Inconsistency of incomplete pairwise comparisons matrices

Dawid Talaga

AGH University of Science and Technology, Kraków, Poland

Abstract

TODO: Abstract, should normally be not longer than 200 words.

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1. Introduction

People have made decisions for ages. Some of them are very simple and come easily but others, more complicated, require deeper analysis. It happens when there are many compared objects, which are complex and the selection criterion is hard to measure clearly. Fortunately, the development of mathematics brought an interesting tool - *The Pairwise Comparisons (PC) Method*. The first case of using the method (in a very simple version) is the election system described by *Ramond Llull* [1] in the thirteenth century. Its rules were based on the fact that the candidates were pairwise compared with each other and the winner was the one who won in the largest number of direct comparisons. The method was reinvented in the eighteenth century by Condorcet and Bord [2] as they proposed it in their voting system. In the twentieth century, the method found the application in the theory of social choice, the main representatives of which were the Nobel prize winners *Keneth Arrow* [3] and *Amartya Sen* [4]. The current shape of the method was influenced by the changes introduced by *Fechner* [Fechner 1966] and then refined by *Thrustone* [5]. However, the breakthrough was the introduction to the method *The Analytic Hierarchy Process (AHP)* by *Saaty* [6], which allowed to compare many more complex objects and create a hierarchical structure.

The *PC* method is based on the assumption that it is not worth comparing all objects at the same time. It is better to compare them in pairs and then gather the results together. Such pairwise comparisons are much more intuitive and natural for a human being. How can one be sure that these judgements are consistent? Or what to do if some comparisons are missing? In such a case, is it worth taking the *PC* method at all?

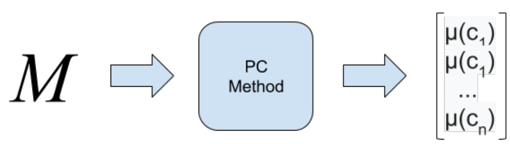
Email address: talagadawid@gmail.com (Dawid Talaga)

The answer to the first question is the concept of inconsistency introduced into the method. This paper tries to answer the next two questions - meaning to examine whether available methods for determining inconsistencies give reliable results when a part of the comparisons are missing. The main aim of this paper is to check which method of calculating the inconsistency is the best in this case. I order to do it, a series of tests was carried out on various known inconsistency indexes, taking into account many different parameters: the matrix size, the amount of missing data, a level of inconsistency. The results of the research are included below.

2. Preliminaries

2.1. The Pairwise Comparisons Method

The PC method is used to choose the best alternative from a set of concepts. However, this goal is achieved by comparing in pairs. A numerical value is assigned to each pair. It not only determines which alternative is preferred but also informs about the intensity of this preference. In this way, the finite set of concepts $C = \{c_{1,}, \ldots, c_n\}$ is transformed into a PC matrix $M = (m_{ij})$, where $m_{i,j} \in R$ i $i, j \in \{1, \ldots, n\}$. It is worth noting that the values m_{ij} and m_{ji} represent the same pair. Therefore, one should expect that $m_{ji} = \frac{1}{m_{ij}}$. If $\forall i, j \in \{1, \ldots, n\} : m_{ij} = \frac{1}{m_{ji}}$, the matrix is called a *reciprocal*. The PC matrix is the basis for calculating the method. It is used in a function $\mu : C \rightarrow R$ that assigns a positive real number to each alternative in the set C. The vector $\mu = [\mu(c_1), \ldots, \mu(c_n)]$ formed in this way is called the weight vector (see Fig.1). It informs which alternative has won.



There are many ways to calculate the vector μ, among the popular ones are the method using the matrix's eigenvalues or the method based on geometric means [7].

