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A simple iterative method for the valuation of American options§

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We introduce a simple iterative method to determine the optimal exercise boundary for American options, allowing us to compute the values of American options and their Greeks quickly and accurately. Following Little, Pant and Hou's idea (2000), we derive a new equation for the optimal exercise boundary containing a single integral. The proposed method is an iterative numerical method for finding its solution. Using it, we can calculate the entire optimal exercise boundary in a non-time-recursive way, in contrast to conventional methods. Extensive numerical results indicate that our method is computationally more efficient than the methods currently available, particularly for hedge ratios.

Keywords: Option pricing; American option; Early exercise boundary; Numerical approach; Iterative method

JEL Classification: C, C6, C63, G, G1, G13

1. Introduction

Since the seminal papers of Black and Scholes (1973) and Merton (1973), it is well known that European options have closed-form valuation formulas, while the majority of American-style options do not. This is mainly because the possibility of the early exercise of American options leads to complications for analytic calculation. Therefore, researchers and practitioners have given much attention to the development of analytical approximation formulas such as those suggested by Parkinson (1977), Johnson (1983), Geske and Johnson (1984), MacMillan (1986), Barone-Adesi and Whaley (1987), Omberg (1987), Bjerksund and Stensland (1993), Broadie and Detemple (1996), Ju (1998), and numerical methods such as those of Brennan and Schwartz (1976, 1978), Boyle (1977), Hull and White (1990), and Longstaff and Schwartz (2001).

McKean (1965) and Van Moerbeke (1976) lay a foundation for American option valuation; they use it to solve a free boundary problem with one free boundary changing in time to maturity (generally called an *optimal exercise boundary*). Following their work, there have been various attempts to value American options more quickly and accurately. Historical reviews of such attempts are found in Karatzas and Shreve (1998) and Barone-Adesi (2005).

As Barone-Adesi (2005) points out, there is some remarkable work that gives new impetus to the quest for the valuation of American-style options. Jamshidian (1990), Kim (1990), Jacka (1991), and Carr *et al.* (1992) provide us with a valuation formula for American options in integral form as a function of the optimal exercise boundary. They postulate that the underlying asset prices follow a lognormal diffusion process. Moreover, Kim (1990) finds an implicit form of integral equation for the optimal exercise boundary. He shows that once the optimal exercise boundary is determined, it is simple to compute the value of American options. Therefore, the important task in pricing American options is determining the optimal exercise boundary efficiently. Kim (1990)

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also suggests the numerical method of solving a number of integral equations equal to the number of discretised time intervals.

We focus on solving the integral equation suggested by Kim (1990). More specifically, we develop a simple iterative method to solve it. Following Little *et al.*'s (2000) idea, we show that the integral equation can be converted into a numerical functional form with respect to the optimal exercise boundary, and develop an iterative method to calculate the boundary as a fixed point of the functional. In contrast, the method suggested by Kim (1990) is to repeat a numerical method solving the implicit integral equation at each point time-recursively. However, his method requires a relatively long computation time and suffers from cumulative numerical errors that are hard to control (see Barone-Adesi 2005). Kim (1990) admits that his method requires improvement with respect to computational efficiency.

The majority of numerical methods for pricing American options, including the binomial method due to Cox *et al.* (1979) and the finite difference method (FDM) by Brennan and Schwartz (1976), are time-recursive; their basic idea is to discretise the lifetime of an option and find its optimal exercise boundary backward in time. Since such time-recursive methods execute calculations repeatedly for every (discretised) time step, they require long computation times and cumulative (thus relatively large) pricing errors, particularly for long-lived options.

One may think that long computation times could be reduced by exploiting approximation methods. However, the methods by Geske and Johnson (1984) and Huang *et al.* (1996) are unlikely to overcome the limitations of time-recursive methods. Approximation methods must be utilised repeatedly to calculate the value of American options with different times to expiration even if all other conditions remain the same. Therefore, inevitably, a great deal of computation time is required to price many such options. In contrast, the approximation methods suggested by Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993) provide relatively short computation time but low accuracy. The randomisation method by Carr (1998) provides a relatively fast algorithm for pricing an American put option without sacrificing accuracy. However, it is not easy to reduce computing error, especially in calculating sensitivity coefficients (Greeks), since it calls for a numerical scheme (such as the Richardson extrapolation method) to obtain a limit value for the series of approximate option values. The result by Sullivan (2000), who utilises a Gaussian quadrature method to approximate the value of an American option, discusses some of the drawbacks of Carr's method. Finally, Ju (1998) proposes a method of American option valuation that involves approximating its early exercise boundary with connecting pieces of exponential functions. In the model, it is necessary to determine several coefficients repeatedly whenever American options with different times to maturity are considered.

Recently, Zhu (2006) has found an exact and explicit solution to the Black–Scholes equation for American put options. He presents the solution as a Taylor series expansion with infinitely many terms, where each term contains three single integrals and two double integrals. His work is an excellent exercise in option pricing theory; however, it seems difficult to implement his solution numerically. The infinite sum with two double integrations is likely to produce many computation errors, so exploiting Zhu's solution for an American option valuation does not seem to be superior to Carr's approximation method.

The main contribution of this paper is to introduce an *iterative* method to calculate the optimal exercise boundary in a non-time-recursive way. We borrow the idea of Little *et al.* (2000), who suggest a new representation of the early exercise boundary containing only a single (or one-dimensional) integral. We show how a slight modification of the representation has led us to develop a simple but powerful iterative method for pricing American options.

