Triangular and Simplex Numbers

An Introduction to Mathematical Thinking

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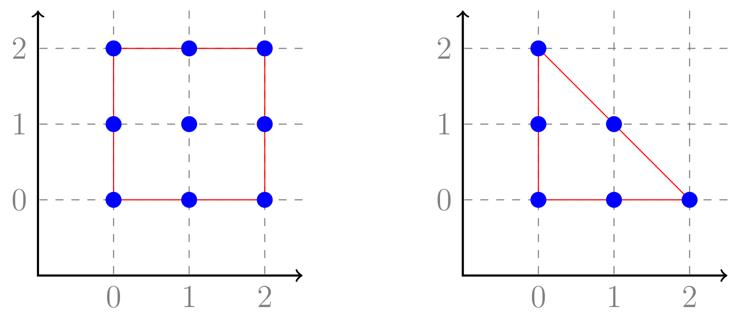
Abstract

Triangular numbers are positive integers with properties related to triangles. Graphically, they can be represented as the number of points in $\mathbb{Z}^2$ bounded by a right isoceles triangle. They can also be viewed as $\sum_{i=1}^{n} i$, which is known to be $\frac{n(n+1)}{2}$. Triangular numbers can be generalized to simplex numbers, which are analogous to triangular numbers in any dimension. Triangular numbers show up often across various areas of mathematics, mostly in combinatorics. Although some of the included results have been known for centuries, I independently arrived at many of the results. This paper describes the results, many of which can be shown though very elegant proofs, and my journey to them. Because triangular numbers are a relatively unknown but not overly complicated area of mathematics, they present an ideal opportunity to start real mathematical thinking.

1 Introduction

The concept of triangular numbers is fundamentally. Most people are very familiar with square numbers (n^2), but we don't put much thought into why a square is chosen as the shape (Figure 1a). Geometrically, there are other very interesting shapes. Two shapes that stand out are the triangle, which has the least number of verticies required to bound an area in two-dimensional space, and the circle, which has an infinite number of vertices. All other polygons are somewhere between the two. So, it would seem that triangles, like squares, should be similarly important. A square number can be thought of as the number of points is $\mathbb{Z}^2$ bounded by a square. They can also be viewed as $\sum_{i=1}^{n} n$ and n^2.



(a) $3^2 = 9$ (b) $T_3 = 6$

Figure 1: Graphic Representations

2 Triangular Numbers

To form a triangular number, we can instead bound the points with a right isoceles triangle. (Figure 1b) What would a triangular number be? If we examine the triangular numbers, we can see that in each successive row, the number of dots increments by one. So, we can express the nth triangular number T_n as the sum of the numbers 1 to n.

$T_n = 1 + 2 + ... + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

With some simple steps, we can transform triangular numbers into something much more well known.

$$
\begin{aligned}
T_n &= \frac{n(n+1)}{2} \\
&= \frac{(n+1)!}{2!(n-1)!} \quad \text{(rewrite using factorials)} \\
&= \frac{k!}{2!(k-2)!} \quad \text{(let k = n+1)} \\
&= \binom{k}{2} \quad \text{(by the definition of a binomial coefficient)} \\
&= \binom{n+1}{2} \quad \text{(substitute)}
\end{aligned}
$$

In plain English, this means that the nth triangular number is the number of combinations of 2 items from a set of n+1 items where order is unimportant and repeats are culled. This countrasts to square numbers, where order matters and repeats are allowed. This fact connects us to the world of combinatorics. However, I decided to not delve into combinatorics to find properties of triangular numbers. It is possible to manipulate already-known properties of binomial coefficients to say something about triangular numbers, but that would mean deviating from the beautiful geometry of triangular numbers. There exist many other properties of triangular numbers that have elegant geometric proofs, which was the path of interest to me.

For example, two important triangular identities are $T_n + T_{n-1} = n^2$ and $T_n - T_{n-1} = n$. These identities are essential to the relationship between triangular numbers and quadratic quantities. It means that any square number, or any integer, can by represented as the sum or difference of two triangular numbers.

