorem 3 in their paper.

Solution to Problem 4: Underbraced are the numbers of independent comparisons for each level of the hierarchy, starting from the top.

$$\underbrace{\frac{3(3-1)}{2}}_{3} + \underbrace{\frac{4(4-1)}{2} + \frac{3(3-1)}{2} + \frac{3(3-1)}{2}}_{12} + \underbrace{9\frac{3(3-1)}{2}}_{27} = 42$$

Solution to Problem 5: Consider the analytic formula of c_3, that is,

$$c_3 = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}}\right).$$

At this point, consider the matrix $\dot{\mathbf{A}}$ with the entry $\dot{a}_{ij}$ missing. The sum contains four transitivities and we can expand it

$$c_3 = 6\Bigg(8 \underbrace{-\frac{\dot{a}_{13}}{\dot{a}_{12}\dot{a}_{23}} - \frac{\dot{a}_{12}\dot{a}_{23}}{\dot{a}_{13}}}_{-\frac{37}{6}} \underbrace{-\frac{\dot{a}_{14}}{\dot{a}_{12}\dot{a}_{24}} - \frac{\dot{a}_{12}\dot{a}_{24}}{\dot{a}_{14}}}_{-\frac{\dot{a}_{14}}{2}-\frac{2}{\dot{a}_{14}}} \underbrace{-\frac{\dot{a}_{14}}{\dot{a}_{13}\dot{a}_{34}} - \frac{\dot{a}_{13}\dot{a}_{34}}{\dot{a}_{14}}}_{-\frac{\dot{a}_{14}}{8}-\frac{8}{\dot{a}_{14}}} \underbrace{-\frac{\dot{a}_{24}}{\dot{a}_{23}\dot{a}_{34}} - \frac{\dot{a}_{23}\dot{a}_{34}}{\dot{a}_{24}}}_{-\frac{13}{6}}\Bigg)$$

$$= 6\left(-\frac{5x}{8} - \frac{10}{x} - \frac{1}{3}\right) = -\frac{15\dot{a}_{14}}{4} - \frac{60}{\dot{a}_{14}} - 2.$$

Let us inspect the first and second derivatives of c_3 in $\dot{a}_{14}$:

$$\frac{\partial c_3}{\partial \dot{a}_{14}} = \frac{60}{(\dot{a}_{14})^2} - \frac{15}{4},$$
$$\frac{\partial^2 c_3}{\partial \dot{a}_{14}^2} = -\frac{120}{(\dot{a}_{14})^3}.$$

The second derivative is strictly negative for positive values of $\dot{a}_{14}$, which means that the function is strictly concave for $\dot{a}_{14} > 0$ and that, if there is a maximum, then it is unique. By equating the first derivative to zero, we recover that $(\dot{a}_{14})^2 = 16$. Of the two solutions we take the positive one, which is $\dot{a}_{14} = 4$.

Solution to Problem 6:

$$\mathbf{C} = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 1/2 & 1 & 1/3 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 1 & 1/2 & 2 \end{pmatrix}$$

Solution to Problem 7:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 1/2 & 1 & 2 & 3/2 \\ 1/4 & 1/2 & 1 & 3/4 \\ 1/3 & 2/3 & 4/3 & 1 \end{pmatrix}$$

Solution to Problem 8: If the matrix is consistent, then any column can act as the priority vector. An alternative method, which is also used to derive the vector from inconsistent matrices, is the arithmetic mean of the rows,

$$u_i = \frac{1}{n} \sum_{j=1}^{n} p_{ij}.$$

In the case of consistent matrices, this method returns exactly the correct vector. The proof is similar to the one used to solve Problem 2.

AKADEMIA GÓRNICZO-HUTNICZA IM. STANISŁAWA STASZICA W KRAKOWIE

**WYDZIAŁ ELEKTROTECHNIKI, AUTOMATYKI,
INFORMATYKI I INŻYNIERII BIOMEDYCZNEJ**

KATEDRA INFORMATYKI STOSOWANEJ

Praca dyplomowa magisterska

Portal ankiet porównawczych
Pairwise comparisons poll portal

Autor: *Karol Wójcik*
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Kraków, 2014

Oświadczam, świadomy(-a) odpowiedzialności karnej za poświadczenie nieprawdy, że niniejszą pracę dyplomową wykonałem(-am) osobiście i samodzielnie i nie korzystałem(-am) ze źródeł innych niż wymienione w pracy.

Serdecznie dziękuję … tu ciąg dalszych podziękowań np. dla promotora, żony, sąsiada itp.

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1. Wprowadzenie

1.1. Cele pracy

Celem poniższej pracy jest przedstawienie portalu ankiet porównawczych. Został on zaimplementowany z myślą o rozpowszechnieniu metody porównań parami jako skutecznego instrumentu ankietowego. Powstał na bazie małego narzędzia porównań parowych, wykorzystanego do przeprowadzenia eksperymentu ewaluacji jakości ośrodków naukowych w Polsce za pomocą metody parowej [21]. Współpracowałem z autorami tej pracy zapewniając wykonanie serwisu i obsługę techniczną. Z jednorazowej aplikacji zrodził się pomysł zbudowania portalu ankiet porównawczych. Ma za zadanie świadczyć wysokiej jakości usługi związane z szeroko rozumianą metodą porównań parami. Brak odpowiednich narzędzi był motywacją do stworzenia aplikacji badawczej. Naturalną konsekwencją było stworzenie takich możliwości, aby kolejne badania tworzyć szybciej i łatwiej, a sam proces przeprowadzania ankiety zrobić przyjazny dla respondentów.

Współpracując z naukowcami zajmującymi się w swoich badaniach metodą porównań parami, udało mi się określić zbiór podstawowych wymagań i zależności. Wszystko po to aby nowy system był w możliwie największym stopniu zgodny z zasadami metody jak i oczekiwaniami potencjalnych odbiorców i użytkowników.

1.2. Zawartość pracy

Na kolejnych kartach tej pracy zostanie przedstawiona metoda porównań parami od strony matematyczno-formalnej. Zaprezentowane będą także główne problemy i zależności z nią związane. Po obszernym wstępie teoretycznej wiedzy i definicji nastąpi prezentacja projektu systemu. Jego zawartość stanowi opis wymagań funkcjonalnych i niefunkcjonalnych wraz z przypadkami użycia. Jako ostatni punkt znajduje się prezentacja implementacji. Zawiera ona informacje o środowisku i opcjach konfiguracji systemu oraz kilka szczególnych punktów zasługujących na poświęcenie im uwagi.

2. Metoda porównań parami

W rozdziale tym przedstawiono metodę porównań parami. Jest to prosta metoda zaliczająca się do metod podejmowania decyzji. Służy do subiektywnej oceny wariantów poprzez zestawienie ich ze sobą w parach i wyboru najlepszego z nich.

Początki takich porównań sięgają zamierzchłych czasów, kiedy pojęcie pieniądza ani nawet liczb jeszcze nie istniało. Z łatwością możemy sobie wyobrazić naszych praprzodków, porównujących wagę ptaka trzymanego w jednej ręce z wagą ryby trzymanej w drugiej ręce. Także na porównaniach ludzie opierali barterową wymianę dóbr. Istniała potrzeba określenia jaka ilość rzeczy typu A jest konieczna do otrzymania przedmiotu typu B, np. ile łososi można dostać w zamian za jednego zająca.

W naukach ścisłych metoda porównań parami jest obecna od XVIII wieku, kiedy jej uproszczoną wersję w postaci teorii wyboru społecznego przedstawił Nicolas de Condorcet w 1785 w [4]. Wykorzystana była do stworzenia sprawnego systemu głosowania zwanego kryterium Condorceta. W 2001 roku jego autorstwo zostało przypisane jednak średniowiecznemu filozofowi, Rajmundowi Llull, gdyż zostały wtedy odkryte jego zagubione manuskrypty pochodzące z XIII w. Warto zaznaczyć iż wielu innych uczonych zajmowało się teorią wyboru społecznego i kontynuowało badania w tej dziedzinie. Byli to między innymi Jean-Charles de Borda (1733–1799), Lewis Carroll (1832–1898) oraz Duncan Black (1908–1991). Kolejnym po Condorcecie lecz zdecydowanie bardziej formalnym użyciem metody porównań parowych była praca autorstwa Fechnera z roku 1860 [5]. Wiek XX przyniósł znaczne ożywienie, najpierw za sprawą opracowania L.L. Thurstone'a pt. *A Law of Comparative Judgments* ([18]), które rozszerzyło metodę porównań z użycia sposobu binarnego (większy/mniejszy) na dowolny wybór stosunku. A następnie za sprawą Thomasa Saatyego w [17] który jako pierwszy zaproponował użycie hierarchicznych struktur danych. Stanowiło to istotną zmianę w rozstrzyganiu problemów złożonych z wielu kryteriów.

Dla problemu mającego n kategorii mamy $\frac{n(n-1)}{2}$ porównań parowych do wykonania. Na przykład: 10 kategorii generuje 45 par, 30 kryteriów daje 435 porównań parowych, a zaledwie 100 obiektów porównywanych podnosi liczbę par do 4950. Jak widać mamy tu do czynienia ze złożonością $O(n^2)$. Znakomita większość ostatnich badań prowadzonych z użyciem porównań parami składała się z nie więcej niż 7 kategorii. Liczba 7 daje 21 par do porównania, co wydaje się być swego rodzaju barierą psychologiczną dla większości ludzi zanim popadną w znużenie lub znudzenie tematem. Dzięki zastosowaniu hierarchii można znacząco obniżyć złożoność obliczeniową, aż do poziomu $O(n \log_n)$. Więcej na ten temat zostanie przedstawione w dalszej części pracy.

Metoda porównań parowych może być wykorzystana do różnorakich pomiarów. Zarówno przedmiotowych, typu odległość, wartość czy też waga, jak i niemierzalnych typu, poziom bezpieczeństwa publicznego czy też stopień zanieczyszczenia. Wartości początkowe dla porównań parami są zazwyczaj określane poprzez pomiary znanych i stałych atrybutów jak długość, cena itd. lub subiektywne preferencje respondentów. (zob. [26]).

Przeprowadzono kilka badań dowodzących przewagę metody porównań parami. W 1998 roku W.W. Koczkodaj w pracy [14] przedstawił wyniki przeprowadzonego eksperymentu, potwierdzające zwiększenie dokładności wyniku przy wykorzystaniu porównań parami. Do tego celu został zaprojektowany eksperyment statystyczny, wykorzystujący metodę Monte Carlo, mający odpowiedzieć na pytanie czy rzeczywiście używanie metody parowej poprawia jakość oszacowania dla czynnika. Eksperyment został zaprojektowany i zaimplementowany w taki sposób aby zminimalizować błędy statystyczne. Uzyskane wyniki po zestawieniu z wynikami otrzymanymi metodą bezpośrednią wykazały znaczny wzrost dokładności.

2.1. Macierz porównań parowych

Metoda porównywania parami wykorzystuje do obliczeń tzw. macierz porównań parowych której elementy określają stosunek dwóch odpowiadających sobie obiektów.

Rozważmy n kategorii będących obiektami porównywanymi. Element $m_{i,j}$ przestawia subiektywną preferencję elementu a_i nad a_j, $i, j = 1, ..., n$ i wynosi $m_{i,j} = \frac{a_i}{a_j}$. Macierz porównań parowych A ma postać

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{21}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{n1}} & \frac{1}{a_{n2}} & \cdots & 1 \end{bmatrix} \tag{2.1}$$

O macierzy $A = [a_{ij}] \in R^{n \times n}$ powiemy że jest macierzą porównań parowych (macierz PC) jeśli $a_{ij} > 0$ dla każdego $i, j = 1, ..., n$. Macierz porównań parowych A jest nazywana *wzajemną* (ang. *reciprocal*) jeśli $a_{i,j} = \frac{1}{a_{i,j}}$ dla każdego $i, j = 1, ..., n$. Na tej podstawie możemy stwierdzić iż $a_{i,i} = 1$ dla każdego $i = 1, ..., n$. Istotnym a także naturalnym jest tutaj założenie niezmienności właściwości. Kiedy jedną z kategorii porównywanych jest np. samochód, to przy każdorazowym porównaniu z inną kategorią mamy na myśli obiekt reprezentujący, a nie konkretny model wybranego producenta. Pomimo intuicyjnej zgodności nie możemy przyjąć że każda macierz spełnia ten warunek. Opisuje to W.W. Koczkodaj w [25]. Obrazowym przykładem może być *ślepy test* win, podczas którego istnieje możliwość iż w naszej ocenie $i - ty$ trunek jest lepszy niż ten sam $i - ty$, podany w innym, nieoznaczonym kieliszku. Dla testującego są to dwie osobne kategorie, kiedy w rzeczywistości był to ten sam napój. Problem ten został poruszony w [23] gdzie autorzy zaprezentowali propozycję zrzucenia domyślnego warunku wzajemności z rozważanych macierzy. Na potrzeby pracy przyjąłem że będę pracował jednak na tych, które posiadają tę właściwość.

Macierz PC A jest nazywana *spójną* (ang. *consistent*) jeśli $a_{ij} * a_{jk} = a_{ik}$ dla każdego $i, j, k = 1, ..., n$. Zachodzi przy tym zależność iż każda macierz spójna, jest także wzajemna. W drugą stronę implikacja ta z reguły nie jest prawdą. Problem spójności zostanie szerzej opisany w kolejnym rozdziale.

2.2. Ranking

Wynikiem którego oczekujemy od badań metodą porównań parowych jest wektor wag badanych obiektów/kategorii. Dzięki niemu wiemy czym kierują się nasi respondenci i co dla nich jest najważniejsze lub ma największą wartość. Do ustalenia priorytetów na podstawie macierzy porównań parowych możemy wykorzystać dwie metody: obliczając wektor własny macierzy bądź średnie geometryczne. Przyjmijmy że mamy daną macierz wzajemną, określoną dla $n = 5$ kategorii:

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 9 \\ \frac{1}{2} & 1 & 2 & 4 & 9 \\ \frac{1}{3} & \frac{1}{2} & 1 & 2 & 8 \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{2} & 1 & 7 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{8} & \frac{1}{7} & 1 \end{bmatrix}$$

Znormalizowany wektor wyznaczony w oparciu o wartości własne macierzy wynosi zatem:

$$\begin{bmatrix} 0.426 \\ 0.281 \\ 0.165 \\ 0.101 \\ 0.027 \end{bmatrix}$$

W konsekwencji możemy stwierdzić że wagi kategorii wyliczone za pomocą metody wartości własnych wynoszą:

$$w(x1) = 0.426, w(x2) = 0.281, w(x3) = 0.165, w(x4) = 0.101, w(x5) = 0.027$$

Niemal identyczny wynik otrzymamy obliczając średnią geometryczną każdego z wierszy macierzy oraz następnie dokonując normalizacji wyników.

Skoro istnieją dwie równoważne metody warto zastanowić się czy są pomiędzy nimi różnice i w końcu - którą z nich stosować? Rozważanie należy zacząć od tego, iż obie metody dają identyczny wynik jeśli obliczenia prowadzimy na macierzy spójnej (a co za tym idzie także wzajemnej), gdyż jej wektor własny jest równy średnim geometrycznym. Różnica pojawia się w samym sposobie wykonywania obliczeń. Znacznie prościej i szybciej jesteśmy w stanie obliczyć średnie geometryczne, nawet macierzy większych rozmiarów typu $n >= 5$, niż wyznaczyć jej wartości własne. Jest to główny argument dla którego w pracy będę wykorzystywał metodę średnich geometrycznych.

Mając wektor V będący rezultatem działania $(\prod_{j=1}^{n} a_{ij})^{1/n}$ obliczonego dla każdego wiersza macierzy, należy wyznaczyć sumę jego elementów. Aby otrzymać znormalizowany wektor wag V' należy podzielić każdy element wektora V przez sumę obliczoną w poprzednim kroku.

Formalnie:

$$V[*] = [(\prod_{j=1}^{n} a_{1j})^{1/n}, (\prod_{j=1}^{n} a_{2j})^{1/n}, (\prod_{j=1}^{n} a_{3j})^{1/n}, ..., (\prod_{j=1}^{n} a_{nj})^{1/n}] \tag{2.2}$$

$$s = \sum_{i=1}^{n} V_(i) \tag{2.3}$$

$$V'[*] = [\frac{v_1}{s}, \frac{v_2}{s}, \frac{v_3}{s}, ..., \frac{v_n}{s}] \tag{2.4}$$

Na podstawie wektora V' można stworzyć poprawioną wersję macierzy źródłowej. Nowy element a'_{ij} jest ilorazem v'_i/v'_j.

3. Niespójność w metodzie porównań parami

Przeciwnicy metody porównywania parami opierają swoje zarzuty głównie na problemie wnioskowania na podstawie niespójnych danych. Mimo iż metoda ma silne fundamenty matematyczne, najsłabszym elementem jest człowiek i jego odpowiedzi na pytania ankiety. Przyczyną jest niespójność danych. Mamy z nią do czynienia w sytuacji kiedy wynik porównania danych dwóch kategorii podany przez respondenta odbiega od wartości która wynika z jego poprzednich porównań których składnikami były te kategorie. Przykładem może być sytuacja gdy trzymając w ręku trzy zapałki mamy za zadanie oszacować ich długość wykorzystując porównania parowe. Porównując pierwszą z drugą, wynikiem jest 1. Pierwszą z trzecią: 1/2. Aby całe porównanie było spójne, wynik zestawienia zapałki drugiej i trzeciej musi zatem wynosić także 1/2.

Przekładając to na język matematyki powiemy że macierz porównań parowych A (2.1) jest *spójna* (lub także *przechodnia*) jeśli $a_{ij} * a_{jk} = a_{ik}$ dla każdego $i, j, k = 1, ..., n$. Macierz spójna odpowiada sytuacji wzorcowej gdzie istnieją dokładne wartości $s_1, ..., s_n$ dla właściwości. Elementy macierzy A zdefiniowane są jako ilorazy $a_{ij} = s_i/s_j$ i tworzą spójną macierz. W praktyce jednak bardzo rzadko zdarza się pracować na idealnych danych, dlatego stosuje się różne metody filtrowania wyników, a nawet ich dostrajania tak aby zapewniały minimum wymaganej dokładności.

Do określania stopnia poprawności danych wprowadzony został tzw. współczynnik niespójności. Analizując jego wartość możemy podejmować decyzję o uwzględnieniu lub odrzuceniu danej macierzy ze zbioru wynikowego. Na przestrzeni lat powstało wiele metod wyznaczania wartości współczynnika. Oprócz prac Saaty'ego i Koczkodaja, opisanych poniżej, także inni uczeni mają swój wkład. W 1993 roku Dodd, Donegan i McMaster przedstawili odwrotny wskaźnik niespójności [6]. Następnie w 1996 roku Monsuur zastosował transformacje maksymalnych wartości własnych [16], a siedem lat później Pelaez i Lamata badali wszystkie trójki elementów macierzy i użyli wyznacznika do określenia niespójności [12], ponadto w 2007 roku Stein oraz Mizzi wprowadzili harmoniczny współczynnik niespójności [20].

Innym typem współczynnika jest odległość od macierzy spójnej. Chu, Kalaba i Spingarn w 1979 roku użyli szacowania błędu metodą najmniejszych kwadratów [1], sześć lat po nich, w 1985 roku Crawford i Williams użyli logarytmicznego szacowania metodą najmniejszych kwadratów [7], a w 2003 roku Aguaron i Moreno-Jimenez zastosowali geometryczny współczynnik niespójności dla logarytmicznego oszacowania błędu metodą najmniejszych kwadratów (metoda średniej geometrycznej wiersza macierzy) [8]. W swojej pracy skupiłem się na publikacjach *A scaling method for priorities in hierarchical structures* [17] oraz *A new definition of consistency for pairwise comparisons* [13] i *Generalization of a new* *definition of consistency for pairwise comparisons* [27], autorstwa dwóch wspomnianych na początku uczonych i zdecydowałem użyć ich metod radzenia sobie z niespójnością danych.

3.1. Metoda wartości własnych - Saaty

Jednym z pierwszych uczonych który podjął się tego tematu jest Thomas Saaty. W 1977 roku zaproponował metodę wyznaczania współczynnika niespójności macierzy porównań parowych [17]. Jego sposób opiera się na analizie wartości własnych macierzy. Wykorzystał w tym celu teorię macierzy rzeczywistych dodatnich Frobeniusa-Perrona [22] mówiącą iż największa wartość własna λ_{max} macierzy A, jest zawsze $\lambda_{max} \geq n$. Zauważył iż w przypadku gdy λ_{max} równa jest n to wtedy i tylko wtedy macierz A jest spójna. Indeks średniej niespójności (CI_n) macierzy jest określony równaniem

$$CI_n = \frac{\lambda_{max} - n}{n-1},$$

Wynika z niego iż indeks jest tak naprawdę przeskalowaniem największej wartości własnej. CI_n jest także zawsze nieujemny, ponieważ $\lambda_{max} \geq n$. Sam w sobie nie reprezentuje jednak żadnej wartości dopóki nie zostanie zestawiony z jakimkolwiek punktem odniesienia. Dzięki temu możemy określić wielkość odchylenia od macierzy spójnej. Punktem referencyjnym zaproponowanym przez Saaty'ego jest losowy współczynnik RI_n.

Przyjmijmy że jest dany zbiór np. 500 losowo wygenerowanych macierzy porównań parowych rozmiaru $n \times n$, takich że każdy element $a_{ij}(i < j)$ został losowo wybrany spośród wartości skali

$$\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9$$

oraz $a_{ji} = \frac{1}{a_{ij}}$. Wtedy współczynnik RI_n oznacza średnią wartość wyliczonych dla tych losowych macierzy indeksów niespójności. Zależy on od rozmiaru macierzy n i wybranej skali. Wyliczone wartości współczynnika zostały przedstawione w tabeli 3.1. Współczynnik niespójności CR_n macierzy A wynosi

$$CR_n = \frac{CI_n}{RI_n}$$

Dla macierzy spójnej $\lambda_{max} = n$, więc $CI_n = 0$ a także $CR_n = 0$. Saaty przyjął że niespójność na poziomie 10% lub mniej może być uznana za akceptowalną, a co za tym idzie macierz może zostać wykorzystana do przeprowadzenia wnioskowania na jej podstawie. Intuicyjne znaczenie reguły dziesięciu procent jest pomijane przez kilku autorów. Reguła została wyjaśniona statystycznie przez Vargas (1982) [19]. W roku 1994, kilkanaście lat po pierwszej publikacji, Saaty poprawił swoją teorię zmieniając maksymalne akceptowalne wartości niespójności na 5% dla macierzy 3×3 oraz 8% dla 4×4.

Istnieje również tzw. reguła kciuka (ang. *thumb rule*) która upraszcza proces badania właściwości macierzy do analizy współczynnika CI. Przyjmuje się że jeżeli $CI < 0.1$ to macierz jest wystarczająco spójna.

Pomimo iż jedyną szeroko akceptowalną i stosowaną zasadą niespójności dla dowolnego rodzaju macierzy jest praca Saaty'ego, jego definicja niespójności ma kilka słabych punktów. Głównym z nich

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RI_n	0	0	0	0.52	0.89	1.11	1.25	1.35	1.4	1.45	1.49	1.51	1.54	1.56	1.57	1.58

Tablica 3.1: Wartości losowego współczynnika niespójności dla różnej ilości kryteriów porównania (n)

wydaje się być zasada 10 procent, określona bardziej na podstawie doświadczeń i obserwacji aniżeli definicji matematycznej. Kolejnym elementem wskazywanym przez krytyków jest traktowanie niespójności w ujęciu globalnym - bez podania nawet przybliżonego jej źródła. Wartości własne są wartościami charakteryzującymi całą macierz, a więc przez ich analizę nie można stwierdzić które elementy macierzy najbardziej wpływają na zwiększenie niespójności.

3.2. Metoda odległościowa - Koczkodaj

Druga wybrana przeze mnie metoda została przedstawiona przez Waldemara Koczkodaja (1993) [13] oraz później przez Koczkodaj i Duszak (1994) [27]. Jak sam napisał w [13]: *"Autor tej pracy z całą mocą wierzy iż zdecydowanie za niska popularność metody porównań parami ma swoje źródła w definicji spójności."*

Każda macierz porównań parowych rozmiaru 3×3 ma 3 obiekty porównywane a, b, c. W celu przedstawienia współczynnika niespójności Koczkodaja rozważmy standardową macierz parową 3×3. Zakładając że macierz jest wzajemna możemy ją zredukować do wektora trzech współrzędnych (a, b, c). W przypadku spójności, równość $b = ac$ jest prawdziwa. Zawsze istnieje możliwość wyznaczenia trzech spójnych, wzajemnych macierzy parowych (reprezentowanych przez trzy wektory) poprzez wyliczenie jednej współrzędnej jako kombinacji pozostałych dwóch współrzędnych. Te trzy wektory są postaci: $\left(\frac{b}{c}, b, c\right)$, (a, ac, c) oraz $\left(a, b, \frac{b}{a}\right)$.

Współczynnik niespójności (nazywany także *miarą spójności* CM) dla macierzy parowej 3×3 zdefiniowany jest przez Koczkodaja jako względna odległość od najbliższej, spójnej macierzy parowej 3×3 reprezentowanej przez jeden z tych trzech wektorów.

Def. 1. *Miara spójności dla standardowej* 3×3 *macierzy porównań parowych wynosi:*

$$CM(a, b, c) = \min\left\{\frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}\left|b - ac\right| \frac{1}{c}\left|c - \frac{b}{a}\right|\right\} \tag{3.1}$$

Współczynnik niespójności macierzy wzajemnej $\boldsymbol{A}$ $n \times n (n > 2)$ *wynosi:*

$$CM(\boldsymbol{A}) = \max\left\{ \min\left\{\left|1 - \tfrac{b}{ac}\right|, \left|1 - \tfrac{ac}{b}\right|\right\} \quad \text{\textit{dla każdej triady (a, b, c) w} } \boldsymbol{A} \right\} \tag{3.2}$$

W przypadku macierzy wyższych rzędów, wskaźnik niespójności elementu macierzy jest równy maksymalnej wartości CM ze wszystkich możliwych triad zawierających ten element.

Ilość wszystkich możliwych triad dla macierzy $n \times n$ porównań parowych wynosi (Duszak i Koczkodaj, 1994):

$$\frac{n(n-1)(n-2)}{3!} \tag{3.3}$$

Do głównych zalet tej metody można zaliczyć:

– łatwość interpretacji (względne odchylenie od macierzy spójnej, która może zostać określona przy zachowaniu dwóch ocen niezmienionych),

– tworzy lepsze podstawy dla doboru progu (np. CM dla $[1, 3, 2]$ lub $[3, 6, 3]$ lub $[3, 9, 4]$, a nawet $[2, 8, 3]$ wynosi 0.25),

– określa położenie spójności (CM jest zawsze połączony z konkretnym elementem macierzy, a nie enigmatycznym globalnym wskaźnikiem jak wartość własna).

W przypadku macierzy 4×4 i skali *od 1 do 5* próg akceptowalnego rozwiązania nie powinien przekraczać $1/3$ [24]. Ze względu jednak na lokalny charakter rozważanej niespójności, podany próg można uznać za prawdziwy i rozważać także dla macierzy innych wymiarów oraz dla innej skali porównawczej. Jak do tej pory nie stwierdzono zależności między rozmiarem, a zakresem skali w obszarze wpływu na wysokość progu akceptowalnego rozwiązania. Potencjalna obecność takiego związku może być przedmiotem kolejnych badań nad metodami porównań parowych.

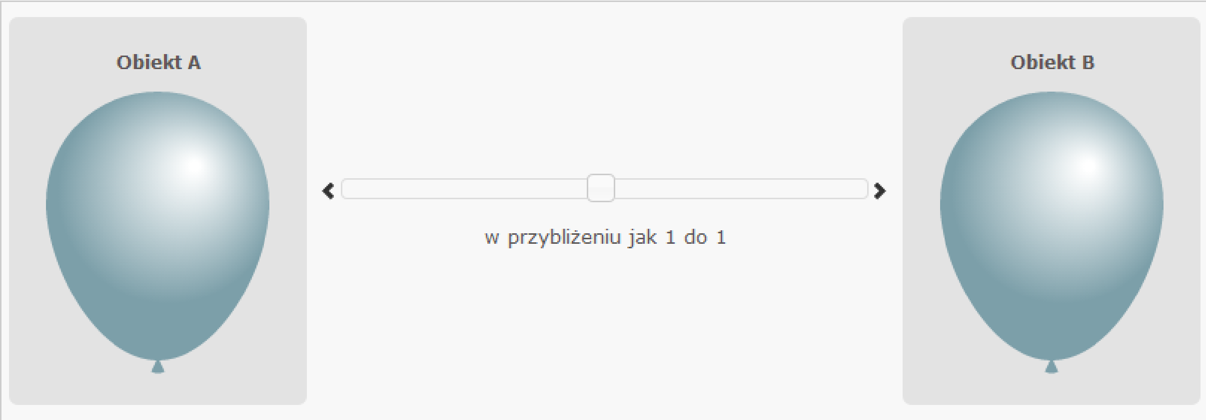
4. Problem doboru skali w porównaniach parowych

Wszystkie teorie i prace traktujące o porównaniach parowych obowiązkowo muszą poruszyć problem skali porównania. Jest to niezwykle istotny element metody porównywania parami. Jak zauważają autorzy w [10] ma wpływ na całość metody w tym także może być przyczyną spływającej na nią krytyki. Przekonują że właściwa skala pozwoli znacznie poprawić jakość danych początkowych, zmniejszając błędy niespójności i tym samym wytrącając argumenty przeciwników metody.

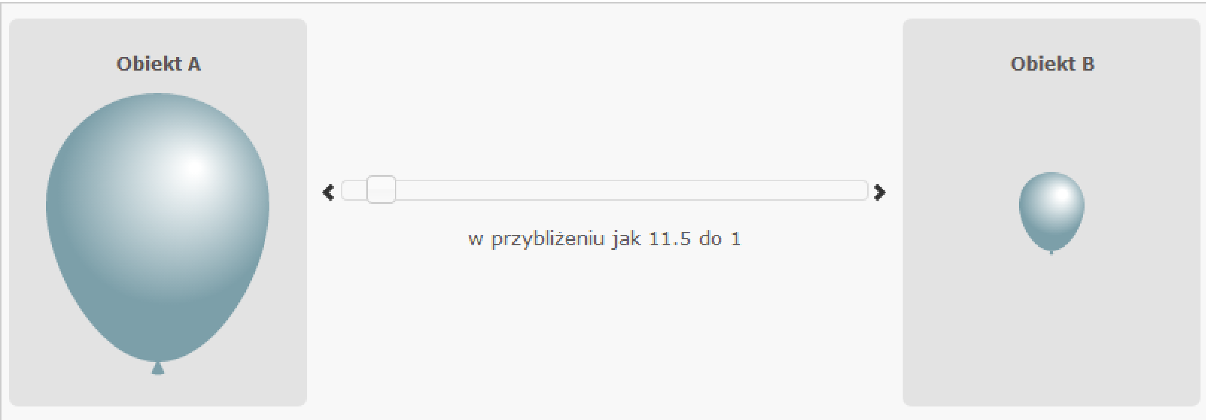
Najprostsza wersja to skala binarna *wygrana/przegrana* (Condorcet [4]). Odpowiada ona na pytanie: *Który?*, rozumiejąc przez to który z dwóch obiektów ma większą wartość w ramach porównywanej cechy: jest większy/mniejszy, tańszy/droższy, itd. Przykładem zastosowania mogą być wybory. Za każde zwycięstwo, kandydat otrzymuje 1 punkt, a za remis 0.5 punktu. Po przeliczeniu głosów otrzymujemy listę priorytetową kandydatów z których pierwszy jest zwycięzcą. Obecnie tego typu skala nie jest już wykorzystywana, lub ma to miejsce niezwykle rzadko. Zastąpiona została przez wielopoziomowe skale o różnym zakresie. Głównym elementem odróżniającym je od wersji podstawowej jest wartość liczbowa. Dzięki niej możemy odpowiedzieć na jeszcze jedno pytanie: *Ile?*. Jest to niezwykle cenna informacja, dająca realny punkt odniesienia. Pytanie może istnieć w dwóch formach, tj. *Ile razy?* oraz *O ile?*. Najczęściej spotykana i wykorzystywana jest forma pierwsza, ale sam problem decyzyjny mógłby posłużyć za inspirację do kolejnej pracy naukowej z pogranicza psychologii i matematyki.

Nie ustają spory wśród naukowców na temat zakresu skali. Wielu autorów badających metodę porównań parowych najczęściej przedstawia swoje propozycje.

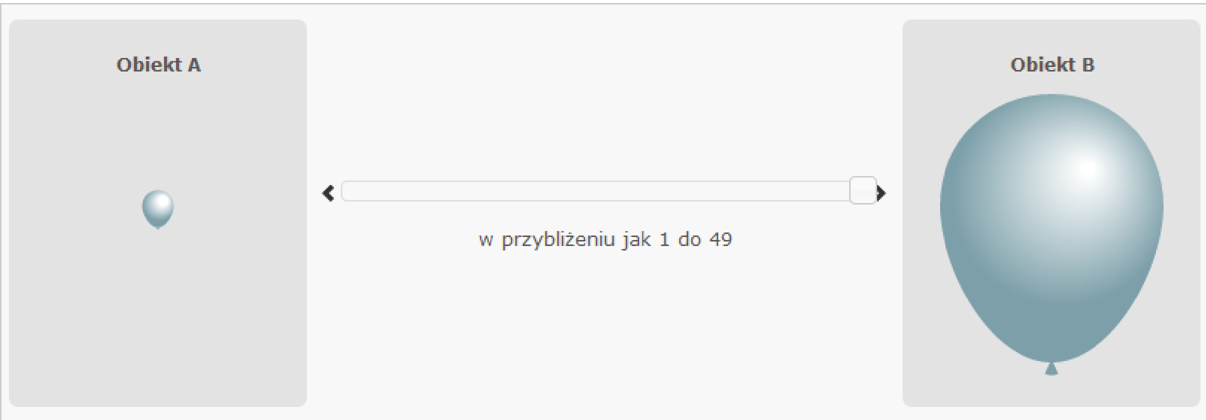
Zgodnie z pracą [9] istnieją teoretyczne podstawy by sądzić iż najlepsza skala powinna być mniejsza od stałej Fülöpa: $\sqrt{\frac{1}{2}(11 + 5\sqrt{5})} \approx 3.330191$. Odpowiednio dobrana skala wraz z adekwatnie prowadzoną stymulacją wzrokową respondentów pozwala na uzyskanie lepszej jakości wyników. Respondenci zwracają większą uwagę na różnice wielkości obrazków niż wartości liczbowych. W roli narzędzia służącego do ustalenia wartości najlepiej spisuje się suwak, który ma swoją pozycję startową na środku osi po której może się poruszać. Ilustracje 4.1 - 4.3 przedstawiają realizację powyższych wytycznych na potrzeby prototypowego narzędzia [21]. Porównywane ze sobą są dwa dowolne obiekty. Zmiana rozmiaru obiektu - przestawienie suwaka na inną pozycję - jest proporcjonalna w stosunku do zmiany wartości porównania.



Rysunek 4.1: Stymulacja wzrokowa skali, stan początkowy



Rysunek 4.2: Ponad jedenastokrotna przewaga obiektu A w stosunku do B



Rysunek 4.3: Przytłaczająca różnica wartości na korzyść obiektu B

4.1. Skala Saaty'ego

Najszerzej znana i stosowana obecnie w porównaniach parowych, praca Saaty'ego ([17]) opiera się na dziewięciostopniowej skali (tabela 4.1). Daje respondentowi bardzo szeroki wybór. Pomiędzy elementami o mocno zdefiniowanym znaczeniu znajdują się punkty pośrednie, niejako zmiękczające znaczenie. Skala przedstawia stopnie w porządku rosnącym. Gdy dokonujemy porównania skala staje się symetryczna, a więc możemy wybrać wartości odwrotne, ułamkowe, gdy w naszej ocenie obiekt główny jest *mniejszy/mniej ważny/itd...* od obiektu z nim zestawionego. Uzyskane z porównań wartości są elementami macierzy PC.

Wartość	Definicja	Znaczenie
1	taka sama ważność	Oba obiekty wpływają w tym samym stopniu na porównanie
3	średnio ważniejsze	Doświadczenie i ocena lekko faworyzuje jeden z obiektów
5	ważniejsze	Doświadczenie i ocena zdecydowanie wyróżnia jeden z obiektów
7	dużo ważniejsze	Doświadczenie i ocena bardzo mocno faworyzuje jeden z obiektów. Jego waga jest wykazana w praktyce
9	nieporównywalnie ważniejsze	Dowód za słusznością jednego z obiektów ma największą możliwą zasadność
2,4,6,8	wartości pośrednie	Kiedy trzeba zdobyć się na pewne ustępstwa

Tablica 4.1: Skala ocen wg Saaty'ego

4.2. Skala Likerta

To pięciostopniowa skala, którą wykorzystuje się w kwestionariuszach ankiet i wywiadach kwestionariuszowych, dzięki której można uzyskać odpowiedź dotyczącą stopnia akceptacji zjawiska czy też poglądu. Jest intensywnie eksploatowana w wielu metodach badań społecznych. Jej nazwa pochodzi od nazwiska Rensisa Likerta, który wynalazł ją w 1932 r.[15].

Skala składa się z pięciu poziomów odpowiedzi, ułożonych w kolejności od stopnia całkowitej akceptacji do całkowitego odrzucenia. Zadaniem ankietowanego jest określić w jakim stopniu zgadza się z danym twierdzeniem. Przykładowe warianty opisane na skali:

1. - zdecydowanie nie zgadzam się
2. - raczej się nie zgadzam
3. - nie mam zdania
4. - raczej się zgadzam
5. - zdecydowanie się zgadzam

Liczba możliwych do wyboru odpowiedzi powinna być nieparzysta i najczęściej jest to 5. Pozwala to na centralne położenie możliwie najbardziej neutralnego stwierdzenia. Badany wybiera tę możliwość, która najbardziej odpowiada jego odczuciom.

Skala powstała ze względu na potrzebę liczenia wskaźników zbudowanych z sumy (lub średniej) pytań kwestionariuszowych i w związku z tym różni się od innych skal porządkowych, że może być i z reguły jest traktowana jako ilościowa.

W badaniach kwestionariuszowych wyznacza się tzw. szczególne upodobanie. Pozwala to za pomocą w/w skali określić paradygmat (najogólniejszy model) zbioru ankietowanych.

4.3. Inne

Autorzy pracy [10] dowodzą iż najlepsza skala to taka, która ma możliwie najmniejszy zakres. Dając respondentowi szeroki przedział wartości, typu $1 - 100$ narażamy się na wysokie błędy niespójności. Ponadto można zaobserwować iż w ocenach dominują wartości w okolicach pełnych dziesiątek $20, 30, 40, ..., 70, 80$

Są 2 opcje pracy ze skalą i wybieranych wartości. Mogą być dyskretyzowane wartości lub dowolne z ustawioną dokładnością.

Przykłady innych skal można znaleźć w publikacji [11].

Jak widać wybór jest szeroki, każda opcja ma swoje wady i zalety. Dlatego w swojej pracy postanowiłem pozostawić tą decyzję twórcy ankiety. Dzięki temu może skorzystać z dowolnej wybranej lub nawet zastosować własną skalę.

5. Grupowanie hierarchiczne

Znakomita większość systemów spotykanych na co dzień posiada strukturę hierarchiczną. Składają się one z małych grup, np. rodzina i rozrastają do różnorakich dużych organizacji jak społeczność miasta, województwa czy też cały naród. Na czele narodu stoi często rząd, który także ma ściśle określoną strukturę.

Teoria Hierarchii jest formą ogólnej teorii systemów [2]. Wyłoniła się jako osobna część w trakcie intensywnych prac nad złożonością. Mająca swoje korzenie w pracach o tematyce ekonomicznej, chemicznej oraz psychologicznej, teoria hierarchii skupia się na poziomach organizacji i problemach skali. Ma bardzo szerokie zastosowanie i jest powszechnie używana nie tylko przez naukowców.

W teorii hierarchii nie ma rozbudowanych i skomplikowanych reguł, wręcz przeciwnie, stosunkowo mały zbiór zasad pozwala uporządkować nawet skomplikowaną strukturę do postaci wielu poziomów. Z matematycznego punktu widzenia hierarchia jest zbiorem częściowo uporządkowanym. Poziomy wyższe są ponad poziomami niższymi oraz relacja w górę jest niesymetryczna względem relacji w dół. Poziom to nic innego jak zbiór elementów charakteryzowany przez ich właściwości, np. podział ludności ze względu na wiek czy kolor oczu.

Technicznie rzecz biorąc, hierarchia to zbiór zależności pomiędzy poziomami. Dla dwóch następujących po sobie poziomów, poziom wyższy jest nazywany *rodzicem*, a niższy *dzieckiem*. Hierarchia może reprezentować struktury organizacyjne, geograficzne lub dowolnego innego typu. Dlatego ważne jest wspieranie różnych rodzajów hierarchii wraz z odpowiednio dużą elastycznością konfiguracji. Dzięki temu możemy tworzyć struktury modelujące wiele różnych procesów.

5.1. Klasyfikator

Klasyfikator hierarchiczny pozwala na przetworzenie danych wejściowych dzięki czemu otrzymujemy odpowiednio pogrupowane wyniki [3]. Proces klasyfikacji rozpoczyna się od najniższego poziomu i do działania wykorzystuje oryginalne dane wejściowe. Następnie ocena poszczególnych jednostek jest sukcesywnie łączona i następuje klasyfikacja na wyższym poziomie. Proces jest kontynuowany do momentu uzyskania jednego, spójnego wyniku. Jest to więc czysty przykład metody bottom-up. Warto odnotować, iż proces klasyfikacji hierarchicznej opiera się nie na możliwościach obliczeniowych wejściowych komponentów, ale na samej strukturze hierarchii. W konsekwencji, dzięki temu hierarchia może być prostą i łatwo rozszerzalną strukturą.

5.2. Zastosowanie hierarchicznego klasyfikatora

Na przestrzeni kilku ostatnich dekad zostało stworzonych wiele zastosowań dla grupowania hierarchicznego. Jednym ze znakomitych przykładów jest cała gałąź rozpoznawania obrazów. Rozpoznawanie obrazów jest uznawane za jeden z najbardziej skomplikowanych problemów, który wymaga znacznych mocy obliczeniowych. Dość powiedzieć, że większość zabezpieczeń anty-robotowych znajdujących się na stronach www opiera się właśnie na rozpoznawaniu obrazów, gdyż nikt tak jak człowiek nie potrafi sobie z nimi poradzić. Na pomoc technologii przychodzi przetwarzanie hierarchiczne, które dobrze współpracuje z rozpoznawaniem obrazów. A to dlatego, iż zdjęcia/obrazy mogą być przedstawione jako zbiór komponentów lub obiektów. Te obiekty z kolei mogą być postrzegane jako zbiór mniejszych kształtów. Na nie składają się zbiory linii itd. Widać wyraźnie, iż takie przetwarzanie jest zbieżne z tym jak działa klasyfikacja hierarchiczna.

Podobnie proces może zachodzić w drugą stronę, przetwarzając kształty i linie w obiekty, przy czym najprawdopodobniej będą musiały zostać wykonane pewne dodatkowe kroki pośrednie. W rezultacie, mając wszystkie elementy ułożone hierarchicznie w skierowanym, acyklicznym grafie, zaczynając od pikseli i pnąc się w górę możemy przeprowadzić wnioskowanie, co znajduje się na obrazku.

Istnieje wiele innych zastosowań, gdzie z powodzeniem wykorzystamy grupowanie hierarchiczne, jak np. rozpoznawanie tekstu pisanego (OCR), czy też samo-świadomość robotów.

5.3. Hierarchia w metodzie PC

Struktury hierarchiczne w metodzie porównań parowych mają swój początek w latach 70. ubiegłego wieku w pracach Thomasa Saaty'ego. Przedstawił on metodę *Analytic Hierarchy Process* (AHP). Jest to wielokryterialna metoda hierarchicznej analizy problemów decyzyjnych. Umożliwia ona przedstawienie skomplikowanego i złożonego problemu decyzyjnego za pomocą prostej struktury hierarchicznej i na jej podstawie utworzenie rankingu dla określonych wariantów. Metoda jest wykorzystywana w wielu dziedzinach, od politologii i socjologii, aż po zarządzanie i transport. Radzi sobie z problemem uszeregowania skończonej ilości kategorii (wariantów) ocenianych pod względem również skończonej liczby kryteriów. Złożoność problemu jest najlepiej widoczna poprzez ilość poziomów w hierarchii po zdekomponowaniu. AHP wykorzystuje algebrę macierzową do poukładania wariantów w sposób optymalny pod względem matematycznym. Jest to sprawdzona metoda, używana do podejmowania decyzji w sprawach o wielomilionowych stawkach.

Najpopularniejszymi obszarami zastosowania metody AHP są:

- klasyfikowanie wymogów i czynników które mają wpływ na procesy powstawania i rozwoju oprogramowania,
- wybieranie najlepszej z możliwych strategii dla poprawy bezpieczeństwa samochodów i innych pojazdów silnikowych,

– szacowanie kosztów i ustalanie opcji dla planowania zapotrzebowania materiałowego (MRP),

– wybieranie najbardziej pożądanych komponentów oprogramowania spośród grona dostawców,

– ocenianie jakości badań rozwojowych lub planów inwestycyjnych.

W obliczeniach, oprócz macierzy wynikowych porównań, wykorzystywane są również właściwe wartości charakterystyczne wariantów jak wartość, cena, ilość, waga, itd. Źródłem błędów metody jest niespójność macierzy parowych, pozostałe etapy przetwarzania oparte są na obliczeniach matematycznych.

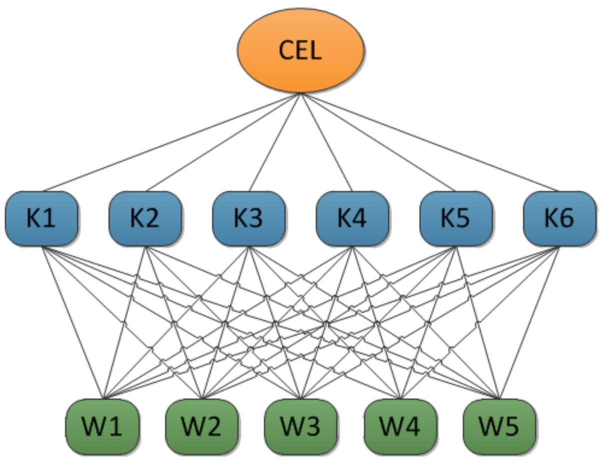
Algorytm metody AHP składa się z czterech etapów:

1. Definicja problemu i określenie celu do którego dążymy.
2. Definicja kategorii i wariantów mających wpływ na cel oraz uporządkowanie ich w strukturze hierarchicznej.
3. Zbieranie danych o preferencjach respondenta.
4. Tworzenie rankingu końcowego.

Niekiedy jako ostatni krok dodaje się tzw. analizę wrażliwości, polegającą na badaniu w jaki sposób poszczególne czynniki wpływają na końcowy rezultat.

Słowo wyjaśnienia dla kroku trzeciego. W tym miejscu wykorzystywane są porównania parowe. Zestawiane są ze sobą warianty na każdym poziomie hierarchii, ale nie pomiędzy poziomami. Porównania wykorzystują skalę Saaty'ego opisaną w rozdziale 4.1. Po dokonaniu porównań ma miejsce wstępne przetwarzanie danych dla każdej macierzy, obliczana jest niespójność (metodą przedstawioną w r. 3.1) oraz tworzony jest ranking.

Rysunek 5.1 przedstawia skondensowaną postać przykładowej struktury dla problemu polegającego na uszeregowaniu pięciu wariantów ocenianych względem sześciu kryteriów.



Rysunek 5.1: Model hierarchicznej struktury procesu decyzyjnego AHP

5.3.1. Przykład

Aby lepiej zobrazować proces oceny posłużę się przykładem.

Wyobraźmy sobie iż ekspert X staje przed wyborem nowego telefonu - smartfona dla siebie. Pod uwagę bierze tylko najnowsze i najlepsze konstrukcje, tzw. flagowce. Zakłada iż budżet w wysokości 2000 zł będzie wystarczający. Wstępne rozeznanie ogranicza jego wybór do 4 modeli (A, B, C, D). Główne cechy, które mają wpływ na ostateczną decyzję to cena, długość pracy na baterii, rozmiar ekranu, ilość pamięci wewnętrznej oraz jakość wbudowanego aparatu fotograficznego. Korzystając z AHP, jako pierwsze należy określić problem. Oznacza to zdefiniowanie struktury najlepiej odpowiadającej porównywaniu smartfonów w ramach wybranych cech. Tabela 5.1 przedstawia zebrane urządzenia i ich cechy które rozważa ekspert X. Są to prawdziwe dane telefonów powstałych w 2014 roku. Cena jest przybliżoną średnią cen z popularnego polskiego serwisu aukcyjnego z dnia 15 marca 2015. Dane o żywotności baterii pochodzą z testów serwisu GsmArena. Dla eksperta X dostępne jest oprogramowanie AHP zajmujące się wszystkimi wyliczeniami. Od niego wymagamy dokonania porównań.

Telefon A - iPhone 6 Plus

Telefon B - Samsung Galaxy S5

Telefon C - Nokia Lumia 930

Telefon D - Sony Xperia Z3

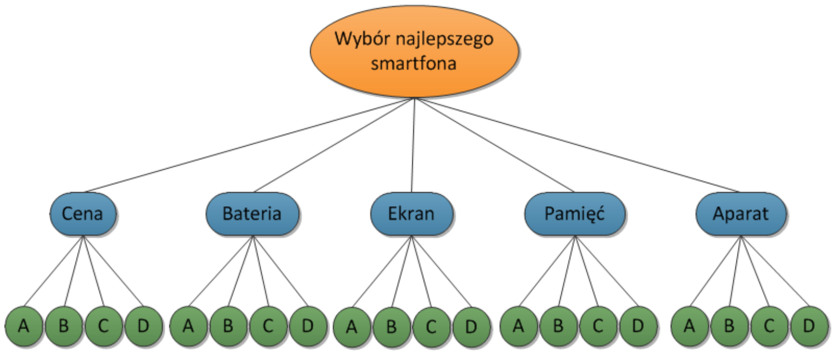
Cecha	A	B	C	D
Cena	3100 zł	1550 zł	1850 zł	1900 zł
Bateria	79h	72h	46h	85h
Ekran	5.5"FHD	5.1 FHD	5"FHD	5.2"FHD
Pamięć	64 GB	16GB	32 GB	16GB
Aparat	8Mpx	16Mpx	20Mpx	20.2 Mpx

Tablica 5.1: Cechy i rozważane alternatywy w ramach wyboru smartfona

Drugim krokiem jest określenie zależności rodzic-dziecko oraz zbudowanie hierarchii kryteriów i alternatyw przedstawionej na rysunku 5.2.

Najwyższy poziom, korzeń, przedstawia cel "wybór najlepszego smartfona". Poziom drugi zawiera cechy produktów w ramach których będą porównywane telefony z poziomu trzeciego.

Dysponując wszystkimi danymi, ekspert przystępuje do trzeciej, najdłuższej części procesu AHP. Musi dokonać szeregu porównań parowych na każdym poziomie oraz w kontekście cechy. Oprogramowanie wykorzystuje skalę Saaty'ego 4.1. Pierwsze porównanie to *cena vs bateria*. Nasz bohater uznaje iż czas pracy na baterii jest nieco ważniejszy od ceny (ocena 3), a cena ma zdecydowanie większy wpływ



Rysunek 5.2: Hierarchia atrybutów i telefonów

na ocenę końcową niż pamięć (ocena 5). Pan X kontynuuje oceny, które zebrane tworzą macierz:

$$\begin{array}{c} \\ cena \\ bateria \\ ekran \\ pamiec \\ aparat \end{array} \begin{array}{c} \begin{array}{ccccc} cena & bateria & ekran & pamiec & aparat \end{array} \\ \begin{pmatrix} 1 & \frac{1}{3} & 5 & 3 & \frac{1}{2} \\ 3 & 1 & 5 & 4 & 2 \\ \frac{1}{5} & \frac{1}{5} & 1 & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & 3 & 1 & 2 \\ 2 & \frac{1}{2} & 6 & \frac{1}{2} & 1 \end{pmatrix} \end{array} \tag{5.1}$$

Dla tak powstałej macierzy przeprowadzane są obliczenia. Najpierw, korzystając z metody wartości własnych (3.1) wyznaczana jest niespójność. Rozmiar macierzy wynosi $n = 5$. Największa wartość własna $\lambda_{max} = 5.559$, a zatem współczynnik $CI_5 = \frac{5.559-5}{5-1} \approx 0.14$. Mając powyższe dane, określany jest współczynnik niespójności

$$CR_5 = \frac{0.14}{1.11} \approx 0.126$$

Wyliczona wartość jest nieco powyżej progu akceptacji wyznaczonego przez Saaty'ego, lecz dalej na poziomie tolerancji, a ze względu na większy rozmiar macierzy i niewielkie odchylenie od normy, ekspert X postanawia zaakceptować te dane i kontynuować pracę. Wyznaczony zostaje wektor wag dla cech:

Cecha	**Priorytet**
Cena	0.194
Bateria	0.420
Ekran	0.045
Pamięć	0.140
Aparat	0.201

Tablica 5.2: Główny wektor wag dla cech

Warto zauważyć że suma priorytetów (wag) sumuje się do wartości 1. Wyniki porównania najważniejszych cech nie pozostawiły wątpliwości - największe znaczenie dla eksperta ma długość pracy na baterii.

Kolejny krok to ocena każdego z wybranych modeli pod względem cech. Kolejność dokonywania porównań nie ma znaczenia, ale pan X zdecydował użyć powyżej zdefiniowanej kolejności. Porównań jest wiele, więc aby nieco ułatwić sobie pracę stworzył tabelkę, przedstawioną na rysunku 5.3, ułatwiającą podejmowanie decyzji.

Długość pracy na baterii (więcej = lepiej)								Ocena pana Jana	
Telefony			Długość pracy		Lepsza bateria	Bateria lepsza jako		Lepszy tel.	Ocena wg skali
#	A	B	A	B		Ilość h	Stosunek		
1	Iphone 6 Plus	Samsung Galaxy S5	79h	72h	A	7h	1.1	A	2
2	Iphone 6 Plus	Nokia Lumia 930	79h	46h	A	33h	1.72	A	7
3	Iphone 6 Plus	Sony Xperia Z3	79h	85h	B	6h	1.08	B	2
4	Samsung Galaxy S5	Nokia Lumia 930	72h	46h	A	26h	1.57	A	6
5	Samsung Galaxy S5	Sony Xperia Z3	72h	85h	B	13h	1.18	B	3
6	Nokia Lumia 930	Sony Xperia Z3	46h	85h	B	39h	1.85	B	8

Rysunek 5.3: Ocena telefonów pod względem długości czasu pracy na baterii

W rezultacie otrzymał kolejny wektor wag:

Alternatywa	**Priorytet lokalny**	**Priorytet globalny**
iPhone 6 Plus	0.300	0.124
Samsung Galaxy S5	0.180	0.076
Nokia Lumia 930	0.040	0.018
Sony Xperia Z3	0.480	0.201
SUMA	**1.000**	**0.420**

Tablica 5.3: Priorytety telefonów z perspektywy czasu pracy na baterii

Priorytet globalny wylicza się jako iloczyn lokalnego oraz iloczynu priorytetów wszystkich poprzednich poziomów. Jako że rozważany przykład ma 2 poziomy, dla baterii używamy wagę 0.420. Jak widać w wierszu podsumowania, suma priorytetów globalnych jest równa wadze nadrzędnej.

Nasz bohater przeprowadza do końca pozostałe porównania, otrzymując następujące priorytety:

Alternatywa	**Priorytet lokalny**	**Priorytet globalny**
iPhone 6 Plus	0.043	0.009
Samsung Galaxy S5	0.142	0.028
Nokia Lumia 930	0.317	0.064
Sony Xperia Z3	0.498	0.100
SUMA	**1.000**	**0.201**

Tablica 5.4: Priorytety telefonów z perspektywy aparatu

Alternatywa	**Priorytet lokalny**	**Priorytet globalny**
iPhone 6 Plus	0.038	0.007
Samsung Galaxy S5	0.524	0.102
Nokia Lumia 930	0.257	0.050
Sony Xperia Z3	0.181	0.035
SUMA	**1.000**	**0.194**

Tablica 5.5: Priorytety telefonów z perspektywy ceny

Alternatywa	**Priorytet lokalny**	**Priorytet globalny**
iPhone 6 Plus	0.664	0.093
Samsung Galaxy S5	0.057	0.008
Nokia Lumia 930	0.221	0.031
Sony Xperia Z3	0.057	0.008
SUMA	**1.000**	**0.140**

Tablica 5.6: Priorytety telefonów z perspektywy ilości pamięci

Alternatywa	**Priorytet lokalny**	**Priorytet globalny**
iPhone 6 Plus	0.050	0.002
Samsung Galaxy S5	0.290	0.013
Nokia Lumia 930	0.490	0.022
Sony Xperia Z3	0.180	0.008
SUMA	**1.000**	**0.045**

Tablica 5.7: Priorytety telefonów z perspektywy ekranu

Tym samym etap trzeci został zakończony, a odpowiedzi przeliczone. Oprogramowanie prezentuje końcowe wyniki w formie tabeli 5.4 wag rozważanych alternatyw i ich cech wraz z podsumowaniem. Suma ostateczna wynosi 1.000, a zatem jest równa priorytetowi celu.

Model	*Bateria*	*Aparat*	*Cena*	*Pamięć*	*Ekran*	Suma
Iphone 6 Plus	*0.124*	*0.009*	*0.007*	*0.093*	*0.002*	0.236
Samsung Galaxy S5	*0.076*	*0.028*	*0.102*	*0.008*	*0.013*	0.228
Nokia Lumia 930	*0.018*	*0.064*	*0.050*	*0.031*	*0.022*	0.184
Sony Xperia Z3	*0.201*	*0.100*	*0.035*	*0.008*	*0.008*	0.353
SUMA	0.420	0.201	0.194	0.140	0.045	**1.000**

Rysunek 5.4: Globalne priorytety dla eksperta X wybierającego telefon

Sony Xperia Z3, posiadająca globalny priorytet 0.353, jest alternatywą mającą największy udział w celu będącym wyborem najlepszego smartfona dla eksperta X. Na drugim miejscu można umiejscowić wspólnie iPhone 6 Plus oraz Samsung Galaxy S5. Różnica ich wyników znajduje się dużo poniżej błędu wynikającego z niespójności wszystkich porównań. Ostatni model, telefon firmy Nokia, znalazł się na końcu stawki lecz jego rezultat pokazuje że nie traci kontaktu z czołówką.

6. Projekt

Podstawowy pomysł na system był następujący: zróbmy internetową aplikację ankietującą, która do badania opinii będzie wykorzystywać metodę porównywania parami. Zgodnie ze stanem obecnej wiedzy nie ma dostępnego podobnego rozwiązania, dzięki czemu istnieje duża szansa na zaistnienie na rynku. Kolejnym potencjalnym zastosowaniem są badania naukowe nad metodą porównań parowych, dowodzenia jej poprawności i popularyzacji zarówno wśród specjalistów jak i użytkowników prywatnych.

6.1. Analiza wymagań

6.1.1. Wymagania funkcjonalne

Głównym zadaniem i celem systemu jest umożliwienie tworzenia ankiet i brania w nich udziału (odpowiadania), a także gromadzenie i przetwarzanie zebranych danych na potrzeby statystyczne. Rysunek 6.1 przedstawia główne przypadki użycia opisujące powyższe funkcjonalności z perspektywy użytkowników systemu.

Na etapie projektowania wyróżniłem trzech podstawowych aktorów, odpowiadają oni rolom użytkowników w aplikacji. Są to:

Administrator	ADMIN
Zarejestrowany użytkownik	USER
Gość	GUEST

Każdy z nich może realizować różne scenariusze. Najważniejsze z nich przedstawiam poniżej.

UC 1	**Załóż konto/rejestruj**
Cel:	Założenie konta w systemie
Główny Aktor:	GUEST
Scenariusz główny:	

1. Gość chce założyć konto w systemie
2. Gość uzupełnia formularz rejestracji podając dane kontaktowe i dostępowe
3. System weryfikuje poprawność i unikalność danych
4. System wysyła wiadomość email z linkiem aktywacyjnym na podany adres
5. Gość odbiera wiadomość i przechodzi na stronę aktywacji konta
6. System weryfikuje poprawność kodu aktywacyjnego i zatwierdza konto
7. Gość może się zalogować do systemu

UC 2	**Weź udział w ankiecie**
Cel:	Udzielenie odpowiedzi i jej zapis w systemie
Poziom:	Funkcja systemowa
Główny Aktor:	GUEST, USER

Scenariusz główny:

1. Użytkownik wybiera interesującą go ankietę z katalogu ankiet: gość widzi tylko ankiety publiczne, zalogowany także te do których wypełnienia został zaproszony
2. Wypełnia formularz danymi podstawowymi, jeśli jest zalogowany, dane są uzupełnione automatycznie
3. Wyraża zgodę na przechowywanie danych
4. Udziela odpowiedzi na wszystkie pytania
5. Zatwierdza wprowadzone dane
6. System przetwarza i zapisuje dane
7. Użytkownik widzi ekran podsumowania zapisanej odpowiedzi

Rozszerzenia i scenariusze alternatywne:

1.a Użytkownik otwiera link który otrzymał w wiadomości e-mail będącej zaproszeniem do wzięcia udziału w ankiecie

3.a Użytkownik nie wyraził zgody, więc nie może wziąć udziału w badaniu

UC 3	**Zarządzaj ankietami**
Cel:	Zarządzanie i organizacja własnych ankiet
Poziom:	Funkcja systemowa
Główny Aktor:	USER
Warunki początkowe:	– Użytkownik posiada konto w systemie – Użytkownik jest zalogowany

Scenariusz główny:

1. Użytkownik otwiera domyślny widok panelu sterowania
2. Przegląda ankiety, może podjąć akcje:
    1. Wyszukiwanie
    2. Rozpoczęcie tworzenia nowej ankiety
    3. Edycja
    4. Usuwanie
    5. Zmiana statusu

UC 4	**Dodaj nową ankietę**
Cel:	Dodanie nowej ankiety w stanie roboczym

Poziom:	Funkcja systemowa
Główny Aktor:	USER
Warunki początkowe:	– Użytkownik posiada konto w systemie – Użytkownik jest zalogowany

Scenariusz główny:

1. Użytkownik inicjuje proces definiowana ankiety
2. Uzupełnia formularz danych podstawowych i zapisuje
3. System weryfikuje poprawność danych i prezentuje główny widok szczegółów ankiety
4. Twórca definiuje obowiązkowe elementy:
   - Przedziały znaczeniowe skali UC 5
   - Porównywane obiekty UC 6
5. Ankieta jest gotowa do aktywacji

Rozszerzenia i scenariusze alternatywne:

3.a Użytkownik nie podał wszystkich wymaganych danych, lub zawierają one błędy, system prezentuje informacje o nich i wraca do pkt. 2

5.a Opcjonalnie użytkownik dodaje respondentów którzy otrzymają zaproszenia do wzięcia udziału w ankiecie

UC 5	**Definiuj przedziały znaczeniowe skali porównawczej**
Cel:	Określenie podziału zakresu wartości na przedziały wraz z opisami ich znaczenia
Poziom:	Funkcja systemowa

Główny Aktor:	USER
Warunki początkowe:	– Użytkownik posiada konto w systemie – Użytkownik jest zalogowany – Istnieje ankieta ze statusem *Nowa* oraz zdefiniowany jest zakres wartości skali (określona jest minimalna i maksymalna wartość skali)

Scenariusz główny:

1. Użytkownik wybiera ankietę i otwiera widok podstawowy
2. Uruchamia definiowanie przedziałów - system prezentuje pasek skali o wartościach z zakresu [min. . . max]
3. Użytkownik dodaje nowy marker
   - Ustawia pozycję na pasku
   - Podaje tekst opisujący marker, może używać symboli zastępczych w miejsca których zostaną wstawione nazwy porównywanych kategorii
4. Użytkownik powtarza poprzedni krok tyle razy ile markerów (przedziałów znaczeniowych) chce utworzyć

UC 6	**Definiuj kategorie**
Cel:	Zdefiniowanie kategorii które będą porównywane w ankiecie oraz ich organizacja
Poziom:	Funkcja systemowa
Główny Aktor:	USER

Warunki początkowe:	– Użytkownik posiada konto w systemie – Użytkownik jest zalogowany – Istnieje ankieta ze statusem *Nowa*

Scenariusz główny:

1. Użytkownik wybiera ankietę i otwiera widok podstawowy
2. Przechodzi do sekcji definiowania kategorii
3. Naciska przycisk dodawania i z dwóch opcji wybiera *Kategoria*
4. Wypełnia formularz nowej kategorii, obowiązkowo podaje nazwę, zapisuje
5. System weryfikuje poprawność danych i po poprawnym zapisie wraca do widoku kategorii
6. Użytkownik powtarza kroki 3 - 5 w celu zdefiniowania wszystkich potrzebnych obiektów

Rozszerzenia i scenariusze alternatywne:

3.a Użytkownik wybrał opcję *Grupa*

1. Podaje nazwę grupy i zapisuje
2. Przenosi wybrane kategorie do grupy aby utworzyć hierarchię
3. Powtarza scenariusz główny (3 - 5) lub alternatywny 3a. do momentu osiągnięcia pożądanej struktury grup i kategorii
4. Zapisuje hierarchię

UC 7	**Aktywuj ankietę**
Cel:	Udostępnienie ankiety respondentom w celu przeprowadzenia badań
Poziom:	Funkcja systemowa

Główny Aktor:	USER
Warunki początkowe:	– Użytkownik posiada konto w systemie – Użytkownik jest zalogowany – Istnieje ankieta ze statusem *Nowa*

Scenariusz główny:

1. Użytkownik wybiera ankietę i otwiera widok podstawowy
2. Naciska przycisk aktywacji
3. System prezentuje widok kompletnej ankiety, w takiej postaci jak ją będą widzieć respondenci
4. Użytkownik zatwierdza widok
5. System zapisuje pytania i prezentuje widok wyboru trybu ankiety
6. Użytkownik wybiera opcję *Standardowy*
7. System zapisuje tryb, publikuje ankietę i wysyła zaproszenia w formie wiadomości e-mail do zdefiniowanych respondentów

Rozszerzenia i scenariusze alternatywne:

6.a Użytkownik wybrał tryb *Częściowy*

1. Użytkownik podaje jak wiele różnych pod-ankiet n chce przygotować i zatwierdza
2. System generuje n ankiet, z których każda jest unikalną permutacją zbioru wszystkich pytań
3. Powtarza krok 6.a.1 jeśli wynik losowania nie jest satysfakcjonujący lub zmienił zdanie w stosunku do wartości n
4. Zatwierdza wybór i rezultat losowania
5. Kontynuacja scenariusza głównego od punktu 7

UC 8	**Przeglądaj statystyki**
Cel:	Odczytanie wyników przeprowadzonych badań
Poziom:	Funkcja systemowa
Główny Aktor:	USER
Warunki początkowe:	– Użytkownik posiada konto w systemie – Użytkownik jest zalogowany – Istnieje ankieta ze statusem *W trakcie* lub *Zakończona*

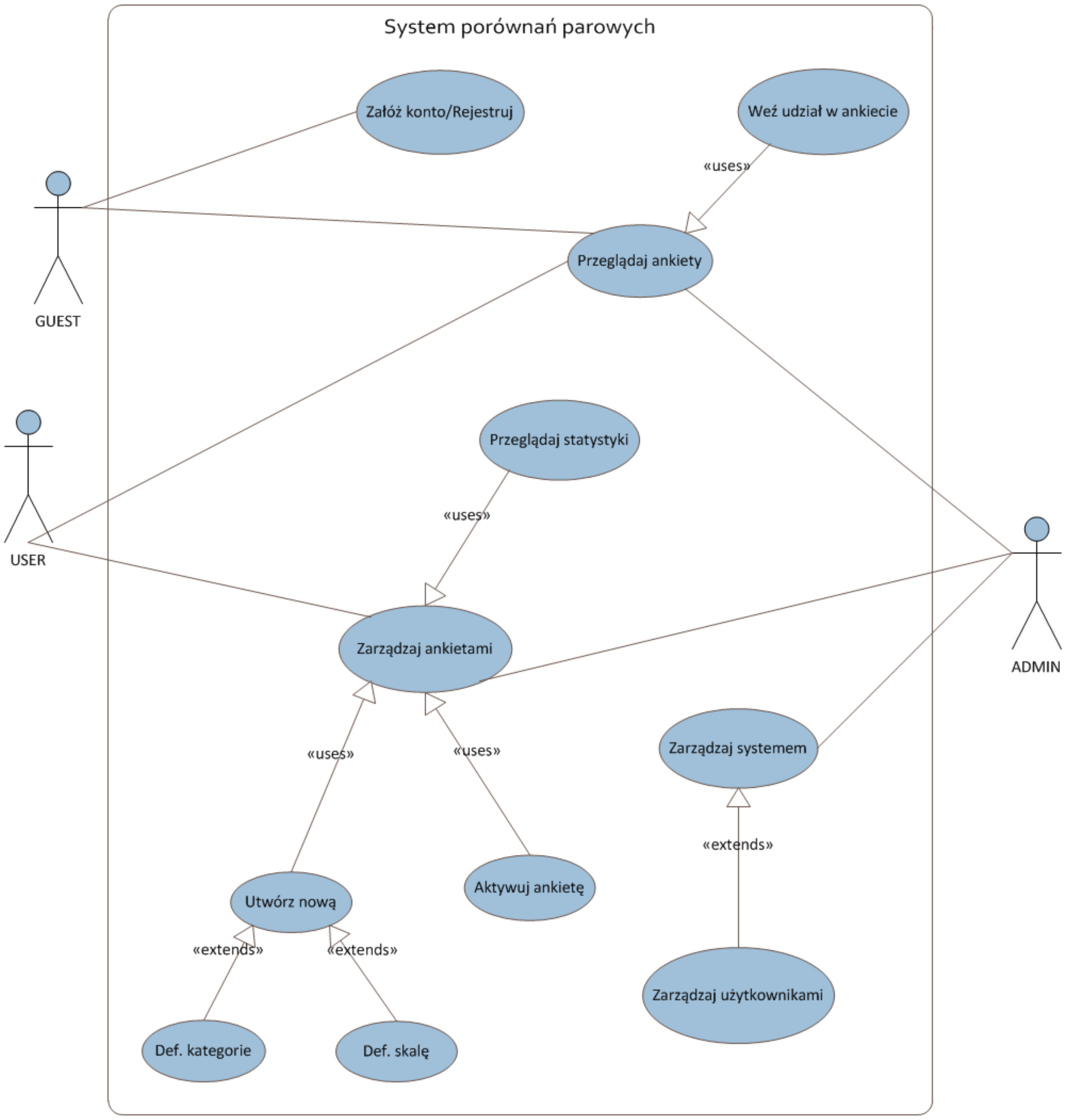
Scenariusz główny:

1. Użytkownik wybiera ankietę i otwiera widok podstawowy
2. Przechodzi do statystyk
3. System prezentuje widok wyników badania wyliczony na podstawie udzielonych odpowiedzi
4. Użytkownik zmienia parametry widoku i zatwierdza
5. System prezentuje wyniki badania z uwzględnieniem parametrów wybranych powyżej

6.1.2. Wymagania niefunkcjonalne

Względem system określiłem następujące wymagania niefunkcjonalne:

1. Zewnętrzny interfejs systemu będzie zbudowany w oparciu o technologię HTML w postaci stron WWW.
2. System będzie dystrybuowany jako oprogramowanie internetowe w modelu SaaS - użytkownik uzyskuje dostęp do całej platformy poprzez założenie konta, bez konieczności instalacji dodatkowego oprogramowania.
3. Wymagana włączona obsługa języka JavaScript w przeglądarce użytkownika, bez tego korzystanie z aplikacji nie będzie możliwe.



Rysunek 6.1: Diagram głównych przypadków użycia dla aplikacji ankietowej

4. Zgodność z przepisami prawa - niedopuszczalne jest aby system w jakikolwiek sposób naruszał prawo, w szczególności prawo autorskie.

5. Stabilność i szybkość działania

   – Łatwa i intuicyjna obsługa.

   – Estetyczny i czytelny wygląd.

   – Sprawne działanie przy 100 użytkownikach pracujących równocześnie - czas odpowiedzi/ładowania strony nie dłuższy niż 5 s.

   – Minimalizacja błędów powstałych przez umyślną bądź nieumyślną działalność użytkownika.

– Zapewnienie poprawnej pracy systemu na dostępnych przeglądarkach WWW.

6. Bezpieczeństwo

   – Identyfikacja i uwierzytelnienie użytkowników - na podstawie loginu oraz hasła. Hasło przechowywane w bazie danych w postaci sumy kontrolnej z solą.
   – Autoryzacja - bazuje na roli użytkownika. Dotyczy dostępu do widoków i funkcjonalności.
   – Brak dostępu do edycji oraz podglądu nie-własnych ankiet i statystyk - nie dotyczy Administratora.
   – Ochrona przeciw SQL Injection - walidacja i binding danych wejściowych.
   – Zapobieganie wprowadzaniu nieprawidłowych danych do systemu.

6.2. Architektura systemu

System został stworzony w języku PHP przy pomocy frameworka *Symfony* w wersji 2. Jest to bardzo rozbudowana platforma pozwalająca na łatwą i szybką implementację pożądanych funkcjonalności wykorzystując programowanie obiektowe. Aplikacja docelowa składa się z modułów, tzw. *bundle* połączonych przez framework. Wykorzystuje wzorzec Model - Widok - Kontroler (*ang. MVC: Model, View, Controller*) dzięki któremu główne warstwy aplikacji są od siebie wyraźnie oddzielone, nie pozwalając na mieszanie ich ze sobą. Pozwala to na utrzymanie wysokiej jakości kodu źródłowego, a także ma duży wpływ na organizację pracy. Łatwiej taką aplikację utrzymywać i rozwijać, dodając w przyszłości nowe funkcje.

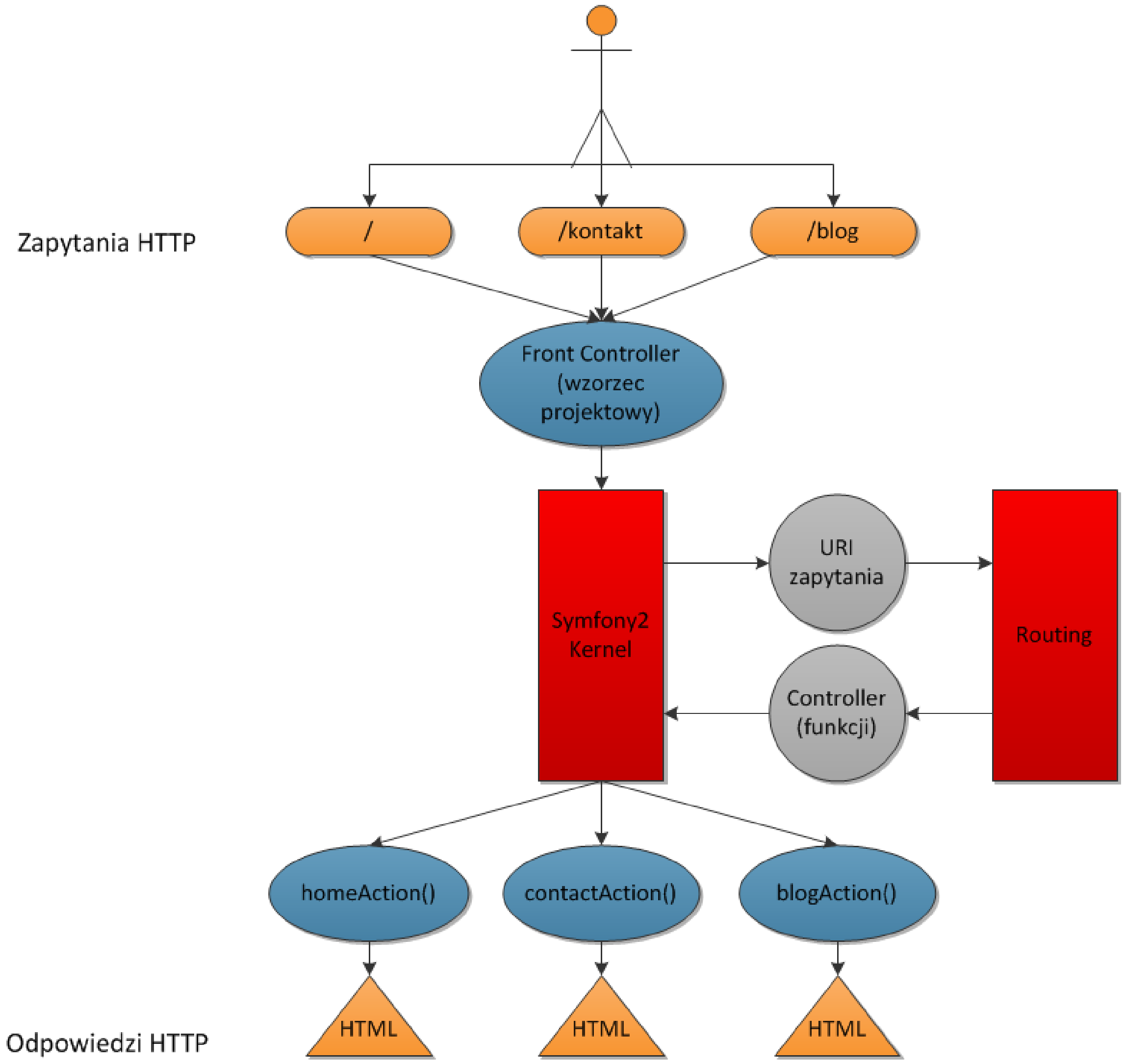
Cała logika biznesowa zaimplementowana jest w *Akcjach* zdefiniowanych w *Kontrolerach*. Połączenie między żądaniami użytkownika, a akcjami jest wynikiem konfiguracji nawigacji (*ang. routing*). Elementy widoczne dla użytkownika, prezentowane w przeglądarce, przygotowywane są przez system szablonów.

Aplikacja zawsze będzie miała za zadanie przetwarzać pewne dane. Aby mogła z nimi pracować potrzebuje *Modelu* - definicji elementów i ich zależności.

Jeśli budowana aplikacja wykorzystuje do działania bazę danych, model jest odwzorowaniem schematu bazy. Mamy wtedy do czynienia z mapowaniem obiektowo-relacyjnym, tzw. *ORM*. Symfony używa Doctrine jako narzędzie ORM oraz bazodanową warstwę abstrakcji *DBAL*. Dzięki temu nie ma znaczenia silnik bazy danych, który wykorzystujemy w naszej aplikacji. Warstwa DBAL potrafi współpracować z większością współczesnych relacyjnych baz danych, np. SQLite, MySQL, PostgreSQL, Oracle, Microsoft SQL, itd. Istnieje także możliwość stworzenia sterownika dla własnego silnika bazy. Aplikacja tworzona przeze mnie używa bazy danych MySQL.

Gdy tworzone rozwiązanie wymaga użycia bazy dokumentowej, można skorzystać z MongoDB ODM (*Object Document Mapper*), który w filozofii i sposobie działania jest bardzo podobny do Doctrine ORM.

Cykl życia aplikacji internetowej, zgodnie z definicją protokołu HTTP, rozpoczyna się w momencie gdy użytkownik wykonuje zapytanie *request*, następnie żądanie jest przetwarzane i serwer wysyła odpowiedź *response*. Dokładnie ten sam schemat jest realizowany w przypadku Symfony:



Rysunek 6.2: Schemat działania aplikacji we frameworku Symfony

Przychodzące zapytania są interpretowane przez routing i przekazywane do funkcji kontrolerów, które zwracają odpowiedź.

Każda *strona* aplikacji jest zdefiniowana w pliku konfiguracyjnym nawigacji (routingu), który mapuje różne adresy URL z funkcjami PHP. Zadaniem każdej funkcji PHP, nazywanej kontrolerem, jest wykorzystanie informacji z zapytania, wraz z wieloma innymi narzędziami które oferuje Symfony, do stworzenia i odesłania obiektu odpowiedzi.

