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ALTERNATIVE MODES OF QUESTIONING IN THE ANALYTIC HIERARCHY PROCESS

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Abstract—The standard mode of questioning in the Analytic Hierarchy Process (AHP) requires the decision maker to complete a sequence of positive reciprocal matrices by answering $n(n-1)/2$ questions for each matrix, each entry being an approximation to the ratio of the weights of the n items being compared. This paper presents two extensions of the eigenvector approach of the AHP which allows the decision maker to say "I don't know" or "I'm not sure" to some of the questions being asked, and to approximate nonlinear functions of the ratios of the weights. In this way, the questioning process can be substantially shortened and better representations of the responses to certain stimuli may be derived.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP) is a decision-aiding method which has received increasing attention in the literature and in application since its development by Saaty [1]. The reader is referred to the recent review paper by Zahedi [2] for a listing of the current literature on this subject. The basis of the AHP is the completion of an $n \times n$ matrix $A = (a_{ij})$ at each level of the decision hierarchy. This matrix A is of the form $a_{ij} = 1/a_{ji}$, $a_{ij} > 0$; i.e. A is a *positive, reciprocal matrix*. The basic theory, as developed by Saaty [1, 3], is based on the fact that a_{ij} is an approximation to the relative weights (w_i/w_j) of the n alternatives under consideration; the value assigned to a_{ij} is typically in the interval $[1/9, 9]$. Given the $n(n-1)/2$ approximations to these weights which the decision maker supplies when completing the matrix A, the weights $\mathbf{w} = (w_i)$ are found by solving the following eigenvector problem:

$$A\mathbf{w} = \lambda_{\max}\mathbf{w}, \tag{1}$$

where $\lambda_{\max}$ is the Perron root or principal eigenvalue of A. A complete discussion of the reasons for using equation (1) to derive the weights $\mathbf{w}$ can be found in Ref. [4].

This paper presents two extensions of the method described above for the elicitation and computation of weights from a set of pairwise comparisons. First, the completion of $n(n-1)/2$ comparisons at each level of the hierarchy can become an onerous task if n is large. Thus, one would like to find a method in which the decision maker could complete less than $n(n-1)/2$ comparisons but still answer enough comparisons in order to derive a meaningful measure of the alternatives' relative weights. Also, it is often the case that a decision maker, when faced with a particular comparison between alternatives i and j, would rather not answer this comparison directly or may simply not yet have a good understanding of his or her preferences for those two alternatives. The first case arises when the elicitation of a_{ij} calls for the decision maker to publically state a tradeoff between two sensitive criteria; e.g. mortality risk vs cost when comparing a set of measures to lessen hazardous materials risks. It may be easier for the decision maker to skip this question and have the judgment being made indirectly through the other responses he or she has provided. The ability to skip certain direct questions may make the decision maker more willing to participate in a structured decision analysis exercise. The final reason for considering the completion of less than $n(n-1)/2$ judgments stems from the fact that the decision maker may not have formed a strong opinion on a particular question and rather than forcing this individual to make an often wild guess or to have the entire process slowed due to one question, one can simply skip that comparison. The next section will describe a method based upon a theory of *nonnegative, quasi-reciprocal matrices* which can be employed to deal with incomplete comparisons.

The second extension considered in this paper is the ability to deal with nonlinear relative preferences. The standard AHP model assumes that a_{ij} is an approximation to w_i/w_j. However, there is substantial evidence in the psychology literature [5, 6] that individuals often respond in a nonlinear fashion to stimuli; e.g. a_{ij} is an approximation to $(w_i/w_j)^\alpha$, where α is a positive scalar. Using a recent result concerning a nonlinear extension of the Perron–Frobenius theorem, a method is developed to deal with nonlinear ratio estimations in the context of the AHP; this result is presented in Section 3.

Thus, this paper presents two methodological extensions of the AHP which should simplify the elicitation process and allow greater flexibility in the modeling of preference structures with nonlinear reciprocal judgments.

2. A METHOD FOR INCOMPLETE COMPARISONS

As was discussed in the previous section, there are three basic reasons why one would want to consider the completion of less than $n(n-1)/2$ pairwise comparisons in the context of the AHP:

—the time to complete all $n(n-1)/2$ comparisons;
—unwillingness to make a direct comparison between two alternatives;
—being unsure of some of the comparisons.

