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Decision Aiding

The geometric consistency index: Approximated thresholds

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Abstract

Crawford and Williams [Journal of Mathematical Psychology 29 (1985) 387] suggested for the Row Geometric Mean Method (RGMM), one of the most extended AHP's priorization procedure, a measure of the inconsistency based on stochastic properties of a subjacent model. In this paper, we formalize this inconsistency measure, hereafter called the Geometric Consistency Index (GCI), and provide the thresholds associated with it. These thresholds allow us an interpretation of the inconsistency tolerance level analogous to that proposed by Saaty [Multicriteria Decision Making: The Analytic Hierarchy Process, New York, 1980] for the Consistency Ratio (CR) used with the Right Eigenvector Method in Conventional-AHP.

Keywords: Multicriteria; AHP; Geometric mean; Consistency; Thresholds

1. Introduction

The Analytic Hierarchy Process (AHP) is a multicriteria decision making (MCDM) technique proposed by Saaty (1977, 1980), that integrates pairwise comparison ratios into a ratio scale. One advantage of this MCDM tool is that it allows us to measure the consistency of the decision maker when eliciting the judgements in a formal and elegant way.

Defining the consistency of a positive reciprocal pairwise comparison matrix, $A = (a_{ij})$, as the cardinal transitivity between the judgements, that is to say, $a_{ij}a_{jk} = a_{ik},\ i, j, k = 1, \ldots, n$, Saaty suggested that the inconsistency in Conventional-AHP, where the Right Eigenvector Method (EVM) is used as priorization procedure, can be captured by a single number ($\lambda_{\max} - n$) which reflects the deviations of all a_{ij} from the estimated ratio of priorities ω_i/ω_j.

In this case, to provide a measure independent of the order of the matrix, n, Saaty proposed the use of the Consistency Ratio (CR). This is obtained by taking the ratio between $\lambda_{\max} - n$ to its expected value over a large number of positive reciprocal matrices of order n, whose entries are randomly chosen in the set of values $\{1/9, \ldots, 1, \ldots, 9\}$. For this consistency measure, he proposed a 10% threshold for the CR (5% and 8% for the 3 by 3 and 4 by 4 matrices, respectively) to accept the estimation of ω (Saaty, 1994). When the CR is greater than 10%, then, in order to improve

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the consistency, most inconsistency judgements, that is to say, those with a greater difference between a_{ij} and ω_i/ω_j, are usually modified and a new ω derived.

There are many other priorization procedures in the literature, but only a few of them present their corresponding indicators to evaluate the inconsistency. Furthermore, when these consistency indexes have been proposed (Crawford and Williams, 1985; Harker, 1987; Golden and Wang, 1989; Wedley, 1991; Takeda, 1993; Takeda and Yu, 1995; Monsuur, 1996; Escobar and Moreno-Jiménez, 1997; Aguarón, 1998), they lack a meaningful interpretation due to the absence of the corresponding thresholds. Obviously, if the priorization procedure is not the EVM, the Saaty approach to evaluate the consistency is not appropriate, by construction, and new consistency measures, related to the priorization procedure, are required.

Recently, and despite the strong defense of the EVM presented by the Saaty school (Saaty, 1990; Vargas, 1994, 1997), the use of the Row Geometric Mean Method (RGMM), or Logarithmic Least Squares Method, as a priorization procedure in AHP has significantly increased (Ramanathan, 1997; Van den Honert, 1998; Levary and Wan, 1999) due fundamentally to its psychological (Gescheider, 1985; Lootsma, 1993; Barzilai and Lootsma, 1997; Brugha, 2000) and mathematical (Narasimhan, 1982; Jensen, 1984; Budescu, 1984; Barzilai, 1997; Aguarón and Moreno-Jiménez, 2000; Escobar and Moreno-Jiménez, 2000; Brugha, 2000) properties.