The method in this paper provides us with a relatively fast and accurate algorithm for calculating the optimal exercise boundary for some time intervals (not a time spot) simultaneously. Thus, the value of American options and their hedge ratios (Greeks) can be calculated directly by the closed-form formulae containing only one summation. Therefore, if the optimal exercise boundary for an American option is calculated once, this method gives relatively quicker and more accurate results for the values of American options with the same strike price and underlying asset for any time to expiration and any underlying asset price, compared with the methods currently available. When it comes to the computation time for finding hedge ratios, our method shows much better performance than other methods. So, one can say that it could be a powerful tool for practitioners if the proposed method is successfully used for hedging purposes in relation to American-style options.

This paper is organised as follows. First, we review the *early exercise premium representation* suggested by Kim (1990) and the corresponding valuation formula for an American put option. Subsequently, the iterative method is explained, and its convergence, computing speed and accuracy are explored. We apply the method to the valuation of other American-style options, and then conclude in the final section.

2. The valuation formula for American puts

In this section, we present the valuation formula for American put options introduced by Kim (1990). His early exercise premium representation has important implications for our iterative method.

Consider an American put option written on an underlying asset (stock) with exercise price K and maturity T, and denote its value at time $t = T - \tau$ by $P(S, \tau)$, where S $(0 < S \leq \infty)$ is the asset price and τ $(0 < \tau \leq T)$ is the time to maturity. We employ the usual conditions: markets are perfect and trading occurs continuously, the asset price follows the lognormal diffusion process, S_t satisfying

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}z_t,$$

for positive constants μ (the expected rate of return of the asset) and σ (the volatility of the asset). Here, z_t is a one-dimensional standard Brownian motion defined on a suitable probability space. We assume that the risk-free interest rate is a constant, r.

As seen in previous literature such as McKean (1965), Merton (1973) and Myneni (1992), the value of an American put is considered the solution to a free boundary problem with a parabolic partial differential equation (PDE). Throughout this paper, we assume that the optimal exercise boundary B_τ of the American put is uniquely determined and that all its sample paths are continuous. Then, the free boundary problem is to find the function $P(S,\tau)$, which is $C^{2,1}$ on $(B_\tau,\infty)\times[0,T]$, satisfying the Black–Scholes PDE

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = \frac{\partial P}{\partial \tau},$$

subject to a terminal condition

$$P(S,0)=0 \quad \text{for all } S\geq B_0,$$

and boundary conditions

$$\lim_{S\uparrow\infty}P(S,\tau)=0,\ \ \lim_{S\downarrow B_\tau}P(S,\tau)=K-B_\tau,\ \ \lim_{S\downarrow B_\tau}\frac{\partial P(S,\tau)}{\partial S}=-1, \tag{1}$$

for all $\tau\in(0,T]$.

Kim (1990) derives a valuation formula for American options that contains an optimal exercise boundary as a function of time to expiration, and an implicit-form integral equation with respect to the optimal exercise boundary. The valuation formula for a *live* American put is, for $S>B_\tau$,

$$P(S,\tau)=p(S,\tau)+\int_0^\tau rK\mathrm{e}^{-r(\tau-\xi)}\aleph(-d_2(S,\tau-\xi;\,B_\xi))\mathrm{d}\xi, \tag{2}$$

where $\aleph(\cdot)$ is the *unit normal distribution function* with

$$d_1(S,\tau;\,B)=\frac{\ln(S/B)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$
$$d_2(S,\tau;\,B)=d_1(S,\tau;\,B)-\sigma\sqrt{\tau},$$

and $p(S,\tau)$ stands for the Black–Scholes formula for the European put option:

$$p(S,\tau)=K\mathrm{e}^{-r\tau}\aleph(-d_2(S,\tau;\,K))-S\aleph(-d_1(S,\tau;\,K)).$$

Then, the integral equation for the optimal exercise boundary B_τ should be

$$K-B_\tau=p(B_\tau,\tau)+\int_0^\tau rK\mathrm{e}^{-r(\tau-\xi)}\aleph(-d_2(B_\tau,\tau-\xi;\,B_\xi))\mathrm{d}\xi. \tag{3}$$

We call this the early exercise premium representation with respect to the optimal exercise boundary B_τ. Note that the right-hand side of equation (3) contains a double integral since $\aleph(\cdot)$ is an integral.

Differentiating both sides of equation (2) with respect to S, we obtain the so-called *delta hedging formula*: if $S>B_\tau$,

$$\begin{aligned}\Delta_\tau &\equiv \frac{\partial P(S,\tau)}{\partial S}\\ &= -\aleph(-d_1(S,\tau;\,K))-\frac{rK}{\sqrt{2\pi}\sigma S}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\\ &\quad\times\exp\left\{-r(\tau-\xi)-\frac{(d_2(S,\tau-\xi;\,B_\xi))^2}{2}\right\}\mathrm{d}\xi,\end{aligned} \tag{4}$$

and, if $0<S\leq B_\tau$ and $\Delta_\tau=-1$.†

As a result, we can calculate the value of the live American put $P(S,\tau)$ in equation (2) and the hedge ratio Δ_τ in equation (4) if we can obtain the optimal boundary B_τ from equation (3). The optimal exercise boundary at expiration should be given as $B_0=K$.