Harker [7] has presented a method to deal with incomplete comparisons in the context of an iterative scheme for the elicitation of the matrix A which is based upon the approximation of the missing elements of A with the data available from the completed comparisons. This approximation of a_{ij} is formed by taking the geometric mean of the intensity of all paths in the directed graph $D(A)$ associated with the partially completed matrix A which connect the alternatives i and j. This approximation scheme in some sense mimics what the decision makers would have to perform if he or she were forced to complete a given comparison.

In this section a more natural approach to dealing with the missing entries a_{ij} will be considered. Instead of approximating the missing entry a_{ij}, which is itself an approximation to the ratio w_i/w_j, let us simply set a_{ij} to be equal to w_i/w_j. In other words, let us complete the missing entries by setting them equal to the value which they seek to approximate. Of course, one does not know *a priori* the value w_i/w_j. The purpose of this section is to derive the necessary theory to deal with the situation in which some a_{ij}'s take on the functional form w_i/w_j instead of as numerical value.

In order to begin, let us reiterate a well-known result in linear algebra and graph theory (see, for example, Ref. [3, Theorem 7.1] for a proof).

Definition 1

A square matrix A is *irreducible* if it cannot be decomposed into the form

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix},$$

where A_1 and A_3 are square matrices and 0 is the zero matrix.

Theorem 1

An $n \times n$ matrix A is irreducible iff its directed graph $D(A)$ is strongly connected.

Therefore, a matrix A is irreducible iff there exists a path between every ordered pair of nodes in the graph of A. In the AHP context, this states that there must exist a direct or indirect comparison between every pair of alternatives under consideration. Given that one always has $a_{ji} = 1/a_{ij}$ when $a_{ij} > 0$ in the AHP, completing the top row of the matrix A will be sufficient to guarantee that A is irreducible.

The following definition will be necessary in what follows.

Definition 2

An $n \times n$ matrix A is called a *nonnegative, quasi-reciprocal matrix* if

$$a_{ij} \geqslant 0 \quad \text{and} \quad a_{ij} > 0 \qquad \text{implies } a_{ji} = 1/a_{ij}, \qquad \forall\, i, j = 1, 2, \ldots, n.$$

Note that all positive, reciprocal matrices are quasi-reciprocal but that the class of quasi-reciprocal matrices allows for zero entries.

Let us now assume that the decision maker has considered a set of n alternatives and has completed some subset of the $n(n-1)/2$ pairwise comparisons to form a matrix $C = (c_{ij})$. For those questions which the decision maker responded, one as $c_{ij} > 0$, $c_{ji} = 1/c_{ij}$. Let us assume that the completed questions form an irreducible matrix. From the above discussion on positive reciprocal matrices and the graph theoretic interpretation of the matrix, it is clear that the completion of the top row of questions is sufficient to make the matrix C irreducible. By definition one has $c_{ii} = 1$, $\forall i = 1, 2, \ldots, n$. For those questions which were not answered, let $c_{ij} = w_i/w_j$. For example, in comparing three alternatives one may have:

$$C = \begin{bmatrix} 1 & 2 & w_1/w_3 \\ 1/2 & 1 & 2 \\ w_3/w_1 & 1/2 & 1 \end{bmatrix}. \tag{2}$$

Computing $C\mathbf{w}$ one obtains the vector $(2w_1 + 2w_2, 1/2w_1 + w_2 + 2w_3, 1/2w_2 + 2w_3)$, which defines a new matrix A where $A\mathbf{w} = C\mathbf{w}$:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{bmatrix}. \tag{3}$$

Thus, the problem of computing the right principal eigenvector $\mathbf{w}$ for the matrix C which contains the functional relations becomes that of computing $\mathbf{w}$ for the nonnegative, quasi-reciprocal matrix A. Thus, the issue of dealing with incomplete pairwise comparisons becomes that of studying the properties of nonnegative, quasi-reciprocal matrices. Let us now formalize this issue.

Let $B = (b_{ij})$ be an $n \times n$ matrix formed from the partially completed matrix C as follows:

$$\begin{aligned} b_{ij} &= c_{ij} && \text{if } c_{ij} \text{ is a real number } > 0 \\ &= 0 && \text{otherwise} \\ b_{ii} &= m_i, && \text{the number of unanswered questions in row } i = 1, 2, \ldots, n. \end{aligned}$$

In our example, B is given by

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1/2 & 1 \end{bmatrix}. \tag{4}$$

and the matrix $(I + B)$ equals A in equation (3). By assumption we have that B will be an irreducible matrix. Defining $A = (I + B)$ to be the nonnegative, quasi-reciprocal matrix formed from the partial pairwise comparisons which will obviously also be irreducible, one has from the Perron–Frobenius theorem that $\lambda_{\max}$ will be a real positive and simple eigenvalue which is not exceeded in modulus by any other eigenvalue of A. Furthermore, the following results are known.