Crawford and Williams (1985) justified the RGMM by means of two different approaches: (1) the minimization of the log quadratic distance of errors (Logarithmic Least Squares Method); and (2) the maximum likelihood estimator of the priorities. The first is a deterministic approach and the second a stochastic one, where a multiplicative model for the perturbations has been supposed ($a_{ij} = (\omega_i/\omega_j)\pi_{ij}$, with π_{ij} independent and log-normal distributions with zero mean and constant variance $\pi_{ij} \sim \mathrm{Lognormal}(0, \sigma)$).

For this priorization procedure (RGMM), Crawford and Williams suggested that the estimator of the variance of the perturbations can be used as a measure of the consistency, where the lower the value, the better the consistency of the judgements. In what follows assuming the proposal of Crawford and Williams, and without entering into the analysis of the validity of the CR as a consistency measure in AHP, we calculate the thresholds that make this measure, called the Geometric Consistency Index (GCI), operative and that allow us to fix a tolerance level with an interpretation analogous to that considered for Saaty's CR.

The paper has been structured as follows: Section 2 presents the two consistency measures considered in this paper (CR and GCI); Section 3 establishes a theoretical relation between the CR and the GCI, the validity of which is tested through a regression analysis; finally, Section 4 closes the paper with some comments about the GCI thresholds.

2. Consistency measures. The Geometric Consistency Index (GCI)

In the Conventional-AHP (Saaty, 1980), the priorities ($\omega_i,\ i = 1, \ldots, n$) are obtained by solving the eigenvector problem

$$A\omega = \lambda_{\max}\omega \sum_{i=1}^{n} \omega_i = 1, \tag{1}$$

where A is a positive pairwise comparison matrix of order n, $\lambda_{\max}$ is the principal eigenvalue of A and ω is the priority vector.

For this priorization procedure, the EVM, Saaty (1980) proposed a measure of the inconsistency in judgements, called the Consistency Index (CI), that is given by

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1}, \tag{2}$$

where $\lambda_{\max}$ is the principal eigenvalue of the judgement matrix and n is its order.

When the reciprocal comparison matrix is consistent, $\lambda_{\max} = n$, and the CI is equal to zero; otherwise, its value is positive. To overcome the order dependency of the CI, Saaty proposed a normalized measure, called the CR, that is given by

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}(n)}, \tag{3}$$

where $\mathrm{RI}(n)$ is the Random (Consistency) Index for matrices of order n. This term is defined as the expected value of the CI corresponding to matrices of order n ($\mathrm{RI} = E[\mathrm{CI}(n)]$), when the judgements are simulated in the set $\{1/9, \ldots, 1, \ldots 9\}$ and the EVM is used as the priorization procedure.

The CR gives a measure of where the judgements in the pairwise comparison matrix lie between totally consistent and totally random. When $\mathrm{CR} = 1$, then $\mathrm{CI} = E[\mathrm{CI}(n)]$ and the judgements are totally random (low precision). High values of CR reflect more inconsistency and thus we are interested in values of CR as low as possible. To accept the consistency of the matrix, Saaty (1980) suggested as a rule of thumb a threshold of 10 percent or less ($\mathrm{CR} \leqslant 0.1$). More recently, Saaty (1994) suggested thresholds of 5% and 8% for 3 by 3 and 4 by 4 matrices, respectively.

Lemma 1. *The CI proposed by Saaty for the EVM can be expressed as an average of the differences between the errors and unity, that is to say*

$$\mathrm{CI} = \frac{1}{n(n-1)} \sum_{i \neq j}^{n} (e_{ij} - 1), \tag{4}$$

where the errors are $e_{ij} = a_{ij}\omega_j/\omega_i$.

Proof. Immediately from definition of CI. □

In this paper we consider the priorization procedure known as the RGMM, where the priorities (without the normalization factor) are given by

$$\omega_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}. \tag{5}$$

For the RGMM, Crawford and Williams (1985) suggested the use of an unbiased estimator of the variance of the perturbations as a measure of the consistency:

$$s^2 = S/\mathrm{d.f.} = \frac{2\sum_{i<j} (\log a_{ij} - \log \omega_i/\omega_j)^2}{(n-1)(n-2)}. \tag{6}$$

The numerator (S) is the squared distance between the log of the judgements a_{ij} and the log of the ratios (ω_i/ω_j), and the denominator (d.f.) is the degrees of freedom that are given as the difference between the judgements included $(n(n-1)/2)$ and the estimated parameters $(n-1)$.