6.3. Projekt interfejsu użytkownika

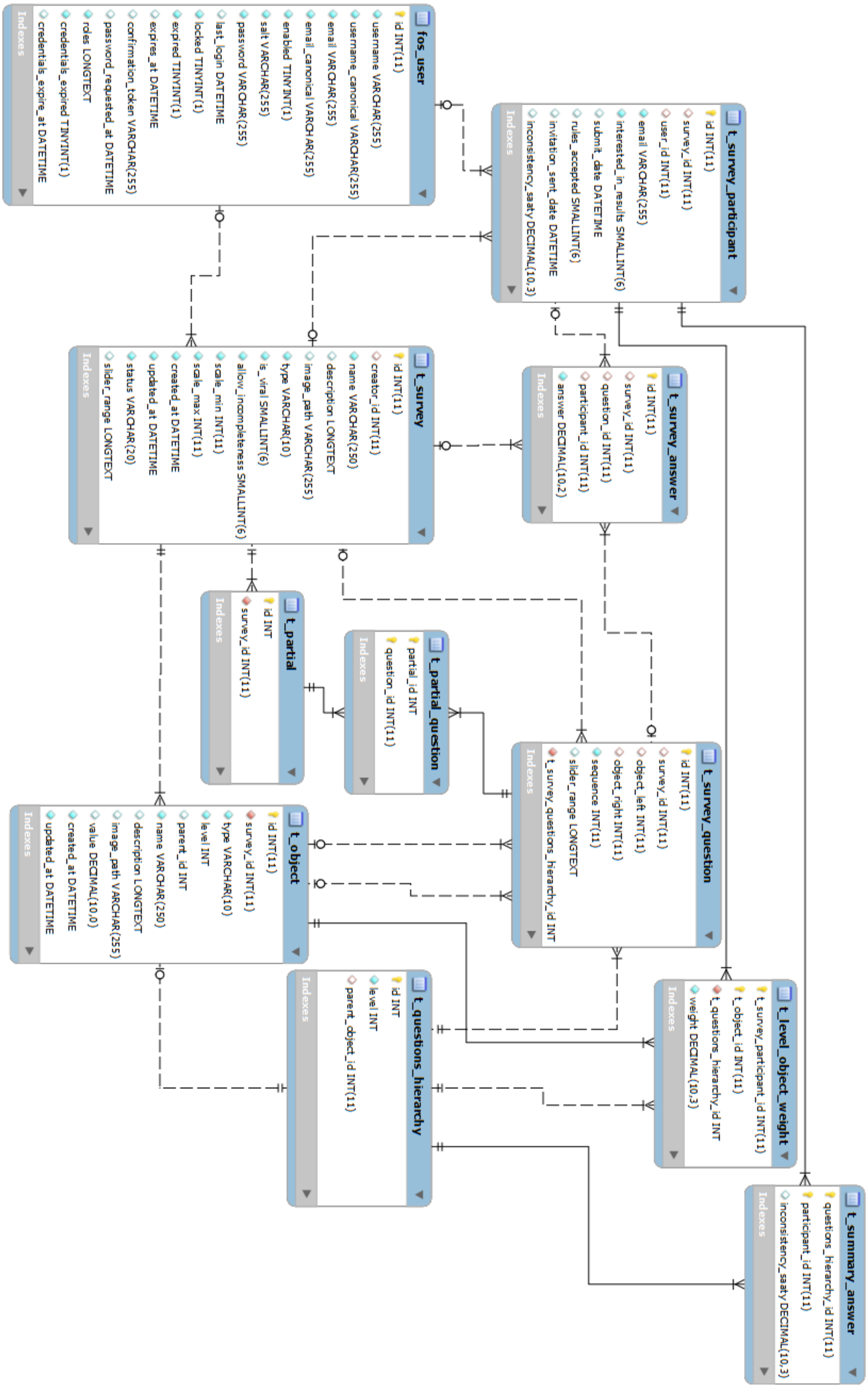
Podstawowym zadaniem systemu jest gromadzenie i przetwarzanie danych podanych przez użytkownika, w tym celu niezbędne są interfejsy umożliwiające ich wprowadzanie. Podział interfejsów opiera się na opisanych powyżej scenariuszach i przedstawia się następująco:

1. Rejestracja
   - E-mail
   - Login
   - Hasło i potwierdzenie hasła
2. Logowanie
   - Login/E-mail
   - Hasło
3. Weź udział w ankiecie
   - E-mail
   - Czy chce otrzymać powiadomienie o wynikach po zakończeniu ankiety
   - Akceptacja warunków
   - Ustawienie pozycji suwaka dla każdego pytania ankiety
4. Dodanie/edycja danych podstawowych ankiety
   - Nazwa
   - Opis
   - Zdjęcie/obrazek - obraz używany w katalogu ankiet
   - Typ - określa czy ankieta jest prywatna, czy też publiczna
   - Minimum skali - dolna granica przedziału skali
   - Maksimum skali - górna granica przedziału skali
   - Czy jest wiralem? - respondenci mogą zapraszać kolejnych do wzięcia udziału
   - Czy dopuszczalny jest zapis niekompletnej odpowiedzi? - respondent nie udziela odpowiedzi na wszystkie, lecz tylko na część pytań
5. Definicja przedziału znaczeniowego
   - Pozycja - wartość skali
   - Opis - wytłumaczenie znaczenia pozycji

6. Dodanie/edycja kategorii
   - Nazwa
   - Opis
   - Zdjęcie/obraz
   - Stała, znana wartość - jeśli znana jest stała wartość którą przyjmuje kategoria w ramach przeprowadzanego badania
   - Pozycja - umiejscowienie kategorii w hierarchii
7. Aktywacja ankiety
   - Tryb - standardowy lub częściowy
   - Ilość pod-ankiet - dla trybu częściowego; ilość mniejszych ankiet które użytkownik chce wygenerować
8. Przeglądanie statystyk
   - Poziom niespójności
   - Poziom hierarchii - dla ankiet hierarchicznych; wybór przeglądanej gałęzi
   - Filtry w oparciu o dane respondentów - wiek, płeć, itd...

6.4. Schemat bazy danych

System zbudowany jest na relacyjnej bazie danych. Jej schemat przedstawiony jest na rysunku 6.3.



Rysunek 6.3: Projekt relacyjnej bazy danych na potrzeby budowanej aplikacji

7. Implementacja systemu

Zbudowany portal ankiet porównawczych dzieli się na dwie logiczne części. Pierwszą jest interfejs zarządzania treścią w systemie, do których użytkownicy mają bezpośredni dostęp. Druga z nich to część analityczna, zajmująca się przetwarzaniem danych zgodnie z przyjętymi zasadami przeprowadzania badań metodą porównań parami. Funkcjonalności poświęcone bezpośredniej interakcji użytkownika z systemem zostały opisane w poprzednim rozdziale. W niniejszy skupiam się na przedstawieniu wymaganych elementów wdrożenia systemu, interfejsu użytkownika a także części analitycznej.

7.1. Konfiguracja

Aplikacja do działania wymaga odpowiednio skonfigurowanego serwera internetowego z obsługą języka PHP w wersji min. 5.4 oraz kilkoma wymaganymi przez Symfony dodatkami, jak np. biblioteka *APC* której używanie może znacznie przyspieszyć działanie aplikacji poprzez wykorzystanie mechanizmów cache.

Dzięki zastosowaniu Symfony, konfiguracja samej aplikacji odbywa się poprzez jeden plik tekstowy. W dodatku konstrukcja frameworka wymusza aby parametry aplikacji znajdowały się tylko na serwerze docelowym. Dlatego nie ma niebezpieczeństwa ich ujawnienia w przypadku udostępnienia kodu źródłowego.

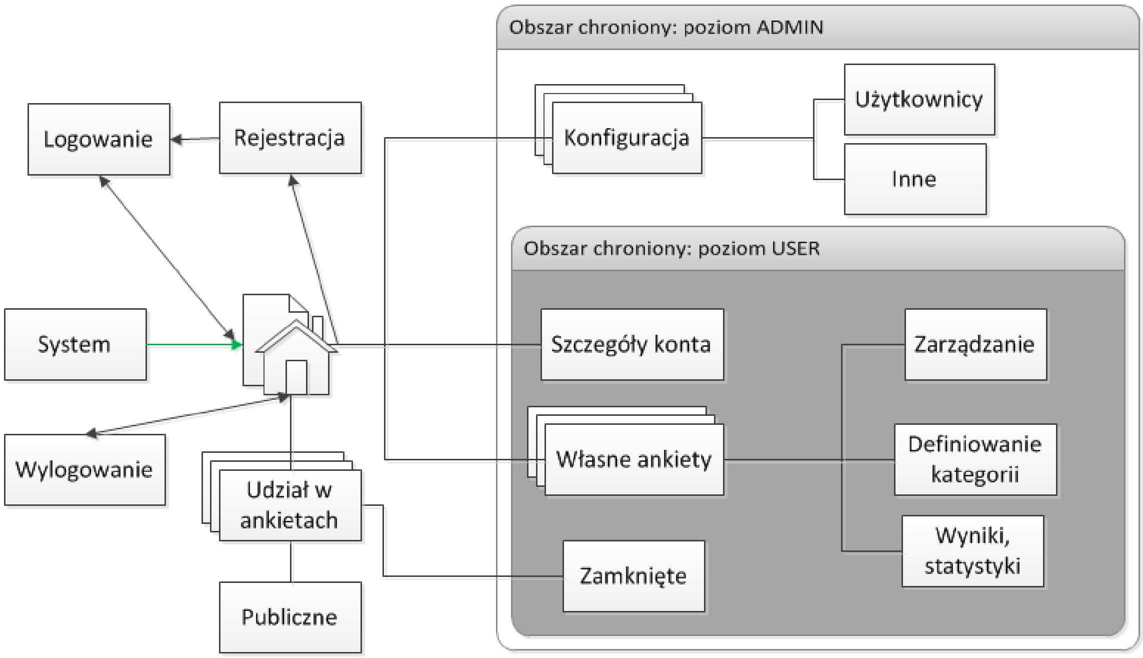
Pierwszą wymaganą rzeczą jest baza danych. Jak wspominałem wcześniej system pracuje na bazie relacyjnej. Na etapie konfiguracji możemy zdefiniować który silnik (MySQL, PostreSQL, itd.) mamy dostępny oraz podajemy dane dostępowe odpowiadające naszej instancji bazodanowej.

System wysyła pocztę elektroniczną. Ma to miejsce np. podczas rejestracji użytkownika czy też zapraszania do wzięcia udziału w ankiecie, a także przy powiadamianiu o wynikach. W związku z tym niezbędne jest podanie konfiguracji konta pocztowego i np. serwera SMTP za którego pośrednictwem będzie można wiadomości wysyłać. W najprostszej opcji można używać konta zarejestrowanego w domenie *gmail.com*. Należy jednak wziąć pod uwagę obowiązujące limity wysyłanych wiadomości. W przypadku gdy system będzie bardzo popularny i zapotrzebowanie na e-maile wzrośnie, można wykupić usługę konta mailowego u wielu różnych dostawców, w tym np. Amazon SES *Amazon Simple Email Service*, lub skonfigurować własny serwer pocztowy. W obu przypadkach należy dbać o to aby nie wysyłać treści nieodpowiednich, a także weryfikować adresy odbiorców i wprowadzić politykę obsługi tzw. *bounce email*, czyli wiadomości które nie mogły zostać dostarczone do adresata z wielu różnych powodów i 'odbiły' się od serwera obsługującego domenę adresata. Najczęściej zdarza się tak z powodu błędnego adresu. Odbicia są poważnym problemem, gdyż mogą spowodować iż nasze wiadomości zaczną być traktowane jak spam i blokowane. Tym samym uniemożliwiając pełne funkcjonowanie. Aplikacja stara się minimalizować ten problem wymagając potwierdzenia adresu mailowego podczas rejestracji. Nie ma natomiast możliwości aby potwierdzić adresy do których wysyłane są zaproszenia. Tutaj pozostaje tylko obsługa 'odbić'.

Użytkownicy mają prawo do przesyłania plików do aplikacji. Na potrzeby identyfikacji ankiety lub kategorii twórca może wykorzystać zdjęcia i obrazy, które zapisze w systemie. Taki scenariusz tworzy potrzebę udostępnienia przestrzeni dyskowej o nie do końca zdefiniowanym rozmiarze. Ze względu na to iż serwery internetowe hostujące aplikacje mają najczęściej niewielką przestrzeń dyskową, lub podnosi ona znacznie koszty utrzymania, zdecydowałem się na integrację aplikacji z usługą przechowywania plików Amazon S3 *Amazon Simple Storage Service*. Dzięki temu serwer aplikacji nie przechowuje plików. Są one zapisywane na dyskach usługi S3, a dostępnym miejsce można w łatwy i szybki sposób zarządzać wedle potrzeb. Usługa ta jest płatna, a do założenia konta wymagane jest podanie karty kredytowej. W pierwszym roku można używać jej za darmo w ramach programu promocyjnego *Free Tier*. Prezentowana aplikacja do współpracy z usługami Amazon potrzebuje kluczy: publicznego i prywatnego, nazwę regionu w którym usługa jest zarejestrowana oraz tzw. 'koszyka' *bucket* który jest pojemnikiem zawierającym przechowywane dane.

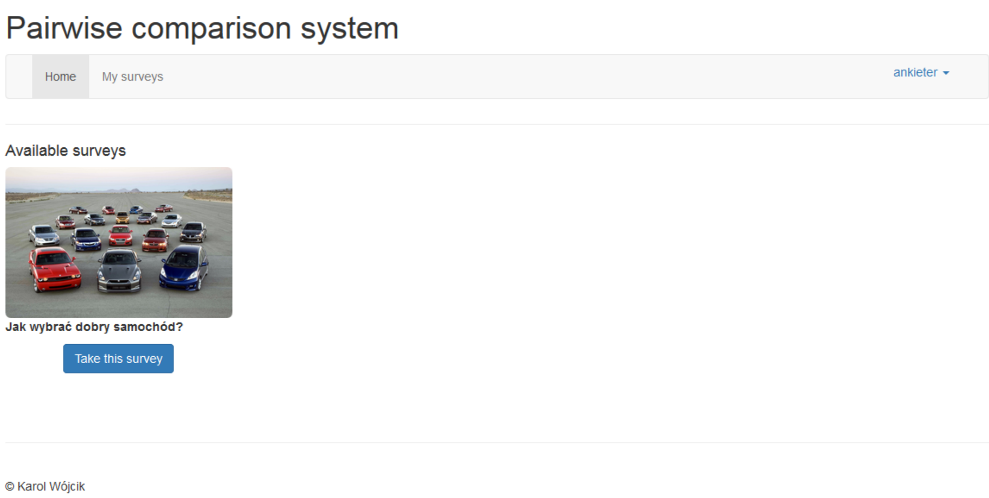
7.2. Interfejs użytkownika

W obecnym podrozdziale prezentuję główne koncepty i najważniejsze widoki z działającej aplikacji, więcej szczegółów można znaleźć na załączonym wirtualnym obrazie systemu. Oprócz całego systemu, zawiera także kilka video-tutoriali będących swoistą instrukcją i prezentacją możliwości systemu.

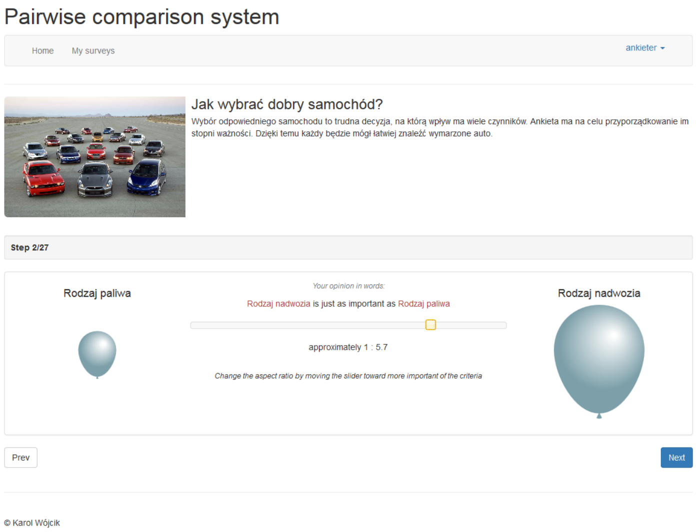


Rysunek 7.1: Schemat interfejsu użytkownika

Interfejs użytkownika został podzielony na trzy główne obszary, odpowiadające poziomom dostępu użytkowników 7.1.. Można wyróżnić cześć dostępną dla wszystkich: stronę główną, ekran logowania, formularz rejestracji, a także publiczny katalog ankiet i możliwość wzięcia udziału w ankiecie jako niezarejestrowany użytkownik systemu.



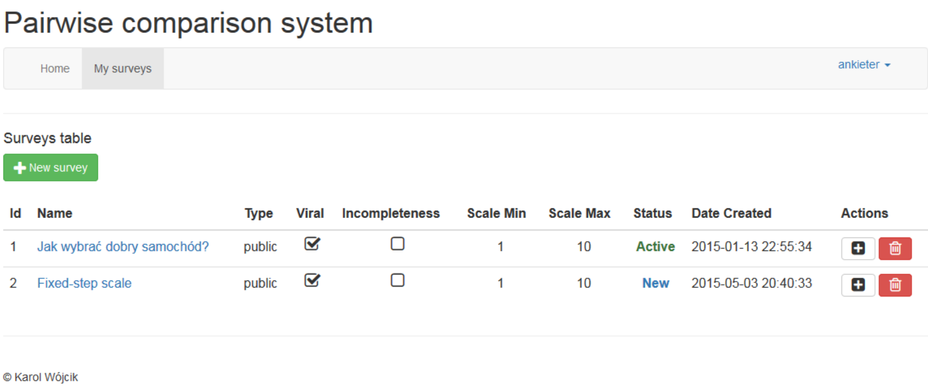
Rysunek 7.2: Strona główna aplikacji - katalog ankiet w których użytkownik może wziąć udział



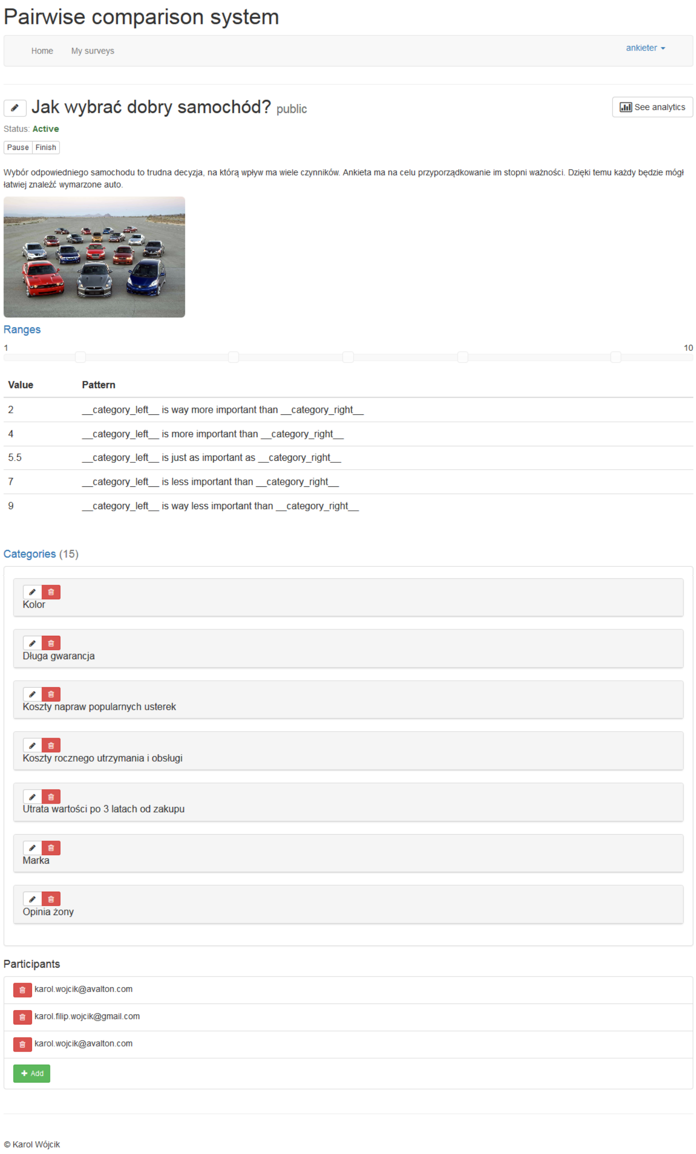
Rysunek 7.3: Każde pytanie ankiety prezentuje spójny widok, rozmiar balonów jest zależny od pozycji suwaka

Następnie każda zalogowana osoba ma dostęp do edycji danych swojego konta, a także widzi własny katalog, wraz z ankietami o ograniczonym dostępie do których został zaproszony, a nie wziął jeszcze w nich udziału.

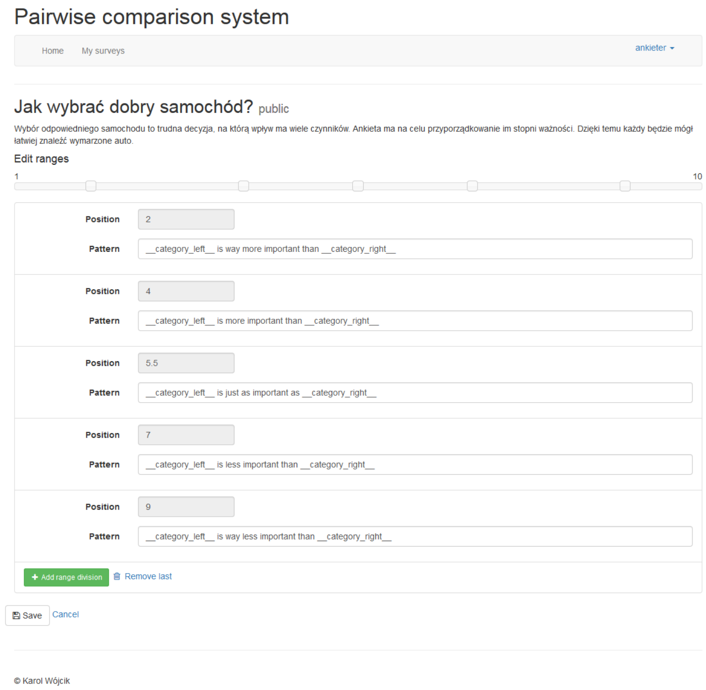
Najważniejsza z funkcjonalności oferowanych posiadaczom kont w systemie jest możliwość własnego ankietowania. Uzyskują oni dostęp do części tworzenia ankiety, definiowania obiektów porównań i hierarchii oraz zapraszania respondentów. Mogą zarządzać treściami które stworzyli, a na końcu przeglądać wyniki, filtrować je i wyciągać wnioski.



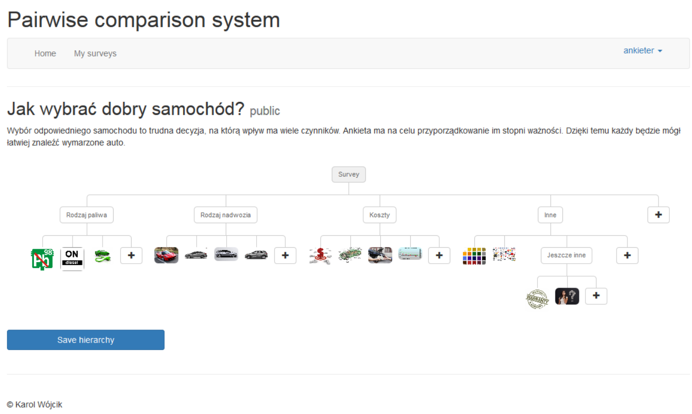
Rysunek 7.4: Lista ankiet zalogowanego użytkownika



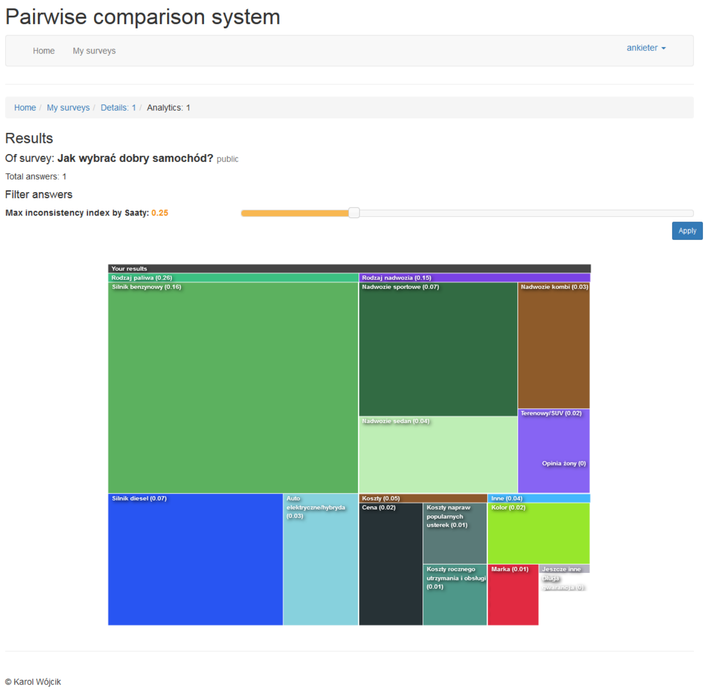
Rysunek 7.5: Widok szczegółów ankiety, ze wszystkimi danymi i zależnościami



Rysunek 7.6: Widok edycji przedziałów znaczeniowych dla skali



Rysunek 7.7: Widok definicji kategorii oraz ich hierarchii



Rysunek 7.8: Podsumowanie wyników - widok dla twórcy ankiety

Najwyższy poziom dostępu zajmuje Administrator, który oprócz wymienionych wcześniej elementów, ma także prawo zarządzać systemem, w szczególności użytkownikami. Schemat interfejsu prezentuje rysunek 7.1

Graficzny interfejs użytkownika został zbudowany w oparciu o framework *Bootstrap*. Jest to darmowy i niezwykle popularny zestaw szablonów HTML i stylów CSS, które sprawiają że projektowanie i budowa interfejsów staje się prostsza i szybsza niż ma to miejsce w przypadku tworzenia od podstaw. Proces wytwarzania interfejsu w Bootstrapie w większości przypadków sprowadza się do układania elementów zdefiniowanych we frameworku. Dostępna jest szczegółowa dokumentacja, a wokół projektu istnieje duża społeczność internetowa. Wszystko to sprawia że praca z Bootstrapem jest przyjemna i efektywna.

Jednak nawet najpiękniejszy szablon jest bezużyteczny gdy nie widać na nim prawdziwej zawartości. Z pomocą przychodzą silniki szablonów, zajmujące się ’ożywianiem’ statycznych elementów i zapełnianiem ich pożądaną zawartością. W aplikacjach budowanych w oparciu o Symfony, rolę takiego silnika pełni *Twig*. Oprócz doskonałej współpracy z frameworkiem Symfony, Twig jest bardzo szybki oraz bezpieczny. Ponadto jego możliwości można dowolnie rozszerzać poprzez tworzenie własnych elementów. W modelu MVC, Twig odpowiada za Widok *View*, zapewniając odpowiednią separację kodu źródłowego aplikacji od szablonów i widoków, przez co kod jest znacznie wyższej jakości.

7.3. Część analityczna

Część analityczna to umownie nazwana grupa funkcjonalności niezbędnych do prawidłowej pracy systemu, ale nie widocznych bezpośrednio dla użytkownika. W jej skład wchodzą przede wszystkim elementy przetwarzania wyników porównań przesyłanych przez respondentów. Poddawane są wstępnej obróbce już w czasie zapisu do bazy danych, np. wyliczane są współczynniki niespójności wg Saatiego i Koczkodaja.

7.3.1. Niespójność

Obliczanie niespójności jest kosztownym procesem. Ciężko wyobrazić sobie możliwość filtrowania wyników względem niespójności w przypadku gdy byłaby ona obliczana każdorazowo dla każdej odpowiedzi. Czas oczekiwania na wynik byłby bardzo duży. Jako że niespójność jest własnością niezmienną (dla niezmiennej macierzy), może być wyznaczona na początku i później używana według potrzeby.

System oferuje dwie metryki niespójności, analizę wektora własnego (metoda Saatyiego) oraz metodę odległościową (wg. W. Koczkodaja). Procedura wyznaczania wektora własnego jest operacją pracochłonną dla komputerów, a także dla programistów ponieważ nie istnieją żadne biblioteki dla języka PHP realizujące tą funkcjonalność. Także i ja stanąłem przed wyborem sposobu obliczania wektora własnego macierzy kwadratowej, dowolnego rozmiaru.

Zdecydowałem się na integrację z popularnym serwisem naukowo-obliczeniowym *WolframAlpha*. Korzystając z publicznego API mam dostęp do większości funkcji, w tym także matematycznych metod obliczeniowych wśród których znajduje się także sposób na obliczanie wektora wartości własnych. W przeciwieństwie do samodzielnej implementacji formuły matematycznej, integracja zapewnia znacznie większą jakość obliczonych wartości przy zachowaniu szybkości działania całej aplikacji. WolframAlpha oferuje swoje API w dwóch wersjach: darmowej i komercyjnej. Darmowa nie może być wykorzystywana do celów komercyjnych, a także ma górne ograniczenie ilości zapytań wynoszące 2000/miesiąc. Jest to więc idealna propozycja dla rozwoju takiego systemu który buduję. W przyszłości, gdy zaistnieje taka potrzeba, można bez problemu pozbyć się ograniczeń przechodząc na licencję komercyjną. W związku z tym iż WolframAlpha jest używany do zapewnienia podstawowej funkcjonalności systemu, wymagana jest konfiguracja dostępu do zdalnych zasobów. Składa się na nią założenie konta developerskiego i stworzenie wirtualnej aplikacji, której identyfikator jest niezbędny do podpisania zapytań i musi być wysyłany każdorazowo przy połączeniu z API.

7.3.2. Ankiety częściowe

Chciałbym rozwinąć myśl przedstawioną w przypadku użycia 6.7. Zaimplementowałem funkcję ankiet w trybie częściowym. Ten tryb polega na tym iż respondent udziela odpowiedzi tylko na wybrane pytania ankiety, ale nie na wszystkie. Ma to zastosowanie w przypadku gdy obiektów porównań jest wiele (duża ilość pytań), a nie chcemy tworzyć hierarchii lecz porównywać wszystkie kategorie na tym samym poziomie. Częściowa odpowiedź może wydawać się niewystarczająca gdyż nie mamy całej macierzy PC, lecz istnieją metody na odpowiednią interpretację lub też zbudowanie pełnej macierzy. Najbardziej naturalnym rozwiązaniem jest wyznaczanie brakujących elementów korzystając z równania spójności macierzy przedstawionego w 2.1. W rezultacie otrzymujemy możliwie najbardziej spójną odpowiedź nie tracąc subiektywizmu respondenta. Istnieją jednak pewne ograniczenia w stosunku do minimalnej ilości pytań. Aby odpowiedź ankietowanego mogła zostać uznana za wartościową i użyta w procesie wnioskowania, musi on odpowiedzieć na przynajmniej n pytań, gdzie n to ilość kategorii. Tylko taka macierz częściowa będzie mogła zostać przetworzona.

Zbudowany system oferuje możliwość wyboru trybu podczas aktywacji ankiety, tj. w momencie gdy wersja robocza jest ukończona i twórca chce rozpocząć ankietowanie. Tryb częściowy jest inicjalizowany dwoma parametrami. Są to: ilość różnych pod-ankiet wygenerowanych z pełnej ankiety, oraz minimalna ilość pytań w pod-ankiecie. Domyślnie drugi parametr przyjmuje wartość odpowiadającą ilości kategorii. Po zatwierdzeniu parametrów, system losowo generuje zbiory pytań tworząc pod-ankiety. Są one następnie prezentowane operatorowi. Jeśli ten uzna że spełniają jego oczekiwania to zatwierdza i ankieta zostaje opublikowana, w przeciwnym przypadku może powtórzyć proces generacji i w rezultacie otrzymać inne permutacje pytań.

8. Podsumowanie i wnioski

Cel pracy jakim było stworzenie i wdrożenie aplikacji internetowej ankiet porównawczych został zrealizowany. Implementacja oparta została na założeniach zdefiniowanych podczas projektowania systemu. System w obecnej formie jest przystosowany do obsługi ankiet parowych tworzonych przez każdego kto tylko posiada konto użytkownika w systemie.

Obecnie nie istnieją żadne nowoczesne narzędzia oferujące kompleksowe usługi z kategorii ankiet parowych. Metoda porównań parami stale się rozwija, obejmując swoim zasięgiem coraz to nowsze obszary nauk, jak np. socjologię. Dlatego potrzebuje dobrych narzędzi które pozwolą w dalszym ciągu się jej rozwijać i popularyzować spośród wszystkich metod podejmowania decyzji.

Moją odpowiedzią na to zapotrzebowanie jest zbudowany system. Może być wykorzystywany zarówno na potrzeby akademickie czy wspierając badania naukowe jak i prywatne podejmowanie decyzji w istotnych dla twórcy ankiety sprawach. Aplikacja jest napisana w popularnym języku PHP i wykorzystuje sprawdzony i polecany framework Symfony 2. Dzięki temu mogła zostać sprawnie zbudowana i dostosowana do potrzeb. Spośród wielu zalet, najważniejszymi które wymienię, są pełna modułowość pozwalająca na łatwą zamianę lub dodanie funkcjonalności poprzez gotowe komponenty, wykorzystanie ORM/ODM do komunikacji z bazą danych czy też silnik szablonów Twig z możliwością łatwego wprowadzenia różnych wersji językowych aplikacji. Z punktu widzenia programisty, przejrzystość kodu źródłowego i możliwość łatwego zrozumienia logiki biznesowej są ogromną wartością projektu, mierzalną np. za pomocą ilości wymaganego czasu do poznania implementacji. W przypadku gdy nad rozwojem aplikacji zaczynać będą pracę nowe osoby.

Plany na dalsze usprawnianie i budowanie systemu są rozległe. Dotyczą obszarów zarówno merytorycznych jak i samego funkcjonowania aplikacji. Najważniejszymi zmianami jest dodanie funkcjonalności z kategorii metody porównań parowych. Wprowadzenie możliwości udzielania niepełnych odpowiedzi i stosowania dla nich heurystyk, dodanie innych metod obliczania niespójności i ich dowolne wybieranie dla tworzonych ankiet, lub nawet definiowanie własnych funkcji obliczeniowych. Oprócz tego chciałbym opracować wspólnie z grafikiem nowy interfejs, który byłby wygodniejszy dla użytkowników, a także zintegrować usługi mediów społecznościowych jak Facebook, Twitter, Wykop itd. Pomoże to w popularyzowaniu portalu.

Gdy wszystkie powyższe elementy będą zrealizowane, chciałbym wprowadzić model subskrypcji na dostęp do serwisu. Podstawowe elementy pozostaną dostępne bezpłatnie dla ograniczonej ilości ankiet, a rozszerzenia i większa ilość prowadzonych badań będzie wymagała wykupienia usługi.

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REDEFINITION OF TRIAD'S INCONSISTENCY AND ITS IMPACT ON THE CONSISTENCY MEASUREMENT OF PAIRWISE COMPARISON MATRIX

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Abstract. There is a theory which meets a prescription of the efficient and effective multicriteria decision making support system called the Analytic Hierarchy Process (AHP). It seems to be the most widely used approach in the world today, as well as the most validated methodology for decision making. The consistency measurement of human judgments appears to be the crucial problem in this concept. This research paper redefines the idea of the triad's consistency within the pairwise comparison matrix (PCM) and proposes a few seminal indices for PCM consistency measurement. The quality of new propositions is then studied with application of computer simulations coded and run in *Wolfram Mathematica* 9.0.

Keywords: *AHP, pairwise comparisons, consistency, judgments, Monte Carlo simulations*

1. Introduction

There is a methodology which meets prescriptions for efficient and effective multiple criteria decision making (MCDM) process. It is called the Analytic Hierarchy Process (AHP) and was developed at the Wharton School of Business by Thomas Saaty [1]. The AHP seems to be the most widely used MCDM approach in the world today, as well the most validated methodology. There are thousands of actual applications in which the AHP results were accepted and used by the competent decision makers (DM). Thorough reviews of the contemporary applications and developments in the AHP can be found for example in [2-5].

The AHP allows DM to set priorities and make choices on the basis of their objectives, knowledge and experiences in a way that is consistent with their intuitive thought process. The process permits accurate priorities to be derived from verbal judgments even though the words themselves may not be very precise. That is why special attention is given to issues associated with consistency of DM judgments. However, inconsistency results not only due to DM inaccuracy in their judgments but also due to existing scales, which must be utilized in order to enable DM to somehow express their fuzzy preferences.

2. Mathematics behind the Analytic Hierarchy Process

The conventional procedure of priorities ranking in AHP is grounded on the well-defined mathematical structure of consistent matrices and their associated right-eigenvector's ability to generate true or approximate weights [1, 5]. It was proved that, if $\boldsymbol{A} = (w_{ij})$, $w_{ij} > 0$, where $i, j = 1,\ldots, n,$ then $\boldsymbol{A}$ has a simple positive eigenvalue $\lambda_{\max}$ called the principal eigenvalue of $\boldsymbol{A}$ and $\lambda_{\max} > |\lambda_k|$ for the remaining eigenvalues of $\boldsymbol{A}$. Furthermore, the principal eigenvector $\boldsymbol{w} = [w_1,\ldots, w_n]^{\mathrm{T}}$ that is a solution of $\boldsymbol{Aw} = \lambda_{\max}\boldsymbol{w}$ has $w_i > 0$, $i = 1,\ldots, n$. If we know the relative weights of a set of activities we can express them in a pairwise comparison matrix (PCM) denoted as $\boldsymbol{A}(w)$. Now, knowing $\boldsymbol{A}(w)$ but not $\boldsymbol{w}$ (vector of priorities) we can use Perron's theorem and solve this problem for $\boldsymbol{w}$.

Definition 1. If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ij} = 1/w_{ji}$ for all $i, j = 1,\ldots, n$, then the matrix $\boldsymbol{A}(w)$ is called *reciprocal*.

Definition 2. If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ik}w_{kj} = w_{ij}$ for all $i, j, k = 1,\ldots, n$, and the matrix is *reciprocal*, then it is called *consistent* or *cardinally transitive*.

Thus, the conventional concept of AHP can be presented as: $\boldsymbol{A}(w)\,\boldsymbol{w} = n\,\boldsymbol{w}$. When AHP is utilized in real life situations we do not have $\boldsymbol{A}(w)$ which would reflect priority weights given by the vector of priorities. Thus, we do not have $\boldsymbol{A}(w)$ but only its estimate $\boldsymbol{A}(x)$. In such a case the consistency property does not hold and the relation between elements of $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$ can be expressed as follows:

$$x_{ij} = e_{ij} w_{ij} \tag{1}$$

where e_{ij} is a perturbation factor nearby unity. In the statistical approach e_{ij} reflects a realization of a random variable with a given probability distribution.

Thus, in order to analyze the consistency of decision makers' judgments, Saaty proposes measuring the inconsistency of data contained in the PCM by a consistency index $CI(n)$ computed according to the following formula:

$$CI(n) = \frac{\lambda_{\max} - n}{n - 1} \tag{2}$$

However, very recent developments in the field exposed the necessity for further analysis in this area because Saaty's index can be a very misleading one.

3. Description of the problem and its solution

It should be realized here that we have three significantly different notions:

– the ***PCM consistency*** perceived from a perspective of the Definition 2, and expressed by the specific inconsistency index value,

– the ***consistency of decision makers***, i.e. their trustworthiness, reflected by the number and size of their judgments discrepancies, and
– the ***PCM applicability*** for estimation of decision makers' priorities in the way that leads to minimization of their estimation errors.

As it seems, the third issue is probably the most important problem in the contemporary arena of the MCDM theory concerning AHP, and the only way to examine that phenomena is through computer simulations.

Saaty insists that the consistency index $CI(n)$ is necessary and sufficient to uniquely capture the consistency inherent in pairwise comparison judgments. However, there are many other procedures devised in order to cope with this problem. They are connected with various other priorities estimation procedures that exist, and can be found in the literature, e.g. Logarithmic Least Squares Method (LLSM) [6] given by the formula:

$$w_{(LLSM)} = \min \sum_{i,j=1}^{n} (\ln a_{ij} - \ln w_i + \ln w_j)^2 \tag{3}$$

and connected with it the consistency index (CI_{LLSM}) given by the formula:

$$CI_{\mathrm{LLSM}} = \frac{2}{(n-1)(n-2)} \sum_{i<j} \log^2\left(\frac{a_{ij} w_j}{w_i}\right) \tag{4}$$

Many of these procedures are optimization based and seek a vector $\boldsymbol{w}$ as a solution of the minimization problem given by the formula:

$$\min \boldsymbol{D}(\boldsymbol{A}(x), \boldsymbol{A}(w)) \tag{5}$$

subject to some assigned constraints such as for example positive coefficients and normalization condition. Because the distance function $\boldsymbol{D}$ measures an interval between matrices $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$, different ways of its definition lead to different prioritization concepts, prioritization results and consistency measures [7]. Furthermore, since the publication of [7] a few other procedures were introduced to the literature along with their consistency measures concepts, i.e.: the procedure based on the goal programming approach [8], and a few based on constrained optimization models [9, 10].

However, there is a consistency index that is not connected with any prioritization procedure, and this index was devised by Koczkodaj [11], which attracts our special attention. In order to understand its essence we must explain the notion of a triad. It is a fact that for any three distinguished decision alternatives there are three meaningful priority ratios (denoted hereafter as: α, β, χ), which have their different locations in a particular PCM (denoted as: $\boldsymbol{A}(x) = [x_{ij}]_{n\mathrm{x}n}$). Now, if $\alpha = a_{ik}$, $\chi = a_{kj}$, $\beta = a_{ij}$ for some different $i \le n, j \le n$, and $k \le n$, the tuple (α, β, χ) is called a *triad*. Certainly, in every consistent PCM, for all triads the following equality holds: $\alpha\chi = \beta$. Thus, equations $1 - \beta/\alpha\chi = 0$ and $1 - \alpha\chi/\beta = 0$ must be true in such circumstances. Applying this principle, Koczkodaj introduced his index denoted as: $TI(\alpha,\beta,\chi)$ intended for measurement of triad's consistency, given by the formula:

$$TI(\alpha,\beta,\chi) = \min\left[\left|1-\frac{\beta}{\alpha\chi}\right|, \left|1-\frac{\alpha\chi}{\beta}\right|\right] \tag{6}$$

and following this idea he proposed the following index $KI(TI)$ designated to measure consistency of any reciprocal PCM:

$$KI(TI) = \max[TI(\alpha,\beta,\chi)] \tag{7}$$

where the maximum value of $TI(\alpha,\beta,\chi)$ is taken from the set of all possible triads in the upper triangle of a given PCM.

The fact that Koczkodaj's index is not connected with any prioritization procedure makes it especially attractive. Furthermore, it is possible to redefine that index and make it less complicated from the viewpoint of mathematical applications.

That is why we propose two new indices for characterization of the triad's consistency which allow us to simplify computations and consider only one component within the index instead of searching for a minimum of two as in the case of $TI(\alpha,\beta,\chi)$. Following the idea, that $\ln(\alpha\chi/\beta) = -\ln(\beta/\alpha\chi)$, instead of $TI(\alpha,\beta,\chi)$, we suggest two seminal formulae intended for measurement of triad's consistency, denoted as $LTI(\alpha,\beta,\chi)$ and $LTI^*(\alpha,\beta,\chi)$, defined by the two following equations:

$$LTI(\alpha,\beta,\chi) = |\ln(\alpha\chi/\beta)| \tag{8}$$

$$LTI^*(\alpha,\beta,\chi) = \ln^2(\alpha\chi/\beta) \tag{9}$$

We can now suggest two new indices for the purpose of consistency measurement of any PCM, given by the following formulae (simplifying, LTI therein denotes alternatively $LTI(\alpha,\beta,\chi)$ or $LTI^*(\alpha,\beta,\chi)$):

$$MLTI(LTI) = \frac{1}{N}\sum_{i=1}^{N}[LTI_i(\alpha,\beta,\chi)] \tag{10}$$

$$DLTI(LTI) = MLTI(LTI) \cdot \sqrt{\frac{1}{N}\sum_{i=1}^{N}(LTI_i(\alpha,\beta,\chi) - MLTI(LTI))^2} \tag{11}$$

where the above measures (10) and (11) can be computed on the basis of all different triads (α,β,χ) not necessarily in the upper triangle of the given PCM which in this case can be reciprocal or nonreciprocal.

In order to verify if the above presented indices qualify for the assessment of the *PCM applicability* for estimation of the decision makers' priorities in the way that leads to minimization of their estimation errors, we would like to verify their performance with the help of Monte Carlo simulations. It is the fact that Monte Carlo simulations are commonly recognized as important and credible source of scientific information [12]. They are applied for examination purposes of various phenomena, e.g.: consequences of decisions made, or different processes subdued to random impact of the particular environment [13].

4. Computer simulations of the selected index performance

In order to evaluate a performance of earlier proposed indices, we designed the following simulation scenario. In agreement with assumptions described in Grzybowski [14] it is possible to design and execute the simulation algorithm comprising the following steps:

Step 1. Randomly generate a priority vector $\boldsymbol{k} = [k_1,\dots, k_n]^T$ of assigned size [nx1] and related perfect PCM(k) = $\boldsymbol{K}(k)$.

Step 2. Randomly choose an element k_{xy} for $x < y$ of $\boldsymbol{K}(k)$ and replace it with $k_{xy}e_B$ where e_B is relatively a significant error which is randomly drawn from the interval D_B with assigned probability distribution π.

Step 3. For each other element k_{ij}, $i < j \le n$ randomly choose a value e_{ij} for the small error in accordance with the given probability distribution π and replace the element k_{ij} with the element $k_{ij}\, e_{ij}$.

Step 4. Round all values of $k_{ij}\, e_{ij}$ for $i < j$ of $\boldsymbol{K}(k)$ to the closest value from a considered scale.

Step 5. Replace all elements k_{ij} for $i > j$ of $\boldsymbol{K}(k)$ with $1/k_{ij}$.

Step 6. After all replacements are done, calculate the value of the examined index as well as the estimates of the vector $\boldsymbol{k}$ denoted as $\boldsymbol{k}^*$(EP) with application of assigned estimation procedure (EP). Then compute estimates errors AE($\boldsymbol{k}^*$(EP), $\boldsymbol{k}$) and RE($\boldsymbol{k}^*$(EP), $\boldsymbol{k}$) denoting the absolute and relative error respectively. Remember values computed in this step as one record.

Step 7. Repeat Steps 2 to 6 N_M times.

Step 8. Repeat Steps 2 to 7 N_R times.

Step 9. Return *all* records organized as one database.

The probability distribution π attributed in Step 3 to the perturbation factor e_{ij} is applied in equal proportions as: *gamma*, *log-normal*, *truncated normal*, and *uniform* distribution. These are four of the distribution types, which are most frequently considered in literature for various implementation purposes.

Our simulation scenario assumes that the perturbation factor e_{ij} will be drawn from the interval $e \in [0.5;1.5]$. Noticeably, in the simulation scenario, parameters of implemented probability distributions are set in such a way that the expected value of e_{ij} equals unity. The latter assumption seems very reasonable because human judgments are not accurate, nevertheless undeniably they circle nearby perfect ones. The "big error", applied in Step 2, has the uniform distribution on the interval $e_B \in [2;4]$.

Due to necessity of diminishing the volume of this paper we will present only results for *MLTI*(*LTI*) and only for $n = 4$. For the same reason we selected only LLSM as the procedure applied for estimation purposes, thus in our simulations EP = LLSM. Our simulation scenario also assumes application of the rounding procedure which in this research operates according to Saaty's scale. It comprises the integers from one (equivalent to the verbal judgment: "equally preferred") to nine (equivalent to the verbal judgment: "extremely preferred") and their reciprocals.

Finally, our scenario takes into account the obligatory assumption in conventional AHP applications, i.e.: the PCM reciprocity condition. In such cases only judgments from the upper triangle of a given PCM are taken into account and those from the lower triangle are replaced by the inverses of the former ones. The outcome of the simulation scenario present Table 1 and Figures 1, 2. Distinguished plots within Figures 1 and 2 present relations between average *MLTI*(*LTI*) and *p*-quantiles or average *AE*(LLSM) with Spearman rank correlation coefficients (SRCC).

Table 1

Performance of the index *MLTI*(*LTI*) in relation to *AE*(LLSM) distribution

Average *MLTI*	*p*-quantiles of *AE*(LLSM)			Average *AE*(LLSM)
	$p = 0.1$	$p = 0.5$	$p = 0.9$	
0.426707	0.0104028	0.0325444	0.0693912	0.0370195
0.750337	0.0157984	0.0411051	0.0791007	0.0451398
0.990685	0.0161317	0.0438768	0.0809734	0.0472575
1.223530	0.0162664	0.0438350	0.0846752	0.0481755
1.449600	0.0159626	0.0475032	0.0887029	0.0511369
1.690140	0.0171854	0.0471083	0.0908819	0.0518714
1.927420	0.0190256	0.0487652	0.0926036	0.0538121
2.159650	0.0183335	0.0483145	0.0951448	0.0540101
2.395130	0.0193643	0.0487246	0.0999413	0.0555574
2.637340	0.0188251	0.0479729	0.1001040	0.0550359
2.876040	0.0197419	0.0505363	0.1067170	0.0584622
3.102830	0.0195147	0.0518240	0.1088690	0.0598924
3.333710	0.0224324	0.0584043	0.1148840	0.0648018
3.575140	0.0192635	0.0577230	0.1220180	0.0646491
4.267770	0.0212392	0.0607142	0.1353990	0.0711039

Note: results based on 40 000 random reciprocal PCMs

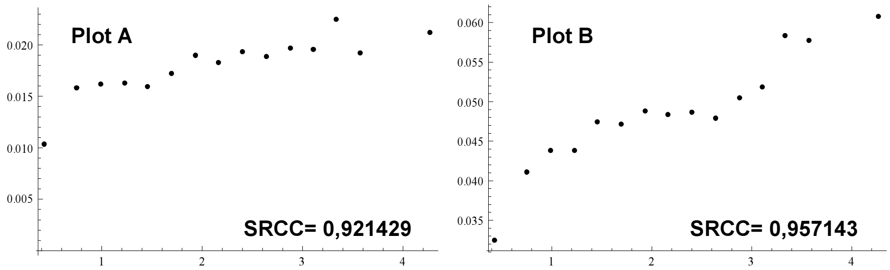

Fig. 1. Performance of the index *MLTI*(*LTI*). Plot of correlation between average values of *MLTI*(*LTI*) and: AE quantiles of order $p = 0.1$ (Plot A) and $p = 0.5$ (Plot B)

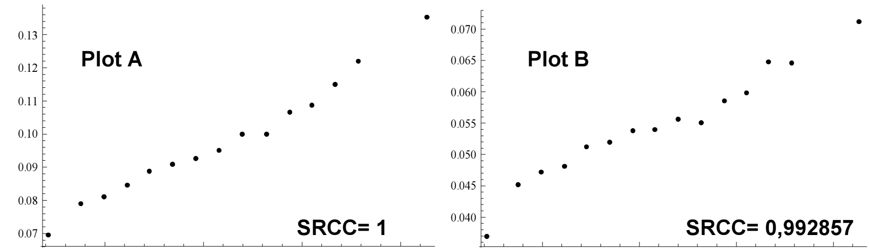

Fig. 2. Performance of the index *MLTI*(*LTI*). Plot of correlation between average values of *MLTI*(*LTI*) and: AE quantiles of order $p = 0.9$ (Plot A) and the average AE (Plot B)

5. Conclusions

We can see that the relation between the average value of *MLTI*(*LTI*) and analyzed statistics is more or less monotonic. This is a very positive message from the perspective of the *PCM applicability* evaluation for estimation of decision makers' priorities in the way that leads to minimization of their estimation errors. Our attention especially attracts the fact that the mean value of *MLTI*(*LTI*) and the quantile of order 0.9 are perfectly monotonic (SRCC = 1). It is worth mentioning here that the same characteristics perform slightly worse when *CI*(LLSM) was examined.

The monotonic relationship between the values of *MLTI*(*LTI*) and the quantiles as well as mean absolute errors (AE) is especially compelling from the perspective of this research study. It is so because these quantiles can be used to accept or reject particular PCM as a good or bad source of information. Thus, both quantiles of order 0.1 and 0.9 provide some knowledge about a prospective outcome we may achieve when the process of estimation is finished.

We feel that new measurements introduced herein concerning the *PCM applicability* evaluation for estimation of decision makers' priorities in the way that leads to the minimization of their estimation errors, either have potential to become a subject of further analysis and research studies. From the perspective of this research study both $LTI(\alpha,\beta,\chi)$ and *MLTI*(*LTI*) are quite good indicators of the trustworthiness of the PCM as a source of information about the priority vector.

Recapitulating, we proposed some new indices for the triad's consistency measurement which, in our opinion, have better prospects than for example *CI*(LLSM). It should be underlined here that they are simpler than Koczkodaj's proposition which makes them especially attractive from the perspective of further more formal, mathematical analysis.

It is crucial to also examine other consistency measures proposed in this paper which were not examined here (due to article's volume constraint) and also find out their performance in relation to other known prioritization procedures and other values for n. Special attention must be given to the issue, if these measures grow in value simultaneously with the growth of estimation errors.

We intend to undertake research studies concerning these issues and hopefully plan to report about their results in the near future.

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The quality of priority ratios estimation in relation to a selected prioritization procedure and consistency measure for a Pairwise Comparison Matrix

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Abstract: An overview of current debates and contemporary research devoted to the modeling of decision making processes and their facilitation directs attention to the Analytic Hierarchy Process (AHP). At the core of the AHP are various prioritization procedures (PPs) and consistency measures (CMs) for a Pairwise Comparison Matrix (PCM) which, in a sense, reflects preferences of decision makers. Certainly, when judgments about these preferences are perfectly consistent (cardinally transitive), all PPs coincide and the quality of the priority ratios (PRs) estimation is exemplary. However, human judgments are very rarely consistent, thus the quality of PRs estimation may significantly vary. The scale of these variations depends on the applied PP and utilized CM for a PCM. This is why it is important to find out which PPs and which CMs for a PCM lead directly to an improvement of the PRs estimation accuracy. The main goal of this research is realized through the properly designed, coded and executed seminal and sophisticated simulation algorithms in *Wolfram Mathematica 8.0.* These research results convince that the embedded in the AHP and commonly applied, both genuine PP and CM for PCM may significantly deteriorate the quality of PRs estimation; however, solutions proposed in this paper can significantly improve the methodology.

Keywords*: pairwise comparisons, priority ratios, consistency, AHP, Monte Carlo simulations*

Introduction

It is agreed that the world is a complex system of interacting elements. It is obvious that human minds have not yet evolved to the point where they can clearly perceive relationships of this global system and solve crucial issues associated with them. In order to deal with complex and fuzzy social, economic, and political issues, people must be supported and guided on their way to order priorities, to agree that one goal out-weighs another from a perspective of certain established criterion, to make tradeoffs in order to be able to serve the greatest common interest (Caballero, Romero & Ruiz 2016; García-Melón et al. 2016).

Obviously, intuition cannot be trusted, although many commonly do so, attempting to devise solutions for complex problems which demand reliable answers. Overwhelming scientific evidence indicates that the unaided human brain is simply not capable of simultaneous analysis of many different competing factors and then synthesizing the results for the purpose of rational decision. It is presumably the principal reason why scientists continuously deal with explanations and modeling of decisional problems in a way to make them widely comprehendible. That is why many supportive methodologies have been elaborated in order to make the decision making process easier, more credible and sometimes even possible. Indeed, numerous psychological experiments (Martin 1973),

including the well-known Miller study (Miller 1956) put forth the notion that humans are not capable of dealing accurately with more than about seven (±2) things at a time (the human brain is limited in its short term memory capacity, its discrimination ability and its bandwidth of perception).

Principles of the analytic thinking process

Humans learn about anything by two means. The first involves examining and studying some phenomenon from the perspective of its various properties, and then synthesizing findings and drawing conclusions. The second entails studying some phenomenon in relation to other similar phenomena and relating them by making comparisons (Saaty 2008). The latter method leads directly to the essence of the matter i.e. judgments regarding the phenomenon. Judgments can be relative and absolute. An absolute judgment is the relation between a single stimulus and some information held in short or long term memory. A relative judgment, on the other hand, can be defined as the identification of some relation between two stimuli both present to the observer (Blumenthal 1977). It is said that humans can make much better relative judgments than absolute ones (Saaty 2000). It is probably so because humans have better ability to discriminate between the members of a pair, than compare one thing against some recollection from long term memory.

For detailed knowledge, the mind structures complex reality into its constituent parts, and these in turn into their elements. The number of parts usually ranges between five and nine. By breaking down reality into homogenous clusters and subdividing those into smaller ones, humans can integrate large amounts of information into the structure of a problem and form a more comprehensive picture of the whole system. Abstractly, this process entails the decomposition of a system into a hierarchy which is a model of a complex reality. Thus, a hierarchy constitutes a structure of multiple levels where the first level is the objective followed successively by levels of factors, criteria, sub-criteria, and so on down to a bottom level of alternatives. The goal of this hierarchy is to evaluate the influence of higher level elements on those of a lower level or alternatively the contribution of elements in the lower level to the importance or fulfillment of the elements in the level above. In this context the latter elements serve as criteria and are called properties.

Generally, a hierarchy can be functional or structural. The latter closely relates to the way a human brain analyzes complexity by breaking down the objects perceived by the senses into clusters and sub-clusters, and so on. Thus, in structural hierarchies, complex systems are structured into their constituent parts in descending order according to their structural properties. In contrast, in functional hierarchies complex systems are decomposed into their constituent parts in accordance to their essential relationships.

A large number of hierarchies in application are available in the literature (Saaty 1993). Supposedly, the hierarchical classification is the most powerful method applied by the human mind during intellectual reasoning and ordering of information and/or observations. Thus, we may agree that an efficient and effective multiple criteria decision making process should encompass the following steps:

– transpose the problem into a hierarchy;
– derive judgments that reflect ideas and feelings or emotions;
– represent these judgments with meaningful numbers values;
– apply those number values for computing priorities for the elements in the hierarchy;

– synthesize the results in order to establish an overall outcome.

There is a multiple criteria decision making support methodology which meets the prescription developed above. It is called the Analytic Hierarchy Process (AHP) and was developed at the Wharton School of Business by Thomas Saaty (1977). Although it is a very popular and widely implemented theory of choice, it is also controversial, thus very often validated and valuated from the perspective of its methodology. From that perspective, most recent papers, such as Grzybowski (2016); Kazibudzki (2016a); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); Aguarón, Escobar & Moreno-Jiménez (2014); Lin, Kou & Ergu (2013); Brunelli, Canal & Fedrizzi (2013), unfold new research areas in this matter which should be thoroughly examined and provoke questions which should be answered, that is:

1) Is the principal right eigenvector (REV), as the prioritization procedure (PP), necessary and sufficient for the AHP?

2) Is the reciprocity of the Pairwise Comparison Matrix (PCM) a reasonable condition leading to the improvement of the priority ratios estimation quality?

3) Are PCM consistency measures, commonly applied and embedded in the AHP, really conducive to the improvement of the priority ratios estimation quality?

Principles of the Analytic Hierarchy Process

Preliminaries

The AHP seems to be the most widely used multiple criteria decision making approach in the world today. Probably, the most recent list of application oriented papers can be found in Grzybowski (2016). Actual applications in which the AHP results were accepted and used by competent decision makers can be found in: Saaty (2008); Ishizaka & Labib (2011); Ho (2008); Vaidya & Kumar (2006); Bhushan & Ria (2004); or Saaty & Vargas (2006). However, regardless of AHP popularity, the genuine methodology is also undeniably the most validated, developed and perfected contemporary methodology, see for example: Kazibudzki (2016b); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); or Aguarón, Escobar & Moreno-Jiménez (2014).

The AHP allows decision makers to set priorities and make choices on the basis of their objectives, knowledge and experience in a way that is consistent with their intuitive thought process. AHP has substantial theoretical and empirical support encompassing the study of human judgmental process by cognitive psychologists. It uses the hierarchical structure of the decision problem, pairwise relative comparisons of the elements in the hierarchy, and a series of redundant judgments. This approach reduces errors and provides a measure of the consistency of judgments. The process permits accurate priorities to be derived from verbal judgments even though the words themselves may not be very precise. Thus, it is possible to use words for comparing qualitative factors and then to derive ratio scale priorities that can be combined with quantitative factors.

To make a proposed solution possible i.e. derive ratio scale priorities on the basis of verbal judgments, a scale is utilized to evaluate the preferences for each pair of items. Apparently, the most popular is Saaty's numerical scale which comprises of the integers from one (equivalent to the verbal judgment - 'equally preferred') to nine (equivalent to the verbal judgment - 'extremely preferred'), and their reciprocals. However, in conventional AHP applications it may be desirable to utilize other scales also i.e. a geometric and/or numerical scale. The former usually consists of the numbers computed in accordance with the formula $2^{n/2}$ where n comprises of the integers from minus eight to eight. The latter involves arbitrary integers from one to n and their reciprocals.

The first step in using AHP is to develop a hierarchy by breaking a problem down into its primary components. The basic AHP model includes the goal (a statement of the overall objective), criteria (the factors that should be considered in reaching the ultimate decision) and alternatives (the feasible alternatives that are available to achieve said ultimate goal). Although the most common and basic AHP structure consists of a goal-criteria-alternatives sequence (Fig.1). AHP can easily support more complex hierarchies. A variety of basic hierarchical structures include:

– goal, criteria, sub-criteria, scenarios, alternatives;
– goal, players, criteria, sub-criteria, alternatives;
– goal, criteria, levels of intensities, many alternatives.

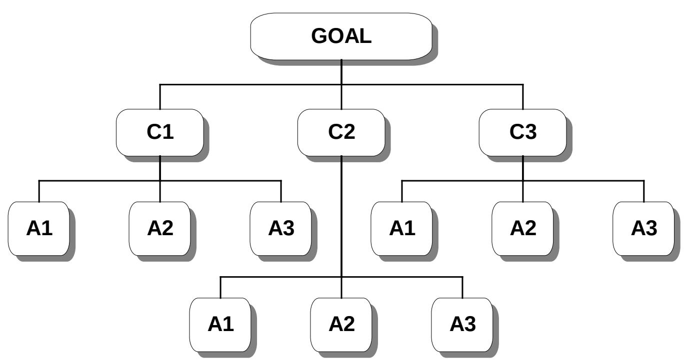


Fig. 1 - Example of a fundamental three level hierarchy encompassing three criteria and three alternatives under each criterion

Mathematics behind the Analytic Hierarchy Process

The conventional procedure of priority ranking in AHP is grounded on the well-defined mathematical structure of consistent matrices and their associated right-eigenvector's ability to generate true or approximate weights.

The German mathematician, Oscar Perron, proved in 1907 that, if $\boldsymbol{A}=(a_{ij})$, $a_{ij}>0$, where i, $j=1,\ldots, n$, then $\boldsymbol{A}$ has a simple positive eigenvalue $\lambda_{\max}$ called the principal eigenvalue of $\boldsymbol{A}$ and $\lambda_{\max}>|\lambda_k|$ for the remaining eigenvalues of $\boldsymbol{A}$. Furthermore, the principal eigenvector $\boldsymbol{w}=[w_1,\ldots, w_n]^T$ that is a solution of $\boldsymbol{A}\boldsymbol{w}=\lambda_{\max}\boldsymbol{w}$ has $w_i>0$, $i=1,\ldots, n$. Thus, the conventional concept of AHP can be presented as follows:

$$\begin{bmatrix} w1/w1 & w1/w2 & w1/w3 & \dots & w1/wn \\ w2/w1 & w2/w2 & w2/w3 & \dots & w2/wn \\ w3/w1 & w3/w2 & w3/w3 & \dots & w3/wn \\ \vdots & \vdots & \vdots & & \vdots \\ wn/w1 & wn/w2 & wn/w3 & \dots & wn/wn \end{bmatrix} \times \begin{bmatrix} w1 \\ w2 \\ w3 \\ \vdots \\ wn \end{bmatrix} = \begin{bmatrix} nw1 \\ nw2 \\ nw3 \\ \vdots \\ nwn \end{bmatrix} \tag{1}$$

If the relative weights of a set of activities are known, they can be expressed as a Pairwise Comparison Matrix (PCM) as shown above $\boldsymbol{A}(w)$. Now, knowing $\boldsymbol{A}(w)$ but not $\boldsymbol{w}$ (vector of priority ratios), Perron's theorem can be applied to solve this problem for $\boldsymbol{w}$. The solution leads to n unique values for λ, with an associated vector $\boldsymbol{w}$ for each of the n values.

PCMs in the AHP reflect relative weights of considered activities (criteria, scenarios, players, alternatives, etc.), so the matrix $\boldsymbol{A}(w)$ has a special form. Each subsequent row of that matrix is a constant multiple of its first row. In this case a matrix $\boldsymbol{A}(w)$ has only one non-zero eigenvalue, and since the sum of the eigenvalues of a positive matrix is equal to the sum of its diagonal elements, the only non-zero eigenvalue in such a case equals the size of the matrix and can be denoted as $\lambda_{\max} = n$.

The norm of the vector $\boldsymbol{w}$ can be written as $\|\boldsymbol{w}\|=\boldsymbol{e}^T\boldsymbol{w}$ where: $e=[1, 1,\dots, 1]^T$ and $\boldsymbol{w}$ can be normalized by dividing it by its norm. For uniqueness, $\boldsymbol{w}$ is referred to in its normalized form.

<u>Theorem 1</u>: A positive n by n matrix has the ratio form $\boldsymbol{A}(w)=(w_i / w_j)$, $i, j=1,\dots, n$, if, and only if, it is consistent.

<u>Theorem 2</u>: The matrix of ratios $\boldsymbol{A}(w)=(w_i / w_j)$ is consistent if and only if n is its principal eigenvalue and $\boldsymbol{Aw}=n\boldsymbol{w}$. Further, $\boldsymbol{w}>0$ is unique to within a multiplicative constant.

<u>Definition 1</u>: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ij}=1/w_{ji}$ for all i, $j=1,\dots, n$ then the matrix $\boldsymbol{A}(w)$ is called *reciprocal*.

<u>Definition 2</u>: The matrix $\boldsymbol{A}(w)$ is called *ordinal transitive* if the following conditions hold:

(A) if for any $i=1,\dots, n$, an element a_{ij} is not less than an element a_{ik} then $a_{ij} \ge a_{ik}$ for $i=1,\dots, n$, and

(B) if for any $i=1,\dots, n$, an element a_{ji} is not less than an element a_{ki} then $a_{ji} \ge a_{ki}$ for $i=1,\dots, n$.

<u>Definition 3</u>: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ik}w_{kj}=w_{ij}$ for all $i, j, k=1,\dots, n$, and the matrix is *reciprocal*, then it is called *consistent* or *cardinal transitive*.

Certainly, in real life situations when AHP is utilized, there is not an $\boldsymbol{A}(w)$ which would reflect weights given by the vector of priority ratios. As was stated earlier, the human mind is not a reliable measurement device. Assignments such as, 'Compare – applying a given ratio scale – your feelings concerning alternative 1 versus alternative 2', do not produce accurate outcomes. Thus, $\boldsymbol{A}(w)$ is not established but only its estimate $\boldsymbol{A}(x)$ containing intuitive judgments, more or less close to $\boldsymbol{A}(w)$ in accordance with experience, skills, specific knowledge, personal taste and even temporary mood or overall disposition. In such case, consistency property does not hold and the relation between elements of $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$ can be expressed as follows:

$$x_{ij} = e_{ij} w_{ij} \quad (2)$$

where e_{ij} is a perturbation factor fluctuating near unity. In the statistical approach e_{ij} reflects a realization of a random variable with a given probability distribution.

It has been shown that for any matrix, small perturbations in the entries imply similar perturbations in the eigenvalues, that is why in order to estimate the true priority vector $\boldsymbol{w}$, conventional AHP utilizes Perron's theorem. The solution of the matrix equation $\boldsymbol{Aw}=\lambda_{\max}\boldsymbol{w}$, gives us $\boldsymbol{w}$ as the Right Principal Eigenvector (REV) associated with $\lambda_{\max}$.

In practice the REV solution is obtained by raising the matrix $\boldsymbol{A}(x)$ to a sufficiently large power, then the rows of $\boldsymbol{A}(x)$ are summed and the resulting vector is normalized in order to receive $\boldsymbol{w}$. This concept can be also delivered in the form of the following formula:

$$w = \lim_{k \to \infty} \left(\frac{A^k \times e}{e^T \times A^k \times e} \right) \quad (3)$$

where: $e=[1, 1,\ldots, 1]^T$.

Description of the first problem

It has been promoted that the REV prioritization procedure (PP) is necessary and sufficient to uniquely establish the ratio scale rank order inherent in inconsistent pairwise comparison judgments (Saaty & Hu 1998). However, there are alternative PPs devised to cope with this problem. Many of them are optimization based and seek a vector w, as a solution of the minimization problem given by the formula:

$$\min \boldsymbol{D}(\boldsymbol{A}(x), \boldsymbol{A}(w)) \quad (4)$$

subject to some assigned constraints such as, for example, positive coefficients and normalization condition. Because the distance function $\boldsymbol{D}$ measures an interval between matrices $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$, different ways of its definition lead to various prioritization concepts and prioritization results. As an example, Choo et al. (2004) describes and compares eighteen estimation procedures for ranking purposes although some authors suggest there are only fifteen that are different. Furthermore, since the publication of the above article, a few additional procedures have been introduced to the literature, see for example: Grzybowski (2012).

Certainly, when the PCM is consistent, all known procedures coincide. However, in real life situations, as was discussed earlier, human judgments produce inconsistent PCMs. The inconsistency is a natural consequence of human brain dynamics described earlier and also a consequence of the questioning methodology, mistaken entering of judgment values, and scaling procedure (i.e. rounding errors). It seems crucial to emphasize here that usually even perfectly consistent PCMs, only because of rounding errors are not error-free. It can be illustrated on the basis of the following hypothetic example.

The genuine priority vector: $\boldsymbol{w}=[7/20, 1/4, 1/4, 3/20]$ is considered and derived from it, $\boldsymbol{A}(w)$ which can be presented as follows:

$$\boldsymbol{A}(w)=\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Now it is considered $\boldsymbol{A}(x)$ produced by a hypothetic decision maker (DM), whose judgments are perfectly consistent. Even if it is assumed that the selected DM is very trustworthy and can express judgments very precisely, DM is still somehow limited by the necessity of expressing judgments on a scale (the example utilizes Saaty's scale). As such, the DM will produce the PCM ($\boldsymbol{A}(x)$) which is not error-free because the entries must be in this case rounded to the closest values of Saaty's scale. Since $\boldsymbol{A}(x)$ must be reciprocal (the fundamental requirement of the AHP) the PCM appears as follows:

$$\boldsymbol{A}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

It may be noticed that the above PCM is perfectly consistent, so this construct seems to be exemplary. However, the hypothetic DM, despite best intentions, is burdened with inescapable estimation errors. In the above situation the priority vector (PV) derived from $\boldsymbol{A}(x)$ by any PP, provides the following priority ratios (PRs): $\boldsymbol{x}$=[2/7, 2/7, 2/7, 1/7] which are not equal to those considered exemplary: $\boldsymbol{w}$=[7/20, 1/4, 1/4, 3/20]. Obviously, the deviation between those PVs can also be expressed by their Mean Absolute Error (MAE), for instance, established by the following formula:

$$MAE(w,x)=\frac{1}{n}\sum_{i=1}^{n}\left|w_i - x_i\right| \tag{5}$$

where n is the number of elements within the particular PV. Noticeably, in the above example, MAE equals 1/28.

From that perspective, Saaty & Hu's (1998) declaration articulating that the REV *is the only valid PP for deriving the PV from a PCM, particularly when the PCM is inconsistent* seems at least questionable. However, they provide an example of a situation where variability in ranks does not occur for each individual judgment matrix, it occurs in the overall ranking of the final alternatives due to the application of different PPs and the multi-criteria process itself. They argue that only the REV possesses a sound mathematical background directly dealing with the question of inconsistency. Furthermore, as they state, only the REV captures the rank order inherent in the inconsistent data in a unique manner. It appears to be time to verify the credibility of these statements utilizing the Monte Carlo simulations.

For that purpose, apart from the REV, four different PPs have been arbitrarily selected ranked as the best within AHP methodology (Kazibudzki & Grzybowski 2013; Lin 2007; Choo & Wedley 2004) – Table 1.

Table 1 – Formulae for the prioritization procedures

The Prioritization Procedure	Formula for the Prioritization Procedure
Logarithmic Utility Approach – LUA –	$w_{(LUA)} = \min \sum_{i=1}^{n} \ln^2 \left(\sum_{j=1}^{n} \frac{a_{ij} w_j}{n w_i} \right)$
Sum of Squared Relative Differences Method – SRDM	$w_{(SRDM)} = \min \sum_{i=1}^{n} \left(\frac{1}{n w_i} \sum_{j=1}^{n} a_{ij} w_j - 1 \right)^2$
Logarithmic Least Squares Method – LLSM –	$w_{(LLSM)} = \min \sum_{i=1}^{n} \sum_{j=1}^{n} \ln^2 \left(a_{ij} \frac{w_j}{w_i} \right)$
Simple Normalized Column Sum – SNCS –	$w_{i(SNCS)} = \frac{1}{n} \sum_{j=1}^{n} \left(a_{ij} \Big/ \sum_{k=1}^{n} a_{kj} \right)$

The first problem study

The objective of this chapter is to verify the above statement i.e. *the REV is the only valid method for deriving the PV from a PCM, particularly when the matrix is inconsistent*.

Taking into account the exemplary study of Saaty & Hu (1998), it seems that the best way to analyze the problem is to examine whether different PPs are really inferior in the estimation of true PVs whose intent is accurate estimation. From that perspective, only computer simulations can illuminate the question, for it is possible to elaborate an algorithm which enables simulation of different kinds of errors which may occur during the process of judgment, and enables assessment which one from the selected PPs delivers better estimates (from a given perspective) of the genuine PV.

Thus, the following simulation algorithm was constructed. Assuming that the decisional problem can be presented in the form of a three level hierarchy (goal, criteria and alternatives – see Figure 1). In order to emulate the problem presented in Saaty & Hu (1998), the hypothetical hierarchy is also designed as a four criteria and four alternatives structure i.e. n=4 and m=4. In agreement with these assumptions, it is possible to elaborate and execute the simulation algorithm **SA|1|** comprising of the following steps:

Step 1. Randomly generate a priority vector $\boldsymbol{k}=[k_1,\ldots, k_n]^T$ of assigned size [n x 1] for criteria and related perfect PCM(k)=$\boldsymbol{K}(k)$

Step 2. Randomly generate exactly n priority vectors $\boldsymbol{a}_\mathbf{n}=[a_{n,1},\ldots, a_{n,m}]$ of assigned size [m x 1] for alternatives under each criterion and related perfect PCMs(a)=$\boldsymbol{A}_\mathbf{n}(a)$

Step 3. Compute a total priority vector $\boldsymbol{w}$ of the size [m x 1] applying the following procedure: $w_x=k_1a_{1,x} + k_2a_{2,x} +\ldots+ k_na_{n,x}$

Step 4. Randomly choose a number e from the assigned interval [α; β] on the basis of assigned probability distribution π

Step 5. Apply separately *Step 5A* and *Step 5B*:

Step 5A – *the case of PCM forced reciprocity implementation;*

replace all elements a_{ij} for $i<j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with ea_{ij} , and all elements k_{ij} for $i<j$ of $\boldsymbol{K}(k)$ with ek_{ij}

Step 5B – *the case of arbitrary PCM acceptance;*

replace all elements a_{ij} for $i\neq j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with ea_{ij} , and all elements k_{ij} for $i\neq j$ of $\boldsymbol{K}(k)$ with ek_{ij}

Step 6. Apply separately *Step 6A* and *Step 6B*:

Step 6A – *when Step 5A is performed;*

round all values of elements a_{ij} for $i<j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$, and all values of elements k_{ij} for $i<j$ of $\boldsymbol{K}(k)$ to the closest values from a considered scale, then replace all elements a_{ij} for $i>j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with $1/a_{ij}$, and all elements k_{ij} for $i>j$ of $\boldsymbol{K}(k)$ with $1/k_{ij}$

Step 6B – *when Step 5B is performed;*

round all values of elements a_{ij} for $i\neq j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$, and all values of elements k_{ij} for $i\neq j$ of $\boldsymbol{K}(k)$ to the closest values from a considered scale

Step 7. On the basis of all perturbed $\boldsymbol{A}_{\mathbf{n}}(a)$ denoted as $\boldsymbol{A}_{\mathbf{n}}(a)^*$ and perturbed $\boldsymbol{K}(k)$ denoted as $\boldsymbol{K}(k)^*$ compute their respective priorities vectors $\boldsymbol{a}_{\mathbf{n}}^*$ and $\boldsymbol{k}^*$ with application of assigned estimation procedure (EP), i.e.: REV, LUA, SRDM, LLSM, and SNCS.

Step 8. Compute a total priority vectors $\boldsymbol{w}^*$(EP) of the size [m x 1] applying the following procedure: $w^*_x=k^*_1a^*_{1,x}+k^*_2a^*_{2,x}+\ldots+k^*_na^*_{n,x}$

Step 9. Calculate *Spearman rank correlation coefficients* – $\mathrm{SRC}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ between all $\boldsymbol{w}^*$(EP) and $\boldsymbol{w}$, as well designated estimation precision characteristics, i.e.: mean relative errors:

$$RE_{\gamma,\chi}(w^*(EP),w)=\frac{1}{m}\sum_{i=1}^{m}\frac{|w_i-w_i^*(EP)|}{w_i} \tag{6}$$

along with mean relative ratios:

$$RR_{\gamma,\chi}(w^*(EP),w)=\frac{1}{m}\sum_{i=1}^{m}\frac{w_i^*(EP)}{w_i} \tag{7}$$

Step 10. Repeat Steps 4 to 9, χ times, where χ denotes a size of the sample

Step 11. Repeat Steps 1 to 9, γ times, where γ denotes a number of considered AHP models

Step 12. Return arithmetic average values of all $\mathrm{SRC}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, $\mathrm{RE}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, and $\mathrm{RR}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ computed during all runs in Steps: 10 and 11, i.e.:

$$MSRC(w^*(EP),w)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}SRC_i(w^*(EP),w) \tag{8}$$

$$MRE(w^*(EP),w)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}RE_i(w^*(EP),w) \tag{9}$$

$$MRR(w^*(EP),w)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}RR_i(w^*(EP),w) \tag{10}$$

where: $\mathrm{MSRC}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, $\mathrm{MRE}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ and $\mathrm{MRR}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ denotes: *mean Spearman rank correlation coefficient*, *average mean relative error* and *average mean relative ratio*, respectively.

In the first experiment, the probability distribution π attributed in Step 4 to the perturbation factor *e* is selected arbitrarily to be the *gamma* or *uniform* distribution. These are two of the distribution types most frequently considered in literature for various implementation purposes (Grzybowski 2016). Usually recommended are such types as: *gamma*, *log-normal*, *truncated normal*, or *uniform*. Apart from these most popular π, one

can find applications of the Couchy, Laplace, or either *triangle* and *beta* probability distributions (see e.g. Dijkstra 2013).

The first simulation scenario also assumes that the perturbation factor e will be drawn from the interval $e \in [0.01; 1.99]$. Noticeably, in each case hereafter, the parameters of different probability distributions applied are set in such a way that the expected value of e in each particular simulation scenario EV(e)=1. It seems a very reasonable assumption, because although human judgments are not accurate, they undeniably aim perfect ones.

Furthermore, the number of alternatives and criteria in a single AHP model will be assigned randomly. By ’randomly‘ – without any other explicit specification – hereafter defined as a process operating under uniform distribution. All simulation scenarios also assume application of the rounding procedure which always operates according to the *geometric* scale described earlier in this paper.

Finally, the first scenario also takes into account the obligatory assumption in conventional AHP applications i.e. the PCM reciprocity condition. In such cases, only judgments from the upper triangle of a given PCM are taken into account and those from the lower triangle are replaced by the inverses of the former.

The outcomes i.e. mean characteristics for 30,000 cases (χ=15 and γ=2000) of the first simulation scenario are presented in Table 2. It may be noticed from Table 2, that the REV can be undeniably classified as the worst PP from the perspective of PRs derived from ranks established on the basis of three different prioritization quality measures i.e. MRE, MSRC, and MRR. The best two PPs from the viewpoint of this classification are LLSM, known also as Geometric Mean Procedure (GM), and LUA. Certainly, the first scenario experiment was designed only to contrast the results presented by Saaty & Hu (1998). It is the intention to establish wider and more fundamental relationships among the presented PPs.

Table 2 – Mean performance measures of arbitrarily selected PPs for 30,000 cases

Scenario Details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean Rank
Geometric Scale	*gamma* distribution	*FR*-PCM	LLSM	0.438438	1	0.682300	2	1.21242	1	1.3(3)
			REV	0.452614	5	**0.668380**	5	1.22051	4	4.6(6)
			LUA	0.447349	2	0.673067	3	1.21792	2	2.3(3)
			SRDM	0.448759	3	0.671380	4	1.21870	3	3.3(3)
			SNCS	0.450734	4	0.692453	1	1.24398	5	3.3(3)
	uniform distribution	*FR*-PCM	LLSM	0.288608	1	0.804860	2	1.12813	1	1.3(3)
			REV	0.302346	4	**0.792580**	5	1.13530	4	4.3(3)
			LUA	0.298401	2	0.795767	3	1.13350	2	2.3(3)
			SRDM	0.299400	3	0.794820	4	1.13400	3	3.3(3)
			SNCS	0.303463	5	0.808333	1	1.15450	5	3.6(6)

Note: *FR*-PCM denotes *forced reciprocity* applied to PCM during simulations

The second simulation scenario was designed to encompass new assumptions not yet taken into account in the literature. First of all, taken into consideration were results obtained not only on the basis of reciprocal PCM, but also the simulation outcomes of nonreciprocal PCM. Secondly, it was decided to implement into simulations new intervals for random errors and apply their new probability distribution. As is known, many

simulation analyses presented in literature assume very non symmetric intervals for a perturbation factor (considering its influence on the particular element of PCM). For example consider the interval for perturbation factor applied in the first simulation scenario i.e. $e \in [0.01;1.99]$. Under this assumption, it becomes apparent that if some entry of PCM is modified *in plus* by the perturbation factor from that particular interval, it is multiplied maximal by the number 1.99, so if the original entry is 3, the modified value will be around 6. However, if some entry of PCM is modified *in minus* by the perturbation factor from that particular interval, it may result that some entry will be multiplied by the number 0.01, so in fact the entry will be divided by 100. Thus, in the situation where the original entry is 9, the modified value will be 0.09, which can be rounded to 1/9 on the Saaty's scale. It may be noticed that this modification practically reverses the preference of DM from e.g. extremely preferred A over B, to extremely preferred B over A (applying the Saaty scale).

It is obvious that this very common assumption is imposed by another very crucial and logical assumption which states that the expected value of e in every particular simulation scenario should equal one i.e. $\mathrm{EV}(e)=1$. It is quite easy to fulfill that requirement on the basis of an asymmetric interval for the perturbation factor (from the perspective of its influence on a particular element of PCM). However, it is rather a challenge to have this assumption implemented with a symmetric interval for the perturbation factor. That is why commonly applied simulation scenarios minimize the range for the perturbation factor in order to achieve at least the delusion of symmetry for $e \in [0.5;1.5]$. Nevertheless, that objective has been attained with the present research, yet to be achieved by other researchers. Presently it seems reasonable to apply symmetric intervals to simulations for the perturbation factor because they better reflect true life situations. Thus, different kinds of probability distributions (PDs) were experimented with and it was discovered that Fisher-Snedecor PD possesses the feature that can be useful in the present analysis. It occurs that for $n_1=14$ and $n_2=40$ degrees of freedom for one thousand randomly generated numbers on the basis of this PD, their mean equals 1.03617, so it is very close to unity, and these numbers fluctuate from 0.174526 to 5.57826. So, with these assumptions, we have $e \in [0.174526;\ 5.57826]$, which gives a very symmetric distribution for the perturbation factor, and $\mathrm{EV}(e) \approx 1$. The results of prioritization quality for different selected PPs and assumed prioritization quality measures i.e. MSRC, MRE, and MRR obtained on the basis of described earlier simulation scenario, are presented in Table 3.

As can be noticed, the REV again is not the dominant PP from the perspective of all simulation scenarios under prescribed frameworks (it takes third place in the total classification order). Certainly, apparent differences in the PV estimation quality in relation to the selected PP are noticeable for nonreciprocal PCMs.

Then, the LUA and SRDM or LLSM dominate over the rest of the selected PPs, especially from the perspective of rank correlations which are the crucial issue from the viewpoint of rank preservation phenomena. These issues will be treated in the section entitled '*Breakthroughs and milestones of this research*'.

