CS 2050 Fall 2023 Homework 5

Due: October 6 @ 11:59 PM

Released: September 29

This assignment is due at **11:59 PM EDT** on **Friday, October 6, 2023**. Submissions submitted at least 24 hours prior to the due date will receive 2.5 points of extra credit. On-time submissions receive no penalty. You may turn it in one day late for a 10-point penalty or two days late for a 25-point penalty. Assignments more than two days late will NOT be accepted. We will prioritize on- time submissions when grading before an exam. You should submit a typeset or *neatly* written PDF on Gradescope. The grading TA

should not have to struggle to read what you've written; if your handwriting is hard to decipher, you will be required to typeset your future assignments. Illegible solutions will be given 0 credit. A 5-point penalty will occur if pages are incorrectly assigned to questions when submitting. You may collaborate

with other students, but any written work should be your own. Write the names of the students you work with on the top of your assignment. Always justify your work, even if the problem doesn't specify it. It can

help the TA's to give you partial credit. Author(s): Kavya Selvakumar, Anthony Zang, Taiki Aiba, Atharva

Gorantiwar

1. (2 points each) Determine whether the following statements are true or false. You do not need to justify your answer.

   (a) $\emptyset \subset \emptyset$

   Solution: False.

   (b) $a \in \{a\}$

   Solution: True.

   (c) $\emptyset \subseteq \{a\}$

   Solution: True.

   (d) $\emptyset \in \{a\}$

   Solution: False.

   (e) $\{\emptyset\} \in \{\emptyset\}$

   Solution: False.

   (f) $\{\{\emptyset\}\} \subseteq \{\{\emptyset\}, \{\emptyset\}\}$

   Solution: True.

2. (3 points each) Determine the cardinality of the following sets. You do not need to justify your answer.

   (a) $\emptyset$

   Solution: 0.

   (b) $\{T\}$

   Solution: 1.

   (c) $\{\emptyset, \{\emptyset\}\}$

   Solution: 2.

   (d) $\{a, \{a\}, \{a, \{a\}\}\}$

   Solution: 3.

3. (3 points each) Find the power set of each of the following sets. You do not need to justify your answer.

   (a) $\{a, b\}$

$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

(b) $\{1, \{2\}, 3\}$

$\{\emptyset, \{1\}, \{\{2\}\}, \{3\}, \{1, \{2\}\}, \{1, 3\}, \{\{2\}, 3\}, \{1, \{2\}, 3\}\}$

(c) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$

$\{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$

4. (2 points each) Find the truth set of each of the following predicates, where the domain is the set of integers. You do not need to justify your answer.

(a) $Q(x): x^2 < 16$

$\{-3, -2, -1, 0, 1, 2, 3\}$

(b) $S(x): 3x - 5 = 9$

$\{\emptyset\}$

5. (3 points each) Let $A = \{1, 2, 3, 4\}$ and $B = \{y, z\}$. Find the following sets. You do not need to justify your answer.

(a) $A \times B$

$\{\{1, y\}, \{1, z\}, \{2, y\}, \{2, z\}, \{3, y\}, \{3, z\}, \{4, y\}, \{4, z\}\}$

(b) $B \times A$

$\{\{y, 1\}, \{y, 2\}, \{y, 3\}, \{y, 4\}, \{z, 1\}, \{z, 2\}, \{z, 3\}, \{z, 4\}\}$

6. (3 points each) Let $A = \{a, b, c, d\}$ and $B = \{a, b, c, d, e, f, g, h, i\}$, and let U represent a universal set $\{a, b, c, d, e, f, g, h, i, j, k, l\}$. Find the following sets. You do not need to justify your answer.