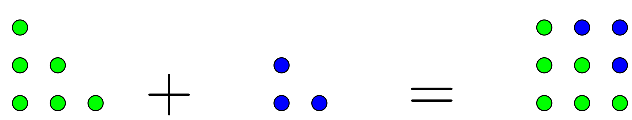


Figure 2: $T_3 + T_2 = 3^2$

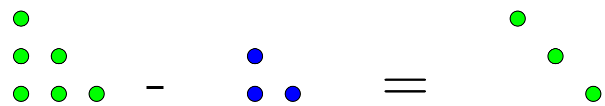


Figure 3: $T_3 - T_2 = 3$

2.1 Properties

Many properties of triangular numbers can be shown geometrically, while others are shown through the manipulation of equations.

For example, the identity $T_{2n} = 3 \cdot T_n + T_{n-1}$ can be shown through a simple graphic.

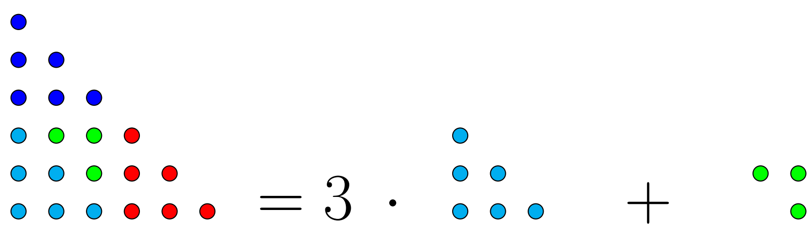


Figure 4: $T_6 = 3 \cdot T_3 + T_2$ [4]

These properties are integral to building the intuition and machinery necessary to derive more complicated identities. Being able to visualize triangular numbers leads one to make non-trivial connections that are not apparent in the algebra. However, some identities do lend themselves to an algebraic derivation.

1. $(T_n)^2 = \sum_{i=1}^{n} i^3$
2. $T_{a+b} = T_a + T_b + 2ab$
3. $T_{ab} = T_a T_b + T_{a-1} T_{b-1}$
4. $T_n - T_{n-k} = \frac{k}{2}(2n + 1 - k)$
5. All even perfect numbers (a number which equals the sum of its proper divisors excluding itself) are triangular numbers. [7]

And , by using (4), it can be shown that the difference of two triangular numbers will never be a perfect square if k is two or greater.

Some other identities are even more obscure. Gauss, with his Eureka Theorem, showed that any integer is the sum of at most three triangular numbers. [1] There is no known elementary proof of this, showing a connection to deeper mathematics.

2.1.1 Reciprocal Infinite Series

There is a nice cleanliness to the reciprocal sum of all triangular numbers. By a simple telescopic series, $\sum_{n=1}^{\infty} \frac{1}{T_n} = 2$.

Since $\frac{1}{T_n} = \frac{2}{n(n+1)}$ can be rewritten as $\frac{2}{n} - \frac{2}{n+1}$ by partial fractions, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = \sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+1}$$

If we write out the first few term, we begin to see a pattern:

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = \left[\frac{2}{1} - \frac{2}{2}\right] + \left[\frac{2}{2} - \frac{2}{3}\right] + \left[\frac{2}{3} - \frac{2}{4}\right] + \dots$$

It appears that the first part of each term cancels the second part of the previous term. Because they cancel out in pairs, we are left with

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2 - \lim_{n\to\infty} \frac{2}{n+1}$$

Because the limit of $\frac{2}{n+1}$ as n goes to infinity is zero, we are left with the exceedingly simple

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2$$

Some mathematicians looked at subseries of this, expanding it to the form T_{mn+r} instead of just T_n. [2] They found that

$$\sum_{n=0}^{\infty} \frac{1}{T_{mn+r}} = \frac{2}{m} \sum_{0<j<m/2} \left\{ \left[\cos\left(\frac{2\pi j(r+1)}{m}\right) - \cos\left(\frac{2\pi jr}{m}\right) \right] \right.$$
$$\cdot \ln\left[2 - \cos\left(\frac{2\pi j}{m}\right)\right] - \left[\sin\left(\frac{2\pi j(r+1)}{m}\right) - \sin\left(\frac{2\pi jr}{m}\right) \right]$$
$$\left. \cdot \frac{\pi(m-2j)}{m} \right\} + 2\delta_{mr} + \varepsilon_m \cdot (-1)^{r+1} 2\ln(2)$$

Note: δ_{mr} is the Kronecker delta function, which is 1 when $m = r$ and 0 otherwise. ε_m returns 1 if m is even, and 0 if m is odd.