Table 3 – Mean performance measures of arbitrarily selected five different ranking procedures for various uniformly drawn 100,000 AHP models – 1,000 hypothetic decisional problems perturbed 100 times each (*)

Scenario Details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean Rank
Geometric Scale	*n*, *m*∈{3, 4…, 7}	FRPCM	LLSM	0.123288	4	0.916281	1	1.04646	3	2.6(6)
			REV	0.123030	1	0.915056	5	1.04546	1	2.3(3)
			LUA	0.123044	3	0.915489	2	1.04699	4	3
			SRDM	0.123038	2	0.915476	3	1.04567	2	2.3(3)
			SNCS	0.132926	5	0.915228	4	1.05865	5	4.6(6)
		APCM	LLSM	0.100511	1	0.930242	4	1.02953	4	3
			REV	0.101523	4	0.930164	5	1.02938	3	4
			LUA	0.100658	2	0.930965	2	1.02926	2	**2**
			SRDM	0.101310	3	0.930510	3	1.02925	1	**2.3(3)**
			SNCS	0.108689	5	0.931026	1	1.04315	5	3.6(6)
	n, *m*∈{8, 9…, 12}	FRPCM	LLSM	0.079748	4	0.931396	1	1.03319	4	3
			REV	0.079110	1	0.928266	5	1.03116	1	2.3(3)
			LUA	0.079321	3	0.928817	2	1.03173	3	2.6(6)
			SRDM	0.079286	2	0.928769	4	1.03166	2	2.6(6)
			SNCS	0.086223	5	0.928799	3	1.03935	5	4.3(3)
		APCM	LLSM	0.063936	4	0.943393	3	1.02252	4	3.6(6)
			REV	0.062735	3	0.942399	5	1.02070	1	3
			LUA	0.061757	1	0.944593	1	1.02109	3	**1.6(6)**
			SRDM	0.061852	2	0.944314	2	1.02105	2	**2**
			SNCS	0.068981	5	0.942764	4	1.02879	5	4.6(6)
Saaty's scale	*n*, *m*∈{3, 4…, 7}	FRPCM	LLSM	0.143650	4	0.911381	1	1.06578	4	3
			REV	0.142967	1	0.911151	4	1.06498	1	2
			LUA	0.143069	3	0.911347	2	1.06520	3	2.6(6)
			SRDM	0.143054	2	0.911320	3	1.06517	2	2.3(3)
			SNCS	0.155694	5	0.910735	5	1.07850	5	5
		APCM	LLSM	0.116095	1	0.927455	1	1.04681	3	**1.6(6)**
			REV	0.116994	4	0.926955	4	1.04705	4	4
			LUA	0.116337	2	0.927129	3	1.04657	1	**2**
			SRDM	0.116962	3	0.926532	5	1.04658	2	3.3(3)
			SNCS	0.127154	5	0.927397	2	1.06051	5	4
	n, *m*∈{8, 9…, 12}	FRPCM	LLSM	0.100279	4	0.917231	1	1.04856	4	3
			REV	0.098084	1	0.915833	4	1.04630	1	2
			LUA	0.098648	3	0.916245	2	1.04695	3	2.6(6)
			SRDM	0.098569	2	0.916193	3	1.04687	2	2.3(3)
			SNCS	0.106674	5	0.915633	5	1.05424	5	5
		APCM	LLSM	0.078464	4	0.938192	3	1.03563	4	3.6(6)
			REV	0.077002	3	0.937837	4	1.03422	1	2.6(6)
			LUA	0.076762	1	0.939669	1	1.03469	3	**1.6(6)**
			SRDM	0.076789	2	0.939415	2	1.03464	2	**2**
			SNCS	0.084307	5	0.937796	5	1.04125	5	5

Average Mean Rank	**LLSM**	**2.958**	***REV***	***2.792***	**LUA**	**2.292**	***SRDM***	***2.417***	**SNCS**	**4.542**
Order	**4**		**3**		**1**		**2**		**5**	

Note: (*) AHP models drawn randomly (uniformly) for assigned set of criteria and alternatives. The scenario assumes application of both: perturbation factor drawn with F-Snedecor probability for n_1=14 and n_2=40

degrees of freedom, and rounding errors associated with a given scale (geometric or Saaty's). It assumes calculation of performance measures either for reciprocal PCMs (FRPCM) or nonreciprocal PCMs (APCM).

Description of the second problem

In the previous two sections of this research, it was determined that the quality of PV estimation depends on the selected PP. This section will focus on the other facet of the problem i.e. how the quality of PV estimation depends on the type of PCM Consistency Measure (PCM-CM) engaged in the prioritization process. The difference between the meaning of consistency of a given PCM and the particular PCM-CM is intentionally stressed at this point. Indeed, there are several PCM-CMs provided in the literature called consistency indices (CIs), nevertheless the scientific meaning of PCM consistency is given by the definition (Definition 3).

The most popular and certainly less intuitive is the PCM-CM proposed by Saaty. He proposed his PCM-CM on the basis of his PP which involves eigenvectors and eigenvalues calculations. Thus, the indication of the fact that for the consistent PCM its $\lambda_{max}= n$, for the purpose of PCM consistency measurement, Saaty proposes his CI be determined by the following formula:

$$CI_{REV} = \frac{\lambda_{\max} - n}{n-1} \tag{11}$$

where n indicates the number of alternatives within the particular PCM. The significant disadvantage of this PCM-CM is the fact it can operate exclusively with reciprocal PCMs. In the case of nonreciprocal PCMs, this measure is useless (its values are meaningless) which in consequence seriously diminishes the value of the whole Saaty approach (Linares et al. 2014).

However, as mentioned earlier, there are a number of additional PCM-CMs. Some of them, as in the case of CI_{REV}, originate from the PPs devised for the purpose of the PV estimation process. Their distinct feature is the fact that all of them can operate equally efficiently in conditions where reciprocal and nonreciprocal PCMs are accepted. A number of them, selected on the basis of their popularity (but not only) and up-to-date nature (Kazibudzki 2016b) are presented in Table 4.

Noticeably, there are few propositions of PCM-CMs which are not connected with any PP and are devised on the basis of the PCM consistency definition (Definition 3). Koczkodaj's (1993) idea is the first to be considered. His PCM-CM is grounded on his concept of triad consistency. The notion of a triad:

Statement 1: For any three distinguished decision alternatives A_1, A_2, and A_3, there are three meaningful priority ratios i.e. α, β, and χ, which have their different locations in a particular $\boldsymbol{A}(w)=[w_{ij}]_{nxn}$

Definition 4: If $\alpha=w_{ik}$, $\chi=w_{kj}$, $\beta=w_{ij}$ for some different $i\leq n$, $j\leq n$, and $k\leq n$, then the tuple (α, β, χ) is called a *triad.*

Definition 5: If the matrix $\boldsymbol{A}(w)=[w_{ij}]_{nxn}$ is consistent, then $\alpha\chi=\beta$ for all triads.

Table 4 – Formulae for the PCM-CMs related to their PPs

Symbol of the PP	Formula for the PCM-CM
LUA	$\mathrm{CI_{LUA}}=\frac{1}{n}\sqrt{\min\sum_{i=1}^{n}\ln^2\left(\sum_{j=1}^{n}\frac{a_{ij}w_j}{nw_i}\right)}$
SRDM	$\mathrm{CI_{SRDM}}=\sqrt{\frac{1}{n}\min\sum_{i=1}^{n}\left(\frac{1}{nw_i}\sum_{j=1}^{n}a_{ij}w_j-1\right)^2}$
LLSM	$\mathrm{CI_{LLSM}}=\frac{2}{(n-1)(n-2)}\sum_{i<j}\log^2\left(\frac{a_{ij}w_j}{w_i}\right)$

In consequence, either of the equations 1–β/αχ=0 and 1–αχ/β=0 have to be true. Taking the above into consideration, Koczkodaj proposed his measure for triad inconsistency by the following formula:

$$TI(\alpha,\beta,\chi)=\min\left[\left|1-\frac{\beta}{\alpha\chi}\right|,\left|1-\frac{\alpha\chi}{\beta}\right|\right] \tag{12}$$

Following his idea, he then proposed the following CM of any reciprocal PCM:

$$K(TI)=\max[TI(\alpha,\beta,\chi)] \tag{13}$$

where the maximum value of TI(α,β,χ) is taken from the set of all possible triads in the upper triangle of a given PCM.

On the basis of Koczkodaj's idea of triad inconsistency, Grzybowski (2016) presented his PCM consistency measure determined by the following formula:

$$A(TI)=\frac{1}{N}\sum_{i=1}^{N}[TI_i(\alpha,\beta,\chi)] \tag{14}$$

Finally, following the idea, that ln(αχ/β) = minus ln(β/αχ), Kazibudzki (2016a) redefined triad inconsistency and proposed:

– two formulae for its measurement -

$$LTI_1(\alpha,\beta,\chi)=|\ln(\alpha\chi/\beta)| \tag{15}$$

$$LTI_2(\alpha,\beta,\chi)=\ln^2(\alpha\chi/\beta) \tag{16}$$

– and one meaningful formula for PCM-CM -

$$A(LTI_i)=\frac{1}{N}\sum_{j=1}^{N}[LTI_{ij}(\alpha,\beta,\chi)] \tag{17}$$

which can be calculated on the basis of triads from the upper triangle of the given PCM when it is reciprocal, or all triads within the given PCM when it is nonreciprocal.

The second problem study

This section begins with the fundamental question which should encourage all researchers who deal with the problem of PR estimation quality to seek appropriate PCM consistency measurement. The question asks:
Does a growth of the PCM consistency directly lead to the betterment of the priority vector estimation quality?

Apparently, the answer to this question seems to be affirmative. Commonly, this is the reason why one strives to keep the consistency of the PCM at the highest possible level. However, *is it a good idea to use universally recognized PCM-CMs for this purpose?* To answer this question a preliminary analysis of the example provided and examined in the section entitled '*Description of the first problem*' can be initiated.

Thus, the genuine PV is reconsidered, $\boldsymbol{w}$=[7/20, 1/4, 1/4, 3/20] and $\boldsymbol{A}(w)$ derived from that PV can be presented as follows:

$$\boldsymbol{A}(w)=\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Now considering two PCMs i.e. $\boldsymbol{R}(x)$ and $\boldsymbol{A}(x)$ produced by a hypothetical DM, whose judgments are rounded to Saaty's scale – DM is very trustworthy and is able to express judgments very precisely. In the first scenario, entries of $\boldsymbol{A}(w)$ are rounded to Saaty's scale and the entries are made reciprocal (a principal condition for a PCM in the AHP) producing:

$$\boldsymbol{R}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

In the second scenario only entries of $\boldsymbol{A}(w)$ are rounded to Saaty's scale (nonreciprocal case)producing:

$$\boldsymbol{A}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

It should be noted that $\boldsymbol{R}(x)$ is perfectly consistent and $\boldsymbol{A}(x)$ is not. Tables 5 and 6 present selected values of the PPs related PCM-CMs (that is CI_{REV}, CI_{LUA}, and CI_{LLSM}) for $\boldsymbol{R}(x)$ and $\boldsymbol{A}(x)$ together with PVs derived from $\boldsymbol{R}(x)$ and $\boldsymbol{A}(x)$; Mean Absolute Errors (MAEs) [Formula (18)], among $\boldsymbol{w}$*(PP) and the genuine $\boldsymbol{w}$ for the case; Spearman Rank Correlation Coefficients (SRCs) among $\boldsymbol{w}$*(PP) and the genuine $\boldsymbol{w}$ for the case.

$$MAE\left(w^*(PP), w\right) = \frac{1}{n}\sum_{i=1}^{n}\left|w_i - w_i^*(PP)\right| \tag{18}$$

Table 5 – Values of the PCM-CMs for $\boldsymbol{R}(x)$ and proposed characteristics of PVs estimates (*) quality in relation to the genuine PV for the case

PP	Estimates	Performance measures		
		CI(*PP*)	MAE	SRC
REV	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
LUA	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
LLSM	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966

(*) derived from $\boldsymbol{R}(x)$ with application of a particular PP

Table 6 – Values of the PCM-CMs for $\boldsymbol{A}(x)$ and proposed characteristics of PVs estimates (*) quality in relation to the genuine PV for the case

PP	Estimates	Performance measures		
		CI(*PP*)	MAE	SRC
REV	$[0.309401, 0.267949, 0.267949, 0.154701]^T$	– 0.0893164	0.0202995	1
LUA	$[0.306135, 0.268645, 0.268645, 0.156576]^T$	0.0344483	0.0219326	1
LLSM	$[0.314288, 0.264284, 0.264284, 0.157144]^T$	0.0400378	0.0178559	1

(*) derived from $\boldsymbol{A}(x)$ with application of a particular PP

Surprisingly, a very interesting phenomenon can be noted on the basis of information provided in Tables 5 and 6. The nonreciprocal version of the analyzed PCM contains non-zero values for the selected PCM-CMs. In cases similar to this example, the value of Saaty's PCM-CM always becomes negative which makes it inexplicable and in consequence useless under such circumstances (as already mentioned earlier). The other two measures are positive and higher than zero which indicates that the particular PCM is not consistent. On the basis of the same indicators in the case of the reciprocal version of the analyzed PCM, its perfect consistency is apparent because all selected PCM-CMs in this case are equal to zero. However, the estimation precision measures (MAE and SRC) i.e. characteristics of the particular PV estimation quality, indicate something quite opposite. Surprisingly, apparent are smaller values of MAEs and perfect correlation of ranks between estimated and genuine PV for nonreciprocal version of the analyzed PCM. Certainly, this conclusion concerns all analyzed PPs and it is very true in the situation when the particular PCM is apparently less consistent (on the basis of selected exemplary PCM-CMs).

It has been suggested that these discoveries inevitably lead to the conclusion that the time has just come to revise the common yet erroneous approach to the PCM consistency measurement which can be described as … *the lower PCM-CM, the better PR estimation quality*.

Therefore, it becomes apparent that there are actually three significantly different consistency notions: (1) the consistency of PCM stated by Definition 3, and reflected by a value of the specific CM which in its way denotes a deviation of the analyzed PCM from its fully consistent counterpart; (2) the consistency of DM i.e. their reliability from the viewpoint of their expertise, measured by a comparison of DM judgments reflected by the particular PCM with judgments made more or less randomly; and (3) the PCM consistency stated by Definition 3 and reflected by a value of the specific CM which denotes the

particular PCM applicability for PRs derivation in the way that minimizes estimation errors.

The third notion is of particular interest from the perspective of the Multiple Criteria Decision Making (MCDM) quality. The key concept of the issue was first presented by Grzybowski (2016) and enhanced by Kazibudzki (2016a). It was decided to examine the phenomenon described therein and further develop it to improve the quality of MCDM. The simulation framework for this purpose was adopted from Kazibudzki (2016a) as the only way to examine said phenomena through computer simulations. The simulation algorithm **SA|2|** thus comprises of the following phases:

Phase 1 Generate randomly a priority vector $\boldsymbol{w}=[w_1,\ldots, w_n]^T$ of assigned size $[n \times 1]$ and related perfect PCM(w)=$\boldsymbol{K}(w)$

Phase 2 Select randomly an element w_{xy} for $x<y$ of $\boldsymbol{K}(w)$ and replace it with $w_{xy}e_B$ where e_B is a relatively significant error, randomly drawn (*uniform* distribution) from the interval $e_B \in [2;4]$. Errors of that magnitude are basically considered as "significant", see e.g.: Grzybowski (2012), Dijkastra (2013), Lee (2007).

Phase 3 For each other element w_{ij}, $i<j\leq n$ select randomly a value e_{ij} for the relatively small error in accordance with the given probability distribution π (applied in equal proportions as: *gamma*, *log-normal*, *truncated normal*, and *uniform* distribution) and replace the element w_{ij} with the element w_{ij} e_{ij} where e_{ij} is randomly drawn (*uniform* distribution) from the interval $e_{ij} \in [0,5;1,5]$

Phase 4 Round all values of $w_{ij}\, e_{ij}$ for $i<j$ of $\boldsymbol{K}(w)$ to the nearest value of a considered scale

Phase 5 Replace all elements w_{ij} for $i>j$ of $\boldsymbol{K}(w)$ with $1/w_{ij}$

Phase 6 After all replacements are done, return the value of the examined index as well as the estimate of the vector $\boldsymbol{w}$ denoted as $\boldsymbol{w}$*(PP) with application of assigned prioritization procedure (PP). Then return the mean absolute error MAE between $\boldsymbol{w}$ and $\boldsymbol{w}$*(PP). Remember values computed in this phase as one record.

Phase 7 Repeat *Phases* from 2 to 6 N_n times.

Phase 8 Repeat *Phases* from 1 to 7 N_m times.

Phase 9 Return *all* records to one database file.

Once again, all parameters of the applied PDs – *gamma*, *log-normal*, *truncated normal*, and *uniform* – in the above simulation framework are set as previously in such a way that the expected value EV(e_{ij})=1.

The simulation begins from n=4, because simulations for n=3 are not interesting due to direct interrelation of considered PCM consistency measures (Bozóki & Rapcsak 2008, Dijkstra 2013). For the sake of objectivity, the simulation data is gathered in the following way: all values of selected consistency measures are split into 15 separate sets designated by the quantiles Q of order p from 1/15 to 14/15. The 15 intervals are defined as: the first is from 0 to the quantile of order 1/15 i.e. $\mathrm{VRCM}_1=[0, Q_{1/15})$, where VRCM represents *a Value Range of the Selected PCM Consistency Measure*; the second denotes $\mathrm{VRCM}_2=[Q_{1/15}, Q_{2/15})$, and so on… to the last one which starts from the quantile of order 14/15 and goes on to infinity i.e. $\mathrm{VRCM}_{15}=[Q_{14/15}, \infty)$. The following variables are examined: Mean VRCM_n, average MAE within VRCM_n between $\boldsymbol{w}$ and $\boldsymbol{w}$***(PP), MAE quantiles of the following orders, 0.05, 0.1, 0.5, 0.9, 0.95, and relations between all of them. In the preliminary simulation program, it was decided that PP=LLSM. The application of the rounding procedure was also assumed which in this preliminary research operates according to Saaty's scale.

Lastly, the scenario takes into account the compulsory assumption in conventional AHP applications i.e. the PCM reciprocity condition. The results are based on N_n=20, and N_m=500, i.e. 10,000 cases.

In the case of a good PCM-CM, one could assume that MAE quantiles of any order should monotonically grow concurrently with the growth of the selected PCM-CM e.g. VRCM index. The same relation should occur for Mean $\mathrm{VRCM_n}$ and average MAE for $\mathrm{VRCM_n}$. The results of the proposed simulation framework, or any other similar simulation scenario which would contradict such a pertinent relationship would unequivocally lead to the conclusion that the examined PCM-CM does not serve its purpose.

An examination from that point of view is in order, the performance of six PCM-CMs selected from among very common or recently proposed (Fig.2): Saaty CI_{REV} – (Plot A), together with Crawford & Williams CI_{LLSM} – (Plot B), and Koczkodaj $K(TI)$ – (Plot C), together with Grzybowski $A(TI)$ – (Plot D), as well as Kazibudzki $A(LTI_1)$ and CI_{LUA} – (Plots E-F).

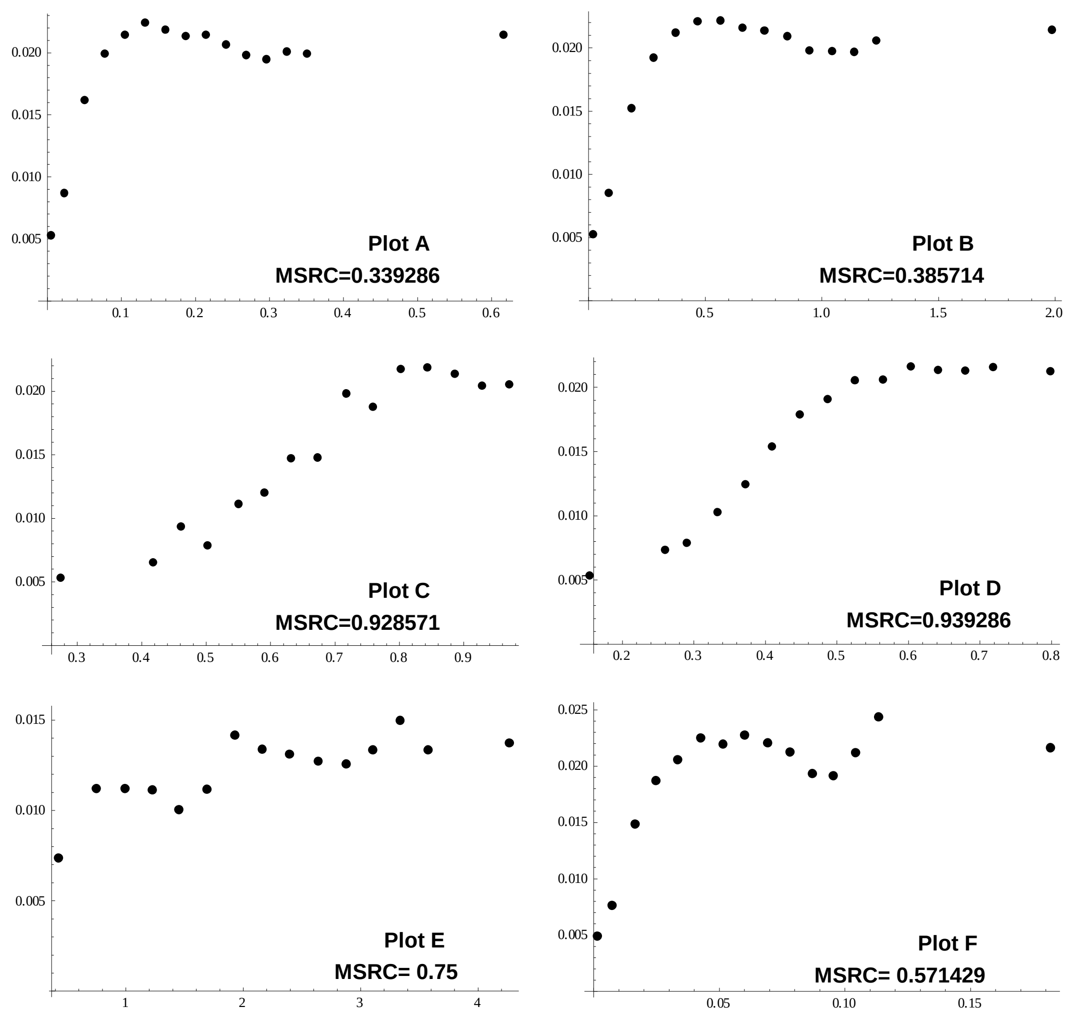


Fig. 2 – Performance of selected PCM-CMs – The plots present the relation between a mean value of a given PCM-CM within a given interval ($\mathrm{VRCM_n}$) and quantiles of order 0.05 of MAEs distribution concerning estimated and genuine PV for the case. The results are generated with application of LLSM as the PP. Plots are based on 10,000 random reciprocal PCMs for n=4. The relation strength MSRC denotes Mean Spearman Rank Correlation Coefficient between analyzed variables.

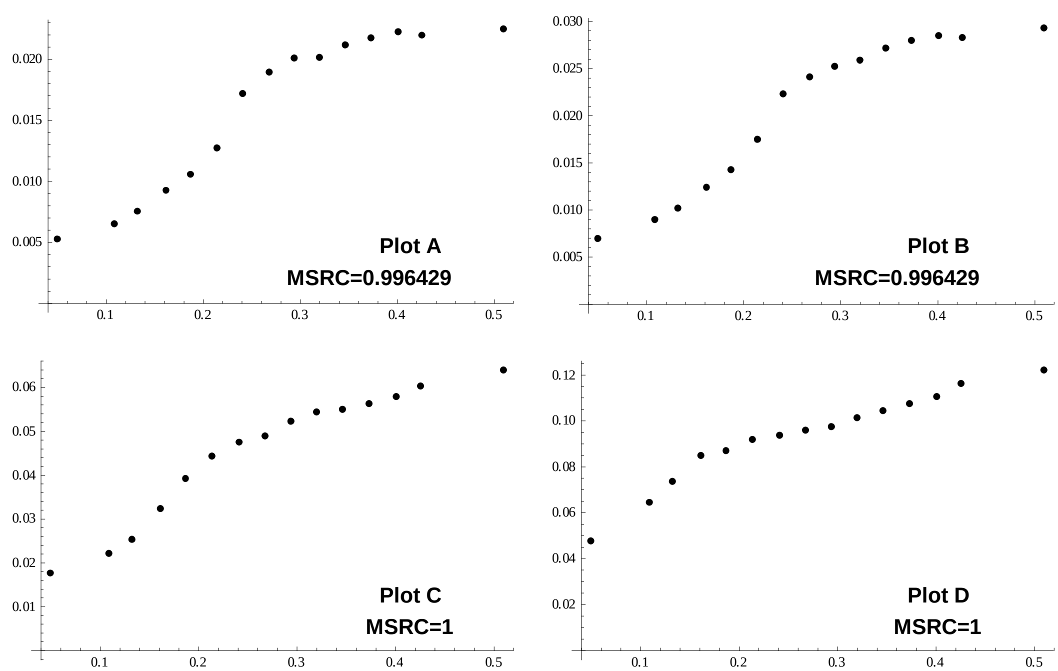
Noticeably, when the quality of PV in MCDM process of AHP is taken into consideration, the presented relations indicate that the analyzed performance of selected PCM-CMs vary more or less from the target. Indeed, the relations indicate that most of the analyzed indices may even misinform DMs about their judgment applicability for the construct of the PV which best converge with the ideal one i.e. obtained from a fully consistent PCM. As seen similarly in the example provided earlier in this paper (Tab. 5 and 6), taking the particular index as the measure of PCM consistency, one can expect both i.e. the betterment of PRs estimation quality (increase of the estimation accuracy) together with the increase of the particular CI (decrease of PCM consistency); and inversely, the deterioration of PRs estimation quality (decrease of the estimation accuracy) together with the descent of the particular CI (improvement of PCM consistency).

Noticeably, the analyzed PCM-CMs are not selected without a reason as they are commonly applied and/or suggested as good solutions in the process of PV estimation on the basis of inconsistent PCMs (for discussion see also Grzybowski 2016). This was the motivation to search for a PCM-CM which relation to PV estimation errors, reflected by SRC, would be very close or equal to 1 (the most desirable situation).

Thus, a seminal solution is proposed in this matter. On the basis of triad inconsistency measure $LTI_2(\alpha,\beta,\chi)=\ln^2(\alpha\chi/\beta)$ introduced by Kazibudzki (2016a), the following PCM-CM is submitted:

$$CM(LTI_2)=\frac{MEAN[LTI_2(\alpha,\beta,\chi)]}{1+MAX[LTI_2(\alpha,\beta,\chi)]} \tag{19}$$

The proposed PCM-CM is denoted as *the Triads Squared Logarithm Corrected Mean* and an examination of its performance on the basis of simulation algorithm **SA|2|** proposed earlier in this paper was carried out.



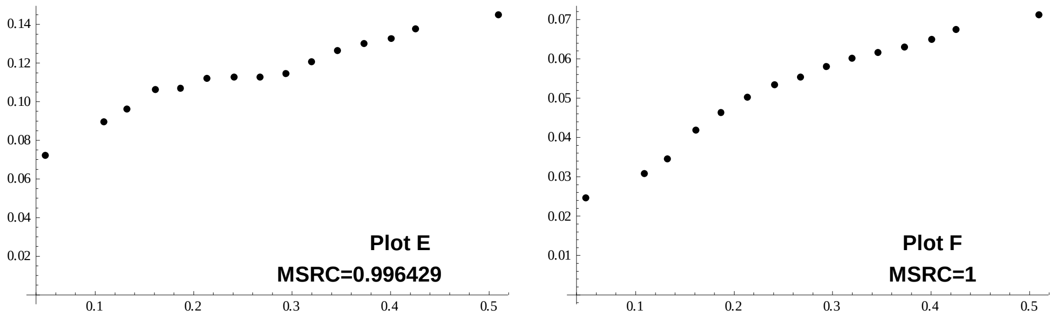


Fig. 3 – Performance of the new PCM-CM: CM(LTI_2) The plots present a relation between a mean value of CM(LTI_2) within a given interval ($VRCM_n$) and quantiles of order 0.05, 0.1, 0.5, 0.9, 0.95 of MAEs distribution as well their average values for estimated and genuine PVs. The results are generated with application of LLSM as the PP. Plots are based on 10,000 random reciprocal PCMs for *n*=4.

As can be noticed, the proposed CM(LTI_2) significantly outperforms the other PCM-CMs analyzed earlier in this paper. It is undeniably a seminal revelation that unquestionably opens a new chapter in MCDM on the basis of AHP – especially because CM(LTI_2) is suitable for both reciprocal and nonreciprocal PCM.

Breakthroughs and milestones of the research

As was said in 1990 by the creator of AHP: … *there is a well-known principle in mathematics that is widely practiced, but seldom enunciated with sufficient forcefulness to impress its importance. A necessary condition that a procedure for solving a problem be a good one is that if it produces desired results, and we perturb the variables of the problem in some small sense, it gives us results that are 'close' to the original ones. (…) An extension of this philosophy in problems where order relations between the variables are important is that on small perturbations of the variables, the procedure produces close, order preserving results* (Saaty 1990, p. 18).

The quality of PR estimation in relation to the selected PP

With said notion in mind, an effort was undertaken to verify the statement of followers of the REV, boldly spreading the idea that so long as inconsistency is accepted, the REV is the paramount theoretical basis for deriving a scale and no other concepts qualify.

It is a fact that in order to support some theory, one must verify it through many experiments to validate its reliability. On the other hand one needs only one example showing it does not work in order to abolish its credibility. Thus, numerous examples were provided indicating that the REV concedes with other devised PP to determine ranking of alternatives. However, although data obtained during simulation experiments are unequivocal, they support the above notion only generally. That is why scientific verification of their meaning is carried out on the basis of the statistical hypothesis testing theory (SHTT).

If $MSRC_{PP}$ and $MSRC_{REV}$ respectively are denoted as mean SRC of selected PP and mean SRC of the REV, their difference significance can be tested using "t" statistics defined by the following formula:

$$t = R\sqrt{\frac{n-2}{1-R^2}} \qquad (20)$$

where R is the difference between particular MSRCs.

This statistic has a distribution of *t–student* with n minus 2 degrees of freedom df, where n equals the size of the sample. The following hypothesis was tested:

$$H_0\text{: } \mathrm{MSRC}_{PP} - \mathrm{MSRC}_{REV} = 0$$

versus

$$H_1\text{: } \mathrm{MSRC}_{PP} - \mathrm{MSRC}_{REV} > 0$$

In order to conform to the example presented by Saaty & Hu (1998), the data gathered in Table 2 was considered. The simulation framework of that case is df=29,998. Thus, for assumed levels of significance α=0.01, α=0.02 or α=0.03, the critical values of *t–student* statistics equal consecutively $t_{0.01}$ = 2.326472, $t_{0.02}$ = 2.053838, or $t_{0.03}$ = 1.880865.

In the situation when a level of tested *t–student* statistics is higher than its critical value for the assumed level of significance, the hypothesis H_0, must be rejected in favor of alternative hypothesis H_1. In the opposite situation, there are no foundations to reject H_0. The selected statistics and their values for the problem evaluation are presented in Table 7.

Clearly, the results of the simulation scenario, designed in accordance with the framework presented in Saaty & Hu (1998), indicate two PPs which on the basis of SHTT always perform better than the REV, regardless of the preselected PD. It should be emphasized that the performance of selected PPs is examined here from the perspective of rank preservation phenomena which is reflected in our research by the MSRC between genuine and perturbed PV. It should be evident that the above conclusions, unlike any other before, are the effect of sound statistical reasoning (rigorous significance level) based on the seminal approach toward AHP methodology evaluation grounded on precisely planned and performed simulation study.

Table 7 – MSRC values and principal statistics for the performance test of the REV versus other selected PPs

Scenario details			Procedure	MSRC	R	R^2	*t–value*	*α–level**
Geometric Scale	*gamma* distribution	*FR*-PCM	LLSM	0.682300	0.01392	0.00019	2.411167969	0.01
			REV	0.668380	><	><	><	><
			LUA	0.673067	0.00469	0.00002	0.811794069	–
			SRDM	0.671380	0.00300	0.00001	0.519600260	–
			SNCS	0.692453	0.02407	0.00058	4.170635557	0.01
	uniform distribution	*FR*-PCM	LLSM	0.804860	0.01228	0.00015	2.127047876	0.02
			REV	0.792580	><	><	><	><
			LUA	0.795767	0.00319	0.00001	0.551988995	–
			SRDM	0.794820	0.00224	0.00001	0.387967421	–
			SNCS	0.808333	0.01575	0.00025	2.728747286	0.01

Note: (*) the closest significance level providing the ground to reject a tested hypothesis

In order to develop the concept further it was decided to expand the simulation program. The results of this endeavor are presented in Table 3. They should be considered as surprising, especially when one realizes that the PP embedded in the AHP merely takes third place in the overall performance ranking. The ranking takes into account not only

MSRC, but MRE and MRR also, the latter never taken into consideration in previous simulation research. The MRR will now be examined to expand its concept and highlight its novelty.

Let's consider a vector $\boldsymbol{k}$ of values to be estimated, $\boldsymbol{k}$=[3, 3, 3, 3], and three of its estimates, $\boldsymbol{k_1}$=[2, 4, 2, 4], $\boldsymbol{k_2}$=[2, 2, 2, 2], $\boldsymbol{k_3}$=[4, 4, 4, 4]. It may be noted that the MREs of all the estimates (given by formula (6)) are the same and equal 1/3. However, MRRs of the estimates (given by formula (7)) are not the same and equal respectively, $\mathrm{MRR}_1(k,k_1)=1$, $\mathrm{MRR}_2(k,k_2)=2/3$, $\mathrm{MRR}_3(k,k_3)=4/3$. Obviously, the goal of estimation is both i.e. to minimize MREs and maintain the MRRs close to unity. This prerequisite is of great importance when one deals with PVs i.e. vectors normalized to unity, as in the case of AHP. Certainly, one can encounter the following three estimation scenarios.

<u>Scenario 1</u> Consider a vector $\boldsymbol{w}$ of genuine PRs trying to estimate $\boldsymbol{w}$=[0.25, 0.25, 0.25, 0.25], and its estimate $\boldsymbol{w_1}$=[0.01, 0.49, 0.05, 0.45]. This scenario gives a rather high MRE of 0.88, which indicates the mean 88% volatility of estimated PRs in relation to their primary values, and MRR=1.

<u>Scenarios 2–3</u>

Consider a vector $\boldsymbol{p}$ of genuine PRs trying to estimate $\boldsymbol{p}$=[0.1, 0.2, 0.3, 0.4], and its two estimates $\boldsymbol{p_1}$=[0.15, 0.3, 0.25, 0.3], and $\boldsymbol{p_2}$=[0.05, 0.1, 0.35, 0.5]. This situation entails a moderate MRE of 0.35425 for both estimates, and two MRRs i.e. $\mathrm{MRR}_1(p,p_1)=1.145$, and $\mathrm{RR}_2(p,p_2)=0.85425$, for the second and third scenario respectively.

Obviously, during the PRs estimation process, it is desirable to avoid situations exemplified by the first and second scenario. Noticeably, they both have something in common. Apart from estimation discrepancies they lead to rank reversal of the initial priorities (emphasis added).

Turning back to Table 3, having in mind the imposed simulation scenario, F-Snedecor PD mean value of a perturbation factor EV(e)=1.03617, we can conclude as follows:

1) the applied measures (MRE, MSRC, MRR) reflecting the quality of PR estimation process within the simulation framework are always better for nonreciprocal PCMs in relation to their reciprocal equivalents;
2) the applied measures of the quality of PR estimation within the simulation framework indicate better estimation results for a relatively higher number of alternatives;
3) both MRE and MRR values indicate that the quality of PR estimation within the simulation framework is better when geometric scale is implemented instead of Saaty's scale for preferences expression of DMs (MRR is then more often less than 1.03617 which indicates less risk of rank reversal);
4) and last but not least, the REV procedure IS NOT a dominating procedure during PR estimation in the simulated framework of the AHP.

The quality of PR estimation in relation to the CM of the PCM

Thus far the alterability of prioritization quality in consequence of the application of preselected PP, preference scale and reciprocal or nonreciprocal PCM in the AHP has been dealt with. This chapter endeavors to focus and conclude the findings concerning the

alterability of prioritization quality in relation to the applied method of the PCM consistency measurement.

Figure 2 demonstrates the basic relation between the distribution of estimation MAEs and values of selected PCM-CMs when LLSM is applied as the PP. The objective was to realize that those measures are not a good indicator of the quality of PR estimation, although the quality of PR estimation should be the core of PCM consistency measurement. Thus, a seminal solution for this problem was introduced i.e. the novel PCM-CM - CM(LTI_2) and depicted its performance in relation to the quality of PR estimation (Fig. 3). As noted, its performance is much better than the PCM-CMs presented earlier (Fig. 2), independently of the MAEs distribution characteristics applied. Below (Tables 8–9), detailed characteristic data is presented for CM(LTI_2) for LLSM and LUA as the PPs, and Saaty's scale as the preferred applied scale.

Table 8 – Performance of the CM(LTI_2) index Statistical characteristics of the MAEs distribution in relation to various $VRCM_i$ for i=1,…,15 of CM(LTI_2) values. The results were generated for n=4 on the basis of **SA|2|** as the simulation algorithm and are based on 10,000 perturbed random reciprocal PCMs. The scenario assumed LLSM as the PP.

i	$VRCM_i$ for CM(LTI_2)	Mean CM(LTI_2) in $VRCM_i$	p–quantiles of MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LLSM)					Average MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0.00, 0.0934)	0.049049	0.0052533	0.0070072	0.0177090	0.0478280	0.0722714	0.0246076
2	[0.0934, 0.12)	0.108368	0.0065427	0.0089647	0.0221881	0.0646356	0.0896474	0.0307832
3	[0.120, 0.147)	0.131977	0.0075604	0.0101952	0.0254009	0.0735568	0.0962640	0.0346824
4	[0.147, 0.173)	0.161289	0.0092812	0.0124076	0.0323969	0.0848569	0.1062240	0.0418424
5	[0.173, 0.200)	0.186567	0.0106050	0.0142825	0.0392350	0.0872419	0.1070400	0.0463689
6	[0.200, 0.227)	0.213651	0.0127312	0.0174795	0.0443425	0.0921171	0.1121160	0.0503101
7	[0.227, 0.253)	0.240868	0.0171655	0.0223103	0.0474780	0.0939184	0.1129280	0.0534051
8	[0.253, 0.280)	0.267645	0.0189530	0.0241065	0.0489027	0.0959089	0.1126270	0.0554222
9	[0.280, 0.307)	0.293803	0.0200809	0.0252443	0.0523480	0.0975035	0.1147230	0.0580895
10	[0.307, 0.333)	0.319702	0.0201740	0.0259357	0.0544712	0.1014610	0.1208420	0.0601639
11	[0.333, 0.360)	0.345876	0.0211796	0.0271488	0.0550490	0.1043660	0.1267140	0.0615576
12	[0.360, 0.387)	0.372744	0.0217402	0.0279791	0.0563253	0.1076280	0.1302670	0.0630527
13	[0.387, 0.413)	0.400500	0.0222736	0.0284786	0.0579657	0.1105020	0.1326590	0.0649738
14	[0.413, 0.440)	0.425325	0.0219914	0.0282637	0.0603546	0.1163910	0.1378310	0.0674297
15	[0.440, ∞)	0.509413	0.0224786	0.0293611	0.0639097	0.1220180	0.1448450	0.0711265

Noted, all statistical characteristics of the MAEs distribution in relation to various $VRCM_i$ for i=1,…,15 of CM(LTI_2) values monotonically grow in both cases. This phenomenon ascertains that the proposed measure of the quality of PR estimation in relation to PCM-CM outperforms other commonly known or recently introduced means of PCM consistency control which were examined during this research. The paramount position of the CM(LTI_2) is additionally strengthened by the fact that its performance improves significantly for higher numbers of alternatives without regard to which PP is employed.

It should be noted that all characteristics presented herein are of great importance in MCDM, because one has to consider the potential of rejecting a "good" PCM, and vice versa i.e. the possibility of acceptance a "bad" PCM, as in the classic SHTT. However, for first time in the course of the AHP development history, the possibility of selecting the level of trustworthiness and basing decisions on statistical facts has been demonstrated. For

instance, considering some hypothetic PCM for n=4, with its CM(LTI_2)≈0.319702 (Tab. 8), one can expect with 95% certainty that MAE should not exceed the value of 0.1208420.

Table 9 – Performance of the CM(LTI_2) index Statistical characteristics of the MAEs distribution in relation to various $VRCM_i$ for i=1,…,15 of CM(LTI_2) values. The results were generated for n=4 on the basis of **SA|2|** as the simulation algorithm and are based on 10,000 perturbed random reciprocal PCMs. The scenario assumed LUA as the PP.

i	$VRCM_i$ for CM(LTI_2)	Mean CM(LTI_2) in $VRCM_i$	p–quantiles of MAEs among ***w*** and ***w****(LUA)					Average MAEs among ***w*** and ***w****(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0.00, 0.0921)	0.0483805	0.0051862	0.0070132	0.0176693	0.0485092	0.0721248	0.0246818
2	[0.0921, 0.119)	0.107336	0.0065804	0.0087362	0.0223436	0.0668610	0.0901757	0.0310714
3	[0.119, 0.145)	0.130827	0.00728515	0.0096983	0.0248230	0.0756282	0.0986153	0.0345387
4	[0.145, 0.172)	0.159831	0.0097014	0.0126492	0.0318836	0.0839675	0.1048160	0.0417584
5	[0.172, 0.199)	0.185763	0.0108996	0.0147705	0.0390121	0.0867685	0.1087370	0.0464742
6	[0.199, 0.226)	0.212789	0.0127518	0.0171452	0.0444749	0.0906489	0.1110220	0.0502253
7	[0.226, 0.252)	0.239711	0.0168641	0.0221950	0.0483727	0.0943307	0.1121290	0.0538373
8	[0.252, 0.279)	0.266664	0.0191223	0.0243966	0.0499312	0.0963741	0.1128810	0.0561933
9	[0.279, 0.306)	0.292923	0.0210745	0.0265733	0.0536876	0.0971178	0.1136750	0.0590709
10	[0.306, 0.332)	0.318738	0.0222280	0.0280330	0.0570706	0.1018000	0.1224680	0.0622836
11	[0.332, 0.359)	0.344798	0.0229873	0.0290093	0.0582741	0.1054570	0.1267530	0.0640174
12	[0.359, 0.386)	0.371865	0.0237677	0.0299580	0.0592460	0.1080910	0.1309080	0.0652984
13	[0.386, 0.412)	0.399489	0.0243569	0.0309199	0.0612529	0.1118710	0.1350460	0.0678424
14	[0.412, 0.439)	0.424271	0.0245079	0.0311770	0.0630208	0.1197740	0.1443030	0.0707793
15	[0.439, ∞)	0.507848	0.0240355	0.0310822	0.0660800	0.1264270	0.1500270	0.0737878

At the same time, one can expect with 95% certainty that it will be higher than 0.0201740. Whether one decide to accept such a PCM or reject it, obviously depends on the quality requirements of PR estimation and the attitude regarding these errors. Indeed, the outcome of the research finally creates the potential for true consistency control in an unprecedented way i.e. directly related to the PR estimation quality.

Consider the following PV as ***w***=[0.345, 0.335, 0.32] of DM preferences for alternatives, A_1, A_2, A_3, respectively. Taking into consideration earlier assumed level of CM(LTI_2)≈0.319702, the order of alternatives ranks i.e. A_1=1, A_2=2, A_3=3, can be very deceptive, and is rather meaningless. Indeed, in such a situation one can expect with 95% certainty that MAE>0.0201740 which makes one aware that the true rank order of examined preferences may look otherwise, due to estimation errors related to DM inconsistency e.g. ***w***=[(0.345–0.04), (0.335+0.01), (0.32+0.03)]=[0.305, 0.345, 0.35], which designates a different order for alternatives ranks, A_1=3, A_2=2, A_3=1.

On the other hand, consider PV as ***w***=[0.6, 0.35, 0.05] of DM preferences for alternatives: A_1, A_2, A_3, consecutively, as previously. Again, assuming CM(LTI_2)≈0.319702, it can be anticipated with 95% certainty that MAE<0.1208420 which insures confidence in the order of alternatives ranks.

In order to conserve the length of the paper, but at the same time enable similar analyses concerning different numbers of alternatives the exemplary generalized (results are averaged for geometric scale and Saaty's scale applied fifty-fifty) characteristics of CM(LTI_2) performance for n>4, and for selected PP in appendices to this article are provided (Tables: A1–A2).

Concluding, this simulation framework a performance of different PCM-CMs in relation to implementation of the most popular PPs, preference scales, and number of alternatives were compared. The research findings can be stated as follows:

1) it is possible to significantly improve the quality of PR estimation when CM(LTI_2) is applied as the PCM-CM;
2) LLSM and LUA as the PP, differ insignificantly from the perspective of CM(LTI_2) performance, the same concerns other examined PP;
3) when the number of alternatives grows, the performance of examined PCM-CMs improves.

Conclusions and further research areas

The objective of the article was to generate answers to the following questions: *Is the REV as the PP necessary and sufficient for the AHP? Is the reciprocity of PCMs a reasonable condition leading to the betterment of the PRs estimation quality? Are PCM-CMs, commonly applied and embedded in the AHP, really conducive to the improvement of the PRs estimation quality?*

The thorough and seminal investigation which significantly upgrades the AHP methodology provides the following answers to these questions:

1) the REV as the PP is not necessary and sufficient for the AHP. Moreover, the research reveals two PP which outperform the REV;
2) the reciprocity of PCM in the AHP is the artificial condition and directly leads to deterioration of the PR estimation quality.
3) the commonly applied PCM-CMs embedded in the AHP, mislead and in consequence often directly lead to deterioration of the PR estimation quality.

Proposed: resign from known PCM-CMs embedded in the AHP in favor of CM(LTI_2) that can operate both types of PCM i.e. reciprocal and nonreciprocal, withhold the PCM reciprocity requirement from the AHP and consider the replacement of the REV as the PP within the AHP in favor of LUA or LLSM.

Certainly, there is a need for further research in the field. Firstly, one should examine the performance of CM(LTI_2) when nonreciprocal PCM are applied. Secondly, one may study its performance from the perspective of relative estimation errors, and last but not least, one could evaluate its performance from the perspective of the entire hierarchy as opposed to a single PCM.

To recapitulate; in conjunction with other contemporary and seminal research papers e.g. Grzybowski (2016); Kazibudzki (2016a, 2016b); García-Melón et al. (2016); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); Aguarón, Escobar & Moreno-Jiménez (2014); Lin, Kou & Ergu (2013); Brunelli, Canal & Fedrizzi (2013), the results of this scientific research enriches the state of knowledge about the true value of the AHP which is widely recognized as an applicable MCDM support system. Hopefully, the results of this freshly finished authentic examination will improve the quality of human's prospective choices.

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Appendices

Table A1 – Performance of CM(LTI_2) index under the action of LLSM as the PP. Statistical characteristics of the MAEs distribution in relation to various levels of CM(LTI_2) within a given VRCM$_i$ for i=1,…,15. The results are based on 10,000 perturbed random reciprocal PCMs (geometric and Saaty's scales applied fifty-fifty), and were generated on the basis of **SA|2|** as the simulation algorithm. The table contains results for $n\in\{5, 6, 7, 8, 9\}$, presented consecutively.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0899)	0.057912	0.0039186	0.0049954	0.0109799	0.0221753	0.0274887	0.0127898
2	[0.0899, 0.107)	0.099124	0.0056158	0.0073876	0.0157136	0.0324243	0.0398139	0.0183201
3	[0.107, 0.124)	0.116088	0.0063140	0.0079525	0.0184299	0.0389673	0.0490687	0.0214159
4	[0.124, 0.142)	0.133907	0.0075132	0.0102233	0.0230429	0.0443459	0.0539668	0.0258898
5	[0.142, 0.159)	0.151127	0.0099921	0.0129535	0.0261046	0.0486044	0.0581258	0.0290851
6	[0.159, 0.176)	0.167911	0.0113191	0.0142543	0.0289546	0.0558904	0.0682777	0.0328936
7	[0.176, 0.193)	0.184671	0.0125612	0.0158052	0.0320054	0.0594491	0.0730399	0.0357402
8	[0.193, 0.211)	0.201896	0.0136853	0.0171375	0.0339101	0.0640703	0.0789391	0.0380755
9	[0.211, 0.228)	0.219329	0.0142803	0.0178080	0.0361548	0.0711273	0.0839402	0.0408705
10	[0.228, 0.245)	0.236371	0.0150518	0.0185369	0.0380656	0.0762136	0.0919801	0.0435024
11	[0.245, 0.262)	0.253302	0.0161087	0.0208189	0.0405464	0.0789105	0.0929572	0.0462684
12	[0.262, 0.280)	0.270523	0.0160427	0.0205586	0.0431223	0.0821647	0.0965329	0.0482168
13	[0.280, 0.297)	0.288211	0.0165698	0.0209757	0.0457022	0.0865715	0.100490	0.0504072
14	[0.297, 0.314)	0.305099	0.0177870	0.0226112	0.0455671	0.0859316	0.100544	0.0507868
15	[0.314, ∞)	0.357080	0.0186614	0.0241816	0.0493007	0.0932224	0.107664	0.0547348
I	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0901)	0.0618775	0.0036042	0.0044511	0.0090509	0.0175099	0.0212066	0.0102634
2	[0.0901, 0.102)	0.096694	0.00584585	0.0072472	0.0147877	0.0255157	0.0297767	0.0158427
3	[0.102, 0.115)	0.109186	0.0071783	0.0088408	0.0167774	0.0304119	0.0360603	0.0186253
4	[0.115, 0.127)	0.121228	0.00831565	0.0102091	0.0192100	0.0349865	0.0420209	0.0214601
5	[0.127, 0.139)	0.133028	0.0088771	0.0109435	0.0208206	0.0393504	0.0481357	0.0236802
6	[0.139, 0.151)	0.144977	0.0097898	0.0118208	0.0225534	0.0439163	0.0538868	0.0259512
7	[0.151, 0.163)	0.156874	0.0101678	0.0126009	0.0248914	0.0500528	0.0613696	0.0288113
8	[0.163, 0.176)	0.169306	0.0113233	0.0138144	0.0274455	0.0552421	0.0656847	0.0317599
9	[0.176, 0.188)	0.181783	0.0120341	0.0147276	0.0297646	0.0587297	0.0700824	0.0339487
10	[0.188, 0.200)	0.193745	0.0124796	0.0157621	0.0317564	0.0613300	0.0720410	0.0356610
11	[0.200, 0.212)	0.205758	0.0137977	0.0167981	0.0329443	0.0622977	0.0721443	0.0368687
12	[0.212, 0.225)	0.218204	0.0140878	0.0175574	0.0347152	0.0652521	0.0774105	0.0386492
13	[0.225, 0.237)	0.230723	0.0140705	0.0177333	0.0369638	0.0672822	0.0764555	0.0402684

14	[0.237, 0.249)	0.242818	0.0146810	0.0186397	0.0381558	0.0692375	0.0786225	0.0413928
15	[0.249, ∞)	0.279499	0.0168309	0.0207854	0.0401272	0.0721349	0.0829652	0.0439267
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.07975)	0.061626	0.00329141	0.0040063	0.0079292	0.0153184	0.017980	0.0089902
2	[0.07975, 0.09)	0.085354	0.00558449	0.0066781	0.0124836	0.0217916	0.0254877	0.0136394
3	[0.09, 0.10)	0.095128	0.00622084	0.0074651	0.0136346	0.0241580	0.0288301	0.0151410
4	[0.10, 0.11)	0.105046	0.00677146	0.0081844	0.0150571	0.0277432	0.0343089	0.0170348
5	[0.11, 0.12)	0.114884	0.00728075	0.0089529	0.0164708	0.0329745	0.0408782	0.0192642
6	[0.12, 0.13)	0.124902	0.00792417	0.0097471	0.0185168	0.0378364	0.0464765	0.0217170
7	[0.13, 0.14)	0.134949	0.00851189	0.0104389	0.0202075	0.0415434	0.0507830	0.0236614
8	[0.14, 0.15)	0.144883	0.00952136	0.0115606	0.0224145	0.0446314	0.0531641	0.0257116
9	[0.15, 0.161)	0.155416	0.0101888	0.0121602	0.0241178	0.0465694	0.0553538	0.0275101
10	[0.161, 0.171)	0.165845	0.0110535	0.0132394	0.0261677	0.0499157	0.0583309	0.0293786
11	[0.171, 0.181)	0.175874	0.0116123	0.0139639	0.0273006	0.0515428	0.0596329	0.0304575
12	[0.181, 0.191)	0.185981	0.0121824	0.0150547	0.0293308	0.0532065	0.0613544	0.0320030
13	[0.191, 0.201)	0.195819	0.0122294	0.0152015	0.0299135	0.0553010	0.0642011	0.0330142
14	[0.201, 0.211)	0.205937	0.0132402	0.0164008	0.0321805	0.0552598	0.0636846	0.0343310
15	[0.211, ∞)	0.235348	0.0147413	0.0179580	0.0321805	0.0586515	0.0682411	0.0363445
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.06861)	0.056493	0.0029930	0.0036359	0.0071723	0.0129253	0.0151100	0.0079193
2	[0.06861, 0.078)	0.073616	0.0047668	0.0056647	0.0098201	0.0165545	0.0197820	0.0107659
3	[0.078, 0.087)	0.082558	0.0051148	0.00615425	0.0108293	0.0189720	0.0232764	0.0121106
4	[0.087, 0.095)	0.090957	0.0054815	0.0067644	0.0120701	0.0230822	0.0289636	0.0139455
5	[0.095, 0.104)	0.0995085	0.0062208	0.0074045	0.0134360	0.0267488	0.0338094	0.0157422
6	[0.104, 0.113)	0.108507	0.0065308	0.0079148	0.0148310	0.0307300	0.0379708	0.0175495
7	[0.113, 0.122)	0.117503	0.0073636	0.0087983	0.0166204	0.0342287	0.0402093	0.0192815
8	[0.122, 0.131)	0.126447	0.0077367	0.00920785	0.0182781	0.0366835	0.0432579	0.0209778
9	[0.131, 0.140)	0.135467	0.0081883	0.0099817	0.0200944	0.0391024	0.0463982	0.0227669
10	[0.140, 0.149)	0.144406	0.00893715	0.0109052	0.0215995	0.0404294	0.0465999	0.0240918
11	[0.149, 0.158)	0.153395	0.0096365	0.0118788	0.0228543	0.0420224	0.0488816	0.0252208
12	[0.158, 0.167)	0.162379	0.0105213	0.0128739	0.0250637	0.0441591	0.0509963	0.0270496
13	[0.167, 0.176)	0.171319	0.0109917	0.0133182	0.0253654	0.0446033	0.0525163	0.0275815
14	[0.176, 0.185)	0.180246	0.0120041	0.0144395	0.0266159	0.0464516	0.0529339	0.0289197
15	[0.185, ∞)	0.205854	0.0127740	0.0155662	0.0283804	0.0479564	0.0549310	0.0304352
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.059795)	0.051197	0.0026372	0.0031677	0.0061278	0.0107141	0.0122502	0.0066588
2	[0.05979, 0.068)	0.064092	0.0040166	0.0047872	0.0079722	0.0133870	0.0158901	0.0087678
3	[0.068, 0.076)	0.072019	0.0044127	0.0052033	0.0089180	0.0154595	0.0189767	0.0099818
4	[0.076, 0.085)	0.080495	0.0046625	0.0055923	0.0098746	0.0183837	0.0234965	0.0114040
5	[0.085, 0.093)	0.089047	0.0052813	0.0062378	0.0110158	0.0221043	0.0279174	0.0129826
6	[0.093, 0.101)	0.097017	0.0056575	0.0067669	0.0124636	0.0261396	0.0326188	0.0147615
7	[0.101, 0.109)	0.105051	0.0061505	0.0074036	0.0138920	0.0290254	0.0358010	0.0164984
8	[0.109, 0.118)	0.113488	0.0066692	0.0079686	0.0153474	0.0308319	0.0365922	0.0177312
9	[0.118, 0.126)	0.122009	0.0073133	0.0087907	0.0171076	0.0330852	0.0388189	0.0193438
10	[0.126, 0.134)	0.129857	0.0076181	0.0092912	0.0186416	0.0343317	0.0401595	0.0204982
11	[0.134, 0.142)	0.137970	0.0083801	0.0102174	0.0199779	0.0355818	0.0416818	0.0216939
12	[0.142, 0.151)	0.146298	0.0091112	0.0107807	0.0212040	0.0376109	0.0430078	0.0229256
13	[0.151, 0.159)	0.154883	0.0097330	0.0118942	0.0219635	0.0378245	0.0435528	0.0237785
14	[0.159, 0.167)	0.162793	0.0102563	0.0125995	0.0228089	0.0390630	0.0443591	0.0244409
15	[0.167, ∞)	0.184864	0.0115601	0.0138072	0.0242891	0.0403012	0.046879	0.0259996

Table A2 – Performance of CM(LTI_2) index under the action of LUA as the PP. Statistical characteristics of the MAEs distribution in relation to various levels of CM(LTI_2) within a given VRCM$_i$ for i=1,…,15. The results are based on 10,000 perturbed random reciprocal PCMs (geometric and Saaty's scales applied fifty-fifty), and were generated on the basis of **SA|2|** as the simulation algorithm. The table contains results for $n\in${5, 6, 7, 8, 9}, presented consecutively.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.08867)	0.057344	0.0040500	0.0051112	0.0109956	0.0223145	0.0281738	0.0129222
2	[0.08867, 0.106)	0.097631	0.0051370	0.0066813	0.0149743	0.0314682	0.0402326	0.0179091
3	[0.106, 0.123)	0.115154	0.0062273	0.0080033	0.0178583	0.0404214	0.0493824	0.0215433
4	[0.123, 0.140)	0.132429	0.0077722	0.0103867	0.0235191	0.0443250	0.0533440	0.0260091
5	[0.140, 0.158)	0.149841	0.0100660	0.0130848	0.0264063	0.0492151	0.0598150	0.0293785
6	[0.158, 0.175)	0.166943	0.0122130	0.0152940	0.0305507	0.0567287	0.0669818	0.0339792
7	[0.175, 0.192)	0.183544	0.0134146	0.0168104	0.0341529	0.0622582	0.0730947	0.0376190
8	[0.192, 0.209)	0.200556	0.0144681	0.0180775	0.0371079	0.0664060	0.0801886	0.0405209
9	[0.209, 0.227)	0.217798	0.0152484	0.0195489	0.0387389	0.0726297	0.0886177	0.0432569
10	[0.227, 0.244)	0.235136	0.0161576	0.0201625	0.0403835	0.0771441	0.0945720	0.0454089
11	[0.244, 0.261)	0.252143	0.0164634	0.0205743	0.0428687	0.0812496	0.0997771	0.0479053
12	[0.261, 0.278)	0.269128	0.0174125	0.0217309	0.0445472	0.0844031	0.1015070	0.0498806
13	[0.278, 0.296)	0.286560	0.0184856	0.0235664	0.0474587	0.0907092	0.1046180	0.0527022
14	[0.296, 0.313)	0.304366	0.0176996	0.0228077	0.0479535	0.0900992	0.1047390	0.0532388
15	[0.313, ∞)	0.354236	0.0192203	0.0244908	0.0503011	0.0929620	0.1098880	0.0556579
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.09033)	0.063185	0.0035880	0.0044267	0.0089867	0.0179120	0.0222610	0.0103740
2	[0.09033, 0.103)	0.097359	0.0063423	0.0078999	0.0155573	0.0273932	0.0319008	0.0168600
3	[0.103, 0.115)	0.109410	0.00722965	0.0091969	0.0177403	0.0325171	0.0386338	0.0196173
4	[0.115, 0.127)	0.121343	0.0084960	0.0107027	0.0210117	0.0381138	0.0455231	0.0233632
5	[0.127, 0.139)	0.133108	0.0094898	0.0116495	0.0229152	0.0420076	0.0524375	0.0257261
6	[0.139, 0.152)	0.145538	0.0108994	0.0132421	0.0253036	0.0481306	0.0607595	0.0287714
7	[0.152, 0.164)	0.157728	0.0114276	0.0139558	0.0271455	0.0539605	0.0656762	0.0310679
8	[0.164, 0.176)	0.169681	0.0121504	0.0150640	0.0292961	0.0575169	0.0691811	0.0336092
9	[0.176, 0.189)	0.182266	0.0128272	0.0159433	0.0313538	0.0612481	0.0726740	0.0356428
10	[0.189, 0.201)	0.194801	0.0138039	0.0173398	0.0328835	0.0629099	0.0736509	0.0370798
11	[0.201, 0.213)	0.206815	0.0140270	0.0173321	0.0347352	0.0651391	0.0772600	0.0387081
12	[0.213, 0.225)	0.218752	0.0145777	0.0184396	0.0366233	0.0669927	0.0785993	0.0400494
13	[0.225, 0.238)	0.231199	0.0152711	0.0185568	0.0383407	0.0700653	0.0825784	0.0420203
14	[0.238, 0.250)	0.243637	0.0158614	0.0199098	0.0378087	0.0715807	0.0841889	0.0422926
15	[0.250, ∞)	0.280974	0.0171267	0.0210891	0.0412814	0.0738420	0.0859174	0.0450109
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.07955)	0.061285	0.0032254	0.0039749	0.0080564	0.0157394	0.0187853	0.0091669
2	[0.07955, 0.09)	0.085237	0.0055756	0.00703635	0.0133354	0.0239340	0.0276372	0.0145973
3	[0.09, 0.10)	0.095217	0.0067076	0.00822896	0.0151476	0.0267136	0.0313365	0.0168334
4	[0.10, 0.11)	0.105055	0.0071842	0.0087319	0.0167347	0.0300692	0.0372910	0.0187971
5	[0.110, 0.120)	0.114948	0.0078851	0.0096590	0.0184274	0.0350956	0.0451092	0.0211768
6	[0.120, 0.130)	0.124976	0.00841585	0.0104132	0.0202074	0.0399982	0.0487099	0.0233008
7	[0.130, 0.140)	0.134961	0.0094352	0.0114042	0.0217320	0.0442261	0.0533524	0.0258727
8	[0.140, 0.150)	0.144948	0.0097585	0.0120015	0.0234560	0.0471241	0.0565387	0.0269760
9	[0.150, 0.160)	0.154896	0.0105670	0.0127710	0.0253768	0.0489510	0.0573372	0.0286380
10	[0.160, 0.171)	0.165237	0.0113250	0.0138414	0.0270817	0.0499840	0.0598609	0.0301303
11	[0.171, 0.181)	0.175843	0.0120019	0.0146386	0.0287811	0.0534214	0.0617005	0.0317433
12	[0.181, 0.191)	0.185759	0.0126595	0.0153651	0.0298787	0.0548531	0.0642158	0.0329712
13	[0.191, 0.201)	0.195725	0.0128283	0.0154763	0.0313703	0.0562006	0.0639533	0.0339822
14	[0.201, 0.211)	0.205744	0.0139552	0.0170698	0.0326421	0.0578949	0.0674790	0.0354460
15	[0.211, ∞)	0.235674	0.0149229	0.0180402	0.0341664	0.0601557	0.0694405	0.0371047
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0688)	0.056836	0.0030787	0.0036205	0.0073926	0.0145693	0.0176789	0.0084662
2	[0.0688, 0.078)	0.073666	0.0051105	0.0061309	0.0109223	0.0189602	0.0222010	0.0121584
3	[0.078, 0.087)	0.082561	0.0055045	0.0066116	0.0121236	0.0213335	0.0264824	0.0136680

4	[0.087, 0.096)	0.091556	0.0059645	0.0072229	0.0133996	0.0253087	0.0332026	0.0154277
5	[0.096, 0.105)	0.100528	0.0063149	0.0077633	0.0145624	0.0296739	0.0375550	0.0170392
6	[0.105, 0.114)	0.109496	0.0069818	0.0085509	0.0161950	0.0334936	0.0413081	0.0189646
7	[0.114, 0.123)	0.118520	0.0075613	0.0090468	0.0175431	0.0357544	0.0434245	0.0204242
8	[0.123, 0.132)	0.127416	0.0079922	0.0097597	0.0192472	0.0395417	0.0473757	0.0223206
9	[0.132, 0.141)	0.136453	0.0086852	0.0105264	0.0209858	0.0407715	0.0482480	0.0238363
10	[0.141, 0.150)	0.145395	0.0097858	0.0118143	0.0228944	0.0419213	0.0491251	0.0254135
11	[0.150, 0.159)	0.154468	0.0100095	0.0125284	0.0241689	0.0438311	0.0507983	0.0265045
12	[0.159, 0.168)	0.163355	0.0106847	0.0132036	0.0259776	0.0454986	0.0526910	0.0279660
13	[0.168, 0.177)	0.172363	0.0111935	0.0138710	0.0271393	0.0471614	0.0543407	0.0291270
14	[0.177, 0.186)	0.181389	0.0124251	0.0149686	0.0273290	0.0474666	0.0540064	0.0294350
15	[0.186, ∞)	0.206400	0.0133439	0.0162347	0.0294684	0.0496426	0.0567018	0.0315851
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.05999)	0.051571	0.0027899	0.0033529	0.0066145	0.0129846	0.0152304	0.00757575
2	[0.05999, 0.068)	0.064108	0.00425595	0.0049956	0.0089923	0.0154707	0.0180262	0.0099654
3	[0.068, 0.076)	0.0720425	0.0045459	0.0053806	0.0096431	0.0170663	0.0215704	0.0110540
4	[0.076, 0.085)	0.080557	0.00484755	0.0059128	0.0107496	0.0200923	0.0264333	0.0124749
5	[0.085, 0.093)	0.089092	0.0053555	0.0064602	0.0118465	0.0241501	0.0307256	0.0140711
6	[0.093, 0.101)	0.096974	0.0059146	0.0070858	0.0130955	0.0270450	0.0340523	0.0154649
7	[0.101, 0.109)	0.105011	0.0064092	0.0077271	0.0143894	0.0307524	0.0372307	0.0172088
8	[0.109, 0.118)	0.113412	0.0070543	0.0083974	0.0158975	0.0323549	0.0386885	0.0184995
9	[0.118, 0.126)	0.121966	0.0076984	0.0092627	0.0180989	0.0344896	0.0405523	0.0204213
10	[0.126, 0.134)	0.129881	0.0081201	0.0097966	0.0194020	0.0364334	0.0431290	0.0216106
11	[0.134, 0.142)	0.137905	0.0088861	0.0106243	0.0206471	0.0371653	0.0437476	0.0225820
12	[0.142, 0.151)	0.146345	0.0094580	0.0114928	0.0216677	0.0380785	0.0443759	0.0235540
13	[0.151, 0.159)	0.154771	0.0101348	0.0124254	0.0231652	0.0396143	0.0456328	0.0248599
14	[0.159, 0.167)	0.162795	0.0108284	0.0129963	0.0234617	0.0396838	0.0458155	0.0250440
15	[0.167, ∞)	0.184832	0.0122591	0.0145491	0.0255388	0.0426019	0.0490175	0.0273476

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A New Definition of Consistency of Pairwise Comparisons

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Abstract—A new definition of consistency is introduced. It allows us to locate the roots of inconsistency and is easy to interpret. It also forms a better basis than the old eigenvalue consistency for selecting a threshold based on common sense. The new definition of consistency is applicable to expert systems and to knowledge acquisition. It is instrumental in applications of fuzzy sets and the theory of evidence where the definitions of the membership and belief functions are fundamental issues.

A method of pairwise comparison introduced by Thurstone [1] in 1927 was a milestone in decision making science. It is comparable to the introduction of derivatives in calculus or eigenvalues in linear algebra. The decision making process nearly always involves some kind of constituency in modern democratic societies. We have various boards of governors or directors, committees, task groups, city councils, panels of experts, etc. Stormy discussion and various ways of dispute, reasoning, and argumentation take place to arrive at certain decisions. Most constituencies have worked out precise and practical policies for running meetings in an orderly and effective way. What we lack, however, is a device for drawing solid conclusions, and very often the loudest individual wins! Unfortunately, loudness does not necessarily go along with wisdom. Casual thinking does not work well in predicting complex outcomes. Casual thinking is partial, fragmentary, and has no effective way to measure intangibles. In the decision making process, many factors must be considered simultaneously and with about the same degree of importance, therefore an approach with more finesse is necessary to obtain a clear and unambiguous conclusion. It has been shown by numerous examples [2,3] that the pairwise comparison method can always be used to draw the final conclusions in a comparatively easy and elegant way. The brilliance of the pairwise comparison could be reduced to a common sense rule: take two at a time if you are unable to handle more than that.

It is intriguing why such a natural and powerful tool has never become widely accepted by decision makers despite its extreme practicality. The author of this paper truly believes that failure of the pairwise comparison method to become more popular has its roots in the consistency definition. The consistency definition is given by the following formula:

$$\mathrm{cf} = \frac{\lambda_{\mathbf{A}} - \mathrm{order}(\mathbf{A})}{(\mathrm{order}(\mathbf{A}) - 1)\,\lambda_{\mathrm{random}}}, \tag{1}$$

for a reciprocal matrix: $\mathbf{A} = [w_{ij}]$, where w_{ij} is a positive element which expresses the relative importance of two stimuli: i and j.

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Each w_{ij} element has the reciprocal property $w_{ij} = 1/w_{ji}$. For consistent reciprocal matrices, we also have $w_{ij} = w_{ik}/w_{jk}$. In the case of three stimuli, our reciprocal matrix reduces to:

$$\mathbf{R}_3 = \begin{bmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{bmatrix},$$

where a expresses a referee's relative preference of stimulus 1 over 2, b expresses preference of stimulus 1 over 3, and c is a relative preference of stimulus 2 over stimulus 3.

We will call matrix $\mathbf{R}_3$ *a basic reciprocal matrix*, since $\mathbf{R}_1$ is a trivial case and $\mathbf{R}_2$ is always consistent. Matrix $\mathbf{R}_3$ is consistent if, and only if, $b = a * c$.

Saaty [4] introduced eigenvalues for handling a gradation in comparative judgments in 1977. Saaty is probably the greatest single contributor to the popularization of the pairwise comparison method. It is no wonder that his heuristic of checking the consistency has not been objectively (or probably even not carefully) looked at. Saaty's consistency is based on 10% of the deviation of the largest eigenvalue of a given matrix from the corresponding eigenvalue of a matrix randomly generated. Its major attraction is universalism: Saaty's consistency definition is good for any order of a matrix. The major drawback of Saaty's consistency definition is the rather unfortunate threshold of 10%. The author of this paper does not believe in rounded numbers like 10% and decided to look at the consistency problem more closely. (His research was dictated by practical problems with consistency of judgments related to ranking hazards of abandoned mines in Canada).

The other major problem is less obvious, but hardly less important. Eigenvalues are surrounded by a certain enigma—not many of us can comprehend the meaning of eigenvalues, but nearly all of us have a certain respect for them, let alone total admiration. In fact, there is clear interpretation of eigenvalues while we have a quite clear view of the consistency of judgments. It will be explained, in the Conclusions, that we do need to have an interpretation of consistency of judgments. This interpretation should be somehow consistent with common sense. The third weakness of Saaty's consistency definition is related to location of inconsistency or rather lack of it. An eigenvalue is a global characteristic of a matrix. By examining it we cannot say which matrix element contributed to the increase of consistency. Costly computations are needed for a matrix of higher than the third order. The above three disadvantages are associated together and a new more intuitive definition of consistency must be introduced.

It is worthwhile to note that for a small (how small is small?) deviation of a matrix, its eigenvalues do not change much, but there is no proof that it works the other way around. We expect that bigger changes in the matrix will cause bigger variations of eigenvalues. However, we cannot be even sure that our reciprocal matrix (which is of a very special shape) will behave this way. Saaty proved that the largest eigenvalue of a reciprocal matrix is equal to the order of the matrix if and only if the matrix is consistent. In lack of any analytic indication that reciprocal matrices behave well as far as eigenvalue change is concern, a brute force method was employed for examining the relationship between consistency and exactness of the solution: all possible cases (4913!) of the reciprocal matrix of the order of three were computed for all judgement values proposed by Saaty from 1 to 9 (and associated with them inverted values $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}$). The results showing the relationship between eigenvalues and the accuracy of corresponding solutions are shown in Figure 1.

Figure 1 shows that there is practical justification for using eigenvalues as a consistency measure. However, the threshold of 10% does not look apparent for any "practical" reasons. The 10% deviation of the biggest eigenvalue from the matrix order calculated by (1), as proposed by Saaty limits the solution accuracy to approximately 30%. The solution accuracy is computed as

$$\sqrt{\left(a - \frac{x_1}{x_2}\right)^2 + \left(b - \frac{x_1}{x_3}\right)^2 + \left(c - \frac{x_2}{x_3}\right)^2}, \tag{2}$$

where (x_1, x_2, x_3) is a normalized solution.

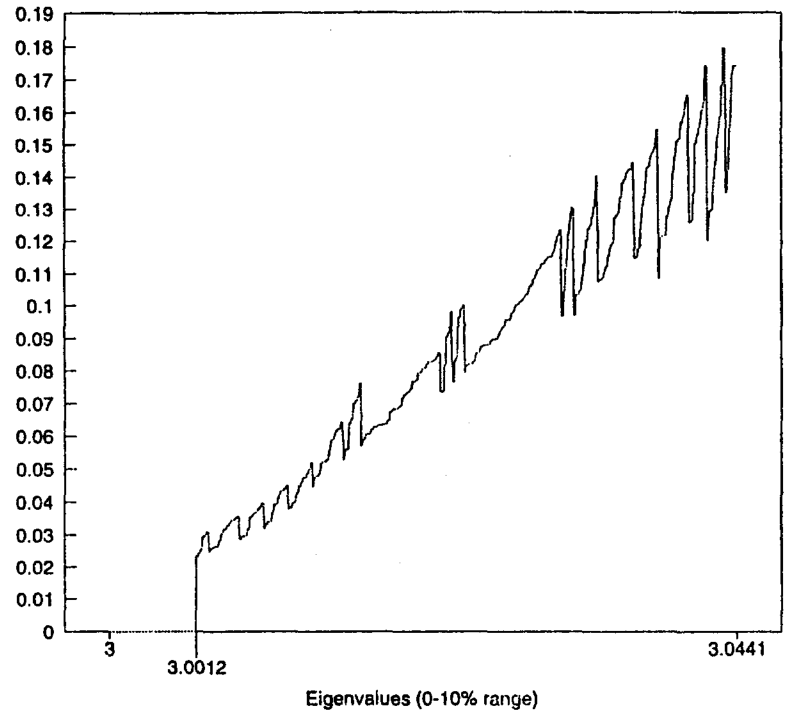

Figure 1. Relationship between eigenvalues and accuracy 0–10% range, as proposed by Saaty.

A careful reader may ask a couple of apparent questions: "Is consistency really so important in the pairwise comparison method?"; "Do we need to look at the consistency of judgments?"

Checking consistency in the pairwise comparison method could be compared in checking that the divisor in a planned division operation is not equal to 0. Simply it does not make sense to divide anything by 0. Similarly all proposed (heuristic) solutions to pairwise comparison models are based on an assumption that the given reciprocal matrix is consistent [4]. Can't we simply assume that the reciprocal matrix is consistent? The answer is definitely not! The power of the pairwise comparison method would be diminished should we ever request that all the judgments are consistent, since we know that most judgements are subjective and nearly always contain some kind of bias. The simplest case of combinatorial consistency was analyzed in [5]. It could be reduced to a case of three stimuli (criteria or attributes) A, B, and C. We assume that $A > B$ and $B > C$, but we also insist on claiming that $C > A$. The definition of a fully consistent reciprocal matrix was introduced in [5], but the only widely accepted measure of inconsistency is due to Saaty.

Our definition of a basic reciprocal matrix $\mathbf{R}_3(a, b, c)$ is based on clear intuition of consistency: it is a measure of deviation from the nearest consistent reciprocal matrix. The interpretation of the consistency measure becomes more apparent when we reduce a basic reciprocal matrix to a vector of three coordinates $[a,\ b,\ c]$. We know that $b = a * c$ holds for each consistent basic reciprocal matrix. We can always produce three consistent basic reciprocal matrices (represented by three vectors) by computing one coordinate from the combination of the remaining two coordinates. These three vectors are: $\left[\frac{b}{c},\ b,\ c\right]$, $[a,\ a*c,\ c]$, and $\left[a,\ b,\ \frac{b}{a}\right]$. The inconsistency measure will be defined as the relative distance to the nearest consistent basic reciprocal matrix represented by one of these three vectors for a given metric. In case of Euclidean (or Chebysheff) metrics, we have:

$$\mathrm{CM} = \min\left(\frac{1}{a}\left|a - \frac{b}{c}\right|,\ \frac{1}{b}\left|b - a\,c\right|,\ \frac{1}{c}\left|c - \frac{b}{a}\right|\right),$$

We can easily extend the above definition to matrices of higher orders. For a given matrix element it can be defined as maximum of CM of all possible triads which include this element. The new relationship is shown in Figure 2.

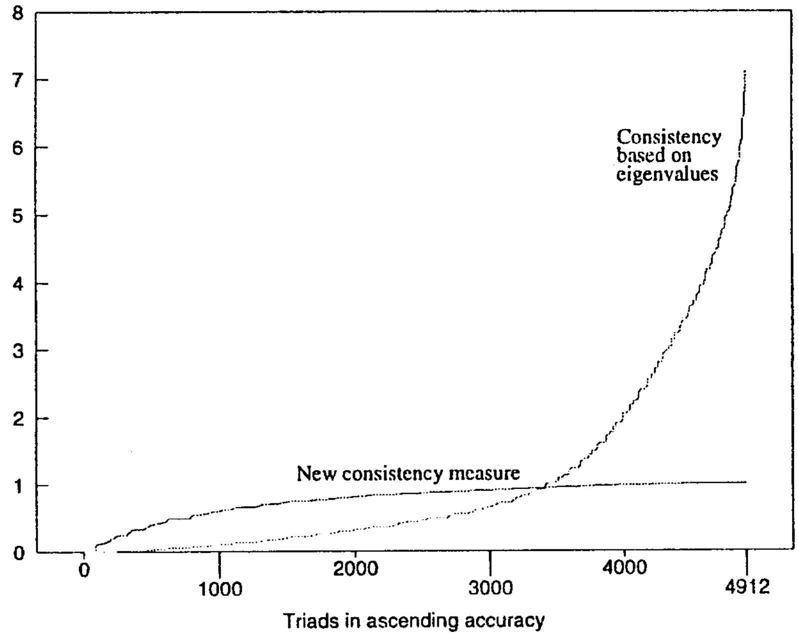

Figure 2. Comparison of the new consistency measure with the consistency based on eigenvalues.

We can easily see that the new consistency measure (CM):

— is easy to interpret (relative deviation from a consistent matrix which may be obtained by keeping two judgments invariant),
— forms a better basis for selecting a threshold based on common sense (e.g., CM of [1, 3, 2] or [3, 6, 3] or [3, 9, 4] or even [2, 8, 3] is 0.25),
— locates the consistency (CM is always associated with a certain matrix element and it is not an enigmatic global matrix characteristic like an eigenvalue).

A very basic question remains: "How is the new consistency definition related to eigenvalues?". The answer to this question is not easy, but the relationship was a surprise even for the author of this paper. With the accuracy of computations, both indicators of consistency remain constant for certain segments or clusters of cases of triads, but the new consistency generates finer segments. For example see Table 1.

The new consistency measure, CM, forms segments with 0.01, 0.2, 0.06, and 0.02 for one corresponding segment of eigenvalues, EV = 3.0012. It is also worthy to note that the accuracy changes with CM rather than with eigenvalues.

It may not be easy to provide a full analytic proof of the above observed regularities, but we can still look at the definition of eigenvalues for a reciprocal matrix. Its characteristic equation is

$$\lambda^3 - 3\lambda^2 + \det(\mathbf{R}_3) = 0,$$

where $\det(\mathbf{R}_3)$ is a determinant of a basic reciprocal matrix which is:

$$\det(\mathbf{R}_3) = \frac{(a\,c - b)^2}{a\,b\,c}.$$

Table 1.

Accuracy	CM	EV
0.024198	0.01	3.0012
0.024198	0.01	3.0012
0.030071	0.01	3.0012
0.030071	0.01	3.0012
0.030924	0.01	3.0012
0.030924	0.01	3.0012
0.02322	0.2	3.0012
0.02322	0.2	3.0012
0.025206	0.06	3.0012
0.025206	0.06	3.0012
0.029384	0.02	3.0012
0.029384	0.02	3.0012
0.025684	0.33	3.0015
0.026003	0.25	3.0015
0.026003	0.25	3.0015
0.026303	0.11	3.0015
0.026303	0.11	3.0015

The largest eigenvalue of $\mathbf{R}_3$ is expressed by the following formula:

$$\lambda_{\max} = 1 + \sqrt[3]{\frac{\det(\mathbf{R}_3)}{2} - \frac{\sqrt{\det(\mathbf{R}_3)}}{2}\sqrt{\det(\mathbf{R}_3)+4} + 1} + \sqrt[3]{\frac{\det(\mathbf{R}_3)}{2} + \frac{\sqrt{\det(\mathbf{R}_3)}}{2}\sqrt{\det(\mathbf{R}_3)+4} + 1}\,.$$

(The author does not want to disappoint the reader, but must admit that the above formulas were obtained by Maple, a system for symbolic computation.) It is obvious that the changes of $\lambda_{\max}$ depends on $\det(\mathbf{RM}_3)$ which in turn depends on the changes of $a * c - b$. The new consistency is based on relationship $b = a * c$. Not only does it imply that there is some kind of alignment between Saaty's definition and the new definition of consistency, but it opens new avenues for future ones.

Conclusions

We hope that the new definition of consistency will refocus the attention of researchers from the race of finding better and better approximation of solutions (in forms of heuristics) for inconsistent matrices to devising heuristics which can influence judgments to be more consistent (but by no means totally consistent). Finding an ideal solution for inconsistent (or very inconsistent) matrices is a mirage. It is a theoretically challenging and exciting task but without much use. It could be compared to an attempt at finding lengths of bars measured by a rule which randomly changes its length (by, for example, extreme temperature or atmospheric pressure). The truth is that no "ideal" solution will help us unless we try to understand the source of our problem, which is the inconsistency of judgments. Certainly it is difficult to change the inconsistency unless we know not only its value but the exact location of it. Our definition allows us to locate the inconsistency. It gives the referee a necessary feedback and opportunity of reconsideration of his judgments by using various techniques (e.g., Delphi method). However, it is not advisable to allow the referee the total flexibility. Improvement of his subjective judgment may change to an unsubstantiated race for total consistency of judgments instead of his unbiased subjective opinions. We may, for example, allow the referee to change only a fixed number of opinions by a factor of a fixed total. For example, in case of a matrix of order 4 we have 6 judgments. In this case we may allow a maximum of three modifications with a restriction that the total of all changes does not exceed say 3. In other words each three judgments may be modified by one up or down, or one judgment may be modified by 3 up or down.

The pairwise comparison method is one of the most amazing and universal approaches to solving difficult problems. In particular, contributions to proving superiority of the pairwise comparison method in terms of higher precision of the solution are awaited. They are crucial for convincing (or at least encouraging) practitioners to employ the pairwise comparison method to decision making processes (e.g., regarding environmental problems). It will be of great benefit of all of us.

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Inconsistency in the ordinal pairwise comparisons method with and without ties

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Abstract

Comparing alternatives in pairs is a well-known method of ranking creation. Experts are asked to perform a series of binary comparisons and then, using mathematical methods, the final ranking is prepared. As experts conduct the individual assessments, they may not always be consistent. The level of inconsistency among individual assessments is widely accepted as a measure of the ranking quality. The higher the ranking quality, the greater its credibility.