2.2. Inconsistency

The second important parameter describing the matrix M is *consistency.* The matrix is consistent if for each i, j, k, where $1 \leq i, j, k \leq n$ occures:

$$m_{ik} = m_{ij} m_{jk} \quad \forall_{i,j,k}. \tag{1}$$

Three numbers that should meet this assumption are called a *triad.*

If the matrix is consistent and the weight vector μ is computed, then for each variables i, j, where $1 \leq i, j \leq n$ meets the equation:

$$a_{ij} = \frac{\mu_i}{\mu_j} \tag{2}$$

In practice, it is very rare for a matrix M to be completely consistent. In the long history of the PC method, many methods have been developed to calculate inconsistencies. Many of them are based directly on the definition of consistency (1), some methods use the eigenvalues of the matrix, others are based on the assumption that each fully consistent matrix fulfills the condition (2).

3. Inconsistency indexes

This section presents sixteen common inconsistency indexes. Their detailed description, including the formulas, is necessary to modify them in the next step so that they can also work for incomplete matrices. Many of them have been described and tested numerically in [8]. In all methods, it is assumed that the PC matrix is reciprocal.

3.1. Saaty index

This is one of the most important and popular indexes and was introduced by *Saaty* [9]. In order to determine inconsistency, the matrix's eigenvalue should be computed. The author used the dependence that the largest eigenvalue of the matrix is equal to its dimension if and only if the given matrix is completely consistent. On this assumption, he based his thoughts and proposed the formula:

$$CI(A) = \frac{\lambda_{max} - n}{n - 1}, \tag{3}$$

where λ_{max} is the principal eigenvalue of the PC matrix and n is its dimension.

3.2. Geometric consistency index

This index on the assumption (2) was proposed by *Craford* and *Williams* [10] and then refined by Aguaròn and Moreno-Jimènez [11]. In this case the priority vector should be calculated using the geometric mean method. Consider (2) one can create a matrix:

$$E = \left[e_{ij} \mid e_{ij} = a_{ij}\frac{w_j}{w_i} \right], \quad i, j = 1, ..., n. \tag{4}$$

The inconsistency index is calculated as follows:

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} ln^2 e_{ij}. \tag{5}$$

3.3. Koczkodaj index

One of the most popular inconsistency indexes was proposed by *Koczkodaj* [12]. It is based directly on the definition of consistency (1). The value of the inconsistency index for one triad (2.2) was defined as:

$$K_{i,j,k} = min\{\frac{1}{a_{ij}} \mid a_{ij} - \frac{a_{ik}}{a_{jk}} \mid, \frac{1}{a_{ij}} \mid a_{ik} - a_{ij}a_{jk} \mid, \frac{1}{a_{jk}} \mid a_{jk} - \frac{a_{ik}}{a_{ij}} \mid\}. \tag{6}$$

This formula has been simplified by *Duszak* and *Koczkodaj* [13] and is given as:

$$K(\alpha, \beta, \gamma) = min\{\mid 1 - \frac{\beta}{\alpha\gamma} \mid, \mid 1 - \frac{\alpha\gamma}{\beta} \mid\}, \quad gdzie\, \alpha = a_{ij}, \beta = a_{ik}, \gamma = a_{jk} \tag{7}$$

Then it was genaralized [13] for $n > 2$. Finally, the inconsistency index has the following form:

$$K = max\{K(\alpha, \beta, \gamma) | 1 \leq i < j < k \leq n\} \tag{8}$$

It is worth noting that not only does the coefficient find the greatest inconsistency but also indicates the place in which it occurs.

3.4. Kazibudzki indexes

Based on the Koczkodaj inconsistency index and observation that $ln(\frac{\alpha\gamma}{\beta}) = -ln(\frac{\beta}{\alpha\gamma})$, *Kazibudzki* proposed several additional inconsistency indexes [14]. Instead of the formula for inconsistency of the triad (7), he introduced two new formulas:

$$LTI(\alpha, \beta\gamma) = \mid ln(\frac{\alpha\gamma}{\beta}) \mid, \tag{9}$$

$$LTI*(\alpha, \beta\gamma) = ln^2(\frac{\alpha\gamma}{\beta}). \tag{10}$$

Based on the above equations, *Kazibudzki* proposed new indexes. The simplest ones use the geometric mean of the triads. Thus, new indexes could be written in the form:

$$MLTI(LTI) = \frac{1}{n} \sum_{i=1}^{n} [LTI_i(\alpha, \beta\gamma)], \tag{11}$$

$$MLTI(LTI*) = \frac{1}{n} \sum_{i=1}^{n} [LTI*_i(\alpha, \beta\gamma)]. \tag{12}$$