From a deterministic point of view, the smaller the s^2, the shorter the distance between the judgements a_{ij} and the ratios ω_i/ω_j. From a stochastic one, the smaller the s^2, the smaller will be the variance of the perturbations π_{ij} and the better will be the fit between the judgements and the priorities vector ω.

Next, we consider the measure of consistency proposed by Crawford and Williams (1985) which, in what follows, we call the GCI.

Definition 1. Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, and the vector of priorities, ω, obtained by the RGMM, let us define the GCI as

$$\mathrm{GCI} = \frac{2}{(n-1)(n-2)} \sum_{i<j} \log^2 e_{ij}, \tag{7}$$

where $e_{ij} = a_{ij}w_j/w_i$ is the error obtained when the ratio ω_i/ω_j is approximated by a_{ij}.

In an interpretation analogous to that given in Lemma 1 for the CI, the GCI can be seen as an average of the squared difference between the log of the errors and the log of unity:

$$\mathrm{GCI} = \frac{2}{(n-1)(n-2)} \sum_{i<j} (\log e_{ij} - \log 1)^2.$$

In this case, the reciprocal property of the errors is considered in the proposed measure, because the errors $e_{ij} > 1$ and $e_{ij} < 1$ contribute with the same amount.

3. Approximated thresholds for the Geometric Consistency Index

As Barzilai (1996) indicates, the value of s^2 (GCI) can be considered as a measure of the goodness of fit. However, the range of values that will give it the operative character required by a measure of these characteristics remains to be established (also see Golden and Wang, 1989). One way of making this measure operative would be to normalize it in a way analogous to that carried out with Saaty's consistency ratio; that is to say, to divide the value that measures the log quadratic distance between the errors e_{ij} and unity (s^2) by its expected value when the judgements are simulated in the interval $\{1/9, \dots, 1, \dots, 9\}$.

However, as we will demonstrate in Lemma 2, the expected value of s^2 is a constant. Therefore, we will follow an indirect procedure to obtain the thresholds associated with the GCI, which allows for an interpretation of the inconsistency tolerance level analogous to that proposed by Saaty for the EVM (CR $\leqslant 0.1$).

Lemma 2. *If the judgements of a pairwise comparisons matrix follow independent, reciprocal and identical distributions, the mean of the GCI is given by*

$$E[\mathrm{GCI}] = \mathrm{Var}(\log a_{ij}). \tag{8}$$

Proof. See Appendix A. □

The indirect procedure consists in establishing the thresholds for the new measure on the basis of its relationship with the CR in its band where we can accept the estimation of the priorities as being good (CR $\leqslant 0.1$). Note, however, that the study has in fact been made in a broader range to guarantee the validity of the conclusions (CR $\leqslant$ 0.2).

In this sense, we first present a theoretical relation between the GCI and the CR that is valid for small errors and that will allow us to estimate the corresponding thresholds.

Theorem 1. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \dots, n$, and the vector of priorities, ω, obtained by the RGMM, it holds that*

$$\mathrm{GCI} = \frac{2n}{n-2}\mathrm{CI} + \mathrm{o}(\varepsilon^3), \tag{9}$$

where $\varepsilon = \max_{ij}\{|\log e_{ij}|\}$ and $e_{ij} = a_{ij}\omega_j/\omega_i$.

