Inconsistency of incomplete pairwise comparisons matrices

Dawid Talaga

AGH University of Science and Technology, Kraków, Poland

Abstract

TODO: Abstract, should normally be not longer than 200 words.

Keywords: pairwise comparisons, inconsistency, incomplete matrices, AHP

1. Introduction

TODO: Opis metody PC

2. Inconsistency problem

TODO: Opis o co chodzi z niespójnością

$$a_{ik} = a_{ij}a_{jk} \quad \forall_{i,j,k} \tag{1}$$

$$a_{ij} = \frac{w_i}{w_j} \tag{2}$$

triad - definicja

Może być wyliczane poprzez wartości własne lub średnie geometryczne

3. Inconsistency indices

In the long history of PC method, a lot of methods have been developed to calculate inconsistencies. Many of them are based directly on the definition of consistency (1), some methods use the eigenvalues of the matrix, others are based on the assumption that each fully consistent matrix fulfills the condition (2).

This section presents twelve common inconsistency indices, which can be relatively easily modified in such a way that they work for incomplete matrices. Many of these coefficients have been described and tested numerically in [1].

saaty idx, potem geometric index,

Email address: `talagadawid@gmail.com` (Dawid Talaga)

indices -> indexes

rozdział 3 - umotywować dokłądny przegląd indeksów

intrduction: 1. narracyjna historia o rozwoju indeksów (ramond lull, kondorse, teoria społecznego wybory - kenetal, amartay sen) - kilka zdań o historii + kilka zdań o tym po co ten artykuł jest. ma być tekstowe - bez wzorów

sekcja 2: Premilinaries: opis PC (alternatywy, porównania) i niespójności -

sekcja 3: inconsistency indexes: jeżeli mamy dużo niespójności, to dobrze jest ją zmierzyć - bo mogą być niewiarygodne - było wiele pomysłów. saaty, geometric index, reszta

sekcja 4: które da się dostosował łatwo i metoda

sekcja 5 - moja lub discussuin: discussion: analiza wyników

conclision: udało się pokazać, że index ... jest ok

3.1. Koczkodaj index

One of the most popular inconsistency indices was proposed by Koczkodaj [2]. It is based directly on the definition of consistency (1). The value of the inconsistency index for one triad (2) was defined as:

$$K_{i,j,k} = min\{\frac{1}{a_{ij}} \mid a_{ij} - \frac{a_{ik}}{a_{jk}} \mid, \frac{1}{a_{ij}} \mid a_{ik} - a_{ij}a_{jk} \mid, \frac{1}{a_{jk}} \mid a_{jk} - \frac{a_{ik}}{a_{ij}} \mid\}. \quad (3)$$

This formula has been simplified by Duszak and Koczkodaj [3] and is given as:

$$K(\alpha, \beta, \gamma) = min\{\mid 1 - \frac{\beta}{\alpha\gamma} \mid, \mid 1 - \frac{\alpha\gamma}{\beta} \mid\}, \quad gdzie\, \alpha = a_{ij}, \beta = a_{ik}, \gamma = a_{jk} \quad (4)$$

Then it was genaralized [3] for $n > 2$. Finally, the inconsistency index has the following form:

$$K = max\{K(\alpha, \beta, \gamma) | 1 \leq i < j < k \leq n\} \quad (5)$$

It is worth noting that not only does the coefficient find the greatest inconsistency but also indicates the place in which it occurs.

3.2. Grzybowski index

Based on the Koczkodaj index, Grzybowski introduced a new one [4]. He resigned from looking for the most inconsistent triad, in favor of averaging the values of all inconsistencies. The new inconsistency index has been defined as $G = Mean(K(\alpha, \beta, \gamma)$, where $Mean$ means the arithmetic average of all triads. Thus, the index could be formulated as follows:

$$G = \frac{1}{n} \sum_{i=1}^{n} K_i(\alpha, \beta, \gamma). \quad (6)$$

3.3. Kazibudzki indices

Based on the Koczkodaj inconsistency index and observation that $ln(\frac{\alpha\gamma}{\beta}) = -ln(\frac{\beta}{\alpha\gamma})$, Kazibudzki proposed several additional inconsistency indices [5]. Instead of the formula for inconsistency of the triad (4), he introduced two new formulas:

$$LTI(\alpha, \beta\gamma) = \mid ln(\frac{\alpha\gamma}{\beta}) \mid, \tag{7}$$

$$LTI*(\alpha, \beta\gamma) = ln^2(\frac{\alpha\gamma}{\beta}). \tag{8}$$

Based on the above equations, Kazibudzki proposed new indices. The simplest ones use the geometric mean of the triads. Thus, new indices could be written in the form:

$$MLTI(LTI) = \frac{1}{n}\sum_{i=1}^{n}\left[LTI_i(\alpha, \beta\gamma)\right], \tag{9}$$

$$MLTI(LTI*) = \frac{1}{n}\sum_{i=1}^{n}\left[LTI*_i(\alpha, \beta\gamma)\right]. \tag{10}$$

After further research [6], Kazibudzki introduces another inconsistency index, again based on (8). It was defined as $CM(LTI*) = \frac{MEAN[LTI*(\alpha,\beta,\gamma)]}{1+MAX[LTI*(\alpha,\beta,\gamma)]}$. Hence,

$$CM(LTI*) = \frac{\frac{1}{n}\sum_{i=1}^{n}[LTI*_i(\alpha, \beta, \gamma)]}{1 + max\{LTI*_i(\alpha, \beta, \gamma)\}}. \tag{11}$$

3.4. Index of determinants

This index was proposed by Pelaez and Lamata [7] and is also based on the concept of triad. The authors noticed that PCM matrices can be construct on the basis of triads. Their determinant is closely related to the consistency of the matrix.

