CS 4320, Question 1

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1. Assume $X \to Y$ and $X \to Y$
$(X)^+ = \{X, Y\}$
$X \in (X)^+$ by Armstrong's axiom of reflexivity (i.e. since $X \subseteq X, X \to X$)
$Y \in (Y)^+$ by the first functional dependency.
$(Y)^+ = \{X, Y\}$
$Y \in (Y)^+$ by Armstrong's axiom of reflexivity (i.e. since $Y \subseteq Y, Y \to Y$)
$X \in (Y)^+$ by the second functional dependency.
Therefore, $(Y)^+ = \{X, Y\} = (X)^+$

For the other direction, assume $(Y)^+ = (X)^+$
$\forall x \in (X)^+, x \in (Y)^+$
Since the closure of $(Y)^+$ is the set of all attributes k such that $(Y \to k) \in F_y^+$, $Y \to x, \forall x \in (X)^+$.
Hence, $Y \to X$.
$\forall y \in (Y)^+, y \in (X)^+$
Since the closure of $(X)^+$ is the set of all attributes k such that $(X \to k) \in F_x^+$, $X \to y, \forall y \in (Y)^+$.
Hence, $X \to Y$.

(b) Assume $X \to Y$ and consider a relation R with attributes A.
Then $X \to XY$ by Armstrong's axiom of reflexivity (i.e. since $X \subseteq XY, X \to XY$)
Since a superkey refers to any $S \subseteq A$ such that $S \to A$, X is a superkey of XY, which is logically equivalent to the statement that X is a superkey of $\pi_{XY}(R)$ since $\pi_{XY}(R)$ is a relation with the attributes XY.

For the other direction, assume X is a superkey of $\pi_{XY}(R)$, which refers to a new relation with the attributes XY. This means that X is a superkey of XY.
Since a superkey refers to any $S \subseteq A$ such that $S \to A$, $X \to XY$. This can be broken into $X \to X$ and $X \to Y$. Hence, $X \to Y$ is proven.

(c) Assume $Z \to Y$ holds on R.
To prove $R = \pi_X(R) \bowtie \pi_Y(R)$, we have to prove:
(i) $R \subseteq \pi_X(R) \bowtie \pi_Y(R)$

(ii) $\pi_X(R) \bowtie \pi_Y(R) \subseteq R$
To prove (i),
$\forall t \in R, t[X] \in \pi_X(R)$
$t[Y] \in \pi_Y(R)$
$t = t[X] \cup t[Y]$
Since $Z \neq \emptyset$, 2 projections of an attribute of t will appear in $\pi_X(R)$ and $\pi_Y(R)$, and hence they will be matched in the join.
Therefore, $t \in \pi_X(R) \bowtie \pi_Y(R)$.

To prove (ii),
$\forall t \in \pi_X(R) \bowtie \pi_Y(R), \exists t1, t2 \in R$ such that $t[Z] = t1[Z]$ and $t[Z] = t2[Z]$.
$Z \rightarrow Y$ holds on R implies that if $t1[Z] = t2[Z] \Rightarrow t1[Y] = t2[Y]$.
Since $t1[Z] = t[Z] = t2[Z]$ and t1 and t2 are matched in a join to form t, this implies that $t[Y] = t1[Y]$.
$\Rightarrow t = t1$
Since $\forall t \in \pi_X(R) \bowtie \pi_Y(R) \exists t1 \in R$ such that $t = t1$, $\pi_X(R) \bowtie \pi_Y(R) \subseteq R$.

Hence, $R = \pi_X(R) \bowtie \pi_Y(R)$.