Proof. See Appendix A. □

Corollary 1. *Under the conditions of Theorem* 1, *it holds that*

$$\mathrm{GCI} = k(n)\mathrm{CR} + \mathrm{o}(\varepsilon^3), \tag{10}$$

where

$$k(n) = \frac{2n}{n-2}E[\mathrm{CI}(n)].$$

Proof. Immediate on the basis of the earlier theorem and $\mathrm{CR} = \mathrm{CI}/\mathrm{RI}(n)$. □

Using this result and removing the term $\mathrm{o}(\varepsilon^3)$, the relationship, $k(n)$, between GCI and CR is given in Table 1, where the $\mathrm{RI}(n) = E[\mathrm{CI}(n)]$ have been obtained through the simulation of 100,000 matrices for each order (n), where the judgements belong to the set of values $\{1/9, \dots, 1, \dots, 9\}$. For $n = 3$, the value of RI(3) = 0.525 is a rounded value of the exact one, RI(3) = 0.5245, obtained by enumerating all possible combinations of judgements.

This result makes clear that if the judgements matrices $A = (a_{ij})$ are close to consistency (small errors), then the two measures, that is to say, the CR of Saaty and the earlier-mentioned GCI, are proportional. We tested this using a regression analysis based on the values of the CR and the GCI obtained from a simulation study where nearly 1,200,000 matrices $A = (a_{ij})$ were generated around the identity (near consistency).

In general terms, it can be noted that the behaviour of the two measures is similar (see Fig. 1, corresponding to $n = 4$, as an example) for the different values of n (high values of CR provide high values of GCI). Nevertheless, it is interesting

Table 1
Values of RI(n) and $k(n)$ for $n = 3, \dots, 16$

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
RI(n)	0.525	0.882	1.115	1.252	1.341	1.404	1.452	1.484	1.513	1.535	1.555	1.570	1.583	1.595
$k(n)$	3.147	3.526	3.717	3.755	3.755	3.744	3.733	3.709	3.698	3.685	3.674	3.663	3.646	3.646

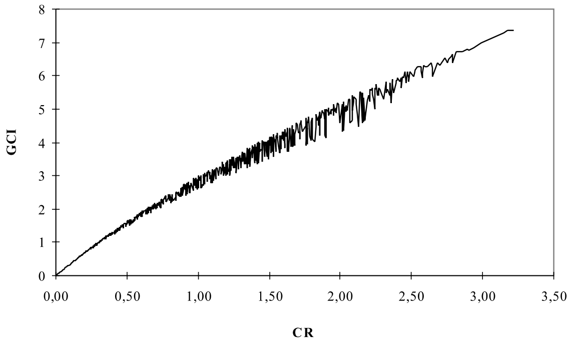

Fig. 1. Relation between GCI and CR for $n = 4$.

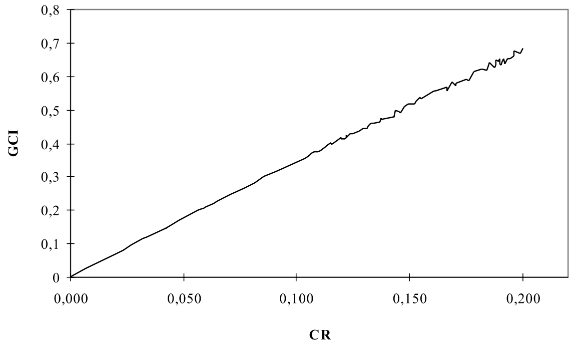

Fig. 2. Relation between GCI and CR for $n = 4$ and $CR < 0.2$.

to carry out a more detailed analysis of the behaviour of the two indices when the CR takes values that are considered as acceptable ($CR \leqslant 0.1$). Moreover, in what follows, we will consider a broader range ($CR \leqslant 0.2$).

Here (Fig. 2), we can see that for low values of Saaty's consistency ratio the relationship is practically linear, as guaranteed by the earlier theorem. This fit is particularly significant when CR is below 0.1. In order to test this fit, we have calculated the regression line of GCI over CR for different intervals of CR. Table 2 presents the number of available observations (N), the slope of the regression line (m), and the goodness of fit (r^2) for different orders of the judgement matrix n and four different consistency intervals (0–0.01, 0–0.05, 0–0.10, 0–0.20).