One way to determine the level of inconsistency among the paired comparisons is to calculate the value of the inconsistency index. One of the earliest and most widespread inconsistency indices is the consistency coefficient defined by Kendall and Babington Smith. In their work, the authors consider binary pairwise comparisons, i.e., those where the result of an individual comparison can only be: better or worse. The presented work extends the Kendall and Babington Smith index to sets of paired comparisons with ties. Hence, this extension allows the decision makers to determine the inconsistency for sets of paired comparisons, where the result may also be "equal." The article contains a definition and analysis of the most inconsistent set of pairwise comparisons with and without ties. It is also shown that the most inconsistent set of pairwise comparisons with ties represents a special case of the more general set cover problem.

Keywords: pairwise comparisons, consistency coefficient, inconsistency, AHP

1. Introduction

The use of pairwise comparisons (PC) to form judgments has a long history. Probably the first who formally defined and used pairwise comparisons for decision making was *Ramon Llull* (the XIII century) [6]. He proposed a voting system based on binary comparisons. The subject of comparisons (alternatives) were people - candidates for office. Voters evaluated the candidates in pairs, deciding which one was better. In the XVIII century, *Llull's* method was rediscovered by *Condorcet* [7], then once again reinvented in the middle of the XX century by *Copeland* [6, 8]. At the beginning of the XX century, *Thurstone* used the pairwise comparisons method (PC method) quantitatively [39]. In this approach, the result returned does not only contain information about who or what is better, but also indicates how strong the preferences

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are. Later, both approaches, ordinal (qualitative), as proposed by *Llull,* and cardinal (quantitative), as used by *Thurstone*, were developed in parallel. Comparing alternatives in pairs plays an important role in research into decision making systems [14, 17, 29], ranking theory [34, 21], social choice theory [38], voting systems [40, 12, 41] and others.

In general, the PC method is a ranking technique that allows the assessment of the importance (relevance, usefulness, competence level etc.) of a number of alternatives. As it is much easier for people to assess two alternatives at a time than handling all of them at once, the PC method assumes that, first, all the alternatives are compared in pairs, then, by using an appropriate algorithm, the overall ranking is synthesized. The choice of the algorithm is not easy and is still the subject of research and vigorous debate [35, 42, 28]. Of course, it also depends on the nature of the comparisons. The cardinal methods use different algorithms [19, 13] than the ordinal ones [21, 6, 20, 40]. Despite the many differences between ordinal and cardinal pairwise comparisons, both approaches have much in common. For example, both approaches use the idea of inconsistency among individual comparisons. The notion of inconsistency introduced by the pairwise comparisons method is based on the natural expectation that every two comparisons of any three different alternatives should determine the third possible comparison among those alternatives.

To better understand the phenomenon of inconsistency, let us assume that we have to compare three alternatives c_1, c_2 and c_3 with respect to the same criterion. If after the comparison of c_1 and c_2 it is clear to us that c_2 is more important than c_2, and similarly, after comparing c_2 and c_3 it is evident that c_3 is more important than c_1 then we may expect that c_3 will also turn out to be more important than c_1. The situation in which c_1 is better than c_3 would raise our surprise and concern. That is because it seems natural to assume that the preferential relationship should be transitive. If it is not, we have to deal with inconsistency. As pairwise comparisons are performed by experts, who, like all human beings, sometimes make mistakes, the phenomenon of inconsistency is something natural. The ranking synthesis algorithm must take it into account. On the other hand, if a large number of such "mistakes" can be found in the set of paired comparisons, one can have reasonable doubts as to the credibility of the ranking obtained from such lower quality data.

Both ordinal and cardinal PC methods developed their own solutions for determining the degree of inconsistency. Research into the cardinal PC method resulted in a number of works on inconsistency indices. Probably the most popular inconsistency index was defined by *Saaty* in his seminal work on *the Analytic Hierarchy Process (AHP)* [34]. His work prompted others to continue the research [27, 32, 1, 37, 3, 5]. The ordinal PC methods also have their own ways of assessing the level of inconsistency. In their seminal work [26] *Kendall* and *Babington Smith* introduced the *inconsistency index* (called by the authors the *consistency coefficient*). Their index allows the inconsistency degree of a set composed of binary pairwise comparisons to be determined. The results obtained by the authors were the inspiration for many other researchers in different fields of science [23, 30, 31, 2, 4, 36].

Although the ordinal pairwise comparisons method is a really powerful and handy tool facilitating the right decision, in practice we very often face the problem that the two options seem to be equally important. In such a situation, we can try to get around the problem by a brute force method of breaking ties. For example, we can do this by *"instructing the judge to toss a mental coin when he cannot otherwise reach a decision; or, allowing him the comfort of reserving judgment, we can let a physical coin decide for him"* [9, p. 94 - 95]. It is clear, however, that instead of relying on more or less arbitrary methods of breaking ties, it is better to accept their existence and incorporate them into the model. Indeed, ties have been inextricably linked with the ranking theory for a long time [6, 25, 9]. The ordinal pairwise comparisons method with ties has its own techniques of synthesizing ranking [15, 10, 40]. In this perspective, research into the inconsistency of ordinal pairwise comparisons with ties is quite poor. In particular, the *consistency coefficient* as defined by [26] is not suitable for determining the inconsistency of PC with ties. The problem was recognized by *Jensen* and *Hicks* [22], and later by *Iida* [18]. These authors also made attempts to patch up this gap in the ranking theory, however, the fundamental question as to what extent the set of PC with ties can be inconsistent still remains unanswered.

The purpose of the present article is to answer this question, and thus to define the inconsistency index for the ordinal PC with ties in the same manner as *Kendall* and *Babington Smith* did [26] for binary PC. The definition of the inconsistency index is accompanied by a thorough study of the most inconsistent sets of pairwise comparisons with and without ties.

The article is composed of eight sections including the introduction and four appendices. The PC with ties is formally introduced in the next section (Sec. 2). For the purpose of modeling PC with ties, a generalized tournament graph has also been defined there. The most inconsistent set of binary PC is studied in (Sec. 3). It is also proven that the number of inconsistent triads in such a graph is determined by Kendall Babington Smith's *consistency coefficient.* The next section (Sec. 4) describes how the most inconsistent set of PC with ties may look. Thus, it contains several theorems describing the quantitative relationship between the elements of the generalized tournament graph. Finally, in (Sec. 5) the most inconsistent set of PC with ties is proposed. The generalized inconsistency index for ordinal PC is also defined (Sec. 6). The penultimate section (Sec. 7) contains a discussion of the subject. In particular, the relationship between the maximally inconsistent set of PC and the *NP-complete* set cover problem [24] is shown. A brief summary is provided in (Sec. 8).

2. Model of inconsistency

Let us suppose we have a number of possible choices (alternatives, concepts) $c_1, \ldots, c_n$ where we are able to decide only whether one is better (more preferred) than the other or whether both alternatives are equally preferred. In the first case, we will write that $c_i \prec c_j$ to denote that c_j is more preferred than c_i, whilst in the second case, to express that two alternatives c_i and c_j are equally preferred we write $c_i \sim c_j$. The preference relationship is total. Hence, for every two c_i and c_j it holds that either $c_i \prec c_j$, $c_j \prec c_i$ or $c_i \sim c_j$. The relationship is reflexive and asymmetric. In particular, we will assume that if $c_i \prec c_j$ then not $c_j \prec c_i$, and $c_i \sim c_i$ for every $i, j = 1, \ldots, n$. It is convenient to represent the relationship of preferences in the form of an $n \times n$ matrix.

Definition 1. *The $n \times n$ matrix $M = [m_{ij}]$ where $m_{ij} \in \{-1, 0, 1\}$ is said to be the* ordinal PC matrix *for n alternatives $c_1, \ldots, c_n$ if a single comparison m_{ij} takes the value 1 when c_i wins with c_j (i.e. $c_i \succ c_j$), -1 if, reversely, c_j is better than c_i (i.e. $c_j \succ c_i$) and 0 in the case of a tie between c_i and c_j ($c_i \sim c_j$). The the diagonal values are 0.*

The *PC matrix* is skew-symmetric except the diagonal, so that for every $i, j = 1, \ldots, n$ it holds that $m_{ij} + m_{ji} = 0$. An example of the ordinal PC matrix for five alternatives is given below (1).

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{pmatrix} \tag{1}$$

The *PC matrix* can be easily represented in the form of a graph.

Definition 2. *A tournament graph* (t-graph) *with n vertices is a pair $T = (V, E_d)$ where $V = c_1, \dots, c_n$ is a set of vertices and $E_d \subset V^2$ is a set of ordered pairs called directed edges, so that for every two distinct vertices c_i and c_j either $(c_i, c_j) \in E_d$ or $(c_j, c_i) \in E_d$.*

Let us expand the definition of a tournament graph so that it can also model the collection of pairwise comparisons with ties.

Definition 3. *The generalized tournament graph* (gt-graph) *with n vertices is a triple $G = (V, E_u, E_d)$ where $V = c_1, \dots, c_n$ is a set of vertices, $E_u \subset 2^V$ is a set of unordered pairs called undirected edges, and $E_d \subset V^2$ is a set of ordered pairs called directed edges, so that for every two distinct vertices c_i and c_j either $(c_i, c_j) \in E_d$ or $(c_j, c_i) \in E_d$ or $\{c_i, c_j\} \in E_u$.*

Wherever it increases the readability of the text the directed and undirected edges (c_i, c_j), (c_j, c_i), $\{c_i, c_j\}$ between $c_i, c_j \in V$ are denoted as $c_i \to c_j, c_j \to c_i$ and $c_i - c_j$ correspondingly.

It is easy to see that every tournament graph can easily be extended to a generalized tournament graph where $E_u = \emptyset$. Therefore, it will be assumed that every *t-graph* is also a *gt-graph,* but not reversely.

Definition 4. *A family of* t-graphs *with n vertices will be denoted as $\mathscr{T}_n^t$, where $\mathscr{T}_n^t = \{(V, E_d)$ is a t-graph, where $|V| = n\}$, and similarly, a family of* gt-graphs *with n vertices will be denoted as $\mathscr{T}_n^g$, where $\mathscr{T}_n^g = \{(V, E_u, E_d)$ is a gt-graph, where $|V| = n\}$*

It is obvious that for every $n > 0$ it holds that $\mathscr{T}_n^t \subsetneq \mathscr{T}_n^g$.

Definition 5. *A family of* gt-graphs *with n vertices and m directed edges will be denoted as $\mathscr{T}_{n,m}^g = \{(V, E_u, E_d)$ is a gt-graph, where $|V| = n$ and $|E_d| = m\}$*

Definition 6. *A* gt-graph *$G_M \in \mathscr{T}_n^g$ is said to correspond to the $n \times n$* ordinal PC matrix *$M = [m_{ij}]$ if every directed edge $(c_i, c_j) \in E_d$ implies $m_{ji} = 1$ and $m_{ij} = -1$, and every undirected edge $\{c_i, c_j\} \in E_u$ implies $m_{ij} = 0$.*

Definition 7. *All three mutually distinct vertices $t = \{c_i, c_k, c_j\} \subseteq V$ are said to be a triad. The vertex c is said to be contained by a triad $t = \{c_i, c_k, c_j\}$ if $c \in t$. A triad $t = \{c_i, c_k, c_j\}$ is said to be covered by the edge $(p, q) \in E_d$ if $p, q \in t$.*

Sometimes it will be more convenient to write a triad $t = \{c_i, c_k, c_j\}$ as the set of edges, e.g. $\{c_i \to c_k, c_k - c_j, c_i - c_j\}$. However, both notations are equivalent, the latter one allows the reader to immediately identify the type of triad.

Definition 8.

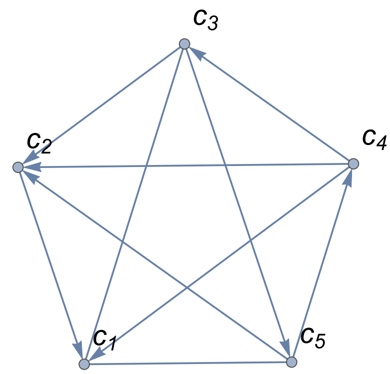
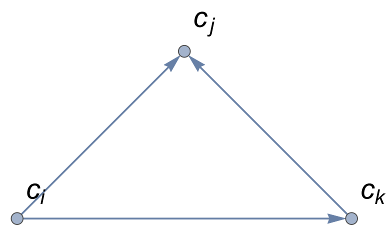
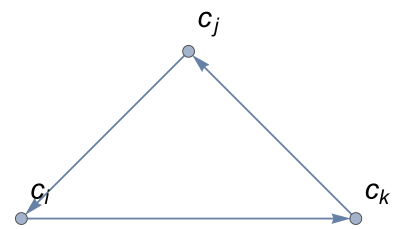


Figure 1: The *gt-graph* corresponding to the matrix M, see (1).



(a) CT_3 - a consistent triad covered by three directed edges



(b) IT_3 - an inconsistent (circular) triad covered by three directed edges

Figure 2: Triads for paired comparisons without ties

In their work, *Kendall and Babington Smith* dealt with the ordinal pairwise comparisons without ties [26]. Hence, in fact, they do not consider the situation in which $c_i \sim c_j$. For the same reason, their *ordinal PC matrices* had no zeros anywhere outside the diagonal[1]. For the purpose of defining the notion of inconsistency in preferences, they adopt the transitivity of the preference relationship. According to this assumption, every triad c_i, c_k, c_j of three different alternatives can be classified as consistent or inconsistent (contradictory). Providing that there are no ties between alternatives, there are two different kinds of triads (it is easy to verify that any other triad can be simply boiled down to one of these two by simple index changing). The first one $c_i \to c_k, c_k \to c_j$ and $c_i \to c_j$ hereinafter referred to as the consistent triad[2] CT_3, and $c_i \to c_k, c_k \to c_j$ and $c_j \to c_i$ termed hereinafter as the inconsistent triad IT_3 (Fig. 2).

Of course, the more inconsistent the triads in the *ordinal PC matrix*, the more inconsistent the set of preferences, hence the less reliable the conclusions drawn from the set of paired comparisons. To determine how inconsistent the given set of paired comparisons is, *Kendall and Babington Smith* [26] provide the maximal number of inconsistent triads in the $n \times n$ *PC*

[1] In fact, those matrices had no zeros as the authors inserted dashes on the diagonal [26].

[2] Index 3 means that this kind of triad is formed by three directed edges.

matrix without ties. Denoting the actual number of inconsistent triads in T_M by $|T_M|_i$, and the maximal possible number of inconsistent triads in $n \times n$ PC matrix M as $\mathcal{I}(n)$, we have [3]:

$$\mathcal{I}(n) = \begin{cases} \frac{n^3-n}{24} & \text{when} \quad \text{n is odd} \\ \frac{n^3-4n}{24} & \text{when} \quad \text{n is even} \end{cases} \tag{2}$$

Therefore, the inconsistency index for M defined in [26] is:

$$\zeta(M) = 1 - \frac{|T_M|_i}{\mathcal{I}(n)} \tag{3}$$

Unfortunately, including ties into consideration significantly complicates the scene. Besides the two types of triads CT_3 and IT_3 we need to take into consideration an additional five:

CT_0 - consistent triad of three equally preferred alternatives c_i, c_k and c_j such that $c_i \sim c_k, c_k \sim c_j$ and $c_i \sim c_j$.

IT_1 - inconsistent triad composed of three alternatives c_i, c_k and c_j such that $c_i \sim c_k, c_k \sim c_j$ and $c_i \prec c_j$.

IT_2 - inconsistent triad composed of three alternatives c_i, c_k and c_j such that $c_i \sim c_k, c_k \prec c_j$ and $c_j \prec c_i$.

CT_{2a} - consistent triad composed of three alternatives c_i, c_k and c_j such that $c_i \sim c_k, c_k \prec c_j$ and $c_i \prec c_j$.

CT_{2b} - consistent triad composed of three alternatives c_i, c_k and c_j such that $c_i \sim c_k, c_j \prec c_k$ and $c_j \prec c_i$.

The above triads can be easily represented as tournament graphs with ties (Fig. 4). With the increased number of different types of triads in a graph, the maximum number of inconsistent triads also increases. For example, according to (2) the maximum number of inconsistent triads in $\mathcal{I}(4)$ without ties is 2. When ties are allowed, the maximal number of inconsistent triads increases to 4, which is the total number of triads in every simple graph (i.e. with only one edge between one pair of vertices) with four vertices.

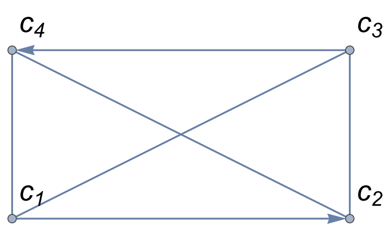
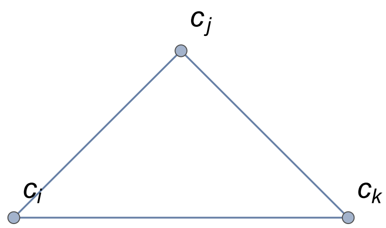


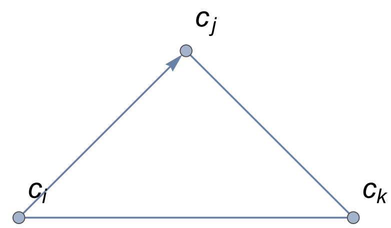
Figure 3: $\mathcal{I}(4)$ with four IT_1 triads

[3] As every $n \times n$ *ordinal PC matrix* M corresponds to some tournament graph T_n^* we also use the notation $|T_n^*|_i$ to express the number of inconsistent triads in it.

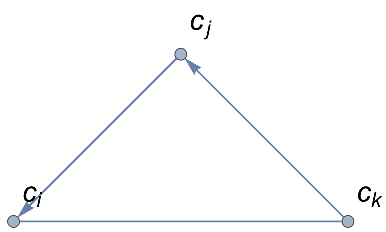
Let us analyze the graph in (Fig 3). It is easy to notice that it contains four IT_1 triads which are: $\{c_1 \rightarrow c_2,\, c_2 \mathbin{—} c_3,\, c_3 \mathbin{—} c_1\}$, $\{c_1 \rightarrow c_2,\, c_2 \mathbin{—} c_4,\, c_4 \mathbin{—} c_1\}$, $\{c_1 \mathbin{—} c_3,\, c_3 \rightarrow c_4,\, c_4 \mathbin{—} c_1\}$, and $\{c_2 \mathbin{—} c_3,\, c_3 \rightarrow c_4,\, c_4 \mathbin{—} c_1\}$. Thus, it is clear that the formulae (2) and (3) cannot be used to estimate inconsistency in preferences when ties are allowed. The desire to extend those concepts to paired comparisons with ties was the main motivation for writing the work.



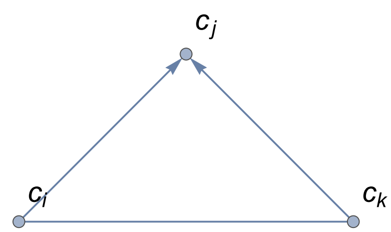
(a) CT_0 - a consistent triad not covered by any directed edge



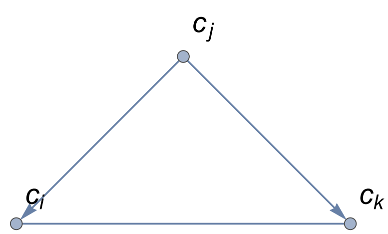
(b) IT_1 - an inconsistent triad covered by one directed edge



(c) IT_2 - an inconsistent triad covered by two directed edges



(d) CT_{2a} - a consistent triad covered by two directed edges. One alternative is more preferred than two others.



(e) CT_{2b} - a consistent triad covered by two directed edges. One alternative is less preferred than two others.

Figure 4: Triads specific for the pairwise comparisons with ties

3. The most inconsistent set of preferences without ties

To construct the most inconsistent set of pairwise preferences without ties, let us introduce a few definitions relating to the degree of vertices. Since every *t-graph* is also a *gt-graph* the definitions are formulated for the *gt-graph*.

Definition 9. *Let $G = (V, E_u, E_d)$ be a gt-graph and $c, d \in V$. Then input degree, output degree, undirected degree and degree of a vertex c are defined as follows: $deg_{in}(c) \stackrel{df}{=} |\{d \in V : d \to c \in E_d\}|$, $deg_{out}(c) \stackrel{df}{=} |\{d \in V : c \to d \in E_d\}|$, $deg_{un}(c) \stackrel{df}{=} |\{d \in V : c — d \in E_u\}|$ and $deg(c) \stackrel{df}{=} deg_{in}(c) + deg_{out}(c) + deg_{un}(c)$.*

Theorem 1. *Let $G = (V, E_u, E_d)$ from $\mathscr{T}^g_n$. Then every vertex $c \in V$, for which $deg_{in}(c) = k$ is contained by at least $\binom{k}{2}$ consistent triads of the type CT_{2a} or CT_3. Those triads are said to be introduced by c.*

Proof. Let $c_1, \ldots, c_k \in V$ be the vertices such that the edges $c_i \to c$ are in E_d. Since T is a *gt-graph* with n vertices, then for every c_i, c_j where $i, j = 1, \ldots, k$ there must exist an edge $c_i \to c_j$, $c_j \to c_i$ in E_d or $c_i — c_j$ in E_u. In the first two cases, the vertices c_i, c, c_j make a consistent triad type CT_{2a}, whilst in the latter case the vertices c_i, c, c_j form a consistent triad type CT_3. Since there are k vertices adjacent via the incoming edge to c there are at least as many different consistent triads containing c as two-element combinations of $c_1, \ldots, c_k$ i.e. $\binom{k}{2}$. See (Fig. 5).

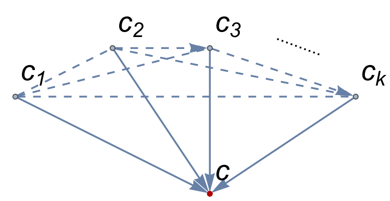


Figure 5: Consistent triads introduced by the vertex $c \in V$ with $\deg_{in}(c) = k$

In general, the given vertex c can form more consistent triads than those indicated in the above theorem. This is due to the fact that there may be two or more edges in the form $c \to c_{k+1}, \ldots, c \to c_{k+r}$. Thus, in T there may also be some number of consistent triads CT_{2b} containing c.

The Theorem 1 is also true for the ordinary tournament graph (without ties). However, since the only consistent triads in such a graph are type CT_3 (i.e. there are no triads of the type CT_{2a} or CT_{2b} containing c), the only consistent triads containing c are those introduced by c. This leads to the following observation:

Corollary 1. Let $T = (V, E_d)$ from $\mathscr{T}^t_n$. Then every vertex $c \in V$, for which $\deg_{in}(c) = k$ is contained by exactly $\binom{k}{2}$ consistent triads of the type CT_3.

Thus, if we would like to construct a tournament graph without ties which has the maximal number of inconsistent triads, we have to minimize the number of consistent triads introduced by the vertices, i.e.

$$|T|_c \stackrel{df}{=} \sum_{c \in V} \binom{\deg_{in}(c)}{2} \tag{4}$$

Since there are no other consistent triads in the tournament graph than those introduced by the vertices, the expression (5) denotes, in fact, the number of inconsistent triads in some

$T \in \mathscr{T}_n^t$. Thus,

$$|T|_i = \binom{n}{3} - \sum_{c \in V} \binom{\deg_{in}(c)}{2} \tag{5}$$

It is commonly known that the sum of degrees in any undirected graph $G = (V, E)$ equals $2|E|$ [11, p. 5]. For the same reason in $T \in \mathscr{T}_n^t$ the sum of incoming edges into vertices is[4] $|E| = \binom{n}{2}$, i.e.:

$$\sum_{c \in V} \deg_{in}(c) = \binom{n}{2} \tag{6}$$

Hence, we would like to minimize (5) providing that the expression (6) holds. Intuitively $|T|_i$ is the largest (5) i.e. $|T|_c$ is the smallest (4) when the input degrees of vertices in a graph are the most evenly distributed[5].

Definition 10. *A* gt-graph *with n vertices is said to be maximal with respect to the number of inconsistent triads, or briefly* maximal *if it has the highest possible number of inconsistent triads among the* gt-graphs *with the size n. The fact that the* gt-graph *is maximal will be denoted $G \in \overline{\mathscr{T}_n^g}$ or $T \in \overline{\mathscr{T}_n^t}$, depending on whether ties are or are not allowed. $\overline{\mathscr{T}_n^t}$ and $\overline{\mathscr{T}_n^g}$ denote families of* gt-graphs *with the highest possible number of inconsistent triads, i.e.*

$$\overline{\mathscr{T}_n^t} = \{T \in \mathscr{T}_n^t \text{ such that } |T|_i = \max_{T_r \in \mathscr{T}_n^t} |T_r|_i\} \tag{7}$$

$$\overline{\mathscr{T}_n^g} = \{G \in \mathscr{T}_n^g \text{ such that } |G|_i = \max_{G_r \in \mathscr{T}_n^g} |G_r|_i\} \tag{8}$$

Before we prove the Theorem (2) about the maximal *t-graph* let us notice that for $r \in \mathbb{N}_+$ it holds that:

$$\binom{2r+1}{2} = r \cdot (2r+1) \tag{9}$$

and

$$\binom{2r}{2} = r \cdot r + r(r-1) \tag{10}$$

The above expression (9) means that by adopting $n = 2r + 1$ as the number of vertices in a graph, we may assign exactly r incoming edges to every vertex c in V when n is odd. Similarly (10), providing that $n = 2r$ is even, we can assign r incoming edges to r vertices and $r - 1$ incoming edges to the next r vertices.

Theorem 2. *The number of inconsistent triads in the t-graph $T = (V, E_d)$ is maximal i.e. $T \in \overline{\mathscr{T}_n^t}$ if and only if*

1. *for every c in V $deg_{in}(c) = r$ when $n = 2r + 1$*
2. *there are r vertices $c_1, \ldots, c_r$ in V such that $deg_{in}(c_i) = r$, and r vertices $c_{r+1}, \ldots, c_n$ such that $deg_{in}(c_j) = r - 1$, where $n = 2r$ and $1 \leq i \leq r < j \leq n$.*

[4] Every directed edge corresponds to one victory.

[5] As it will be explained latter the input degrees are the most evenly distributed if for two different vertices c, d holds that $|\deg_{in}(c) - \deg_{in}(d)| \leq 1$.

Proof. To prove the theorem, it is enough to show that (4) is minimized by the distributions of the vertex degrees mentioned in the thesis of the theorem. Let us suppose that $n = 2r + 1$ and (4) is minimal but not all the vertices have input degrees equal r. Thus, there must be at least one $c_i \in V$ such that $\deg_{in}(c_i) \neq r$. Let us suppose that $\deg_{in}(c_i) = p > r$ (the second case is symmetric). Formulae (6) and (9) imply that there must also be at least one $c_j \in V$ such that $\deg_{in}(c_j) = q < r$. Therefore we can decrease p and increase q by one without changing the sum (6) just by replacing $c_j \to c_i$ to $c_i \to c_j$. Since $p + q = z$ and z is constant, the sum of consistent triads introduced by c_i and c_j (Theorem 1) is given as:

$$\binom{p}{2} + \binom{q}{2} = \binom{p}{2} + \binom{z-p}{2} = p(p-z) + \frac{z(z-1)}{2} \tag{11}$$

Since $z(z-1)/2$ is constant let

$$f(p) \overset{df}{=} p(p-z) + \frac{z(z-1)}{2} \tag{12}$$

The value $f(p)$ decreases alongside a decreasing p if

$$f(p) - f(p-1) > 0 \tag{13}$$

which is true if and only if

$$2p > (z-1) \tag{14}$$

Since $p > q$ and $p + q = z$ the last statement is true, which implies that, by decreasing $\deg_{in}(c_i)$ and increasing $\deg_{in}(c_j)$ by one, we can decrease the expression (4). This fact is contrary to the assumption that (4) is minimal, but not all the vertices have input degrees equal r.

The proof for $n = 2r$ is analogous to the case when $n = 2r + 1$ except the fact that as c_i we should adopt such a vertex for which $\deg_{in}(c_i) \neq r$ and $\deg_{in}(c_i) \neq r - 1$. Note that there must be one if we reject the second statement of the thesis and, at the same time, we claim that (4) is minimal. □

The proof of (Theorem 2) also suggests an algorithm that converts any tournament graph into a graph with the maximal number of inconsistent triads. In every step of such an algorithm, it is enough to find a vertex c_i whose input degree differs from r (when n is odd) or differs from r and $r-1$ (when n is even) and decreases (or increases) its input degree in parallel with increases (or decreases) in the input degree of c_j. If it is impossible to find such a pair (c_i, c_j) this means that the graph is maximal. The algorithm satisfies the stop condition as with every iteration the number of inconsistent triads in a graph gets higher whilst the total number of triads in a graph is bounded by $\binom{n}{3}$.

Kendall and *Babington Smith* [26] suggest a way of constructing the most inconsistent graph that brings to mind *circulant graphs* [33]. Namely, first add to a graph the cycle $c_1 \to c_2 \to c_3 \to \ldots \to c_n \to c_1$ then the cycle $c_1 \to c_3 \to c_5 \to \ldots \to c_n \to c_2 \to \ldots$ if n is even or two cycles $c_1 \to c_3 \to \ldots \to c_{n-1} \to c_1$ and $c_2 \to c_4 \to \ldots \to c_n \to c_2$ if n is odd, and so on. Adding cycles with more and more skips needs to be continued until the insertion of all $\binom{n}{2}$ edges. An example of the maximally inconsistent graphs $T_X \in \mathscr{T}_6^t$ and $T_Y \in \mathscr{T}_7^t$ can be found in (Fig. 6). Those graphs correspond to the matrices X and Y (15).

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 \end{pmatrix} \tag{15}$$

The Theorem 2 clearly indicates the form of the most inconsistent tournament graph, but it does not specify the number of inconsistent triads in such a graph. This number, however, can be easily computed using the formula (2). To see that the results obtained so far are consistent with (2) as defined in [26] let us prove the following theorem.

Theorem 3. *For every t-graph $T = (V, E_d)$ where $T \in \overline{\mathscr{T}^t_n}$, $n \geq 3$ which has the form defined by the Theorem 2 it holds that*

$$|T|_i = \mathcal{I}(n) \tag{16}$$

PROOF. According to (5)

$$|T|_i = \binom{2r+1}{3} - \sum_{c \in V} \binom{\deg_{in}(c)}{2} \tag{17}$$

Let $n = 2r + 1$ and $r \in \mathbb{N}_+$. Then due to (Theorem 2)

$$|T|_i = \binom{2r+1}{3} - \left(\underbrace{\binom{r}{2} + \ldots + \binom{r}{2}}_{2r+1} \right) \tag{18}$$

$$|T|_i = \frac{r(2r-1)(2r+1)}{3} - \frac{(r-1)r(2r+1)}{2} \tag{19}$$

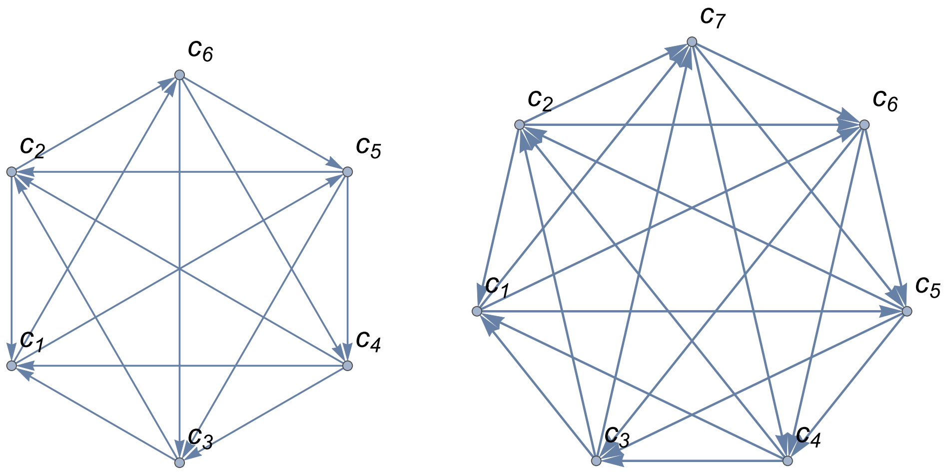
$$|T|_i = \frac{r\left(2r^2+3r+1\right)}{6} = \frac{(2r+1)^3 - (2r+1)}{24} \tag{20}$$

$$|T|_i = \frac{(2r+1)^3 - (2r+1)}{24} = \frac{n^3 - n}{24} = \mathcal{I}(n) \tag{21}$$

Similarly, when $n = 2r$ and $r \in \mathbb{N}_+$. Then due to (Th. 2)

$$|T|_i = \binom{2r}{3} - \left(\underbrace{\binom{r}{2} + \ldots + \binom{r}{2}}_{r} \right) - \left(\underbrace{\binom{r-1}{2} + \ldots + \binom{r-1}{2}}_{r} \right) \tag{22}$$

$$|T|_i = \frac{r(2r-2)(2r-1)}{3} - \frac{(r-1)r^2}{2} - \frac{(r-2)(r-1)r}{2} \tag{23}$$



(a) $T_X \in \overline{\mathscr{T}_6^t}$ (b) $T_Y \in \overline{\mathscr{T}_7^t}$

Figure 6: An example of the most inconsistent tournament graphs with six and seven vertices

$$|T|_i = \frac{r\left(r^2-1\right)}{3} = \frac{(2r)^3 - 4(2r)}{24} = \frac{n^3 - 4n}{24} = \mathcal{I}(n) \tag{24}$$

which completes the proof of the theorem. □

The above theorem shows that the number of inconsistent triads in the tournament graph in which input degrees of their vertices are most evenly distributed is expressed by the formula provided by *Kendall* and *Babington Smith* [26]. This result, of course, is the natural consequence of the fact that such a graph is maximal as regards the number of inconsistent triads, as proven in (Theorem 2).

4. Properties of the most inconsistent set of preferences with ties

The graph representation of the set of paired comparisons with ties is the *gt-graph*. As it may contain two different types of edges, and hence, essentially more different kinds of triads (Fig. 4), the problem of finding the maximum number of inconsistent triads in such a graph is appropriately more difficult. The reasoning presented in this section is composed of three parts. In the first part, the properties of the *gt-graph* are discussed. Next, the maximally inconsistent *gt-graph* is proposed, and then, we prove that the proposed graph is indeed maximal with respect to the number of inconsistent triads.

The most straightforward example of the fully consistent *gt-graph* is a complete undirected graph of n vertices (undirected n-clique). It contains only undirected edges, thus all the triads contained in it are type CT_0. At first glance it seems that by successive replacing of undirected edges into directed ones we can make the graph more and more inconsistent. At the beginning, we will try to choose isolated edges i.e. those which are not adjacent to any directed edge. It is easy to observe that such edges alone cover $n-2$ different triads. Hence, by replacing isolated undirected edges into directed ones we increase the number of inconsistent triads by

$n-2$. Unfortunately, we can insert at most $\lfloor \frac{n}{2} \rfloor$ isolated directed edges (every isolated edge needs two vertices out of n only for itself). Then we have to replace not isolated undirected edges into directed ones, and finally, we decide to make such replacements, which results in increasing the number of inconsistent triads in a graph, but also increases input degrees for some vertices. After several experiments carried out according to the above scheme, one may observe that it is not easy to choose the edge to replace. However, studying the above greedy algorithm is not useless. The first thing to notice is the fact that every *gt-graph* containing more than a certain number of edges should always have some number of consistent triads. Another finding is the observation that when constructing a maximal *gt-graph* one should strive to put at least one directed edge in each triad. Otherwise, the triad remains consistent, increasing the chance that the resulting *gt-graph* is not maximal. Both intuitive observations lead to the conclusion that the construction of the maximal *gt-graph* is a matter of finding a balance between too many directed edges resulting in the appearance of consistent triads of the type CT_{2a} and CT_{2b} and too few directed edges resulting in the existence of consistent triads of the type CT_0. Let us try to formulate this conclusion in a more formal way.

Theorem 4. *Each gt-graph $G \in \mathscr{T}^g_{n,m}$ contains at least $\mathcal{C}(n,m)$ consistent triads of the type CT_{2a} or CT_3 where*

$$\mathcal{C}(n,m) = \frac{1}{2} \left\lfloor \frac{m}{n} \right\rfloor \left(2m - n \left\lfloor \frac{m}{n} \right\rfloor - n \right) \tag{25}$$

Proof. The theorem is a straightforward consequence of (Theorem 1 and 2). The first of them estimates the number of triads CT_{2a} or CT_3 for a given vertex, whilst the second one shows that the sum of triads CT_{2a} or CT_3 introduced by the vertices is minimal when the input degrees are evenly distributed. As we would like to determine the lower bound for the number of consistent triads in G, we therefore have to assume that the input degrees are evenly distributed. Since there are m directed edges in G (it occurs that m times one alternative is better than the other), then the sum of input degrees of vertices is m. Therefore, adopting an even distribution postulate, every vertex has at least $\lfloor \frac{m}{n} \rfloor$ victories assigned (their input degree is at least $\lfloor \frac{m}{n} \rfloor$). Of course, the input degree of some of them may be larger by one. In other words, in the considered *gt-graph* there are p vertices whose input degree is $\lfloor \frac{m}{n} \rfloor$ and $n-p$ vertices whose input degree might be $\lfloor \frac{m}{n} \rfloor + 1$. According to (Theorem 1) such a graph has at least $\mathcal{C}(n,m)$ consistent triads, where

$$\mathcal{C}(n,m) = p \binom{\lfloor \frac{m}{n} \rfloor}{2} + (n-p) \binom{\lfloor \frac{m}{n} \rfloor + 1}{2} \tag{26}$$

We know that the sum of input degrees of vertices is m, so

$$p \left\lfloor \frac{m}{n} \right\rfloor + (n-p) \left(\left\lfloor \frac{m}{n} \right\rfloor + 1 \right) = m \tag{27}$$

Hence,

$$p = n \left(\left\lfloor \frac{m}{n} \right\rfloor + 1 \right) - m \tag{28}$$

Therefore (26) can be written as

$$\mathcal{C}(n,m) = \left(n \cdot \left(\left\lfloor \frac{m}{n} \right\rfloor + 1 \right) - m \right) \cdot \binom{\lfloor \frac{m}{n} \rfloor}{2} + \left(m - n \cdot \left\lfloor \frac{m}{n} \right\rfloor \right) \cdot \binom{\lfloor \frac{m}{n} \rfloor + 1}{2} \tag{29}$$

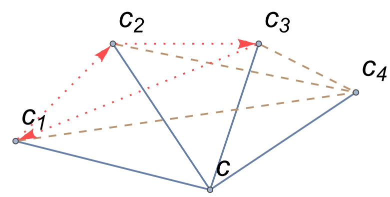


Figure 7: Vertex c where $\deg_{un}(c) = 4$ is contained by 6 different triads. Three of them are CT_0 (dashed edges), the other three are IT_1 (dotted edges).

which, after appropriate transformations leads to (25). $\square$

The immediate consequence of (Lemma 4) is the following corollary:

Corollary 2. Each *gt-graph* $G \in \mathscr{T}^g_{n,m}$ contains at most

$$\binom{n}{3} - \mathcal{C}(n, m) \tag{30}$$

inconsistent triads.

For the purpose of further consideration, let us denote by $\mathcal{T}$ a set of all the triads in the *gt-graph* and by $\mathcal{T}_i$ - a set of triads covered by $i = 0, \ldots, 3$ directed edges. For brevity, we denote the sum $\mathcal{T}_i \cup \mathcal{T}_j$ as $\mathcal{T}_{i,j}$. In particular, it holds that $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_{2,3}$. This allows the formulation of a quite straightforward but useful observation.

Corollary 3. As every two sets out of $\mathcal{T}_0, \ldots, \mathcal{T}_3$ are mutually disjoint, then for every gt-graph $G \in \mathscr{T}^g_n$ it is true that

$$\binom{n}{3} = |\mathcal{T}_0| + |\mathcal{T}_1| + |\mathcal{T}_{2,3}| \tag{31}$$

Another important piece of information about the *gt-graph* follows from the number of undirected edges adjacent to particular vertices. Such edges may form the triads CT_0 but may also form the triads IT_1 (Fig. 7). This observation allows the number of both triad types to be estimated.

Lemma 1. *For every* gt-graph $G \in \mathscr{T}^g_n$ *where* $G = (V, E_u, E_d)$ *it holds that*

$$\sum_{c \in V} \binom{deg_{un}(c)}{2} = 3\,|\mathcal{T}_0| + |\mathcal{T}_1| \tag{32}$$

Proof. Let $c_1 — c, \ldots, c_k — c$ be the undirected edges in E_u adjacent to some $c \in V$. There are $\binom{k}{2}$ triads that contain c. The type of triad depends on the edge (c_i, c_j). If $(c_i, c_j) \in E_u$ then the triad belongs to $\mathcal{T}_0$ whilst if $(c_i, c_j) \in E_d$ then the triad is in $\mathcal{T}_1$. While calculating the sum $\sum_{c \in V} \binom{\deg_{un}(c)}{2}$ every uncovered triad is counted three times as there are three vertices adjacent to two undirected edges forming the triad. For the same reason, the triads covered by one directed edge are taken into account only once. $\square$

Similarly as before, we try to generalize the result (32) to all the graphs that have m directed edges.

Lemma 2. *For each gt-graph* $G \in \mathscr{T}^g_{n,m}$ *where* $G = (V, E_u, E_d)$ *it holds that*

$$\mathcal{D}(n,m) \leq 3\,|\mathcal{T}_0| + |\mathcal{T}_1| \tag{33}$$

where

$$\mathcal{D}(n,m) = \frac{1}{2}\left(n - \left\lfloor \frac{2m}{n} \right\rfloor - 2\right)\left(n^2 + n\left(\left\lfloor \frac{2m}{n} \right\rfloor - 1\right) - 4m\right) \tag{34}$$

Proof. Similarly as in (Lemma 4) the left side of (32) is minimal if undirected degrees are evenly distributed among the vertices. As for every $c \in V$ it holds that $\deg_{un}(c) = \deg(c) - \deg_{in}(c) - \deg_{out}(c)$ then $\deg_{un}(c) = n - 1 - (\deg_{in}(c) + \deg_{out}(c))$. Thus, undirected degrees of vertices are evenly distributed if and only if the number of directed edges adjacent to the vertices are evenly distributed.

It is easy to see that in a *gt-graph* having m directed edges the sum of input and output degrees is $2m$. Thus, for every graph that minimizes the left side of (32) it holds that:

$$p\left\lfloor \frac{2m}{n} \right\rfloor + (n-p)\left(\left\lfloor \frac{2m}{n} \right\rfloor + 1\right) = 2m \tag{35}$$

The above equality means in particular that in such a graph there are $p \leq n$ vertices $c_1, \ldots, c_p$ for which $\deg_{in}(c_i) + \deg_{out}(c_i) = \left\lfloor \frac{2m}{n} \right\rfloor$ and $1 \leq i \leq p$, and $n-p$ vertices $c_{p+1}, \ldots, c_n$ for which $\deg_{in}(c_j) + \deg_{out}(c_j) = \left\lfloor \frac{2m}{n} \right\rfloor + 1$ and $p+1 \leq j \leq n$. This statement also implies that in every graph that minimizes the left side of (32) there are p vertices $c_1, \ldots, c_p$ for which $\deg_{un}(c_i) = n - 1 - \left\lfloor \frac{2m}{n} \right\rfloor$ and $1 \leq i \leq p$, and also $n-p$ vertices $c_{p+1}, \ldots, c_n$ for which $\deg_{un}(c_j) = n - 2 - \left\lfloor \frac{2m}{n} \right\rfloor$ and $p+1 \leq j \leq n$.

Thus, for every $G \in \mathscr{T}^g_{n,m}$ the lower bound of $3\,|\mathcal{T}_0| + |\mathcal{T}_1|$ is:

$$\mathcal{D}(n,m) = p\binom{n - 1 - \left\lfloor \frac{2m}{n} \right\rfloor}{2} + (n-p)\binom{n - 2 - \left\lfloor \frac{2m}{n} \right\rfloor}{2} \tag{36}$$

Since from (35) p equals

$$p = n\left(\left\lfloor \frac{2m}{n} \right\rfloor + 1\right) - 2m \tag{37}$$

Thus,

$$\begin{aligned}\mathcal{D}(n,m) = &\left(n\left(\left\lfloor \frac{2m}{n} \right\rfloor + 1\right) - 2m\right)\binom{n - 1 - \left\lfloor \frac{2m}{n} \right\rfloor}{2} \\ &+ \left(2m - n\left\lfloor \frac{2m}{n} \right\rfloor\right)\binom{n - 2 - \left\lfloor \frac{2m}{n} \right\rfloor}{2}\end{aligned} \tag{38}$$

The above expression simplifies to

$$\mathcal{D}(n,m) = \frac{1}{2}\left(-\left\lfloor \frac{2m}{n} \right\rfloor + n - 2\right)\left(n\left\lfloor \frac{2m}{n} \right\rfloor - 4m + (n-1)n\right) \tag{39}$$

which completes the proof of the theorem. □

Through the analysis of the degree of vertices we can also estimate the value $|\mathcal{T}_{2,3}|$.

Lemma 3. *For every* gt-graph $G \in \mathscr{T}_n^g$ *where* $G = (V, E_u, E_d)$ *it holds that*

$$\frac{1}{3}\sum_{c\in V}\binom{deg_{in}(c) + deg_{out}(c)}{2} \le |\mathcal{T}_{2,3}| \tag{40}$$

Proof. Let $c_1 \to c, c \to c_2, \dots, c_k \to c$ be the directed edges in E_d adjacent to some $c \in V$. There are $\binom{k}{2}$ triads that contain c where $k = \deg_{in}(c) + \deg_{out}(c)$, which are covered by two or three directed edges. While calculating the sum $\sum_{c\in V}\binom{\deg_{in}(c)+\deg_{out}(c)}{2}$ triads covered by two directed edges are counted once, whilst all the triads covered by three directed edges are counted three times. In the worst case scenario, all the considered triads are covered by three directed edges. Thus, $\frac{1}{3}\sum_{c\in V}\binom{\deg_{in}(c)+\deg_{out}(c)}{2}$ is the lower bound for $|\mathcal{T}_{2,3}|$. This observation completes the proof. □

Similarly as before, let us extend the above Lemma to all *gt-graphs* that have n vertices and m directed edges.

Lemma 4. *For each gt-graph* $G \in \mathscr{T}_{n,m}^g$ where $G = (V, E_u, E_d)$ *it holds that*

$$\mathcal{E}(n, m) \le |\mathcal{T}_{2,3}| \tag{41}$$

where

$$\mathcal{E}(n, m) = \frac{1}{6}\left\lfloor\frac{2m}{n}\right\rfloor\left(4m - n\left(\left\lfloor\frac{2m}{n}\right\rfloor + 1\right)\right) \tag{42}$$

Proof. Similarly as in (Lemma 2) the left side of (40) is minimal if the sum of input and output degrees of the vertices are evenly distributed. It is easy to see that in a *gt-graph* that has m directed edges the sum of input and output degrees is $2m$. Thus, for every graph that minimizes the left side of (40) it holds that (35). This implies that in the *gt-graph* which minimizes the left side of (40) there should be p vertices adjacent to $\left\lfloor\frac{2m}{n}\right\rfloor$ directed edges and $n - p$ vertices adjacent to $\left\lfloor\frac{2m}{n}\right\rfloor + 1$ directed edges. Based on (40) we conclude that

$$\mathcal{E}(n, m) = \frac{1}{3}\left(p\binom{\left\lfloor\frac{2m}{n}\right\rfloor}{2} + (n-p)\binom{\left\lfloor\frac{2m}{n}\right\rfloor + 1}{2}\right) \tag{43}$$

Applying (37) we obtain

$$\begin{aligned}\mathcal{E}(n, m) =& \frac{1}{3}\left\{\left[n\left(\left\lfloor\frac{2m}{n}\right\rfloor + 1\right) - 2m\right]\binom{\left\lfloor\frac{2m}{n}\right\rfloor}{2}\right.\\ &\left. + \left[n - \left(n\left(\left\lfloor\frac{2m}{n}\right\rfloor + 1\right) - 2m\right)\right]\binom{\left\lfloor\frac{2m}{n}\right\rfloor + 1}{2}\right\}\end{aligned} \tag{44}$$

Hence,

$$\mathcal{E}(n, m) = \frac{1}{3}\left\{\left(n\left\lfloor\frac{2m}{n}\right\rfloor + n - 2m\right)\binom{\left\lfloor\frac{2m}{n}\right\rfloor}{2} + \left(2m - n\left\lfloor\frac{2m}{n}\right\rfloor\right)\binom{\left\lfloor\frac{2m}{n}\right\rfloor + 1}{2}\right\} \tag{45}$$

The above equation simplifies to

$$\mathcal{E}(n,m) = \frac{1}{6}\left\lfloor\frac{2m}{n}\right\rfloor\left(4m - n\left\lfloor\frac{2m}{n}\right\rfloor - n\right) \tag{46}$$

Which completes the proof of the Lemma. □

The Corollary (3) and Lemmas (1 - 4) allow us to estimate the minimal number of consistent triads which are not covered by any directed edge.

Theorem 5. *For each gt-graph $G \in \mathscr{T}^g_{n,m}$ where $G = (V, E_u, E_d)$ holds that*

$$\mathcal{F}(n,m) \leq |\mathcal{T}_0| \tag{47}$$

where

$$\mathcal{F}(n,m) = \frac{1}{2}\left(\mathcal{D}(n,m) + \mathcal{E}(n,m) - \binom{n}{3}\right) \tag{48}$$

which is equivalent to

$$\mathcal{F}(n,m) = \frac{1}{6}\left(-2n\left\lfloor\frac{2m}{n}\right\rfloor^2 + (8m-2n)\left\lfloor\frac{2m}{n}\right\rfloor + (n-2)((n-1)n-6m)\right) \tag{49}$$

Proof. According to (Corollary 3)

$$\binom{n}{3} = |\mathcal{T}_0| + |\mathcal{T}_1| + |\mathcal{T}_{2,3}| \tag{50}$$

Due to (Lemma 2) it holds that

$$\mathcal{D}(n,m) - 3\,|\mathcal{T}_0| \leq |\mathcal{T}_1| \tag{51}$$

Therefore it is true that

$$\binom{n}{3} \geq |\mathcal{T}_0| + (\mathcal{D}(n,m) - 3\,|\mathcal{T}_0|) + |\mathcal{T}_{2,3}| = \mathcal{D}(n,m) + |\mathcal{T}_{2,3}| - 2\,|\mathcal{T}_0| \tag{52}$$

As we know (Lemma 4) that $\mathcal{E}(n,m) \leq |\mathcal{T}_{2,3}|$ it is true that

$$\binom{n}{3} \geq \mathcal{D}(n,m) + \mathcal{E}(n,m) - 2\,|\mathcal{T}_0| \tag{53}$$

Hence,

$$|\mathcal{T}_0| \geq \frac{1}{2}\left(\mathcal{D}(n,m) + \mathcal{E}(n,m) - \binom{n}{3}\right) \tag{54}$$

which, after simplifying, leads to

$$|\mathcal{T}_0| \geq \frac{1}{6}\left((8m-2n)\left\lfloor\frac{2m}{n}\right\rfloor - 2n\left\lfloor\frac{2m}{n}\right\rfloor^2 + (n-2)((n-1)n-6m)\right) \tag{55}$$

Which completes the proof of the theorem. □

One can easily check that for fixed n the values of $\mathcal{F}(n,m)$ decrease to 0 then become negative, whilst $|\mathcal{T}_0|$ is always a positive integer. Hence, the inequality (47) can also be written as:

$$\max\{0, \lceil \mathcal{F}(n,m) \rceil\} \leq |\mathcal{T}_0| \tag{56}$$

Both theorems 4 and 5 provide estimations for the minimal number of consistent triads in a *gt-graph*. Theorem 4 provides the lower bound $\mathcal{C}(n,m)$ for the number of triads CT_{2a} and CT_3, whilst Theorem 5 provides the lower bound for the number of consistent triads CT_0. Hence, the number of consistent triads in the *gt-graph* $T \in \mathscr{T}^g_{n,m}$ cannot be lower than $\mathcal{G}(n,m)$ where

$$\mathcal{G}(n,m) \overset{df}{=} \mathcal{C}(n,m) + \max\{0, \lceil \mathcal{F}(n,m) \rceil\} \tag{57}$$

Of course, its number could be even higher as we do not care about triads CT_{2b}. The immediate consequence of the above expression is the observation that the number of inconsistent triads in the *gt-graph* cannot be higher than $\mathcal{H}(n,m)$ where:

$$\mathcal{H}(n,m) \overset{df}{=} \binom{n}{3} - \mathcal{G}(n,m) \tag{58}$$

In particular, the most inconsistent *gt-graph* $G \in \overline{\mathscr{T}^g_n}$ with some fixed $n \geq 3$ can have as many inconsistent triads as the maximal value of the upper bounding function $\mathcal{H}(n,m)$, i.e.

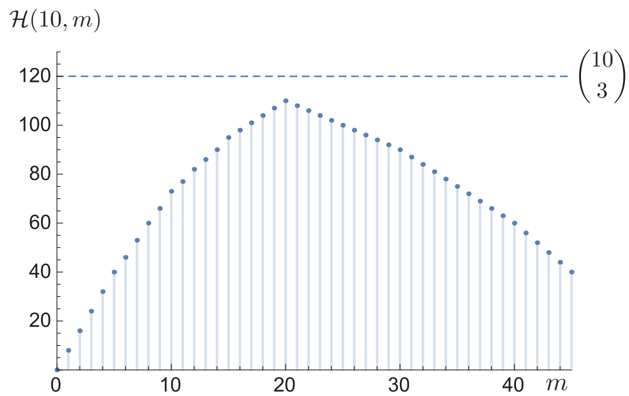
$$|G|_i \leq \max_{0 \leq m \leq \binom{n}{2}} \mathcal{H}(n,m) \tag{59}$$

Reversely, a *gt-graph* $G \in \mathscr{T}^g_n$, which fits that maximum must be maximal i.e. wherever $|G|_i = \max_{0 \leq m \leq \binom{n}{2}} \mathcal{H}(n,m)$ then $G \in \overline{\mathscr{T}^g_n}$. Through the experimental analysis of the upper bounding function $\mathcal{H}(n,m)$ we can see that for every fixed n it has one distinct maximum (Fig. 8).

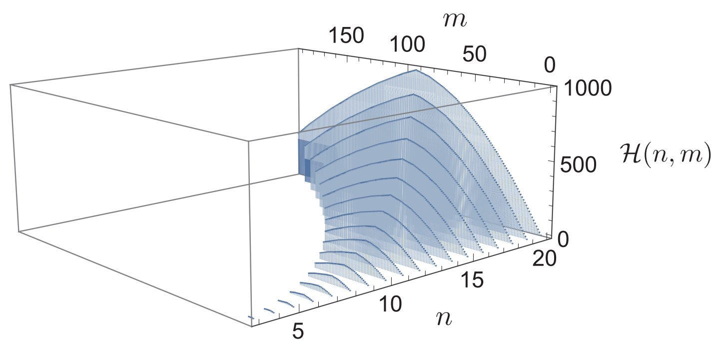
In the next section we propose the graph which fits the maximum of $\mathcal{H}(n,m)$ and formally prove indispensable theorems.

5. The most inconsistent set of preferences with ties

In order to find the maximal gt-graph, let us try to look at the function $\mathcal{H}(n,m)$ and the two functions $\mathcal{C}(n,m)$ and $\mathcal{F}(n,m)$ of which it is composed (Fig. 9). $\mathcal{C}(n,m)$ determines the minimal number of consistent triads covered by more than one directed edge. The more directed edges, the greater the number of consistent triads in a graph. Hence, for some small number of directed edges $\mathcal{C}$ equals 0, then slowly begins to grow. The function $\mathcal{F}(n,m)$ indicates the minimal number of triads not covered by any directed edge. Those triads are also consistent. With the increase in the number of directed edges, their quantity decreases and eventually reaches 0. Since for the positive ordinates $\mathcal{F}$ decreases faster than $\mathcal{C}$ grows, then the function $\mathcal{H}$ reaches the maximum when $\mathcal{F}$ becomes 0. This indicates that in the optimal *gt-graph* all the triads should be covered by at least one directed edge. This requires the introduction of so many directed edges that the number of triads will become consistent thereby. However, the slope of both functions $\mathcal{F}$ and $\mathcal{C}$ indicates that it is more important to cover each triad CT_0 than not to create too many consistent triads CT_{2a}, CT_{2b} or CT_3.



(a) $\mathcal{H}(n, m)$ for $n = 10$ and $m = 0, \ldots, \binom{10}{2}$



(b) $\mathcal{H}(n, m)$ for $n = 3, \ldots, 20$ and $m = 0, \ldots, \binom{n}{2}$

Figure 8: The upper bounding function $\mathcal{H}(n, m)$

The considerations in the previous section also indicate that directed edges should be evenly distributed. Otherwise, the *gt-graph* may not be maximal. The above somewhat intuitive considerations, based on the viewing functions in the figure, lead to the definition of the most inconsistent *gt-graph*.

Definition 11. *A double tournament graph (hereinafter referred to as* dt-graph*), is a* gt-graph $G = (V_1 \cup V_2, E_{d_1} \cup E_{d_2}, E_u)$ *such that* (V_1, E_{d_1}) *and* (V_2, E_{d_2}) *are* t-graphs*, where* $V_1 \cap V_2 = \emptyset$ *and* $E_u = \{\{c, d\} \ : \ c \in V_1 \wedge d \in V_2\}$.

It is easy to observe that in every *dt-graph* all triads are covered by directed edges (Lemma 6). Thus, for every *dt-graph* it holds that $\max\{0, \lceil \mathcal{F}(n, m) \rceil\} = 0$. This does not guarantee, however, the minimality of $\mathcal{C}(n, m)$. Let us propose an improved version of the *dt-graph,* which, as will be shown later, indeed contains the maximal number of inconsistent triads.

Proposition 1. *The* dt-graph $T = (V_1 \cup V_2, E_{d_1} \cup E_{d_2}, E_u)$ *is the maximal* dt-graph *if* (V_1, E_{d_1}) *and* (V_2, E_{d_2}) *are maximal* t-graphs *where* $|V_1| = \left\lfloor \frac{n}{2} \right\rfloor$ *and* $|V_2| = \left\lceil \frac{n}{2} \right\rceil$.

In other words, we suppose that the *dt-graph* with n vertices composed of two maximal *t-graphs* whose numbers of vertices are identical (when n is even) or differ by one (when n is

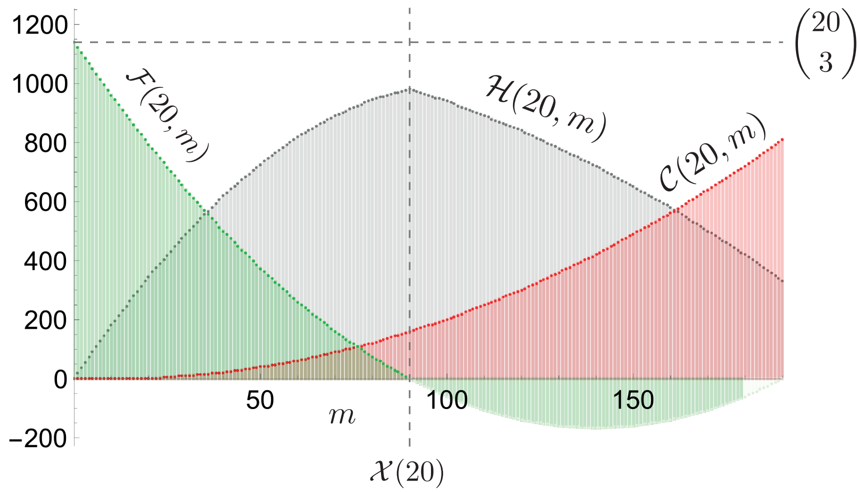


Figure 9: Bounding of consistent and inconsistent triads for *gt-graph* with $n = 20$ vertices.

odd) is *maximal.* Examples of such maximal *dt-graph* candidates can be found at (Fig. 10). The matrices that correspond to the graphs G_{X^*} and G_{Y^*} are given as (60).

$$X^* = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \quad Y^* = \begin{pmatrix} 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \tag{60}$$

Let us denote the number of directed edges in a maximal *dt-graph* candidate by $\mathcal{X}(n)$. It is easy to see that:

$$\mathcal{X}(n) = \binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2} \tag{61}$$

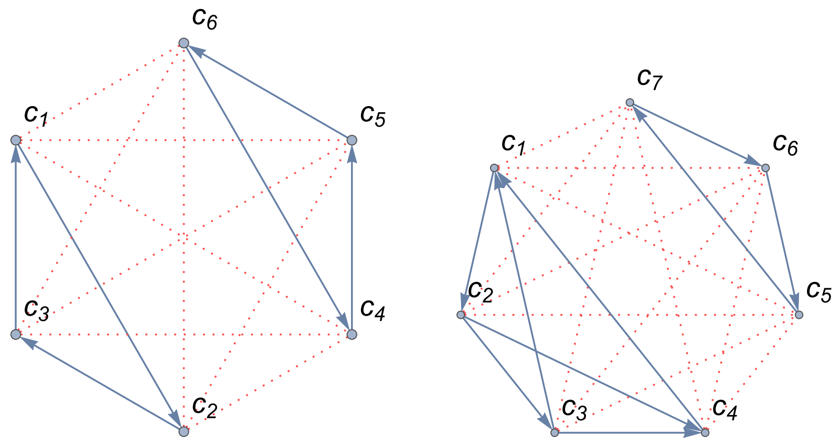
Corollary 4. It can be easily calculated that when n is even i.e. $n = 2q$ and $q \in \mathbb{N}_+$ it holds that

$$\mathcal{X}(2q) = q(q-1) \tag{62}$$

whilst when n is odd i.e. $n = 2q + 1$ and $q \in \mathbb{N}_+$ it holds that

$$\mathcal{X}(2q+1) = q^2 \tag{63}$$

To determine the number of consistent/inconsistent triads in this *"maximal gt-graph candidate"* let us observe that all the consistent triads are in the two maximal tournament subgraphs. This observation can be written in the form of a short Lemma.



(a) $G_{X^*} \in \overline{\mathcal{J}_6^g}$ (b) $G_{Y^*} \in \overline{\mathcal{J}_7^g}$

Figure 10: Two examples of the maximal dt-graphs (undirected edges were dotted).

Lemma 5. *For every* dt-graph $G = (V_1 \cup V_2, E_{d_1} \cup E_{d_2}, E_u)$ *and a triad* $t = \{v_i, v_k, v_j\}$ *if* $t \cap V_1 \neq \emptyset$ *and* $t \cap V_2 \neq \emptyset$ *then* t *is inconsistent.*

Proof. Since $t \cap V_1 \neq \emptyset$ and $t \cap V_2 \neq \emptyset$, there are two vertices from t in one of the two sets V_1 and V_2 and one vertex from t in the other set. Let us suppose that $v_i, v_k \in V_1$ and $v_j \in V_2$. Since (V_1, E_{d_1}) is a *t-graph* then the edge between v_i and v_k is directed. Due to the definition of *dt-graph* both edges (v_i, v_j) and (v_k, v_j) are undirected, hence t is IT_1. □

The immediate conclusion can be written as the Lemma

Lemma 6. *The dt-graph does not contain uncovered triads*

Proof. Let us consider the *dt-graph* $G = (V_1 \cup V_2, E_{d_1} \cup E_{d_2}, E_u)$ and a triad $t = \{v_i, v_k, v_j\}$. If $v_i, v_k \in V_1$ and $v_j \in V_2$ then t is inconsistent (Lemma 5), hence it cannot be uncovered. If all $v_i, v_k, v_j \in V_1$ then all three edges are spanned between v_i, v_k and v_j. Hence, t is covered. The proof is completed as all the other cases are similar. □

It is also easy to determine the number of inconsistent triads in the candidate graph. Due to (Theorem 3) the number of consistent triads in the maximal tournament sub-graphs are $\binom{\lfloor \frac{n}{2} \rfloor}{3} - \mathcal{I}\left(\lfloor \frac{n}{2} \rfloor\right)$ and $\binom{\lceil \frac{n}{2} \rceil}{3} - \mathcal{I}\left(\lceil \frac{n}{2} \rceil\right)$ correspondingly. Since there are no consistent triads in double tournament graphs, except those that are fully enclosed in the maximal tournament sub-graphs (Lemma 5), the number of inconsistent triads in the maximal *gt-graph* candidate is given as:

$$\mathcal{Y}(n) = \binom{n}{3} - \left(\binom{\lfloor \frac{n}{2} \rfloor}{3} - \mathcal{I}\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right) - \left(\binom{\lceil \frac{n}{2} \rceil}{3} - \mathcal{I}\left(\left\lceil \frac{n}{2} \right\rceil\right)\right) \tag{64}$$

To confirm that a *dt-graph* (Proposition 1) is indeed maximal we need to prove that

- the function $\mathcal{H}(n,m)$ reaches the maximum when the number of directed edges in a graph equals $m = \mathcal{X}(n)$, and
- the maximum of $\mathcal{H}(n,m)$ equals $\mathcal{Y}(n)$

Therefore to make the Proposition 1 a fully fledged claim we prove (Theorem 6). However, before we start (Theorem 6) let us prove a couple of Lemmas which formally confirm what we have seen at (Fig. 9). The aim of the first Lemma (7) is a formal confirmation of the shape of the function $\mathcal{F}$. In particular, it confirms that $\mathcal{F}$ crosses the x-axis at the same point where $\mathcal{H}$ reaches the maximum i.e. for every fixed $n \geq 3$, $\mathcal{F}$ is positive when $0 \leq m < \mathcal{X}(n)$, equals 0 when $m = \mathcal{X}(n)$ and it is non-positive for $\mathcal{X}(n) \leq m \leq \binom{n}{2}$.

Lemma 7. *For every $n \in \mathbb{N}_+, n \geq 3$ and $k \in \mathbb{N}_+$ it holds that:*

$$\mathcal{F}(n, \mathcal{X}(n)) = 0 \tag{65}$$

$$\mathcal{F}(n, \mathcal{X}(n) - k) \geq 1, \;\; \textit{where} \;\; 0 < k < \mathcal{X}(n) \tag{66}$$

$$\mathcal{F}(n, \mathcal{X}(n) + k) \leq 0, \;\; \textit{where} \;\; 0 < k \leq \binom{n}{2} - \mathcal{X}(n) \tag{67}$$

Proof. Proof of the Lemma, consisting of elementary but time consuming operations, can be found in (Appendix A).

The aim of the next Lemma is to show that $\mathcal{C}$ is strictly increasing for every m not smaller than n and obviously not greater than the maximal number of edges in a *gt-graph* i.e. $\binom{n}{2}$ (Fig. 9). Thus, by adding more directed edges than n we may only increase the minimal number of consistent triads of the types CT_{2a} or CT_3.

Lemma 8. *For every $n \in \mathbb{N}_+, n \geq 3$ the function $\mathcal{C}$*

1. is constant and equals $\mathcal{C}(n,m) = 0$ for every m such that $0 \leq m < n$
2. is strictly increasing for every $m \in \mathbb{N}_+$ such that $n \leq m \leq \binom{n}{2}$, i.e.

$$\mathcal{C}(n, m+1) - \mathcal{C}(n,m) > 0 \tag{68}$$

Proof. Proof of the Lemma, consisting of elementary but time consuming operations, can be found in (Appendix B).

In every gt-graph with n vertices and m directed edges there are at least $\mathcal{C}(n,m)$ consistent triads CT_{2a} or CT_3. This means that in this graph there are at most $\binom{n}{3} - \mathcal{C}(n,m)$ inconsistent triads. In particular the Lemma 9 shows that there is no *gt-graph* with n vertices and $\mathcal{X}(n)$ directed edges which has more inconsistent triads than the maximal *gt-graph* defined in (Proposition 1).

Lemma 9. *For every $n \in \mathbb{N}_+, n \geq 3$ it holds that*

$$\mathcal{Y}(n) = \binom{n}{3} - \mathcal{C}(n, \mathcal{X}(n)) \tag{69}$$

Proof. Proof of the Lemma, composed of elementary but time consuming operations, can be found in (Appendix C).

The next Lemma shows that the minimal number of consistent triads in a *gt-graph* decreases along with adding the next directed edges. Such a decrease continues as long as the number of directed edges does not reach the value $\mathcal{X}(n)$. In other words, following the increasing number of directed edges (until there are less than $\mathcal{X}(n)$) the number of inconsistent triads also increases.

Lemma 10. *For every $n \in \mathbb{N}_+, n \geq 3$ the function $\mathcal{G}$ is strictly decreasing for every $m \in \mathbb{N}_+$ such that $1 \leq m \leq \mathcal{X}(n)$, i.e.*

$$\mathcal{G}(n,m) - \mathcal{G}(n,m+1) > 0 \ \ where \ \ 1 \leq m < \mathcal{X}(n) \tag{70}$$

Proof. Proof of the Lemma, composed of elementary but time consuming operations, can be found in (Appendix D).

For every fixed $n \geq 3$ the function $\mathcal{H}$ determines the maximal possible number of inconsistent triads in every *gt-graph*.

The aim of the theorem below is to confirm that, indeed, the proposed *dt-graph* (Proposition 1) is a *maximal gt-graph.*

Theorem 6. *For every* dt-graph $G = (V_1 \cup V_2, E_{d_1} \cup E_{d_2}, E_u)$ *with n vertices where (V_1, E_{d_1}) and (V_2, E_{d_2}) are maximal* t-graphs and $|V_1| = \left\lfloor \frac{n}{2} \right\rfloor$ *and* $|V_2| = \left\lceil \frac{n}{2} \right\rceil$ *and $n > 3$ it holds that:*

1. *$\mathcal{X}(n) = m$ maximizes $\mathcal{H}(n,m)$, i.e.*

$$\mathcal{H}(n, \mathcal{X}(n)) = \max_{0 \leq m \leq \binom{n}{2}} \mathcal{H}(n,m) \tag{71}$$

2. *$\mathcal{Y}(n)$ is a maximum of $\mathcal{H}(n,m)$*

$$\mathcal{H}(n, \mathcal{X}(n)) = \mathcal{Y}(n) \tag{72}$$

Proof. As (58) then the first claim of the theorem is equivalent to

$$\mathcal{G}(n, \mathcal{X}(n)) = \min_{0 \leq m \leq \binom{n}{2}} \mathcal{G}(n,m) \tag{73}$$

As (57) then the function $\mathcal{G}$ is the sum of $\mathcal{C}(n,m)$ and $\max\{0, \lceil \mathcal{F}(n,m) \rceil\}$. From (Lemma 8) we know that $\mathcal{C}$ does not decrease with respect to m. On the other hand, due to the (Lemma 7) $\mathcal{F}(n, \mathcal{X}(n) + k) \leq 0$ for every $0 < k \leq \binom{n}{2} - \mathcal{X}(n)$, which translates to the observation that for every $m \geq \mathcal{X}(n)$ it holds that $\max\{0, \lceil \mathcal{F}(n,m) \rceil\} = 0$. Hence, for every $m \geq \mathcal{X}(n)$ the function $\mathcal{G}$ does not decrease and boils down to $\mathcal{G}(n,m) = \mathcal{C}(n,m)$. In other words

$$\mathcal{G}(n, \mathcal{X}(n)) \leq \mathcal{G}(n, \mathcal{X}(n) + 1) \leq \ldots \leq \mathcal{G}(n, \binom{n}{2}) \tag{74}$$

This fact, coupled with (Lemma 10) i.e.

$$\mathcal{G}(n, 0) > \mathcal{G}(n, 1) > \ldots > \mathcal{G}(n, \mathcal{X}(n)) \tag{75}$$

implies that indeed

$$\mathcal{G}(n, \mathcal{X}(n)) = \min_{0 \leq m \leq \binom{n}{2}} \mathcal{G}(n, m) \tag{76}$$

which completes the proof of the first claim (71) of the Theorem 6. To prove the second claim it is enough to recall that for every $m \geq \mathcal{X}(n)$ it holds that $\mathcal{G}(n,m) = \mathcal{C}(n,m)$. Thus, in particular

$$\mathcal{H}(n, \mathcal{X}(n)) = \binom{n}{3} - \mathcal{C}(n, \mathcal{X}(n)) \tag{77}$$

which satisfies the second claim (72) of the Theorem 6, and which thereby confirms the Proposition 1. □

6. Inconsistency indices in paired comparisons with ties

As shown in (Section 2) the inconsistency index (called there *"coefficient of consistence"*) defined by *Kendall* and *Babington Smith* [26, p. 330] cannot be used in the context of ordinal pairwise comparisons with ties. Thus, in (3) $\mathcal{I}(n)$ needs to be replaced by $\mathcal{Y}(n)$ - the maximal number of triads in the case when ties are allowed. The generalized inconsistency index that covers pairwise comparisons with ties finally takes the form

$$\zeta_g(M) = 1 - \frac{|G_M|_i}{\mathcal{Y}(n)} \tag{78}$$

where M is an ordinal PC matrix with ties of the size $n \times n$ (Def. 1) and G is a gt-graph corresponding to M. The formula (78), although concise, may not be handy in practice. This is due to the use in (64) of the floor $\lfloor x \rfloor$ and ceiling $\lceil x \rceil$ operations as well as binomial symbol $\binom{x}{y}$. For this reason, let us simplify (64) depending on whether n and $n/2$ are odd or even. There are four cases that need to be considered:

$$\mathcal{Y}(n) = \begin{cases} \frac{13n^3-24n^2-16n}{96} & \text{when } n = 4q \text{ for } q = 1,2,3,\ldots \\ \frac{13n^3-24n^2-19n+30}{96} & \text{when } n = 4q+1 \text{ for } q = 1,2,3,\ldots \\ \frac{13n^3-24n^2-4n}{96} & \text{when } n = 4q+2 \text{ for } q = 1,2,3,\ldots \\ \frac{13n^3-24n^2-19n+18}{96} & \text{when } n = 4q+3 \text{ for } q = 0,1,2,\ldots \end{cases} \tag{79}$$

For example, to compute the inconsistency index for the ordinal PC matrix M (1) (see Fig. 1) first it is necessary to compute the number of inconsistent triads in M. Since (1) has five inconsistent triads: (A_1, A_2, A_3), (A_1, A_2, A_5), (A_1, A_3, A_5), (A_1, A_4, A_5) and (A_3, A_4, A_5) then $|T_M| = 5$. On the other hand, $5 = 4 \cdot 1 + 1$ hence, the value $\mathcal{Y}(5)$ is obtained by replacing n with 5 in the expression $1/96 \cdot \left(13n^3 - 24n^2 - 19n + 30\right)$, i.e. $\mathcal{Y}(5) = 10$. In other words, in the considered *gt-graph* (Fig. 1) five triads out of ten possible ones are inconsistent. The generalized consistency index for M takes the form:

$$\zeta_g(M) = 1 - \frac{5}{10} = \frac{1}{2} \tag{80}$$

Hence the inconsistency level for M (1) is 50%.

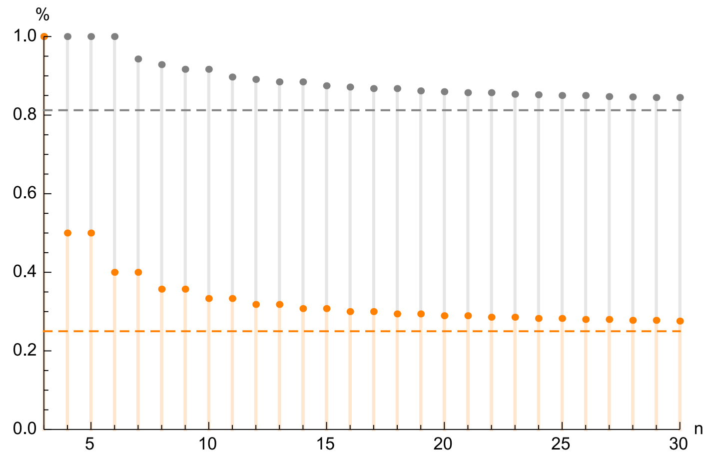


Figure 11: The maximal values of $\eta(M)$ for *t-graph* and *gt-graph*

As every *t-graph* is also a *gt-graph* but not reversely (see Def. 2 and 3) then the generalized inconsistency index ζ_g can also be used to estimate the inconsistency level of paired comparisons without ties. Conversely it is not possible.

Both inconsistency indices ζ and ζ_g compare the number of inconsistent triads in M with the maximal number of such triads in a matrix of the same size as M. Hence, for the maximally inconsistent matrix the index functions will return 1, whilst the inconsistency index for a fully consistent matrix is 0. The maximal value of the inconsistency index, of course, does not automatically imply that all the triads in the given matrix are inconsistent. To capture this phenomenon, let us define the *absolute inconsistency index* η as a ratio of the number of inconsistent triads to the number of all possible triads in the $n \times n$ matrix M.

$$\eta(M) \stackrel{df}{=} \frac{|G_M|_i}{\binom{n}{3}} \tag{81}$$

Of course, $0 \leq \eta(M) \leq 1$. If, for example, $\eta(M) = 0.4$ then it would mean that M contains 60% consistent triads and 40% inconsistent triads. The maximal value that $\eta(M)$ may take is limited by $\mathcal{I}(n)/\binom{n}{3}$ and $\mathcal{Y}(n)/\binom{n}{3}$ for *t-graphs* and *gt-graphs* correspondingly. Thus, for the larger matrices $\eta(M)$ may never reach 1. Let us consider the first few values of $\mathcal{I}(n)/\binom{n}{3}$ and $\mathcal{Y}(n)/\binom{n}{3}$ (Fig. 11).

We can see that for small graphs the percentage of inconsistent triads is higher than for the larger graphs. In particular, for $n = 3, \ldots, 6$ there are such *gt-graphs* that have all triads inconsistent. However, there is only one *t-graph* which has all triads inconsistent. It is just a single triad. Although the percentage of inconsistent triads for both *t-graph* and *gt-graph* decrease, they seem to never drop below certain values. It is easy to compute that[6]:

$$\lim_{n\to\infty} \frac{\mathcal{I}(n)}{\binom{n}{3}} = 0.25 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathcal{Y}(n)}{\binom{n}{3}} = 0.8125 \tag{82}$$

In other words, although in the larger *t-graphs* ($n > 3$) and *gt-graphs* ($n > 6$), there must always be consistent triads. Hence, it is impossible to create a completely inconsistent set of paired comparisons when the alternatives are more than 3 (without ties) and 6 (when ties are allowed). As we can see very often, consistent triads must exist. However, it should be remembered that the "guaranteed" number of consistent triads is limited. The expression (82) implies that at most 75% of triads are "guaranteed" to be consistent without ties, and at most 18.75% of triads are "guaranteed" to be consistent when ties are allowed.

Figuratively speaking, the possibility of a tie allows us to be much more inconsistent. However, we rarely have a chance to be completely inconsistent - only when there are *"sufficiently few"* alternatives. Fortunately, there is no limit to the number of consistent triads in a *gt-graph*. Hence, we can be as consistent (and as frequently) in our views as we want.

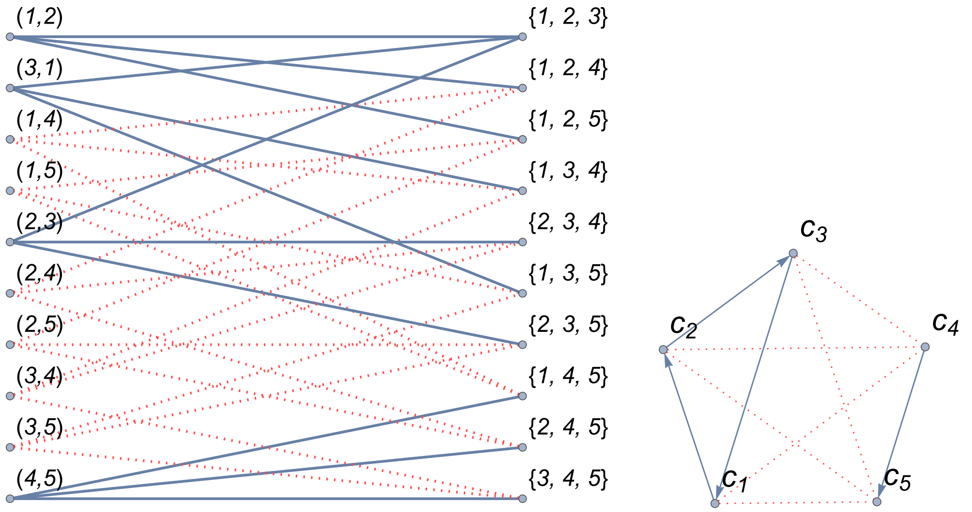
7. Discussion and remarks

To calculate the inconsistency index ζ or the generalized inconsistency index ζ_g for some ordinal PC M $n \times n$ matrix we need to determine the number of inconsistent triads in M. The most straightforward method is to consider every single triad and decide whether it is consistent or not. Since in every complete set of paired comparisons for n alternatives there are $\binom{n}{3} = \frac{n(n-1)(n-2)}{3}$ different triads, then the running time of such a procedure is $O(n^3)$. For *t-graphs*, however, there is a faster way to determine the number of inconsistent triads in a graph. As mentioned earlier, (5) denotes the number of inconsistent triads $|T|_i$ in some *t-graph* $T = (V, E_d)$. To compute (5) $|T|_i$ we need to visit every vertex $c \in V$ and determine its input degree. Computing $\deg_{in}(c)$ for every $c \in V$ requires visiting every edge $(c_i, c_j) \in E_d$ twice. The first time when calculating $\deg_{in}(c_i)$, the second time when $\deg_{in}(c_j)$ is calculated. Thus, determining $\deg_{in}(c_1), \ldots, \deg_{in}(c_n)$ requires $2\,|E_d|$ operations. As $|E_d| = \frac{n(n-1)}{2}$ then the actual running time of computation for (5) is $O(n(n-1)) = O(n^2)$. For this reason the inconsistency index ζ can be determined faster than ζ_g.

Looking at the different types of triads occurring in a *gt-graph* (Fig. 4), one may notice that a triad not covered by any directed edge is consistent, whilst a triad covered by one directed edge is always inconsistent (see Def. 7). Therefore the question arises as to whether it is possible to cover all triads by one directed edge. If not, what is the minimal number of directed edges covering all triads? Let us try to formally address this question. Denote the set of directed edges of some *gt-graph* by $E_d = \{(c_1, c_2), (c_1, c_3), \ldots, (c_{n-1}, c_n)\}$ and the set of triads by $\mathcal{T} = \{\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \ldots, \{c_{n-2}, c_{n-1}, c_n\}\}$. Of course, $|E_d| = \binom{n}{2}$ and $|\mathcal{T}| = \binom{n}{3}$. Then, let $B = (V, E)$ be a bipartite graph such that $V = E_d \cup \mathcal{T}$ and $E = \{(e, t) \mid (e, t) \in E_d \times \mathcal{T} \text{ and } e \text{ covers } t\}$. Hence, we would like to select the minimal subset of edges from E_d whose elements cover (i.e. are connected to) every triad in $\mathcal{T}$.

[6]Expression $\lim_{n\to\infty} \mathcal{I}(n)/\binom{n}{3} = 0.25$ means that both $\lim_{n\to\infty} \left(\frac{n^3-n}{24}\right)/\binom{n}{3} = \lim_{n\to\infty} \left(\frac{n^3-4n}{24}\right)/\binom{n}{3} = 0.25$. Similarly $\lim_{n\to\infty} \frac{\mathcal{Y}(n)}{\binom{n}{3}} = 0.8125$ means that all four limits (see 79) equal 0.8125.