After further research *Kazibudzki* introduces another inconsistency index [15], again based on (10). It was defined as $CM(LTI*) = \frac{MEAN[LTI*(\alpha,\beta,\gamma)]}{1+MAX[LTI*(\alpha,\beta,\gamma)]}$. Hence,

$$CM(LTI*) = \frac{\frac{1}{n}\sum_{i=1}^{n}[LTI*_i(\alpha, \beta, \gamma)]}{1 + max\{LTI*_i(\alpha, \beta, \gamma)\}}. \tag{13}$$

3.5. Index of deteminants

This index was proposed by *Pelaez* and *Lamata* [16] and is also based on the concept of triad. The authors noticed that PCM matrices can be construct on the basis of triads. Their determinant is closely related to the consistency of the matrix.

For every triad (a_{ik}, a_{ij}, a_{jk}) one can build a matrix in the form:

$$T_{ijk} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ \frac{1}{a_{ij}} & 1 & a_{jk} \\ \frac{1}{a_{ik}} & \frac{1}{a_{jk}} & 1 \end{pmatrix}, \quad gdzie\, i < j < k. \tag{14}$$

The determinant of this matrix is:

$$det(A) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2. \tag{15}$$

If the matrix is fully consistent, then $det(A) = 0$, else $det(A) > 0$. Based on the above considerations, the authors introduced the new inconsistency index that can be formulated as follows:

$$CI* = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2\right). \tag{16}$$

3.6. Kułakowski and Szybowski indexes

Kułakowski and *Szybowski* proposed two further inconsictency indexes [17], which are also based on triads. They use the fact that the number of triads that can be found in a PCM matrix is

$$\binom{n}{3} = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{6}. \tag{17}$$

The index is formulated as follows:

$$I_1 = \frac{6\sum_{t\in T} K(t)}{n(n-1)(n-2)}, \tag{18}$$

where $K(t)$ is the *Koczkodaj* index for triad $t = (\alpha, \beta, \gamma)$ of the set of all triads T.

The second inconsistency index is similar:

$$I_2 = \frac{6\sqrt{\sum_{t\in T} K^2(t)}}{n(n-1)(n-2)}. \tag{19}$$

indexes can be combined with each other to create new coefficients. In this way *Kułakowski* and *Szybowski* proposed two new indexes. The first one is based on (8) and (18). This index allows to choose what effect on the result should the greatest inconsistency found have and what the average inconsistency of all triads. The new inconsistency index looks as follows:

$$I_\alpha = \alpha K + (1-\alpha)I_1, \tag{20}$$

where $0 \leq \alpha \leq 1$.

The second index expands the first one by (19):19

$$I_{\alpha,\beta} = \alpha K + \beta I_1 + (1-\alpha-\beta)I_2. \tag{21}$$

3.7. Harmonic consistency index

Index introduced by *Stein* and *Mizzi* and it presents a completely new method of inconsistency counting [18]. At the beginning it requires the creation of an auxiliary vector $s = (s_1, ..., s_n)^T$, where n is the dimension of the matrix A, for which the index will be calculated. Each element of the vector s is the sum of values in one column of the matrix A. Hence,

$$s_j = \sum_{i=1}^{n} a_{ji} \quad \forall j. \tag{22}$$

The authors proved that if the matrix A is consistent, then $\sum_{j=1}^{n} s_j^{-1} = 1$. The formula for the mean harmonic looks as follows [19]:

$$HM = \frac{n}{\sum_{j=1}^{n} \frac{1}{s_j}}. \tag{23}$$

The final formula for inconsistency index was obtained by normalizing the above equation (23):