Plugging in specific m and r:

- $\sum_{n=0}^{\infty} \frac{1}{T_{2n+2}} = 2 - 2\ln 2$
- $\sum_{n=0}^{\infty} \frac{1}{T_{3n+1}} = \frac{2\pi\sqrt{3}}{9}$
- $\sum_{n=0}^{\infty} \frac{1}{T_{4n+1}} = \frac{\pi}{4} + \frac{3}{2}\ln 2$
- $\sum_{n=0}^{\infty} \frac{1}{T_{4n+2}} = \frac{\pi}{4} - \frac{3}{2}\ln 2$
- $\sum_{n=0}^{\infty} \frac{1}{T_{4n+3}} = -\frac{\pi}{4} + \frac{5}{2}\ln 2$

These results cannot be proven without complex analysis and other specialized tools, revealing a deeper connection to higher mathematics that is not intuitively obvious. See Appendix A for an explanation of how they follow from the given formula. However, a brute-force numerical analysis I carried out confirms them to my satisfaction.

2.1.1.1 Example - T_{3n+1}

$$\sum_{n=0}^{\infty} \frac{1}{T_{3n+1}} = \frac{2}{3} \sum_{0<j<3/2} \left\{ \left[\cos\left(\frac{2\pi j(1+1)}{3}\right) - \cos\left(\frac{2\pi j(1)}{3}\right) \right] \right.$$
$$\cdot \ln\left[2 - \cos\left(\frac{2\pi j}{3}\right)\right] - \left[\sin\left(\frac{2\pi j(1+1)}{3}\right) - \sin\left(\frac{2\pi j(1)}{3}\right)\right]$$
$$\left. \cdot \frac{\pi(3-2j)}{3} \right\} + 2\delta_{3,1} + \varepsilon_3 \cdot (-1)^{1+1} 2\ln(2)$$

$$= \frac{2}{3}\left\{0 - [-\sqrt{3}] \cdot \frac{\pi}{3}\right\} + 2 \cdot 0 + 0 \cdot 2\ln 2$$
$$= \frac{2\pi\sqrt{3}}{9}$$

At the very least, it is remarkable that there are several similar terms in the results.

3 Simplex Numbers

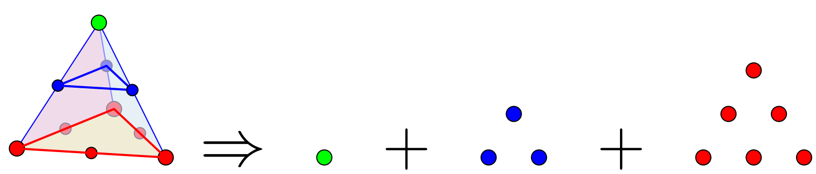


Figure 5: $3^{\Delta 3} = 10$

Triangular numbers can be generalized to any dimension by turning them into simplexes. A simplex is a generalization of the idea of a triangle or tetrahedron to any dimension. For example, a triangle is a 2-simplex, meaning a simplex in two dimensions. A tetrahedron is a 3-simplex - and the list goes on. I will use the notation $n^{\Delta d}$ to signify the nth d-simplex number.

3.1 Tetrahedral Numbers

We have cube numbers, (n^3 or $\sum_{i=1}^{n} n^2$) — what would tetrahedral numbers be? (Figure 5) In the same way that a cube is made up of squares, a tetrahedral number is the sum of triangular numbers.