Let us consider the problem for $n = 5$ (Fig. 12a).



(a) Bipartite graph corresponding to triads cover problem where $n = 5$

(b) Maximal dt-graph with 5 vertices (undirected edges are dotted)

Figure 12: Triads cover problem

In such a case $E_d = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$ and $\mathcal{T} = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}\}$. As every edge covers three different triads we may form the set $S = \{\{t_i, t_j, t_k\} \mid t, t_j, t_k \in \mathcal{T}, \exists e \in E_d$ that covers $t_i, t_j, t_k\}$. For example, a tripleton $\{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}\}$ is an element of S as all its elements are covered by edges $(1,2)$ etc. Thus, the question about the minimal subset of $|E_d|$ whose elements cover all the elements in $|\mathcal{T}|$, can be reformulated as follows: what is the minimal subset of S such that the union of its elements equals $\mathcal{T}$?

In general, we can not provide a satisfactory answer to such a question. The problem we formulate is called a *set cover problem*[7] and is one of *Karp's* 21 *NP-complete* problems formulated in 1972 [24]. Fortunately, we are not dealing with a *set cover problem* as such, but with its special instance that can be called *a "triads cover problem"*. In the latter case, a *maximal dt-graph* comes to the rescue (1). The number of directed edges in the *maximal dt-graph* is $\mathcal{X}(n)$. Due to (Lemma 7) we know that every *gt-graph* that has less than $\mathcal{X}(n)$ directed edges must contain at least one triad of the type CT_0. On the other hand, any maximal *dt-graph* does not contain uncovered triads (Lemma 6). This means that a *maximal dt-graph* is a minimal graph covering all triads by directed edges.

Let us consider the maximal *dt-graph* for $n = 5$. According to (Proposition 1), such a graph should be composed of two maximal subgraphs having $\lfloor \frac{5}{2} \rfloor = 3$ and $\lceil \frac{5}{2} \rceil = 2$ vertices. An instance of the first subgraph can be a triad (c_1, c_2), (c_2, c_3) and (c_3, c_1) whilst the second subgraph is just a single edge (c_4, c_5). As the maximal *dt-graph* with 5 vertices provides a minimal edge covering of triads in 5-clique then the minimal subset of S that covers the

[7] *Wikipedia may serve as a quick reference:* ***https://en.wikipedia.org/wiki/Set_cover_problem***

entire $\mathcal{T}$ is, for example, $\{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}\}$, $\{\{1,2,3\}, \{1,3,4\}, \{1,3,5\}\}$ $\{\{1,2,3\}, \{2,3,4\}, \{2,3,5\}\}$ and $\{\{1,4,4\}, \{2,4,5\}, \{3,4,5\}\}$ (Fig. 12).

8. Summary

In the presented article, the inconsistency index proposed by *Kendall and Babington Smith* [26] has been extended to cover pairwise comparisons with ties. For this purpose, the most inconsistent sets of pairwise comparisons with and without ties have been analyzed. To model pairwise comparisons with ties a generalized tournament graph has been defined. An additional *absolute consistency index* η for pairwise comparisons with and without ties has also been proposed. The relationship between the maximally inconsistent set of pairwise comparisons with ties and the set cover problem has also been shown.

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Appendix A. Proof of Lemma 7

THESIS.

For every $n \in \mathbb{N}_+, n \geq 3$ and $k \in \mathbb{N}_+$ it holds that:

$$\mathcal{F}(n, \mathcal{X}(n)) = 0 \tag{65}$$

$$\mathcal{F}(n, \mathcal{X}(n) - k) \geq 1, \text{ where } 0 < k \leq \mathcal{X}(n) \tag{66}$$

$$\mathcal{F}(n, \mathcal{X}(n) + k) \leq 0, \textit{ where } 0 < k \leq \binom{n}{2} - \mathcal{X}(n) \tag{67}$$

PROOF. EQUATION (65), PART 1.

Let n be even i.e. $n = 2q$ where $q \in \mathbb{N}_+$. Thus, let us insert to (49) as n the value $2q$ and as m the value $\mathcal{X}(2q)$. After a series of elementary transformations applied to (48) we obtain:

$$\mathcal{F}(2q, \mathcal{X}(2q)) = \frac{1}{3}(-2)q\left(\lfloor q \rfloor^2 + (1-2q)\lfloor q \rfloor + (q-1)q\right) \tag{A.1}$$

Since $q \in \mathbb{N}_+$ then

$$\lfloor q \rfloor = q \tag{A.2}$$

Thus,

$$\mathcal{F}(2q, \mathcal{X}(2q)) = \frac{1}{3}(-2)q\left(q^2 + (q-1)q + (1-2q)q\right) \tag{A.3}$$

Which after reduction leads to

$$\mathcal{F}(2q, \mathcal{X}(2q)) = 0 \tag{A.4}$$

PROOF. EQUATION (65), PART 2.

Let n be odd i.e. $n = 2q+1$ where $q \in \mathbb{N}_+$. Similarly, let us replace n in (49) by $2q+1$ and m by $\mathcal{X}(2q+1)$. After elementary transformations we obtain:

$$\begin{aligned}\mathcal{F}(2q+1, \mathcal{X}(2q+1)) = &-\frac{1}{3}(2q+1)\left\lfloor \frac{2q^2}{2q+1} \right\rfloor^2 \\ &+\frac{1}{3}\left(4q^2-2q-1\right)\left\lfloor \frac{2q^2}{2q+1} \right\rfloor \\ &+\frac{1}{3}\left(-2q^2+3q-1\right)q\end{aligned} \tag{A.5}$$

Since $q \in \mathbb{N}_+$, we can bound $2q^2/(2q+1)$ from above

$$\frac{2q^2}{2q+1} < \frac{2q^2}{2q} = q \tag{A.6}$$

and below

$$q-1 = \frac{2\left(q-1\right)^2}{2\left(q-1\right)} < \frac{2\left(q-1\right)^2}{2q+1} = \frac{2q^2-2q+2}{2q+1} \leq \frac{2q^2}{2q+1} \tag{A.7}$$

Therefore, when q is a positive integer it is true that

$$\left\lfloor \frac{2q^2}{2q+1} \right\rfloor = (q-1) \tag{A.8}$$

By applying (A.8) to (A.5) we obtain

$$\begin{aligned} \mathcal{F}(2q+1, \mathcal{X}(2q+1)) = &\frac{1}{3}\left(4q^2-2q-1\right)(q-1) \\ &+ \frac{1}{3}q\left(-2q^2+3q-1\right) \\ &- \frac{1}{3}(2q+1)\left(q-1\right)^2 \end{aligned} \tag{A.9}$$

Then, after making further transformations it is easy to verify that:

$$\mathcal{F}(2q+1, \mathcal{X}(2q+1)) = 0 \tag{A.10}$$

which completes the proof of (65).

Proof. Equation (66), part 1.

Let n be even i.e. $n = 2q$ where $q \in \mathbb{N}_+$. Thus, to prove that $\mathcal{F}(n, \mathcal{X}(n)-k)$ is greater than 0 it is enough to show that for every $q \geq 2$ and $1 \leq k < q(q-1)$ it holds that $\mathcal{F}(n, \mathcal{X}(n)-k) > 1$. Thus, let us insert to (49) as n the value $2q$. After a series of elementary transformations applied to (48) we obtain:

$$\mathcal{F}(2q, \mathcal{X}(2q)-k) = \frac{2}{3}\left(-q\left\lceil\frac{k}{q}\right\rceil^2 + (2k+q)\left\lceil\frac{k}{q}\right\rceil + k(q-1)\right) \tag{A.11}$$

Let us observe that for the positive integer $p = 1, 2, \ldots$ if $p \cdot q \leq k < (p+1)q - 1$ then $\left\lceil\frac{k}{q}\right\rceil = p$. In order to analyze $\mathcal{F}$ let us replace $\left\lceil\frac{k}{q}\right\rceil$ by p and define h such that

$$h(q,k) = \frac{2}{3}\left(p(2k+q) + k(q-1) - qp^2\right) \tag{A.12}$$

where $p \cdot q \leq k < (p+1)q-1$ for every $p = 1, 2, \ldots, q-2$. Of course, when $p \cdot q \leq k < (p+1)q-1$ it holds that

$$\mathcal{F}(2q, \mathcal{X}(2q)-k) = h(q,k) \tag{A.13}$$

As h is linear with respect to k then in order to check whether $h(k) > 0$ it is enough to check whether h is greater than 0 at both ends of the considered interval. So,

$$h(q, p \cdot q) = \frac{2}{3}pq(p+q) \tag{A.14}$$

and

$$h(q,(p+1)q-1) = \frac{1}{3}\left(2p^2q + 2pq^2 + 4pq - 4p + 2q^2 - 4q + 2\right) \tag{A.15}$$

Since for $p, q = 1, 2, \ldots$ it holds that $4pq \geq 4p$ and $2p^2q + 2pq^2 \geq 4q$ then

$$h(q,(p+1)q-1) \geq \frac{1}{3}\left(2q^2 + 2\right) \geq \frac{1}{3}(2+2) > 1 \tag{A.16}$$

Thus, for every $p \cdot q \leq k < (p+1)q - 1$ where $p = 1, 2, \ldots, q-2$, $h(k) > 0$. We just need to check h for $k = q(q-1)$. In such a case $\left\lceil \frac{k}{q} \right\rceil = q - 1$. Thus $h(q(q-1))$ takes the form:

$$h(q, q(q-1)) = \frac{2}{3}q\left(2q^2 - 3q + 1\right) \tag{A.17}$$

As $q \geq 2$ then it is easy to verify that $h(q, q(q-1)) > 0$.

Since $h(q,k) > 0$ for every $p = 1, 2, \ldots, q-2$, where $p \cdot q \leq k < (p+1)q - 1$ and for $k = q(q-1)$ then also $\mathcal{F}(2q, \mathcal{X}(2q) - k) > 0$ for $n = 2q$ and $1 \leq k < q(q-1)$, which completes the first part of the proof.

Proof. Equation (66), part 2.

Let n be even i.e. $n = 2q + 1$ where $q \in \mathbb{N}_+$. Thus, let us insert to (49) as n the value $2q + 1$ and $\mathcal{X}(2q+1) - k$, where this time $1 \leq k \leq q^2$ (see 63). After a series of elementary transformations applied to (48) we obtain:

$$\begin{aligned}\mathcal{F}(n, \mathcal{X}(n) - k) = &\frac{1}{3}\left((4k + 2q + 1)\left\lceil \frac{2\left(k - q^2\right)}{2q+1} \right\rceil + 4q^2 \left\lfloor \frac{2\left(q^2 - k\right)}{2q+1} \right\rfloor - \right. \\ &\left. (2q+1)\left\lfloor \frac{2\left(q^2 - k\right)}{2q+1} \right\rfloor^2 + (2q-1)\left(3k - q^2 + q\right)\right)\end{aligned} \tag{A.18}$$

Since for every $x \in \mathbb{R}$ it holds [8] [16] that $-\lceil x \rceil = \lfloor -x \rfloor$, and $\mathcal{X}(n) = \mathcal{X}(2q+1) = q^2$ then

$$\begin{aligned}\mathcal{F}(2q+1, q^2 - k) = &\frac{1}{3}\left(-(4k + 2q + 1)\left\lfloor \frac{2\left(q^2 - k\right)}{2q+1} \right\rfloor + 4q^2 \left\lfloor \frac{2\left(q^2 - k\right)}{2q+1} \right\rfloor - \right. \\ &\left. (2q+1)\left\lfloor \frac{2\left(q^2 - k\right)}{2q+1} \right\rfloor^2 + (2q-1)\left(3k - q^2 + q\right)\right)\end{aligned} \tag{A.19}$$

It is easy to observe the relationship between $\left\lfloor \frac{2\left(q^2-k\right)}{2q+1} \right\rfloor$ and k is:

$\left\lfloor \frac{2\left(q^2-k\right)}{2q+1} \right\rfloor = 0$ if and only if $0 \leq 2\left(q^2 - k\right) < 2q + 1$, in other words, we require that $q^2 - q - \frac{1}{2} \leq k < q^2$

[8] A quick reference is `https://en.wikipedia.org/wiki/Floor_and_ceiling_functions`

$\left\lfloor \frac{2\left(q^2-k\right)}{2q+1} \right\rfloor = 1$ if and only if $2q+1 \le 2\left(q^2-k\right) < 2(2q+1)$ which translates to the interval: $\frac{1}{2}\left(2q^2 - 2\left(2q+1\right)\right) \le k < \frac{1}{2}\left(2q^2 - 1\left(2q+1\right)\right)$

$\left\lfloor \frac{2\left(q^2-k\right)}{2q+1} \right\rfloor = 2$ if and only if $2(2q+1) \le 2\left(q^2-k\right) < 3(2q+1)$, hence $\frac{1}{2}\left(2q^2 - 3\left(2q+1\right)\right) \le k < \frac{1}{2}\left(2q^2 - 2\left(2q+1\right)\right)$

and in general, $r \stackrel{df}{=} \left\lfloor \frac{2\left(q^2-k\right)}{2q+1} \right\rfloor$ if and only if $(r-1)(2q+1) \le 2\left(q^2-k\right) < r(2q+1)$, which translates to the interval for k: $\frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right) \le k < \frac{1}{2}\left(2q^2 - (r-1)\left(2q+1\right)\right)$.

Thus, instead of analyzing $\mathcal{F}$ with respect to k over the whole domain i.e. $1 \le k \le q^2$ and $q \ge 2$ we can analyze it in the subsequent intervals, in which the value $\left\lfloor \frac{2\left(q^2-k\right)}{2q+1} \right\rfloor$ is known and fixed.

Let us introduce the auxiliary function h :

$$h(q,k,r) \stackrel{df}{=} \mathcal{F}(2q+1, q^2-k) \tag{A.20}$$

defined for k such that $\frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right) \le k < \frac{1}{2}\left(2q^2 - (r-1)\left(2q+1\right)\right)$. Hence,

$$h(q,k,r) = \frac{1}{3}\left(-(4k+2q+1)r + 4q^2r - (2q+1)r^2 + (2q-1)\left(3k - q^2 + q\right)\right) \tag{A.21}$$

Moreover, r is the highest when k is 1. Thus, due to (A.8) it holds that $\left\lfloor \frac{2\left(q^2-1\right)}{2q+1} \right\rfloor \le \left\lfloor \frac{2q^2}{2q+1} \right\rfloor = q-1$. Therefore, we know that $r \le q-1$. Hence, instead of showing that $\mathcal{F}(2q+1, q^2-k) > 1$ for every $0 \le k \le q^2$, we prove that $h(q,k,r) > 1$ when $\frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right) \le k < \frac{1}{2}\left(2q^2 - (r-1)\left(2q+1\right)\right)$ for every $0 \le r \le q-1$.

Let us observe that $h(q,k,r)$ is a decreasing function with respect to k. That is because

$$h(q,k,r) - h(q,k-1,r) = 2q - \frac{4r}{3} + 1 \tag{A.22}$$

where $r \le q-1$. In particular, it is easy to verify that always $2q+1 > \frac{4r}{3}$ for $r \le q-1$. The above equalities justify the following estimation:

$$h(q,k,r) > h(q,k-1,r) > \ldots > h(q, \frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right), r) \tag{A.23}$$

Thus, to prove that $h(q,k,r) > 0$ for all admissible values of q,k,r we need to check whether $h(q, \frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right), r) > 0$ for $0 \le r \le q-1$.

So, applying the lower bound for k, i.e. $k = \frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right)$ to (A.21) we obtain

$$h(q, \frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right), r) = \frac{1}{6}(2q+1)\left(4q^2 - 6qr - 2q + 2r^2 + r\right) \tag{A.24}$$

Let us denote $h_2(q,r) \stackrel{df}{=} h(q, \frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right), r)$. It is easy to observe that h_2 is a parabola with respect to r. Since $\frac{\partial^2 h_2}{\partial r^2} = \frac{2}{3}(2q+1)$ is greater than 0 for $q \ge 2$, thus $h_2(q,r)$

has the minimum with respect to r when $\frac{\partial h_2}{\partial r} = 0$. I.e.

$$\frac{\partial h_2}{\partial r} = -\frac{1}{6}(2q+1)(6q-4r-1) = 0 \tag{A.25}$$

i.e., when

$$r = \frac{1}{4}(6q-1) \tag{A.26}$$

In other words, h_2 decreases for $r = 1, 2, \ldots$, then reaches the minimum[9] at $r = \frac{1}{4}(6q-1)$, next starts to increase for $r \geq \left\lceil \frac{1}{4}(6q-1) \right\rceil$. However, h, h_2 are defined for $r \leq q-1$. Thus, it is clear that within the interval $0 \leq r \leq q-1$ the function h_2 is strictly decreasing with respect to r. Moreover, it is easy to verify that $q - 1 < \left\lfloor \frac{1}{4}(6q-1) \right\rfloor$. Thus, to determine the minimal value of h_2 it is enough to check their value for $r = q-1$.

Thus h_2:

$$h_2(q, q-1) = \frac{1}{6}\left(2q^2 + 3q + 1\right) \tag{A.27}$$

Since, $q \geq 2$ then it is easy to verify that $h_2(q, q-1) > 0$. This implies that $h(q, k, r) > 0$ for every $0 \leq r \leq q-1$ and k such that $\frac{1}{2}\left(2q^2 - r\left(2q+1\right)\right) \leq k < \frac{1}{2}\left(2q^2 - \left(r-1\right)\left(2q+1\right)\right)$. Hence, also $\mathcal{F}(n, \mathcal{X}(n) - k) > 0$ for $n = 2q+1$ where $1 \leq k \leq q^2$, which completes the proof of (66).

Proof. Equation (67), part 1.

Let n be even i.e. $n = 2q$ where $q \in \mathbb{N}_+$. Since (4) to prove that $\mathcal{F}(n, \mathcal{X}(n) + k)$ is smaller than 0 it is enough to show that for every integer q, k such that $q \geq 2$ and $1 \leq k \leq \binom{n}{2} - \mathcal{X}(n)$ where $\binom{n}{2} - \mathcal{X}(n) = \binom{2q}{2} - q(q-1) = q^2$ it holds that $\mathcal{F}(2q, q(q-1)+k) \leq 0$. After a series of elementary transformations applied to (48) we obtain that:

$$\mathcal{F}(2q, q(q-1)+k) = -\frac{2}{3}\left(q\left\lfloor\frac{k}{q}\right\rfloor^2 + (q-2k)\left\lfloor\frac{k}{q}\right\rfloor + k(q-1)\right) \tag{A.28}$$

Let us consider the relationship between k and $\left\lfloor\frac{k}{q}\right\rfloor$. When $1 \leq k < q$ it holds that $\left\lfloor\frac{k}{q}\right\rfloor = 0$, when $q \leq k < 2q$ it holds that $\left\lfloor\frac{k}{q}\right\rfloor = 1$ and similarly, $2q \leq k < 3q$ then it holds that $\left\lfloor\frac{k}{q}\right\rfloor = 2$. In general, when $rq \leq k < (r+1)q$ then $\left\lfloor\frac{k}{q}\right\rfloor = r$. Of course, since $k \leq q^2$ then $r \leq q$. Hence, instead of considering the function $\mathcal{F}$ at once, we may analyze it in the intervals in which $\left\lfloor\frac{k}{q}\right\rfloor$ is known and constant. Let us define:

$$f(q, k, r) \stackrel{df}{=} qr^2 + (q-2k)r + k(q-1) \tag{A.29}$$

It is easy to see that $f(q, k, r) = -\frac{3}{2} \cdot \mathcal{F}(2q, q(q-1)+k)$ if $rq \leq k < (r+1)q$ for $r = 0, \ldots, q-1$. Hence, instead of analyzing $\mathcal{F}$ we will focus on the auxiliary function f.

The first observation is that f is linear with respect to k providing that q and r are known and fixed. Thus, the minimal value of f with respect to k within the interval $rq \leq k < (r+1)q$

[9] In fact, due to the diophantic nature of h_2, its minimum is either at $\left\lfloor \frac{1}{4}(6q-1) \right\rfloor$ or $\left\lceil \frac{1}{4}(6q-1) \right\rceil$.

is $\min\{f(q,rq,r), f(q,(r+1)q,r)\}$. In other words, it is enough to check that f is greater than 0 at both edges of the interval for k. Let us consider f at the lower bound, i.e. for $k = rq$.

$$f(q,rq,r) = qr(q-r) \tag{A.30}$$

It is easy to verify that for every $0 < r < q$ and $q \geq 2$ the value $f(q,rq,r) > 0$. The function $f(q,rq,r)$ reaches 0 when $r = 0$. Thus, $f(q,rq,r) \geq 0$ for every r such that $0 \leq r \leq q$.

Let us consider f at the other end of interval, i.e. for $k = (r+1)q - 1$.

$$f(q,(r+1)q-1,r) = q^2(r+1) - q\left(r^2 + 2r + 2\right) + 2r + 1 \tag{A.31}$$

Similarly as above, we would like to show that for every admissible r the function $f(q,(r+1)q-1,r) \geq 0$. Hence, let us rewrite f with respect to r.

$$f(q,(r+1)q-1,r) = -qr^2 + r\left(q^2 - 2q + 2\right) + \left(q^2 - 2q + 1\right) \tag{A.32}$$

When considering f as a polynomial with respect to r one may notice that the coefficient at r^2 is negative ($-q < 0$) which means that f is concave.

Let us denote $f_2(q,r) \overset{df}{=} f(q,(r+1)q-1,r)$. It is easy to compute that $\frac{\partial f_2}{\partial r} = 0$ when $r = \frac{q^2-2q+2}{2q}$. Since $\frac{\partial^2 f_2}{\partial r^2} = -2q > 0$, thus f_2 reaches the maximum[10] for $r = \frac{q^2-2q+2}{2q}$. Since the interval of r is $0 \leq r < q$ and also $0 \leq \frac{q^2-2q+2}{2q} < q$ therefore the minimum of f_2 for $0 \leq r < q$ is the smaller of the two $f_2(q,0)$ and $f_2(q,q-1)$.

Hence

$$f_2(q,0) = q^2 - 2q + 1, \quad f_2(q,q-1) = q - 1 \tag{A.33}$$

Since for every $q \geq 2$ it holds that $\min\{f_2(q,0), f_2(q,q-1)\} \geq 0$ then $f_2(q,r) \geq 0$ for every fixed $q \geq 2$ and $0 \leq r < q$, which implies that also for $k = (r+1)q-1$, $f(q,k,r) \geq 0$. Therefore $f(q,k,r) \geq 0$ for every $rq \leq k < (r+1)q$ for $r = 0, \ldots, q$.

As $f(q,k,r) = -\frac{3}{2} \cdot \mathcal{F}(2q, q(q-1)+k)$ when $rq \leq k < (r+1)q$, then due to the arbitrary choice of r it holds that $\mathcal{F}(n, \mathcal{X}(n)+k) \leq 0$ for $n = 2q$ and $0 \leq k < q^2$. As one may observe, the above reasoning does not cover $k = q^2$. This is the last "point interval" that needs to be considered. For $k = q^2$ we have

$$\mathcal{F}(2q, q(q-1)+q^2) = \frac{1}{3}(-2)q\left(\lfloor 2q \rfloor^2 + (1-4q)\lfloor 2q \rfloor + 2(2q-1)q\right) \tag{A.34}$$

Since $q \in \mathbb{N}_+$ then $\lfloor 2q \rfloor = 2q$. Hence it is easy to verify that

$$\mathcal{F}(2q, q(q-1)+q^2) = 0 \tag{A.35}$$

Which completes the first part of the proof of (67).

PROOF. EQUATION (67), PART 2.

Let n be odd i.e. $n = 2q+1$ where $q \in \mathbb{N}_+$. Since (4) to prove that $\mathcal{F}(n, \mathcal{X}(n)+k)$ is smaller than 0 it is enough to show that for every integer q, k such that $q \geq 2$ and $1 \leq k \leq$

[10] In fact, due to the diophantine nature of f it reaches the maximum for $r = \left\lfloor \frac{q^2-2q+2}{2q} \right\rfloor$ or $r = \left\lceil \frac{q^2-2q+2}{2q} \right\rceil$.

$\binom{2q}{2} - q^2 - 1 = q^2 - q - 1$ it holds that $\mathcal{F}(2q+1, q^2+k) \leq 0$. After a series of elementary transformations applied to (48) we obtain:

$$\mathcal{F}(2q+1, q^2+k) = -\frac{1}{3}\Bigg((2q+1)\left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor^2 - \left(4k+4q^2-2q-1\right)\left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor + (2q-1)(3k+(q-1)q)\Bigg) \tag{A.36}$$

Since $1 \leq k \leq q^2 - q - 1$ we may estimate the upper and the lower bound for $\left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor$ as

$$q - 1 \leq \left\lfloor \frac{2q^2}{2q+1} \right\rfloor + \left\lfloor \frac{2k}{2q+1} \right\rfloor \leq \left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor \tag{A.37}$$

and

$$\left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor \leq \left\lfloor \frac{2\left(q^2+q^2-q-1\right)}{2q+1} \right\rfloor \leq \left\lfloor \frac{4q^2}{2q} - \frac{2q+2}{2q+1} \right\rfloor = \left\lfloor 2q - \frac{2q+2}{2q+1} \right\rfloor = \lfloor 2q-2 \rfloor = 2q-2 \tag{A.38}$$

Let us denote $r \stackrel{df}{=} \left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor$. Thus, $q-1 \leq r \leq 2q-2$. Let us consider the relationship between k and r. It holds that $\left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor = r$ wherever $r \leq \frac{2\left(q^2+k\right)}{2q+1} < r+1$. Thus it is easy to determine that $\left\lfloor \frac{2\left(q^2+k\right)}{2q+1} \right\rfloor = r$ wherever $\frac{1}{2}\left(2qr + r - 2q^2\right) \leq k < \frac{1}{2}\left((r+1)(2q+1) - 2q^2\right)$.

Let us consider the function $\mathcal{F}(2q+1, q^2+k)$ for $k \in \mathbb{N}_+$ such that $\frac{1}{2}\left(2qr + r - 2q^2\right) \leq k < \frac{1}{2}\left((r+1)(2q+1) - 2q^2\right)$. For this purpose, let us define f

$$f(q,k,r) \stackrel{df}{=} (2q+1)r^2 - r\left(4k+4q^2-2q-1\right) + (2q-1)(3k+(q-1)q) \tag{A.39}$$

It is easy to verify that

$$\mathcal{F}(2q+1, q^2+k) = -\frac{1}{3} f(q,k,r) \tag{A.40}$$

providing that $q, r \in \mathbb{N}_+$, $\frac{1}{2}\left(2qr + r - 2q^2\right) \leq k < \frac{1}{2}\left((r+1)(2q+1) - 2q^2\right)$, $q - 1 \leq r \leq 2q-2$ and $q \geq 2$. Hence, wherever $f(q,k,r) \geq 0$ then $\mathcal{F}(2q+1, q^2+k) \leq 0$. Let us observe that f is linear with respect to k. Therefore it is enough to check the value of $f(q,k,r)$ at the edges of the admissible interval for k, and prove that those values are above 0 in any possible interval determined by r. For this purpose let us define

$$f_2(q,r) \stackrel{df}{=} f(q, \frac{1}{2}\left(2qr + r - 2q^2\right), r) \tag{A.41}$$

for the lower bound, and

$$f_3(q,r) \stackrel{df}{=} f(q, \frac{1}{2}\left((r+1)(2q+1) - 2q^2\right) - 1, r) \tag{A.42}$$

for the upper bound. Hence

$$f_2(q,r) = -\frac{1}{2}(2q+1)\left(4q^2 - 6qr - 2q + 2r^2 + r\right) \tag{A.43}$$

$$f_3(q,r) = -4q^3 + 6q^2(r+1) - q\left(2r^2 + 2r + 5\right) + \frac{1}{2}\left(-2r^2 + 3r + 3\right) \tag{A.44}$$

Let us reorganize the above equations with respect to r:

$$f_2(q,r) = -(2q+1)\,r^2 + \left(2q + 6q^2 - \frac{1}{2}\right) r - 4q^3 + q \tag{A.45}$$

$$f_3(q,r) = -(2q+1)\,r^2 + \left(6q^2 - 2q + \frac{3}{2}\right) r - 4q^3 + 6q^2 - 5q + \frac{3}{2} \tag{A.46}$$

Since both f_2 and f_3 have second degree polynomials with respect to r, and the coefficients nearby r^2 are negative, then f_2 and f_3 are concave parabolas. Therefore f_2 and f_3 are not smaller than 0 within the interval $q-1 \leq r \leq 2q-2$ if they are not negative at both ends of the interval i.e. $q-1$ and $2q-2$. As the estimation (A.37) is not perfect, let us assume for a moment that r is in $q \leq r \leq 2q-2$, whilst the case $r = q-1$ we handle separately.

Let us examine (A.45).

$$f_2(q,r) = q^2 + \frac{q}{2} \;\; when \;\; r = q \tag{A.47}$$

and

$$f_2(q,r) = (2q-3)(2q+1) \;\; when \;\; r = 2q-2 \tag{A.48}$$

Since $q \geq 2$ both of the above equations are greater than 0. For (A.46) it is enough to assume that $q-1 \leq r \leq 2q-2$. Thus,

$$f_3(q,r) = q^2 - \frac{3q}{2} - 1 \;\; when \;\; r = q-1 \tag{A.49}$$

and

$$f_3(q,r) = 2q^2 + 2q - \frac{11}{2} \;\; when \;\; r = 2q-2 \tag{A.50}$$

Similarly, it is easy to verify that both of the above expressions are non negative as $q \geq 2$.

At the end, let us explicitly calculate

$$f(q,k,q-1) = 2kq + k \tag{A.51}$$

As k is always non negative, then also in this case f is non negative 0. Thereby for every $1 \leq k \leq q^2 - q - 1$ it holds that $\mathcal{F}(2q+1, q^2+k) \leq 0$ which completes the proof of the Lemma 7 □

Appendix B. Proof of the Lemma 8

Thesis.

For every $n \in \mathbb{N}_+, n \geq 3$ the function $\mathcal{C}$:

1. is constant and equals $\mathcal{C}(n,m) = 0$ for every m such that $0 \leq m < n$
2. is strictly increasing for every $m \in \mathbb{N}_+$ such that $n \leq m \leq \binom{n}{2}$, i.e.

$$\mathcal{C}(n,m+1) - \mathcal{C}(n,m) > 0 \tag{68}$$

Proof. Claim 1.

The first claim that $\mathcal{C}(n,m) = 0$ for every m such that $0 \leq m < n$ is a direct consequence of the equation (25). It is enough to note that the right side of expression (25) is the product where the first part is $\frac{1}{2}\left\lfloor \frac{m}{n} \right\rfloor$. Hence, wherever $m < n$ the product often equals 0.

Proof. Claim 2.

Due to (Theorem 4) it holds that

$$\begin{aligned}\mathcal{C}(n,m+1) - \mathcal{C}(n,m) =& \frac{1}{2}\left(\left\lfloor \frac{m}{n} \right\rfloor \left(n\left\lfloor \frac{m}{n} \right\rfloor - 2m + n\right) - \right.\\ &\left.\left\lfloor \frac{m+1}{n} \right\rfloor \left(n\left\lfloor \frac{m+1}{n} \right\rfloor - 2m + n - 2\right)\right)\end{aligned} \tag{B.1}$$

It is easy to observe that for some positive integer $p = 1, 2, \ldots$ when $m = np-1$ then $\left\lfloor \frac{m}{n} \right\rfloor = p-1, \left\lfloor \frac{m+1}{n} \right\rfloor = p$. Next, by increasing m by one we get $m = np$ and $\left\lfloor \frac{m}{n} \right\rfloor = p, \left\lfloor \frac{m+1}{n} \right\rfloor = p$. Then, for $m = n(p+1) - 1$ the values of our floored expressions change to $\left\lfloor \frac{m}{n} \right\rfloor = p, \left\lfloor \frac{m+1}{n} \right\rfloor = p+1$, and then by increasing m by one we get $\left\lfloor \frac{m}{n} \right\rfloor = p+1, \left\lfloor \frac{m+1}{n} \right\rfloor = p+1$. Hence, there are two different intervals with respect to the values $\left\lfloor \frac{m}{n} \right\rfloor$ and $\left\lfloor \frac{m+1}{n} \right\rfloor$. The first one in which both expressions have the same value, and the other one (composed of one point) in which their values differ by one. In general, we may observe that:

wherever $m = np - 1$ then $\left\lfloor \frac{m}{n} \right\rfloor = p-1, \left\lfloor \frac{m+1}{n} \right\rfloor = p$, and wherever $np \leq m < n(p+1) - 1$ then $\left\lfloor \frac{m}{n} \right\rfloor = p, \left\lfloor \frac{m+1}{n} \right\rfloor = p$.

Let us define the auxiliary function h by replacing in (B.1) $\left\lfloor \frac{m}{n} \right\rfloor$ by r and $\left\lfloor \frac{m+1}{n} \right\rfloor$ by t:

$$h(n,m,r,t) \stackrel{df}{=} \frac{1}{2}\left(r\left(nr - 2m + n\right) - t\left(nt - 2m + n - 2\right)\right) \tag{B.2}$$

The function h can be rewritten with respect to m, so

$$h(n,m,r,t) = \frac{1}{2}nr^2 + m\left(t - r\right) + \frac{1}{2}nr - \frac{1}{2}nt^2 - \frac{1}{2}nt + t \tag{B.3}$$

It is easy to observe that

$$\mathcal{C}(n, m+1) - \mathcal{C}(n, m) = h(n, m, r, t) \tag{B.4}$$

where $r = \left\lfloor \frac{m}{n} \right\rfloor$ and $t = \left\lfloor \frac{m+1}{n} \right\rfloor$. Thus, instead of analyzing $h(n, m, r, t)$ for m such that $n \leq m \leq \binom{n}{2}$ we analyze $h(n, m, r, t)$ in two intervals $m = np - 1$ and $np \leq m < n(p+1) - 1$. This, due to the arbitrary choice of p, would apply to $\mathcal{C}(n, m+1) - \mathcal{C}(n, m)$ over the whole interval $n \leq m \leq \binom{n}{2}$.

Let us observe that h is linear with respect to m. Thus to prove that $h(n, m, r, t) > 0$ when n, r, t are constant, one needs only to verify the value of h at the ends of both intervals to which m may belong. Thus, let us consider the first "point" interval $m = np - 1$. In this interval $\left\lfloor \frac{m}{n} \right\rfloor = p - 1$, $\left\lfloor \frac{m+1}{n} \right\rfloor = p$, thus:

$$h(n, np-1, p-1, p) = p - 1 \tag{B.5}$$

As $m \geq n$, and $m = np - 1$ thus $p \geq 2$. Hence,

$$h(n, np-1, p-1, p) \geq 2 - 1 = 1 \tag{B.6}$$

This supports the thesis of the theorem, i.e. $np \leq m < n(p+1) - 1$, where $\left\lfloor \frac{m}{n} \right\rfloor = p$, $\left\lfloor \frac{m+1}{n} \right\rfloor = p$. For both its ends we have:

$$h(n, np, p, p) = p \tag{B.7}$$

$$h(n, n(p+1) - 1, p, p) = p \tag{B.8}$$

As $m \geq n$ and $np \leq m$ then $p \geq 1$. Thus in both cases h is strictly greater than 0. Hence, for every $np - 1 \leq m \leq n(p+1) - 1$ it holds that

$$\mathcal{C}(n, m+1) - \mathcal{C}(n, m) > 0 \tag{B.9}$$

Due to the arbitrary choice of p this statement completes the proof of the theorem. □

Appendix C. Proof of the Lemma 9

Thesis.

For every $n \in \mathbb{N}_+, n \geq 3$ it holds that

$$\binom{n}{3} - \mathcal{C}(n, \mathcal{X}(n)) = \mathcal{Y}(n) \tag{69}$$

Proof. Part 1.

Let $n = 4q$ (n is even, and $\left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil = 2q$ is even), $n \geq 4$, hence $q \geq 1$ and $\mathcal{X}(4q) = 2q(2q-1)$. Thus to prove (69) for even numbers we show that

$$\binom{4q}{3} - \mathcal{C}(4q, 2q(2q-1)) - \mathcal{Y}(4q) = 0 \tag{C.1}$$

Since (64) reduces to:

$$\mathcal{Y}(4q) = \binom{4q}{3} - \left(\binom{2q}{3} - \frac{q\left(q^2-1\right)}{3}\right) - \left(\binom{2q}{3} - \frac{q\left(q^2-1\right)}{3}\right) \tag{C.2}$$

by elementary transformations one may show that (C.1) is equivalent to

$$2q\left(\left\lceil \frac{1}{2} - q \right\rceil + q - 1\right)^2 = 0 \tag{C.3}$$

The above is true as $\lceil \frac{1}{2} - q \rceil = 1 - q$ for every $q \in \mathbb{N}_+$.

Proof. Part 2.

Let $n = 4q+1$ (n is odd, $\lfloor \frac{n}{2} \rfloor = 2q$ is even, and $\lceil \frac{n}{2} \rceil = 2q+1$ is odd), $n \geq 4$, hence $q \geq 1$ and $\mathcal{X}(4q+1) = \binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2} = \binom{2q}{2} + \binom{2q+1}{2} = 4q^2$. Thus to prove (69) for $n = 4q+1$ we show that

$$\binom{4q+1}{3} - \mathcal{C}(4q+1, 4q^2) - \mathcal{Y}(4q+1) = 0 \tag{C.4}$$

Since (64) reduces to:

$$\mathcal{Y}(4q+1) = \binom{4q+1}{3} - \left(\binom{2q}{3} - \frac{q\left(q^2-1\right)}{3}\right) - \left(\binom{2q+1}{3} - \frac{q\left(2q^2+3q+1\right)}{6}\right) \tag{C.5}$$

by elementary transformations one may show that (C.4) is equivalent to

$$\frac{1}{2}\left((4q+1)\left\lfloor \frac{4q^2}{4q+1} \right\rfloor^2 + \left(-8q^2+4q+1\right)\left\lfloor \frac{4q^2}{4q+1} \right\rfloor + q\left(4q^2-5q+1\right)\right) = 0 \tag{C.6}$$

Let us note that for every $q \geq 1$ it holds[11] that $\left\lfloor \frac{4q^2}{4q+1} \right\rfloor = q - 1$. Thus, the above equation can be written in the form

$$\frac{1}{2}\left(\left(-8q^2+4q+1\right)(q-1) + \left(4q^2-5q+1\right)q + (4q+1)(q-1)^2\right) = 0 \tag{C.7}$$

which can be easily verified as true.

[11] compare with (A.8).

Proof. Part 3.

Let $n = 4q+2$ (n is even, $\lfloor \frac{n}{2} \rfloor = 2q+1$ is odd, and $\lceil \frac{n}{2} \rceil = 2q+1$ is odd) and $\mathcal{X}(4q+2) = \binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2} = \binom{2q+1}{2} + \binom{2q+1}{2} = 2q(2q+1)$ Thus, to prove (69) for $n = 4q+2$ we show that

$$\binom{4q+2}{3} - \mathcal{C}(4q+2, 2q(2q+1)) - \mathcal{Y}(4q+2) = 0 \tag{C.8}$$

Since (64) reduces to:

$$\mathcal{Y}(4q+2) = \binom{4q+2}{3} - 2\left(\binom{2q+1}{3} - \frac{q\left(2q^2+3q+1\right)}{6}\right) \tag{C.9}$$

by elementary transformations one may show that (C.8) is equivalent to

$$(2q+1)\left(\lfloor q \rfloor^2 + (1-2q)\lfloor q \rfloor + (q-1)q\right) = 0 \tag{C.10}$$

As q is an integer it is easy to show that (C.10) is true.

Proof. Part 4.

Let $n = 4q+3$ (n is odd $\lfloor \frac{n}{2} \rfloor = 2q+1$ is odd, and $\lceil \frac{n}{2} \rceil = 2q+2$ is even) and $\mathcal{X}(4q+3) = \binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2} = \binom{2q+1}{2} + \binom{2q+2}{2} = (2q+1)^2$. Thus, to prove (69) for $n = 4q+3$ we show that

$$\binom{4q+3}{3} - \mathcal{C}(4q+3, (2q+1)^2) - \mathcal{Y}(4q+3) = 0 \tag{C.11}$$

by elementary transformations one may show that (C.11) is equivalent to:

$$\begin{aligned} &\frac{1}{2}\left(\left(-8q^2-4q+1\right)\left\lfloor \frac{(2q+1)^2}{4q+3} \right\rfloor + \right. \\ &\left. (4q+3)\left\lfloor \frac{(2q+1)^2}{4q+3} \right\rfloor^2 + \left(4q^2+q-1\right)q\right) = 0 \end{aligned} \tag{C.12}$$

Since[12] $\left\lfloor \frac{(2q+1)^2}{4q+3} \right\rfloor = \lfloor q \rfloor = q$ then the above expression can be written as:

$$\frac{1}{2}\left((4q+3)q^2 + \left(4q^2+q-1\right)q + \left(-8q^2-4q+1\right)q\right) = 0 \tag{C.13}$$

which can easily be verified as true. This also completes the proof of the Lemma 9.

□

[12]Let us notice that $\left\lfloor \frac{(2q+1)^2}{4q+3} \right\rfloor = \left\lfloor \frac{4q^2+4q+1}{4q+3} \right\rfloor = \ldots = \left\lfloor q + \frac{q+1}{4q+3} \right\rfloor$. The fact that for $q = 0, 1, \ldots$ the expression $\frac{q+1}{4q+3}$ is always smaller than 1, implies that $\left\lfloor \frac{(2q+1)^2}{4q+3} \right\rfloor = \lfloor q \rfloor$.

Appendix D. Proof of the Lemma 10

THESIS.

For every $n \in \mathbb{N}_+, n \geq 3$ the function $\mathcal{G}$ is strictly decreasing for every $m \in \mathbb{N}_+$ such that $1 \leq m \leq \mathcal{X}(n)$, i.e.

$$\mathcal{G}(n,m) - \mathcal{G}(n,m+1) > 0 \ \textit{where} \ 1 \leq m < \mathcal{X}(n) \tag{70}$$

PROOF OF (70), PART 1 (FOR EVEN NUMBERS)

Let $n = 2q$ (even), $n \geq 3$, hence $q \geq 2$, and $m, m+1 \leq \mathcal{X}(2q) = q(q-1)$. Note that, in particular, the last assumption implies that $m \leq q(q-1)-1$. Hence (70) can be written as:

$$\begin{aligned}
3\left(\mathcal{G}(n,m) - \mathcal{G}(n,m+1)\right) = &-2q\left\lfloor \frac{m}{q} \right\rfloor^2 + (4m-2q)\left\lfloor \frac{m}{q} \right\rfloor + 2q\left\lfloor \frac{m+1}{q} \right\rfloor^2 \\
&-3q\left\lfloor \frac{m}{2q} \right\rfloor^2 + 3q\left\lfloor \frac{m+1}{2q} \right\rfloor^2 - 4m\left\lfloor \frac{m+1}{q} \right\rfloor \\
&+2q\left\lfloor \frac{m+1}{q} \right\rfloor - 4\left\lfloor \frac{m+1}{q} \right\rfloor + 3(m-q)\left\lfloor \frac{m}{2q} \right\rfloor \\
&-3m\left\lfloor \frac{m+1}{2q} \right\rfloor + 3q\left\lfloor \frac{m+1}{2q} \right\rfloor - 3\left\lfloor \frac{m+1}{2q} \right\rfloor + 6q - 6
\end{aligned} \tag{D.1}$$

Let us denote $r_1 = \left\lfloor \frac{m}{q} \right\rfloor, r_2 = \left\lfloor \frac{m}{2q} \right\rfloor, r_3 = \left\lfloor \frac{m+1}{q} \right\rfloor, r_4 = \left\lfloor \frac{m+1}{2q} \right\rfloor$. This allows us to denote

$$\begin{aligned}
3\left(\mathcal{G}(n,m) - \mathcal{G}(n,m+1)\right) = &-2qr_1^2 + (4m-2q)r_1 + 2qr_3^2 - 3qr_2^2 \\
&+3qr_4^2 - 4mr_3 + 2qr_3 - 4r_3 + 3(m-q)r_2 \\
&-3mr_4 + 3qr_4 - 3r_4 + 6q - 6
\end{aligned} \tag{D.2}$$

Let us introduce the auxiliary function h such that

$$\begin{aligned}
h(q,m,r_1,r_2,r_3,r_4) \stackrel{df}{=} &r_1(4m-2q) + 3r_2(m-q) - 4mr_3 - 3mr_4 \\
&-2qr_1^2 - 3qr_2^2 + 2qr_3^2 + 3qr_4^2 + 2qr_3 \\
&+3qr_4 + 6q - 4r_3 - 3r_4 - 6
\end{aligned} \tag{D.3}$$

It is easy to verify that

$$3\left(\mathcal{G}(n,m) - \mathcal{G}(n,m+1)\right) = h(q,m,r_1,r_2,r_3,r_4) \tag{D.4}$$

Let us try to investigate changes in the values r_1, r_2, r_3 and r_4. To do so, let us create the following table:

interval of m	$\left\lfloor \frac{m}{q} \right\rfloor$	$\left\lfloor \frac{m}{2q} \right\rfloor$	$\left\lfloor \frac{m+1}{q} \right\rfloor$	$\left\lfloor \frac{m+1}{2q} \right\rfloor$
$0q \leq m < 1q-1$	0	0	0	0
$1q-1 = m$	0	0	1	0
$1q \leq m < 2q-1$	1	0	1	0
$2q-1 = m$	1	0	2	1
$2q \leq m < 3q-1$	2	1	2	1
$3q-1 = m$	2	1	3	1
$3q \leq m < 4q-1$	3	1	3	1
$4q-1 = m$	3	1	4	2
$4q \leq m < 5q-1$	4	2	4	2

As we can see, there are four kinds of interval (hereinafter referred to as cases) that need to be considered with respect to m. Every analyzed interval is parametrized by the auxiliary variable $s \in \mathbb{N} \cup \{0\}$. By choosing arbitrarily $s = 0, 1, 2, 3, \ldots$ we are able to analyze the function h, and as follows $\mathcal{G}(n,m) - \mathcal{G}(n,m+1)$, for every interesting m. The cases we need to consider are:

Case	interval of m	$\left\lfloor \frac{m}{q} \right\rfloor$	$\left\lfloor \frac{m}{2q} \right\rfloor$	$\left\lfloor \frac{m+1}{q} \right\rfloor$	$\left\lfloor \frac{m+1}{2q} \right\rfloor$
1a	$2sq \leq m < (2s+1)q-1$	$2s$	s	$2s$	s
2a	$(2s+1)q-1 = m$	$2s$	s	$2s+1$	$s+1$
3a	$(2s+1)\,q \leq m < (2s+2)q-1$	$2s+1$	s	$2s+1$	s
4a	$(2s+1)q-1 = m$	$2s$	s	$2s+1$	s

Case 1a

Let $2sq \leq m < (2s+1)q-1$. As $m \leq q(q-1)-1$, then the candidate for the highest value of s is the smallest integer for which $q(q-1)-1 < (2s+1)q-1$, hence $\frac{q-2}{2} < s$. This means that $\left\lfloor \frac{q-2}{2} \right\rfloor + 1 = s$, hence $\frac{q-2}{2} + 1 \geq s$. On the other hand, as $2sq \leq m$ and $m \leq q(q-1)-1$ then $s \leq \frac{q(q-1)-1}{2q}$. Since the second condition is more restrictive[13] we assume that $s \leq \frac{q(q-1)-1}{2q}$. Let us denote

$$h(q, m, r_1, r_2, r_3, r_4) = h(q, m, 2s, s, 2s, s) \tag{D.5}$$

Hence,

$$h(q, m, 2s, s, 2s, s) = 6q - 11s - 6 \tag{D.6}$$

The highest possible value of s is $\frac{q(q-1)-1}{2q}$, hence the minimal value of h providing this constraint is $6(q-1) - 11\frac{q(q-1)-1}{2q}$ i.e.

$$h(q, m, 2s, s, 2s, s) \geq 6(q-1) - 11\frac{q(q-1)-1}{2q} \tag{D.7}$$

Which is equivalent to

[13]Note that $\left(\frac{q-2}{2} + 1\right) - \frac{q(q-1)-1}{2q} = \frac{1+q}{2q}$

$$h(q,m,2s,s,2s,s) \geq \frac{q^2-q+11}{2q} \tag{D.8}$$

Hence, it is clear that for $q \geq 2$ the right side of the above equation is always greater than 0.

Case 2a

Let $(2s+1)q-1=m$. Since $m \leq q(q-1)-1$ then s cannot be higher than the maximal integer which meets the inequality $(2s+1)q-1 \leq q(q-1)-1$, i.e. $s \leq \frac{q-2}{2}$. Let us calculate h, for $m=(2s+1)q-1$, $r_1=2s, r_2=s, r_3=2s+1$ and $r_4=s+1$.

$$h(q,m,r_1,r_2,r_3,r_4)=9q-11s-6 \tag{D.9}$$

As the maximal $s=\frac{q-2}{2}$ then

$$h(q,m,r_1,r_2,r_3,r_4) \geq 9q-11\frac{q-2}{2}-6 \tag{D.10}$$

which is equivalent to

$$h(q,m,r_1,r_2,r_3,r_4) \geq \frac{7q}{2}+5 \tag{D.11}$$

It is clear that for $q \geq 2$ the right side of the above equation is always greater than 0.

Case 3a

Let $(2s+1)\,q \leq m < (2s+2)q-1$

Since $m \leq q(q-1)-1$ then s is not higher than the maximal integer which meets the inequality $q(q-1)-1 < (2s+2)q-1$, i.e. $\frac{q-3}{2} < s$. Thus, $s=\left\lfloor\frac{q-3}{2}\right\rfloor+1$, hence $s \leq \frac{q-3}{2}+1$. On the other hand, also $(2s+1)\,q \leq m$ and $m \leq q(q-1)-1$. Thus s should meet $(2s+1)\,q \leq q(q-1)-1$, i.e. $s \leq \frac{1}{2}\left(\frac{q(q-1)-1}{q}-1\right)$. The second condition is more restrictive[14] hence we assume that $s \leq \frac{1}{2}\left(\frac{q(q-1)-1}{q}-1\right)$. Let us calculate h assuming $r_1=2s+1, r_2=s, r_3=2s+1$, and $r_4=s$. So,

$$h(q,m,r_1,r_2,r_3,r_4)=h(q,m,2s+1,s,2s+1,s) \tag{D.12}$$

and thus,

$$h(q,m,2s+1,s,2s+1,s)=6q-11s-10 \tag{D.13}$$

The highest allowed value of s is $\frac{1}{2}\left(\frac{q(q-1)-1}{q}-1\right)$, thus it is true that

$$h(q,m,2s+1,s,2s+1,s) \geq 6q-\frac{11}{2}\left(\frac{q(q-1)-1}{q}-1\right)-10 \tag{D.14}$$

which is equivalent to

$$h(q,m,2s+1,s,2s+1,s) \geq \frac{1}{2}\left(q+\frac{11}{q}+2\right) \tag{D.15}$$

[14] as $\left(\frac{q-3}{2}+1\right)-\frac{1}{2}\left(\frac{q(q-1)-1}{q}-1\right)=\frac{q+1}{2q}$

It is clear that for $q \geq 2$ the above equation is always greater than 0.

CASE 4A

Let $(2s+1)q-1 = m$

Since $m \leq q(q-1)-1$ then s cannot be higher than the maximal integer which meets the inequality $(2s+1)q-1 \leq q(q-1)-1$, i.e. $s \leq \frac{q-2}{2}$. Let us calculate h, by the assumptions that $m = (2s+1)q-1$, $r_1 = 2s, r_2 = s, r_3 = 2s+1$ and $r_4 = s$.

$$h(q,m,r_1,r_2,r_3,r_4) = 6q - 11s - 6 \tag{D.16}$$

Since the maximal s is $\frac{q-2}{2}$ then

$$h(q,m,r_1,r_2,r_3,r_4) \geq 6q - 11\left(\frac{q-2}{2}\right) - 6 \tag{D.17}$$

which is equivalent to

$$h(q,m,r_1,r_2,r_3,r_4) \geq \frac{q}{2} + 5 \tag{D.18}$$

It is clear that for $q \geq 2$ the above equation is always greater than 0. This remark completes the proof for $n = 2q$.

PROOF OF (70), PART 2 (FOR ODD NUMBERS)

Let $n = 2q+1$ (odd), $n \geq 3$, hence $q \geq 1$ and $0 \leq m, m+1 \leq \mathcal{X}(2q+1) = q^2$. In particular, the last assumption implies that $0 \leq m \leq q^2-1$. When $n = 2q+1$ it holds that:

$$\begin{aligned}
6\left(\mathcal{G}(n,m) - \mathcal{G}(n,m+1)\right) = &- 6 + 12q + (6m-6q-3)\left\lfloor \frac{m}{2q+1} \right\rfloor \\
&- 3(2q+1)\left\lfloor \frac{m}{2q+1} \right\rfloor^2 + (8m-4q-2)\left\lfloor \frac{2m}{2q+1} \right\rfloor \\
&- 2(2q+1)\left\lfloor \frac{2m}{2q+1} \right\rfloor^2 - 3\left\lfloor \frac{m+1}{2q+1} \right\rfloor \\
&- 6m\left\lfloor \frac{m+1}{2q+1} \right\rfloor + 6q\left\lfloor \frac{m+1}{2q+1} \right\rfloor + 3\left\lfloor \frac{m+1}{2q+1} \right\rfloor^2 \\
&+ 6q\left\lfloor \frac{m+1}{2q+1} \right\rfloor^2 - 6\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor - 4q\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor^2 \\
&+ 2\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor^2 + 8m\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor + 4q\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor
\end{aligned} \tag{D.19}$$

Let us denote $r_1 = \left\lfloor \frac{2m}{2q+1} \right\rfloor$, $r_2 = \left\lfloor \frac{m}{2q+1} \right\rfloor$, $r_3 = \left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor$ and $r_4 = \left\lfloor \frac{m+1}{2q+1} \right\rfloor$. This allows us to simplify the above equation to

$$\begin{aligned}
6\left(\mathcal{G}(n,m) - \mathcal{G}(n,m+1)\right) = &- 6 + 12q + r_1(8m-4q-2) \\
&- 2(2q+1)r_1^2 + r_2(6m-6q-3) \\
&- 8mr_3 - 6mr_4 + 3(2q+1)r_2^2 + 4qr_3^2 + 6qr_4^2 \\
&+ 4qr_3 + 6qr_4 + 2r_3^2 + 3r_4^2 - 6r_3 - 3r_4
\end{aligned} \tag{D.20}$$

Let us define:

$$\begin{aligned} h(q,m,r_1,r_2,r_3,r_4) = &-6+12q+r_1(8m-4q-2)-2(2q+1)r_1^2 \\ &+r_2(6m-6q-3)-8mr_3-6mr_4 \\ &+3(2q+1)r_2^2+4qr_3^2+6qr_4^2 \\ &+4qr_3+6qr_4+2r_3^2+3r_4^2-6r_3-3r_4 \end{aligned} \tag{D.21}$$

It is clear that

$$6\left(\mathcal{G}(n,m)-\mathcal{G}(n,m+1)\right)>0 \Leftrightarrow h(q,m,r_1,r_2,r_3,r_4)>0 \tag{D.22}$$

Let us try to investigate changes in the values r_1, r_2, r_3 and r_4. To do so, let us write down a few cases of each in the form of a table:

interval	$\left\lfloor \frac{2m}{2q+1} \right\rfloor$
$0 \le m < \frac{1}{2}(2q+1)$	0
$\frac{1}{2}(2q+1) \le m < \frac{2}{2}(2q+1)$	1
$\frac{2}{2}(2q+1) \le m < \frac{3}{2}(2q+1)$	2
$\frac{3}{2}(2q+1) \le m < \frac{4}{2}(2q+1)$	3
$\frac{4}{2}(2q+1) \le m < \frac{5}{2}(2q+1)$	4

interval	$\left\lfloor \frac{m}{2q+1} \right\rfloor$
$0 \le m < 2q+1$	0
$2q+1 \le m < 2(2q+1)$	1
$2(2q+1) \le m < 3(2q+1)$	2
$3(2q+1) \le m < 4(2q+1)$	3
$4(2q+1) \le m < 5(2q+1)$	4

interval	$\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor$
$0 \le m < \frac{1}{2}(2q+1)-1$	0
$\frac{1}{2}(2q+1)-1 \le m < \frac{2}{2}(2q+1)-1$	1
$\frac{2}{2}(2q+1)-1 \le m < \frac{3}{2}(2q+1)-1$	2
$\frac{3}{2}(2q+1)-1 \le m < \frac{4}{2}(2q+1)-1$	3
$\frac{4}{2}(2q+1)-1 \le m < \frac{5}{2}(2q+1)-1$	4

interval	$\left\lfloor \frac{m+1}{2q+1} \right\rfloor$
$0 \le m < (2q+1)-1$	0
$(2q+1)-1 \le m < 2(2q+1)-1$	1
$2(2q+1)-1 \le m < 3(2q+1)-1$	2
$3(2q+1)-1 \le m < 4(2q+1)-1$	3
$4(2q+1)-1 \le m < 5(2q+1)-1$	4

As we can see, there are four kinds of interval (hereinafter referred to as cases) that need to be considered with respect to m. Every analyzed interval is parametrized by the auxiliary variable $s \in \mathbb{N} \cup \{0\}$. By choosing arbitrarily $s = 0, 1, 2, 3, \ldots$ we are able to analyze the function h, and as follows $\mathcal{G}(n, m) - \mathcal{G}(n, m+1)$, for every interesting m. The cases we need to consider are:

Case	interval of m	$\left\lfloor \frac{2m}{2q+1} \right\rfloor$	$\left\lfloor \frac{m}{2q+1} \right\rfloor$	$\left\lfloor \frac{2(m+1)}{2q+1} \right\rfloor$	$\left\lfloor \frac{m+1}{2q+1} \right\rfloor$
1b	$\frac{2s}{2}(2q+1) \le m < \frac{2s+1}{2}(2q+1) - 1$	$2s$	s	$2s$	s
2b	$m = \frac{2s+1}{2}(2q+1) - 1$	$2s$	s	$2s+1$	s
3b	$\frac{2s+1}{2}(2q+1) \le m < \frac{2s+2}{2}(2q+1) - 1$	$2s+1$	s	$2s+1$	s
4b	$m = \frac{2s+2}{2}(2q+1) - 1$	$2s+1$	s	$2s+2$	$s+1$

Case 1b

Let $\frac{2s}{2}(2q+1) \le m < \frac{2s+1}{2}(2q+1) - 1$.

In general $0 \le m \le q^2 - 1$, thus $0 \le \frac{2s}{2}(2q+1)$ and $q^2 - 1 < \frac{2s+1}{2}(2q+1) - 1$ which implies (providing that $s \in \mathbb{N} \cup \{0\}$) that $0 \le s$ and s should not be greater than the smallest integer that meets the inequality $s > \frac{q^2}{2q+1} - 1$. This implies that $s = \left\lfloor \frac{q^2}{2q+1} - 1 \right\rfloor + 1$, thus $s \le \frac{q^2}{2q+1}$. On the other hand, $\frac{2s}{2}(2q+1) \le m$ and $m \le q^2 - 1$. This suggests that $\frac{2s}{2}(2q+1) \le q^2 - 1$, i.e. $s \le \frac{q^2-1}{2q+1}$. Since the second constraint is more restrictive[15] then we adopt $s \le \frac{q^2-1}{2q+1}$

Thus, let us consider $h(q, m, r_1, r_2, r_3.r_4)$ where, following the assumptions of case 1, $r_1 = 2s$, $r_2 = s$, $r_3 = 2s$ and $r_4 = s$. It is easy to calculate that

$$h(q, m, 2s, s, 2s, s) = 12q - 22s - 6 \tag{D.23}$$

The highest possible s is $\frac{q^2-1}{2q+1}$, hence it holds that

$$h(q, m, 2s, s, 2s, s) \ge 6(2q-1) - 22\left(\frac{q^2-1}{2q+1}\right) \tag{D.24}$$

which is true if and only if

$$h(q, m, 2s, s, 2s, s) \ge \frac{2\left(q^2+8\right)}{2q+1} \tag{D.25}$$

It is clear that the above expression is strictly higher than 0 for $q \ge 1$.

Case 2b

Let $m = \frac{2s+1}{2}(2q+1) - 1$

The highest possible value of m is $q^2 - 1$ thus $m = \frac{2s+1}{2}(2q+1) - 1 \le q^2 - 1$, hence, $s \le \frac{1}{2}\left(\frac{2q^2}{2q+1} - 1\right)$.

Let us consider $h(q, m, r_1, r_2, r_3.r_4)$ where (see case 2) $r_1 = 2s$, $r_2 = s$, $r_3 = 2s+1$, $r_4 = s$ and denote:

$$\widehat{h}(q, m, r_1, r_2, r_3.r_4) \stackrel{df}{=} h(q, \frac{2s+1}{2}(2q+1) - 1, 2s, s, 2s+1, s) \tag{D.26}$$

[15] as $\frac{q^2}{2q+1} - \frac{q^2-1}{2q+1} = \frac{1}{2q+1}$

Thus, we may calculate that

$$\widehat{h}(q,m,r_1,r_2,r_3.r_4) = 12q - 22s - 6 \tag{D.27}$$

Adopting the upper bound of $s = \frac{1}{2}\left(\frac{2q^2}{2q+1} - 1\right)$ we obtain

$$\widehat{h}(q,m,r_1,r_2,r_3.r_4) \geq 12q - 22\left(\frac{1}{2}\left(\frac{2q^2}{2q+1} - 1\right)\right) - 6 \tag{D.28}$$

which is equivalent to

$$\widehat{h}(q,m,r_1,r_2,r_3.r_4) \geq \frac{2q^2 + 22q + 5}{2q+1} \tag{D.29}$$

It is clear that the right side of the above expression is strictly higher than 0 for $q \geq 1$.

Case 3b

Let $\frac{2s+1}{2}(2q+1) \leq m < \frac{2s+2}{2}(2q+1) - 1$. The highest possible value of m is $q^2 - 1$, thus the highest possible value of s cannot be greater than the smallest positive integer for which $q^2 - 1 < \frac{2s+2}{2}(2q+1) - 1$. Hence $\frac{q^2}{2q+1} - 2 < s$, which implies that $\left\lfloor \frac{q^2}{2q+1} - 2 \right\rfloor + 1 = s$. Therefore $\frac{q^2}{2q+1} - 1 \geq s$. On the other hand, $\frac{2s+1}{2}(2q+1) \leq m$ and $m \leq q^2 - 1$. This suggests that $\frac{1}{2}\left(\frac{2\left(q^2-1\right)}{2q+1} - 1\right) \geq s$. Since the first condition is more restrictive[16] then we assume that $\frac{q^2}{2q+1} - 1 \geq s$.

Let us consider $h(q,m,r_1,r_2,r_3.r_4)$ where (following case 2) $r_1 = 2s+1$, $r_2 = s$, $r_3 = 2s+1$ and $r_4 = s$. It is easy to calculate that

$$h(q,m,2s+1,s,2s+1,s) = 2(6q - 11s - 7) \tag{D.30}$$

The upper bound for s is $\frac{q^2}{2q+1} - 1$, thus

$$h(q,m,2s+1,s,2s+1,s) \geq 2\left(6q - 11\left(\frac{q^2}{2q+1} - 1\right) - 7\right) \tag{D.31}$$

which is equivalent to

$$h(q,m,2s+1,s,2s+1,s) \geq \frac{2\left(q^2 + 14q + 4\right)}{2q+1} \tag{D.32}$$

It is clear that the above expression is strictly higher than 0 for $q \geq 1$.

Case 4b

Let $m = \frac{2s+2}{2}(2q+1) - 1$. The highest possible value of m is $q^2 - 1$. Thus $m = \frac{2s+2}{2}(2q+1) - 1 \leq q^2 - 1$, which is equivalent to $s \leq \frac{1}{2}\left(\frac{q^2}{2q+1} - 1\right)$.

Let us consider $h(q,m,r_1,r_2,r_3.r_4)$ where (see case 4) $r_1 = 2s+1$, $r_2 = s$, $r_3 = 2s+2$, $r_4 = s+1$ and denote:

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[16] as $\frac{1}{2}\left(\frac{2\left(q^2-1\right)}{2q+1} - 1\right) - \left(\frac{q^2}{2q+1} - 1\right) = \frac{2q-1}{4q+2}$

$$\widehat{h}(q, m, r_1, r_2, r_3.r_4) \overset{df}{=} h(q, \frac{2s+2}{2}(2q+1) - 1, 2s+1, s, 2s+2, s+1) \tag{D.33}$$

It is easy to calculate that

$$\widehat{h}(q, m, r_1, r_2, r_3.r_4) = 2(6q - 11s - 7) \tag{D.34}$$

As the highest possible value of s is $\frac{1}{2}\left(\frac{q^2}{2q+1} - 1\right)$ then

$$\widehat{h}(q, m, r_1, r_2, r_3.r_4) \geq 2\left(6q - 11\left(\frac{1}{2}\left(\frac{q^2}{2q+1} - 1\right)\right) - 7\right) \tag{D.35}$$

Which is equivalent to

$$\widehat{h}(q, m, r_1, r_2, r_3.r_4) \geq \frac{13q^2 + 6q - 3}{2q+1} \tag{D.36}$$

It is easy to verify that the above expression is strictly greater than 0 for $q \geq 1$. The last observation completes the proof of the lemma. $\square$

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The new triad based inconsistency indices for pairwise comparisons

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Abstract

Pairwise comparisons are widely recognized method supporting decision making process based on the subjective judgments. The key to this method is the notion of inconsistency that has a significant impact on the reliability of results. Inconsistency is expressed by means of inconsistency indices. Depending on their construction, such indices may pay attention to different aspects of the set of pairwise comparisons.

The family of indices proposed in this article tries to combine the advantages coming from different indices, thereby increases the expressiveness of the family elements. The newly introduced notion of equivalence can help in comparing the indices and identifying their common properties.

1. Introduction

Pairwise comparisons is a method of comparing objects in pairs in order to decide their importance. It is especially useful when intangible attributes of objects need to be compared. Due to their intangibility and abstractness objects (alternatives) need to be compared by human experts. Since, it is easier for a human to compare two objects at a time, first individual comparisons are made, then the result is synthesized as the ordered (ranking) list of objects. The list is used by decision makers to make a decision on objects.

Pairwise comparisons play an important role in well-known decision analysis methods such as *AHP* (Analytic Hierarchy Process) and its generalization *ANP* (Analytic Network Process)[23]. The effectiveness of the method was confirmed several times in practice[10].

A desirable property of a set of comparisons is consistency. In other words if there are two experts opinions that the object a is two times more important than b, and b is three times more important than c, there should be also an opinion according to which a is six times more important than c. If the opinion on the pair (a, c) is different, this means there is a lack of consistency (existence of inconsistency). Since objects are compared in pairs by fallible humans, very often with regard to the abstract, intangible criteria, an inconsistency in judgments happens very often. If experts have a deep insight into the problem domain and act rationally, one can expect that the inconsistency is not to high, i.e. the

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judgment as regards the pair (a, c) does not differ to much from what could be inferred from two other comparisons (a, b) and (b, c). If it is high either the experts are not competent enough, they do not act rationally or the problem (criterion) is so complex and difficult that even two essentially different judgments about the same thing are justified. From the perspective of a person who has to make a decision based on the highly inconsistent expert judgments, each of these situations is bad. The decision becomes impossible. Therefore from the decision maker point of view it is important to have a possibly consistent set of comparative opinions.

Since the inconsistency (or consistency) is so important, the question arises how to represent it in the appropriate and meaningful way. The search for answers led to define a number of inconsistency (consistency) indices including the most known *Saaty's* Eigenvalue Based Index[22], Geometric Consistency Index proposed *Aguarï¿œn* and *Moreno-Jimenï¿œz*[1], *Barzilai's* Relative Error[2], *Pelï¿œez* and *Lamata's* index[20], *Koczkodaj's* inconsistency index[13] and others. Most of the proposed indices try to answer the question about the average level of inconsistency. Since every single disturbance contributes to the average, usually a single, local improvement in inconsistency between three different objects entails decreasing an inconsistency index. Thus, in the case of a change the customer quickly gets a useful feedback. On the other hand, it may happen that, despite the fact that the average inconsistency is relatively low, there are a small subset of comparisons for which inconsistency is high. The index defined by *Koczkodaj*[13] informs about the maximal local inconsistency. Therefore, it is able to protect the client against the local large disturbances of data. Whilst focusing on the worst local disturbance it does not take into account the average inconsistency. Thus, a number of changes (for better and for worse) may pass unnoticed.

The aim of the paper is to define a family of inconsistency indices that combine both the above-mentioned characteristics - sensitivity to any local changes and taking into account the worst local inconsistency spot. As a starting point the inconsistency introduced by a triad of objects, which is used to define more complex concepts including *Koczkodaj's* index, was chosen. This provides a basis for defining a family of inconsistency indices that, on the one hand, reflects the level of an average inconsistency but, on the other hand, provides information about the worst local disturbance.

2. Preliminaries

The first use of the set of pairwise comparisons for further result synthesis is attributed to *Llull* - a medieval scholar, mathematician, philosopher, alchemist[7,24]. Again, the use of this approach can be found in the *Condorcet's Theory of Voting*[27]. The pairwise comparisons method began to be regularly studied and used in the twentieth century. A number of works including *Thurstone*[26], *Bradley* et al.[4] or *Saaty*[22] significantly influenced on the development of the method. The latter introduced a hierarchy, which provides an easy way to handle the large number of criteria.

In time, the method has become more and more popular. The increased interest among researchers and practitioners has resulted in a number of publications dealing with the theory and practice of the method[12,21,14]. The result of scientific explorations are, for example, the *Rough Set* approach[9], fuzzy *PC* relation handling[19], incomplete *PC* relation[8], non-numerical rankings[11] and a ranking with a reference set[17,18]. A lot of research is devoted to the problem of inconsistency measuring. In works[16,6,3,5] authors analyze various properties of different inconsistency indices trying to find important regularities in their behavior.

The input data to the pairwise comparison method is a matrix $M = (m_{ij}) \wedge m_{ij} \in \mathbb{R}_+ \wedge i, j \in \{1, \ldots, n\}$ that contains the numerical values of expert judgments, so that m_{ij} means the relative importance of object c_i with respect to the object c_j.

Definition 1. *A matrix M is said to be reciprocal if for all $i, j \in \{1, \ldots, n\} : m_{ij} = 1/m_{ji}$, and M is said to be consistent if for all $i, j, k \in \{1, \ldots, n\} : m_{ij} \cdot m_{jk} = m_{ik}$.*

Very often it is assumed that the matrix M is reciprocal, which is in line with the natural intuition according to which if c_i is two times larger than c_j, thus also c_j is two times smaller than c_i. Of course, there are exceptions to this rule[15]. On the other hand, the matrices M, as containing subjective opinions of humans, are usually inconsistent. Thus, there are triads in M in the form m_{ij}, m_{jk}, m_{ik} for which $m_{ij} \cdot m_{jk} \neq m_{ik}$. Moreover, the more consistent triad m_{ij}, m_{jk}, m_{ik} the closer $m_{ij} \cdot m_{jk}$ and m_{ik}. This leads to the observation that along with the decreasing inconsistency in the triad m_{ij}, m_{jk}, m_{ik} the ratio $m_{ij} \cdot m_{jk}/m_{ik}$ tends to 1. It is easy to see that the following is always true:

$$\frac{m_{ij}m_{jk}}{m_{ik}} \leq 1 \quad \text{or} \quad \frac{m_{ik}}{m_{ij}m_{jk}} \leq 1 \tag{1}$$

Let the distance between the smaller ratio out of the two defined above and 1 be the triad inconsistency. Formally:

Definition 2. *The triad inconsistency is:*

$$\mathscr{K}(t) \stackrel{df}{=} min\left\{\left|1-\frac{m_{ij}}{m_{ik}m_{kj}}\right|,\left|1-\frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\} \quad (2)$$

where t is the triad (m_{ik}, m_{ij}, m_{kj}) *and* $i < k < j$.

The above definition allows for easy formulation of the Koczkodaj's inconsistency index [13]. Let T be a set of triads in the form $t = (m_{ik}, m_{ij}, m_{kj})$ where $i < k < j$, composed from the entries of M. Thus,

Definition 3. The *Koczkodaj* inconsistency index is

$$\mathscr{K}(M) \stackrel{df}{=} \max_{t\in T}\mathscr{K}(t) \quad (3)$$

It is easy to see that (Def. 3) detects the worst case of triad inconsistency. Thus, the changes in other triads, except one with the maximal inconsistency, are not reflected in the value of $\mathscr{K}(M)$.

According to [20,6] one can define the *Pelï¿œez-Lamata* triad inconsistency as follows:

$$\mathscr{PL}(t) \stackrel{df}{=} \frac{m_{ij}}{m_{ik}m_{kj}} + \frac{m_{ik}m_{kj}}{m_{ij}} - 2 \quad (4)$$

where t is the triad (m_{ik}, m_{ij}, m_{kj}) and $i < k < j$. This leads to the definition of the index for the entire matrix M.

Definition 4. *The Pelï¿œez-Lamata inconsistency index is*

$$\mathscr{PL}(M) \stackrel{df}{=} \frac{6\sum_{t\in T}\mathscr{PL}(t)}{n(n-1)(n-3)}. \quad (5)$$

3. The new inconsistency indices

The triad inconsistency (Def. 2) can also be used to define inconsistency indices that will have opposite properties than (Def. 3). That is, every improvement of a triad inconsistency will entail decreasing the index, although it may happen that even the high triad inconsistency remains undetected when the rest of the triads are sufficiently consistent. The natural candidates for such indices are the average values of the triad inconsistencies.

Before defining them let us look closer to $\mathscr{K}(t)$. It is easy to see that $\mathscr{K}(t) < 1$. That is because $|1-x| < 1$ for $x \in (0,1]$, and $|1-\frac{1}{x}| < 1$ for $x > 1$. On the other hand, a single triad (m_{ik}, m_{ij}, m_{kj}) corresponds to the one set $\{i, k, j\}$, where $i \neq j$, $j \neq k$, $i \neq k$, $i, j, k \in \{1, \ldots, n\}$ and n is the number of objects for comparison. Such sets could be selected in $\binom{n}{3}$ ways thus, the total number of distinct triads is $\binom{n}{3}$. This observation leads to the following estimation:

$$\sum_{t\in T}\mathscr{K}(t) < \binom{n}{3} = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{6} \quad (6)$$

where M is $n \times n$. Let the new inconsistency indices be

$$I_1(M) \stackrel{df}{=} \frac{6\sum_{t\in T}\mathscr{K}(t)}{n(n-1)(n-2)} \quad (7)$$

and

$$I_2(M) \stackrel{df}{=} \frac{6\sqrt{\sum_{t\in T}\mathscr{K}^2(t)}}{n(n-1)(n-2)} \quad (8)$$

Since $\mathscr{K}(t) < \mathscr{K}(M)$ for each $t \in T$, it is easy to see that

$$0 \leq I_2(M) \leq I_1(M) \leq \mathscr{K}(M) < 1. \quad (9)$$

In particular,

Remark 1. If M is consistent, then

$$I_2(M) = I_1(M) = \mathscr{K}(M) = 0. \tag{10}$$

The attentive reader may notice some similarity between the $I_1(M), I_2(M)$ inconsistency indices and the idea coming from *Pelï¿œez-Lamata*[20]. As it will be shown later, this similarity is superficial and indices I_1 and I_2 cannot be easily replaced by $\mathscr{PL}(M)$.

Similarly to the definition of strong equivalence of metrics[25] let us introduce the notion of equivalence of indices.

Definition 5. *Two inconsistency indices I and I′ are called equivalent if there exist positive constants α,β such that for every n × n pairwise comparisons matrix M holds*

$$\alpha I(M) \leq I'(M) \leq \beta I(M). \tag{11}$$

One can easily notice that

$$\mathscr{K}(M) = \sqrt{\mathscr{K}^2(M)} \leq \sqrt{\sum_{t\in T} \mathscr{K}^2(t)}, \tag{12}$$

thus,

$$\mathscr{K}(M) \leq \binom{n}{3} I_2(M). \tag{13}$$

(9) and (13) imply that $\mathscr{K}(M)$, $I_1(M)$ and $I_2(M)$ are equivalent.

Notice, that definitions of $\mathscr{PL}(M)$, $I_1(M)$ and $I_2(M)$ look similar. However, for a given triad $t \in T$ expression $m_{ij}/m_{ik}m_{kj}$ is unbounded, so the index $\mathscr{PL}(M)$ may be arbitrary large. Thus, it is not equivalent to $\mathscr{K}(M)$, $I_1(M)$ or $I_2(M)$.

When we take the convex combinations of these indices we may obtain the whole family of indices:

Definition 6. *Let* $\alpha, \beta, \alpha + \beta \in (0, 1)$. *Put*

$$I_\alpha(M) = \alpha\mathscr{K}(M) + (1 - \alpha)I_1(M), \tag{14}$$

$$I_{\alpha,\beta}(M) = \alpha\mathscr{K}(M) + \beta I_1(M) + (1 - \alpha - \beta)I_2(M). \tag{15}$$

Of course,

$$0 \leq \alpha\mathscr{K}(M) < I_{\alpha,\beta}(M) < I_\alpha(M) \leq \mathscr{K}(M) < 1, \tag{16}$$

for each $\alpha, \beta, \alpha + \beta \in (0, 1)$, so $\mathscr{K}(M)$, $I_\alpha(M)$ and $I_{\alpha,\beta}(M)$ are also equivalent. In other words, most theorems proven for $\mathscr{K}(M)$ also hold for $I_\alpha(M)$ ($I_{\alpha,\beta}(M)$) and some α sufficiently close to 1 (and β close to 0).

Example. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 60 & 1 \\ 1 & 1 & 2 & \frac{1}{2} \\ \frac{1}{60} & \frac{1}{2} & 1 & \frac{1}{4} \\ 1 & 2 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 3 & \frac{5}{8} \\ 1 & 1 & 2 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{8}{5} & 2 & 4 & 1 \end{bmatrix}.$$

The rounded results of their indices' calculations are given in the table:

M	$\mathscr{K}(M)$	$\mathscr{PL}(M)$	$I_1(M)$	$I_2(M)$	$I_{\frac{1}{2}}(M)$	$I_{\frac{1}{2},\frac{1}{3}}(M)$
A	0.97	10.4	0.6	0.36	0.78	0.74
B	0.33	0.06	0.18	0.11	0.25	0.24

Hence,

$$0 < I_2(A) < I_1(A) < I_{\frac{1}{2},\frac{1}{3}}(A) < I_{\frac{1}{2}}(A) < \mathscr{K}(A) < 1 < \mathscr{PL}(A)$$

and

$$0 < \mathscr{PL}(B) < I_2(B) < I_1(B) < I_{\frac{1}{2},\frac{1}{3}}(B) < I_{\frac{1}{2}}(B) < \mathscr{K}(B) < 1.$$

4. Conclusion

Although the indices I_1 and I_2 are the opposite of *Koczkodaj's* index because they neglect the maximal triad inconsistency and follow every single change of M, the indices $I_\alpha(M)$ and $I_{\alpha,\beta}(M)$ combine properties of their predecessors. In the case of the local growth of $\mathscr{K}(t)$ for some $t \in T$ they are growing, and reversely, if $\mathscr{K}(t)$ drops down for some fixed $t \in T$ (and $\mathscr{K}(r)$ for $r \neq t$ and $r \in T$ is not rising) they are decreasing. On the other hand, these indices will never be smaller than $\alpha\mathscr{K}(M)$. Thus, they provide a guarantee that even a single but large triad inconsistency will not be ignored.

The parameters α, β and can be treated as the priority coefficients that determine the importance of the individual sub-indices. Thus, depending on the situation the customer can increase α at the expense of other factors, thereby increasing the importance of the maximal triad inconsistency, or decrease α assigning the average inconsistency of greater importance.

The concept of equivalence of inconsistency indices can also be used in the context of other indices than concerned. In particular it seems to be interesting to examine the equivalence classes introduced by this concept. Identification of mutual relationships between these classes can be interesting areas of further investigations.

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A New Measure of Consistency for Positive Reciprocal Matrices

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Abstract—The analytic hierarchy process (AHP) provides a decision maker with a way of examining the consistency of entries in a pairwise comparison matrix and the hierarchy as a whole through the consistency ratio measure. It has always seemed to us that this commonly used measure could be improved upon. The purpose of this paper is to present an alternative consistency measure and demonstrate how it might be applied in different types of matrices.

Keywords—Reciprocal matrices, Consistency index, Eigenvalues, Preferences.

1. INTRODUCTION

The traditional eigenvector method for estimating weights in the analytic hierarchy process (AHP) (see [1]) yields a way of measuring the consistency of a decision maker's preferences arranged in the form of a reciprocal pairwise comparison matrix. The *consistency index* (CI) is given by

$$\mathrm{CI} \equiv \frac{\lambda_{\max} - n}{n-1}, \tag{1}$$

where $\lambda_{\max}$ is the largest eigenvalue of the $n \times n$ reciprocal pairwise comparison matrix.

In [1], Saaty showed that if a decision maker is perfectly consistent (i.e., $a_{ik} = a_{ij}\,a_{jk}$ for all $i, j, k = 1, \dots, n$), $\lambda_{\max} = n$ (CI $= 0$) and if the decision maker is not perfectly consistent, then $\lambda_{\max} > n$. To measure this consistency, Saaty proposed a consistency ratio defined as

$$\mathrm{CR} \equiv \frac{\mathrm{CI}}{\mathrm{RI}}, \tag{2}$$

where RI is the average value of CI obtained from 500 positive reciprocal pairwise comparison matrices whose entries were randomly generated using the 1 to 9 scale. Saaty considers that a

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Table 1. Values of the random index for different matrix orders.

N	1–2	3	4	5	6	7
RI	0	0.58	0.90	1.12	1.24	1.32

value of CR under 0.10 indicates that the decision maker is sufficiently consistent. Table 1 gives values of the average RI for different values of n.

This consistency measure is a reasonable measure but, at the same time, somewhat arbitrary [2–7]. Several questions come to mind.

(1) Why ten percent?
(2) Should the cut-off rule be a function of the matrix size?
(3) It is possible to use the CI in other types of reciprocal matrices, e.g., $a_{ik} = a_{ij} \oplus a_{jk}$?

In an attempt to answer these questions, we have developed a measure of consistency [4] that is

(1) easy to use,
(2) a function of the matrix size,
(3) applicable to other types of reciprocal matrices.

The purpose of this work is to develop and demonstrate an alternative measure of consistency. The paper is laid out as follows. In Section 2, we present the new consistency index (CI*); in Section 3, we recommend a critical value for the new consistency index (RC*); in Section 4, we applied the alternative consistency measure to other types of reciprocal matrices; in Section 5, we illustrate it with some examples, and finally in Section 6, we show the conclusion.