$$HCI = \frac{(HM(s) - n)(n+1)}{n(n-1)}. \tag{24}$$

3.8. Golden and Wang index

This index was introduced by *Golden* and *Wang* [20]. It assumes that the priority vector was calculated using the geometric mean method, then normalized to add up to 1. In this way vector $g* = [g_{1,}^*, ..., g_n^*]$ was obtained, where n is the dimension of the matrix A. The next step is to normalize each column of the matrix A. After this, the sum of the elements of each column in matrix A is 1. The obtained matrix is marked with the symbol A^*. The inconsistency index is defined as follows:

$$GW = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mid a_{ij}^* - g_i^* \mid . \tag{25}$$

3.9. Salo and Hamalainen index

The index proposed by *Salo* and *Hamalainen* [21, 14] uses the definition of inconsistency (1), however it requires the creation of an auxiliary matrix, in which each element is the smallest and largest discrepancy from consistency based on formula (1). The index takes all triads into account:

$$R = (r_{ij})_{nxn} = \begin{pmatrix} [\underline{r}_{11}, \overline{r}_{11}] & \dots & [\underline{r}_{1n}, \overline{r}_{1n}] \\ \vdots & \ddots & \vdots \\ [\underline{r}_{n1}, \overline{r}_{n1}] & \dots & [\underline{r}_{nn}, \overline{r}_{nn}] \end{pmatrix}, \tag{26}$$

where $\underline{r_{ij}} = min\,\{a_{ik}a_{kj} \mid k = 1, \dots, n\}$, $\overline{r_{ij}} = max\,\{a_{ik}a_{kj} \mid k = 1, \dots, n\}$ and n is the dimension of the tested matrix A. A numerical example was presented in [19]. Based on the resulting matrix R, the authors proposed the following inconsistency index:

$$CM = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\overline{r}_{ij} - \underline{r}_{ij}}{(1 + \overline{r}_{ij})\left(1 + \underline{r}_{ij}\right)}. \tag{27}$$

3.10. Cavallo and D'Apuzzo index

The authors Cavallo and D'Apuzzo based their index on triads but they conducted studies on a new path, generalizing them for linear, ordered abelian groups [22, 23]. Thanks to this, the index can be used also with other relations [8]. Index for relation max can be presented in the form of a formula:

$$I_{CD} = \prod_{i=1}^{n-2} \prod_{j=i+1}^{n-2} \prod_{k=j+1}^{n} \left(max \left\{ \frac{a_{ik}}{a_{ij}a_{jk}}, \frac{a_{ij}a_{jk}}{a_{ik}} \right\} \right)^{\frac{1}{\binom{n}{3}}}. \tag{28}$$

3.11. Relative error

This index, proposed by Barzaili [24], requires calculation of the weight vector using the arithmetic mean method for each row and creation of two additional matrices. Thus, the weight vector is $w_i = \frac{1}{n} \sum_{j=1}^{n} a_{ij}$, where n is the dimension of the matrix. The two auxiliary matrices are calculated according to the formulas:

$$C = (c_{ij}) = (w_i - w_j) \tag{29}$$

$$E = (e_{ij}) = (a_{ij} - c_{ij}) \tag{30}$$

Ultimately, the formula for the Relative error is following:

$$RE(A) = \frac{\sum_{ij} e_{ij}^2}{\sum_{ij} a_{ij}^2}. \tag{31}$$

4. Inconsistency indexes for incomplete matrices

There are no inconsistency indexes for incomplete matrices. Those presented in chapter (3) could be use in such cases. It requires usually a slight modification of the index definition or calculation only for selected data. Below the ways in which the examined indexes have been adjusted to be able to deal with incomplete matrices are presented.

Saaty index: The input matrix is modified using the method proposed by Harker [25]. It means that values $c + 1$, where c is the number of non-empty elements in a given row, are places on the diagonal.