3. The iterative method

We introduce a new early exercise premium representation with only a *single* integral by exploiting Little *et al.*'s (2000) idea.‡ When the stock price is below the early exercise boundary, it is optimal to exercise the option. In this case, we can let $P_\tau=K-S_\tau$ in equation (2); hence, we obtain

$$K-S_\tau=p(S_\tau,\tau)+\int_0^\tau rK\mathrm{e}^{-r(\tau-\xi)}\aleph(-d_2(S_\tau,\tau-\xi;\,B_\xi))\mathrm{d}\xi.$$

Now, we substitute S_τ as ϵB_τ with $\epsilon\in(0,\,1]$, then,

$$K-\epsilon B_\tau=p(\epsilon B_\tau,\tau)+\int_0^\tau rK\mathrm{e}^{-r(\tau-\xi)}\aleph(-d_2(\epsilon B_\tau,\tau-\xi;\,B_\xi))\mathrm{d}\xi.$$

By differentiating both sides with respect to ε, we have

$$\begin{aligned}&B_\tau\aleph(d_1(\epsilon B_\tau,\tau;\,K))+\epsilon B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1(\epsilon B_\tau,\tau;\,K)^2\right\}\\ &=\frac{1}{\sigma\sqrt{2\pi\tau}}K\exp\left\{-\left[r\tau+\frac{1}{2}d_2(\epsilon B_\tau,\tau;\,K)^2\right]\right\}\\ &\quad+rK\int_0^\tau\frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}r\\ &\quad\times\exp\left\{-\{r(\tau-\xi)+\frac{1}{2}d_2(\epsilon B_\tau,\tau;\,B_\xi)^2\}\right\}\mathrm{d}\xi.\end{aligned}$$

†Differentiating equation (2) with respect to other variables, we also obtain explicit formulas for other Greeks, and the argument for the delta can be applied to them in the same way.

‡It does not have the same form as in Little *et al.* (2000). We found that our analysis did not go well using the representation in Little *et al.* (2000).

Taking the limit of $\epsilon \uparrow 1$ and rearranging the equation above, we obtain the following:

$$\begin{aligned} B_\tau = &\left[\aleph(d_1(B_\tau,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1(B_\tau,\tau;K)^2\right\}\right]^{-1} \\ &\times\left[\frac{1}{\sigma\sqrt{2\pi\tau}}K\exp\left\{-\left(r\tau+\frac{1}{2}d_2(B_\tau,\tau;K)^2\right)\right\}\right. \\ &+ rK\int_0^\tau \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}r \\ &\left.\times\exp\left\{-\left(r[\tau-\xi]+\frac{1}{2}d_2(B_\tau,\tau;B_\xi)^2\right)\right\}\mathrm{d}\xi\right]. \end{aligned} \tag{5}$$

Equation (5) provides us with an implicit definition of the optimal exercise boundary B_τ.† Viewed differently, it can be interpreted as an equation in which the left-hand side B_τ is determined by the right-hand side. Therefore, we consider the right-hand side of equation (5) as a functional form of our iterative method for calculating B_τ. More specifically, we begin with the function

$$B_\tau^0 = K,$$

which can be used on the right-hand side to obtain the left-hand side as the first-round approximation denoted by B_τ^1. Fortunately, the first-round approximation can be derived explicitly:

$$\begin{aligned} B_\tau^1 = K&\left[\aleph(d_1(K,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1(K,\tau;K)^2\right\}\right]^{-1} \\ &\times\left[\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\left[r\tau+\frac{1}{2}d_2(K,\tau;K)^2\right]\right\}\right. \\ &\left.+\frac{2\sigma r}{2r+\sigma^2}\left\{2\aleph\left[\frac{(2r+\sigma^2)\sqrt{\tau}}{2\sigma}\right]-1\right\}\right] \end{aligned} \tag{6}$$

and $B_0^1 = K$.

The first-round approximation B_τ^1 is substituted on the right-hand side to obtain the second-round approximation B_τ^2. This procedure is repeated until convergence is obtained. In each round, we set the approximate optimal boundary at $\tau = 0$ to be K. We can use any method of numerical integration, e.g. the *trapezoidal rule* or the *Gaussian quadrature rule*, to approximate the integral in equation (5) after the first round.

In this paper, we use the *Gauss–Kronrod rule*, which is one of the most prevalent methods for calculating numerical integrations.‡ The Gauss–Kronrod rule is an adaptive Gaussian quadrature rule for numerical integration with error estimation based on evaluation at special *Kronrod points*. The details for this numerical scheme appear in the original paper by Kronrod (1964) and the survey paper by Gautschi (1999).

Our iterative method is conceptually very simple, easy to implement, and accurate as long as the number of nodes for time to maturity is sufficiently large. This implies that computation time taken for long-dated put options may be extremely long; therefore, it would be useful to develop an accelerated method to compute the optimal exercise boundary rapidly. We use *polynomial interpolation* for the optimal exercise boundary in order to accelerate our method.§

Specifically, we approximate B_τ by a polynomial of degree n that interpolates all points in the set of $\{B_{\tau_i}^k\}_{0\le i\le n}$ at each kth-round iteration. We believe that this polynomial interpolation provides us with a dramatic reduction in computation time without sacrificing any accuracy.

To sum up, we implement the iterative method according to the following procedure.

Step 0: Set $(n+1)$ to be the number of nodes for time to expiration and k the number of iterations. Denote the maturity of the option by T.

Step 1: Calculate B_τ^1 using equation (6).

Step 2: Calculate B_τ^k $(k = 2, 3, \ldots)$.

> **Step 2-1:** Calculate the value $\{B_{t_i}^k\}_{0\le i\le n}$ of the approximate optimal exercise boundary by replacing the function B_τ in the right-hand side of equation (5) with the function B_τ^{k-1} in Step 1. Here, $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ and $t_i - t_{i-1} = T/n$, and we use the Gauss–Kronrod rule to calculate the integration in equation (5).
>
> **Step 2-2:** Construct the function B_τ^k by interpolating the values of $\{B_{t_i}^k\}_{0\le i\le n}$ with a polynomial of order n.
>
> **Step 2-3:** Repeat Steps 2-1 and 2-2 until sufficient accuracy is obtained.

Step 3: Calculate the value of the option (or the hedge ratio) in formula (2) (respectively, equation 4). The Gauss–Kronrod rule is used to calculate the integration.