   (a) $A \cup B \cup \emptyset$

   $\{a, b, c, d, e, f, g, h, i\}$

   (b) $A - B$

   $\{\emptyset\}$

   (c) $B - A$

   $\{e, f, g, h, i\}$

   (d) A^C

   $\{e, f, g, h, i, j, k, l\}$

7. (2 points each) Suppose that A is the set of Swifties at Georgia Tech and B is the set of students who are majoring in computer science (CS). Express each of these sets in terms of A and B. This should be done with set operations, not set-builder notation. You do not need to justify your answer.

   (a) The set of Swifties majoring in CS at Georgia Tech.

   $A \cap B$

   (b) The set of Swifties at Georgia Tech who are not majoring in CS.

   $A - B$

   (c) The set of students at Georgia Tech who are not Swifties and not majoring in CS.

   $\overline{A} \cap \overline{B}$

   (d) The set of student at Georgia Tech who are either Swifties or majoring in CS.

   $A \cup B$

8. (4 points each) Prove or disprove the following statements, for all sets A, B, and C such that A, B, and C are pairwise disjoint sets. You may use set-builder notation and laws of logical equivalence; you are not permitted to use set identities.

   (a) $A \cup (B \cup A) = A$

I proceed with a direct proof to prove $A \cup (B \cup A) = A$.

$Statement$	(Reason)
1) $A \cup (B \cup A) = \{x \mid x \in A \cup (B \cup A)\}$	(1. given)
2) $A \cup (B \cup A) = \{x \mid x \in A \vee x \in (B \cup A)\}$	(2. defintion of union)
3) $A \cup (B \cup A) = \{x \mid x \in A \vee (x \in B \vee x \in A)\}$	(3. definition of union)
4) $A \cup (B \cup A) = \{x \mid (x \in A \vee x \in A) \vee x \in B\}$	(4. commutative and associative law)
5) $A \cup (B \cup A) = \{x \mid x \in A \vee x \in B\}$	(5. idempotent law)
6) $A \cup (B \cup A) = \{x \mid x \in (A \cup B)\}$	(6. definition of set)
7) $A \cup (B \cup A) = A \cup B$	(7. definition of set and set builder)

We have proven that $A \cup (B \cup A) = A$ is equal to $A \cup B$. Since it is given that the two sets are disjoint this means that $A \cup B$ is not equal to A. Thus, we have direcly proven that $A \cup (B \cup A)$ is not equal to A.

(b) $\mathbb{P}(A) \cup \mathbb{P}(B) = \mathbb{P}(A \cup B)$

I proceed with a direct proof showing that both sides are subsets of each other. Let A and B be sets.

$Statement$	(Reason)
1) $P(A \cup B)$	(1. Given)
2) $Let x \in P(A \cup B)$	(2. def of arbitrary element)
3) $x \subseteq A \cup B$	(3. def of power set)
4) $x \subseteq A \vee x \subseteq B$	(4. def of union)
5) $x \in P(A) \vee x \in P(B)$	(5. def of power set)
6) $x \in P(A) \cup P(B)$	(6. def of union)
7) $P(A \cup B) \subseteq P(A) \cup P(B)$	(7. def of subset)

This, by assuming $x \in P(A \cup B)$, we have proved directly that $P(A \cup B) \subseteq P(A) \cup P(B)$.

Statement	(Reason)
1) $P(A) \cup P(B)$	(1. Given)
2) $Let x \in P(A) \cup P(B)$	(2. def of arbitrary element)
3) $x \in P(A) \vee x \in P(B)$	(3. def of union)
4) $x \subset A \vee x \in P(B)$	(4. def of power set)
5) $x \subseteq A \vee x \subseteq B$	(5. def of power set)
6) $x \subseteq (A \cup B)$	(6. definition of union)
7) $x \subseteq P(A \cup B)$	(7. def of power set)
8) $P(A) \cup P(B) \subseteq P(A \cup B)$	(8. def of subset)

This, by assuming $x \in P(A) \cup P(B)$, we have proved directly that $P(A) \cup P(B) \subseteq P(A \cup B)$. Because both the sets $P(A) \cup P(B)$ and $P(A \cup B)$ are subsets of each other, we have directly proven that they are equal.