Note that in all cases the goodness of fit is very high, in such a way that we can accept an almost linear relation between the two measures. Nevertheless, as the range considered for the inconsistency increases, the slopes estimated through regression decrease, due to the small concavity of the relation. The maximum differences between these values and the approximated ones obtained from Corollary 1 have been of 1% for $CR \leqslant 0.01$, 2.5% for $CR \leqslant 0.05$, 4.7% for $CR \leqslant 0.1$ and 7.9% for $CR \leqslant 0.2$. Therefore, and despite the fact that the approximated values overestimate the true ones, we can consider the thresholds of Table 3 as corresponding with Saaty's consistency ratio. For the $CR = 0.1$, the associated GCI are: $GCI = 0.3147$ for $n = 3$; $GCI = 0.3526$ for $n = 4$ and $GCI = 0.370$ for $n > 4$.

Table 2
Estimations of the regression coefficient (m) between GCI and CR ($GCI = mCR$)

n	0–0.01			0–0.05			0–0.10			0–0.20		
	N	m	r^2	N	m	r^2	N	m	r^2	N	m	r^2
3	13,934	3.147	1.000	34,177	3.137	1.000	45,142	3.128	1.000	55,616	3.111	1.000
4	13,599	3.518	1.000	22,857	3.477	1.000	39,063	3.436	1.000	56,023	3.364	0.999
5	14,259	3.702	1.000	18,273	3.637	0.999	36,369	3.570	0.999	58,114	3.466	0.997
6	14,313	3.739	1.000	15,676	3.664	0.999	34,161	3.584	0.998	59,750	3.470	0.996
7	14,166	3.737	1.000	14,404	3.662	0.999	32,669	3.578	0.998	60,578	3.460	0.996
8	13,769	3.726	1.000	13,565	3.653	0.999	31,936	3.570	0.998	61,144	3.449	0.997
9	13,563	3.715	1.000	13,037	3.643	0.999	31,377	3.563	0.999	61,489	3.439	0.997
10	13,240	3.693	1.000	12,605	3.623	0.999	30,941	3.543	0.999	61,588	3.421	0.997
11	12,925	3.681	1.000	12,303	3.613	0.999	30,701	3.536	0.999	61,621	3.413	0.997
12	12,550	3.668	1.000	11,934	3.603	0.999	30,479	3.526	0.999	61,617	3.404	0.998
13	12,275	3.659	1.000	11,589	3.596	0.999	30,304	3.519	0.999	61,597	3.398	0.998
14	12,002	3.648	1.000	11,333	3.588	0.999	30,291	3.511	0.999	61,524	3.391	0.998
15	11,737	3.639	1.000	11,126	3.581	0.999	30,248	3.504	0.999	61,478	3.384	0.998
16	11,549	3.632	1.000	10,813	3.576	0.999	30,162	3.498	0.999	61,462	3.380	0.998

Table 3
Approximated thresholds

CR	0.01	0.05	0.1	0.15
GCI ($n = 3$)	0.0314	0.1573	0.3147	0.4720
GCI ($n = 4$)	0.0352	0.1763	0.3526	0.5289
GCI ($n > 4$)	~0.037	~0.185	~0.370	~0.555

The interpretation of the GCI, the inconsistency measure used for the RGMM, is analogous to that proposed by Saaty for the CR used with the EVM. When the value of the GCI is greater than the corresponding threshold, the most inconsistent judgement (that with larger e_{ij}) has to be modified in the sense of approximating it (a_{ij}) to ω_i/ω_j.

4. Conclusions

In recent years there has been a move towards using geometric mean synthesis of AHP-type scores, for example, the RGMM. This priorization procedure provides estimations that are very close to the priorities of the traditional EVM. Moreover, it presents more desirable analytical properties and requires less computational effort.

In this paper, we have formalised the inconsistency measure proposed for the RGMM by Crawford and Williams (1985), calling it the GCI.

Following an indirect method, due to the GCI's independence order for this measure, we have computed thresholds that allow us an interpretation of the inconsistency level which is analogous to that proposed by Saaty for the EVM (see Table 3).

