Andrew Gleeson
October 28, 2013

Deductive proof of the tetrahedral number formula

A tetrahedral number of size n is the sum of the triangular numbers from 1 to n.

Proof

1. Let $g(n)$ be the nth triangular number.
   $g(n) = \frac{n(n+1)}{2}$
2. Let $F(n)$ be the nth tetrahedral number.
   $F(n) = \sum_{i=1}^{n} g(i)$
3. Substitute:
   $F(n) = \sum_{i=1}^{n} \frac{i(i+1)}{2}$
4. Distribute:
   $F(n) = \sum_{i=1}^{n} \frac{i^2}{2} + \frac{i}{2}$
5. Split:
   $F(n) = \sum_{i=1}^{n} \frac{i^2}{2} + \sum_{i=1}^{n} \frac{i}{2}$
6. Take out constants:
   $F(n) = \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i$
7. Substitute known formulas for sums:
   $F(n) = \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right)$
8. Add:
   $F(n) = \frac{1}{2} \left(\frac{n(n+1)(2n+1)+3n(n+1)}{6} \right)$
9. Factor:
   $F(n) = \frac{1}{2} \left(\frac{n(n+1)((2n+1)+3))}{6} \right)$
10. Simplify:
    $F(n) = \frac{1}{2} \left(\frac{n(n+1)(2n+4)}{6} \right)$
11. Simplify:
    $F(n) = \frac{1}{2} \left(\frac{2n(n+1)(n+2)}{6} \right)$

$$\therefore F(n) = \frac{n(n+1)(n+2)}{6}$$