Theorem 2 [3, Theorem 8-5]

If B is a nonnegative irreducible matrix of order n we have $(I + B)^{n-1} > 0$; i.e. $A = (I + B)$ is a primitive matrix.

Theorem 3 [3, Theorem 7-13]

For a primitive matrix A

$$\lim_{k \to \infty} \frac{A^k \mathbf{e}}{\mathbf{e}\mathrm{T} A^k \mathbf{e}} = \mathbf{c}\mathbf{w}, \tag{5}$$

where $\mathbf{e}$ is the unit vector, c is a constant and $\mathbf{w}$ is the principal eigenvector of A.

Therefore, the matrix B formed from the partial comparison information does lead to a primitive matrix A, and the same convergence result (5) as in the case of positive, reciprocal matrices holds. Equation (5) thus becomes the means by which one computes $\mathbf{w}$, just as in the current theory of the AHP. For example, the matrix B in equation (4) leads to the following matrix A:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1/2 & 1 \end{bmatrix}. \tag{6}$$

The limiting value of iterations (5) leads to the Perron eigenvector $\mathbf{w} = (4/7, 2/7, 1/7)$ and root $\lambda_{\max} = 3$.

Therefore, nonnegative, quasi-reciprocal matrices can be used in exactly the same manner as positive, reciprocal matrices. The only question remaining is the relationship between $\lambda_{\max}$ and n for this class of matrices. For positive, reciprocal matrices it is known that $\lambda_{\max} \geqslant n$ and that $\lambda_{\max} = n$ iff the matrix A is consistent; i.e. $a_{ij}a_{jk} = a_{ik}$, $\forall\, i,j,k$. The following theorem establishes this same result for quasi-reciprocal matrices.

Theorem 4

Let A be a nonnegative, irreducible, quasi-reciprocal matrix. Then the Perron root of A, $\lambda_{\max}$, is $\geqslant n$, the rank of A, and $\lambda_{\max} = n$ iff A is consistent; i.e. $a_{ij}a_{kj} = a_{ik}$, $\forall\, i,j,k$, with a_{ij}, a_{jk}, a_{ik} positive.

Proof. It is well-known that the trace of A, $\operatorname{tr}(A)$, equals the sum of the eigenvalues of A:

$$\operatorname{tr}(A) = n + \sum_i m_i = \sum_i \lambda_i, \tag{7}$$

where the summations are over $i = 1, 2, \ldots, n$. Given that A is irreducible, we know from the Perron–Frobenius theorem that $w_i > 0$, $\forall\, i = 1, 2, \ldots, n$, and hence one can divide the ith row of $A\mathbf{w} = \lambda_{\max}\mathbf{w}$ to form

$$\lambda_{\max} = \sum_j a_{ij} w_j / w_i \tag{8}$$

$$= (1 + m_i) + \sum_{j \neq i} a_{ij} w_j / w_i. \tag{9}$$

Summing equation (9) over $i = 1, 2, \ldots, n$, yields

$$n\lambda_{\max} = n + \sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} [a_{ij}(w_j/w_i) + a_{ji}(w_i/w_j)]. \tag{10}$$

Defining $\mu = (\lambda_{\max} - n)/(n - 1)$ and placing equation (10) into this definition, one obtains

$$\mu = -1 + [n(n-1)]^{-1}\left[\sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} a_{ij}(w_j/w_i) + a_{ji}(w_i/w_j)\right]. \tag{11}$$

Defining $a_{ij} = (w_i/w_j)(1 + \delta_{ij})$, $\delta_{ij} > -1$ if $a_{ij} > 0$, zero otherwise, and placing this into equation (11) yields, after some manipulation:

$$\mu = -1 + [n(n-1)]^{-1}\left[\sum_i m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} 2 + (\delta_{ij})^2/(1 + \delta_{ij})\right] \tag{12}$$

$$= -1 + n(n-1)[n(n-1)]^{-1} + \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} (\delta_{ij})^2/(1 + \delta_{ij}) \tag{13}$$

$$= \sum_{\substack{1 \leqslant i < j \leqslant n \\ a_{ij} \neq 0}} (\delta_{ij})^2/(1 + \delta_{ij}). \tag{14}$$

Since $\delta_{ij} > -1$, one has $\mu \geqslant 0$ and $\mu = 0$ ($\lambda_{\max} = n$) iff $\delta_{ij} = 0$, $\forall\, i, j$, with nonzero a_{ij}. Thus, $\mu = 0$ iff A is consistent. Q.E.D.