To obtain these thresholds, we have proved an analytical relation between the GCI and the CR that is valid for small inconsistencies, but that slightly overestimates these values, as we have seen through a regression analysis. The approximated thresholds,

$$k(n) = \frac{2n}{n-2}\mathrm{RI}(n),$$

computed from this relation are given in Table 1, where the values of the $\mathrm{RI}(n)$ used here have been obtained through the simulation of 100,000 matrices for each order (to the best of our knowledge, the most complete study in this sense).

Finally, we should highlight that, assuming the small discrepancies obtained for $\mathrm{CR} \leqslant 0.1$, the practical values (associated rounded values) of the GCI corresponding to the usual value of the $\mathrm{CR} = 10\%$ are: $\mathrm{GCI} = 0.31$ for $n = 3$; $\mathrm{GCI} = 0.35$ for $n = 4$ and $\mathrm{GCI} = 0.37$ for $n > 4$.

From a practical point of view, the interpretation of the *GCI* is analogous to that proposed by Saaty for the Consistency Ratio used with the Eigenvector Method in Conventional AHP. In our case, if we use the Row Geometric Mean Method as the priorization procedure in AHP and the judgements of the $(n \times n)$ pairwise comparison matrix belong to the fundamental scale of Saaty ($\{1/9, \ldots, 9\}$), then, when the value of the $GCI(n)$ is greater than its corresponding threshold (for example, $GCI > 0.1573$ for $n = 3$), the most inconsistent judgement would have to be modified and a new priority vector calculated. Otherwise, the estimations of the priorities given by the RGMM are accepted.

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Appendix A

Proof of Lemma 2

$$E[\mathrm{GCI}] = E\left[\frac{2}{(n-1)(n-2)}\sum_{i<j}\log^2 e_{ij}\right] = \frac{2}{(n-1)(n-2)}\sum_{i<j}E\left[\log^2 e_{ij}\right]. \tag{A.1}$$

The terms $\log^2 e_{ij}$ can be expressed as

$$\log^2 e_{ij} = \log^2\left[a_{ij}\frac{\omega_j}{\omega_i}\right] = \log^2\left[a_{ij}\left(\prod_{k=1}^{n}\frac{a_{jk}}{a_{ik}}\right)^{1/n}\right] = \log^2\left[a_{ij}^{1-2/n}\left(\prod_{k\neq i,j}\frac{a_{jk}}{a_{ik}}\right)^{1/n}\right].$$

As all these terms have an identical distribution, the expression (A.1) holds as

$$E[\mathrm{GCI}] = \frac{2}{(n-1)(n-2)}\frac{n(n-1)}{2}E\left[\log^2 e_{ij}\right] = \frac{n}{n-2}E\left[\log^2 e_{rs}\right] \tag{A.2}$$

for any r, s. To calculate the expected value of $\log^2 e_{rs}$, we operate as follows:

$$\log^2 e_{rs} = \log^2\left[a_{rs}^{1-2/n}\left(\prod_{k\neq r,s}\frac{a_{sk}}{a_{rk}}\right)^{1/n}\right] = \left[\log a_{rs}^{1-2/n} + \sum_{k\neq r,s}\log a_{sk}^{1/n} - \sum_{k\neq r,s}\log a_{rk}^{1/n}\right]^2.$$

When developing the square of this parenthesis, we have two kinds of terms: those that include a square logarithm (i.e. $\log^2 a_{rs}$) and those that include the product of two logarithms (i.e. $\log a_{rs}\log a_{rk}$). In the second case, as the judgements a_{ij} are reciprocal and independent, the expected value is zero, so we only consider the terms in $\log^2$:

$$E[\log^2 e_{rs}] = E\left[\log^2 a_{rs}^{1-2/n} + \sum_{k\neq r,s}\log^2 a_{sk}^{1/n} + \sum_{k\neq r,s}\log^2 a_{rk}^{1/n}\right] = E\left[\left(\frac{n-2}{n}\right)^2\log^2 a_{rs} + \sum_{k\neq r,s}\frac{1}{n^2}\log^2 a_{sk} + \sum_{k\neq r,s}\frac{1}{n^2}\log^2 a_{rk}\right].$$