(c) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

We proceed with a direct proof to prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$ assuming A and B are sets.

Statement	(Reason)
1) $\overline{A \cup B} = \{x \mid x \in \overline{A \cup B}\}$	(1. Given)
2) $\overline{A \cup B} = \{x \mid x \notin A \cup B\}$	(2. def of complement)
3) $\overline{A \cup B} = \{x \mid \neg(x \in A \cup B)\}$	(3. def of not element in)
4) $\overline{A \cup B} = \{x \mid \neg(x \in A \vee x \in B)\}$	(4. def of union)
5) $\overline{A \cup B} = \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$	(5. De Morgan's Law)
6) $\overline{A \cup B} = \{x \mid x \notin A \wedge \neg(x \in B)\}$	(6. def of not element in)
7) $\overline{A \cup B} = \{x \mid x \notin A \wedge x \notin B\}$	(7. def of not element in)
8) $\overline{A \cup B} = \{x \mid x \in \overline{A} \wedge x \notin B\}$	(8. def of complement)
9) $\overline{A \cup B} = \{x \mid x \in \overline{A} \wedge x \in \overline{B}\}$	(9. def of complement)
10) $\overline{A \cup B} = \overline{A} \cap \overline{B}$	(10. def of intersection)

This, we have directly proven that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

9. (5 points) List all the elements of $\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset))) - \mathbb{P}(\mathbb{P}(\emptyset)) \cup \mathbb{P}(\emptyset) - \emptyset$. You do not need to justify your answer.

$P(\emptyset) = \{\emptyset\}$
$P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
$P(P(\emptyset)) \cup P(\emptyset) = \{\emptyset, \{\emptyset\}\}$
$P(P(P(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
$P(P(P(\emptyset))) - P(P(\emptyset)) \cup P(\emptyset) - \emptyset = \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

10. (8 points) Prove that: $\{5a+6b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$

In order to prove that $\{5a+6b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$, we will prove that each are subsets of each other.

Statement	(Reason)
1)$\{5a+6b \mid a, b \in \mathbb{Z}\}$	(1. Given)
2)$j = 5a+6b$ for some j $\in \mathbb{Z}$	(2. closure of add and mult in Z)
3)Since j $\in \mathbb{Z}$, $\{5a+6b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$	(3. def of subset and combine (1) and (2))
Statement	(Reason)
1)$j = 5a+6b$for some j $\in \mathbb{Z}$	(1. Given)
2)Because $j \in \mathbb{Z}$, $\mathbb{Z} \subseteq \{5a+6b \mid a, b \in \mathbb{Z}\}$	(2. def of subset with (1))

Because both sets are subsets of each other, they are equivalent and thus we have directly proven that $\{5a+6b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$.

11. (2 points each) For each of the following maps, determine whether f is onto, one-to-one, and/or a function. $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, $C = \{1, 2, 3\}$, and $D = \{1, 2, 3, 4\}$. You do not need to justify your answer.

(a) $f : A \to C$, $f(a) = 2, f(b) = 3, f(c) = 1$

Solution: Function, one to one, and onto.

(b) $f : A \to D$, $f(a) = 1, f(b) = 2, f(c) = 3$

Solution: One-to-one and a function.

(c) $f : B \to C$, $f(a) = 3, f(b) = 2, f(c) = 2, f(d) = 1$

Solution: Function, onto.

(d) $f : B \to D$, $f(a) = 1, f(a) = 2, f(b) = 3, f(b) = 4$

Solution: Not a function, onto.

12. (2 points each) For each of the following, find the inverse function f^{-1} or state why it does not exist. You do not need to justify your answer.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x}$

Solution: The inverse doesn't exist because $f(0)$ is undetermined.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 5x - 6$

$f^-1(x) = \frac{x+6}{5}$

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

Solution: Inverse is not a function as a absolute value will cause the same input value to have multiple outpue values.