Therefore, the theory developed by Saaty [1, 3] for positive reciprocal matrices follows completely when one considers incomplete pairwise comparisons in the context of quasi-reciprocal matrices.

In order to illustrate the workings of this partial comparison method, consider the distance to Philadelphia example from Ref. [3] which is reproduced in Table 1. Table 2 illustrates the matrix A when the minimum number of questions (five) are answered, and Table 3 presents the results of sequentially answering more questions. As one can see, the weights derived from partial questioning are fairly accurate at about 10–11 responses; i.e. one typically does not need to answer a large number of questions to get fairly accurate weightings.

3. NONLINEAR RESPONSES IN THE AHP

The second extension to the AHP which we shall consider in this paper deals with nonlinear responses to the pairwise comparisons. In the standard theory it is assumed that a_{ij} is an approximation to the ratio w_i/w_j. However, one could have situations in which a_{ij} is an approximation to some function of this ratio $f(w_i/w_j)$. For example, the function $f = (w_i/w_j)^\alpha$ has been widely studied in the psychology literature for various values of the power α with respect to differing stimuli; Table 4 lists several of these values. Saaty [3, Theorem 7-28] studied the use of the power function when it is assumed that the responses by the decision maker are *perfectly consistent*. In this section, the requirement of consistency will be dropped and a recent result concerning the Perron–Frobenius theorem will be used to study the general functional form $f(w_i/w_j)$.

Table 1. Distance from Philadelphia example [3, p. 42]

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	1	9	3	3	9
Chicago	1/8	1/9	1	1/6	1/5	2
San Francisco	1/3	1/3	6	1	1/3	6
London	1/3	1/3	5	3	1	6
Montreal	1/7	1/9	1/2	1/6	1/6	1
Distance (miles)	5729	7449	660	2732	3658	400
Relative distance	0.2777	0.3611	0.0320	0.1324	0.1773	0.0194

Table 2. Distance example with five questions answered

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	5	0	0	0	0
Chicago	1/8	0	5	0	0	0
San Francisco	1/3	0	0	5	0	0
London	1/3	0	0	0	5	0
Montreal	1/7	0	0	0	0	5
Eigenvector $\mathbf{w}$	0.20265	0.60796	0.02533	0.06755	0.6755	0.02895
$\lambda_{\max}$	6.000					

Table 3. Results of the distance example with incomplete questioning

	Eigenvector $\mathbf{w}$						
No. of questions	w_1	w_2	w_3	w_4	w_5	w_6	$\lambda_{\max}$
5	0.20265	0.60796	0.02553	0.06755	0.06755	0.02895	6.0000
6	0.23761	0.53684	0.04345	0.07498	0.07498	0.03214	6.0563
7	0.25471	0.47756	0.04052	0.11649	0.07751	0.03322	6.0954
8	0.26563	0.43977	0.03873	0.11097	0.11097	0.03392	6.1188
9	0.27265	0.42794	0.03848	0.11008	0.11008	0.04077	6.1215
10	0.26919	0.41975	0.03235	0.13368	0.10582	0.03921	6.1701
11	0.26793	0.41690	0.02985	0.12797	0.11860	0.03875	6.1832
12	0.26650	0.41041	0.03553	0.13374	0.12291	0.03092	6.2810
13	0.26331	0.40137	0.03437	0.10901	0.16290	0.02904	6.4223
14	0.26178	0.39729	0.03338	0.11613	0.16502	0.02640	6.4538
15	0.26185	0.39749	0.03343	0.11639	0.16424	0.02660	6.4536

If one assumes that a_{ij} is an approximation to $(w_i/w_j)^\alpha$ for any positive value of α, the eigenvector problem can be written as

$$A\mathbf{w}^\alpha = \lambda_{\max}\mathbf{w}^\alpha, \tag{15}$$

where $\mathbf{w}^\alpha = (w_1^\alpha, w_2^\alpha, \ldots, w_n^\alpha)$. Defining $\mathbf{v}$ to be equal to $\mathbf{w}^\alpha$ one can immediately see that equation (15) becomes our standard eigenvector problem in the AHP which can be dealt with in the standard fashion or by the incomplete comparison method derived in the previous section. Thus, equation (15) can be solved with any matrix A; *consistency is not a requirement for the power law to be applicable.* One need only convert from $\mathbf{v}$ to $\mathbf{w}$ by taking the αth root of each component of the vector $\mathbf{v}$.