$n^{\Delta 3} = 1^{\Delta 2} + 2^{\Delta 2} + ... + n^{\Delta 2} = \sum_{i=1}^{n} i^{\Delta 2} = \frac{n(n+1)(n+2)}{6}$

I spent a great amount of time working with tetrahedral numbers, because the concreteness of the graphical interpretation was very helpful for me. However, it does have its limits – for example, I attempted to visualize the multiplication of a triangular number and a scalar as a prism, which turned out to be fruitless.

3.2 Fourth dimension and beyond

Many of the properties of triangular and tetrahedral numbers hold true for any arbitrary dimension. It was natural for me to ask the same questions as I did about triangular numbers. For example, there is a pattern that the simplex number in each dimension is a sum of the smaller dimensions. Geometrically, a triangle is made out of three lines. A tetrahedron is bounded by four triangles. It would make sense, then, that a 4-simplex would in some sense be comprised of tetrahedra. Numerically, we saw that $n^{\Delta 2}$ is the sum from 1 to n, and that $n^{\Delta 3}$ was the sum of the first n triangular numbers. This carries over:

$$n^{\Delta d} = \sum_{i=1}^{n} i^{\Delta d-1}$$

The combinatoric identity also generalizes:

$$n^{\Delta d} = \binom{n+d-1}{d}$$

The reciprocal infinite sum generalizes wonderfully, giving us

$$\sum_{n=1}^{\infty} \frac{1}{n^{\Delta d}} = \frac{d}{d-1}$$

However, some properties do not translate easily. The square identity, $n^2 = T_n + T_{n-1}$, is very simple. Unfortunately, one identity derived graphically is significantly more complex for cubes:

$$n^3 = n^{\Delta 3} + (2n-2)^{\Delta 3} - 3(n-2)^{\Delta 3}$$

This can be shown by visualizing tetrahedral numbers as the sum of triangular numbers, and cubes as the sums of square numbers. Although the proof is relatively simple, find this property took a considerable amount of graphical intuition and induction. On the other hand, an algebraic combination of

$$T_n + T_{n-1} = n^2 \tag{1}$$
$$T_n - T_{n-1} = n \tag{2}$$

by multiplication gives us:

$$(T_n)^2 - (T_{n-1})^2 = n^3$$

Unfortunately, multiplication of triangular number does not lend itself to a convenient graphical interpretation.

Symbolic Proof

$$\begin{aligned}
&n^{\Delta 3} + (2n-2)^{\Delta 3} - 3(n-2)^{\Delta 3} = \\
&= \frac{(n)(n+1)(n+2)}{6} + \frac{(2n-2)(2n-1)(2n)}{6} - \frac{3(n-2)(n-1)(n)}{6} \\
&= \frac{n^3+3n^2+2n}{6} + \frac{8n^3-12n^2+4n}{6} - \frac{3n^3-9n^2+6n}{6} \\
&= \frac{n^3+3n^2+2n+8n^3-12n^2+4n-3n^3+9n^2-6n}{6} \\
&= \frac{6n^3}{6} = n^3
\end{aligned}$$

4 Conclusion

Hopefully, you agree that triangular and simplex numbers are interesting. Although the concept of a triangular number is exceedingly simple, they have a surprisingly broad scope and a powerful geometric appeal. Many of the results have a certain symmetry that falls from the underlying simplicity of the concept. Additionally, and more informally, following the triangular trail was a great deal of fun. Because triangular and simplicial numbers are relatively obscure, I was only able to find citations for many results after I knew exactly what I was looking for – that is, I independently derived them. This research required me to do a great deal of mathematical thinking, which will not be found in most math classes. The beauty of triangular numbers lies in the ease with which they lend themselves to the independent derivation of properties. In much of mathematics, you must spend years taking classes, building intuition and machinery, before being able to discover results for yourself without knowing the answer ahead of time, or even if an answer exists. The machinery of triangular numbers is sticks and stones – algebra is enough to understand most of the properties. This allowed me to jump into the discovery phase very quickly. In a sense, the obscurity of triangular numbers is a boon. There are no books to read about triangular numbers – you bring all your tools to the table yourself. You need to wrangle results out yourself. Isn't that what mathematical thinking is? The plug-and-chug method does no one any favors – a computer will always outperform a human in raw computation and rote memorization. Mathematical advances are driven by imagination, and the heuristic process. I think working with triangular and simplex numbers helps to further this ablity.