2. A NEW MEASURE OF CONSISTENCY

So that inconsistency exists in the judgments, we need to at least compare three alternatives, because when we compare two alternatives the judgments are always perfect (for $n \times n$ matrices with $n < 3$ there is no inconsistency). Comparing three alternatives, it is possible that inconsistency exists when

(i) there exists a cycle between the alternatives ($a_i > a_j > a_k > a_i$),
(ii) the intensity with value a_{ik} is different to the product of the arc a_{ij}, a_{jk} (see Figure 1).

Therefore, the existent relationship among three alternatives (Figure 1) is defined as transitivity (Γ) [8], for us being the minimal element of consistency [6].

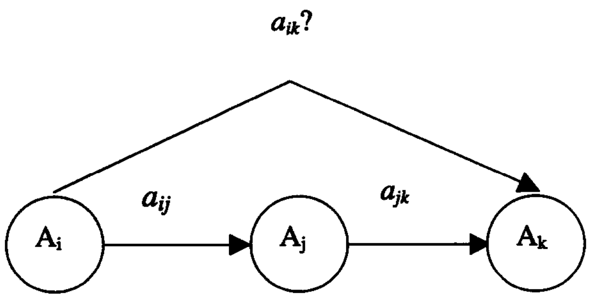

Figure 1. Transitivity (Γ).

DEFINITION 1. *(See [8].) A preference structure on a set A is a triplet $\{P, I, R\}$ where*

- *P is a preference binary relation (asymmetric);*
- *I is an indifference binary relation (reflexive and symmetric);*
- *R is a binary relation representing no preference (irreflexive and symmetric);*
- *$P \cup I \cup R$ is a strongly complete binary relation;*
- *$P \cap I = \emptyset$, $I \cap R = \emptyset$, $P \cap R = \emptyset$.*

DEFINITION 2. *(See [8].) A preference structure $\{P, I, R\}$ is a weak order if and only if*

$$R = \emptyset,$$
$$P \text{ is transitive},$$
$$I \text{ is transitive}.$$

DEFINITION 3. *(See [8].) A transitivity Γ is a weak order preference structure on a set of three alternatives $A = \{A_i, A_j, A_k\}$.*

DEFINITION 4. *Two transitivities, Γ_i and Γ_j, are different, if they have at least one different element.*

By them, in the AHP the $M_{3\times3}$ reciprocal matrix *is the minimal element of consistency* [4].

C	A_i	A_j	A_k
A_i	1	a_{ij}	a_{ik}
A_j	$\frac{1}{a_{ij}}$	1	a_{jk}
A_k	$\frac{1}{a_{ik}}$	$\frac{1}{a_{jk}}$	1

Another way to measure consistency is by using the determinant of the matrix.

THEOREM 1. *The reciprocal pairwise comparison matrix $M_{3\times3}$ is perfect if and only if* $\det(M_{3\times3}) = 0$.

PROOF.

$$\det(M_{3\times3}) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2.$$

If the judgments are perfect, then $a_{ik} = a_{ij}a_{jk}$ and $\det(M_{3\times3}) = 0$. ∎

COROLLARY 1. *The judgments in a pairwise comparison matrices $M_{3\times3}$ are not perfect if* $\det(M) > 0$.

Since a number and its inverse sum is higher or equal to 2, then $\det(M_{3\times3}) > 0$.

Then, to measure the consistency of an $n \times n$ matrix, we measure all the different transitivities. The number of different transitivities (NT) of an $n \times n$ matrix is given by

$$\mathrm{NT}(M_{n\times n}) = \begin{cases} 0, & \text{if } n < 3, \\ \dfrac{n!}{(n-3)!3!}, & \text{otherwise.} \end{cases}$$

We defined the consistency index, and we denoted it as CI* to distinguish it from Saaty's index [1].

DEFINITION 5. *The consistency index CI* of an $M_{n\times n}$ matrix is given by the average of the consistency index of the matrix transitivities.*

$$\mathrm{CI}^*(M_{n\times n}) = \begin{cases} 0, & \textit{if } n < 3, \\ \det(M_{n\times n}), & \textit{if } n = 3, \\ \dfrac{1}{\mathrm{NT}(M_{n\times n})} \sum\limits_{i=1}^{\mathrm{NT}(M_{n\times n})} \mathrm{CI}^*(\Gamma_i), & \textit{if } n > 3, \end{cases}$$

where $\mathrm{NT}(M_{n\times n})$ is the number of different transitivities.

3. A CRITICAL VALUE

In the first section, we asked if the ten percent rule proposed by Saaty [1] to accept or reject judgments is reasonable and whether or not it should be a function of the matrix size. To answer these questions, we first study the relationship between the consistency index CI* and Saaty's consistency ratio. Table 2 shows the value of the new consistency index CI* corresponding to matrices with a value of Saaty's consistency ratio less than or equal to ten percent. As has already been pointed out by other authors [2], there are more than 25 percent of the 3-by-3 reciprocal matrices with a consistency ratio less than or equal to ten percent. As the matrix size increases, this percentage decreases dramatically. This shows that to uniformly accept or reject paired comparison matrices, the critical value should be a function of the matrix size.

Table 2. Values of the average CI* and percent of reciprocal matrices with CR ≤ 0.10.

n	3	4	5	6	7	8	9
CI*	1.132	1.208	1.258	1.284	1.329	1.354	1.379
Percent	25.88	4.62	4.60	0.20	0*	0*	0*

*Less than 0.1 percent.

In Table 2, 25.88 percent corresponds to the ten percent rule, and it is the 25.88 percentile of the distribution of Saaty's consistency ratio. The equivalent value of the new consistency index for 3-by-3 matrices would be 1.132 also given in Table 2. To develop a critical value for the new consistency index CI* we use a percentile of the distribution of CI*. These values will allow us to accept the same percentage of matrices for all values of n. Table 3 provides the 25.88 percentiles of CI* and Figure 2 gives the corresponding graphical plots for samples of size 100,000. These plots show that there appears to be a linear relationship between CI* and n for each corresponding percentile.

Table 3. 25.88 percentiles of CI*.

n	3	4	5	6	7	8	9
CI*	1.132	5.239	10.234	16.329	19.699	23.755	27.223

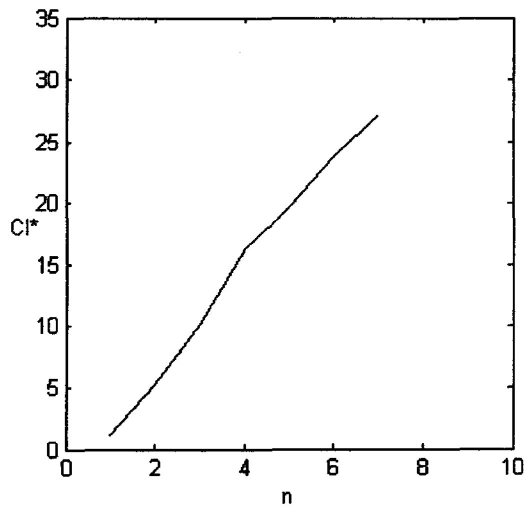

Figure 2. 25.88 percentile.

Table 4 *gives different values of* CI* for matrices of size three to nine and for different percentiles.

Figure 3 shows that the consistency index CI* is a function of the size of the matrix. This suggests that we should select a percentile of the distribution as the critical value that in turn yields the corresponding value of CI* for each value of n.

Table 4. CI* values for different matrices and percentiles (100,000 simulations).

Percentiles	Matrix Size						
	3	4	5	6	7	8	9
0.01	0.00	0.406	1.697	3.238	4.698	6.158	7.618
0.05	0.05	1.301	3.350	5.842	8.641	11.44	14.239
0.10	0.166	2.098	4.857	8.303	11.668	15.028	18.388
0.15	0.355	2.885	6.387	10.571	14.117	17.663	21.209
0.20	0.694	3.679	7.915	12.816	16.496	20.176	23.856
0.25	1.033	4.918	10.099	16.002	19.389	23.662	26.996
0.30	1.432	5.695	11.701	17.305	20.648	23.991	27.331
0.35	2.25	6.908	13.874	19.541	22.778	26.015	29.25
0.40	2.722	8.514	16.629	22.040	24.927	27.814	30.701
0.45	3.52	10.453	19.535	24.654	26.791	28.928	31.065
0.50	4.28	12.954	22.886	27.063	28.962	30.860	32.758

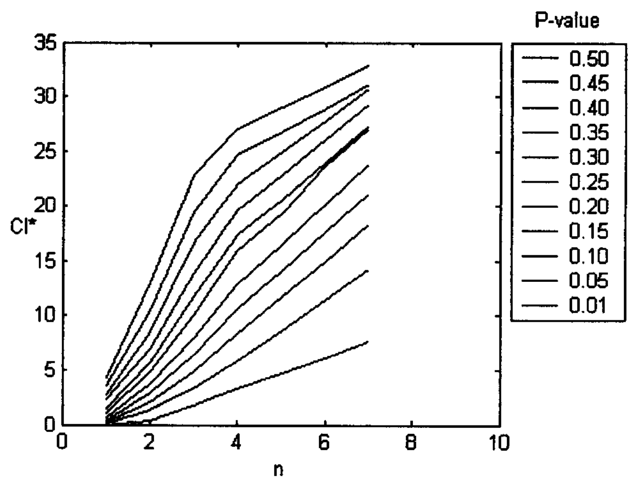

Figure 3. Percentiles of CI* as a function of n from Table 3.

4. USE OF THE NEW CONSISTENCY INDEX IN OTHER TYPES OF RECIPROCAL MATRICES

The attractiveness of the new consistency index is due to its potential use in fuzzy set theory [9–11]. Fuzzy sets are used in decision theory where the preference relation among the alternatives is additive instead of multiplicative. In the AHP, the preference relations satisfy the condition $a_{ij}a_{ji} = 1$, while in fuzzy set theory we have $a_{ji} = 1 - a_{ij}$, where $a_{ij} \in [0,1]$, and indifference corresponds to the value 0.5. Table 5 gives a 3-by-3 additive reciprocal matrix and its corresponding consistency index.

For these types of matrices we cannot use $\lambda_{\max}$ to measure inconsistency, and hence define a consistency index, because it is not a monotone function of the entries of the matrix. However, we

Table 5. A 3-by-3 additive reciprocal matrix.

C	A_i	A_j	A_k
A_i	0.5	0.6	0.8
A_j	0.4	0.5	0.8
A_k	0.2	0.2	0.5

CI* = 0.005

can use CI*. In this context, a consistent matrix is a matrix whose entries satisfy the condition for additive preference matrices defined by Lamata-Peláez [4]

$$a_{ik} = (a_{ij} - 0.5) + a_{jk}, \qquad \text{for all } i, j, k,$$

where a_{ij} ($a_{ij} \in [0,1]$) represents how much more preferred alternative A_i is than alternative A_j, and the indifference value is given by $a_{ii} = 0.5$. This definition of consistency is consistent with the definition of the new consistency measure given. We have the following.

THEOREM 2. *An additive reciprocal pairwise comparison matrix $M_{3\times 3}$ is consistent if and only if* $\det(M_{3\times 3}) = 0$. *If the matrix is inconsistent,* $\det(M_{3\times 3}) > 0$.

5. EXAMPLES

We illustrate the behavior of the new consistency measure with some 4-by-4 multiplicative reciprocal matrices. Table 6a shows a matrix considered inconsistent according to the CR criterion, but the new measure of inconsistency considers it consistent (see Table 2). The matrix in Table 6b is still inconsistent according to CR but not with the index. The matrices in Tables 6c and 6d are consistent under both criteria.

Table 6. Preference matrices with their consistency measures.

(a)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{5}$	
A_2	7	1	$\frac{1}{2}$	$\frac{1}{3}$	$\lambda_{\max} = 4.828$
A_3	7	2	1	$\frac{1}{9}$	CR = 0.306
A_4	5	3	9	1	CI* = 4.446

(b)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{9}$	
A_2	5	1	4	$\frac{1}{8}$	$\lambda_{\max} = 4.38$
A_3	3	$\frac{1}{4}$	1	$\frac{1}{9}$	CR = 0.14
A_4	9	8	9	1	CI* = 1.664

(c)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{9}$	
A_2	3	1	$\frac{1}{2}$	$\frac{1}{5}$	$\lambda_{\max} = 4.275$
A_3	7	2	1	$\frac{1}{7}$	$\mathrm{CR} = 0.102$
A_4	9	5	7	1	$\mathrm{CI}^* = 1.268$

(d)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{9}$	
A_2	3	1	$\frac{1}{2}$	$\frac{1}{5}$	$\lambda_{\max} = 4.167$
A_3	7	2	1	$\frac{1}{5}$	$\mathrm{CR} = 0.061$
A_4	9	5	5	1	$\mathrm{CI}^* = 0.741$

6. CONCLUSIONS

In this paper, we have presented a alternative measure to examine the consistency of entries in a pairwise comparison matrix. This measure is based in the determinant of the matrix and it measure the minimal element of consistency: the transitivities.

The advantages of this measure of consistency are:

(a) It is easy to use;
(b) it is a function of the matrix size; and finally,
(c) it is applicable to other types of reciprocal matrices.

Also in this work, we have proposed a new critical value that it is based in this measure and we have applied this measure in several examples.

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Theory and Methodology

Preference programming through approximate ratio comparisons

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Abstract

In the context of hierarchical weighting, this paper operationalizes interval judgments which allow the decision maker to enter ambiguous preference statements by indicating the relative importance of factors as intervals of values on a ratio scale. Through such judgments the decision maker can capture the subjective uncertainty in his preferences and thus avoid the often cumbersome elicitation of exact ratio estimates. After each new statement the interval judgments are synthesized into dominance relations on the alternatives by solving a series of linear programming problems. This leads to an interactive process of preference programming which provides more detailed results as the decision maker gradually enters a more specific preference description. Moreover, the overall effort of preference elicitation is smaller than in the analytic hierarchy process because the most preferred alternative can usually be identified before all possible comparisons between pairs of factors have been completed.

Keywords: Decision analysis; Analytic hierarchy process; Multiple criteria programming

1. Introduction

Methods of hierarchical weighting typically decompose the overall objectives of a problem into their lower level subobjectives until the resulting hierarchy provides a sufficiently detailed framework for the analysis. Within such a framework, the analytic hierarchy process (AHP) (Saaty, 1980) elicits preferences through pairwise comparisons in which the decision maker (DM) considers the relative importance of two factors at a time with respect to a common higher level criterion. For each comparison the DM indicates the intensity of preference of one factor over another as a point estimate on a ratio scale.

Several researchers have acknowledged the difficulties in eliciting exact ratio estimates. Van Laarhoven and Pedrycz (1983), Buckley (1985) and Boender et al. (1989) address this problem by suggesting fuzzy sets for the assessment and analysis of pairwise comparisons. Saaty and Vargas (1987), on the other hand, propose *interval judgments* which allow the DM to make approximate ratio statements as intervals of values on a ratio scale. Arbel (1989) interprets such judgments as linear constraints which at each criterion define a non-empty set of local priorities called the *feasible region*.

The present paper builds on Arbel's (1989,

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1991) approach by developing efficient algorithms for synthesizing interval judgments into dominance relations on the alternatives. These relations resemble those employed in multiattribute utility models (see, e.g. Bana e Costa, 1990; Hazen, 1986; Insua and French, 1991; Moskowitz et al., 1992; Weber, 1987), and because they are revised after each new preference statement, the DM is involved in an interactive decision support process which offers intermediate results even before most pairwise comparisons have been addressed.

The proposed approach to preference programming seems to have substantial practical potential due to the interactiveness of its decision support. It holds particular promise for the support of group decisions (Hämäläinen et al., 1992) since conflicting opinions can be modeled through composite intervals which contain the different views in the group. At the same time, the approach can be easily implemented into interactive decision aids such as INPRE (Salo and Hämäläinen, 1992c) because only linear programming is needed to compute the results.

This paper is organized as follows. Section 2 summarizes some of the earlier work on imprecise judgments in hierarchical weighting. Section 3 discusses dominance relations and develops algorithms for their computation. In Section 4, the consequences of the earlier judgments are analyzed to help the DM preserve the consistency of his judgments. Section 5 arranges the computations in the form of an algorithm, Section 6 addresses topics in sensitivity analysis, and Section 7 illustrates the methodology in the context of an energy policy problem.

2. Earlier work

In the framework of fuzzy analysis, van Laarhoven and Pedrycz (1983) propose triangular membership functions for the imprecise elicitation of pairwise comparisons and the computation of corresponding fuzzy weights. Buckley (1985) and Boender et al. (1989) extend these results to more general membership functions, and, like van Laarhoven and Pedrycz (1983), employ the logarithmic least squares method to compute the local priorities. Along with the many approximations involved, the main weakness of these approaches is perhaps the lack of clear-cut rules for converting the fuzzy weights into dominance results.

Following another line of investigation, Saaty and Vargas (1987) propose interval judgments for the AHP as a way to model the subjective uncertainty in the DM's preferences. With interval judgments, the DM can make statements such as 'the i-th subelement is at least weakly and at most strongly more important than the j-th subelement'. Using numerical counterparts for the verbal expressions of strength of preference gives the equivalent statement 'the i-th subelement is three to five times more important than the j-th subelement'; this is abbreviated as $I_{ij} = [l_{ij}, u_{ij}] = [3, 5]$.

In terms of interval judgments, the comparison matrix can be written as

$$\begin{pmatrix} 1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \cdots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \cdots & 1 \end{pmatrix}, \tag{1}$$

where the lower and upper bounds satisfy a reciprocity condition analogous to that of usual comparison matrices (Saaty and Vargas, 1987). That is, if the DM considers the i-th subelement to be more important than the j-th subelement ($l_{ij} = 1$), then the j-th subelement has to be less important than the i-th subelement ($u_{ji} = 1$). Extending this argument to the general case shows that $l_{ij}u_{ji} = 1$ for $i \neq j$.

Saaty and Vargas (1987) consider the derivation of local priorities from the matrix representation (1), but conclude that the problem of determining all the right principal eigenvectors of those reciprocal matrices whose elements belong to the intervals I_{ij} is computationally a relatively intractable task. In particular, the eigenvector is a nonlinear function of the matrix elements so that there are no straightforward techniques to determine exact bounds for its components. One possibility to approximate these bounds is to use Monte Carlo simulation to generate eigenvectors from samples of repreciprocal matrices whose elements belong to the intervals in (1).

Yoon (1988) studies how sensitive local priorities and final weights are to possible errors in the comparison matrix; however, in order to avoid complicated algebraic calculations, he approximates local priorities using the normalized row sum of the comparison matrix. Zahir (1991), on the other hand, characterizes perturbations in the right principal eigenvector due to uncertainty about the elements of the comparison matrix.

Instead of analyzing the properties of the interval matrix (1), Arbel (1989, 1991) interprets interval judgments as linear constraints on the local priorities. Emphasizing the definition of interval judgments, he points out that a given local priority vector $w=(w_1,\dots,w_n)$ is consistent with the judgment $I_{ij}=[l_{ij}, u_{ij}]$ only if it satisfies the constraints

$$l_{ij}w_j \le w_i \le u_{ij}w_j.$$

He then defines the *feasible region* as the set of those local priorities which satisfy all such constraints. Thus the feasible region can be written as

$$S=Q^n \cap \left\{w \mid l_{ij}w_j \le w_i \le u_{ij}w_j\right\}, \tag{2}$$

where

$$Q^n=\left\{(w_1,\dots,w_n) \mid w_i \ge 0,\ \sum_{i=1}^{n} w_i=1\right\}$$

and l_{ij}, u_{ij} correspond to those interval bounds that the DM has specified. According to this definition, the feasible region is well-defined even if some or even all of the bounds in (1) are missing, but it may become empty if the judgments are inconsistent. Fig. 1 shows the feasible region $S \in Q^3$ based on the judgments $I_{12}=[1, 2]$, $I_{13}=[1, 3]$.

Arbel and Vargas (1992) formulate maximization and minimization problems for establishing bounds for the components of right principal eigenvectors when the elements of a reciprocal matrix are constrained to the intervals in (1). They also characterize the alternatives' weight intervals as solutions to non-linear programs in which all the local priorities in the hierarchy are included as decision variables. As a result, the time needed to solve these optimization problems increases with the size of the hierarchy, and therefore this approach, which in the consistent case would produce weight intervals similar to ours, is unappealing for the development of interactive decision support.

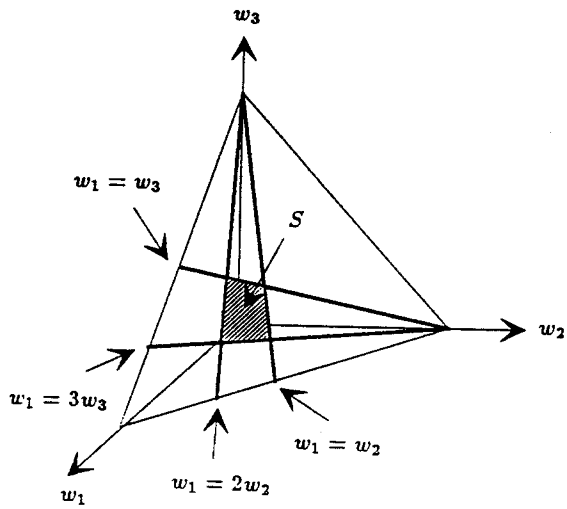

Fig. 1. A feasible region.

The method of Solymosi and Dombi (1986) converts ordinal preference statements into a feasible region from which the centroid vector is then chosen to represent the DM's preferences. Olson and Dorai (1992) observe that the centroid method often gives approximately similar results as the AHP even if it requires far fewer inputs. However, the centroid method may be somewhat unreliable because it does not allow the DM to enter ratio or other judgments that are more informative than ordinal comparisons.

The present paper focuses on the development of computationally efficient mechanisms for determining the preferred alternatives when the local priorities are constrained to the feasible sets (2). Furthermore, it extends earlier results on the derivation of weight intervals (Salo and Hämäläinen, 1992a) in that it proposes a less restrictive dominance criterion, considers the computation of ranges of criteria weights, and suggests an index for measuring the amount of imprecision in the DM's judgments.

The methodology of this paper can be regarded as a generalization of the conventional AHP. The main difference is that in addition to exact ratio comparisons the DM may specify interval judgments as well. This results in a natural extension of the AHP (Kress, 1991) as the consistency bounds of Section 4 help the DM to maintain the consistency of the preference model.

3. Processing interval judgments

This section considers two dominance concepts for synthesizing non-empty feasible regions into dominance relations for the alternatives. These concepts, following those used in imprecisely specified multiattribute utility models (see Bana e Costa, 1990; Hazen, 1986; Salo and Hämäläinen, 1992b; Weber, 1987), are based on an analysis of the weights that feasible parameter values assign to the alternatives. Computationally, the dominance results are determined from a decomposition scheme of linear programs.

3.1. Weight intervals and absolute dominance

Each combination of local priorities from the feasible regions gives a unique weight to each alternative. Therefore, as the local priorities are allowed to vary over the feasible regions, every alternative receives an interval of weights. Like the weights of the alternatives in the conventional AHP, these weight intervals convey information about which alternatives are preferred to others.

More specifically, if $V(x)$, the weight interval of alternative x, lies above that of alternative y (i.e. $r > s$ whenever $r \in V(x)$, $s \in V(y)$), then any feasible combination of local priorities assigns to x a weight greater than that of y. In such a situation x is said to dominate y according to the *absolute dominance* criterion, which is defined by

$$x \succ_{\mathrm{A}} y \Leftrightarrow \min_{r \in V(x)} r > \max_{s \in V(y)} s. \tag{3}$$

Clearly, if $x \succ_{\mathrm{A}} y$, then x is preferred to y because for all feasible local priorities the alternative x has the higher weight. The absolute dominance relation can be displayed to the DM conveniently through the weight intervals (see Fig. 10 in Section 7).

In principle, it is possible to approximate weight intervals by applying interval arithmetic to process bounds for the components of the local priorities (see, e.g. Moore, 1966). However, this approach is unsatisfactory because the components of the local priority vector, which add up to one, are not independent. For instance, if in a hierarchy of two alternatives and three criteria all feasible regions are convex combinations of the points $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$, then according to interval arithmetic the weight intervals for the alternatives are bounded from above by

$$\tfrac{3}{4} \times \tfrac{3}{4} + \tfrac{3}{4} \times \tfrac{3}{4} = \tfrac{9}{8} > 1.$$

This meaningless result demonstrates that interval arithmetic has to be rejected.

Instead, tight bounds for the weights of the alternatives can be found by solving optimization problems in which the alternative's weight is maximized/minimized subject to the DM's statements. By taking advantage of the principle of hierarchical composition, which guarantees that the weights of upper level criteria are independent of the judgments on the lower levels, these problems can decomposed into a series of linear programming problems over the feasible regions.

This decomposition is presented in the following notation which has been adopted from Saaty (1980). The hierarchy $\boldsymbol{H} = \boldsymbol{C} \cup \boldsymbol{A}$ consists of a set of criteria $\boldsymbol{C}$ and the set of alternatives $\boldsymbol{A}$. For any criterion $x \in \boldsymbol{C}$ the set x^-, which cannot be empty, contains those elements of $\boldsymbol{H}$ which are structured directly under x. Conversely, for any element x the set x^+ consists of the criteria $y \in \boldsymbol{C}$ such that x is structured directly under y.

The hierarchy $\boldsymbol{H}$ can be partitioned into levels $L_1, L_2, \dots, L_h$ such that L_h is the set of alternatives, L_1 consists of the topmost criterion b, and for any other criterion $x \in L_i$ the criteria in x^+ belong to L_{i-1}. If x is an alternative then x^+ is the set of criteria y for which $y^- \subset \boldsymbol{A}$. For such an x, the set x^+ contains the criteria in L_{h-1}, but it can also contain criteria on the higher levels of the hierarchy.

The feasible region at criterion y is denoted by S_y. If $w^y \in S_y$ and $x \in y^-$, when w_x^y is the component of w^y that corresponds to x (in the sequel the superscript is usually omitted). By definition the weight of the topmost element is one, i.e. $v(b) = 1$. For the other elements the weights are derived from the feasible local priorities $w^y \in S_y$, $y \in \boldsymbol{C}$ recursively by

$$v(x) = \sum_{y \in x^+} v(y) w_x^y. \tag{4}$$

Theorem 1 decomposes the computation of weight intervals into a sequence of linear programming problems over the feasible regions. Intuitively, this theorem is based on the following argument which resembles the principle of optimality in dynamic programming. First, if the alternative x is structured under the criterion y, then y gives the largest share of its weight to x when the component w_x^y attains its maximum over the feasible region S_y. At the same time, this maximum $\bar{v}_y(x)$, called the *absolute upper bound*, is the highest weight that x can have in the subhierarchy of $\boldsymbol{H}$ rooted at y (see Fig. 2).

Next, take a criterion $z \in y^+$. The weight that x receives from z comes through the criteria in z^-. For these criteria, including y, the absolute upper bounds indicate how much of their weight they can give to x. These absolute bounds can therefore be employed at z as the coefficients of a linear maximization problem the solution of which gives the maximum for the weight of x in the subhierarchy rooted at z. The resulting linear program at z can be viewed as the problem of allocating weight among the already solved subhierarchies.

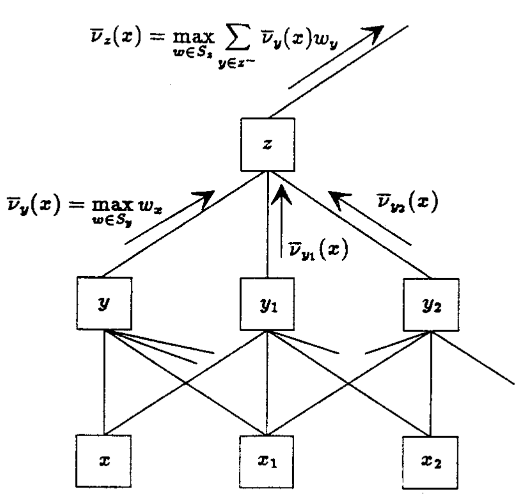

Fig. 2. Computation of weight intervals.

The above procedure can now be repeated until at the topmost element b the absolute bound $\bar{v}_b(x)$ equals the maximum for the weight of x in the entire hierarchy. In the same way, the minimum for the weight of x can be found by computing the *absolute lower bounds* $\underline{v}_y(x)$ which are equal to the smallest weights that x can receive in the subhierarchies rooted at $y \in \boldsymbol{C}$.

Theorem 1. *Let $x \in \boldsymbol{A}$ be a decision alternative. For $y \in \boldsymbol{C}$ such that $y^- \subset \boldsymbol{A}$ define the absolute bounds*

$$\bar{v}_y(x) = \max_{w \in S_y} w_x, \tag{5}$$

$$\underline{v}_y(x) = \min_{w \in S_y} w_x. \tag{6}$$

Proceed successively from level L_{h-2} upwards by recursively defining the absolute bounds

$$\bar{v}_y(x) = \max_{w \in S_y} \sum_{z \in y^-} \bar{v}_z(x) w_z, \tag{7}$$

$$\underline{v}_y(x) = \min_{w \in S_y} \sum_{z \in y^-} \underline{v}_z(x) w_z \tag{8}$$

for the criteria $y \in L_i$, $1 \le i < h-1$ ($y^- \not\subset \boldsymbol{A}$). Then $V(x) = [\underline{v}_b(x), \bar{v}_b(x)]$ is the set of weights assigned to x by feasible combinations of local priorities.

Proof. See the Appendix.

The recursive structure of Theorem 1 affords a constructive algorithm for finding the weight intervals for the alternatives. Once the absolute bounds at some level L_i of the hierarchy have been computed, the problems (7)–(8) can be solved at the adjacent higher level criteria by using the absolute bounds of level L_i. Since at each criterion two linear programs must be solved for every alternative, the total number of linear programs is $2|C||A|$ (here $|\cdot|$ denotes the number of elements in a set).

As the local priorities vary over the feasible regions, the Eq. (4) assigns weight intervals to the different criteria as well. Such intervals help the DM identify which criteria are the most important ones, and they, too, can be computed by solving a series of linear programs. Specifically, the weight intervals for the criteria in L_i can be found by considering the levels L_1 through L_i of the hierarchy; this subhierarchy is then processed with Theorem 1 by treating the elements of L_i as if they were alternatives.

3.2. Pairwise dominance

Even if the weight intervals of alternatives x and y overlap, it can happen that for all fixed combinations of feasible local priorities the weight of x is higher than that of y. In such a situation the DM cannot tighten the constraints on the feasible regions so that y would receive a weight higher than or equal to the weight of x. Formally this criterion for determining preferred alternatives is defined through the *pairwise dominance* relation $\succ_P$,

$$x \succ_P y \iff \min[v(x) - v(y)] > 0, \tag{9}$$

where the minimization is taken over all the feasible regions in the hierarchy. This relation, too, is transitive, for if $x \succ_P y$ and $y \succ_P z$, then

$$\begin{aligned} 0 &< \min[v(x) - v(y)] + \min[v(y) - v(z)] \\ &\leq \min[v(x) - v(y) + v(y) - v(z)] \\ &= \min[v(x) - v(z)]. \end{aligned}$$

There are several reasons for using pairwise dominance in the choice of preferred alternatives. First, the iterative elicitation of interval judgments can be viewed as a process of gradually eliminating the local priorities that are incompatible with the DM's preferences. Thus, the DM's 'true' preference vectors are contained somewhere within the feasible regions. But if for all combinations of feasible priorities x has the higher weight, then this must be the case for the DM's true preference vector as well.

Second, the interval judgments can be seen as constraints on the probability distributions which can be thought to represent the DM's local preferences. In a natural way, these distributions define weight distributions in which only those weights that result from some combination of feasible priorities have positive probabilities (Saaty and Vargas, 1987). In this context, pairwise dominance of x over y means that for all possible outcomes the alternative x has more weight than y and is therefore preferred to y.

If the precise form of the distributions were known, then also other dominance concepts, based, e.g. on the comparison of expected weights, could be employed. However, the elicitation and the synthesis of such distributions would be more time-consuming than the analysis of interval judgments.

Pairwise dominance among the alternatives can be graphically displayed as a *domination digraph* (see Fig. 10), in which the alternatives correspond to the nodes and the relation $x \succ_P y$ is shown as an arc from x to y (Sage and White, 1984). In this graph, the non-dominated alternatives have no incoming arcs so that the most preferred alternative is found when only one such node remains.

Since absolute dominance in (3) clearly implies (9), pairwise dominance need be computed only for alternatives which have overlapping weight intervals. More specifically, the possible pairwise dominance of x over y must be checked only if the absolute bounds satisfy the inequalities

$$\bar{v}_b(x) > \bar{v}_b(y) \geq \underline{v}_b(x) > \underline{v}_b(y).$$

The transitivity of the relation $\succ_P$ can be exploited to further reduce the number of pairs for which pairwise dominance has to be determined. If there are only two alternatives the two relations coincide, because in this case the relation $x \succ_P y$ gives the inequality $v(x) > v(y)$ which together with $v(x) + v(y) = 1$ implies that $v(y) < \frac{1}{2} < v(x)$; hence $x \succ_A y$.

The following argument demonstrates that pairwise dominance, too, can be computed from a series of linear programming problems. If $z \in L_{h-1}$, then alternative x dominates y in the subhierarchy rooted at z only if $w_x - w_y > 0$ for all $y \in S_z$. The minimum for this difference,

$$\pi_z(x, y) = \min_{w \in S_z} (w_x - w_y),$$

is called the *pairwise bound* for the weight difference of x and y at criterion z (see Fig. 3).

If $t \in z^+$, then x dominates y with respect to $t \in z^+$ only if

$$\sum_{z \in t^-} w_z^t (w_x^z - w_y^z) > 0$$

for all feasible priorities $w^t \in S_t$, $w^z \in S_z$. Since the coefficients w_z^t are non-negative, the minimum of this sum is attained when the differences $w_x^z - w_y^z$ are replaced by the pairwise bounds for the criteria in t^-. For t the pairwise bound is therefore obtained as a solution to a linear programming problem in which the lower level pairwise bounds appear as coefficients for the components of w^t. In this way the bounds can be propagated upwards until the topmost element b is reached.

Theorem 2. *Fix $x, y \in A$. For $z \in C$ such that $z^- \subset A$ define the pairwise bounds*

$$\pi_z(x, y) = \min_{w \in S_z} (w_x - w_y). \tag{10}$$

Proceed successively from level L_{h-2} upwards by recursively defining the pairwise bounds

$$\pi_z(x, y) = \min_{w \in S_z} \sum_{t \in z^-} \pi_t(x, y) w_t \tag{11}$$

for the criteria $z \in L_i$, $1 \le i < h-1$ ($z^- \not\subset A$). Then x dominates y, i.e. $x \succ_P y$, if and only if $\pi_b(x, y) > 0$.

Proof. See the Appendix.

Due to the hierarchical composition, changes in the feasible region at criterion x affect the absolute and pairwise bounds only at those criteria y under which x is either directly or indirectly structured. Therefore updated results can be computed by first revising the bounds at x and then at x^+, thus moving progressively from x towards the topmost element.

The validity of Theorems 1 and 2 does not depend on the particular structure of the feasible

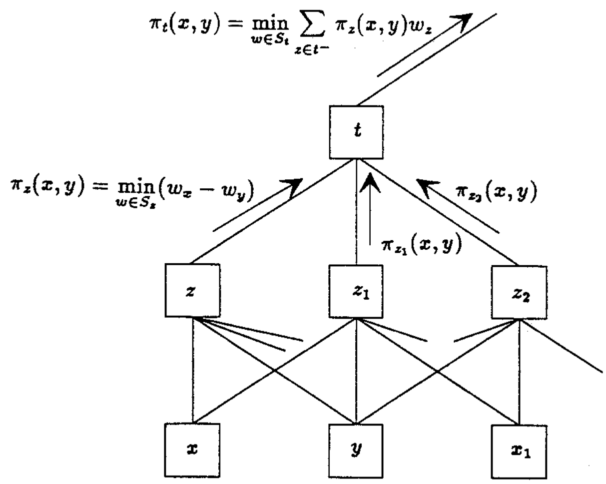

Fig. 3. Computation of pairwise dominance.

regions, provided that these regions are convex and closed sets. As a result even other types of constraints, such as bounds on the components of the local priorities (e.g. $0.25 \le w_x \le 0.80$), could be employed to characterize the DM's preferences and the dominance relations could still be computed as before. In the sequel, however, it is assumed that the DM describes his preferences through interval judgments. Other types of constraints are excluded in favor of pairwise comparisons which have been found an efficient approach to preference assessment both in the AHP and ratio-based techniques of multiattribute value measurement.

4. Support for refining interval judgments

To prevent the feasible regions from becoming empty, the DM needs guidance in the specification of new comparisons and tightening of earlier judgments. For this purpose, this section summarizes some results from Salo and Hämäläinen (1992b) where interval judgments are analyzed under the assumption

$$x \in z^- \;\Rightarrow\; \exists w \in S_z \text{ such that } w_x > 0. \tag{12}$$

This assumption can be justified by noting that if it did not hold, x would not receive any weight from z because the x-component of w^z would be zero for all feasible local priorities, meaning that x would be irrelevant to z. Yet is worth noting that (12) does not require all feasible priorities to have positive x-components.

At criterion z the impact of the DM's earlier judgments can be characterized by the *consistency intervals* $\hat{I}_{xy} = [\hat{l}_{xy}, \hat{u}_{xy}]$. The bounds of these intervals are defined by

$$\hat{u}_{xy} = \max_{w \in S_z} w_x / w_y \tag{13}$$

$$\hat{l}_{xy} = 1/\hat{u}_{yx}, \tag{14}$$

where x and y are subelements of z, and the ratio in (13) is taken to be ∞ if $w_x > 0$, $w_y = 0$ and 0 if $w_x = w_y = 0$. From (12) it follows that $w_x > 0$ for some $w \in S_z$ so that $\hat{u}_{xy} > 0$. The consistency bounds can be computed e.g. with linear fractional algorithms (see Bazaraa and Shetty, 1979) or the algorithm of Potter and Anderson (1980). Alternatively, they can be found by inspecting the extreme points of the feasible region. In Fig. 1, for instance, the interval $\hat{I}_{23} = [\frac{1}{2}, 3]$ shows the consistent ratios for the relative importance of the second and the third subelements that have not been compared yet.

The DM can tighten the constraints on the feasible regions either through new comparisons or by narrowing the bounds of the earlier interval judgments. Both modifications can be modeled by assuming that the DM changes the interval I_{xy} to $I'_{xy} \subset I_{xy}$ (here $\subset$ denotes proper set inclusion) and that after this change the modified feasible region becomes $S'_z \subseteq S_z$. In such a situation, the modified feasible region inherits the property (12) and becomes a proper subset of the earlier feasible region only if the intersection of the intervals I'_{xy} and $\hat{I}_{xy}$ is non-empty and a proper subset of $\hat{I}_{xy}$. In this way, the consistency intervals help the DM see the impact of the earlier judgments and allow him to effectively refine the preference description (see Fig. 7).

If the DM does want to enter an inconsistent judgment, he needs to relax earlier judgments until the feasible region becomes large enough to contain local priorities which satisfy the constraints of such a judgment. For instance, if the DM would like to enter a new lower bound l'_{xy} larger than $\hat{u}_{xy}$, he should first relax some of the upper bounds (these can be identified with the algorithm of Potter and Anderson, 1980) to increase the value of $\hat{u}_{xy}$.

During the analysis, the DM may wish to locate the criteria at which the characterization of his preferences is less precise than elsewhere. To this end, the *ambiguity index* is defined from the consistency intervals through the formula

$$\mathrm{AI}(S_z) = \frac{1}{n(n-1)} \sum_{x,y \in z^-} \frac{\hat{u}_{xy} - \hat{l}_{xy}}{\left(1 + \hat{u}_{xy}\right)\left(1 + \hat{l}_{xy}\right)}, \tag{15}$$

where the ratio is taken to be $1/(1 + \hat{l}_{xy})$ when $\hat{u}_{xy} = \infty$. Because of the reciprocity property of the bounds in (13)–(14), the terms corresponding to the intervals $\hat{I}_{xy}$, $\hat{I}_{yx}$ are equal.

To each criterion the ambiguity index assigns a value in the zero to one range. The index has the value zero only if the feasible region consists of a single local priority vector, and attains the maximum of one only if the DM has not entered any preference statements. The ambiguity index is also monotonous since the feasible region S_z has a greater value than any of its proper subsets. The feasible region in Fig. 1, for instance, has the value

$$\mathrm{AI}(S) = \frac{2}{3 \times 2}\left(\frac{3-1}{4 \times 2} + \frac{2-1}{3 \times 2} + \frac{3 - \frac{1}{2}}{4 \times \frac{3}{2}}\right)$$
$$= \frac{5}{18} = 0.28.$$

5. Computational issues

The computations of Sections 3 and 4 can be sequenced following the algorithm of Fig. 4. In this diagram, the two loops indicate that the consistency intervals can be computed in parallel with the dominance relations. However, if the computer environment does not permit parallelism, it may be advisable to start by updating the consistency intervals and the ambiguity indexes; these help the DM make the next judgment while updated dominance results are being revised.

The DM can perform the pairwise comparisons either a top-down or bottom-up fashion, or in some other order that he finds convenient. In practice, the DM may prefer to start with the comparisons that are easiest to make and move towards the more difficult judgments. During the analysis the DM can identify which judgments would tighten the results most by examining the ambiguity indexes and the intervals of criteria weights.

The algorithm in Fig. 4 has been implemented into a mouse-driven program called INPRE (Salo

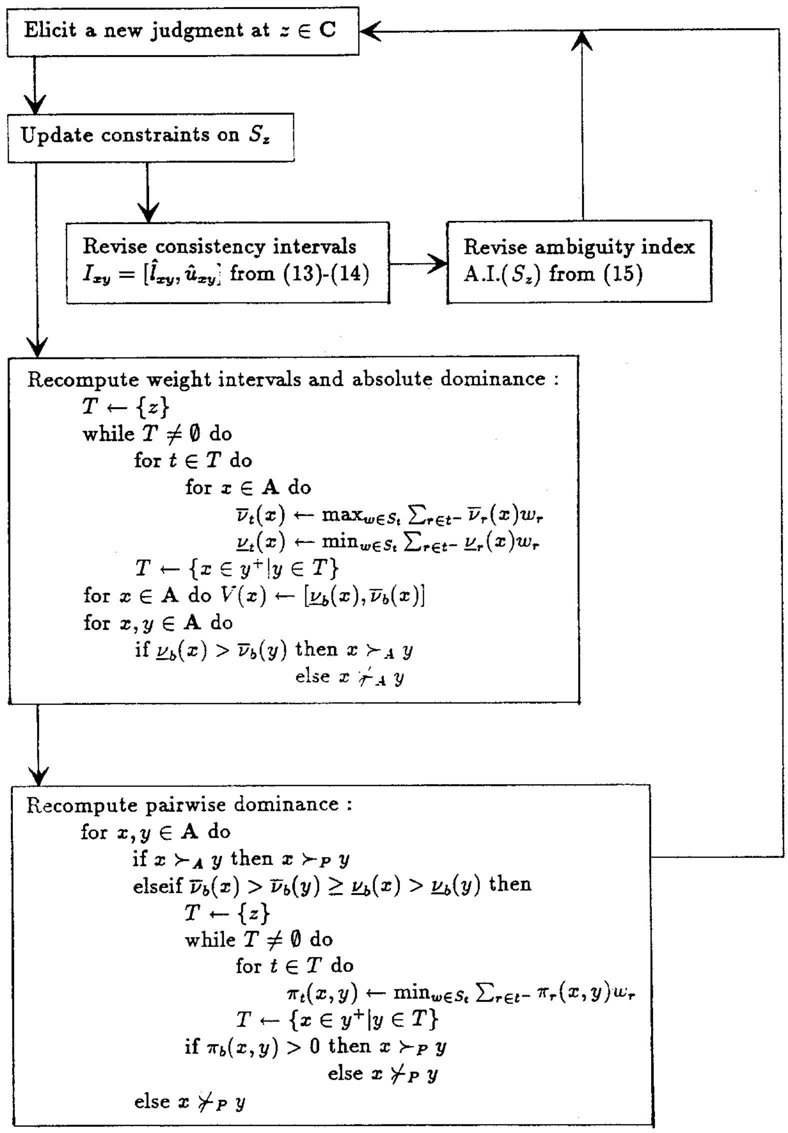

Fig. 4. An algorithm for revising the results.

and Hämäläinen, 1992c). For hierarchies of approximately dozen factors, INPRE computes revised results within a second or two on an AT-compatible microcomputer. This demonstrates that preference programming can indeed be implemented into functional decision support tools.

6. Sensitivity analysis

The absolute and pairwise bounds for the dominance relations are determined as solutions to linear programs over the feasible regions. Therefore, in order to estimate the sensitivity of these relations to individual judgments, it is of interest to identify those bounds in (1) for which a linear functional of the form

$$g_z(w) = \sum_{t \in z^-} \alpha_t w_t^z, \quad w \in S_z, \tag{16}$$

no longer attains its maximum

$$\bar{g}_z = \max_{w \in S_z} g_z(w)$$

if such a bound is tightened by some small amount (the minimization problems (6), (8), (10)–(11) can be adapted to this formulation by converting them into maximization problems by reversing signs). Due to the reciprocity of the bounds l_{xy}, u_{xy}, there is no loss of generality in restricting the attention to the case where the upper bound u_{xy} is modified to $u'_{xy} < u_{xy}$.

If $u_{xy} > \hat{u}_{xy}$, then by the results of Section 4 the bound u_{xy} can be tightened to $\hat{u}_{xy}$ without affecting the feasible region at all. Thus the bound u_{xy} must be equal to $\hat{u}_{xy}$ if after any $u'_{xy} < u_{xy}$ the maximum of $g_z(\cdot)$ over the modified feasible region,

$$S'_z = S_z \cap \{w_x \le u'_{xy} w_y\},$$

is going to be smaller than the maximum over S_z.

Proposition 1. *Assume $\hat{l}_{xy} < u_{xy} = \hat{u}_{xy} \le \infty$. Then $u'_{xy} < u_{xy}$ implies*

$$\max_{w \in S'_z} g_z(w) < \max_{w \in S_z} g_z(w)$$

if and only if $w_x / w_y = \hat{u}_{xy}$ for all extreme points w of S_z such that $g_z(w) = \max_{w \in S_z} g_z(w)$.

Proof. See the Appendix.

Proposition 1 gives necessary and sufficient conditions for identifying the bounds u_{xy} the tightening of which decreases the maximum of $g_z(\cdot)$ over the feasible region. Nevertheless, the maximum of the functional $g_b(\cdot)$, whose coefficients come from the adjacent lower level, can still remain unchanged. This is possible if on the upper levels there exists a feasible combination of local priorities which does not give any weight to z.

7. Example

This section illustrates the application of preference programming in the context of the hierarchy which Hämäläinen (1988, 1990, 1991) employed to help Parliamentarians compare energy production alternatives. In this hierarchy, shown in Fig. 5, the overall benefit of the society has been decomposed into its economic, environmental and political dimensions. The three alternatives in the problem are: no big power plant (A_1), two big coal power plants (A_2) and a nuclear power plant (A_3).

The preference statements in this example approximate those of Mr. Vennamo who at the time of the energy policy decision was the minister of finance and the leader of the rural party (see Hämäläinen, 1990). The interval judgments have been chosen so that the feasible regions contain Mr. Vennamo's local priorities in the conventional AHP analysis. In part, this choice demonstrates how inconsistencies in exact preference statements can be imbedded into interval judgments.

The criteria are numbered consecutively from left to right starting from the top of the hierarchy; hence C_1 refers to the overall benefit of the society and C_4 represents political factors. In the subscripts the elements of the hierarchy are indicated by their indexes so that S_i, for instance, is the feasible region at criterion C_i. If C_j (or A_j) has been structured under C_i and $w \in S_i$ is a feasible local priority vector at C_i, then w_j is the share of the weight of C_i that goes to C_j (or A_j).

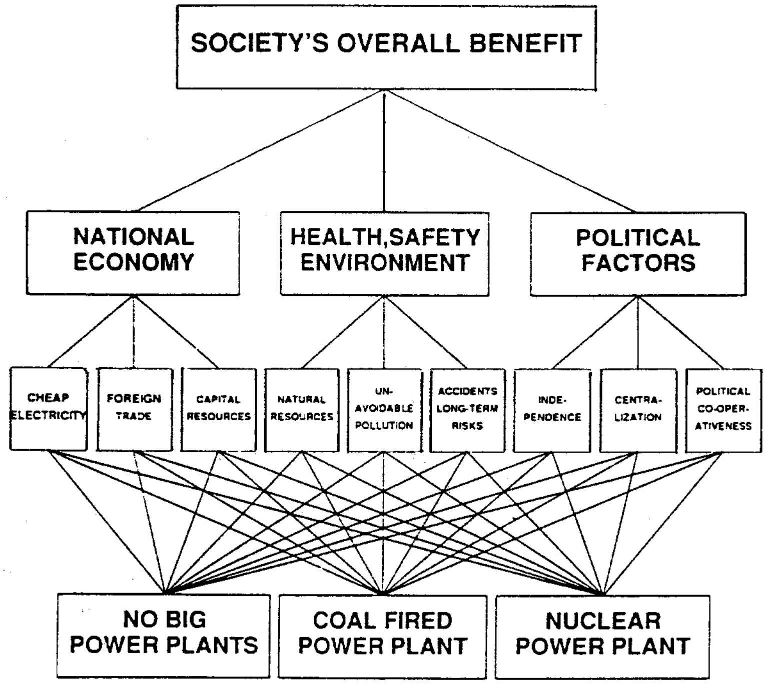

Fig. 5. The energy decision hierarchy.

At the lowest level criteria Mr. Vennamo's judgments were converted into feasible regions by forming the convex hulls of the priority vectors defined by pairs of ratio comparisons (see Salo, 1993). At criterion capital resources, for instance, the pairwise comparisons of the matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 7 & 4 \\ \frac{1}{7} & 1 & \frac{1}{5} \\ \frac{1}{4} & 5 & 1 \end{pmatrix} \end{array}. \tag{17}$$

correspond to three linear equality constraints which do not have a common solution due to the inconsistency of the matrix (17). Yet any two constraints do define a unique local priority vector: the entries a_{12}, a_{13}, for example, correspond to the constraints $w_1 = 7w_2$ and $w_1 = 4w_3$ which together with the normalization requirement give the vector

$$\tfrac{1}{39}(28, 4, 7) = (0.72, 0.10, 0.18).$$

From such vectors, the feasible region in Fig. 6 was defined as the set of local priorities $w =$

Table 1
Interval judgments at third level criteria

	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
I_{21}	$[-, \frac{1}{3}]$	$[-, 3]$	$[-, \frac{1}{7}]$	[1,'1]	$[-, 5]$	$[-, \frac{1}{7}]$	$[\frac{1}{5}, -]$	$[-, \frac{1}{8}]$	$[\frac{1}{5}, -]$
I_{31}	$[\frac{1}{5}, -]$	$[\frac{1}{5}, -]$	$[\frac{1}{4}, -]$	$[1, 1]$	$[\frac{1}{2}, -]$	$[\frac{1}{2}, -]$	$[-, \frac{1}{5}]$	$[\frac{1}{7}, -]$	$[-, \frac{1}{4}]$
i_{32}	$[-, 1]$	$[-, \frac{1}{6}]$	$[-, 5]$	$[-, -]$	$[-, \frac{1}{5}]$	$[-, 6]$	$[\frac{1}{3}, -]$	$[-, 3]$	$[1, -]$

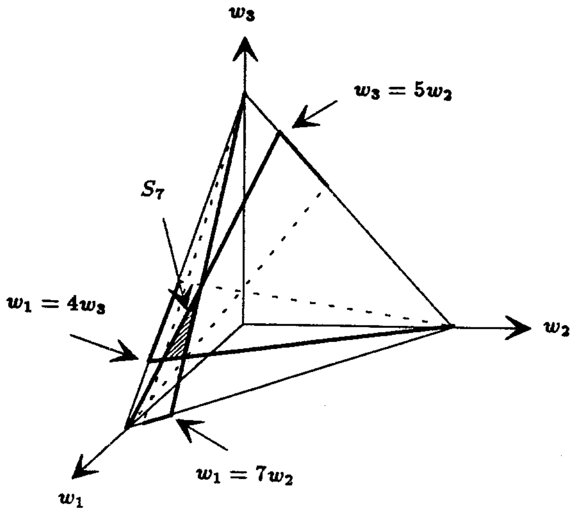

Fig. 6. The feasible region at criterion capital resources.

(w_1, w_2, w_3) which satisfy the inequalities $w_1 \geq 7w_2$, $w_1 \leq 4w_3$ and $w_3 \leq 5w_2$. In the representation (1) these constraints can be written as

$$\begin{array}{c c} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & [7, -] & [-, 4] \\ [-, \frac{1}{7}] & 1 & [\frac{1}{5}, -] \\ [\frac{1}{4}, -] & [-, 5] & 1 \end{pmatrix} \end{array}, \tag{18}$$

where the dashes indicate the entries that do not impose constraints on the feasible region. A similar analysis of the other comparison matrices gave the intervals in Table 1.

For the feasible region in Fig. 6, the consistency bounds of (13) and (14) are

$$\begin{array}{c c} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & [7, 20] & [\frac{7}{5}, 4] \\ [\frac{1}{20}, \frac{1}{7}] & 1 & [\frac{1}{5}, \frac{4}{7}] \\ [\frac{1}{4}, \frac{5}{7}] & [\frac{7}{4}, 5] & 1 \end{pmatrix} \end{array}. \tag{19}$$

In view of Section 4, these bounds show that the feasible region would not change if the DM were to state that the first alternative is better than the third ($l'_{13} = 1 < \frac{7}{5} = \hat{l}_{13}$), and that the DM would be inconsistent if he preferred the second alternative to the third ($\hat{u}_{23} = \frac{4}{7} < 1 = l'_{23}$). From (15) and (19) the ambiguity index of the feasible region in Fig. 6 is found as

$$\begin{aligned} \mathrm{AI}(S_7) = \frac{2}{3 \times 2} \Bigg(& \frac{20-7}{(20+1) \times (7+1)} \\ & + \frac{4 - \frac{7}{5}}{(4+1) \times (\frac{7}{5}+1)} \\ & + \frac{\frac{4}{7} - \frac{1}{5}}{(\frac{4}{7}+1) \times (\frac{1}{5}+1)} \Bigg) = 0.16. \end{aligned}$$

For each pairwise comparison the interval judgments I and the consistency intervals $\hat{I}$ contain up to four different bounds some of which may not be integers in the one to nine range. Instead of the two separate matrix representations these bounds can be displayed together as

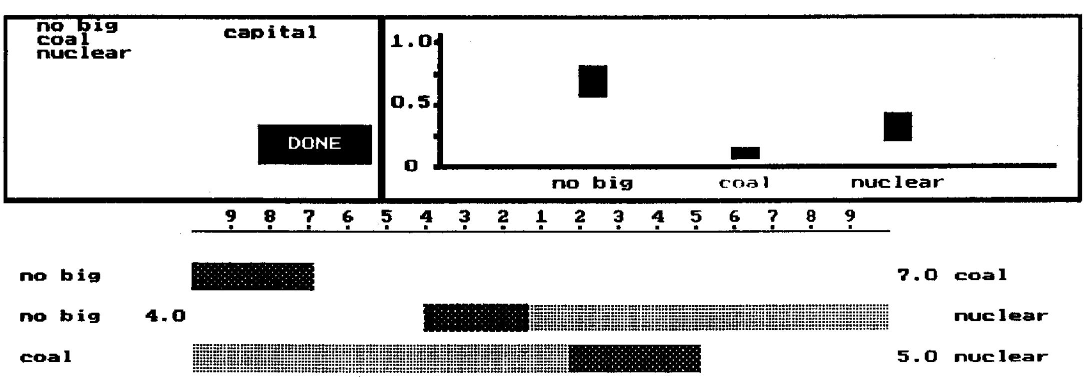

Fig. 7. Interval judgments and consistency bounds.

Table 2
Absolute bounds $\bar{v}$ and $\underline{v}$ at third level criteria

Alt.	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
$\bar{v}$;									
A_1	0.71	0.42	0.77	0.33	0.25	0.63	0.79	0.84	0.71
A_2	0.22	0.71	0.10	0.33	0.77	0.09	0.33	0.10	0.17
A_3	0.20	0.11	0.38	0.33	0.14	0.43	0.14	0.25	0.17
$\underline{v}$:									
A_1	0.60	0.22	0.54	0.33	0.14	0.50	0.56	0.67	0.67
A_2	0.14	0.50	0.04	0.33	0.63	0.05	0.14	0.04	0.14
A_3	0.13	0.05	0.18	0.33	0.08	0.30	0.05	0.11	0.14

horizontal bars. For instance, in Fig. 7 the intervals I in (18) are shown as the outer, shaded parts of the bars, and the inner darker parts correspond to the consistency intervals $\hat{I}$ in (19).

7.1. Phase one of preference analysis

From the interval judgments of Table 1, the weight intervals for the alternatives are found by first computing the absolute bounds at the third level of the hierarchy (see Table 2). For example, solving (5)–(6) over the feasible region in Fig. 6 gives the bounds

$$\bar{v}_7(A_1) = \max_{w \in S_7} w_1 = 0.77,$$

$$\underline{v}_7(A_1) = \min_{w \in S_7} w_1 = 0.54,$$

$$\bar{v}_7(A_2) = \max_{w \in S_7} w_2 = 0.10,$$

$$\underline{v}_7(A_2) = \min_{w \in S_7} w_2 = 0.04,$$

$$\bar{v}_7(A_3) = \max_{w \in S_7} w_3 = 0.38,$$

$$\underline{v}_7(A_3) = \min_{w \in S_7} w_3 = 0.18.$$

Because no judgments have been made on the higher levels of the hierarchy, the feasible regions of the first and second level criteria contain the local priority vectors (1, 0, 0), (0, 1, 0), (0, 0, 1). As a result, the bounds for the alternatives' weight intervals

$$V(A_1) = [0.14, 0.84], \tag{20a}$$

$$V(A_2) = [0.04, 0.77], \tag{20b}$$

$$V(A_3) = [0.05, 0.43], \tag{20c}$$

are found as the maximum and minimum elements on the rows of Table 2. Since intervals overlap absolute dominance does not hold for any pair of alternatives. Solving the minimization problem (10) to check the possible pairwise dominance of the first alternative over the second and the third gives the pairwise bounds of Table 3. The zeros under the column C_8 show that the first alternative is no better than the others with respect to the criterion natural resources. But since C_8 can receive all the weight from the topmost criterion at this point, pairwise dominance does not hold yet.

7.2. Phase two of preference analysis

Suppose that at the second level criteria the DM makes the preference statements

Table 3
Pairwise dominance π at third level criteria

Alt.	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
A_1, A_2	0.40	−0.48	0.46	0.00	−0.62	0.43	0.22	0.58	0.50
A_1, A_3	0.40	0.11	0.15	0.00	0.00	0.07	0.44	0.42	0.50

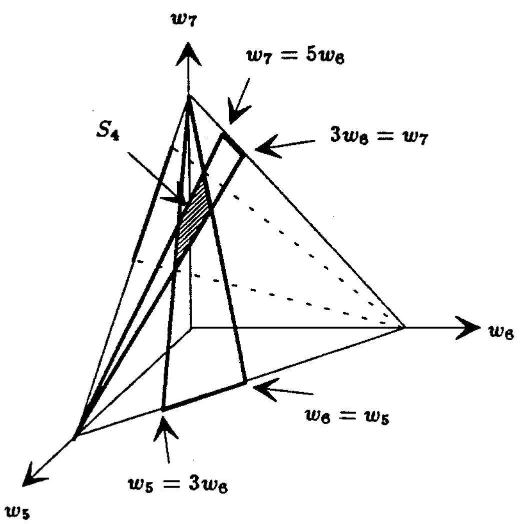

Fig. 8. The feasible region at criterion economic factors.

- W.r.t. economic factors, capital resources are three to five times more important than foreign trade, and cheap electricity is up to three times more important than foreign trade.
- W.r.t. to environmental factors, accidents are up to three times more important than pollution, and at least five times more important than the use of natural resources.
- W.r.t. political factors, centralization is up to two times more important than independence, and four to six times more important than co-operation.

In the computation of revised weight intervals, the absolute bounds are updated first at the second level criteria (see Table 4) and then at the topmost criterion. For example, at criterion economic factors the above interval judgments $I_{76} = [3, 5]$, $I_{56} = [1, 3]$ impose the constraints $3w_6 \leq w_7 \leq 5w_6$ and $w_6 \leq w_5 \leq 3w_6$. Thus the feasible region S_2 has the extreme points (0.14, 0.14, 0.71), (0.20, 0.20, 0.60), (0.43, 0.14, 0.43), (0.33, 0.11, 0.56), which together with Table 2 give the absolute bounds

$$\bar{v}_2(A_1) = \max_{w \in S_2} \sum_{C_i \in C_2^-} \bar{v}_i(A_1) w_i = 0.71 \times 0.14 + 0.42 \times 0.14 + 0.77 \times 0.71 = 0.71,$$

$$\underline{v}_2(A_1) = \min_{w \in S_2} \sum_{C_i \in C_2^-} \underline{v}_i(A_1) w_i = 0.60 \times 0.20 + 0.22 \times 0.20 + 0.54 \times 0.60 = 0.49.$$

Since no judgments have been made at the topmost criterion, the solutions to (7)–(8) at the top of the hierarchy are the maximum and minimum entries in the rows of Table 4, i.e.

$$V(A_1) = [0.32, 0.81], \tag{21a}$$

$$V(A_2) = [0.08, 0.43], \tag{21b}$$

$$V(A_3) = [0.09, 0.36]. \tag{21c}$$

Combining the extreme points of S_2 with the second row of Table 3 gives the revised pairwise bound

$$\pi_2(A_1, A_3) = \min_{w \in S_2} \sum_{C_i \in C_2^-} \pi_i(A_1, A_3) w_i = 0.40 \times 0.14 + 0.11 \times 0.14 + 0.15 \times 0.71 = 0.18 > 0.$$

The other second level pairwise bounds in Table 5 are positive, too, and consequently the big power plant alternative is preferred to the nuclear power plant. On the other hand, the negative pairwise bound

$$\pi_1(A_1, A_2) = \pi_3(A_1, A_2) = -0.09$$

indicates that the second alternative can receive

Table 4
Absolute bounds $\bar{v}$ and $\underline{v}$ at second level criteria

Alt.	$\bar{v}$			$\underline{v}$		
	C_2	C_3	C_4	C_2	C_3	C_4
A_1	0.71	0.54	0.81	0.49	0.32	0.62
A_2	0.25	0.43	0.21	0.12	0.20	0.08
A_3	0.32	0.36	0.21	0.14	0.19	0.09

Table 5
Pairwise bounds π at second level criteria

Alt.	c_2	C_3	C_4
A_1, A_2	0.25	−0.09	0.41
A_1, A_3	0.18	0.03	0.43

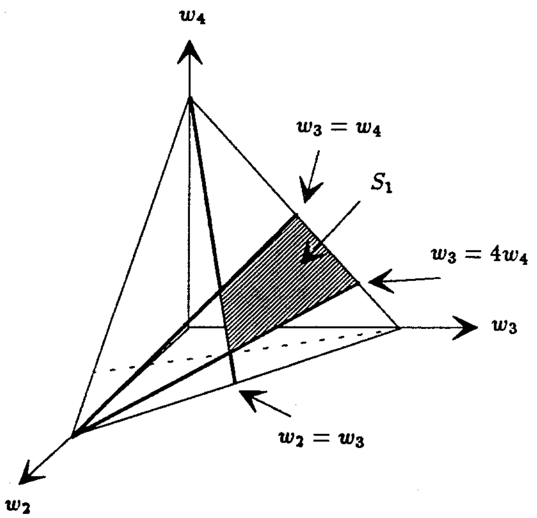

Fig. 9. The feasible region at the topmost criterion.

more weight than the first so that pairwise dominance does not hold for these alternatives.

7.3. Phase three of preference analysis

Finally, assume that the DM considers environmental factors to be more important than economic factors, and up to four times more important than political factors. These statements define the feasible region in Fig. 9 which is characterized by the linear constraints $w_2 \leq w_3$ and $w_4 \leq w_3 \leq 4w_4$ and the extreme points (0.00, 0.50, 0.50), (0.00, 0.80, 0.20), (0.33, 0.33, 0.33), (0.44, 0.44, 0.11).

The revised weight intervals are computed by using the entries of Table 4 as coefficients in the linear programs (7)–(8) over the modified feasible region at the topmost criterion. For example, the bounds for the weight of the first alternative are

$$\begin{aligned}\bar{v}_1(A_1) &= \max_{w \in S_1} \sum_{C_i \in C_1^-} \bar{v}_i(A_1) w_i \\ &= 0.71 \times 0.33 + 0.54 \times 0.33 + 0.81 \times 0.33 \\ &= 0.69, \\ \underline{v}_1(A_1) &= \min_{w \in S_1} \sum_{C_i \in C_1^-} \underline{v}_i(A_1) w_i \\ &= 0.49 \times 0.00 + 0.32 \times 0.80 + 0.62 \times 0.20 \\ &= 0.38,\end{aligned}$$

and a similar analysis of the two other alternatives gives the weight intervals

$$V(A_1) = [0.38, 0.69],$$
$$V(A_2) = [0.13, 0.38],$$
$$V(A_3) = [0.14, 0.33].$$

The weight interval $V(A_1)$ lies above $V(A_3)$ and thus the first alternative dominates the third according to both dominance concepts. Although the intervals $V(A_1)$ and $V(A_2)$ overlap, the first, no big power plant alternative dominates the second one as well, because minimizing (11) over S_1 using the first row of Table 5 gives the positive pairwise bound

$$\begin{aligned}&\pi_1(A_1, A_2) \\ &= \min_{w \in S_1} \sum_{C_i \in C_1^-} \pi_i(A_1, A_2) \omega_i \\ &= 0.25 \times 0.00 - 0.09 \times 0.80 + 0.41 \times 0.20 \\ &= 0.01 > 0. \qquad (22)\end{aligned}$$

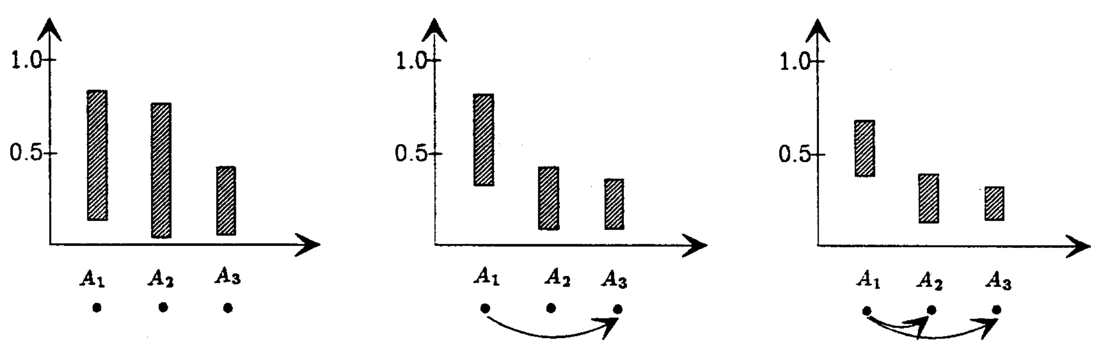

Fig. 10. Changes in weight intervals and pairwise dominance.

Fig. 10 shows the weight intervals and the pairwise dominance relation in the three phases of preference elicitation.

In the last phase the coal plant alternative A_2 achieves its largest weight 0.38 only at the extreme point (0.00, 0.80, 0.20) of the feasible region S_1. Thus, by Proposition 1, the maximum weight for the coal plant decreases if the DM goes on to tighten the bound $u_{34} = 4$. In fact, if this bound is set to three, the revised weight intervals become $V(A_1) = [0.39, 0.69]$ and $V(A_2) = [0.13, 0.37]$; thus the second alternative becomes absolutely dominated by the first one.

In this example the most preferred alternative was found even though many of the interval judgments were left unspecified. For instance, no judgments about the relative importance of economic and political factors were made at the topmost criterion, and at the lowest criterion level only one of the bounds was specified for most interval judgments. This illustrates that preference programming can indeed reduce the effort of preference elicitation.

8. Conclusion

The proposed approach to preference programming is based on computationally efficient mechanisms for processing interval-valued ratio comparisons in hierarchical weighting. The DM's statements are interpreted as linear constraints which after each new preference statement are processed into revised results about nondominated alternatives. These results become gradually more informative as the DM specifies the preference model in increasing detail. The consequences of the earlier judgments are also presented throughout the analysis to help the DM enter statements that lead to more conclusive results.

In comparison to the conventional AHP, the interactive decision support derived from preference programming supplies information about the desirability of the alternatives already in the early stages of the analysis. This feature is likely to reduce the amount of comparison effort since the preferred option can typically be found before all the pairwise comparisons have been completed. On the other hand, in a group setting the DMs can profit from the interval description by specifying aggregate statements that are loose enough to contain their individual judgments. In view of these advantages, it appears that the interactive analysis of interval judgments, as described in this paper, holds substantial promise for real-life applications.

9. Appendix

Proof of Theorem 1. $V(x)$ is a closed set because $v(x)$ is a continuous function of the local priorities w^y in the compact sets S_y, $y \in \boldsymbol{C}$. To prove that it is convex, choose $\overline{w}^y$ from S_y such that $v(x)$ is maximized and $\underline{w}^y$ such that $v(x)$ is minimized. For $z \in \{x\} \cup \boldsymbol{C}$ define the functions $f_z : [0, 1] \to [0, 1]$ recursively by $f_b(t) = 1$ and

$$f_z(t) = \sum_{y \in z^+} \left[t\overline{w}_z^y + (1-t)\underline{w}_z^y \right] f_y(t).$$

Fix

$$r \in \left[\min_{p \in V(x)} p,\ \max_{q \in V(x)} q \right] = \left[f_x(0), f_x(1) \right].$$

Since f_x is continuous there exists $t^r \in [0, 1]$ such that $f_x(t^r) = r$. The local priorities

$$t^r\overline{w}^y + (1-t^r)\underline{w}^y,$$

which by the convexity of S_y are feasible, give the weight r to x.

To prove that the bounds $\overline{v}_b(x)$, $\underline{v}_b(x)$ are tight, fix $w^y \in S_y$, $y \in \boldsymbol{C}$. Then for $1 < i \le h-1$,

$$\begin{aligned} \sum_{y \in L_i} v(y)\overline{v}_y(x) &= \sum_{y \in L_i} \left[\sum_{z \in y^+} v(z) w_y^z \right] \overline{v}_y(x) \\ &= \sum_{\substack{z \in L_{i-1} \\ z^- \not\subset A}} v(z) \left[\sum_{y \in z^-} \overline{v}_y(x) w_y^z \right] \\ &\le \sum_{\substack{z \in L_{i-1} \\ z^- \not\subset A}} v(z)\overline{v}_z(x). \end{aligned}$$

Applying this inequality repeatedly gives

$$v(x) = \sum_{y \in x^+} v(y) w_x^y$$

$$\le \sum_{y \in L_{h-1}} v(y)\bar{v}_y(x) + \sum_{\substack{y \in x^+ \\ L(y)<h-1}} v(y)\bar{v}_y(x)$$

$$\le \sum_{\substack{y \in L_{h-2} \\ y^- \not\subset A}} v(y)\bar{v}_y(x) + \sum_{\substack{y \in x^+ \\ L(y)<h-1}} v(y)\bar{v}_y(x)$$

$$= \sum_{y \in L_{h-2}} v(y)\bar{v}_y(x) + \sum_{\substack{y \in x^+ \\ L(y)<h-2}} v(y)\bar{v}_y(x)$$

$$\vdots$$

$$\le \sum_{y \in L_1} v(y)\bar{v}_y(x) = v(b)\bar{v}_b(x) = \bar{v}_b(x).$$

For those local priorities which maximize (5) and (7) the above inequalities become equalities, and thus

$$\max_{r \in V(x)} r = \bar{v}_b(x).$$

The proof for the lower bound is similar. □

Proof of Theorem 2. Fix $w^z \in S_z$, $z \in C$. Then for $1 < i \le h-1$,

$$\sum_{z \in L_i} v(z)\pi_z(x, y)$$

$$= \sum_{z \in L_i} \left[\sum_{t \in z^+} v(t) w_z^t \right] \pi_z(x, y)$$

$$= \sum_{\substack{t \in L_{i-1} \\ t^+ \not\subset A}} v(t) \left[\sum_{z \in t^-} \pi_z(x, y) w_z^t \right]$$

$$\ge \sum_{\substack{t \in L_{i-1} \\ t^- \not\subset A}} v(t)\tau_t(x, y).$$

Thus

$$v(x) - v(y)$$

$$= \sum_{z \in x^+} v(z) w_x^z - \sum_{z \in y^+} v(z) w_y^z$$

$$\ge \sum_{\substack{z \in C \\ z^- \subset A}} v(z)\pi_z(x, y)$$

$$= \sum_{z \in L_{h-1}} v(z)\pi_z(x, y)$$

$$+ \sum_{\substack{L(z)<h-1 \\ z^- \subset A}} v(z)\pi_z(x, y)$$

$$\ge \sum_{\substack{z \in L_{h-2} \\ z^- \not\subset A}} v(z)\pi_z(x, y)$$

$$+ \sum_{\substack{L(z)<h-1 \\ z^- \subset A}} v(z)\pi_z(x, y)$$

$$= \sum_{z \in L_{h-2}} v(z)\pi_z(x, y)$$

$$+ \sum_{\substack{L(z)<h-2 \\ z^+ \subset A}} v(z)\pi_z(x, y)$$

$$\vdots$$

$$\ge \sum_{z \in L_1} v(z)\pi_z(x, y) = \pi_b(x, y).$$

For those local priorities which minimize (10)–(11) the above inequalities become equalities. This completes the proof. □

Proof of Proposition 1. $\Rightarrow$: If $\exists\, w \in \text{ext } S$ such that $g_z(w) = \bar{g}_z$ and $w_x/w_y < \hat{u}_{xy}$, then the constraint u_{xy} can be tightened to some $u'_{xy} < u_{xy}$ so that $w_x \le u'_{xy} w_y$. But then $w \in S'$ and

$$\max_{w \in S'_z} g_z(w) = \bar{g}_z.$$

$\Leftarrow$: Let $\hat{l}_{xy} \le u'_{xy} < \hat{u}_{xy}$ and assume that $\exists\, w \in S' \subseteq S$ such that $g_z(w) = \bar{g}_z$. Since S is a polytope $\exists\, \lambda_k > 0$, $w^k \in \text{ext } S$ such that

$$\sum_{k=1}^{K} \lambda_k = 1 \text{ and } w = \sum_{k=1}^{K} \lambda_k w^k.$$

From the linearity of $g_z(\cdot)$ and $g_z(w^k) \le \bar{g}_z$ it follows that $g_z(w^k) = \bar{g}_z$. By assumption $w_x^k / w_y^k = \hat{u}_{xy}$ so that $w_x^k > u'_{xy} w_y^k$ for both finite and infinite $\hat{u}_{xy}$. Multiplying these inequalities by λ_k and summing them gives $w_x > u'_{xy} w_y$, which implies $w \notin S'$, a contradiction. □

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On the Measurement of Preferences in the Analytic Hierarchy Process

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ABSTRACT

In this paper we apply multiattribute value theory as a framework for examining the use of pairwise comparisons in the analytic hierarchy process (AHP). On one hand our analysis indicates that pairwise comparisons should be understood in terms of preference differences between pairs of alternatives. On the other hand it points out undesirable effects caused by the upper bound and the discretization of any given ratio scale. Both these observations apply equally well to the SMART procedure which also uses estimates of weight ratios. Furthermore, we demonstrate that the AHP can be modified so as to produce results similar to those of multiattribute value measurement; we also propose new balanced scales to improve the sensitivity of the AHP ratio scales. Finally we show that the so-called supermatrix technique does not eliminate the rank reversal phenomenon which can be attributed to the normalizations in the AHP. © 1997 John Wiley & Sons, Ltd.

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KEY WORDS: hierarchical weighting; analytic hierarchy process; multiattribute value measurement; ratio scales

1. INTRODUCTION

The AHP (Saaty, 1980) has been very successful in gaining the acceptance of practitioners, possibly owing to the helpfulness of the hierarchical problem representations and the appeal of pairwise comparisons in preference elicitation. At present, popular software products such as Expert Choice and HIPRE 3+ continue to promote the AHP as an easy method of multicriteria decision analysis (e.g. Buede, 1992). The range of reported practical applications is extensive (e.g. Vargas, 1990).

Despite the popularity of the AHP, many authors have expressed concern over certain issues in the AHP methodology. The possibility of rank reversals, for instance, has been regarded as unacceptable. Other much-debated problems include the meaning of pairwise comparisons, the relationship between scores and criteria weights, the properties of the 1–9 ratio scale and the prohibitive complexity of the supermatrix approach (Kamenetzky, 1982; Watson and Freeling, 1982; Belton and Gear, 1983; Belton, 1986; Dyer, 1990; Weber and Borcherding, 1993; Schenkerman, 1994).

This paper addresses each of the above points from the perspective of multiattribute value measurement and, in doing so, aims to provide a constructive summarizing analysis. In particular we demonstrate what restrictions multiattribute value representations place on the use of ratio statements in pairwise comparisons. We also show how the elicitation procedures in the AHP could be carried out so that the results are in accordance with multiattribute value measurement. Thus our results can be seen as a step towards the reconciliation of the two methodologies and the resolution of issues raised earlier in the literature.

At present, alternative methods are often portrayed as rivalling approaches and the emphasis tends to be placed on the differences rather than on the similarities. Against this background there is a continuing need for comparative research which seeks to clarify interrelationships between alternative methods, thus helping practitioners in the choice of well-suited approaches to the problems they are facing. Indeed, we believe that comparative research, together with the possible convergence of methodologies, contributes significantly to the important goal of improving the practice of decision analysis.

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2. ASPECTS OF MEASUREMENT

Multiattribute value measurement is based on the primitive notion of an underlying preference relation. If this relation satisfies a number of conditions, formalized as axioms, then it has a functional representation which attaches a real number to each consequence. The construction of this value representation provides decision support by allowing preference statements to be inferred even for those consequences about which no explicit judgements have been made. The value representation also ensures that the inferred statements comply with the underlying axioms which are regarded as maxims of rational choice behaviour.

More specifically, let $\succeq$ denote the decision maker's (DM's) preference relation defined on the set of n-attribute consequences $\mathbf{x} = (x_1, \ldots, x_n) \in X$. For practical purposes, consequences can be thought of as real or hypothetical alternatives, i.e. the consequence space contains all the relevant alternatives. If the relation $\succeq$ has the required properties (e.g. transitivity, completeness, mutual preferential independence; see Krantz *et al.* (1971) and Keeney and Raiffa (1976)), then there exists a real-valued numerical representation, i.e. a value function on $v(x)$ such that

$$\begin{aligned} &\mathbf{x} \succeq \mathbf{y} \iff \\ &v(\mathbf{x}) = \sum_{i=1}^{n} v_i(x_i) \geqslant \sum_{i=1}^{n} v_i(y_i) = v(\mathbf{y}) \end{aligned} \tag{1}$$

That is, consequence **x** is preferred to consequence **y** if and only if $v(\mathbf{x})$, the value attached to **x**, is greater than $v(\mathbf{y})$, the value attached to **y**. In (1) the value representation is unique up to positive affine transformations; that is, if $v(\mathbf{x}) = \Sigma_{i=1}^{n} v_i(x_i)$ is a model of the DM's preference relation, then for any $\alpha > 0$ and real β the function $v'(\mathbf{x}) = \alpha v(\mathbf{x}) + \beta$ is an equivalent representation of the relation $\succeq$.

Dyer and Sarin (1979) specify conditions for the DM's preference relation $\succeq^*$, defined on differences (or exchanges) in consequence pairs of $X^* = \{\mathbf{xy} | \mathbf{x} \succeq \mathbf{y}\}$, such that the additive representation in (1) has the property

$$\begin{aligned} &\mathbf{x}^1\mathbf{x}^2 \succeq^* \mathbf{y}^1\mathbf{y}^2 \iff \\ &v(\mathbf{x}^1) - v(\mathbf{x}^2) \geqslant v(\mathbf{y}^1) - v(\mathbf{y}^2) \end{aligned}$$

In other words, value difference $v(\mathbf{x}^1) - v(\mathbf{x}^2)$ is greater than value difference $v(\mathbf{y}^1) - v(\mathbf{y}^2)$ if and only if the improvement from alternative $\mathbf{x}^2$ to $\mathbf{x}^1$ is judged to be greater than the improvement from alternative $\mathbf{y}^2$ to $\mathbf{y}^1$. This property gives these *measurable value functions* a strength-of-preference interpretation which allows them to be elicited from comparisons of value improvements in pairs of actual or hypothetical alternatives (e.g. Farquhar and Keller, 1989).

Once a suitable range of achievement levels $[x_i^\circ, x_i^*]$ has been defined for each attribute, it is customary to normalize the value function representation so that the values $v(\mathbf{x}^\circ) = v(x_1^\circ, \ldots, x_n^\circ) = 0$ and $v(\mathbf{x}^*) = v(x_1^*, \ldots, x_n^*) = 1$ are assigned to the worst and best conceivable consequences respectively. This range needs to be wide enough to cover the alternatives' achievement levels (or 'performances') on the ith attribute. By normalizing the component value functions onto the [0,1] range, the additive representation can be written as

$$\begin{aligned} v(\mathbf{x}) &= \sum_{i=1}^{n} v_i(x_i) \\ &= \sum_{i=1}^{n} [v_i(x_i) - v_i(x_i^\circ)] \\ &= \sum_{i=1}^{n} [v_i(x_i^*) - v_i(x_i^\circ)] \frac{v_i(x_i) - v_i(x_i^\circ)}{v_i(x_i^*) - v_i(x_i^\circ)} \\ &= \sum_{i=1}^{n} w_i s_i(x_i) \end{aligned} \tag{2}$$

where $s_i(x_i) = [v_i(x_i) - v_i(x_i^\circ)]/[v_i(x_i^*) - v_i(x_i^\circ)] \in [0, 1]$ is the normalized score of **x** on the ith attribute and $w_i = v_i(x_i^*) - v_i(x_i^\circ)$ is the weight of the ith attribute.

Formally, the additive expression (2) resembles the way in which the AHP uses the equation

$$w(x) = \sum_{i=1}^{n} w_i w_i(x) \tag{3}$$

for preference aggregation; here $w(x)$ and w_i stand for the overall weights (or priorities) of alternative x and the ith criterion respectively and $w_i(x)$ is the x-component of the local priority vector at the ith criterion. In view of the apparent similarity of (2) and (3), it is appropriate to ask under what conditions the results of the two approaches coincide.

3. PREFERENCE ELICITATION IN THE AHP

Pairwise comparisons

The questions that the AHP uses to elicit preference information about the alternatives are typically of the form 'Which of the alternatives, Mercedes or Honda, is better with respect to quality and by how much?'. However, Watson and Freeling (1982), Belton (1986) and Dyer (1990), among others, have argued that such value comparisons do not constitute an acceptable procedure of preference elicitation.