The effect of employing a power different from 1 is illustrated in Table 5 which is based on the matrix from Saaty's [3, p. 19] optics example:

$$A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ & 1 & 4 & 6 \\ & & 1 & 4 \\ & & & 1 \end{bmatrix}.$$

As Table 5 illustrates, a value of $\alpha < 1$ tends to amplify the differences between the various alternatives, and $\alpha > 1$ tends to dampen these differences. Thus, one might wish to employ some power of the eigenvector in order to see greater differences or to satisfy a decision maker who tends to favor equality of weights rather than stark contrasts—the "even keel" mentality. There is a great amount of empirical research which should be performed in order to ascertain the appropriate values of α in various decision contexts; e.g. risk assessments, assessment of cost items, assessment of probabilities, etc. Through this line of research one may be able to "tune" the AHP to the various stimuli which a decision maker is likely to face.

One can generalize the functional forms to be considered in the nonlinear AHP by using recent results in functional analysis which deal with a nonlinear extension of the Perron–Frobenius theorem by Kohlberg [8] (which is based upon the earlier work by Kohlberg and Pratt [9]) and Rath [10]. In order to summarize these results, consider a continuous mapping $F: R_+^n \to R_+^n$ which satisfies

homogeneity of degree 1: $F(\eta\mathbf{x}) = \eta F(\mathbf{x})$

primitive: for some integer $l > 0$, $\mathbf{x} \geqslant \mathbf{y}$ implies $F^l(\mathbf{x}) > F^l(\mathbf{y})$, where F^l denotes the lth application of F.

Using these two concepts Kohlberg [8] has shown the following.

Table 4. Values of power α for various stimuli [3, p. 189]

Stimuli	Power α
Loudness	0.3
Brightness	0.33–0.5
Length	1.1
Duration	1.15
Numerousness	1.34
Heaviness	1.45
Velocity	1.77
Electric shock	4.0

Table 5. Eigenvector values for various powers α

Power α	w_1	w_2	w_3	w_4
0.1	0.999937	0.000063	0.000000	0.000000
0.2	0.991985	0.007898	0.000115	0.000002
0.4	0.907923	0.081015	0.009761	0.001301
0.6	0.792956	0.158341	0.038626	0.010078
0.8	0.694322	0.207404	0.071992	0.026282
1.0	0.618669	0.235323	0.100934	0.045074
1.2	0.561702	0.251002	0.123972	0.063324
1.4	0.518344	0.259875	0.141963	0.079818
1.6	0.484705	0.264915	0.156077	0.094302
1.8	0.458070	0.267740	0.167292	0.106899
2.0	0.436572	0.269252	0.176338	0.117839
5.0	0.321306	0.264825	0.223581	0.190288
10.0	0.284770	0.258532	0.237548	0.219150

Theorem 5

Let $F: R^n_+ \to R^n_+$ be a continuous mapping which is homogeneous of degree 1 and primitive, then:

(a) there exists a vector $\mathbf{x}^0 > 0$ which is unique up to proportionality such that $F(\mathbf{x}^0) = \lambda_0 \mathbf{x}^0$ for some $\lambda_0 > 0$;

(b) $\lim_{k\to\infty} F^l(\mathbf{x})/\|F^l(\mathbf{x})\| = c\mathbf{x}^0, \forall\, \mathbf{x} \geqslant 0$;
i.e. the nonlinear map will converge for any starting point $\mathbf{x}$ and $\|\cdot\|$ is any norm in R^n.

Let us now consider the case where a_{ij} represents an approximation of an arbitrary function $f(w_i/w_j)$ instead of the special case of $(w_i/w_j)^\alpha$ which is discussed above. For example, $f(w_i/w_j)$ could take the form $\exp[\beta(w_i/w_j)]/\exp(\beta)$; i.e. the response of the decision maker is an exponential function of the weights on the alternatives, or in other words, the ratio of the alternatives' weights varies with the logarithm of a_{ij}:

$$w_i/w_j = 1 + (1/\beta)\ln a_{ij}.$$

Which exact functional forms one should employ will not be discussed in this paper; this issue is left for future research.