4. Convergence, accuracy and speed

Figure 1 shows the graph of the calculated optimal exercise boundary as a function of time to maturity τ until the fifth-round iteration. We use the iterative method with parameters $r = 0.05$, $\sigma = 0.2$, $K = \text{US\$}45$, $T = 0.5$ year and the number of nodes $(n+1) = 17$. The calculated optimal exercise boundary does not change much even after four iterations. Since the first-round approximation B_τ^1 can be obtained explicitly in equation (6), it seems that three iterations are enough to obtain a suitably accurate optimal exercise boundary for this case. This is true for

†Equation (5) can also be obtained from equation (3). Usually, whatever the underlying asset process, we can utilise our iterative method to price an American option if we can obtain its early exercise premium representation. Using the results in Cox (1975) and Kim and Yu (1996), we can obtain an early exercise premium representation where the underlying asset price follows a constant elasticity of variance (CEV) model. However, we did not find such a representation for Heston-type stochastic volatility models until now.

‡It is possible to use a more advanced numerical integration scheme in order to enhance accuracy.

§Other interpolation methods, such as interpolation using piecewise polynomials (e.g. cubic splines) can be used to eliminate unwanted oscillations. However, we could not find any evidence that such methods can provide us with superior results.

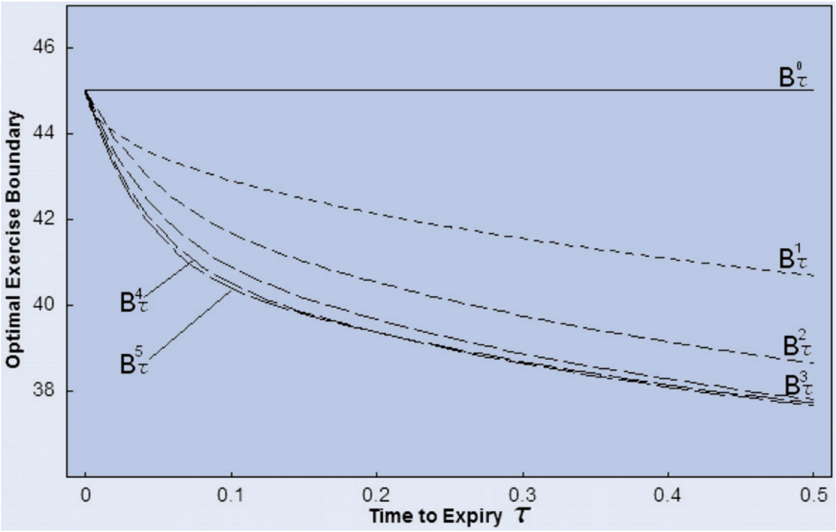

Figure 1. The convergence of the optimal exercise boundary. The parameters are $r = 0.05$, $\sigma = 0.2$, $K = \text{US\$45}$, $T = 0.5$ year and $n = 16$.

various other parameter values. In this section, we will show that the values of American put options for various parameters, which are obtained by sixth-round iteration, have relatively small computation errors compared with the results obtained by other numerical methods.

In table 1, we display seven examples to show the convergence property of the iterative method for a wide range of parameters. The table shows the calculated values of 10 time-iterated optimal exercise boundaries B_τ^{10} at the initial time $\tau = T$ and the corresponding American put option value when the number of nodes n doubles. The 'Benchmark' results are obtained using the binomial tree method due to Cox *et al*. (1979) with 10 000 time steps, and we consider these results to be the exact values of the American put options. It is remarkable that the calculated values for American put options using more than eight node points have errors of less than 10^{-2} throughout the entire number of cases. We can obtain the approximate value of an American put with an error of less than 10^{-2} simply by using the polynomial interpolation of order eight.†

Table 2 reports the computation times and root mean squared errors (RMSEs) when we calculate the values of 601 American put options with six-month maturity by various methods. We consider the results obtained from the binomial tree method due to Cox *et al*. (1979) with 10 000 time steps as the exact values, so the RMSEs are the distances from these values.‡ As table 2 shows, the computation times by the iterative method and the randomisation method of Carr (1998) are almost the same, but the iterative method is more accurate. If we compare the results obtained from the iterative method with those from Ju's (1998) method, we find that the iterative method provides a faster computation time under similar RMSE conditions. On the other hand, both the computation time and the RMSE of the iterative method are superior to those of the binomial tree method. Compared with the approximation methods of Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993), the iterative method is more accurate, but requires more computation time. However, it is well known that the two approximation methods have a critical weak point: there is no way of improving their accuracy.

Table 3 reports the computation times and RMSEs when calculating the values of 1601 American put options with five-year maturity. We can obtain the same implications as those in table 2, except that here, the iterative method gives us strictly better performance than Ju's (1998) approximation method.

Tables 4 and 5 report the computation times and RMSEs when calculating the values of, respectively, 520 American put options with various short maturities (from three months to six months) and 4020 American put options with long maturities (from three years to five years) by various methods.§ The results in the tables also show better performance from the iterative method.

Tables 6 and 7 report the computation times and RMSEs when calculating the hedge ratios (deltas) of, respectively, 401 American put options with six-month maturity and 701 American put options with five-year maturity. Throughout this paper, we calculate deltas for the iterative method by using the relationship of equation (4), and for the approximation methods by using the

†Although results from the numerical simulations indicate that they are convergent to the exact values, the proof of convergence remains open.
‡We calculate the RMSEs in tables 3–5 in the same way.
§We remove the results obtained using Ju's (1998) approximation method, since it takes a great deal of computation time to obtain similar accuracy to our results.

Table 1. The convergence of the calculated values of the optimal exercise boundary B_τ^{10} at $\tau = T$ and the corresponding values of the American put when n doubles.