For every triad (a_{ik}, a_{ij}, a_{jk}) one can build a matrix in the form:

$$T_{ijk} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ \frac{1}{a_{ij}} & 1 & a_{jk} \\ \frac{1}{a_{ik}} & \frac{1}{a_{jk}} & 1 \end{pmatrix}, \quad gdzie\, i < j < k. \tag{12}$$

The determinant of this matrix is:

$$det(A) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2. \tag{13}$$

If the matrix is fully consistent, then $det(A) = 0$, else $det(A) > 0$. Based on the above considerations, the authors introduced the new inconsistency index that can be formulated as follows:

$$CI* = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2\right). \tag{14}$$

3.5. Kułakowski and Szybowski indices

Kułakowski and Szybowski proposed two further inconsictency indices [8], which are also based on triads. They use the fact that the number of triads that can be found in a PCM matrix is $\binom{n}{3} = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{6}$. The index is formulated as follows:

$$I_1 = \frac{6\sum_{t\in T} K(t)}{n(n-1)(n-2)}, \tag{15}$$

where $K(t)$ is the Koczkodaj index for triad $t = (\alpha, \beta, \gamma)$ of the set of all triads T.

The second inconsistency index is similar:

$$I_2 = \frac{6\sqrt{\sum_{t\in T} K^2(t)}}{n(n-1)(n-2)}. \tag{16}$$

Indices can be combined with each other to create new coefficients. In this way Kułakowski and Szybowski proposed two new indices. The first one is based on (5) and (15). This index allows to choose what effect on the result should the greatest inconsistency found have and what the average inconsistency of all triads. The new inconsistency index looks as follows:

$$I_\alpha = \alpha K + (1-\alpha) I_1, \tag{17}$$

where $0 \leq \alpha \leq 1$.

The second index expands the first one by (16):

$$I_{\alpha,\beta} = \alpha K + \beta I_1 + (1-\alpha-\beta) I_2. \tag{18}$$

3.6. Geometric consistency index

One of the indices which are based on the assumption (2) was proposed by Craford and Williams [9] and then refined by Aguaròn and Moreno-Jimènez [10]. In this case the priority vector should be calculated using the geometric mean method. Consider (2) one can create a matrix:

$$E = \left[e_{ij} \mid e_{ij} = a_{ij}\frac{w_j}{w_i} \right], \quad i, j = 1, ..., n. \tag{19}$$

The inconsistency index is calculated as follows:

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} ln^2 e_{ij}. \tag{20}$$

3.7. Harmonic consistency index

Index introduced by Stein and Mizzi and it presents a completely new method of inconsistency counting [11]. At the beginning it requires the creation of an auxiliary vector $s = (s_1, ..., s_n)^T$, where n is the dimension of the matrix

A, for which the index will be calculated. Each element of the vector s is the sum of values in one column of the matrix A. Hence,

$$s_j = \sum_{i=1}^{n} a_{ji} \quad \forall j. \tag{21}$$

The authors proved that if the matrix A is consistent, then $\sum_{j=1}^{n} s_j^{-1} = 1$. The formula for the mean harmonic looks as follows [12]:

$$HM = \frac{n}{\sum_{j=1}^{n} \frac{1}{s_j}}. \tag{22}$$

The final formula for inconsistency index was obtained by normalizing the above equation (22):

$$HCI = \frac{(HM(s) - n)(n+1)}{n(n-1)}. \tag{23}$$

3.8. Golden and Wang index

This index was introduced by Golden and Wang [13]. It assumes that the priority vector was calculated using the geometric mean method, then normalized to add up to 1. In this way vector $g* = [g_{1,}^{*}, ..., g_n^*]$ was obtained, where n is the dimension of the matrix A. The next step is to normalize each column of the matrix A. After this, the sum of the elements of each column in matrix A is 1. The obtained matrix is marked with the symbol A^*. The inconsistency index is defined as follows:

$$GW = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mid a_{ij}^* - g_i^* \mid . \tag{24}$$