In view of the properties of the value representation (1), this is indeed the case, for if $v_i(\cdot)$ is a component value function, then $v_i'(\cdot) = v_i(\cdot) + \beta$ is an equivalent representation of preferences, and yet, if the achievement levels x_i and y_i are not equally preferred, the ratio $[v_i(x_i) + \beta]/[v_i(y_i) + \beta]$ assumes different values depending on the choice of β. That is, the result of the comparison depends on a parameter (β) whose value may be selected arbitrarily.

In contrast, positive value differences can be legitimately measured on a ratio scale because the expression

$$\frac{v_i'(x_i^1) - v_i'(x_i^2)}{v_i'(y_i^1) - v_i'(y_i^2)} = \frac{[\alpha v_i(x_i^1) + \beta] - [\alpha v_i(x_i^2) + \beta]}{[\alpha v_i(y_i^1) + \beta] - [\alpha v_i(y_i^2) + \beta]} = \frac{v_i(x_i^1) - v_i(x_i^2)}{v_i(y_i^1) - v_i(y_i^2)}$$

remains constant for all admissible choices of parameters α and β. Together with (2), this suggests a preference elicitation procedure where the DM is asked to make ratio statements about value differences $v_i(x_i^k) - v_i(x_i^\circ)$ defined by the alternatives' achievement levels x_i^k and criterion-specific reference points x_i°. On each criterion these reference points must be less preferred than the actual alternatives' achievement levels. In other words, the DM should be asked questions such as 'Which of the alternatives, Mercedes or Honda, gives the greater quality improvement over BadQualityCar (i.e. the poor-quality reference car)?' and, assuming that the reply is Mercedes, 'How many times greater is the quality improvement from BadQualityCar to Mercedes than the quality improvement from BadQualityCar to Honda?' (see Watson and Freeling (1982) for an analogous phrasing).

The need for reference points can be easily understood, for example, in the case of measuring distances between cities (see the example in Saaty (1980)). We need to designate one city (i.e. Philadelphia) as a reference point from which the distances are measured. It would be pointless to ask questions such as 'Which city is farther away, London or San Francisco?'. In the context of intangibles it is equally imperative to provide reference points relative to which the pairwise comparisons are performed.

The AHP has two measurement modes, absolute and relative, which differ in that in relative measurement the alternatives' local priorities are normalized so that they add up to one, whereas in absolute measurement no such normalization is applied to the alternatives (Saaty, 1994).

With regard to the different measurement modes, a further advantage of the value difference interpretation is that they need not be treated as inherently different types of information. For example, it has been suggested that the comparison matrix should be built from ratios of absolute measurements whenever such measurements are available (Schoner *et al.*, 1993; Schenkerman, 1994). However, this approach cannot always be recommended because it presumes that (i) the absolute measurements and (ii) the subjective values associated with these measurements are linearly related to each other, even though in reality the relationship may well be non-linear. For example, a 2 week holiday need not be twice as attractive as a 1 week holiday in terms of its duration. Rather than looking at ratios of durations, one should compare the benefit, or value increment, of having a 2 week holiday (as opposed to none) with that obtained from a 1 week holiday (as opposed to none).

Criteria weights

In their discussion of criteria weights, Belton and Gear (1983), Saaty *et al.* (1983), Schoner and Wedley (1989a) and Schoner *et al.* (1993) assert that in the AHP the criteria weights should be proportional to the average (or total) contribution of the alternatives on the respective criteria. In the value difference framework the correctness of this assertion can be demonstrated as follows. Given that the pairwise comparisons among the alternatives $\mathbf{x}^1, \ldots, \mathbf{x}^m$ are correctly linked to the value differences $v_i(x_i^j) - v_i(x_i^\circ)$, $j = 1, \ldots, m$, then the value representation (2) can be rewritten as

$$v(\mathbf{x}^k) = \sum_{i=1}^{n} [v_i(x_i^k) - v_i(x_i^\circ)] = \sum_{i=1}^{n} \sum_{j=1}^{m} [v_i(x_i^j) - v_i(x_i^\circ)] \frac{v_i(x_i^k) - v_i(x_i^\circ)}{\sum_{j=1}^{m} [v_i(x_i^j) - v_i(x_i^\circ)]} \tag{4}$$

A comparison with (3) now shows that w_i, the weight of the ith criterion, should indeed be proportional to the value difference $\Sigma_{j=1}^{m} [v_i(x_i^j) - v_i(x_i^\circ)]$. This logical requirement should be followed in other hierarchical weighting procedures as well (see Edwards and Barron (1994) for related comment about SMART), yet it can be easily forgotten in cases of practical decision support, as the following example demonstrates.

A company is operating in countries C_1 and C_2. It has two investment plans A_1 and A_2 which produce revenues as follows:

	C_1	C_2	Total	
A_1	3	4	7	
A_2	1	8	9	= best
Total	4	12	16	

The manager decides to use a simple weighting model (Figure 1) to compare the plans. In this model the overall revenue is divided by the country criteria. The easy mistake in weighting is to assign equal weights (0.5) to the countries, based on the thinking that profits are equally important irrespective of their origin. This leads to the choice of plan A_1 although the total profits from plan A_2 are higher:

$$w(A_1) = \tfrac{1}{2} \times \tfrac{3}{4} + \tfrac{1}{2} \times \tfrac{4}{12} = \tfrac{13}{24} = \text{best}$$
$$w(A_2) = \tfrac{1}{2} \times \tfrac{1}{4} + \tfrac{1}{2} \times \tfrac{8}{12} = \tfrac{11}{24}$$

The correct weighting of the country criteria should take into account the relative amount of revenues coming from the respective countries (criteria). This leads to the criteria weights $w_1 = \frac{4}{16} = \frac{1}{4}$ and $w_2 = \frac{12}{16} = \frac{3}{4}$, which in turn produce the correct overall weights:

$$w(A_1) = \tfrac{1}{4} \times \tfrac{3}{4} + \tfrac{3}{4} \times \tfrac{4}{12} = \tfrac{7}{16}$$
$$w(A_2) = \tfrac{1}{4} \times \tfrac{1}{4} + \tfrac{3}{4} \times \tfrac{8}{12} = \tfrac{9}{24} = \text{best}$$

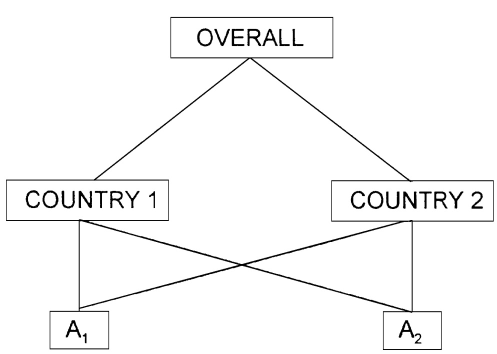


Figure 1. Prioritizing two investment plans A_1 and A_2 with revenues from two different countries

Local priorities can be normalized in other ways as well provided that the interpretation of criteria weights is modified accordingly (Belton and Gear, 1983; Schoner and Wedley, 1989a; Schoner *et al.*, 1993). For instance, if under each criterion the local priority of the kth alternative is set to one, then (2) becomes

$$\sum_{i=1}^{n} v_i(x_i) - v_i(x_i^\circ) = \sum_{i=1}^{n} [v_i(x_i^k) - v_i(x_i^\circ)] \frac{v_i(x_i) - v_i(x_i^\circ)}{v_i(x_i^k) - v_i(x_i^\circ)}$$

which together with (3) shows that the corresponding criteria weights should now be proportional to the value differences $v_i(x_i^k) - v_i(x_i^\circ)$.

Once the local priorities $w_i(x)$ in (3) are available, questions for the assessment of criteria weights can be phrased in terms of any of the alternatives. For example, if the DM estimates that the value difference from the reference point x_1° to the kth alternative under the first criterion is r times larger than the difference from x_2° to the lth alternative under the second criterion, then the ratio of criteria weights can be written as

$$\frac{w_1}{w_2} = \frac{\sum_{j=1}^{m} [v_1(x_1^j) - v_1(x_1^\circ)]}{\sum_{j=1}^{m} [v_2(x_2^j) - v_2(x_2^\circ)]}$$

$$= \frac{[v_1(x_1^k) - v_1(x_1^\circ)]\left\{1 + \sum_{j \neq k} \frac{[v_1(x_1^j) - v_1(x_1^\circ)]}{[v_1(x_1^k) - v_1(x_1^\circ)]}\right\}}{[v_2(x_2^l) - v_2(x_2^\circ)]\left\{1 + \sum_{j \neq l} \frac{[v_2(x_2^j) - v_2(x_2^\circ)]}{[v_2(x_2^l) - v_2(x_2^\circ)]}\right\}}$$

$$= r\frac{1 + \sum_{j \neq k} w_1(j)/w_1(k)}{1 + \sum_{j \neq l} w_2(j)/w_2(l)}$$

The resulting alternative ways of estimating criteria weights can be used to check the consistency of the DM's responses.

On the higher levels of tree-shaped hierarchies the normalized value representation (4) implies that the weight of a criterion must be proportional to the average value improvement that the alternatives bring over the reference points in the entire subhierarchy of which the criterion is the topmost element. In more general AHP hierarchies, which do not have a tree structure, the criteria weights do not have this clear-cut interpretation because they are not uniquely related to the weights of the lowest-level criteria.

With the above interpretation of ratio comparisons the AHP can be expected to give results similar to those of multiattribute value measurement. However, in practice the comparison of AHP against other preference elicitation methodologies is problematic because different procedures of multiattribute value assessment produce divergent results; for instance, the observed violations of procedural invariance imply that there is no single benchmark for comparisons (Weber and Borcherding, 1993). Consequently, future empirical research should compare the performance of the conventional AHP and its variants with a number of multiattribute evaluation techniques. Among other things, such research should investigate how decision makers understand elicitation questions and to what extent ratios of value differences give results that are insensitive to the choice of reference points and in keeping with other preference assessment methods.

4. DISCRETIZATION OF RATIO SCALES

In their empirical comparison of ratio scales, Schoner and Wedley (1989b) requested subjects to judge the proportion of different colours in displays of varying fuzziness using Stevens' magnitude estimation and the 1–9 scale of the AHP. They found that the AHP produced consistently less accurate estimates, especially when the colours covered roughly equal proportions of the display.

As illustrated in Figure 2, a plausible explanation for this finding lies in the uneven dispersion of the local priorities $w = (w_1, w_2)$ that are supported by the ratios $r = \frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9$ through the mapping $w_1 = 1/(r+1)$, $w_2 = r/(r+1)$ (see Figure 2). The effect of replacing the ratio 1 by 2, for instance, is 15 times greater than the local priority difference between the ratios 8 and 9. Similar results hold for a larger number of alternatives, although the procedures for resolving inconsistencies in a complete set of pairwise comparisons may moderate the effect.

For a given set of priority vectors the corresponding ratios can be computed from the inverse relationship $r = w/(1-w)$. In particular, by choosing priority vectors which are equally far apart from each other, we can define so-called *balanced scales*: for instance, the priorities 0.1, 0.15, 0.2, ..., 0.80, 0.85, 0.9, for example, lead to the scale 1, 1.22, 1.50, 1.86, 2.33, 3.00, 4.00, 5.67, 9.00, whereas the scale 1, 1.27, 1.62, 2.09, 2.78,

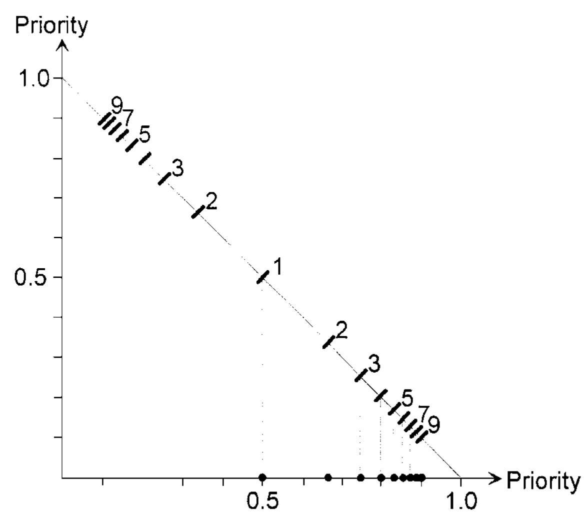
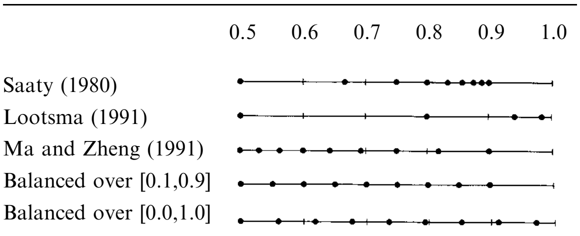


Figure 2. Dispersion of local priorities as a function of ratio estimate

Table I. Priority distributions in discretized ratio scales



3.86, 5.80, 10.3, 33.3 is obtained by covering the unit range [0,1] with 17 priority vectors such that each represents priorities in segments of equal length (see Table I). The 9/9, 9/8, ..., 9/2, 9/1 scale of Ma and Zheng (1991) also gives more uniformly distributed priorities than the 1–9 scale, while the priorities in Lootsma's (1991) geometrically progressing 1, 4, $4^2, \ldots$ scale are sparse over most of the priority range.

A shortcoming of the 1–9 scale (or, for that matter, of any scale with a finite upper bound M) is that the upper bound restricts the range of local priority vectors. Specifically, if the $n \times n$ comparison matrix is consistent (so that its largest eigenvalue is n), then the maximum for the first component is achieved at

$$
\begin{aligned}
& w_1 + \sum_{i=2}^{n} M w_i = n w_1 \\
\iff \quad & w_1 + M(1 - w_1) = n w_1 \\
\iff \quad & w_1(n + M - 1) = M
\end{aligned}
$$

i.e. no component of a local priority vector can exceed the bound $w_{\max} = M/(n + M - 1)$. The corresponding lower bound $w_{\min} = 1/[M(n-1)+1]$ follows from the equality

$$
\begin{aligned}
& w_1 + \sum_{i=2}^{n} \frac{1}{M} w_i = n w_1 \\
\iff \quad & M w_1 + (1 - w_1) = M n w_1 \\
\iff \quad & w_1[M(n-1) + 1] = 1
\end{aligned}
$$

Thus the existence of a finite upper bound imposes the fundamental restrictions

$$w_{\max} = \frac{M}{n + M - 1} \tag{5}$$

$$w_{\min} = \frac{1}{M(n-1) + 1} \tag{6}$$

which are illustrated in Figure 3.

It is worth noting that the above restrictions apply to any ratio procedure in which the DM

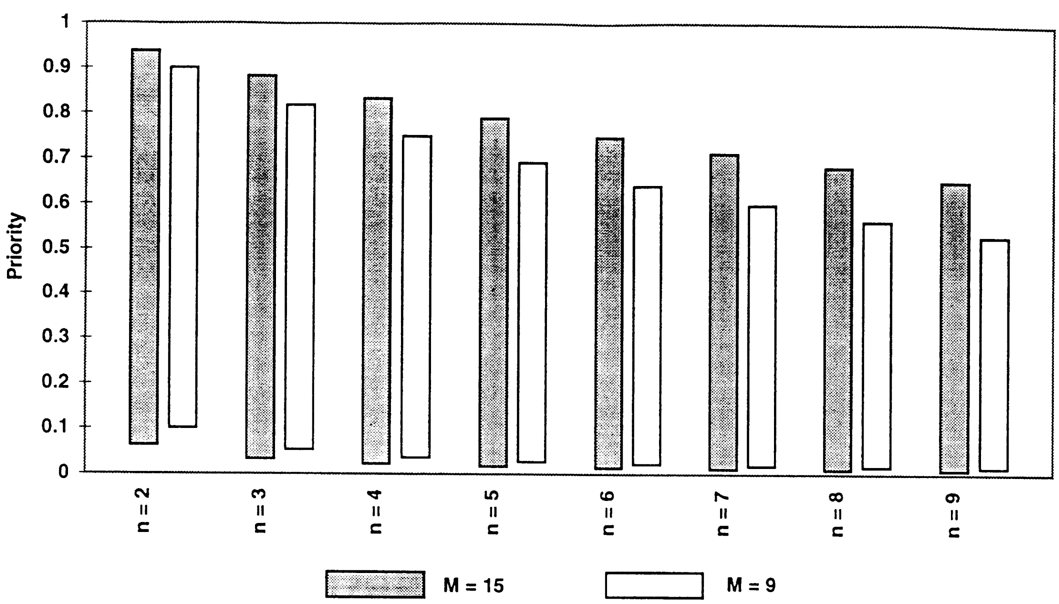


Figure 3. Range of possible priority as a function of maximum ratio (M) and number of subcriteria (n)

(perhaps unknowingly) places an upper bound for the submitted ratios. For example, if the DM uses 10 as the lowest reference score in SMART and, at the same time, implicitly assumes that 100 is the highest possible, he/she effectively puts an upper bound of 10 on the ratio comparisons. In this case the restrictions (5) and (6) hold with M = 10.

To some extent the above restrictions can be relaxed by introducing additional criterion levels so that each criterion has fewer subcriteria. For instance, a single criterion with eight subcriteria may be divided into two intermediate subcriteria ($n_1 = 2$) which in turn have four subcriteria ($n_2 = 4$) each (see Figure 4). In the modified hierarchy the maximum priority for the initial subcriteria, $[M/(n_1 + M - 1)][M/(n_2 + M - 1)]$, is larger than the corresponding upper bound in the single-level hierarchy, i.e. $M/(n_1 n_2 + M - 1)$.

Table II shows how the maximum overall priorities become higher when $n = n_1 n_2$ subcriteria are divided under n_1 intermediate criteria with n_2 subcriteria for each. Exchanging the roles of n_1 and n_2 has no effect on the upper bounds; for instance, the eight subcriteria in Figure 4 could be structured under four (instead of 2) intermediate subcriteria and the upper bound would still remain the same. The behaviour of these theoretical bounds parallels empirical findings according to which larger priority ratios are observed in steeper hierarchies (e.g. Stillwell *et al.*, 1987). However, the influence of the hierarchical structure on the results of weighting procedures still calls for further research (Pöyhönen and Hämäläinen, 1996).

The consistency ratio (CR) in the AHP is computed as the ratio between the consistency index $(\lambda_{max} - n)/(n - 1)$ of the given comparison matrix (with dimension n and principal eigenvalue λ_{max}) and the average of similar indices computed

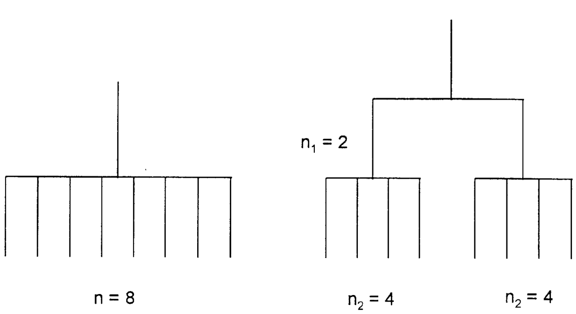


Figure 4. Conversion of a one-level hierarchy into a hierarchy with two intermediate subcriteria

Table II. Effect of decomposition on maximum local priorities for $M = 9$

n	4	6	8	10
Maximum priority	0.75	0.64	0.56	0.50
$n_2(n_1 = 2)$	2	3	4	5
Maximum priority	0.81	0.74	0.68	0.62

for randomly generated reciprocal comparison matrices. Thus by construction the consistency ratio is a meaningful measure only on condition that the same scale has been employed both in the assessment of the actual comparison matrix and in the generation of the random matrices, but this in turn suggests that scale-invariant measures should be preferred to those that depend on a particular discretization of the ratio scale or on a given probability distribution on the points included in the discretization.

One way of deriving scale-invariant consistency measures is to transform the inconsistent replies into a non-empty set of feasible priorities, whereafter the properties of the resulting set can be used for measuring the inconsistency of the original comparison matrix. Specifically, if $a(i, j)$ denotes the element in the ith row and jth column of the comparison matrix, then the corresponding *extended bound* can be defined as $\bar{r}(i, j) = \max_k a(i, k)a(k, j)$ (Salo, 1993). Using these extended bounds, a scale-invariant consistency measure (CM) is obtained through the equation

$$\mathrm{CM} = \frac{2}{n(n-1)} \sum_{i>j} \frac{\bar{r}(i,j) - \underline{r}(i,j)}{[1 + \bar{r}(i,j)][1 + \underline{r}(i,j)]} \qquad (7)$$

where $\underline{r}(i, j)$ stands for the inverse of $\bar{r}(j, i)$. The above measure is in essence an indicator of the size of the *extended region*, i.e. the set of local priorities such that $w_i \leqslant \bar{r}(i, j)w_j$ for all $i, j \in \{1, \ldots, n\}$ (Salo, 1993), which in turn grows as the entries of the comparison matrix become more inconsistent.

The consistency measure (7) is incorporated in the HIPRE 3+ software, which, among other things, allows its users to employ different scales in hierarchical preference elicitation (Hämäläinen and Lauri, 1995).

There are situations in which the 1–9 scale seems to work well, such as in estimating distances from Philadelphia to several other cities in the example of Harker and Vargas (1987). However, the choice of another set of cities could well have favoured some other scale. The ratios to be included in a discretized scale should be selected with regard to entire sets of objects about which ratio comparisons may be made. In a recent experiment we discovered interpersonal differences in the interpretation of verbal statements, which complicates the use of such statements in preference elicitation. In addition, we found that the balanced scale in Table I outperforms the 1–9 scale in capturing the subjects' understanding of verbal expressions. All in all, there is a need for a better understanding of what people really understand by expressions such as 'weakly more important' or 'very strongly more important', especially since verbal statements sometimes produce greater inconsistencies without improving the accuracy of the results (Lund and Palmer, 1986). Related results have been reported in the literature on interpersonal differences in verbal probability assessment (Beyth-Marom, 1982) and in the extensive psychological research on scale problems (Birnbaum, 1978).

Apart from criterion-specific concerns, one also needs to examine to what extent the interpretation of verbal statements depends on the problem context, for if the verbal expressions are understood differently from one criterion to the next, then their automated use in preference elicitation becomes problematic indeed. Here new and interesting approaches are provided by methods which allow ambiguity or incompleteness in preference statements. In these approaches, verbal statements can be mapped into real-valued intervals in order to cover a range of interpretations and preferences (Hämäläinen *et al*., 1991; Salo and Hämäläinen, 1992a, 1995; Hämäläinen and Pöyhönen, 1996).

5. THE SUPERMATRIX APPROACH

The supermatrix technique (Saaty, 1980, Chap. 8) has been suggested as a remedy to the rank reversals in the AHP (Harker and Vargas, 1987). Unfortunately, there are no clear-cut rules for determining when supermatrices should be used (Dyer, 1990); moreover, the supermatrix technique requires that the DM answers a much larger number of questions. These questions may also be quite complex, e.g. 'Given an alternative and a criterion, which of the two alternatives influences the given criterion more and how much more than another alternative?' (Saaty and Takizawa, 1986). In fact, the complexity of questions such as this may in part explain the scarcity of reported supermatrix applications. Hämäläinen and Seppäläinen (1986) were the first to call the method the analytic network process (ANP) and were indeed able to apply it in a policy problem with interrelationships between two planning horizons.

Despite claims to the contrary, the supermatrix technique does not eliminate rank reversals. For instance, assume that the DM makes the following statements about a new, third alternative in the example of Harker and Vargas (1987, p. 1395):

- w.r.t. C_1, A_3 is two times better than A_1;
- w.r.t. C_2, A_1 is four times better than A_3;
- w.r.t. C_3, A_1 and A_3 are equally preferred;
- w.r.t. A_3, C_2 is seven times more important than C_3;
- w.r.t. A_3, C_1 is two times more important than C_3.

After these statements the supermatrix takes the form

$$W = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0.3 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0.3 & 0.5 & 0.7 \\ 0 & 0 & 0 & 0.4 & 0.2 & 0.1 \\ 0.25 & 0.364 & 0.444 & 0 & 0 & 0 \\ 0.25 & 0.545 & 0.111 & 0 & 0 & 0 \\ 0.5 & 0.091 & 0.444 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{8}$$

and straightforward computations now show that these statements change the feedback weights of the alternatives from (0.554, 0.446) to (0.352, 0.360, 0.287); in other words, the introduction of A_3 reverses the ranks of alternatives A_1 and A_2. There is also experimental evidence which suggests that reversals of this kind are too frequent to be classified as artefacts of numerical manipulation (Salo and Hämäläinen, 1992b).

The feedback technique suffers from rank reversals because it fails to ensure that the entries of the supermatrix are correctly linked to value differences between the alternatives. These links can be created by requiring, as in (2), that the local priority of the ith alternative on the kth criterion is related to the value differences by $w_k(i) = [v_k(x_k^i) - v_k(x_k^\circ)]/\Sigma_{j=1}^m [v_k(x_k^j) - v_k(x_k^\circ)]$. At the same time, feedback priorities need to be selected so that correct criteria weights are obtained when they are combined with the alternatives' weights; thus the relative importance of the kth criterion at the ith alternative, $w_i(k)$, should be such that the expression $w_i(k)\Sigma_{j=1}^{m}[v_j(x_j^i)-v_j(x_j^0)]$ is proportional to the value difference $v_k(x_k^i)-v_k(x_k^0)$ or, more specifically,

$$\frac{w_i(k)}{w_i(l)}=\frac{v_k(x_k^i)-v_k(x_k^\circ)}{v_l(x_l^i)-v_l(x_l^\circ)}=\frac{w_k w_k(i)}{w_l w_l(i)} \tag{9}$$

If these compatibility restrictions on the criteria weights and local priorities do not hold, the DM's judgements are not in agreement with an underlying value representation.

A direct implication of the above remarks is that the feedback questions, when correctly posed, are essentially similar to those employed in the assessment of criteria weights; for instance, the question 'Given an alternative (e.g. Mercedes), which criterion (e.g. style or quality) is more important in the overall choice of best car?' (Harker and Vargas, 1987) should in fact be rephrased as 'Which is preferred, the style improvement from BadCostCar to Mercedes or the quality improvement from BadQualityCar to Mercedes?'. Another reason why complete supermatrices seem excessively laborious is that the feedback replies at any one alternative provide estimates for the criteria weights (Schoner and Wedley, 1989a).

6. RANK REVERSALS

It has been argued that the relative measurement mode in the AHP is superior to the methods of value and utility theory because it incorporates preference reversals which are sometimes exhibited by unaided decision making (e.g. Saaty, 1994; Vargas, 1994). From a normative perspective, however, this line of reasoning is debatable: because one of the main objectives of the analysis is to contribute to decision quality, it is not clear why the results of the analysis should retain the questionable features of unaided human choice behaviour.

Instead, one should seek to determine whether the decision maker wishes to receive guidance in which preferences for pairs of alternatives are independent of what other alternatives are contained in the choice set. If this form of independence exists (which means that pairs of alternatives may be evaluated in isolation from the rest of the choice set), the relative measurement mode of the AHP is unsuitable because rank reversals must not occur. In the absence of such independence the preference relation as such is an elaborate construct which to our knowledge has not been characterized in terms of axioms. Even if a satisfactory axiomatization were available, the assessment of the preference representation would be a complex task because comparisons between pairs of alternatives would have to involve other alternatives as well. In contrast, the AHP elicits preferences without reference to other alternatives, wherefore the occasional rank reversals may be side-effects of the normalization procedure rather than credible results of the modelling procedure.

From the descriptive point of view the fact that rank reversals occur in the AHP does not imply that the AHP is good at predicting preference reversals. The appraisal of this type of predictive ability calls for empirical experiments with carefully selected control groups. Also, on condition that the AHP is found a successful predictor of preference reversals and unaided decision making at large, one may question what the aims of an AHP analysis are if unsupported judgements would produce essentially similar outcomes.

In the relative measurement mode of the AHP, any alternative that is not dominated by some convex combination of other alternatives may become the best one as a result of the introduction or deletion of other alternatives. For example, if the jth alternative is non-dominated in the above sense, there exists a set of constants, c_i, $i=1,\ldots,n$, such that the inequalities

$$\sum_{i=1}^{n} c_i w_i(j) \geqslant \sum_{i=1}^{n} c_i w_i(k)$$

hold for all $k\neq j$. Now, if the priorities for the new $(n+1)$th alternative are chosen so that the term $1/[1+w_i(n+1)]$ is proportional to c_i/w_i (where w_i, $i=1,\ldots,n$, denote initial criterion weights), the revised weight of the jth alternative becomes

$$\sum_{i=1}^{n} w_i \frac{w_i(j)}{1+w_i(n+1)} \propto \sum_{i=1}^{n} w_i \frac{c_i}{w_i} w_i(j) = \sum_{i=1}^{n} c_i w_i(j)$$

Since the same proportionality relation holds for the other alternatives as well, it follows that the introduction of the $(n+1)$th alternative causes the jth alternative to become the best one.

Thus the fundamental mathematical reason for the occurrence of rank reversals in the relative measurement mode is that the local priorities at the lowest level of the hierarchy are normalized so that they add up to one. When new alternatives are added or deleted, the local priorities associated with the other alternatives inevitably change and as a result the final ranking of the alternatives may also change. Any practitioner using normalized priorities should be aware of this possibility.

We feel that the normative and descriptive uses of the AHP, or of any decision theory for that matter, must be understood within their appropriate domains. The claim that some decision support technique is both normatively acceptable and descriptively adequate is in itself contradictory. Why would a support methodology be needed which describes how unaided decisions are made and, at the same time, prescribes how these decisions should be made?

7. CONCLUSIONS

Starting from the foundations of multiattribute value measurement, we have demonstrated that the pairwise comparisons in ratio estimation should be interpreted in terms of value differences between pairs of underlying alternatives. This interpretation is general and applies to all methods (including the AHP and SMART) which make use of ratio statements in the elicitation of hierarchical models. In fact, when the questions in the AHP are rephrased according to the value difference interpretation, the AHP can be regarded as a variant of multiattribute value measurement. While it is still unclear to what extent the DM's intuitive responses to the standard AHP questions conform to the value difference interpretation, we feel that AHP practitioners could improve their analyses by stating the pairwise comparison questions accordingly.

The other topics, i.e. the choice of the scale and whether or not to use normalizations, are important issues which should be seen as practical procedural choices whose consequences need to be understood. Although discretized ratio scales such as the 1–9 scale of the AHP can be very helpful in preference elicitation, they are nevertheless problematic as they severely restrict the range and distribution of possible priority vectors. The balanced scales proposed in this paper provide an essential improvement in this matter. Even so, the assumption that verbal expressions can be mapped onto numbers in the same way, no matter who is responding and in what context, must be regarded with due caution. The implication of scale selections must be considered explicitly, especially if the results are to be used in a normative sense. Risks associated with scale selection can be mitigated through software tools which allow the practitioner to compare results based on different scales.

Often hierarchical weighting procedures such as the AHP are used to create an improved problem understanding and to support communication among a group of decision makers with little interest in the details of deriving numerical results. Even in this kind of case the analyst should make every effort to explain techniques. The decision makers need to understand that both the structure of the hierarchy and the criteria weights need to reflect the set of decision alternatives and their differences.

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Decision Support

The harmonic consistency index for the analytic hierarchy process

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Abstract

A new consistency measure, the harmonic consistency index, is obtained for any positive reciprocal matrix in the analytic hierarchy process. We show how this index varies with changes in any matrix element. A tight upper bound is provided for this new consistency measure when the entries of matrix are at most 9, as is often recommended. Using simulation, the harmonic consistency index is shown to give numerical values similar to the standard consistency index but it is easier to compute and interpret. In addition, new properties of the column sums of reciprocal matrices are obtained.

Keywords: Multiple criteria analysis; AHP; Consistency index; Reciprocal matrices

1. Introduction

The analytic hierarchy process (AHP) is a popular decision tool used to rank items based on subjective pairwise comparisons (Saaty, 1977, 1980, 1986). A decision maker is asked to state pairwise preferences among n items. Comparison of items i and j results in a positive number a_{ij} giving the strength of preference of item i over item j; the larger the number the greater the preference in favor of item i while a value of 1 indicates indifference. The strength of preference need only be obtained for the $n(n-1)/2$ pairs (i,j) with $i < j$. From those a_{ij} measures we can obtain the strength of preference a_{ji} of item j over i by setting $a_{ji} = 1/a_{ij}$. The values can be displayed in a $n \times n$ matrix $A = (a_{ij})$ that has all positive values with 1's on the main diagonal and satisfies the property $a_{ji} = 1/a_{ij}$. This type of matrix is called a reciprocal matrix (also called a symmetrically reciprocal

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matrix by Fichtner, 1983). If $a_{ij} = a_{ik}a_{kj}$ for all i, j and $k = 1, \ldots, n$ then A is said to be consistent.

The goal of AHP analysis is to use the empirically obtained matrix A of pairwise comparisons to obtain implied relative preferences $w = (w_1, \ldots, w_n)^{\mathrm{T}}$, called the priority vector. Suppose that each item i has a priority value w_i and that the decision maker uses these values to form the A matrix from the ratio $a_{ij} = w_i/w_j$. If so, the A matrix is consistent and any column will recover the w values, which are unique up to a positive scalar multiple. In that case column j will be of the form $(w_1/w_j, \ldots, w_n/w_j)^{\mathrm{T}}$ which is proportional to w. In general, the responses of the decision maker will not be consistent. Given a reciprocal matrix A, not necessarily consistent, the central problem in AHP is to determine the vector w, that in some sense fits the responses in the A matrix. Saaty (1977) proposed the initial solution method based on solving an eigenvalue problem. One way is to choose w so that the matrix $W = (w_i/w_j)$ is close to the observed A using an appropriate metric. Two other popular methods adopt a specific error measures: the least-squares approach chooses w to minimize the sum of squared differences; the row geometric mean method minimizes the sum of squares of differences of the logarithms of the values. Gass and Rapcsák (2004) contains an excellent overview of the more common methods. Many other solution methods have been proposed in the AHP literature and their relative performance evaluated by simulation (e.g., Choo and Wedley, 2004).

We are not concerned here with the entire range of solution methods. Instead, we take the viewpoint that Saaty's method is appropriate and widely used. In addition, a simple column normalization procedure ("additive normalization") has received attention as a convenient approximation, which avoids computing eigenvectors. We concern ourselves with obtaining an appropriate consistency index to be used with the additive normalization procedure. This numerical value measures how close the decision maker is to making consistent decisions throughout the entire matrix. Saaty (1977) has produced the standard consistency index (CI) based on his eigenvector solution to the problem. Our new method can be viewed as an approximation to the CI but simpler to compute and to understand.

In Section 2, the eigenvector solution is discussed, Section 3 contains results on the properties of the column sums, Section 4 introduces the harmonic consistency index, Section 5 investigates how the harmonic consistency index changes with A, Section 6 obtains a bound on the harmonic consistency index in terms of a bound on A. Section 7 summarizes the results.

2. The standard solution

Perron's Theorem (e.g., Saaty, 1987) shows that any positive matrix A has a largest eigenvalue $\lambda_{\max}$ that is real and positive. The corresponding eigenvalue problem $Ax = \lambda_{\max}x$ has a solution x with $x_i > 0$ for all i which is unique to within multiplication by a scalar. Uniqueness will be obtained by requiring $\sum x_i = 1$. The eigenvalue $\lambda_{\max}$ always satisfies $\lambda_{\max} \geqslant n$ with $\lambda_{\max} = n$ if and only if consistency holds. The eigenvector solution was the original solution proposed to the AHP problem (Saaty, 1977). For any reciprocal matrix, not necessarily consistent, this vector x can be thought of as an estimate of a "true" priority vector. In this case, a matrix with (i,j) element given by x_i/x_j is an approximation to A. Saaty and Hu (1998) and Saaty (2003) argue that this is the only sensible solution. Only in the consistent case does a priority vector exist that recovers the A matrix exactly.

Since we will refer to column normalization frequently in this paper, let s_k be the sum of column k of the reciprocal matrix A. Divide each column k of A by s_k with the resulting matrix denoted by $\tilde{A}$. The matrix $\tilde{A}$ can be used to verify consistency of A as the following theorem shows. Saaty (1977) proved this result using eigenvectors, but we provide a direct proof in Appendix A.

Theorem 1. *An $n \times n$ reciprocal matrix A is consistent if and only if all columns of $\tilde{A}$ are identical.*

The reciprocals of the column sums $(s_1^{-1}, \ldots, s_n^{-1})$ appear on the diagonal of $\tilde{A}$ since the diagonal of A contains 1's. Furthermore, if A is consistent, then each column of $\tilde{A}$ will equal $(s_1^{-1}, \ldots, s_n^{-1})^{\mathrm{T}}$.

Example 1

$$A_1 = \begin{bmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.3 & 0.3 & 0.3 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$$

and $s_1^{-1} + s_2^{-1} + s_3^{-1} = 1$.

The matrix A_1 is consistent since $a_{13} = a_{12}a_{23}$ and all columns of $\tilde{A}$ are identical.

In elementary expositions of the AHP, it is common to avoid mention of the eigenvalue problem and instead use a simpler method: the arithmetic mean within each row of $\tilde{A}$ (Anderson et al., 2004; Evans and Olson, 2003; Taylor, 2004; Winston and Albright, 2001). The origin of this approximation can be traced back to Saaty (1977, p. 239). This method was called additive normalization (AN) by Srdjevic (2005). The resulting vector will be fairly close to the eigenvector solution if the matrix A is close to consistent. In simulation experiments, the AN method has performed as well as any of the many methods that have been invented including Saaty's eigenvector method (Zahedi, 1986; Choo and Wedley, 2004; Srdjevic, 2005).

We note that the AN method, like the eigenvector method are deterministic methods; they take the data in the A matrix as given constants rather than observations on random variables. The lack of consistency in a decision maker's responses is assumed due to psychological limitations rather than statistical error.

Example 2. We now change a_{13}, and its reciprocal a_{31}, so that the matrix A_2 is not consistent:

$$A_2 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 3 \\ 1/4 & 1/3 & 1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0.571 & 0.600 & 0.500 \\ 0.286 & 0.300 & 0.375 \\ 0.143 & 0.100 & 0.125 \end{bmatrix}$$

and $s_1^{-1} + s_2^{-1} + s_3^{-1} = \frac{279}{280}$.

The AN solution is $(0.5572, 0.3202, 0.1226)^{\mathrm{T}}$. The eigenvector solution is $x = (0.5584, 0.3196, 0.1220)^{\mathrm{T}}$, $\lambda_{\max} = 3.0183$.

In Example 2 the AN solution and the eigenvector solution are virtually identical. This is due to the fact that the matrix A_2 is close to consistent as can be seen since $\lambda_{\max}$ is very close to 3. We will show in Section 4 that this similarity is not surprising as the AN method can be viewed as an approximation to the eigenvector method.

3. Properties of the column sums

In order to investigate the column sums of A, we will need three versions of Jensen's inequality (e.g., DeGroot, 1970) given in Proposition 1. The proofs are found in Appendix A.

Proposition 1

(a) *Assume,* $\sum p_i \leqslant 1$, *and* φ *is a concave function on the non-negative real line with* $\varphi(0) = 0$. *Then* $\sum p_i \varphi(x_i) \leqslant \varphi[\sum p_i x_i]$ *holds for all* $x_i > 0$.

(b) *If* $\sum p_i < 1$ *and* φ *is strictly increasing and concave with* $\varphi(0) = 0$ *then there does not exist* $x_1, \ldots, x_n$ *satisfying* $\sum p_i \varphi(x_i) = \varphi[\sum p_i x_i]$.

(c) *If* $\sum p_i = 1$, *all* $p_i > 0$ *and* φ *is strictly increasing and concave with* $x_1, \ldots, x_n$ *satisfying* $\sum p_i \varphi(x_i) = \varphi[\sum p_i x_i]$. *Then all* x_i *are equal.*

In the consistent case, $(s_1^{-1}, \ldots, s_n^{-1})$ is the solution given by any reasonable solution method. The next theorem shows that the sum over $(s_1^{-1}, \ldots, s_n^{-1})$ is at most 1. (All summations in this article without explicit indexing indicated are assumed to range from 1 to n.)

Theorem 2. *Let* A_n *be a* $n \times n$ *reciprocal matrix with* s_j *the sum of column j. Then* $\sum s_j^{-1} \leqslant 1$.

Proof. The proof is by induction on n. If $n = 1$ then $s_1 = 1$ and the result holds. Now assume the result holds for n and show it holds for $n + 1$. Let A_{n+1} be any $(n + 1) \times (n + 1)$ reciprocal matrix and partition it by separating out the last row and column:

$$A_{n+1} = \left(\begin{array}{ccc|c} & & & b_1 \\ & A_n & & \vdots \\ & & & b_n \\ \hline 1/b_1 & \cdots & 1/b_n & 1 \end{array} \right)$$

The $\{b_i\}$ are arbitrary positive elements in column $n+1$ which also determine row $n+1$. Let $s_1, \ldots, s_n$ be the column sums of the submatrix A_n. Let R be the sum of the reciprocals of the column sums of A_{n+1}. Then we need to show that for all $b_i > 0$ we have $R \leqslant 1$, where

$$R = \sum [s_j + b_j^{-1}]^{-1} + [1+b]^{-1} \tag{1}$$

and $b = \sum b_i$. Let $p_j = s_j^{-1}$ so that $R = \sum p_j \left[\frac{b_j/p_j}{1+b_j/p_j}\right] + \frac{1}{1+b}$. Change the ariables from b_j to $x_j = b_j/p_j$. R now becomes $R = \sum p_j \frac{x_j}{1+x_j} + \frac{1}{1+\sum p_i x_i}$. $R \leqslant 1$ is equivalent to

$$\sum p_j \frac{x_j}{1+x_j} \leqslant \frac{\sum p_i x_i}{1+\sum p_i x_i}. \tag{2}$$

Consider the concave function $\varphi(x) = x/(1+x)$. This satisfies Proposition 1(a) and with $\sum p_j \leqslant 1$ from the induction hypothesis we conclude (2) holds for all $x_i > 0$. Therefore the proof by induction is complete. □

We now connect the value of $\sum s_j^{-1}$ with consistency. For motivation, we note that in Examples 1 and 2 we have $(s_1^{-1}, \ldots, s_n^{-1})$ on the main diagonal of $\tilde{A}_1$ and $\tilde{A}_2$. In the consistent case of Example 1, the $\sum s_j^{-1} = 1$ and all columns of $\tilde{A}_1$ were $(s_1^{-1}, \ldots, s_n^{-1})$. In Example 2, $\sum s_j^{-1} < 1$ and we had an inconsistent matrix.

Theorem 3. *The $n \times n$ reciprocal matrix A_n is consistent if and only if $\sum s_j^{-1} = 1$.*

Proof. ($\Rightarrow$) If A_n is consistent, then from the proof of Theorem 1 there exist positive weights $w_1, \ldots, w_n$ such that column j is given by $a_{ij} = w_i/w_j$ for $i = 1, \ldots, n$. Therefore, $s_j = \sum (w_i/w_j) = \frac{1}{w_j} \sum w_i$ and thus $\sum s_j^{-1} = 1$.

($\Leftarrow$) The proof is by induction on n. If $n = 1$ then, $s_1 = 1$ and the result holds. Now assume the result holds for n and show it holds for $n+1$. Let A_{n+1} be any $(n+1) \times (n+1)$ reciprocal matrix and partition it by separating out the last row and column as in the proof of Theorem 2. Let R be the sum of the reciprocals of the column sums of A_{n+1}. We assume $R = 1$ and want to show A_{n+1} is consistent. Proceeding as in the proof of Theorem 2, we have

$$\sum p_j \frac{x_j}{1+x_j} = \frac{\sum p_i x_i}{1+\sum p_i x_i}, \tag{3}$$

with $p_j = s_j^{-1}$. First suppose $\sum p_j < 1$. From Proposition 1(b) with $\varphi(x) = x/(1+x)$ we see that (3) cannot hold. From Theorem 2, the only other possibility is that $\sum p_j = 1$ and by the induction hypothesis we conclude the submatrix A_n is consistent. By Theorem 1, all columns of $\tilde{A}_n$ are the same and each column is $(s_1^{-1}, \ldots, s_n^{-1})^{\mathrm{T}}$ from which we conclude column 1 of A_n can be written as $(1, s_1 p_2, \ldots, s_1 p_n)^{\mathrm{T}}$.

We now proceed to find all solutions $\{x_i\}$ of (3). Since $\varphi(x) = x/(1+x)$ is strictly increasing and concave we apply Proposition 1(c) to conclude that the only solution of (3) is a constant solution $x_i = \kappa$ for all i. Therefore, with $b_j = x_j/s_j = \kappa p_j$, column $n+1$ of A_{n+1} is given by $(\kappa p_1, \ldots, \kappa p_n, 1)^{\mathrm{T}}$ and, since $p_1 + \cdots + p_n = 1$, when normalized the column is $(\kappa p_1, \ldots, \kappa p_n, 1)^T/(\kappa+1)$. The reciprocal of b_1 appears in element $(n+1, 1)$ of A_{n+1}. Using the result from the previous paragraph, the first column of A_{n+1} becomes $(1, s_1 p_2, \ldots, s_1 p_n, s_1/\kappa)^{\mathrm{T}}$ and after normalization this is $(\kappa p_1, \ldots, \kappa p_n, 1)^{\mathrm{T}}/(\kappa+1)$. And the same is true for any of the first n columns of A_{n+1}. This is the same vector we found for the normalized column $n+1$. Therefore, all columns of $\tilde{A}_{n+1}$ are identical and by Theorem 1 we conclude A_{n+1} is consistent. □

4. The harmonic consistency index

Saaty (1980) defined a consistency index for a reciprocal matrix as $\mathrm{CI} = (\lambda_{\max} - n)/(n-1)$, $\mathrm{CI} \geqslant 0$ with $\mathrm{CI} = 0$ if and only if the matrix is consistent. Other consistency indices have been created: Monsuur (1996) used a transformation of $\lambda_{\max}$; Peláez and Lamata (2003) examined all triples of elements and used the determinant to measure consistency and Koczkodaj (1993) also required the analysis of all triples of elements. Another type of consistency measure is the distance from a specific consistent matrix. Crawford and Williams (1985) used the sum of squared deviations of the log of the elements of a matrix from the log of the matrix elements generated by the row geometric mean solution. See Aguarón and

Moreno-Jiménez (2003) for a discussion of additional consistency indices.

We now define an index based on Theorem 2 using the harmonic mean. Denote the harmonic mean of $s = (s_1, \ldots, s_n)$ by HM(s). The condition $\sum s_j^{-1} \leqslant 1$ of Theorem 2 is equivalent to $\mathrm{HM}(s) \geqslant n$. Like the CI, it is desirable to scale this quantity as a function of n. We could use $[\mathrm{HM}(s) - n]/(n-1)$ by analogy with CI but we choose a slight adjustment of this quantity, based on simulation results, to make it more comparable to the CI values (see Table 1). Therefore, we define the *harmonic consistency index*, by

$$\mathrm{HCI} = \frac{[\mathrm{HM}(s) - n](n+1)}{n(n-1)}. \tag{4}$$

Returning to Example 2, we use $\lambda_{\max} = 3.0183$ to compute $\mathrm{CI} = 0.0091$, and the $\mathrm{HM} = 280/93 = 3.0107$ from which it follows that $\mathrm{HCI} = 0.0072$.

Fig. 1 shows a plot of CI versus HCI from a simulation of 500 4×4 matrices. The simulation procedure to construct a random matrix was as follows. We only need to specify the elements of the upper triangular portion of the A matrix and the other half will be filled in with reciprocals. Saaty (1977) has suggested that a maximum value of 9 be used in the A matrix. Using integer values 1 to 9 and their reciprocals, the upper triangular elements of A were randomly filled with elements from $\{1, 2, \ldots, 9, 1/2, 1/3, \ldots, 1/9\}$ (Tummala and Ling, 1998). The results showed that on average

Table 1
Random consistency index (RI)

	n										
	3	4	5	6	7	8	9	10	15	20	25
RI	0.525	0.882	1.115	1.252	1.341	1.404	1.452	1.484	1.583	1.630	1.654
HRI	0.550	0.859	1.061	1.205	1.310	1.381	1.437	1.484	1.599	1.650	1.675

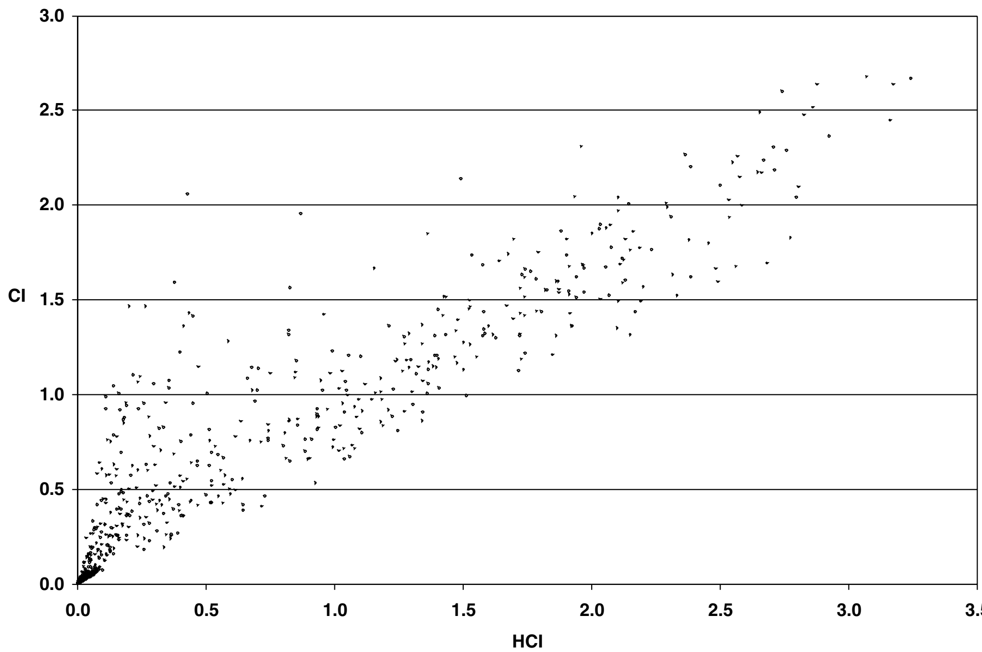

Fig. 1. The harmonic consistency index versus the consistency index for 500 simulated 4×4 A matrices.

CI and HCI were close and the correlation coefficient for $n = 4$ was 0.90.

Since CI tends to increase with matrix size, Saaty (1980, p. 21) suggested a method to try to remove the dependence on n. The same type of random matrices is used as above. The average of the CI values for the these random matrices for each n, called the random consistency index (RI), are given in Table 1 up to $n = 15$ (Aguarón and Moreno-Jiménez, 2003). Since values for size 20 and 25 are not available in the literature, we conducted a simulation to obtain the RI (accurate to within 0.004). The RI is used as a baseline measure. If we divide the CI for a problem by the appropriate RI from Table 1 the result is called the consistency ratio (CR). The rule-of-thumb generally used is that if a matrix has a CR of at most 0.10 (0.05 for $n = 3$ and 0.08 for $n = 4$) then it is sufficiently close to being consistent that the eigenvector priority vector is meaningful (Saaty, 1994).

We repeated this simulation process for the HCI. Generating random matrices and averaging the HCI gave what we call the harmonic random consistency index (HRI). Replications of up to 100,000 were made for each n to obtain accuracy 0.004. We see that the formula for HCI as defined in (4) is sufficiently accurate to track the CI as n changes. We should divide the HCI in a given problem by the value of HRI in Table 1 to remove the effect of matrix size. This results in the harmonic consistency ratio (HCR). From the results displayed in Fig. 1 and Table 1, we conclude that the HCI tracks the CI quite closely and therefore, pending detailed simulations, the HCR should be used with the same 0.10 rule-of-thumb as the CR.

HCI is recommended as a consistency measure if the AN solution method is used. The reason is that there is a natural connection between the HCI and the AN. To see this, first compute the matrix power A^k and then normalize any of its columns to sum to 1. As k increases, this column vector converges to the dominant eigenvector of A (Saaty, 1986, Theorem 14). This result can be stated as

Limiting result

Start with any n-vector $x_0 > 0$ and define $A^k x^0$, $k \geqslant 1$. Define the vector norm $\|y\| \equiv \sum |y_i|$. Then $A^k x_0 / \|A^k x_0\|$ converges to the dominant eigenvector and $\|A^k x_0\| / \|A^{k-1} x_0\|$ converges to $\lambda_{\max}$.

This is equivalent to the power method of linear algebra (Saaty, 1977; Reiter, 1990) which is the standard way to iteratively compute the dominant eigenvalue of a matrix and its corresponding eigenvector. The only difference is that after each iteration we normalize the x_k vector to sum to 1.

Power method

Start with any initial vector $x_0 > 0$, $\|x_0\| = 1$. Define a sequence of scalars $\lambda_1, \lambda_2, \ldots$ and vectors $x_1, x_2, \ldots$ with unit norm that satisfy the approximate eigenvalue problem $Ax_{k-1} = \lambda_k x_k$. This is accomplished by defining $x_k = Ax_{k-1} / \|Ax_{k-1}\|$ and $\lambda_k = \|Ax_{k-1}\|$. This provides an iterative method to solve the eigenvalue problem $Ax = \lambda_{\max} x$ since $x_k \to x$ and $\lambda_k \to \lambda_{\max}$ as k increases.

To connect this with the AN method, define x_0 as a column n-vector with element i given by $(ns_i)^{-1}$. By defining $x_1 = Ax_0$ we obtain a vector x_1 which gives the averages in each row of the column normalized A matrix. Thus, x_1 is the AN solution vector. It will always sum to 1 since $(1, \ldots, 1) A[(ns_1)^{-1} \ \cdots \ (ns_n)^{-1}]^{\mathrm{T}} = (s_1, \ldots, s_n)[(ns_1)^{-1} \ \ldots \ (ns_n)^{-1}]^{\mathrm{T}} = 1$. Since $\|x_0\| \neq 1$ we see that the AN solution is the first iteration using the limiting result above. Therefore AN is the initial step in an approximation of Saaty's eigenvector solution to the problem. Furthermore, for $k = 1$, $\|A^k x_0\| / \|A^{k-1} x_0\| = \|x_1\| / \|x_0\| = 1/\|x_0\| = \left[\sum s_i^{-1}/n\right]^{-1} = \mathrm{HM}(s)$. Thus the initial approximation to $\lambda_{\max}$ is the harmonic mean of the column sums of the A matrix. Using the power method viewpoint we obtain the same conclusion: Now we start with a normalized with element i given by $\mathrm{HM}(s)/(ns_i)$. Then $x_1 = Ax_0 / \|Ax_0\|$ is still the AN solution since multiplying the previous x_0 by a scalar will cancel out in this ratio. Also, $\lambda_1 = \|Ax_0\| = \mathrm{HM}(s)(1, \ldots, 1) A[(ns_1)^{-1} \ \cdots \ (ns_n)^{-1}]^{\mathrm{T}} = \mathrm{HM}(s)$.

The AN method is a simple and quick approximation to Saaty's eigenvector method. The harmonic mean of the column sums is the corresponding approximation to $\lambda_{\max}$. Returning to Example 2, we see $\mathrm{HM}(s) \approx \lambda_{\max}$ and the AN solution is very close to the eigenvector solution.

If we want a measure of consistency based on the AN, then the HCI is the natural measure. In addition, AN and HCI are easier to compute than the eigenvector and CI plus they have the added benefit of being easier to understand.

5. How the HCI changes as the A matrix changes

Aupetit and Genest (1993) investigate how the CI varies as an element of the A matrix changes. Of course, the corresponding reciprocal element is also forced to change. They showed that as any one element increases the CI will: (i) always increase, (ii) always decrease or (iii) decrease to a minimum and then increase. The latter case is the usual one as we would expect to locate a choice for the element of A, which minimizes the inconsistency relative to the rest of the matrix. If there were multiple local minima in the CI function it would cast doubt on using it as a meaningful measure of consistency. We now prove the analogous result for the HCI.

Theorem 4. *As a function of any one element of the $n \times n$ reciprocal A matrix, $n \geqslant 3$, the HCI has exactly one minimum and no maximum.*

Proof. Given any reciprocal matrix A with $s_1, \ldots, s_n$ as the column sums, we multiply the element in position (j,k) by any $\varepsilon > 0$. Let the initial value of a_{jk} be denoted a. The sum of the elements in column k becomes $a_{1k} + \cdots + a\varepsilon + \cdots + a_{nk}$ and this can be written as $s_k + a(\varepsilon - 1)$ where $s = (s_1, \ldots, s_n)$ are the original column sums. The value in position (k,j) now changes from $1/a$ to $1/a\varepsilon$. Let $s(\varepsilon) = (s_1(\varepsilon), \ldots, s_n(\varepsilon))$ denote the vector of column sums after the 2 cells are adjusted. The sum of column j becomes $s_j(\varepsilon) = s_j + 1/a\varepsilon - 1/a$. The partial derivative of $\mathrm{HM}(s(\varepsilon))$ with respect to ε is $\frac{\mathrm{HM}(s(\varepsilon))^2}{n}\left[\frac{\partial s_j(\varepsilon)}{\partial \varepsilon}\frac{1}{s_j^2(\varepsilon)} + \frac{\partial s_k(\varepsilon)}{\partial \varepsilon}\frac{1}{s_k^2(\varepsilon)}\right]$. Since we are only interested in the sign of this derivative we only need to examine the quantity in brackets. That simplifies to

$$\frac{a}{[s_k + a(\varepsilon - 1)]^2} - \frac{a}{[a\varepsilon s_j + 1 - \varepsilon]^2}.$$

The quantities inside the squared terms are always positive. Therefore, this expression is positive if and only if $\varepsilon > [s_k-(a+1)]/[as_j-(a+1)]$. Note that the numerator is always positive since $n \geqslant 3$ implies there is at least one other (positive) element in column k of A in addition to a and 1. Likewise the denominator is strictly positive since $s_j > 1 + 1/a$. Therefore there is exactly one stationary point as ε ranges over the positive reals:

$$\varepsilon^* = [s_k - (a+1)]/[as_j - (a+1)]. \tag{5}$$

This corresponds to a minimum in HM since the derivative is positive for all $\varepsilon > \varepsilon^*$ and negative for all $\varepsilon < \varepsilon^*$. $\square$

The proof of the previous theorem shows that for any inconsistent reciprocal matrix A, there is a unique adjustment for each given element to make the consistency the best it can be (minimize the HCI) by using ε^*. Note that if $n = 2$ then ε^* might be 0 or undefined (infinite). That is, HM might increase or decrease for all ε just as $\lambda_{\max}$ does.

6. The maximum HCI if $1/S \leqslant a_{ij} \leqslant S$

Saaty (1980) and others have suggested that the response of decision makers be limited to a maximum value of 9, for psychological reasons. Aupetit and Genest (1993) relied upon classical matrix theory results to find the maximum value of $\lambda_{\max}$ and hence CI under this assumption. We will obtain the same upper bound for the HM as was found for $\lambda_{\max}$. Assume S is the maximum value for the elements of A. Consider this form of a reciprocal matrix:

$$A = \begin{bmatrix} 1 & S & 1/S & S & \cdots \\ 1/S & 1 & S & 1/S & \\ S & 1/S & 1 & S & \\ 1/S & S & 1/S & 1 & \\ \vdots & & & & \ddots \end{bmatrix}. \tag{6}$$

We will prove that HCI assumes its maximum value for this matrix. Aupetit and Genest (1993) showed $\lambda_{\max}$ and CI also achieved their maximum values for this same matrix. From a graph theoretic viewpoint, Genest et al. (1993) showed this matrix is 'maximally intransitive' in the sense that it has the maximum number of circular triads (cycles of length 3 where $a_{ij} > 1$, $a_{jk} > 1$, $a_{ki} > 1$). Vargas (1980) relates the definition of CI to cycles of various lengths.

Theorem 5. *Let A be any reciprocal matrix with bounded elements $1/S \leqslant a_{ij} \leqslant S$. The harmonic mean of the column sums satisfies. $n \leqslant \mathrm{HM}(s) \leqslant 1 + 0.5(n-1)(S + 1/S)$. Therefore $\mathrm{HCI} \leqslant (S + 1/S - 2)(n+1)/(2n)$.*

Proof. First we demonstrate that the maximum value of the HM will be achieved with an A matrix having only S and $1/S$ elements off the diagonal. This follows from Theorem 3 which shows that HM must assume its maximum on the boundary of the interval $[1/S, S]$. Thus we can always replace any element a, $1/S < a < S$, by either S or $1/S$ and thereby increase the HM.

Consider any reciprocal matrix consisting of only S and $1/S$ off diagonal elements. Since each S is paired with an $1/S$ element, we will have $n(n-1)/2$ S elements and the same number of $1/S$ elements in the entire matrix. Therefore, the sum of all the elements is given by $n + 0.5n(n-1)(S + 1/S)$ so the column average is $\sum s_i/n = 1 + 0.5(n-1)(S + 1/S) \equiv B$. We now use the well-known harmonic mean-arithmetic mean inequality to obtain $\mathrm{HM}(s) \leqslant B$. This provides an upper bound for all n.

Now assume n is odd and consider the matrix in (6). By direct computation all column sums are equal to B and thus the HM will be equal to B. So this matrix for any odd n maximizes HM. By Proposition 1(c), the only kind of matrix for which the HM will reach this maximal value will have column sums that are identical (and equal to B). This implies an equal number of S and $1/S$ elements in each column (and row). This suggests matrix (6) is not unique in maximizing HM as can be easily seen by example. In addition, if n is even it will not be possible to find a matrix with $\mathrm{HM} = B$. The closest we can get to equal column sums will be $n/2$ of the S elements and $n/2 - 1$ of the $1/S$ or vice-versa. (Half the columns will be of each type.) We can easily compute the harmonic mean for that type of matrix. For n even, this will provide a tight upper bound for HM that is slightly less than the previous bound of B:

$$B - (S - 1/S)^2/(4B). \qquad \square \tag{7}$$

It is reasonable that any measure of inconsistency should reach its maximum value at the maximally intransitive matrix (6). CI and HCI have passed this test. In fact, for odd n, $\lambda_{\max}$ and HM assume the same value, B, for this matrix.

7. Summary

New results have been obtained concerning properties of reciprocal matrices. In particular, since $\sum s_j^{-1} \leqslant 1$, we were able to define a new consistency measure for a reciprocal matrix, the harmonic consistency index (HCI) and the companion harmonic consistency ratio (HCR). Simulation results indicate HCI has values similar to the standard CI measure as illustrated in Fig. 1 and Table 1. Fig. 1 shows that while HCI and CI are similar on average, HCI is far from a simple transformation of CI. Since the CI and CR are so commonly used, we scaled HCI and HCR so that, on average, HCI $\approx$ CI and HCR $\approx$ CR. This should simplify the use and interpretation of HCI and HCR values.

Theorems 4 and 5 demonstrated how HCI varies as matrix elements change. Somewhat surprisingly, HM and $\lambda_{\max}$ satisfied the same bounds.

The authors are not advocating use of the AN method; other procedures may be preferred. The authors view three approaches as superior in both a practical and theoretical sense: the standard eigenvector approach, the row geometric mean (Crawford and Williams, 1985) and the singular value decomposition (Gass and Rapcsák, 2004). The AN method has gained favor as a simple but effective solution method since it only requires column normalization and row averages. The simple nature of AN has led to its increased use, especially as a pedagogical tool. However, the CI method is then used along with the AN method to measure consistency. Therefore computation of the dominant eigenvalue is still required which defeats the purpose of using a simple approach. When we showed that the AN is an approximation of the eigenvalue method (via the power method), the harmonic mean arose as an approximation to $\lambda_{\max}$. Thus the HCI provides a natural and appropriate consistency measure when using AN that is simple to obtain and interpret.

Additional theoretical and applied analysis concerning the properties of the harmonic consistency index is in progress.

Appendix A

This appendix contains the proofs of Theorem 1 and Proposition 1.

Proof of Theorem 1. ($\Rightarrow$) If we have consistency, then any column of A can be obtained from column 1: $a_{ij} = a_{i1}a_{1j}$ for all i and j. Summing both sides over i gives $s_j = s_1 a_{1j}$. By definition, $\tilde{a}_{ij} = a_{ij}/s_j = a_{i1}a_{1j}/s_j = a_{i1}/s_1 = \tilde{a}_{i1}$. So all columns of $\tilde{A}$ are the same.

($\Leftarrow$) Let $p_k = s_k^{-1}$. Let w be any of the identical columns of $\tilde{A}$. Then $w_i = a_{ij}/s_j = a_{ij}p_j$ which implies $w_i = p_i$ since $a_{ii} = 1$. Thus $a_{ij} = p_i/p_j$ and $a_{ik}a_{kj} = (p_i/p_k)(p_k/p_j) = p_i/p_j = a_{ij}$. $\square$

Proof of Proposition 1. (a) Let $x^* = \sum p_i x_i$. By properties of a concave function the tangent line to the curve at the point $x = x^*$ will satisfy $\varphi(x) \leqslant \alpha + \beta x$ for all x. The curve and the line meet at $x^* : \varphi(x^*) = \alpha + \beta x^*$. Note that $\varphi(0) = 0$ implies $\alpha \geqslant 0$. Therefore $\varphi(x_i) \leqslant \alpha + \beta x_i$ implies $\sum p_i\varphi(x_i) \leqslant \alpha \sum p_i + \beta \sum p_i x_i \leqslant \alpha + \beta x^*$ since $\alpha \geqslant 0$ and $\sum p_j \leqslant 1$. Therefore we conclude $\sum p_i \varphi(x_i) \leqslant \varphi(x^*)$.

(b) $x^* = \sum p_i x_i$. By properties of a strictly increasing concave function the tangent line to the curve at the point $x = x^*$ will satisfy $\varphi(x) < \alpha + \beta x$ for all $x \neq x^*$. The curve and the line meet at x^* so that $\varphi(x^*) = \alpha + \beta x^*$. Note that $\varphi(0) = 0$ implies $\alpha > 0$. Thus $\sum p_i \varphi(x_i) \leqslant \alpha \sum p_i + \beta \sum p_i x_i < \alpha + \beta x^*$ since $\alpha > 0$ and $\sum p_i < 1$. Therefore we conclude $\sum p_i \varphi(x_i) < \varphi(x^*)$.

(c) Partition $\{1, \ldots, n\}$ into the set $E = \{i : x_i = x^*\}$ and its complement. As in the proof of (b), $\varphi(x_i) < \alpha + \beta x_i$ for $i \notin E$ and $\varphi(x_i) = \alpha + \beta x_i$ for $i \in E$. Therefore, we can write $\sum p_i \varphi(x_i) = \sum_{i \in E} p_i \varphi(x_i) + \sum_{i \notin E} p_i \varphi(x_i)$ where $\sum_{i \notin E} p_i \varphi(x_i) < \alpha \sum_{i \notin E} p_i + \beta \sum_{i \notin E} p_i x_i$ while $\sum_{i \in E} p_i \varphi(x_i) = \alpha \sum_{i \in E} p_i + \beta \sum_{i \in E} p_i x_i$. The sum of the right hand sides of the previous two relations reduces to $\varphi(x^*)$. Therefore if $E = \{1, \ldots, n\}$ then $\sum p_i \varphi(x_i) = \varphi(x^*)$ while if even one $i \notin E$, then $\sum p_i \varphi(x_i) < \varphi(x^*)$. So we conclude $\sum p_i \varphi(x_i) = \varphi(x^*)$ implies all the x_i must be the same value. (Note that we know all the p_i are positive. If only $p_i \geqslant 0$ we would conclude all the $x_i = x^*$ except for x_i with $p_i = 0$.) $\square$

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The quality of priority ratios estimation in relation to a selected prioritization procedure and consistency measure for a Pairwise Comparison Matrix

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Abstract: An overview of current debates and contemporary research devoted to the modeling of decision making processes and their facilitation directs attention to the Analytic Hierarchy Process (AHP). At the core of the AHP are various prioritization procedures (PPs) and consistency measures (CMs) for a Pairwise Comparison Matrix (PCM) which, in a sense, reflects preferences of decision makers. Certainly, when judgments about these preferences are perfectly consistent (cardinally transitive), all PPs coincide and the quality of the priority ratios (PRs) estimation is exemplary. However, human judgments are very rarely consistent, thus the quality of PRs estimation may significantly vary. The scale of these variations depends on the applied PP and utilized CM for a PCM. This is why it is important to find out which PPs and which CMs for a PCM lead directly to an improvement of the PRs estimation accuracy. The main goal of this research is realized through the properly designed, coded and executed seminal and sophisticated simulation algorithms in *Wolfram Mathematica 8.0.* These research results convince that the embedded in the AHP and commonly applied, both genuine PP and CM for PCM may significantly deteriorate the quality of PRs estimation; however, solutions proposed in this paper can significantly improve the methodology.

***Keywords**: pairwise comparisons, priority ratios, consistency, AHP, Monte Carlo simulations*

Introduction

It is agreed that the world is a complex system of interacting elements. It is obvious that human minds have not yet evolved to the point where they can clearly perceive relationships of this global system and solve crucial issues associated with them. In order to deal with complex and fuzzy social, economic, and political issues, people must be supported and guided on their way to order priorities, to agree that one goal out-weighs another from a perspective of certain established criterion, to make tradeoffs in order to be able to serve the greatest common interest (Caballero, Romero & Ruiz 2016; García-Melón et al. 2016).

Obviously, intuition cannot be trusted, although many commonly do so, attempting to devise solutions for complex problems which demand reliable answers. Overwhelming scientific evidence indicates that the unaided human brain is simply not capable of simultaneous analysis of many different competing factors and then synthesizing the results for the purpose of rational decision. It is presumably the principal reason why scientists continuously deal with explanations and modeling of decisional problems in a way to make them widely comprehendible. That is why many supportive methodologies have been elaborated in order to make the decision making process easier, more credible and sometimes even possible. Indeed, numerous psychological experiments (Martin 1973),

including the well-known Miller study (Miller 1956) put forth the notion that humans are not capable of dealing accurately with more than about seven (±2) things at a time (the human brain is limited in its short term memory capacity, its discrimination ability and its bandwidth of perception).

Principles of the analytic thinking process

Humans learn about anything by two means. The first involves examining and studying some phenomenon from the perspective of its various properties, and then synthesizing findings and drawing conclusions. The second entails studying some phenomenon in relation to other similar phenomena and relating them by making comparisons (Saaty 2008). The latter method leads directly to the essence of the matter i.e. judgments regarding the phenomenon. Judgments can be relative and absolute. An absolute judgment is the relation between a single stimulus and some information held in short or long term memory. A relative judgment, on the other hand, can be defined as the identification of some relation between two stimuli both present to the observer (Blumenthal 1977). It is said that humans can make much better relative judgments than absolute ones (Saaty 2000). It is probably so because humans have better ability to discriminate between the members of a pair, than compare one thing against some recollection from long term memory.

For detailed knowledge, the mind structures complex reality into its constituent parts, and these in turn into their elements. The number of parts usually ranges between five and nine. By breaking down reality into homogenous clusters and subdividing those into smaller ones, humans can integrate large amounts of information into the structure of a problem and form a more comprehensive picture of the whole system. Abstractly, this process entails the decomposition of a system into a hierarchy which is a model of a complex reality. Thus, a hierarchy constitutes a structure of multiple levels where the first level is the objective followed successively by levels of factors, criteria, sub-criteria, and so on down to a bottom level of alternatives. The goal of this hierarchy is to evaluate the influence of higher level elements on those of a lower level or alternatively the contribution of elements in the lower level to the importance or fulfillment of the elements in the level above. In this context the latter elements serve as criteria and are called properties.

Generally, a hierarchy can be functional or structural. The latter closely relates to the way a human brain analyzes complexity by breaking down the objects perceived by the senses into clusters and sub-clusters, and so on. Thus, in structural hierarchies, complex systems are structured into their constituent parts in descending order according to their structural properties. In contrast, in functional hierarchies complex systems are decomposed into their constituent parts in accordance to their essential relationships.

A large number of hierarchies in application are available in the literature (Saaty 1993). Supposedly, the hierarchical classification is the most powerful method applied by the human mind during intellectual reasoning and ordering of information and/or observations. Thus, we may agree that an efficient and effective multiple criteria decision making process should encompass the following steps:

– transpose the problem into a hierarchy;
– derive judgments that reflect ideas and feelings or emotions;
– represent these judgments with meaningful numbers values;
– apply those number values for computing priorities for the elements in the hierarchy;

– synthesize the results in order to establish an overall outcome.

There is a multiple criteria decision making support methodology which meets the prescription developed above. It is called the Analytic Hierarchy Process (AHP) and was developed at the Wharton School of Business by Thomas Saaty (1977). Although it is a very popular and widely implemented theory of choice, it is also controversial, thus very often validated and valuated from the perspective of its methodology. From that perspective, most recent papers, such as Grzybowski (2016); Kazibudzki (2016a); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); Aguarón, Escobar & Moreno-Jiménez (2014); Lin, Kou & Ergu (2013); Brunelli, Canal & Fedrizzi (2013), unfold new research areas in this matter which should be thoroughly examined and provoke questions which should be answered, that is:

1) Is the principal right eigenvector (REV), as the prioritization procedure (PP), necessary and sufficient for the AHP?

2) Is the reciprocity of the Pairwise Comparison Matrix (PCM) a reasonable condition leading to the improvement of the priority ratios estimation quality?

3) Are PCM consistency measures, commonly applied and embedded in the AHP, really conducive to the improvement of the priority ratios estimation quality?

Principles of the Analytic Hierarchy Process

Preliminaries

The AHP seems to be the most widely used multiple criteria decision making approach in the world today. Probably, the most recent list of application oriented papers can be found in Grzybowski (2016). Actual applications in which the AHP results were accepted and used by competent decision makers can be found in: Saaty (2008); Ishizaka & Labib (2011); Ho (2008); Vaidya & Kumar (2006); Bhushan & Ria (2004); or Saaty & Vargas (2006). However, regardless of AHP popularity, the genuine methodology is also undeniably the most validated, developed and perfected contemporary methodology, see for example: Kazibudzki (2016b); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); or Aguarón, Escobar & Moreno-Jiménez (2014).

The AHP allows decision makers to set priorities and make choices on the basis of their objectives, knowledge and experience in a way that is consistent with their intuitive thought process. AHP has substantial theoretical and empirical support encompassing the study of human judgmental process by cognitive psychologists. It uses the hierarchical structure of the decision problem, pairwise relative comparisons of the elements in the hierarchy, and a series of redundant judgments. This approach reduces errors and provides a measure of the consistency of judgments. The process permits accurate priorities to be derived from verbal judgments even though the words themselves may not be very precise. Thus, it is possible to use words for comparing qualitative factors and then to derive ratio scale priorities that can be combined with quantitative factors.

To make a proposed solution possible i.e. derive ratio scale priorities on the basis of verbal judgments, a scale is utilized to evaluate the preferences for each pair of items. Apparently, the most popular is Saaty's numerical scale which comprises of the integers from one (equivalent to the verbal judgment - 'equally preferred') to nine (equivalent to the verbal judgment - 'extremely preferred'), and their reciprocals. However, in conventional AHP applications it may be desirable to utilize other scales also i.e. a geometric and/or numerical scale. The former usually consists of the numbers computed in accordance with the formula $2^{n/2}$ where n comprises of the integers from minus eight to eight. The latter involves arbitrary integers from one to n and their reciprocals.

The first step in using AHP is to develop a hierarchy by breaking a problem down into its primary components. The basic AHP model includes the goal (a statement of the overall objective), criteria (the factors that should be considered in reaching the ultimate decision) and alternatives (the feasible alternatives that are available to achieve said ultimate goal). Although the most common and basic AHP structure consists of a goal-criteria-alternatives sequence (Fig.1). AHP can easily support more complex hierarchies. A variety of basic hierarchical structures include:
– goal, criteria, sub-criteria, scenarios, alternatives;
– goal, players, criteria, sub-criteria, alternatives;
– goal, criteria, levels of intensities, many alternatives.

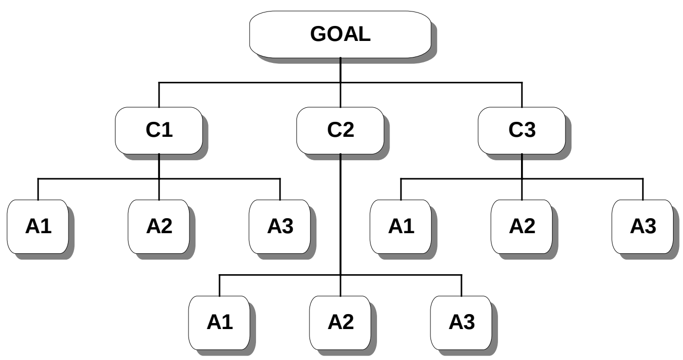


Fig. 1 - Example of a fundamental three level hierarchy encompassing three criteria and three alternatives under each criterion

Mathematics behind the Analytic Hierarchy Process

The conventional procedure of priority ranking in AHP is grounded on the well-defined mathematical structure of consistent matrices and their associated right-eigenvector's ability to generate true or approximate weights.

The German mathematician, Oscar Perron, proved in 1907 that, if $\boldsymbol{A}=(a_{ij})$, $a_{ij}>0$, where i, $j=1,\ldots,n$, then $\boldsymbol{A}$ has a simple positive eigenvalue $\lambda_{\max}$ called the principal eigenvalue of $\boldsymbol{A}$ and $\lambda_{\max}>|\lambda_k|$ for the remaining eigenvalues of $\boldsymbol{A}$. Furthermore, the principal eigenvector $\boldsymbol{w}=[w_1,\ldots,w_n]^T$ that is a solution of $\boldsymbol{Aw}=\lambda_{\max}\boldsymbol{w}$ has $w_i>0$, $i=1,\ldots,n$. Thus, the conventional concept of AHP can be presented as follows:

$$\begin{bmatrix} w1/w1 & w1/w2 & w1/w3 & \dots & w1/wn \\ w2/w1 & w2/w2 & w2/w3 & \dots & w2/wn \\ w3/w1 & w3/w2 & w3/w3 & \dots & w3/wn \\ \vdots & \vdots & \vdots & & \vdots \\ wn/w1 & wn/w2 & wn/w3 & \dots & wn/wn \end{bmatrix} \times \begin{bmatrix} w1 \\ w2 \\ w3 \\ \vdots \\ wn \end{bmatrix} = \begin{bmatrix} nw1 \\ nw2 \\ nw3 \\ \vdots \\ nwn \end{bmatrix} \quad (1)$$

If the relative weights of a set of activities are known, they can be expressed as a Pairwise Comparison Matrix (PCM) as shown above $\boldsymbol{A}(w)$. Now, knowing $\boldsymbol{A}(w)$ but not $\boldsymbol{w}$ (vector of priority ratios), Perron's theorem can be applied to solve this problem for $\boldsymbol{w}$. The solution leads to n unique values for λ, with an associated vector $\boldsymbol{w}$ for each of the n values.

PCMs in the AHP reflect relative weights of considered activities (criteria, scenarios, players, alternatives, etc.), so the matrix $\boldsymbol{A}(w)$ has a special form. Each subsequent row of that matrix is a constant multiple of its first row. In this case a matrix $\boldsymbol{A}(w)$ has only one non-zero eigenvalue, and since the sum of the eigenvalues of a positive matrix is equal to the sum of its diagonal elements, the only non-zero eigenvalue in such a case equals the size of the matrix and can be denoted as $\lambda_{max} = n$.

The norm of the vector $\boldsymbol{w}$ can be written as $\|\boldsymbol{w}\|=\boldsymbol{e}^T\boldsymbol{w}$ where: $e=[1, 1,\dots, 1]^T$ and $\boldsymbol{w}$ can be normalized by dividing it by its norm. For uniqueness, $\boldsymbol{w}$ is referred to in its normalized form.

Theorem 1: A positive n by n matrix has the ratio form $\boldsymbol{A}(w)=(w_i / w_j)$, $i, j=1,\dots, n$, if, and only if, it is consistent.

Theorem 2: The matrix of ratios $\boldsymbol{A}(w)=(w_i / w_j)$ is consistent if and only if n is its principal eigenvalue and $\boldsymbol{Aw}=n\boldsymbol{w}$. Further, $\boldsymbol{w}>0$ is unique to within a multiplicative constant.

Definition 1: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ij}=1/w_{ji}$ for all i, $j=1,\dots, n$ then the matrix $\boldsymbol{A}(w)$ is called *reciprocal*.

Definition 2: The matrix $\boldsymbol{A}(w)$ is called *ordinal transitive* if the following conditions hold:
(A) if for any $i=1,\dots, n$, an element a_{ij} is not less than an element a_{ik} then $a_{ij} \geq a_{ik}$ for $i=1,\dots, n$, and
(B) if for any $i=1,\dots, n$, an element a_{ji} is not less than an element a_{ki} then $a_{ji} \geq a_{ki}$ for $i=1,\dots, n$.

Definition 3: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ik}w_{kj}=w_{ij}$ for all $i, j, k=1,\dots, n$, and the matrix is *reciprocal*, then it is called *consistent* or *cardinal transitive*.

Certainly, in real life situations when AHP is utilized, there is not an $\boldsymbol{A}(w)$ which would reflect weights given by the vector of priority ratios. As was stated earlier, the human mind is not a reliable measurement device. Assignments such as, 'Compare – applying a given ratio scale – your feelings concerning alternative 1 versus alternative 2', do not produce accurate outcomes. Thus, $\boldsymbol{A}(w)$ is not established but only its estimate $\boldsymbol{A}(x)$ containing intuitive judgments, more or less close to $\boldsymbol{A}(w)$ in accordance with experience, skills, specific knowledge, personal taste and even temporary mood or overall disposition. In such case, consistency property does not hold and the relation between elements of $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$ can be expressed as follows:

$$x_{ij} = e_{ij} w_{ij} \tag{2}$$

where e_{ij} is a perturbation factor fluctuating near unity. In the statistical approach e_{ij} reflects a realization of a random variable with a given probability distribution.

It has been shown that for any matrix, small perturbations in the entries imply similar perturbations in the eigenvalues, that is why in order to estimate the true priority vector $\boldsymbol{w}$, conventional AHP utilizes Perron's theorem. The solution of the matrix equation $\boldsymbol{Aw}=\lambda_{max}\boldsymbol{w}$, gives us $\boldsymbol{w}$ as the Right Principal Eigenvector (REV) associated with λ_{max}.

In practice the REV solution is obtained by raising the matrix $\boldsymbol{A}(x)$ to a sufficiently large power, then the rows of $\boldsymbol{A}(x)$ are summed and the resulting vector is normalized in order to receive $\boldsymbol{w}$. This concept can be also delivered in the form of the following formula:

$$w = \lim_{k\to\infty}\left(\frac{A^k \times e}{e^T \times A^k \times e}\right) \tag{3}$$

where: $e=[1, 1,\ldots, 1]^T$.

Description of the first problem

It has been promoted that the REV prioritization procedure (PP) is necessary and sufficient to uniquely establish the ratio scale rank order inherent in inconsistent pairwise comparison judgments (Saaty & Hu 1998). However, there are alternative PPs devised to cope with this problem. Many of them are optimization based and seek a vector w, as a solution of the minimization problem given by the formula:

$$\min \boldsymbol{D}(\boldsymbol{A}(x), \boldsymbol{A}(w)) \tag{4}$$

subject to some assigned constraints such as, for example, positive coefficients and normalization condition. Because the distance function $\boldsymbol{D}$ measures an interval between matrices $\boldsymbol{A}(x)$ and $\boldsymbol{A}(w)$, different ways of its definition lead to various prioritization concepts and prioritization results. As an example, Choo et al. (2004) describes and compares eighteen estimation procedures for ranking purposes although some authors suggest there are only fifteen that are different. Furthermore, since the publication of the above article, a few additional procedures have been introduced to the literature, see for example: Grzybowski (2012).