Parameters	n	B_T^{10}	Put value	Benchmark
I. Baseline case				
$\sigma = 0.2$, $K =$ US\$45, $T = 1$ year	4	36.3704	2.7520	
	8	36.3881	2.7434	
	16	36.3922	2.7411	
	32	36.3933	2.7405	2.7406
II. Change in σ				
$\sigma = 0.15$, $K =$ US\$45, $T = 1$ year	4	39.0978	1.9149	
	8	39.1124	1.9071	
	16	39.1160	1.9051	
	32	39.1170	1.9045	1.9047
$\sigma = 0.25$, $K =$ US\$45, $T = 1$ year	4	33.6764	3.6009	
	8	33.6949	3.5915	
	16	33.6991	3.5890	
	32	33.6982	3.5883	3.5885
III. Change in K				
$\sigma = 0.2$, $K =$ US\$43, $T = 1$ year	4	34.7539	1.9075	
	8	34.7708	1.8999	
	16	34.7748	1.8979	
	32	34.7759	1.8973	1.8975
$\sigma = 0.2$, $K =$ US\$47, $T = 1$ year	4	37.9868	3.7993	
	8	38.0053	3.7903	
	16	38.0097	3.7880	
	32	38.0108	3.7874	3.7876
IV. Change in T				
$\sigma = 0.2$, $K =$ US\$45, $T = 0.5$ year	4	37.7490	2.1011	
	8	37.7602	2.0963	
	16	37.7629	2.0950	
	32	37.7624	2.0947	2.0950
$\sigma = 0.2$, $K =$ US\$45, $T = 3$ year	4	34.2922	3.9477	
	8	34.3191	3.9269	
	16	34.3256	3.9214	
	32	34.3274	3.9201	3.9197

Notes: We take the default parameters of $r = 0.05$ and $S =$ US\$45. The results of the 'Benchmark' case are obtained using the binomial tree method due to Cox *et al*. (1979) with 10 000 time steps.

Table 2. The computation times and RMSEs for the values of 601 American puts with a short maturity (six months).

		Method						
		Numerical		Approximation				
		Binomial	IFDM	BW	BS	Carr	Ju	Iterative
$\sigma = 0.2$	Computation time (sec)	53.882	4.008	0.982	1.202	2.043	4.431	1.903
	RMSE	6.98e−3	7.65e−3	2.72e−2	6.28e−2	4.48e−3	8.39e−4	1.29e−3
$\sigma = 0.4$	Computation time (sec)	54.74	4.041	0.968	1.155	2.09	4.46	1.872
	RMSE	1.9e−2	3.56e−2	2.06e−2	6.96e−2	1.11e−2	1.49e−3	1.92e−3

Notes: The results are the computation times and RMSEs for the values of 601 American put option contracts with 601 underlying stock prices (from US\$90 to US\$120, *in* US\$0.05 steps). The default parameters are $r = 0.05$ *and* $K =$ US\$100\$. We obtained the results for the benchmark case (the exact values) using the binomial tree method with 10 000 time steps introduced by Cox *et al*. (1979). The results in the third column were obtained using the binomial tree method with 100 time steps due to Cox *et al*. (1979), those in the fourth column using the implicit FDM in Brennan and Schwartz (1976) with 100 time steps and 3000 underlying state steps, those in the fifth column using the approximation method in Barone-Adesi and Whaley (1987), those in the sixth column using the approximation method in Bjerksund and Stensland (1993), those in the seventh column using the randomisation method in Carr (1998) with five-point Richardson extrapolation, and those in the eighth column using the multi-piece exponential function method in Ju (1998) with three-point Richardson extrapolation. The ninth column shows the results obtained using the iterative method in this paper with six iterations and ten node points (i.e. polynomial interpolation of order nine). Computation time is the time required to compute the values for all 601 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

Table 3. The computation times and RMSEs for the values of 1601 American puts with a long maturity (five years).

		Method						
		Numerical		Approximation				
		Binomial	IFDM	BW	BS	Carr	Ju	Iterative
$\sigma = 0.2$	Computation time (sec)	145.268	10.608	2.388	3.386	5.428	10.186	6.302
	RMSE	3.26e−2	1.48e−2	1.75e−1	5.55e−2	4.23e−3	3.31e−3	2.45e−3
$\sigma = 0.4$	Computation time (sec)	144.861	10.64	2.246	3.135	5.32	10.187	6.192
	RMSE	1.06e−1	2.66e−1	5.35e−1	1.60e−1	1.96e−2	7.27e−3	3.32e−3

Notes: The results are the computation times and RMSEs for the values of 1601 American put option contracts with 1601 underlying stock prices (from US\$80 to US\$160, in US\$0.05 steps). The default parameters are $r = 0.05$ and $K = \text{US}\$100$. We obtained the results for the benchmark case (the exact values) using the binomial tree method with 10 000 time steps introduced by Cox *et al*. (1979). The results in the third column were obtained using the binomial tree method with 100 time steps due to Cox *et al*. (1979), those in the fourth column using the implicit FDM in Brennan and Schwartz (1976) with 100 time steps and 6000 underlying state steps, those in the fifth column using the approximation method in Barone-Adesi and Whaley (1987), those in the sixth column using the approximation method in Bjerksund and Stensland (1993), those in the seventh column using the randomisation method in Carr (1998) with five-point Richardson extrapolation, and those in the eighth column using the multi-piece exponential function method in Ju (1998) with three-point Richardson extrapolation. The ninth column shows the results obtained using the iterative method in this paper with six iterations and 25 node points (i.e. polynomial interpolation of order 24). Computation time is the time required to compute the values for all 1601 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

Table 4. The computation times and RMSEs for the values of 520 American puts with various short maturities (three to six months).