Geometric consistency index: During calculating the weight vector by the geometric mean, empty values are omitted. Additionally, in the formula (5) only non-empty elements e_{ij} are used. The reason for this exclusion is that the domain of the logarithmic function is R^+.

Koczkodaj index, Kazibudzki indexes, Index of determinants: Only those triads which do not contain empty values are taken into account.

Kułakowski and Szybowski indexes: Only those triads which do not contain empty values are taken into account. In addition, the number of triads is no longer calculated according to the formula (17) but determined directly by counting the number of triads.

Harmonic consistency index: No modification.

Golden and Wang index: During calculating the weight vector by the geometric mean empty values are omitted.

Salo and Hamalainen: No modification.

Cavallo and D'Appuzo: During calculating the product (28) empty values are omitted.

Relative index: No modification.

5. Discussion

The presented inconsistency indexes have been tested. Their aim was to select those indexes which will give reliable results for incomplete matrices. Therefore, it was decided that the measure of the indexes' quality would be a relative error (expressed as a percentage), which took into account the value of the index for a full, inconsistent matrix and the value of the index for the same matrix after partial decomposition. To be sure that the results were fair, all indexes were tested on the same set of matrices. The different sizes of the matrices, the levels of incompleteness and the levels of inconsistency were taken into account. Then, in order to compare the indexes easily and to select the best ones, the results were averaged using the arithmetic mean. While building the algorithm to solve the problem [15] was used.

The algorithm of test of inconsistency indexes:.

1. Randomly generate a vector $w = [w_1, ..., w_n]$ and a consistent PCM matrix associated with it $PCM = (m_{ij})$, where $m_{ij} = \frac{w_i}{w_j}$.
2. Disrupt the matrix by multiplying its elements (excluding the diagonal) by the value of d, randomly selected from the range $\left(\frac{1}{x}, x\right)$.
3. Replace values m_{ij}, where $i < j$ by values m_{ji}.
4. Calculate values of index with all methods for the created matrix.
5. Remove some values from the matrix by removing some of values. The level of incompleteness should be $g\%$.
6. Calculate the values of inconsistencies by all methods for the decomposed matrix.
7. Calculate the relative error for each index.
8. Repeat steps 1 to 10 X_1 times.
9. Calculate the average relative error for each inconsistency index for the PCM matrix.
10. Repeat steps 1 to 10 X_2 times.
11. Calculate the average relative error for each index by averaging the values obtained in step 9.

The above algorithm was carried out for values $X_1 = 100$, $X_2 = 100$. Tests were started for values d in the range $(1.1, 1.2, ..., 4)$ and then the results were averaged. It means that the average relative error of one index was calculated on the basis of 4000 matrices, each of which decomposed randomly 100 times. It gave together 400000 tests how good the index was.

In addition, tests were carried out for various sizes of matrices. The results are divided into two parts:

1. A constant degree of incompleteness, different size of the matrix.
2. Different degrees of incompleteness, constant size of the matrix.

The aim of such a division is to pay attention to how the inconsistency indexes behave when the size of the matrix and the degree of incompleteness are changing. The results of the research are presented below.

Relative error of inconsistency indexes for incomplete matrices with constant degrees of incompleteness $g = 15\%$ and variable matrix size.

Index	n=4	n=7	n=8	n=10	n=15	mean
saaty	33,41	19,82	18,78	19,16	17,37	21,71
geometric	616,68	124,73	77,94	68,62	39,13	185,42
koczkodaj	13,86	3,69	2,14	1,62	0,80	**4,42**
kazibudzkiLTI1	24,80	10,21	6,62	4,97	2,73	9,87
kazibudzkiLTI2	42,31	17,93	11,88	9,03	5,03	17,24
kazibudzkiCMLTI2	35,40	17,07	13,26	11,20	6,81	16,75
pelaeLamata	44,65	19,90	13,46	10,36	5,84	18,84
kulakowskiSzybowski	20,34	7,68	4,88	3,63	1,96	7,70
kulakowskiSzybowski2	44,61	26,05	27,12	29,64	28,46	31,18
kulakowskiSzybowskiIa	16,47	5,18	3,09	2,27	1,16	5,63
kulakowskiSzybowskiIab	17,40	4,89	2,81	2,04	1,01	5,63
harmonic	9 573,02	1 577,49	1 127,33	1 066,35	866,00	2 842,04
goldenWang	115,92	54,37	43,90	43,16	36,26	58,72
saloHamalainen	381,57	205,06	176,11	160,06	136,55	211,87
cavalloDapuzzo	16,94	6,85	4,46	3,36	1,87	6,70
relativeError	1 792,64	226 313,60	746,21	100,87	20,42	45 794,75