Because all the judgements have the same distribution, the expected values of the terms $\log^2 a_{ij}$ coincide with that of $\log^2 a_{rs}$, so it holds that

$$E[\log^2 e_{rs}] = \left[\left(\frac{n-2}{n}\right)^2 + 2\frac{n-2}{n^2}\right]E[\log^2 a_{rs}] = \frac{n-2}{n}E[\log^2 a_{rs}]. \tag{A.3}$$

Using this result in expression (A.2), and taking into account that the expected value of the log of the reciprocal distributions is zero, we have

$$E[\mathrm{GCI}] = E[\log^2 a_{rs}] = \mathrm{Var}(\log a_{rs}). \qquad \square$$

Proof of Theorem 1. Let us consider the matrix $E = (e_{ij})$ with $e_{ij} = a_{ij}\omega_j/\omega_i$ where $\omega = (\omega_i)$ is the weights vector obtained by applying the row geometric mean method. For such a matrix it holds that

$$\prod_{j=1}^{n} e_{ij} = 1, \quad i = 1,\dots,n. \tag{A.4}$$

Taking $\varepsilon_{ij} = \log e_{ij}$, we have that

$$\sum_{j=1}^{n}\varepsilon_{ij} = 0, \quad i = 1,\dots,n. \tag{A.5}$$

If the matrix is consistent, the errors e_{ij} have the value one and the values of the ε_{ij} are null. If the inconsistency is small, by continuity (Saaty, 1980), the ε_{ij} will be found relatively close to zero. We take $\varepsilon = \max_{ij}\{|\varepsilon_{ij}|\}$.

Let $v = (v_i)$, $i = 1,\dots,n$, be the priorities vector obtained by applying the right eigenvector method on the matrix E. We know that in the consistent case the two methods coincide, in such a way that, if the inconsistency is low it can be considered that the vector v will be close to that obtained by the geometric mean (equal to the unit vector as a consequence of expressions (A.4)). Let us take $v_i = 1 + d_i$ with $\sum_i d_i = 0$. With this we can verify

$$Ev = \lambda_{\max} v \tag{A.6}$$

developing

$$\begin{pmatrix} 1 & e_{12} & \cdots & e_{1n} \\ e_{21} & 1 & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1+d_1 \\ 1+d_2 \\ \vdots \\ 1+d_n \end{pmatrix} = \lambda_{\max}\begin{pmatrix} 1+d_1 \\ 1+d_2 \\ \vdots \\ 1+d_n \end{pmatrix} \tag{A.7}$$

and, therefore,

$$\sum_{j=1}^{n} e_{ij}(1+d_j) = \lambda_{\max}(1+d_i). \tag{A.8}$$

On the basis of this expression, we have

$$\lambda_{\max}(1+d_i) \leqslant \max_{ij} e_{ij}\sum_{j=1}^{n}(1+d_j) = \mathrm{e}^{\varepsilon}n \tag{A.9}$$

and, therefore, ($\lambda_{\max} \geqslant n$)

$$d_i \leqslant \frac{e^{\varepsilon} n}{\lambda_{\max}} - 1 \leqslant e^{\varepsilon} - 1. \tag{A.10}$$

As $\sum_j d_j = 0$, $d_i = -\sum_{j\neq i} d_j$ and, on the basis of the above inequality, we have that $d_i \geqslant -(n-1)(e^{\varepsilon}-1)$. If the values of d_i are delimited between $-(n-1)(e^{\varepsilon}-1)$ and $e^{\varepsilon}-1$, then they are, at the very most, of the order $e^{\varepsilon}-1 = o(\varepsilon)$. This shows us that for small inconsistencies, the differences between the two priorities vectors are of the same order as the errors e_{ij} of the matrix.

