Q2.1.4

Let $P = \{p_1, p_2, \dots p_n\}$
$q = \{q_1, q_2 \dots, q_n\}$ be 2 sets of corresponding points in R^d

To find: rotation matrix R and translation vector t such that

$$(R, t) = \underset{\substack{R \in SO(d) \\ t \in R^d}}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \| q_i - (Rp_i + t) \|^2$$

where w_i and weights for each point pair

To find t:

$$\frac{dF}{dt} = \sum_{i=1}^{n} 2w_i (Rp_i + t - q_i) = 0$$

$$\therefore \quad 2t\left(\sum_{i=1}^{n} w_i\right) + 2R\left(\sum_{i=1}^{n} w_i p_i\right) - 2\sum_{i=1}^{n} w_i q_i =$$

Taking $\bar{p} = \dfrac{\sum_{i=1}^{n} w_i p_i}{\sum_{i=1}^{n} w_i}$, $\bar{q} = \dfrac{\sum_{i=1}^{n} w_i q_i}{\sum_{i=1}^{n} w_i}$, we

$$t - \bar{q} + R\bar{p} = 0$$

$$\boxed{t = \bar{q} - R\bar{p}}$$

Putting t into F,

$$F(R, t) = \sum_{i=1}^{n} w_i \| q_i - Rp_i - \bar{q} + R\bar{p} \|^2$$

$$= \sum_{i=1}^{n} w_i \| R(p_i - \bar{p}) - (q_i - \bar{q}) \|^2$$

Restate problem with zero translation

$$x_i = p_i - \bar{p} \quad , \quad y_i = q_i - \bar{q}$$

Hence,

$$R = \underset{R \in SO(d)}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \| R x_i - y_i \|^2$$

Now, $\| R x_i - y_i \|^2 = (R x_i - y_i)^T (R x_i - y_i)$

$= (x_i^T R^T - y_i^T)(R x_i - y_i)$

$= x_i^T x_i - y_i^T R x_i - x_i^T R^T y_i + y_i^T y_i$

Because $R^T R = I$

$x_i^T R^T y_i$ is a scalar : $1 \times d \quad d \times d \quad d \times 1 = 1 \times$

and hence is equal to Transpose

$\therefore \quad x_i^T R^T y_i = (x_i^T R^T y_i)^T = y_i^T R x_i$

$\therefore \quad \| R x_i - y_i \|^2 = x_i^T x_i - 2 y_i^T R x_i + y_i^T y_i$

$$\therefore \quad \underset{R \in SO(d)}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \| R x_i - y_i \|^2$$

$$\underset{R \in SO(d)}{\operatorname{argmin}} \left(\sum_{i=1}^{n} w_i x_i^T x_i - 2 \sum_{i=1}^{n} w_i y_i^T R x_i + \sum_{i=1}^{n} w_i y_i^T y_i \right)$$

$$\underset{R \in SO(d)}{\operatorname{argmin}} \; -2 \sum_{i=1}^{n} w_i y_i^T R x_i$$

$$\underset{R \in SO(d)}{\operatorname{argmax}} \sum_{i=1}^{n} w_i y_i^T R x_i$$

Now, vectorising $\sum_{i=1}^{n} w_i y_i^T R x_i$, we get

$$\sum_{i=1}^{n} w_i y_i^T R x_i = t(W Y^T R X) \text{ , where}$$

$$W = \begin{bmatrix} w_1 & & \\ & w_2 & \\ & & \ddots w_n \end{bmatrix} \qquad Y = \begin{bmatrix} -y_1^T \\ -y_2^T \\ \end{bmatrix}_{d \times n}$$

$$= \mathrm{diag}(w_1, \dots, w_n)$$

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix}_{d \times n}$$

$\therefore$ we need rotation R which maximizes $tr(WY$
$tr(WY^T RX) = tr((WY^T)(RX)) = tr(RXWY^T$

$[tr(AB) = tr(BA)]$

Let $d \times d$ covariance matrix $S = XWY^T$. Taking s
$S = V \Sigma V^T$

$\therefore tr(RXWY^T) = tr(RS) = tr(RV\Sigma U^T) = tr(\Sigma U$

Since V, R, U are orthogonal $\Rightarrow M = V^T R U$ is orthogon

$\therefore$ For each column M_j of M, $M_j^T M_j = 1$

Hence, all number M_{ij} are magnitude $\leq$

$$M_j^T M_j = 1 \Rightarrow \sum_{i=1}^{d} M_{ij}^2 = 1 \Rightarrow M_{ij}^2 \leq 1 \Rightarrow |M_{ij}| \leq$$

$$\therefore tr(\Sigma M) = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \sigma_d \end{bmatrix} M = \sum_{i=1}^{d} \sigma_i M_{ii} \leq$$

where Σ is a diagonal matrix

and $\sigma_1, \sigma_2, \dots \sigma_d \geq 0$

$\therefore$ To maximize $tr(\Sigma M)$; $M_{ii} = 1$

Since M is orthogonal $\Rightarrow$ to maximize $tr(\Sigma M)$,

$$M = I$$

$\therefore$ $M = U^T R V = I \Rightarrow V = RU$

$$R = VU^T$$

Hence proved mathematically that Procrustes alignment gives the best aligning transform between point clouds with known correspondence.