A Proof of the Tetrahedral Number Formula

Inductive proof of the tetrahedral number formula

Let $F(n)$ be the nth tetrahedral number, and $g(n)$ be the nth trianhular number. By definition, $F(n) = g(1) + g(2) + g(3) + \ldots + g(n)$

Claim

$$F(n) = \frac{n(n+1)(n+2)}{6}$$

Proof: Induction on n

n=1

$$F(1) = g(1) = 1$$
$$F(1) = \frac{(1)(2)(3)}{6} = 1$$

Inductive Step

Assume $F(k) = \frac{k(k+1)(k+2)}{6}$. We want to show that $F(k+1) = \frac{(k+1)(k+2)(k+3)}{6}$

$$F(k+1) = g(1) + g(2) + g(3) + \cdots + g(k) + g(k+1)$$
$$F(k+1) = F(k) + g(k+1)$$
$$F(k+1) = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$
$$F(k+1) = \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6}$$
$$F(k+1) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6}$$
$$F(k+1) = \frac{(k+1)(k+2)(k+3)}{6}$$

$\therefore F(n) = \frac{n(n+1)(n+2)}{6}$

Deductive proof of the tetrahedral number formula

Proof

1. Let $g(n)$ be the nth triangular number.

   $g(n) = \frac{n(n+1)}{2}$

2. Let $F(n)$ be the nth tetrahedral number.

   $F(n) = \sum_{i=1}^{n} g(i)$

3. Substitute:

   $F(n) = \sum_{i=1}^{n} \frac{i(i+1)}{2}$

4. Distribute:

   $F(n) = \sum_{i=1}^{n} \frac{i^2}{2} + \frac{i}{2}$

5. Split:

   $F(n) = \sum_{i=1}^{n} \frac{i^2}{2} + \sum_{i=1}^{n} \frac{i}{2}$

6. Take out constants:

   $F(n) = \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i$

7. Substitute known formulas for sums:

   $F(n) = \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right)$

8. Add:

   $F(n) = \frac{1}{2} \left(\frac{n(n+1)(2n+1)+3n(n+1)}{6} \right)$

9. Factor:

   $F(n) = \frac{1}{2} \left(\frac{n(n+1)((2n+1)+3))}{6} \right)$

10. Simplify:

    $F(n) = \frac{1}{2} \left(\frac{n(n+1)(2n+4)}{6} \right)$

11. Simplify:

$$F(n) = \frac{1}{2}\left(\frac{2n(n+1)(n+2)}{6}\right)$$

$$\therefore F(n) = \frac{n(n+1)(n+2)}{6}$$

Bibliography

[1] Bell, Eric Temple. "Gauss, the Prince of Mathematicians". In Newman, James R. The World of Mathematics I. Simon & Schuster. (1956): 295339.

[2] Bruckman, Paul, Joseph B. Dence, Thomas P. Dence, and Justin Young. "Series of Reciprocal Triangular Numbers." The College Mathematics Journal 44.3 (2013): 177-84.

[3] Fang, Jin-Hui. "Nonexistence of a Geometrical Progression That Contains Four Triangular Numbers." Electronic Journal of Combinatorial Number Theory 7 (2007).

[4] Nelsen, Roger B. Proofs without Words: Exercises in Visual Thinking. Washington, D.C.: Mathematical Association of America, 1993.

[5] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A000217.

[6] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A000292.

[7] Tanton, James. "A Dozen Questions about Triangular Numbers." Math Horizons 13 (2005).

[8] Ulas, Maciej. "On Certain Diophantine Equations Related to Triangular and Tetrahedral Numbers." `arXiv:0811.2477v1 [math.NT]`