		Method					
		Numerical		Approximation			
		Binomial	IFDM	BW	BS	Carr	Iterative
$\sigma = 0.2$	Computation time (sec)	48.36	1.842	0.919	1.123	1.934	1.669
	RMSE	1.88e−3	2.27e−3	9.18e−3	1.93e−2	1.39e−3	9.95e−4
$\sigma = 0.4$	Computation time (sec)	47.783	1.84	0.905	1.046	1.841	1.654
	RMSE	6.52e−3	2.91e−3	8.03e−3	2.11e−2	3.63e−3	1.51e−3

Notes: The results are the computation times and RMSEs for the values of 520 American put option contracts with 20 underlying stock prices (from US\$41 to US\$60, in US\$1 steps) and 26 maturities (from 0.25 year to 0.50 year, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = \text{US}\$45$. We obtained the results for the benchmark case (the exact values) using the binomial tree method with 10 000 time steps introduced by Cox *et al*. (1979). The results in the third column were obtained using the binomial tree method with 100 time steps due to Cox *et al*. (1979), those in the fourth column using the implicit FDM in Brennan and Schwartz (1976) with 500 time steps and 200 underlying state steps, those in the fifth column using the approximation method in Barone-Adesi and Whaley (1987), those in the sixth column using the approximation method in Bjerksund and Stensland (1993), and those in the seventh column using the randomisation method in Carr (1998) with five-point Richardson extrapolation. The eighth column shows the results obtained using the iterative method in this paper with six iterations and ten node points (i.e. polynomial interpolation of order nine). Computation time is the time required to compute the values for all 520 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

Table 5. The computation times and RMSEs for the values of 4020 American puts with various long maturities (three to five years).

		Method					
		Numerical		Approximation			
		Binomial	IFDM	BW	BS	Carr	Iterative
$\sigma = 0.2$	Computation time (sec)	371.469	11.575	6.038	7.566	14.008	11.31
	RMSE	1.36e−2	2.11e−3	6.85e−2	3.29e−2	2.28e−3	2.06e−3
$\sigma = 0.4$	Computation time (sec)	370.486	11.403	5.709	7.379	13.838	12.901
	RMSE	4.05e−2	2.81e−2	1.79e−1	7.31e−2	9.36e−3	1.81e−3

Notes: The results are the computation times and RMSEs for the values of 4020 American put option contracts with 20 underlying stock prices (from US\$41 to US\$60, in US\$1 steps) and 201 maturities (from 3 years to 5 years, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = \text{US}\$45$. We obtained the results for the benchmark case (the exact values) using the binomial tree method with 10 000 time steps introduced by Cox *et al*. (1979). The results in the third column were obtained using the binomial tree method with 100 time steps due to Cox *et al*. (1979), those in the fourth column using the implicit FDM in Brennan and Schwartz (1976) with 500 time steps and 500 underlying state steps, those in the fifth column using the approximation method in Barone-Adesi and Whaley (1987), those in the sixth column using the approximation method in Bjerksund and Stensland (1993), and those in the seventh column using the randomisation method in Carr (1998) with five-point Richardson extrapolation. The eighth column shows the results obtained using the iterative method in this paper with six iterations and 25 node points (i.e. polynomial interpolation of order 24). Computation time is the time required to compute the values for all 4020 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

Table 6. The computation times and RMSEs for the hedge ratios (deltas) of 401 American puts with a short maturity (six months).

		Method					
		Numerical		Approximation			
		Binomial	IFDM	BW	BS	Carr	Iterative
$\sigma=0.2$	Computation time (sec)	3.931	1.309	1.45	1.637	5.351	1.279
	RMSE	4.69e−4	3.16e−3	3.31e−3	3.96e−4	5.48e−5	1.11e−4
$\sigma=0.4$	Computation time (sec)	3.9	1.325	1.528	1.653	5.366	1.295
	RMSE	4.81e−4	2.24e−3	1.60e−3	3.48e−4	3.61e−5	6.74e−5

Notes: The results are the computation times and RMSEs for the hedge ratios of 401 American put option contracts with 401 underlying stock prices (from US\$40 to US\$60, in US\$0.05 steps). The default parameters are $r=0.05$, $T=0.5$ and $K=$ US\$45. We obtained the results for the benchmark case (the exact values) using the implicit FDM with 5000 time steps and 10 000 underlying state steps, introduced by Brennan and Schwartz (1976). The results in the third column were obtained using the implicit FDM in Brennan and Schwartz (1976) with 100 time steps and 2000 underlying state steps, those in the fourth column using the approximation method in Barone-Adesi and Whaley (1987), those in the fifth column using the approximation method in Bjerksund and Stensland (1993), and those in the sixth column using the randomisation method in Carr (1998) with four-point Richardson extrapolation, and those in the seventh column using the multipiece exponential function method in Ju (1998) with three-point Richardson extrapolation. The eighth column shows the results obtained using the iterative method in this paper with six iterations and ten node points (i.e. polynomial interpolation of order nine). Computation time is the time required to compute the values for all 401 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

Table 7. The computation times and RMSEs for the hedge ratios (deltas) of 701 American puts with a long maturity (five years).