3.9. Salo and Hamalainen index

The index proposed by Salo and Hamalainen [14, 5] uses the definition of inconsistency (1), however it requires the creation of an auxiliary matrix, in which each element is the smallest and largest discrepancy from consistency based on formula (1). The index takes all triads into account:

$$R = (r_{ij})_{nxn} = \begin{pmatrix} [\underline{r}_{11}, \overline{r}_{11}] & \dots & [\underline{r}_{1n}, \overline{r}_{1n}] \\ \vdots & \ddots & \vdots \\ [\underline{r}_{n1}, \overline{r}_{n1}] & \dots & [\underline{r}_{nn}, \overline{r}_{nn}] \end{pmatrix}, \tag{25}$$

where $\underline{r_{ij}} = min\{a_{ik}a_{kj} \mid k = 1, \dots, n\}$, $\overline{r_{ij}} = max\{a_{ik}a_{kj} \mid k = 1, \dots, n\}$ and n is the dimension of the tested matrix A. A numerical example was presented in [12]. Based on the resulting matrix R, the authors proposed the following inconsistency index:

$$CM = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\overline{r}_{ij} - \underline{r}_{ij}}{(1 + \overline{r}_{ij})\left(1 + \underline{r}_{ij}\right)}. \tag{26}$$

4. Inconsistency of incomplete pairwise comparisons matrices

5. Verification of indices for incomplete matrices

6. Conclusion

References

[1] M. Brunelli, L. Canal, M. Fedrizzi, Inconsistency indices for pairwise comparison matrices: a numerical study, Annals of Operations Research 211 (1) (2013) 493–509. doi:10.1007/s10479-013-1329-0. URL https://doi.org/10.1007/s10479-013-1329-0

[2] W. Koczkodaj, A new definition of consistency of pairwise comparisons, Mathematical and Computer Modelling 18 (7) (1993) 79 – 84. doi:https://doi.org/10.1016/0895-7177(93)90059-8. URL http://www.sciencedirect.com/science/article/pii/0895717793900598

[3] Z. Duszak, W. W. Koczkodaj, Generalization of a new definition of consistency for pairwise comparisons, Information Processing Letters 52 (5) (1994) 273 – 276. doi:https://doi.org/10.1016/0020-0190(94)00155-3. URL http://www.sciencedirect.com/science/article/pii/0020019094001553

[4] A. Z. Grzybowski, New results on inconsistency indices and their relationship with the quality of priority vector estimation, Expert Systems with Applications 43 (2016) 197 – 212. doi:https://doi.org/10.1016/j.eswa.2015.08.049. URL http://www.sciencedirect.com/science/article/pii/S0957417415006065

[5] P. Kazibudzki, Redefinition of triad's inconsistency and its impact on the consistency measurement of pairwise comparison matrix 2016 (2016) 71–78.

[6] P. T. Kazibudzki, The quality of priority ratios estimation in relation to a selected prioritization procedure and consistency measure for a pairwise comparison matrix, CoRR abs/1704.01944. arXiv:1704.01944. URL http://arxiv.org/abs/1704.01944

[7] J. PelÃ¡ez, M. Lamata, A new measure of consistency for positive reciprocal matrices, Computers & Mathematics with Applications 46 (12) (2003) 1839 – 1845. doi:https://doi.org/10.1016/S0898-1221(03)90240-9. URL http://www.sciencedirect.com/science/article/pii/S0898122103902409

[8] K. Kulakowski, J. Szybowski, The new triad based inconsistency indices for pairwise comparisons, Procedia Computer Science 35 (2014) 1132 – 1137, knowledge-Based and Intelligent Information & Engineering Systems 18th Annual Conference, KES-2014 Gdynia, Poland, September 2014 Proceedings. doi:https://doi.org/10.1016/j.procs.2014.08.205. URL http://www.sciencedirect.com/science/article/pii/S1877050914011703

[9] G. Crawford, C. Williams, A note on the analysis of subjective judgment matrices, Journal of Mathematical Psychology 29 (4) (1985) 387 – 405. doi:https://doi.org/10.1016/0022-2496(85)90002-1.
URL http://www.sciencedirect.com/science/article/pii/0022249685900021

[10] J. Aguaron, J. M. Moreno-Jimenez, The geometric consistency index: Approximated thresholds, European Journal of Operational Research 147 (1) (2003) 137 – 145. doi:https://doi.org/10.1016/S0377-2217(02)00255-2.
URL http://www.sciencedirect.com/science/article/pii/S0377221702002552

[11] W. E. Stein, P. J. Mizzi, The harmonic consistency index for the analytic hierarchy process, European Journal of Operational Research 177 (1) (2007) 488 – 497. doi:https://doi.org/10.1016/j.ejor.2005.10.057.
URL http://www.sciencedirect.com/science/article/pii/S0377221705009288

[12] M. Brunelli, Introduction to the Analytic Hierarchy Process, 2015.

[13] B. L. Golden, Q. Wang, An Alternate Measure of Consistency, Springer Berlin Heidelberg, Berlin, Heidelberg, 1989, pp. 68–81.

[14] A. A. Salo, R. P. Hamalainen, Preference programming through approximate ratio comparisons, European Journal of Operational Research 82 (3) (1995) 458 – 475. doi:https://doi.org/10.1016/0377-2217(93)E0224-L.
URL http://www.sciencedirect.com/science/article/pii/0377221793E0224L