If we add the identities (A.8) in i, we obtain

$$\sum_{i,j} e_{ij}(1+d_j) = n\lambda_{\max}. \tag{A.11}$$

Taking into account that $e_{ij} = e^{\varepsilon_{ij}}$ and using the development in powers,

$$\begin{aligned} n\lambda_{\max} &= \sum_{ij}\left(1+\varepsilon_{ij}+\frac{1}{2}\varepsilon_{ij}^2+\frac{1}{6}\varepsilon_{ij}^3+o(\varepsilon_{ij}^3)\right)(1+d_j) \\ &= \sum_{ij}\Bigg(1+\varepsilon_{ij}+\frac{1}{2}\varepsilon_{ij}^2+\frac{1}{6}\varepsilon_{ij}^3+d_j+d_j\varepsilon_{ij} \\ &\qquad +\frac{1}{2}d_j\varepsilon_{ij}^2+\frac{1}{6}d_j\varepsilon_{ij}^3\Bigg)+o(\varepsilon^3) \\ &= n^2+\sum_{ij}\varepsilon_{ij}+\frac{1}{2}\sum_{ij}\varepsilon_{ij}^2+\frac{1}{6}\sum_{ij}\varepsilon_{ij}^3+\sum_{ij}d_j \\ &\qquad +\sum_{ij}d_j\varepsilon_{ij}+\frac{1}{2}\sum_{ij}d_j\varepsilon_{ij}^2+o(\varepsilon^3), \end{aligned}$$

where we have taken into account that the terms $d_j\varepsilon_{ij}^3$ are of order $o(\varepsilon^3)$. Furthermore,

$$\sum_{ij} d_j\varepsilon_{ij} = \sum_j\left(d_j\sum_i \varepsilon_{ij}\right) = 0.$$

Additionally, $\varepsilon_{ij} = -\varepsilon_{ji}$, so that $\sum_{ij}\varepsilon_{ij}^3 = 0$. With all this, we have that

$$\begin{aligned} n\lambda_{\max} &= n^2+0+\frac{1}{2}\sum_{ij}\varepsilon_{ij}^2+0+0+0 \\ &\qquad +\frac{1}{2}\sum_{ij}d_j\varepsilon_{ij}^2+o(\varepsilon^3) \end{aligned}$$

from where

$$\lambda_{\max} = n+\frac{1}{2n}\sum_{ij}\varepsilon_{ij}^2+\frac{1}{2n}\sum_{ij}d_j\varepsilon_{ij}^2+o(\varepsilon^3). \tag{A.12}$$

We can now see that the values d_i are of order $o(\varepsilon)$. Returning to expression (A.8)

$$\begin{aligned} \lambda_{\max}(1+d_i) &= \sum_j e_{ij}(1+d_j) = \sum_j e^{\varepsilon_{ij}}(1+d_j) \\ &= \sum_j (1+\varepsilon_{ij}+o(\varepsilon))(1+d_j) \\ &= \sum_j \left(1+\varepsilon_{ij}+d_j+d_j\varepsilon_{ij}\right)+o(\varepsilon) \\ &= n+\sum_j \underbrace{d_j\varepsilon_{ij}}_{o(\varepsilon)}+o(\varepsilon) = n+o(\varepsilon) \end{aligned}$$

and, therefore, taking into account that on the basis of (A.12) $\lambda_{\max} = n+o(\varepsilon)$, the value of d_i can be approximated as

$$d_i = \frac{n+o(\varepsilon)}{n+o(\varepsilon)} - 1 = o(\varepsilon). \tag{A.13}$$

Thus, the terms $d_j\varepsilon_{ij}^2$ of expression (A.12) are of order $o(\varepsilon^3)$ and we obtain

$$\lambda_{\max} = n+\frac{1}{2n}\sum_{ij}\varepsilon_{ij}^2+o(\varepsilon^3). \tag{A.14}$$

The value of the consistency index will be

$$\mathrm{CI} = \frac{\lambda_{\max}-n}{n-1} = \frac{1}{2n(n-1)}\sum_{ij}\varepsilon_{ij}^2+o(\varepsilon^3) \tag{A.15}$$

and, as the value of the geometric consistency index is given by

$$\frac{1}{(n-1)(n-2)}\sum_{ij}\varepsilon_{ij}^2,$$

we have

$$\mathrm{GCI} = \frac{2n}{n-2}\mathrm{CI}+o(\varepsilon^3). \qquad \square$$

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