		Method					
		Numerical	Approximation				
		IFDM	BW	BS	Carr	Ju	Iterative
$\sigma=0.2$	Computation time (sec)	7.644	2.136	2.48	4.679	9.032	3.509
	RMSE	4.83e−4	5.61e−3	9.12e−4	1.90e−4	1.79e−4	1.55e−4
$\sigma=0.4$	Computation time (sec)	7.489	2.105	2.559	4.71	9.173	3.338
	RMSE	1.57e−2	6.62e−3	8.93e−3	9.51e−3	9.43e−3	9.39e−3

Notes: The results are the computation times and RMSEs for the hedge ratios of 701 American put option contracts with 701 underlying stock prices (from US\$35 to US\$70, in US\$0.05 steps). The default parameters are $r=0.05$, $T=5$ and $K=$ US\$45. We obtained the results for the benchmark case (the exact values) using the implicit FDM with 5000 time steps and 10 000 underlying state steps, introduced by Brennan and Schwartz (1976). The results in the third column were obtained using the implicit FDM in Brennan and Schwartz (1976) with 200 time steps and 2000 underlying state steps, those in the fourth column using the approximation method in Barone-Adesi and Whaley (1987), those in the fifth column using the approximation method in Bjerksund and Stensland (1993), and those in the sixth column using the randomisation method in Carr (1998) with five-point Richardson extrapolation, and those in the seventh column using the multipiece exponential function method in Ju (1998) with three-point Richardson extrapolation. The eighth column shows the results obtained using the iterative method in this paper with six iterations and 25 node points (i.e. polynomial interpolation of order 24). Computation time is the time required to compute the values for all 701 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

central difference form with change in state $\Delta S=0.01$, meaning that

$$\text{Delta at state } S=\frac{\left\{\begin{array}{l}\text{(put value at state } S+0.01)\\ \quad-\text{(put value at state } S-0.01)\end{array}\right\}}{0.02}.$$

We also use the central difference for the implicit FDM of Brennan and Schwartz (1976) and note that the change in state ΔS is automatically determined for this method. We consider the results obtained from the implicit FDM with 5000 time steps and 10 000 underlying state steps as the exact values, so the RMSEs are the distances from these values.†

The results in the two tables show that the iterative method gives us strictly better performances compared with all four methods except Ju (1998). Compared with Ju's (1998) method, it seems that the iterative method is much faster with little loss of accuracy.

In table 8, we display the computation times and RMSEs when calculating the hedge ratios (deltas) for 520 American put option contracts with various short maturities (from three months to six months). All five methods provide us with similar computational speeds, but the iterative method gives us the most accurate results.

The results shown in table 9 take on a different aspect for longer maturities. The implicit FDM, the randomisation method by Carr (1998), and the iterative method give us similar RMSEs, but the iterative method has the shortest computation time. In particular, the computation time of the iterative method is four or five times shorter

†We calculate the RMSEs in tables 7–9 in the same way.

Table 8. The computation times and RMSEs for the hedge ratios (deltas) of 520 American puts with various short maturities (three to six months).

		Method				
		Numerical	Approximation			
		IFDM	BW	BS	Carr	Iterative
$\sigma = 0.2$	Computation time (sec)	1.871	1.544	1.997	2.06	1.358
	RMSE	2.07e−4	2.77e−3	2.94e−3	3.75e−4	1.66e−4
$\sigma = 0.4$	Computation time (sec)	1.918	1.653	1.982	2.137	1.356
	RMSE	1.35e−4	1.92e−3	1.49e−3	3.68e−4	1.05e−4

Notes: The results are the computation times and RMSEs for the hedge ratios of 520 American put option contracts with 20 underlying stock prices (from US\$41 to US\$60, in US\$1 steps) and 26 maturities (from 0.25 year to 0.50 year, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = $ US\$45. We obtained the results for the benchmark case (the exact values) using the implicit FDM with 5000 time steps and 5000 underlying state steps, introduced by Brennan and Schwartz (1976). The results in the third column were obtained using the implicit FDM in Brennan and Schwartz (1976) with 500 time steps and 200 underlying state steps, those in the fourth column using the approximation method in Barone-Adesi and Whaley (1987), those in the fifth column using the approximation method in Bjerksund and Stensland (1993), and those in the sixth column using the randomisation method in Carr (1998) with four-point Richardson extrapolation. The seventh column shows the results obtained using the iterative method in this paper with six iterations and ten node points (i.e. polynomial interpolation of order nine). Computation time is the time required to compute the values for all 520 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

Table 9. The computation times and RMSEs for the hedge ratios (deltas) of 4020 American puts with various long maturities (three to five years).

		Method				
		Numerical	Approximation			
		IFDM	BW	BS	Carr	Iterative
$\sigma = 0.2$	Computation time (sec)	18.422	12.043	14.29	26.956	6.052
	RMSE	8.36e−5	6.00e−3	1.12e−3	8.96e−5	2.58e−4
$\sigma = 0.4$	Computation time (sec)	18.268	11.544	14.742	27.113	5.663
	RMSE	1.44e−2	3.92e−3	1.25e−3	1.59e−3	1.46e−3

Notes: The results are the computation times and RMSEs for the hedge ratios of 4020 American put option contracts with 20 underlying stock prices (from US\$41 to US\$60, in US\$1 steps) and 201 maturities (from 3 years to 5 years, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = $ US\$45. We obtained the results for the benchmark case (the exact values) using the implicit FDM with 5000 time steps and 5000 underlying state steps, introduced by Brennan and Schwartz (1976). The results in the third column were obtained using the implicit FDM in Brennan and Schwartz (1976) with 2000 time steps and 500 underlying state steps, those in the fourth column using the approximation method in Barone-Adesi and Whaley (1987), those in the fifth column using the approximation method in Bjerksund and Stensland (1993), and those in the sixth column using the randomisation method in Carr (1998) with five-point Richardson extrapolation. The seventh column represents the results obtained using the iterative method in this paper with six iterations and 25 node points (i.e. polynomial interpolation of order 24). Computation time is the time required to compute the values for all 4020 contracts; all routines were programmed using the MATHEMATICA language and run on a 3.0 GHz Pentium computer.

than Carr's (1998) method. This is because the iterative method allows us to calculate the hedge ratios using the exact formula (see equation 4), while the others do not. Recall that if we utilise the implicit FDM by Brennan and Schwartz (1976) and the randomisation method by Carr (1998), we calculate hedge ratios using a difference form (e.g. central difference).