Given the type of pairwise function $f(w_i/w_j)$ described above, the ith component of the mapping $F: R^n_+ \to R^n_+$ will take the form

$$F_i(\mathbf{w}) = w_i \sum_j a_{ij} [f(w_i/w_j)]^{-1}. \tag{16}$$

In order to illustrate the meaning of this relationship in the context of the AHP, consider $F_i(\mathbf{w}) = \lambda_0 \mathbf{w}$. Remember that a_{ij} is defined to be an approximation to the function $f(w_i/w_j)$. If a_{ij} were exactly equal to $f(w_i/w_j)$, then the sum in equation (16) and hence λ_0 would equal n, the number of alternatives; i.e. perfect consistency is achieved. In the special case of $f(w_i/w_j) = (w_i/w_j)^\alpha$, the eigenvector problem $F(\mathbf{w}) = \lambda\mathbf{w}$ defined by equation (16) would simplify to equation (15). Thus, equation (16) is the natural representation for the use of general functional forms in the AHP.

If $f(\cdot)$ is a continuous, positive-valued function of degree 0, then the function $F(\mathbf{w})$ defined by equation (16) will be continuous and homogeneous of degree 1. The proof that $F(\mathbf{w})$ is primitive for certain classes of functions $f(\cdot)$ is very involved and depends upon the relative values of the a_{ij}s. It suffices to state that with general functional forms $f(w_i/w_j)$, one need only try the iterative scheme in Theorem 5 and if convergence is achieved, one has obtained the Perron vector for this nonlinear map. Further research is necessary to ascertain if there exist any functional forms beyond the power function $(w_i/w_j)^\alpha$ which are primitive and are empirically useful.

In order to illustrate the general nonlinear mapping, define

$$f(w_i/w_j) = \exp[\beta(w_i/w_j)]/\exp(\beta) \tag{17}$$

and consider two alternatives with $a_{11} = a_{22} = 1, a_{12} = 2, a_{21} = 0.4$. Note that a general functional form such as equation (17) need no longer obey the reciprocal property of the AHP and hence, the entire matrix must be completed. In fact, Saaty [3, Theorem 7-28] has shown that a power function $(w_i/w_j)^\alpha$ is the only form of $f(w_i/w_j)$ which retains the reciprocal property of the standard AHP. There exists some ancedotal evidence that decision makers may not always obey strict reciprocity. This fact points to the need for further research in understanding if nonreciprocal judgments are empirically meaningful and if so, which functional forms $f(w_i/w_j)$ best represent these nonreciprocal judgments. For the moment, let us assume that one can derive meaningful functions of the form $F_i(\mathbf{w})$, as in equation (16). By Theorem 5 one knows that an iterative scheme will converge if $F(\mathbf{w})$ is primitive. For example, Table 6 lists the results of the iterative scheme for the nonlinear map (17) for various values of β. As one can see, the primitivity of the map depends upon the relative values of the parameters (β) and that in this case large β tends to smooth the weights and small β tends to accent the differences between the alternatives.

In summary, there exists a method for dealing with situations in which a_{ij} is an approximation of some function of the weights $\mathbf{w}$. It is a very interesting research question to ascertain which functional relationship are usable and meaningful in the context of the AHP.

Table 6. Results of the nonlinear example

β	w_1	w_2
0.2	0.89094	0.10906
0.4	0.80983	0.19017
0.6	0.75088	0.24912
0.8	0.70797	0.29203
1.0	0.67620	0.32380
1.2	0.65213	0.34787
1.4	0.63346	0.36654
1.6	0.61866	0.38134
1.8	0.60668	0.39332
2.0	0.59682	0.40318
2.2	*did not converge*	*imprimitive map*

4. CONCLUSIONS

This paper has presented two extensions of the AHP methodology to deal with incomplete pairwise comparisons and nonlinear ratio scales. These two extensions should both speed up the elicitation process and provide the analyst with greater flexibility in the modeling of the decision maker's responses to the stimuli of comparing decision alternatives. However, several interesting research questions remain. First, what are the appropriate values of α in the power function approach and if this power law is not applicable, what other functional forms $f(w_i/w_j)$ can be employed in the AHP context? Also, the question as to how the techniques discussed in this paper can be extended to deal more efficiently and effectively with the overall hierarchical structure rather than with a single matrix also remains for future research.

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