Relative error of inconsistency indexes for incomplete matrices with varying degrees of incompleteness and constant matrix size $n = 8$.

Index	g=4%	g=7%	g=14%	g=25%	g=50%	mean
saaty	4,71	9,40	18,78	32,89	65,56	26,27
geometric	23,60	48,44	86,61	135,68	207,99	100,46
koczkodaj	0,48	0,99	2,17	4,52	16,41	**4,92**
kazibudzkiLTI1	2,90	4,31	6,64	10,05	23,09	9,40
kazibudzkiLTI2	5,12	7,71	11,91	18,08	40,77	16,72
kazibudzkiCMLTI2	5,16	8,05	13,34	22,03	61,16	21,95
pelaeLamata	5,73	8,72	13,52	20,54	45,64	18,83
kulakowskiSzybowski	2,17	3,18	4,91	7,43	17,30	7,00
kulakowskiSzybowski2	5,90	12,22	27,12	56,71	202,27	60,84
kulakowskiSzybowskiIa	1,20	1,88	3,12	5,13	13,97	5,06
kulakowskiSzybowskiIab	1,00	1,65	2,84	4,74	12,97	4,64
harmonic	291,74	544,60	1 152,26	1 962,25	3 995,58	1 589,29
goldenWang	14,23	25,23	46,18	68,83	98,24	50,54
saloHamalainen	88,40	137,70	180,17	182,54	148,74	147,51
cavalloDapuzzo	1,95	2,91	4,46	6,81	16,11	6,45
relativeError	18,99	20,81	206,74	68,98	1 056,96	274,50

Analyzing the above results, one can draw several conclusions.

Certainly, the tests managed to show that the error increases with a groth of the level of incompleteness, which should not be a big surprise. At the same time, it decreases when the size of the matrix increases. However, the most important question was about which indexes cope well with incomplete matrices. The Koczkodaj index (3.3) won in 9 out of 10 tests and its average error in both cases turn out to be the lowest (below 5%). The next places are occupied by two indexes introduced by Kułakowski and Szybowski (20, 21) and Cavallo and D'Apuzzo index (3.10). It is worth noting that all of these indexes are based on triads.

A question about what makes the Koczkodaj index giving such good results and whether is it worth using, may arise. One should return to the definition of this index (8) and notice that it is equal to the value of the most inconsistency triad. Therefore, if the level of incompleteness is low, there is a good chance that after deletion some values from the matrix and recalculating the index the value of it will not change at all. It will only change if the element included in the most inconsistent triad is removed. However, in many cases, the examination of the full matrix and th matrix after incompletion give exactly the same results (the error is 0%). This is the only index of this kind among those presented in the paper.

If one uses the Koczkodaj index, one may be worried that the removed comparison belonged to the most inconsistent triad. In such a case, it is difficult to predict what error will be contained in the index of the incomplete matrix. It seems that one should pay particular attention to the indexes proposed by Kułakowski and Szybowski. First of them (18) averages the inconsistencies of the triads. Therefore, it is safe and gives good results (in both tests it took the sixth place and achieved an error below 8%). From this perspective, another index suggested by the same authors (20) turns out to be very interesting. In the tests it took the third place. It has the parameter α allowing to determine the effect of the greatest inconsistency of the triad (α), and the average $(1 - \alpha)$. In the tests carried out, the parameter α was 0.4.

6. Summary

TODO

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