5. An extension to American options with dividends

We apply our iterative method to the valuation of American put options with proportional dividends. We are convinced that the proposed method can be used to price any derivative securities with American features if they have an early exercise premium representation.

Consider an American put option written on an asset that pays continuous proportional dividends at a rate of $\alpha > 0$ and all other circumstances are the same as those in the second section. For this case, the price dynamics for the underlying asset can be represented as

$$\mathrm{d}S_t = (\mu - \alpha)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}z_t.$$

The Black–Scholes PDE for the live American put takes the form

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - \alpha)S\frac{\partial P}{\partial S} - rP = \frac{\partial P}{\partial \tau},$$

subject to a terminal condition

$$P(S, 0) = \max\{0, K - S\} \quad \text{for all } S \geq B_0,$$

and the boundary conditions in equation (1). Then, as in Kim (1990), the valuation formula for the live American put is, for $S > B_\tau$,

$$P(S,\tau) = \tilde{p}(S,\tau) + \int_0^\tau \Big[rK\mathrm{e}^{-r(\tau-\xi)}\aleph(-\tilde{d}_2(S,\tau-\xi;\,B_\xi)) - \alpha S\mathrm{e}^{-\alpha(\tau-\xi)}\aleph(-\tilde{d}_1(S,\tau-\xi;\,B_\xi))\Big]\mathrm{d}\xi,$$

and the early exercise premium representation should be

$$K - B_\tau = \tilde{p}(B_\tau,\tau) + \int_0^\tau \Big[rK\mathrm{e}^{-r(\tau-\xi)}\aleph(-\tilde{d}_2(B_\tau,\tau-\xi;\,B_\xi)) - \alpha B_\tau\mathrm{e}^{-\alpha(\tau-\xi)}\aleph(-\tilde{d}_1(B_\tau,\tau-\xi;\,B_\xi))\Big]\mathrm{d}\xi, \tag{7}$$

where

$$\tilde{d}_1(S,\tau;\,B) = \frac{\ln(S/B) + (r-\alpha+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$
$$\tilde{d}_2(S,\tau;\,B) = \tilde{d}_1(S,\tau;\,B) - \sigma\sqrt{\tau},$$

and $\tilde{p}$, the value of a European put option, in equation (7) is defined as

$$\tilde{p}(S,\tau) = K\mathrm{e}^{-r\tau}\aleph(-\tilde{d}_2(S,\tau;\,K)) - S\mathrm{e}^{-\alpha\tau}\aleph(-\tilde{d}_1(S,\tau;\,K)).$$

As already mentioned in Kim (1990), the early exercise boundary at maturity should be given by $B_0 = \min\{K, rK/\alpha\}$.

An iterative method stems from the following equation, which is essentially equivalent to equation (7):

$$\begin{aligned} B_\tau = &\Bigg[\frac{1}{\sigma\sqrt{2\pi\tau}}K\exp\left\{-\left[r\tau + \frac{1}{2}\tilde{d}_2(B_\tau,\tau;\,K)^2\right]\right\} \\ &+ rK\int_0^\tau \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}r \\ &\times\exp\left\{-\left[r(\tau-\xi) + \frac{1}{2}\tilde{d}_2(B_\tau,\tau-\xi;\,B_\xi)^2\right]\right\}\mathrm{d}\xi\Bigg] \\ &\times\Bigg[\mathrm{e}^{-\alpha\tau}\aleph(\tilde{d}_1(B_\tau,\tau;\,K)) + \frac{1}{\sigma\sqrt{2\pi\tau}} \\ &\times\exp\left\{-\left[\alpha\tau + \frac{1}{2}\tilde{d}_1(B_\tau,\tau;\,K)^2\right]\right\} \\ &+ \alpha\int_0^\tau \mathrm{e}^{-r(\tau-\xi)}\aleph(\tilde{d}_1(B_\tau,\tau-\xi;\,B_\xi)) + \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}r \\ &\times\exp\left\{-\left[r(\tau-\xi) + \frac{1}{2}\tilde{d}_1(B_\tau,\tau-\xi;\,B_\xi)^2\right]\right\}\mathrm{d}\xi\Bigg]^{-1}. \end{aligned}$$

We begin the iteration with the function

$$B_\tau^0 = \min\left\{K, \frac{r}{\alpha}K\right\}.$$

The first-round approximation B_τ^1 and the valuation formula for the hedge ratio Δ_τ can be easily derived, so the specific derivations are left to the reader.

The iterative method also can be used to price American call options with proportional dividends. This can be done using the early exercise premium representation for American call options found by Kim (1990). On the other hand, we can also use the parity property suggested by McDonald and Schröder (1998). They verify that the value of an American call option with underlying asset price S, exercise price K, risk-free interest rate r, and dividend rate α is equal to the value of an American put option with underlying asset price K, exercise price S, risk-free interest rate α, and dividend rate r. We can use this to obtain the value of an American call option with proportional dividends by calculating the value of an American put option satisfying the parity property.

6. Conclusion

In this paper, we provide a simple but innovative iterative method to obtain the optimal exercise boundary of American options, and we show that this allows us to compute the values of American options and their hedge ratios quickly and accurately. Furthermore, the iterative method represents a new way of computing the values of American options.

We derive a new early exercise premium representation with only a single integral by following the idea of Little *et al.* (2000). By exploiting it, we suggest a simple and efficient iterative numerical method. According to the numerical results, our iterative method is computationally superior to the methods currently available, including the binomial method due to Cox *et al.* (1979), the implicit FDM of Brennan and Schwartz (1976), and the randomisation method of Carr (1998).

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