Certainly, when the PCM is consistent, all known procedures coincide. However, in real life situations, as was discussed earlier, human judgments produce inconsistent PCMs. The inconsistency is a natural consequence of human brain dynamics described earlier and also a consequence of the questioning methodology, mistaken entering of judgment values, and scaling procedure (i.e. rounding errors). It seems crucial to emphasize here that usually even perfectly consistent PCMs, only because of rounding errors are not error-free. It can be illustrated on the basis of the following hypothetic example.

The genuine priority vector: $\boldsymbol{w}=[7/20, 1/4, 1/4, 3/20]$ is considered and derived from it, $\boldsymbol{A}(w)$ which can be presented as follows:

$$\boldsymbol{A}(w)=\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Now it is considered $\boldsymbol{A}(x)$ produced by a hypothetic decision maker (DM), whose judgments are perfectly consistent. Even if it is assumed that the selected DM is very trustworthy and can express judgments very precisely, DM is still somehow limited by the necessity of expressing judgments on a scale (the example utilizes Saaty's scale). As such, the DM will produce the PCM ($\boldsymbol{A}(x)$) which is not error-free because the entries must be in this case rounded to the closest values of Saaty's scale. Since $\boldsymbol{A}(x)$ must be reciprocal (the fundamental requirement of the AHP) the PCM appears as follows:

$$\boldsymbol{A}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

It may be noticed that the above PCM is perfectly consistent, so this construct seems to be exemplary. However, the hypothetic DM, despite best intentions, is burdened with inescapable estimation errors. In the above situation the priority vector (PV) derived from $\boldsymbol{A}(x)$ by any PP, provides the following priority ratios (PRs): $\boldsymbol{x}$=[2/7, 2/7, 2/7, 1/7] which are not equal to those considered exemplary: $\boldsymbol{w}$=[7/20, 1/4, 1/4, 3/20]. Obviously, the deviation between those PVs can also be expressed by their Mean Absolute Error (MAE), for instance, established by the following formula:

$$MAE(w,x)=\frac{1}{n}\sum_{i=1}^{n}\left|w_i-x_i\right| \tag{5}$$

where n is the number of elements within the particular PV. Noticeably, in the above example, MAE equals 1/28.

From that perspective, Saaty & Hu's (1998) declaration articulating that the REV *is the only valid PP for deriving the PV from a PCM, particularly when the PCM is inconsistent* seems at least questionable. However, they provide an example of a situation where variability in ranks does not occur for each individual judgment matrix, it occurs in the overall ranking of the final alternatives due to the application of different PPs and the multi-criteria process itself. They argue that only the REV possesses a sound mathematical background directly dealing with the question of inconsistency. Furthermore, as they state, only the REV captures the rank order inherent in the inconsistent data in a unique manner. It appears to be time to verify the credibility of these statements utilizing the Monte Carlo simulations.

For that purpose, apart from the REV, four different PPs have been arbitrarily selected ranked as the best within AHP methodology (Kazibudzki & Grzybowski 2013; Lin 2007; Choo & Wedley 2004) – Table 1.

Table 1 – Formulae for the prioritization procedures

The Prioritization Procedure	Formula for the Prioritization Procedure
Logarithmic Utility Approach – LUA –	$w_{(LUA)} = \min \sum_{i=1}^{n} \ln^2 \left(\sum_{j=1}^{n} \frac{a_{ij} w_j}{n w_i} \right)$
Sum of Squared Relative Differences Method – SRDM	$w_{(SRDM)} = \min \sum_{i=1}^{n} \left(\frac{1}{n w_i} \sum_{j=1}^{n} a_{ij} w_j - 1 \right)^2$
Logarithmic Least Squares Method – LLSM –	$w_{(LLSM)} = \min \sum_{i=1}^{n} \sum_{j=1}^{n} \ln^2 \left(a_{ij} \frac{w_j}{w_i} \right)$
Simple Normalized Column Sum – SNCS –	$w_{i(SNCS)} = \frac{1}{n} \sum_{j=1}^{n} \left(a_{ij} \Big/ \sum_{k=1}^{n} a_{kj} \right)$

The first problem study

The objective of this chapter is to verify the above statement i.e. *the REV is the only valid method for deriving the PV from a PCM, particularly when the matrix is inconsistent*.

Taking into account the exemplary study of Saaty & Hu (1998), it seems that the best way to analyze the problem is to examine whether different PPs are really inferior in the estimation of true PVs whose intent is accurate estimation. From that perspective, only computer simulations can illuminate the question, for it is possible to elaborate an algorithm which enables simulation of different kinds of errors which may occur during the process of judgment, and enables assessment which one from the selected PPs delivers better estimates (from a given perspective) of the genuine PV.

Thus, the following simulation algorithm was constructed. Assuming that the decisional problem can be presented in the form of a three level hierarchy (goal, criteria and alternatives – see Figure 1). In order to emulate the problem presented in Saaty & Hu (1998), the hypothetical hierarchy is also designed as a four criteria and four alternatives structure i.e. n=4 and m=4. In agreement with these assumptions, it is possible to elaborate and execute the simulation algorithm **SA|1|** comprising of the following steps:

Step 1. Randomly generate a priority vector $\boldsymbol{k}=[k_1,\ldots, k_n]^T$ of assigned size [n x 1] for criteria and related perfect PCM(k)=$\boldsymbol{K}(k)$

Step 2. Randomly generate exactly n priority vectors $\boldsymbol{a_n}=[a_{n,1},\ldots, a_{n,m}]$ of assigned size [m x 1] for alternatives under each criterion and related perfect PCMs(a)=$\boldsymbol{A_n}(a)$

Step 3. Compute a total priority vector $\boldsymbol{w}$ of the size [m x 1] applying the following procedure: $w_x=k_1a_{1,x} + k_2a_{2,x} +\ldots+ k_na_{n,x}$

Step 4. Randomly choose a number e from the assigned interval [α; β] on the basis of assigned probability distribution π

Step 5. Apply separately *Step 5A* and *Step 5B*:

<u>Step 5A</u> – *the case of PCM forced reciprocity implementation;*

replace all elements a_{ij} for $i<j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with ea_{ij}, and all elements k_{ij} for $i<j$ of $\boldsymbol{K}(k)$ with ek_{ij}

<u>Step 5B</u> – *the case of arbitrary PCM acceptance;*

replace all elements a_{ij} for $i\neq j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with ea_{ij}, and all elements k_{ij} for $i\neq j$ of $\boldsymbol{K}(k)$ with ek_{ij}

Step 6. Apply separately *Step 6A* and *Step 6B*:

<u>Step 6A</u> – *when Step 5A is performed;*

round all values of elements a_{ij} for $i<j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$, and all values of elements k_{ij} for $i<j$ of $\boldsymbol{K}(k)$ to the closest values from a considered scale, then replace all elements a_{ij} for $i>j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$ with $1/a_{ij}$, and all elements k_{ij} for $i>j$ of $\boldsymbol{K}(k)$ with $1/k_{ij}$

<u>Step 6B</u> – *when Step 5B is performed;*

round all values of elements a_{ij} for $i\neq j$ of all $\boldsymbol{A}_{\mathbf{n}}(a)$, and all values of elements k_{ij} for $i\neq j$ of $\boldsymbol{K}(k)$ to the closest values from a considered scale

Step 7. On the basis of all perturbed $\boldsymbol{A}_{\mathbf{n}}(a)$ denoted as $\boldsymbol{A}_{\mathbf{n}}(a)^*$ and perturbed $\boldsymbol{K}(k)$ denoted as $\boldsymbol{K}(k)^*$ compute their respective priorities vectors $\boldsymbol{a}_{\mathbf{n}}^*$ and $\boldsymbol{k}^*$ with application of assigned estimation procedure (EP), i.e.: REV, LUA, SRDM, LLSM, and SNCS.

Step 8. Compute a total priority vectors $\boldsymbol{w}^*$(EP) of the size $[m \times 1]$ applying the following procedure: $w^*_x=k^*_1a^*_{1,x}+k^*_2a^*_{2,x}+\ldots+k^*_na^*_{n,x}$

Step 9. Calculate *Spearman rank correlation coefficients* – $\mathrm{SRC}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ between all $\boldsymbol{w}^*$(EP) and $\boldsymbol{w}$, as well designated estimation precision characteristics, i.e.: mean relative errors:

$$RE_{\gamma,\chi}\left(w^*(EP),w\right)=\frac{1}{m}\sum_{i=1}^{m}\frac{\left|w_i-w_i^*(EP)\right|}{w_i} \tag{6}$$

along with mean relative ratios:

$$RR_{\gamma,\chi}\left(w^*(EP),w\right)=\frac{1}{m}\sum_{i=1}^{m}\frac{w_i^*(EP)}{w_i} \tag{7}$$

Step 10. Repeat Steps 4 to 9, χ times, where χ denotes a size of the sample

Step 11. Repeat Steps 1 to 9, γ times, where γ denotes a number of considered AHP models

Step 12. Return arithmetic average values of all $\mathrm{SRC}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, $\mathrm{RE}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, and $\mathrm{RR}_{\gamma,\chi}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ computed during all runs in Steps: 10 and 11, i.e.:

$$MSRC\left(w^*(EP),w\right)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}SRC_i\left(w^*(EP),w\right) \tag{8}$$

$$MRE\left(w^*(EP),w\right)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}RE_i\left(w^*(EP),w\right) \tag{9}$$

$$MRR\left(w^*(EP),w\right)=\frac{1}{\gamma\times\chi}\sum_{i=1}^{\gamma\times\chi}RR_i\left(w^*(EP),w\right) \tag{10}$$

where: $\mathrm{MSRC}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$, $\mathrm{MRE}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ and $\mathrm{MRR}(\boldsymbol{w}^*(\mathrm{EP}),\boldsymbol{w})$ denotes: *mean Spearman rank correlation coefficient*, *average mean relative error* and *average mean relative ratio*, respectively.

In the first experiment, the probability distribution π attributed in Step 4 to the perturbation factor *e* is selected arbitrarily to be the *gamma* or *uniform* distribution. These are two of the distribution types most frequently considered in literature for various implementation purposes (Grzybowski 2016). Usually recommended are such types as: *gamma*, *log-normal*, *truncated normal*, or *uniform*. Apart from these most popular π, one

can find applications of the Couchy, Laplace, or either *triangle* and *beta* probability distributions (see e.g. Dijkstra 2013).

The first simulation scenario also assumes that the perturbation factor e will be drawn from the interval $e \in [0.01; 1.99]$. Noticeably, in each case hereafter, the parameters of different probability distributions applied are set in such a way that the expected value of e in each particular simulation scenario EV(e)=1. It seems a very reasonable assumption, because although human judgments are not accurate, they undeniably aim perfect ones.

Furthermore, the number of alternatives and criteria in a single AHP model will be assigned randomly. By 'randomly' – without any other explicit specification – hereafter defined as a process operating under uniform distribution. All simulation scenarios also assume application of the rounding procedure which always operates according to the *geometric* scale described earlier in this paper.

Finally, the first scenario also takes into account the obligatory assumption in conventional AHP applications i.e. the PCM reciprocity condition. In such cases, only judgments from the upper triangle of a given PCM are taken into account and those from the lower triangle are replaced by the inverses of the former.

The outcomes i.e. mean characteristics for 30,000 cases (χ=15 and γ=2000) of the first simulation scenario are presented in Table 2. It may be noticed from Table 2, that the REV can be undeniably classified as the worst PP from the perspective of PRs derived from ranks established on the basis of three different prioritization quality measures i.e. MRE, MSRC, and MRR. The best two PPs from the viewpoint of this classification are LLSM, known also as Geometric Mean Procedure (GM), and LUA. Certainly, the first scenario experiment was designed only to contrast the results presented by Saaty & Hu (1998). It is the intention to establish wider and more fundamental relationships among the presented PPs.

Table 2 – Mean performance measures of arbitrarily selected PPs for 30,000 cases

Scenario Details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean Rank
Geometric Scale	*gamma* distribution	*FR*-PCM	LLSM	0.438438	1	0.682300	2	1.21242	1	1.3(3)
			REV	0.452614	5	**0.668380**	5	1.22051	4	4.6(6)
			LUA	0.447349	2	0.673067	3	1.21792	2	2.3(3)
			SRDM	0.448759	3	0.671380	4	1.21870	3	3.3(3)
			SNCS	0.450734	4	0.692453	1	1.24398	5	3.3(3)
	uniform distribution	*FR*-PCM	LLSM	0.288608	1	0.804860	2	1.12813	1	1.3(3)
			REV	0.302346	4	**0.792580**	5	1.13530	4	4.3(3)
			LUA	0.298401	2	0.795767	3	1.13350	2	2.3(3)
			SRDM	0.299400	3	0.794820	4	1.13400	3	3.3(3)
			SNCS	0.303463	5	0.808333	1	1.15450	5	3.6(6)

Note: *FR*-PCM denotes *forced reciprocity* applied to PCM during simulations

The second simulation scenario was designed to encompass new assumptions not yet taken into account in the literature. First of all, taken into consideration were results obtained not only on the basis of reciprocal PCM, but also the simulation outcomes of nonreciprocal PCM. Secondly, it was decided to implement into simulations new intervals for random errors and apply their new probability distribution. As is known, many

simulation analyses presented in literature assume very non symmetric intervals for a perturbation factor (considering its influence on the particular element of PCM). For example consider the interval for perturbation factor applied in the first simulation scenario i.e. $e \in [0.01;1.99]$. Under this assumption, it becomes apparent that if some entry of PCM is modified *in plus* by the perturbation factor from that particular interval, it is multiplied maximal by the number 1.99, so if the original entry is 3, the modified value will be around 6. However, if some entry of PCM is modified *in minus* by the perturbation factor from that particular interval, it may result that some entry will be multiplied by the number 0.01, so in fact the entry will be divided by 100. Thus, in the situation where the original entry is 9, the modified value will be 0.09, which can be rounded to 1/9 on the Saaty's scale. It may be noticed that this modification practically reverses the preference of DM from e.g. extremely preferred A over B, to extremely preferred B over A (applying the Saaty scale).

It is obvious that this very common assumption is imposed by another very crucial and logical assumption which states that the expected value of e in every particular simulation scenario should equal one i.e. EV(e)=1. It is quite easy to fulfill that requirement on the basis of an asymmetric interval for the perturbation factor (from the perspective of its influence on a particular element of PCM). However, it is rather a challenge to have this assumption implemented with a symmetric interval for the perturbation factor. That is why commonly applied simulation scenarios minimize the range for the perturbation factor in order to achieve at least the delusion of symmetry for $e \in [0.5;1.5]$. Nevertheless, that objective has been attained with the present research, yet to be achieved by other researchers. Presently it seems reasonable to apply symmetric intervals to simulations for the perturbation factor because they better reflect true life situations. Thus, different kinds of probability distributions (PDs) were experimented with and it was discovered that Fisher-Snedecor PD possesses the feature that can be useful in the present analysis. It occurs that for n_1=14 and n_2=40 degrees of freedom for one thousand randomly generated numbers on the basis of this PD, their mean equals 1.03617, so it is very close to unity, and these numbers fluctuate from 0.174526 to 5.57826. So, with these assumptions, we have $e \in [0.174526; 5.57826]$, which gives a very symmetric distribution for the perturbation factor, and EV(e)≈1. The results of prioritization quality for different selected PPs and assumed prioritization quality measures i.e. MSRC, MRE, and MRR obtained on the basis of described earlier simulation scenario, are presented in Table 3.

As can be noticed, the REV again is not the dominant PP from the perspective of all simulation scenarios under prescribed frameworks (it takes third place in the total classification order). Certainly, apparent differences in the PV estimation quality in relation to the selected PP are noticeable for nonreciprocal PCMs.

Then, the LUA and SRDM or LLSM dominate over the rest of the selected PPs, especially from the perspective of rank correlations which are the crucial issue from the viewpoint of rank preservation phenomena. These issues will be treated in the section entitled '*Breakthroughs and milestones of this research*'.

Table 3 – Mean performance measures of arbitrarily selected five different ranking procedures for various uniformly drawn 100,000 AHP models – 1,000 hypothetic decisional problems perturbed 100 times each (*)

Scenario Details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean Rank
Geometric Scale	*n*, *m*∈{3, 4…, 7}	FRPCM	LLSM	0.123288	4	0.916281	1	1.04646	3	2.6(6)
			REV	0.123030	1	0.915056	5	1.04546	1	2.3(3)
			LUA	0.123044	3	0.915489	2	1.04699	4	3
			SRDM	0.123038	2	0.915476	3	1.04567	2	2.3(3)
			SNCS	0.132926	5	0.915228	4	1.05865	5	4.6(6)
		APCM	LLSM	0.100511	1	0.930242	4	1.02953	4	3
			REV	0.101523	4	0.930164	5	1.02938	3	4
			LUA	0.100658	2	0.930965	2	1.02926	2	**2**
			SRDM	0.101310	3	0.930510	3	1.02925	1	**2.3(3)**
			SNCS	0.108689	5	0.931026	1	1.04315	5	3.6(6)
	n, *m*∈{8, 9…, 12}	FRPCM	LLSM	0.079748	4	0.931396	1	1.03319	4	3
			REV	0.079110	1	0.928266	5	1.03116	1	2.3(3)
			LUA	0.079321	3	0.928817	2	1.03173	3	2.6(6)
			SRDM	0.079286	2	0.928769	4	1.03166	2	2.6(6)
			SNCS	0.086223	5	0.928799	3	1.03935	5	4.3(3)
		APCM	LLSM	0.063936	4	0.943393	3	1.02252	4	3.6(6)
			REV	0.062735	3	0.942399	5	1.02070	1	3
			LUA	0.061757	1	0.944593	1	1.02109	3	**1.6(6)**
			SRDM	0.061852	2	0.944314	2	1.02105	2	**2**
			SNCS	0.068981	5	0.942764	4	1.02879	5	4.6(6)
Saaty's scale	*n*, *m*∈{3, 4…, 7}	FRPCM	LLSM	0.143650	4	0.911381	1	1.06578	4	3
			REV	0.142967	1	0.911151	4	1.06498	1	2
			LUA	0.143069	3	0.911347	2	1.06520	3	2.6(6)
			SRDM	0.143054	2	0.911320	3	1.06517	2	2.3(3)
			SNCS	0.155694	5	0.910735	5	1.07850	5	5
		APCM	LLSM	0.116095	1	0.927455	1	1.04681	3	**1.6(6)**
			REV	0.116994	4	0.926955	4	1.04705	4	4
			LUA	0.116337	2	0.927129	3	1.04657	1	**2**
			SRDM	0.116962	3	0.926532	5	1.04658	2	3.3(3)
			SNCS	0.127154	5	0.927397	2	1.06051	5	4
	n, *m*∈{8, 9…, 12}	FRPCM	LLSM	0.100279	4	0.917231	1	1.04856	4	3
			REV	0.098084	1	0.915833	4	1.04630	1	2
			LUA	0.098648	3	0.916245	2	1.04695	3	2.6(6)
			SRDM	0.098569	2	0.916193	3	1.04687	2	2.3(3)
			SNCS	0.106674	5	0.915633	5	1.05424	5	5
		APCM	LLSM	0.078464	4	0.938192	3	1.03563	4	3.6(6)
			REV	0.077002	3	0.937837	4	1.03422	1	2.6(6)
			LUA	0.076762	1	0.939669	1	1.03469	3	**1.6(6)**
			SRDM	0.076789	2	0.939415	2	1.03464	2	**2**
			SNCS	0.084307	5	0.937796	5	1.04125	5	5

Average Mean Rank	**LLSM**	**2.958**	***REV***	***2.792***	**LUA**	**2.292**	***SRDM***	***2.417***	**SNCS**	**4.542**
Order	**4**		**3**		**1**		**2**		**5**	

Note: (*) AHP models drawn randomly (uniformly) for assigned set of criteria and alternatives. The scenario assumes application of both: perturbation factor drawn with F-Snedecor probability for n_1=14 and n_2=40

degrees of freedom, and rounding errors associated with a given scale (geometric or Saaty's). It assumes calculation of performance measures either for reciprocal PCMs (FRPCM) or nonreciprocal PCMs (APCM).

Description of the second problem

In the previous two sections of this research, it was determined that the quality of PV estimation depends on the selected PP. This section will focus on the other facet of the problem i.e. how the quality of PV estimation depends on the type of PCM Consistency Measure (PCM-CM) engaged in the prioritization process. The difference between the meaning of consistency of a given PCM and the particular PCM-CM is intentionally stressed at this point. Indeed, there are several PCM-CMs provided in the literature called consistency indices (CIs), nevertheless the scientific meaning of PCM consistency is given by the definition (Definition 3).

The most popular and certainly less intuitive is the PCM-CM proposed by Saaty. He proposed his PCM-CM on the basis of his PP which involves eigenvectors and eigenvalues calculations. Thus, the indication of the fact that for the consistent PCM its $\lambda_{max}= n$, for the purpose of PCM consistency measurement, Saaty proposes his CI be determined by the following formula:

$$CI_{REV} = \frac{\lambda_{\max} - n}{n-1} \tag{11}$$

where n indicates the number of alternatives within the particular PCM. The significant disadvantage of this PCM-CM is the fact it can operate exclusively with reciprocal PCMs. In the case of nonreciprocal PCMs, this measure is useless (its values are meaningless) which in consequence seriously diminishes the value of the whole Saaty approach (Linares et al. 2014).

However, as mentioned earlier, there are a number of additional PCM-CMs. Some of them, as in the case of CI_{REV}, originate from the PPs devised for the purpose of the PV estimation process. Their distinct feature is the fact that all of them can operate equally efficiently in conditions where reciprocal and nonreciprocal PCMs are accepted. A number of them, selected on the basis of their popularity (but not only) and up-to-date nature (Kazibudzki 2016b) are presented in Table 4.

Noticeably, there are few propositions of PCM-CMs which are not connected with any PP and are devised on the basis of the PCM consistency definition (Definition 3). Koczkodaj's (1993) idea is the first to be considered. His PCM-CM is grounded on his concept of triad consistency. The notion of a triad:

<u>Statement 1</u>: For any three distinguished decision alternatives A_1, A_2, and A_3, there are three meaningful priority ratios i.e. α, β, and χ, which have their different locations in a particular $\boldsymbol{A}(w)=[w_{ij}]_{nxn}$

<u>Definition 4</u>: If $\alpha=w_{ik}$, $\chi=w_{kj}$, $\beta=w_{ij}$ for some different $i\leq n$, $j\leq n$, and $k\leq n$, then the tuple (α, β, χ) is called a *triad.*

<u>Definition 5</u>: If the matrix $\boldsymbol{A}(w)=[w_{ij}]_{nxn}$ is consistent, then $\alpha\chi=\beta$ for all triads.

Table 4 – Formulae for the PCM-CMs related to their PPs

Symbol of the PP	Formula for the PCM-CM
LUA	$\mathrm{CI_{LUA}}=\frac{1}{n}\sqrt{\min\sum_{i=1}^{n}\ln^2\left(\sum_{j=1}^{n}\frac{a_{ij}w_j}{nw_i}\right)}$
SRDM	$\mathrm{CI_{SRDM}}=\sqrt{\frac{1}{n}\min\sum_{i=1}^{n}\left(\frac{1}{nw_i}\sum_{j=1}^{n}a_{ij}w_j-1\right)^2}$
LLSM	$\mathrm{CI_{LLSM}}=\frac{2}{(n-1)(n-2)}\sum_{i<j}\log^2\left(\frac{a_{ij}w_j}{w_i}\right)$

In consequence, either of the equations 1–β/αχ=0 and 1–αχ/β=0 have to be true. Taking the above into consideration, Koczkodaj proposed his measure for triad inconsistency by the following formula:

$$TI(\alpha,\beta,\chi)=\min\left[\left|1-\frac{\beta}{\alpha\chi}\right|,\left|1-\frac{\alpha\chi}{\beta}\right|\right] \tag{12}$$

Following his idea, he then proposed the following CM of any reciprocal PCM:

$$K(TI)=\max[TI(\alpha,\beta,\chi)] \tag{13}$$

where the maximum value of TI(α,β,χ) is taken from the set of all possible triads in the upper triangle of a given PCM.

On the basis of Koczkodaj's idea of triad inconsistency, Grzybowski (2016) presented his PCM consistency measure determined by the following formula:

$$A(TI)=\frac{1}{N}\sum_{i=1}^{N}[TI_i(\alpha,\beta,\chi)] \tag{14}$$

Finally, following the idea, that ln(αχ/β) = minus ln(β/αχ), Kazibudzki (2016a) redefined triad inconsistency and proposed:

– two formulae for its measurement -

$$LTI_1(\alpha,\beta,\chi)=|\ln(\alpha\chi/\beta)| \tag{15}$$

$$LTI_2(\alpha,\beta,\chi)=\ln^2(\alpha\chi/\beta) \tag{16}$$

– and one meaningful formula for PCM-CM -

$$A(LTI_i)=\frac{1}{N}\sum_{j=1}^{N}[LTI_{ij}(\alpha,\beta,\chi)] \tag{17}$$

which can be calculated on the basis of triads from the upper triangle of the given PCM when it is reciprocal, or all triads within the given PCM when it is nonreciprocal.

The second problem study

This section begins with the fundamental question which should encourage all researchers who deal with the problem of PR estimation quality to seek appropriate PCM consistency measurement. The question asks:
Does a growth of the PCM consistency directly lead to the betterment of the priority vector estimation quality?

Apparently, the answer to this question seems to be affirmative. Commonly, this is the reason why one strives to keep the consistency of the PCM at the highest possible level. However, *is it a good idea to use universally recognized PCM-CMs for this purpose?* To answer this question a preliminary analysis of the example provided and examined in the section entitled '*Description of the first problem*' can be initiated.

Thus, the genuine PV is reconsidered, ***w***=[7/20, 1/4, 1/4, 3/20] and ***A***(*w*) derived from that PV can be presented as follows:

$$\boldsymbol{A}(w)=\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Now considering two PCMs i.e. ***R***(*x*) and ***A***(*x*) produced by a hypothetical DM, whose judgments are rounded to Saaty's scale – DM is very trustworthy and is able to express judgments very precisely. In the first scenario, entries of ***A***(*w*) are rounded to Saaty's scale and the entries are made reciprocal (a principal condition for a PCM in the AHP) producing:

$$\boldsymbol{R}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

In the second scenario only entries of ***A***(*w*) are rounded to Saaty's scale (nonreciprocal case)producing:

$$\boldsymbol{A}(x)=\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

It should be noted that ***R***(*x*) is perfectly consistent and ***A***(*x*) is not. Tables 5 and 6 present selected values of the PPs related PCM-CMs (that is CI_{REV}, CI_{LUA}, and CI_{LLSM}) for ***R***(*x*) and ***A***(*x*) together with PVs derived from ***R***(*x*) and ***A***(*x*); Mean Absolute Errors (MAEs) [Formula (18)], among ***w****(PP) and the genuine ***w*** for the case; Spearman Rank Correlation Coefficients (SRCs) among ***w****(PP) and the genuine ***w*** for the case.

$$MAE\left(w^*(PP), w\right) = \frac{1}{n}\sum_{i=1}^{n}\left|w_i - w_i^*(PP)\right| \quad (18)$$

Table 5 – Values of the PCM-CMs for ***R***(*x*) and proposed characteristics of PVs estimates (*) quality in relation to the genuine PV for the case

PP	Estimates	Performance measures		
		CI(*PP*)	MAE	SRC
REV	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
LUA	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
LLSM	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966

(*) derived from ***R***(*x*) with application of a particular PP

Table 6 – Values of the PCM-CMs for ***A***(*x*) and proposed characteristics of PVs estimates (*) quality in relation to the genuine PV for the case

PP	Estimates	Performance measures		
		CI(*PP*)	MAE	SRC
REV	$[0.309401, 0.267949, 0.267949, 0.154701]^T$	– 0.0893164	0.0202995	1
LUA	$[0.306135, 0.268645, 0.268645, 0.156576]^T$	0.0344483	0.0219326	1
LLSM	$[0.314288, 0.264284, 0.264284, 0.157144]^T$	0.0400378	0.0178559	1

(*) derived from ***A***(*x*) with application of a particular PP

Surprisingly, a very interesting phenomenon can be noted on the basis of information provided in Tables 5 and 6. The nonreciprocal version of the analyzed PCM contains non-zero values for the selected PCM-CMs. In cases similar to this example, the value of Saaty's PCM-CM always becomes negative which makes it inexplicable and in consequence useless under such circumstances (as already mentioned earlier). The other two measures are positive and higher than zero which indicates that the particular PCM is not consistent. On the basis of the same indicators in the case of the reciprocal version of the analyzed PCM, its perfect consistency is apparent because all selected PCM-CMs in this case are equal to zero. However, the estimation precision measures (MAE and SRC) i.e. characteristics of the particular PV estimation quality, indicate something quite opposite. Surprisingly, apparent are smaller values of MAEs and perfect correlation of ranks between estimated and genuine PV for nonreciprocal version of the analyzed PCM. Certainly, this conclusion concerns all analyzed PPs and it is very true in the situation when the particular PCM is apparently less consistent (on the basis of selected exemplary PCM-CMs).

It has been suggested that these discoveries inevitably lead to the conclusion that the time has just come to revise the common yet erroneous approach to the PCM consistency measurement which can be described as … *the lower PCM-CM, the better PR estimation quality*.

Therefore, it becomes apparent that there are actually three significantly different consistency notions: (1) the consistency of PCM stated by Definition 3, and reflected by a value of the specific CM which in its way denotes a deviation of the analyzed PCM from its fully consistent counterpart; (2) the consistency of DM i.e. their reliability from the viewpoint of their expertise, measured by a comparison of DM judgments reflected by the particular PCM with judgments made more or less randomly; and (3) the PCM consistency stated by Definition 3 and reflected by a value of the specific CM which denotes the

particular PCM applicability for PRs derivation in the way that minimizes estimation errors.

The third notion is of particular interest from the perspective of the Multiple Criteria Decision Making (MCDM) quality. The key concept of the issue was first presented by Grzybowski (2016) and enhanced by Kazibudzki (2016a). It was decided to examine the phenomenon described therein and further develop it to improve the quality of MCDM. The simulation framework for this purpose was adopted from Kazibudzki (2016a) as the only way to examine said phenomena through computer simulations. The simulation algorithm **SA|2|** thus comprises of the following phases:

Phase 1 Generate randomly a priority vector $\boldsymbol{w}=[w_1,\ldots, w_n]^{\mathrm{T}}$ of assigned size $[n \times 1]$ and related perfect PCM(w)=$\boldsymbol{K}(w)$

Phase 2 Select randomly an element w_{xy} for $x<y$ of $\boldsymbol{K}(w)$ and replace it with $w_{xy}e_B$ where e_B is a relatively significant error, randomly drawn (*uniform* distribution) from the interval $e_B\in[2;4]$. Errors of that magnitude are basically considered as "significant", see e.g.: Grzybowski (2012), Dijkastra (2013), Lee (2007).

Phase 3 For each other element w_{ij}, $i<j\leq n$ select randomly a value e_{ij} for the relatively small error in accordance with the given probability distribution π (applied in equal proportions as: *gamma*, *log-normal*, *truncated normal*, and *uniform* distribution) and replace the element w_{ij} with the element w_{ij} e_{ij} where e_{ij} is randomly drawn (*uniform* distribution) from the interval $e_{ij}\in[0{,}5;1{,}5]$

Phase 4 Round all values of $w_{ij}\, e_{ij}$ for $i<j$ of $\boldsymbol{K}(w)$ to the nearest value of a considered scale

Phase 5 Replace all elements w_{ij} for $i>j$ of $\boldsymbol{K}(w)$ with $1/w_{ij}$

Phase 6 After all replacements are done, return the value of the examined index as well as the estimate of the vector $\boldsymbol{w}$ denoted as $\boldsymbol{w}$*(PP) with application of assigned prioritization procedure (PP). Then return the mean absolute error MAE between $\boldsymbol{w}$ and $\boldsymbol{w}$*(PP). Remember values computed in this phase as one record.

Phase 7 Repeat *Phases* from 2 to 6 N_n times.

Phase 8 Repeat *Phases* from 1 to 7 N_m times.

Phase 9 Return *all* records to one database file.

Once again, all parameters of the applied PDs – *gamma*, *log-normal*, *truncated normal*, and *uniform* – in the above simulation framework are set as previously in such a way that the expected value EV(e_{ij})=1.

The simulation begins from n=4, because simulations for n=3 are not interesting due to direct interrelation of considered PCM consistency measures (Bozóki & Rapcsak 2008, Dijkstra 2013). For the sake of objectivity, the simulation data is gathered in the following way: all values of selected consistency measures are split into 15 separate sets designated by the quantiles Q of order p from 1/15 to 14/15. The 15 intervals are defined as: the first is from 0 to the quantile of order 1/15 i.e. $\mathrm{VRCM}_1=[0, Q_{1/15})$, where VRCM represents *a Value Range of the Selected PCM Consistency Measure*; the second denotes $\mathrm{VRCM}_2=[Q_{1/15}, Q_{2/15})$, and so on… to the last one which starts from the quantile of order 14/15 and goes on to infinity i.e. $\mathrm{VRCM}_{15}=[Q_{14/15}, \infty)$. The following variables are examined: Mean VRCM_n, average MAE within VRCM_n between $\boldsymbol{w}$ and $\boldsymbol{w}$***(PP), MAE quantiles of the following orders, 0.05, 0.1, 0.5, 0.9, 0.95, and relations between all of them. In the preliminary simulation program, it was decided that PP=LLSM. The application of the rounding procedure was also assumed which in this preliminary research operates according to Saaty's scale.

Lastly, the scenario takes into account the compulsory assumption in conventional AHP applications i.e. the PCM reciprocity condition. The results are based on N_n=20, and N_m=500, i.e. 10,000 cases.

In the case of a good PCM-CM, one could assume that MAE quantiles of any order should monotonically grow concurrently with the growth of the selected PCM-CM e.g. VRCM index. The same relation should occur for Mean VRCM_n and average MAE for VRCM_n. The results of the proposed simulation framework, or any other similar simulation scenario which would contradict such a pertinent relationship would unequivocally lead to the conclusion that the examined PCM-CM does not serve its purpose.

An examination from that point of view is in order, the performance of six PCM-CMs selected from among very common or recently proposed (Fig.2): Saaty CI_{REV} – (Plot A), together with Crawford & Williams CI_{LLSM} – (Plot B), and Koczkodaj $K(TI)$ – (Plot C), together with Grzybowski $A(TI)$ – (Plot D), as well as Kazibudzki $A(LTI_1)$ and CI_{LUA} – (Plots E-F).

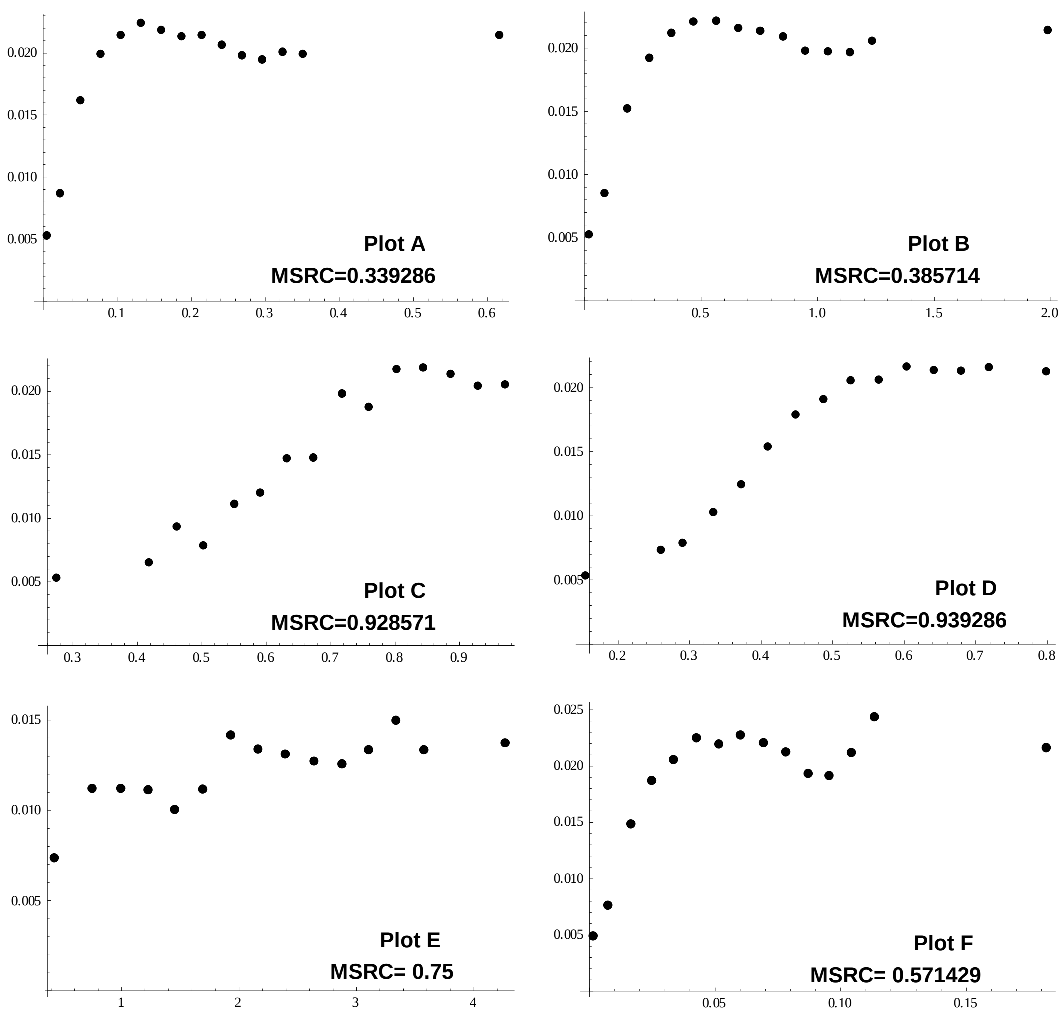


Fig. 2 – Performance of selected PCM-CMs – The plots present the relation between a mean value of a given PCM-CM within a given interval (VRCM_n) and quantiles of order 0.05 of MAEs distribution concerning estimated and genuine PV for the case. The results are generated with application of LLSM as the PP. Plots are based on 10,000 random reciprocal PCMs for n=4. The relation strength MSRC denotes Mean Spearman Rank Correlation Coefficient between analyzed variables.

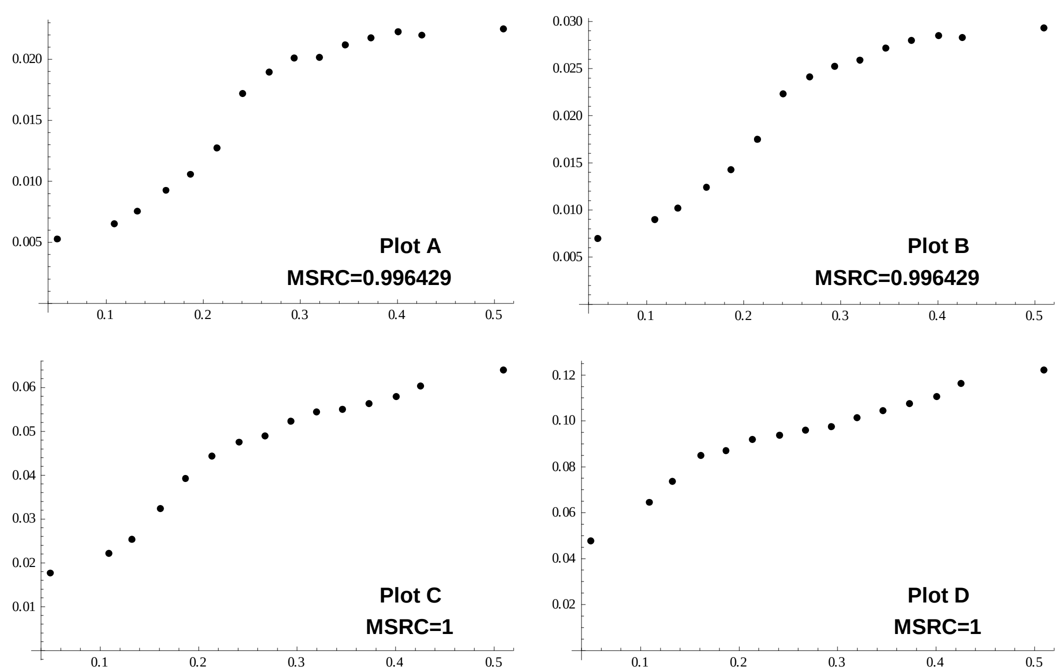
Noticeably, when the quality of PV in MCDM process of AHP is taken into consideration, the presented relations indicate that the analyzed performance of selected PCM-CMs vary more or less from the target. Indeed, the relations indicate that most of the analyzed indices may even misinform DMs about their judgment applicability for the construct of the PV which best converge with the ideal one i.e. obtained from a fully consistent PCM. As seen similarly in the example provided earlier in this paper (Tab. 5 and 6), taking the particular index as the measure of PCM consistency, one can expect both i.e. the betterment of PRs estimation quality (increase of the estimation accuracy) together with the increase of the particular CI (decrease of PCM consistency); and inversely, the deterioration of PRs estimation quality (decrease of the estimation accuracy) together with the descent of the particular CI (improvement of PCM consistency).

Noticeably, the analyzed PCM-CMs are not selected without a reason as they are commonly applied and/or suggested as good solutions in the process of PV estimation on the basis of inconsistent PCMs (for discussion see also Grzybowski 2016). This was the motivation to search for a PCM-CM which relation to PV estimation errors, reflected by SRC, would be very close or equal to 1 (the most desirable situation).

Thus, a seminal solution is proposed in this matter. On the basis of triad inconsistency measure $LTI_2(\alpha,\beta,\chi)=\ln^2(\alpha\chi/\beta)$ introduced by Kazibudzki (2016a), the following PCM-CM is submitted:

$$CM(LTI_2)=\frac{MEAN[LTI_2(\alpha,\beta,\chi)]}{1+MAX[LTI_2(\alpha,\beta,\chi)]} \tag{19}$$

The proposed PCM-CM is denoted as *the Triads Squared Logarithm Corrected Mean* and an examination of its performance on the basis of simulation algorithm **SA|2|** proposed earlier in this paper was carried out.



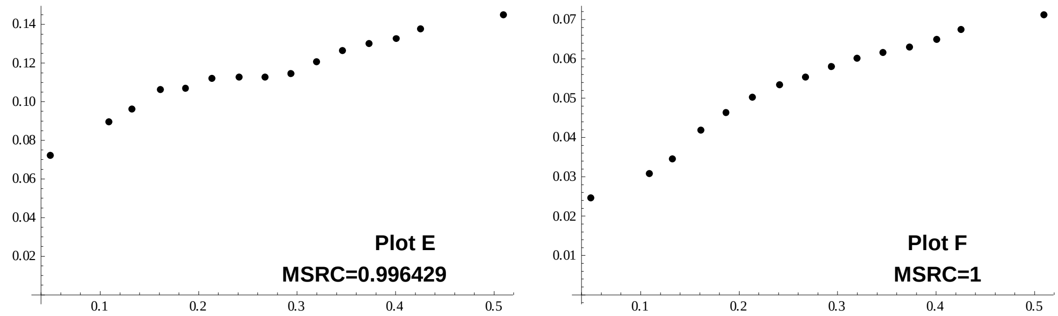


Fig. 3 – Performance of the new PCM-CM: CM(LTI_2) The plots present a relation between a mean value of CM(LTI_2) within a given interval ($VRCM_n$) and quantiles of order 0.05, 0.1, 0.5, 0.9, 0.95 of MAEs distribution as well their average values for estimated and genuine PVs. The results are generated with application of LLSM as the PP. Plots are based on 10,000 random reciprocal PCMs for *n*=4.

As can be noticed, the proposed CM(LTI_2) significantly outperforms the other PCM-CMs analyzed earlier in this paper. It is undeniably a seminal revelation that unquestionably opens a new chapter in MCDM on the basis of AHP – especially because CM(LTI_2) is suitable for both reciprocal and nonreciprocal PCM.

Breakthroughs and milestones of the research

As was said in 1990 by the creator of AHP: … *there is a well-known principle in mathematics that is widely practiced, but seldom enunciated with sufficient forcefulness to impress its importance. A necessary condition that a procedure for solving a problem be a good one is that if it produces desired results, and we perturb the variables of the problem in some small sense, it gives us results that are 'close' to the original ones. (…) An extension of this philosophy in problems where order relations between the variables are important is that on small perturbations of the variables, the procedure produces close, order preserving results* (Saaty 1990, p. 18).

The quality of PR estimation in relation to the selected PP

With said notion in mind, an effort was undertaken to verify the statement of followers of the REV, boldly spreading the idea that so long as inconsistency is accepted, the REV is the paramount theoretical basis for deriving a scale and no other concepts qualify.

It is a fact that in order to support some theory, one must verify it through many experiments to validate its reliability. On the other hand one needs only one example showing it does not work in order to abolish its credibility. Thus, numerous examples were provided indicating that the REV concedes with other devised PP to determine ranking of alternatives. However, although data obtained during simulation experiments are unequivocal, they support the above notion only generally. That is why scientific verification of their meaning is carried out on the basis of the statistical hypothesis testing theory (SHTT).

If $MSRC_{PP}$ and $MSRC_{REV}$ respectively are denoted as mean SRC of selected PP and mean SRC of the REV, their difference significance can be tested using "t" statistics defined by the following formula:

$$t = R\sqrt{\frac{n-2}{1-R^2}} \tag{20}$$

where R is the difference between particular MSRCs.

This statistic has a distribution of *t–student* with n minus 2 degrees of freedom df, where n equals the size of the sample. The following hypothesis was tested:

$$H_0\text{: MSRC}_{PP} - \text{MSRC}_{REV} = 0$$

versus

$$H_1\text{: MSRC}_{PP} - \text{MSRC}_{REV} > 0$$

In order to conform to the example presented by Saaty & Hu (1998), the data gathered in Table 2 was considered. The simulation framework of that case is df=29,998. Thus, for assumed levels of significance α=0.01, α=0.02 or α=0.03, the critical values of *t–student* statistics equal consecutively $t_{0.01}$ = 2.326472, $t_{0.02}$ = 2.053838, or $t_{0.03}$ = 1.880865.

In the situation when a level of tested *t–student* statistics is higher than its critical value for the assumed level of significance, the hypothesis H_0, must be rejected in favor of alternative hypothesis H_1. In the opposite situation, there are no foundations to reject H_0. The selected statistics and their values for the problem evaluation are presented in Table 7.

Clearly, the results of the simulation scenario, designed in accordance with the framework presented in Saaty & Hu (1998), indicate two PPs which on the basis of SHTT always perform better than the REV, regardless of the preselected PD. It should be emphasized that the performance of selected PPs is examined here from the perspective of rank preservation phenomena which is reflected in our research by the MSRC between genuine and perturbed PV. It should be evident that the above conclusions, unlike any other before, are the effect of sound statistical reasoning (rigorous significance level) based on the seminal approach toward AHP methodology evaluation grounded on precisely planned and performed simulation study.

Table 7 – MSRC values and principal statistics for the performance test of the REV versus other selected PPs

Scenario details			Procedure	MSRC	R	R^2	*t–value*	*α–level**
Geometric Scale	*gamma* distribution	*FR*-PCM	LLSM	0.682300	0.01392	0.00019	2.411167969	0.01
			REV	0.668380	><	><	><	><
			LUA	0.673067	0.00469	0.00002	0.811794069	–
			SRDM	0.671380	0.00300	0.00001	0.519600260	–
			SNCS	0.692453	0.02407	0.00058	4.170635557	0.01
	uniform distribution	*FR*-PCM	LLSM	0.804860	0.01228	0.00015	2.127047876	0.02
			REV	0.792580	><	><	><	><
			LUA	0.795767	0.00319	0.00001	0.551988995	–
			SRDM	0.794820	0.00224	0.00001	0.387967421	–
			SNCS	0.808333	0.01575	0.00025	2.728747286	0.01

Note: (*) the closest significance level providing the ground to reject a tested hypothesis

In order to develop the concept further it was decided to expand the simulation program. The results of this endeavor are presented in Table 3. They should be considered as surprising, especially when one realizes that the PP embedded in the AHP merely takes third place in the overall performance ranking. The ranking takes into account not only

MSRC, but MRE and MRR also, the latter never taken into consideration in previous simulation research. The MRR will now be examined to expand its concept and highlight its novelty.

Let's consider a vector ***k*** of values to be estimated, ***k***=[3, 3, 3, 3], and three of its estimates, $\boldsymbol{k_1}$=[2, 4, 2, 4], $\boldsymbol{k_2}$=[2, 2, 2, 2], $\boldsymbol{k_3}$=[4, 4, 4, 4]. It may be noted that the MREs of all the estimates (given by formula (6)) are the same and equal 1/3. However, MRRs of the estimates (given by formula (7)) are not the same and equal respectively, $\mathrm{MRR}_1(k,k_1)=1$, $\mathrm{MRR}_2(k,k_2)=2/3$, $\mathrm{MRR}_3(k,k_3)=4/3$. Obviously, the goal of estimation is both i.e. to minimize MREs and maintain the MRRs close to unity. This prerequisite is of great importance when one deals with PVs i.e. vectors normalized to unity, as in the case of AHP. Certainly, one can encounter the following three estimation scenarios.

Scenario 1 — Consider a vector ***w*** of genuine PRs trying to estimate ***w***=[0.25, 0.25, 0.25, 0.25], and its estimate $\boldsymbol{w_1}$=[0.01, 0.49, 0.05, 0.45]. This scenario gives a rather high MRE of 0.88, which indicates the mean 88% volatility of estimated PRs in relation to their primary values, and MRR=1.

Scenarios 2–3

Consider a vector ***p*** of genuine PRs trying to estimate ***p***=[0.1, 0.2, 0.3, 0.4], and its two estimates $\boldsymbol{p_1}$=[0.15, 0.3, 0.25, 0.3], and $\boldsymbol{p_2}$=[0.05, 0.1, 0.35, 0.5]. This situation entails a moderate MRE of 0.35425 for both estimates, and two MRRs i.e. $\mathrm{MRR}_1(p,p_1)=1.145$, and $\mathrm{RR}_2(p,p_2)=0.85425$, for the second and third scenario respectively.

Obviously, during the PRs estimation process, it is desirable to avoid situations exemplified by the first and second scenario. Noticeably, they both have something in common. Apart from estimation discrepancies they lead to rank reversal of the initial priorities (emphasis added).

Turning back to Table 3, having in mind the imposed simulation scenario, F-Snedecor PD mean value of a perturbation factor EV(e)=1.03617, we can conclude as follows:

1) the applied measures (MRE, MSRC, MRR) reflecting the quality of PR estimation process within the simulation framework are always better for nonreciprocal PCMs in relation to their reciprocal equivalents;
2) the applied measures of the quality of PR estimation within the simulation framework indicate better estimation results for a relatively higher number of alternatives;
3) both MRE and MRR values indicate that the quality of PR estimation within the simulation framework is better when geometric scale is implemented instead of Saaty's scale for preferences expression of DMs (MRR is then more often less than 1.03617 which indicates less risk of rank reversal);
4) and last but not least, the REV procedure IS NOT a dominating procedure during PR estimation in the simulated framework of the AHP.

The quality of PR estimation in relation to the CM of the PCM

Thus far the alterability of prioritization quality in consequence of the application of preselected PP, preference scale and reciprocal or nonreciprocal PCM in the AHP has been dealt with. This chapter endeavors to focus and conclude the findings concerning the

alterability of prioritization quality in relation to the applied method of the PCM consistency measurement.

Figure 2 demonstrates the basic relation between the distribution of estimation MAEs and values of selected PCM-CMs when LLSM is applied as the PP. The objective was to realize that those measures are not a good indicator of the quality of PR estimation, although the quality of PR estimation should be the core of PCM consistency measurement. Thus, a seminal solution for this problem was introduced i.e. the novel PCM-CM - CM(LTI_2) and depicted its performance in relation to the quality of PR estimation (Fig. 3). As noted, its performance is much better than the PCM-CMs presented earlier (Fig. 2), independently of the MAEs distribution characteristics applied. Below (Tables 8–9), detailed characteristic data is presented for CM(LTI_2) for LLSM and LUA as the PPs, and Saaty's scale as the preferred applied scale.

Table 8 – Performance of the CM(LTI_2) index Statistical characteristics of the MAEs distribution in relation to various VRCM$_i$ for i=1,…,15 of CM(LTI_2) values. The results were generated for n=4 on the basis of **SA|2|** as the simulation algorithm and are based on 10,000 perturbed random reciprocal PCMs. The scenario assumed LLSM as the PP.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LLSM)					Average MAEs among $\boldsymbol{w}$ and $\boldsymbol{w}$*(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0.00, 0.0934)	0.049049	0.0052533	0.0070072	0.0177090	0.0478280	0.0722714	0.0246076
2	[0.0934, 0.12)	0.108368	0.0065427	0.0089647	0.0221881	0.0646356	0.0896474	0.0307832
3	[0.120, 0.147)	0.131977	0.0075604	0.0101952	0.0254009	0.0735568	0.0962640	0.0346824
4	[0.147, 0.173)	0.161289	0.0092812	0.0124076	0.0323969	0.0848569	0.1062240	0.0418424
5	[0.173, 0.200)	0.186567	0.0106050	0.0142825	0.0392350	0.0872419	0.1070400	0.0463689
6	[0.200, 0.227)	0.213651	0.0127312	0.0174795	0.0443425	0.0921171	0.1121160	0.0503101
7	[0.227, 0.253)	0.240868	0.0171655	0.0223103	0.0474780	0.0939184	0.1129280	0.0534051
8	[0.253, 0.280)	0.267645	0.0189530	0.0241065	0.0489027	0.0959089	0.1126270	0.0554222
9	[0.280, 0.307)	0.293803	0.0200809	0.0252443	0.0523480	0.0975035	0.1147230	0.0580895
10	[0.307, 0.333)	0.319702	0.0201740	0.0259357	0.0544712	0.1014610	0.1208420	0.0601639
11	[0.333, 0.360)	0.345876	0.0211796	0.0271488	0.0550490	0.1043660	0.1267140	0.0615576
12	[0.360, 0.387)	0.372744	0.0217402	0.0279791	0.0563253	0.1076280	0.1302670	0.0630527
13	[0.387, 0.413)	0.400500	0.0222736	0.0284786	0.0579657	0.1105020	0.1326590	0.0649738
14	[0.413, 0.440)	0.425325	0.0219914	0.0282637	0.0603546	0.1163910	0.1378310	0.0674297
15	[0.440, ∞)	0.509413	0.0224786	0.0293611	0.0639097	0.1220180	0.1448450	0.0711265

Noted, all statistical characteristics of the MAEs distribution in relation to various VRCM$_i$ for i=1,…,15 of CM(LTI_2) values monotonically grow in both cases. This phenomenon ascertains that the proposed measure of the quality of PR estimation in relation to PCM-CM outperforms other commonly known or recently introduced means of PCM consistency control which were examined during this research. The paramount position of the CM(LTI_2) is additionally strengthened by the fact that its performance improves significantly for higher numbers of alternatives without regard to which PP is employed.

It should be noted that all characteristics presented herein are of great importance in MCDM, because one has to consider the potential of rejecting a “good” PCM, and vice versa i.e. the possibility of acceptance a “bad” PCM, as in the classic SHTT. However, for first time in the course of the AHP development history, the possibility of selecting the level of trustworthiness and basing decisions on statistical facts has been demonstrated. For

instance, considering some hypothetic PCM for n=4, with its CM(LTI_2)≈0.319702 (Tab. 8), one can expect with 95% certainty that MAE should not exceed the value of 0.1208420.

Table 9 – Performance of the CM(LTI_2) index Statistical characteristics of the MAEs distribution in relation to various $VRCM_i$ for i=1,…,15 of CM(LTI_2) values. The results were generated for n=4 on the basis of **SA|2|** as the simulation algorithm and are based on 10,000 perturbed random reciprocal PCMs. The scenario assumed LUA as the PP.

i	$VRCM_i$ for CM(LTI_2)	Mean CM(LTI_2) in $VRCM_i$	p–quantiles of MAEs among ***w*** and ***w****(LUA)					Average MAEs among ***w*** and ***w****(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0.00, 0.0921)	0.0483805	0.0051862	0.0070132	0.0176693	0.0485092	0.0721248	0.0246818
2	[0.0921, 0.119)	0.107336	0.0065804	0.0087362	0.0223436	0.0668610	0.0901757	0.0310714
3	[0.119, 0.145)	0.130827	0.00728515	0.0096983	0.0248230	0.0756282	0.0986153	0.0345387
4	[0.145, 0.172)	0.159831	0.0097014	0.0126492	0.0318836	0.0839675	0.1048160	0.0417584
5	[0.172, 0.199)	0.185763	0.0108996	0.0147705	0.0390121	0.0867685	0.1087370	0.0464742
6	[0.199, 0.226)	0.212789	0.0127518	0.0171452	0.0444749	0.0906489	0.1110220	0.0502253
7	[0.226, 0.252)	0.239711	0.0168641	0.0221950	0.0483727	0.0943307	0.1121290	0.0538373
8	[0.252, 0.279)	0.266664	0.0191223	0.0243966	0.0499312	0.0963741	0.1128810	0.0561933
9	[0.279, 0.306)	0.292923	0.0210745	0.0265733	0.0536876	0.0971178	0.1136750	0.0590709
10	[0.306, 0.332)	0.318738	0.0222280	0.0280330	0.0570706	0.1018000	0.1224680	0.0622836
11	[0.332, 0.359)	0.344798	0.0229873	0.0290093	0.0582741	0.1054570	0.1267530	0.0640174
12	[0.359, 0.386)	0.371865	0.0237677	0.0299580	0.0592460	0.1080910	0.1309080	0.0652984
13	[0.386, 0.412)	0.399489	0.0243569	0.0309199	0.0612529	0.1118710	0.1350460	0.0678424
14	[0.412, 0.439)	0.424271	0.0245079	0.0311770	0.0630208	0.1197740	0.1443030	0.0707793
15	[0.439, ∞)	0.507848	0.0240355	0.0310822	0.0660800	0.1264270	0.1500270	0.0737878

At the same time, one can expect with 95% certainty that it will be higher than 0.0201740. Whether one decide to accept such a PCM or reject it, obviously depends on the quality requirements of PR estimation and the attitude regarding these errors. Indeed, the outcome of the research finally creates the potential for true consistency control in an unprecedented way i.e. directly related to the PR estimation quality.

Consider the following PV as ***w***=[0.345, 0.335, 0.32] of DM preferences for alternatives, A_1, A_2, A_3, respectively. Taking into consideration earlier assumed level of CM(LTI_2)≈0.319702, the order of alternatives ranks i.e. A_1=1, A_2=2, A_3=3, can be very deceptive, and is rather meaningless. Indeed, in such a situation one can expect with 95% certainty that MAE>0.0201740 which makes one aware that the true rank order of examined preferences may look otherwise, due to estimation errors related to DM inconsistency e.g. ***w***=[(0.345–0.04), (0.335+0.01), (0.32+0.03)]=[0.305, 0.345, 0.35], which designates a different order for alternatives ranks, A_1=3, A_2=2, A_3=1.

On the other hand, consider PV as ***w***=[0.6, 0.35, 0.05] of DM preferences for alternatives: A_1, A_2, A_3, consecutively, as previously. Again, assuming CM(LTI_2)≈0.319702, it can be anticipated with 95% certainty that MAE<0.1208420 which insures confidence in the order of alternatives ranks.

In order to conserve the length of the paper, but at the same time enable similar analyses concerning different numbers of alternatives the exemplary generalized (results are averaged for geometric scale and Saaty's scale applied fifty-fifty) characteristics of CM(LTI_2) performance for n>4, and for selected PP in appendices to this article are provided (Tables: A1–A2).

Concluding, this simulation framework a performance of different PCM-CMs in relation to implementation of the most popular PPs, preference scales, and number of alternatives were compared. The research findings can be stated as follows:

1) it is possible to significantly improve the quality of PR estimation when CM(LTI_2) is applied as the PCM-CM;
2) LLSM and LUA as the PP, differ insignificantly from the perspective of CM(LTI_2) performance, the same concerns other examined PP;
3) when the number of alternatives grows, the performance of examined PCM-CMs improves.

Conclusions and further research areas

The objective of the article was to generate answers to the following questions: *Is the REV as the PP necessary and sufficient for the AHP? Is the reciprocity of PCMs a reasonable condition leading to the betterment of the PRs estimation quality? Are PCM-CMs, commonly applied and embedded in the AHP, really conducive to the improvement of the PRs estimation quality?*

The thorough and seminal investigation which significantly upgrades the AHP methodology provides the following answers to these questions:

1) the REV as the PP is not necessary and sufficient for the AHP. Moreover, the research reveals two PP which outperform the REV;
2) the reciprocity of PCM in the AHP is the artificial condition and directly leads to deterioration of the PR estimation quality.
3) the commonly applied PCM-CMs embedded in the AHP, mislead and in consequence often directly lead to deterioration of the PR estimation quality.

Proposed: resign from known PCM-CMs embedded in the AHP in favor of CM(LTI_2) that can operate both types of PCM i.e. reciprocal and nonreciprocal, withhold the PCM reciprocity requirement from the AHP and consider the replacement of the REV as the PP within the AHP in favor of LUA or LLSM.

Certainly, there is a need for further research in the field. Firstly, one should examine the performance of CM(LTI_2) when nonreciprocal PCM are applied. Secondly, one may study its performance from the perspective of relative estimation errors, and last but not least, one could evaluate its performance from the perspective of the entire hierarchy as opposed to a single PCM.

To recapitulate; in conjunction with other contemporary and seminal research papers e.g. Grzybowski (2016); Kazibudzki (2016a, 2016b); García-Melón et al. (2016); Chen et al. (2015); Pereira & Costa (2015); Linares et al. (2014); Moreno-Jiménez et al. (2014); Aguarón, Escobar & Moreno-Jiménez (2014); Lin, Kou & Ergu (2013); Brunelli, Canal & Fedrizzi (2013), the results of this scientific research enriches the state of knowledge about the true value of the AHP which is widely recognized as an applicable MCDM support system. Hopefully, the results of this freshly finished authentic examination will improve the quality of human's prospective choices.

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Appendices

Table A1 – Performance of CM(LTI_2) index under the action of LLSM as the PP. Statistical characteristics of the MAEs distribution in relation to various levels of CM(LTI_2) within a given VRCM$_i$ for i=1,…,15. The results are based on 10,000 perturbed random reciprocal PCMs (geometric and Saaty's scales applied fifty-fifty), and were generated on the basis of **SA|2|** as the simulation algorithm. The table contains results for $n \in \{5, 6, 7, 8, 9\}$, presented consecutively.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0899)	0.057912	0.0039186	0.0049954	0.0109799	0.0221753	0.0274887	0.0127898
2	[0.0899, 0.107)	0.099124	0.0056158	0.0073876	0.0157136	0.0324243	0.0398139	0.0183201
3	[0.107, 0.124)	0.116088	0.0063140	0.0079525	0.0184299	0.0389673	0.0490687	0.0214159
4	[0.124, 0.142)	0.133907	0.0075132	0.0102233	0.0230429	0.0443459	0.0539668	0.0258898
5	[0.142, 0.159)	0.151127	0.0099921	0.0129535	0.0261046	0.0486044	0.0581258	0.0290851
6	[0.159, 0.176)	0.167911	0.0113191	0.0142543	0.0289546	0.0558904	0.0682777	0.0328936
7	[0.176, 0.193)	0.184671	0.0125612	0.0158052	0.0320054	0.0594491	0.0730399	0.0357402
8	[0.193, 0.211)	0.201896	0.0136853	0.0171375	0.0339101	0.0640703	0.0789391	0.0380755
9	[0.211, 0.228)	0.219329	0.0142803	0.0178080	0.0361548	0.0711273	0.0839402	0.0408705
10	[0.228, 0.245)	0.236371	0.0150518	0.0185369	0.0380656	0.0762136	0.0919801	0.0435024
11	[0.245, 0.262)	0.253302	0.0161087	0.0208189	0.0405464	0.0789105	0.0929572	0.0462684
12	[0.262, 0.280)	0.270523	0.0160427	0.0205586	0.0431223	0.0821647	0.0965329	0.0482168
13	[0.280, 0.297)	0.288211	0.0165698	0.0209757	0.0457022	0.0865715	0.100490	0.0504072
14	[0.297, 0.314)	0.305099	0.0177870	0.0226112	0.0455671	0.0859316	0.100544	0.0507868
15	[0.314, ∞)	0.357080	0.0186614	0.0241816	0.0493007	0.0932224	0.107664	0.0547348
I	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0901)	0.0618775	0.0036042	0.0044511	0.0090509	0.0175099	0.0212066	0.0102634
2	[0.0901, 0.102)	0.096694	0.00584585	0.0072472	0.0147877	0.0255157	0.0297767	0.0158427
3	[0.102, 0.115)	0.109186	0.0071783	0.0088408	0.0167774	0.0304119	0.0360603	0.0186253
4	[0.115, 0.127)	0.121228	0.00831565	0.0102091	0.0192100	0.0349865	0.0420209	0.0214601
5	[0.127, 0.139)	0.133028	0.0088771	0.0109435	0.0208206	0.0393504	0.0481357	0.0236802
6	[0.139, 0.151)	0.144977	0.0097898	0.0118208	0.0225534	0.0439163	0.0538868	0.0259512
7	[0.151, 0.163)	0.156874	0.0101678	0.0126009	0.0248914	0.0500528	0.0613696	0.0288113
8	[0.163, 0.176)	0.169306	0.0113233	0.0138144	0.0274455	0.0552421	0.0656847	0.0317599
9	[0.176, 0.188)	0.181783	0.0120341	0.0147276	0.0297646	0.0587297	0.0700824	0.0339487
10	[0.188, 0.200)	0.193745	0.0124796	0.0157621	0.0317564	0.0613300	0.0720410	0.0356610
11	[0.200, 0.212)	0.205758	0.0137977	0.0167981	0.0329443	0.0622977	0.0721443	0.0368687
12	[0.212, 0.225)	0.218204	0.0140878	0.0175574	0.0347152	0.0652521	0.0774105	0.0386492
13	[0.225, 0.237)	0.230723	0.0140705	0.0177333	0.0369638	0.0672822	0.0764555	0.0402684

14	[0.237, 0.249)	0.242818	0.0146810	0.0186397	0.0381558	0.0692375	0.0786225	0.0413928
15	[0.249, ∞)	0.279499	0.0168309	0.0207854	0.0401272	0.0721349	0.0829652	0.0439267
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.07975)	0.061626	0.00329141	0.0040063	0.0079292	0.0153184	0.017980	0.0089902
2	[0.07975, 0.09)	0.085354	0.00558449	0.0066781	0.0124836	0.0217916	0.0254877	0.0136394
3	[0.09, 0.10)	0.095128	0.00622084	0.0074651	0.0136346	0.0241580	0.0288301	0.0151410
4	[0.10, 0.11)	0.105046	0.00677146	0.0081844	0.0150571	0.0277432	0.0343089	0.0170348
5	[0.11, 0.12)	0.114884	0.00728075	0.0089529	0.0164708	0.0329745	0.0408782	0.0192642
6	[0.12, 0.13)	0.124902	0.00792417	0.0097471	0.0185168	0.0378364	0.0464765	0.0217170
7	[0.13, 0.14)	0.134949	0.00851189	0.0104389	0.0202075	0.0415434	0.0507830	0.0236614
8	[0.14, 0.15)	0.144883	0.00952136	0.0115606	0.0224145	0.0446314	0.0531641	0.0257116
9	[0.15, 0.161)	0.155416	0.0101888	0.0121602	0.0241178	0.0465694	0.0553538	0.0275101
10	[0.161, 0.171)	0.165845	0.0110535	0.0132394	0.0261677	0.0499157	0.0583309	0.0293786
11	[0.171, 0.181)	0.175874	0.0116123	0.0139639	0.0273006	0.0515428	0.0596329	0.0304575
12	[0.181, 0.191)	0.185981	0.0121824	0.0150547	0.0293308	0.0532065	0.0613544	0.0320030
13	[0.191, 0.201)	0.195819	0.0122294	0.0152015	0.0299135	0.0553010	0.0642011	0.0330142
14	[0.201, 0.211)	0.205937	0.0132402	0.0164008	0.0321805	0.0552598	0.0636846	0.0343310
15	[0.211, ∞)	0.235348	0.0147413	0.0179580	0.0321805	0.0586515	0.0682411	0.0363445
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.06861)	0.056493	0.0029930	0.0036359	0.0071723	0.0129253	0.0151100	0.0079193
2	[0.06861, 0.078)	0.073616	0.0047668	0.0056647	0.0098201	0.0165545	0.0197820	0.0107659
3	[0.078, 0.087)	0.082558	0.0051148	0.00615425	0.0108293	0.0189720	0.0232764	0.0121106
4	[0.087, 0.095)	0.090957	0.0054815	0.0067644	0.0120701	0.0230822	0.0289636	0.0139455
5	[0.095, 0.104)	0.0995085	0.0062208	0.0074045	0.0134360	0.0267488	0.0338094	0.0157422
6	[0.104, 0.113)	0.108507	0.0065308	0.0079148	0.0148310	0.0307300	0.0379708	0.0175495
7	[0.113, 0.122)	0.117503	0.0073636	0.0087983	0.0166204	0.0342287	0.0402093	0.0192815
8	[0.122, 0.131)	0.126447	0.0077367	0.00920785	0.0182781	0.0366835	0.0432579	0.0209778
9	[0.131, 0.140)	0.135467	0.0081883	0.0099817	0.0200944	0.0391024	0.0463982	0.0227669
10	[0.140, 0.149)	0.144406	0.00893715	0.0109052	0.0215995	0.0404294	0.0465999	0.0240918
11	[0.149, 0.158)	0.153395	0.0096365	0.0118788	0.0228543	0.0420224	0.0488816	0.0252208
12	[0.158, 0.167)	0.162379	0.0105213	0.0128739	0.0250637	0.0441591	0.0509963	0.0270496
13	[0.167, 0.176)	0.171319	0.0109917	0.0133182	0.0253654	0.0446033	0.0525163	0.0275815
14	[0.176, 0.185)	0.180246	0.0120041	0.0144395	0.0266159	0.0464516	0.0529339	0.0289197
15	[0.185, ∞)	0.205854	0.0127740	0.0155662	0.0283804	0.0479564	0.0549310	0.0304352
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LLSM)					Average MAEs among **w** and **w***(LLSM)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.059795)	0.051197	0.0026372	0.0031677	0.0061278	0.0107141	0.0122502	0.0066588
2	[0.05979, 0.068)	0.064092	0.0040166	0.0047872	0.0079722	0.0133870	0.0158901	0.0087678
3	[0.068, 0.076)	0.072019	0.0044127	0.0052033	0.0089180	0.0154595	0.0189767	0.0099818
4	[0.076, 0.085)	0.080495	0.0046625	0.0055923	0.0098746	0.0183837	0.0234965	0.0114040
5	[0.085, 0.093)	0.089047	0.0052813	0.0062378	0.0110158	0.0221043	0.0279174	0.0129826
6	[0.093, 0.101)	0.097017	0.0056575	0.0067669	0.0124636	0.0261396	0.0326188	0.0147615
7	[0.101, 0.109)	0.105051	0.0061505	0.0074036	0.0138920	0.0290254	0.0358010	0.0164984
8	[0.109, 0.118)	0.113488	0.0066692	0.0079686	0.0153474	0.0308319	0.0365922	0.0177312
9	[0.118, 0.126)	0.122009	0.0073133	0.0087907	0.0171076	0.0330852	0.0388189	0.0193438
10	[0.126, 0.134)	0.129857	0.0076181	0.0092912	0.0186416	0.0343317	0.0401595	0.0204982
11	[0.134, 0.142)	0.137970	0.0083801	0.0102174	0.0199779	0.0355818	0.0416818	0.0216939
12	[0.142, 0.151)	0.146298	0.0091112	0.0107807	0.0212040	0.0376109	0.0430078	0.0229256
13	[0.151, 0.159)	0.154883	0.0097330	0.0118942	0.0219635	0.0378245	0.0435528	0.0237785
14	[0.159, 0.167)	0.162793	0.0102563	0.0125995	0.0228089	0.0390630	0.0443591	0.0244409
15	[0.167, ∞)	0.184864	0.0115601	0.0138072	0.0242891	0.0403012	0.046879	0.0259996

Table A2 – Performance of CM(LTI_2) index under the action of LUA as the PP. Statistical characteristics of the MAEs distribution in relation to various levels of CM(LTI_2) within a given VRCM$_i$ for i=1,…,15. The results are based on 10,000 perturbed random reciprocal PCMs (geometric and Saaty's scales applied fifty-fifty), and were generated on the basis of **SA|2|** as the simulation algorithm. The table contains results for $n\in\{5, 6, 7, 8, 9\}$, presented consecutively.

i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.08867)	0.057344	0.0040500	0.0051112	0.0109956	0.0223145	0.0281738	0.0129222
2	[0.08867, 0.106)	0.097631	0.0051370	0.0066813	0.0149743	0.0314682	0.0402326	0.0179091
3	[0.106, 0.123)	0.115154	0.0062273	0.0080033	0.0178583	0.0404214	0.0493824	0.0215433
4	[0.123, 0.140)	0.132429	0.0077722	0.0103867	0.0235191	0.0443250	0.0533440	0.0260091
5	[0.140, 0.158)	0.149841	0.0100660	0.0130848	0.0264063	0.0492151	0.0598150	0.0293785
6	[0.158, 0.175)	0.166943	0.0122130	0.0152940	0.0305507	0.0567287	0.0669818	0.0339792
7	[0.175, 0.192)	0.183544	0.0134146	0.0168104	0.0341529	0.0622582	0.0730947	0.0376190
8	[0.192, 0.209)	0.200556	0.0144681	0.0180775	0.0371079	0.0664060	0.0801886	0.0405209
9	[0.209, 0.227)	0.217798	0.0152484	0.0195489	0.0387389	0.0726297	0.0886177	0.0432569
10	[0.227, 0.244)	0.235136	0.0161576	0.0201625	0.0403835	0.0771441	0.0945720	0.0454089
11	[0.244, 0.261)	0.252143	0.0164634	0.0205743	0.0428687	0.0812496	0.0997771	0.0479053
12	[0.261, 0.278)	0.269128	0.0174125	0.0217309	0.0445472	0.0844031	0.1015070	0.0498806
13	[0.278, 0.296)	0.286560	0.0184856	0.0235664	0.0474587	0.0907092	0.1046180	0.0527022
14	[0.296, 0.313)	0.304366	0.0176996	0.0228077	0.0479535	0.0900992	0.1047390	0.0532388
15	[0.313, ∞)	0.354236	0.0192203	0.0244908	0.0503011	0.0929620	0.1098880	0.0556579
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.09033)	0.063185	0.0035880	0.0044267	0.0089867	0.0179120	0.0222610	0.0103740
2	[0.09033, 0.103)	0.097359	0.0063423	0.0078999	0.0155573	0.0273932	0.0319008	0.0168600
3	[0.103, 0.115)	0.109410	0.00722965	0.0091969	0.0177403	0.0325171	0.0386338	0.0196173
4	[0.115, 0.127)	0.121343	0.0084960	0.0107027	0.0210117	0.0381138	0.0455231	0.0233632
5	[0.127, 0.139)	0.133108	0.0094898	0.0116495	0.0229152	0.0420076	0.0524375	0.0257261
6	[0.139, 0.152)	0.145538	0.0108994	0.0132421	0.0253036	0.0481306	0.0607595	0.0287714
7	[0.152, 0.164)	0.157728	0.0114276	0.0139558	0.0271455	0.0539605	0.0656762	0.0310679
8	[0.164, 0.176)	0.169681	0.0121504	0.0150640	0.0292961	0.0575169	0.0691811	0.0336092
9	[0.176, 0.189)	0.182266	0.0128272	0.0159433	0.0313538	0.0612481	0.0726740	0.0356428
10	[0.189, 0.201)	0.194801	0.0138039	0.0173398	0.0328835	0.0629099	0.0736509	0.0370798
11	[0.201, 0.213)	0.206815	0.0140270	0.0173321	0.0347352	0.0651391	0.0772600	0.0387081
12	[0.213, 0.225)	0.218752	0.0145777	0.0184396	0.0366233	0.0669927	0.0785993	0.0400494
13	[0.225, 0.238)	0.231199	0.0152711	0.0185568	0.0383407	0.0700653	0.0825784	0.0420203
14	[0.238, 0.250)	0.243637	0.0158614	0.0199098	0.0378087	0.0715807	0.0841889	0.0422926
15	[0.250, ∞)	0.280974	0.0171267	0.0210891	0.0412814	0.0738420	0.0859174	0.0450109
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.07955)	0.061285	0.0032254	0.0039749	0.0080564	0.0157394	0.0187853	0.0091669
2	[0.07955, 0.09)	0.085237	0.0055756	0.00703635	0.0133354	0.0239340	0.0276372	0.0145973
3	[0.09, 0.10)	0.095217	0.0067076	0.00822896	0.0151476	0.0267136	0.0313365	0.0168334
4	[0.10, 0.11)	0.105055	0.0071842	0.0087319	0.0167347	0.0300692	0.0372910	0.0187971
5	[0.110, 0.120)	0.114948	0.0078851	0.0096590	0.0184274	0.0350956	0.0451092	0.0211768
6	[0.120, 0.130)	0.124976	0.00841585	0.0104132	0.0202074	0.0399982	0.0487099	0.0233008
7	[0.130, 0.140)	0.134961	0.0094352	0.0114042	0.0217320	0.0442261	0.0533524	0.0258727
8	[0.140, 0.150)	0.144948	0.0097585	0.0120015	0.0234560	0.0471241	0.0565387	0.0269760
9	[0.150, 0.160)	0.154896	0.0105670	0.0127710	0.0253768	0.0489510	0.0573372	0.0286380
10	[0.160, 0.171)	0.165237	0.0113250	0.0138414	0.0270817	0.0499840	0.0598609	0.0301303
11	[0.171, 0.181)	0.175843	0.0120019	0.0146386	0.0287811	0.0534214	0.0617005	0.0317433
12	[0.181, 0.191)	0.185759	0.0126595	0.0153651	0.0298787	0.0548531	0.0642158	0.0329712
13	[0.191, 0.201)	0.195725	0.0128283	0.0154763	0.0313703	0.0562006	0.0639533	0.0339822
14	[0.201, 0.211)	0.205744	0.0139552	0.0170698	0.0326421	0.0578949	0.0674790	0.0354460
15	[0.211, ∞)	0.235674	0.0149229	0.0180402	0.0341664	0.0601557	0.0694405	0.0371047
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.0688)	0.056836	0.0030787	0.0036205	0.0073926	0.0145693	0.0176789	0.0084662
2	[0.0688, 0.078)	0.073666	0.0051105	0.0061309	0.0109223	0.0189602	0.0222010	0.0121584
3	[0.078, 0.087)	0.082561	0.0055045	0.0066116	0.0121236	0.0213335	0.0264824	0.0136680

4	[0.087, 0.096)	0.091556	0.0059645	0.0072229	0.0133996	0.0253087	0.0332026	0.0154277
5	[0.096, 0.105)	0.100528	0.0063149	0.0077633	0.0145624	0.0296739	0.0375550	0.0170392
6	[0.105, 0.114)	0.109496	0.0069818	0.0085509	0.0161950	0.0334936	0.0413081	0.0189646
7	[0.114, 0.123)	0.118520	0.0075613	0.0090468	0.0175431	0.0357544	0.0434245	0.0204242
8	[0.123, 0.132)	0.127416	0.0079922	0.0097597	0.0192472	0.0395417	0.0473757	0.0223206
9	[0.132, 0.141)	0.136453	0.0086852	0.0105264	0.0209858	0.0407715	0.0482480	0.0238363
10	[0.141, 0.150)	0.145395	0.0097858	0.0118143	0.0228944	0.0419213	0.0491251	0.0254135
11	[0.150, 0.159)	0.154468	0.0100095	0.0125284	0.0241689	0.0438311	0.0507983	0.0265045
12	[0.159, 0.168)	0.163355	0.0106847	0.0132036	0.0259776	0.0454986	0.0526910	0.0279660
13	[0.168, 0.177)	0.172363	0.0111935	0.0138710	0.0271393	0.0471614	0.0543407	0.0291270
14	[0.177, 0.186)	0.181389	0.0124251	0.0149686	0.0273290	0.0474666	0.0540064	0.0294350
15	[0.186, ∞)	0.206400	0.0133439	0.0162347	0.0294684	0.0496426	0.0567018	0.0315851
i	VRCM$_i$ for CM(LTI_2)	Mean CM(LTI_2) in VRCM$_i$	p–quantiles of MAEs among **w** and **w***(LUA)					Average MAEs among **w** and **w***(LUA)
			p=0.05	p=0.1	p=0.5	p=0.9	p=0.95	
1	[0, 0.05999)	0.051571	0.0027899	0.0033529	0.0066145	0.0129846	0.0152304	0.00757575
2	[0.05999, 0.068)	0.064108	0.00425595	0.0049956	0.0089923	0.0154707	0.0180262	0.0099654
3	[0.068, 0.076)	0.0720425	0.0045459	0.0053806	0.0096431	0.0170663	0.0215704	0.0110540
4	[0.076, 0.085)	0.080557	0.00484755	0.0059128	0.0107496	0.0200923	0.0264333	0.0124749
5	[0.085, 0.093)	0.089092	0.0053555	0.0064602	0.0118465	0.0241501	0.0307256	0.0140711
6	[0.093, 0.101)	0.096974	0.0059146	0.0070858	0.0130955	0.0270450	0.0340523	0.0154649
7	[0.101, 0.109)	0.105011	0.0064092	0.0077271	0.0143894	0.0307524	0.0372307	0.0172088
8	[0.109, 0.118)	0.113412	0.0070543	0.0083974	0.0158975	0.0323549	0.0386885	0.0184995
9	[0.118, 0.126)	0.121966	0.0076984	0.0092627	0.0180989	0.0344896	0.0405523	0.0204213
10	[0.126, 0.134)	0.129881	0.0081201	0.0097966	0.0194020	0.0364334	0.0431290	0.0216106
11	[0.134, 0.142)	0.137905	0.0088861	0.0106243	0.0206471	0.0371653	0.0437476	0.0225820
12	[0.142, 0.151)	0.146345	0.0094580	0.0114928	0.0216677	0.0380785	0.0443759	0.0235540
13	[0.151, 0.159)	0.154771	0.0101348	0.0124254	0.0231652	0.0396143	0.0456328	0.0248599
14	[0.159, 0.167)	0.162795	0.0108284	0.0129963	0.0234617	0.0396838	0.0458155	0.0250440
15	[0.167, ∞)	0.184832	0.0122591	0.0145491	0.0255388	0.0426019	0.0490175	0.0273476