Documentation for isotropic HW shell models

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1 Overview of equations

The GOY shell model is implemented for the Hasegawa Wakatani system of equations

$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right) \nabla^2 \phi - C(\phi - n) = D_\phi$$
$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right) n + \kappa \frac{\partial \phi}{\partial y} - C(\phi - n) = D_n.$$

The Shell model was introduced to capture the richardson cascades in turbulent systems in a quick way. After representing the eqauation(s) in Fourier space,

$$\frac{\partial}{\partial t}\phi_k(t) = L[n_k, \phi_k] + \frac{1}{2} \sum_{k=k'+k''} \phi_{k'}(t)\phi_{k''}(t)$$
$$\frac{\partial}{\partial t} n_k(t) = L[n_k, \phi_k] + \frac{1}{2} \sum_{k=k'+k''} n_{k'}(t)\phi_{k''}(t)$$

where $L[\,]$ is the linear part of an equation and the nonlinear term is written as a sum over all possible fourier components. The dynamical variables $\Phi_n = \left[\frac{1}{k_n^2}\int_{k_n}^{k_{n+1}} \mathrm{d}k \int \mathrm{d}\alpha_k \left|\Phi_{\mathbf{k}}\right|^2 k^3\right]^{1/2}$ and $n_n = \left[\int_{k_n}^{k_{n+1}} \mathrm{d}k \int \mathrm{d}\alpha_k \left|n_{\mathbf{k}}\right|^2 k\right]^{1/2}$ are introduced as an integral in Fourier space over angles α_k for a fiven scale k and over all k vales between k_n and $k_n + dk$ with dk scaling geometrically, $dk = g^n$. The nonlinear interaction is a sum over all possible k', k'' where $k = k' + k''$. The second approximation in the model is to assume only the triad interaction where Φ_n interacts only with Φ_{n-2}, Φ_{n-1}, Φ_{n+1}, and Φ_{n+2} resulting in

$$\frac{\mathrm{d}\Phi_n}{\mathrm{d}t} = a_n \phi^*_{n-1}\phi^*_{n-2} + b_n \phi^*_{n-1}\phi^*_{n+1} + c_n \phi^*_{n+1}\phi^*_{n+2} + \frac{C(\Phi_n - n_n)}{k_n^2} - (\nu_\Phi k_n^{-6} + \nu'_\Phi) \tag{1a}$$
$$\begin{aligned}\frac{\mathrm{d}n_n}{\mathrm{d}t} &= (\phi^*_{n-2}n^*_{n-1} - \phi^*_{n-1}n^*_{n-2}) + b'_n(\phi^*_{n+1}n^*_{n-1} - \phi^*_{n-1}n^*_{n+1}) + c'_n(\phi^*_{n+2}n^*_{n+1} - \phi^*_{n+1}n^*_{n+2}) \\ &\quad C(\Phi_n - n_n) + i\kappa k_n \Phi_n - (\nu k_n^{-6} + \nu' k_n^4) n_n.\end{aligned} \tag{1b}$$

where a_n, b_n, and c_n and a'_n, b'_n, and c'_n are derived using conservations of kinetic energy $E_K = \sum_n k_n^2 \Phi_n^2$, enstropy $W = \sum_n k_n^4 \Phi_n^2$, cross helicity $H = \sum_n k_n^2 \Phi_n n_n$, and internal energy $E_I = \sum_n n_n^2$. For the case of Hasegawa Wakatani equations, the equations become:

$$\frac{\mathrm{d}\Phi_n}{\mathrm{d}t} = \alpha k_n^2 (g^2 - 1)\left[\frac{1}{g^7}\Phi^*_{n-1}\Phi^*_{n-2} - \frac{g^2+1}{g^3}\Phi^*_{n-1}\Phi^*_{n+1} + g^3 \Phi^*_{n+1}\Phi^*_{n+2}\right] + \frac{C(\Phi_n - n_n)}{k_n^2} - (\nu_\Phi k_n^{-6} + \nu'_\Phi k_n^4)\Phi_n \tag{2a}$$
$$\begin{aligned}\frac{\mathrm{d}n_n}{\mathrm{d}t} &= \alpha k_n^2 \left[\frac{1}{g^3}(\Phi^*_{n-2}n^*_{n-1} - \Phi^*_{n-1}n^*_{n-2}) - \frac{1}{g}(\Phi^*_{n-1}n^*_{n+1} - \Phi^*_{n+1}n^*_{n-1}) + g(\Phi^*_{n+1}n^*_{n+2} - \Phi^*_{n+2}n^*_{n+1})\right] \\ &\quad + C(\Phi_n - n_n) + i\kappa k_n \Phi_n - (\nu k_n^{-6} + \nu' k_n^4) n_n.\end{aligned} \tag{2b}$$

2 Specifics of the code

The working file work.cpp, the functions in HW_iso.cpp and the header file HW_iso.h are compiled with the GSL library and openMP to create the executive shell_iso_HW. This executive solves the set of ODE equations for the Hasegawa Wakatani specifically where the paramters specific to the HW system can be specifed in the input file INPUT as discussed in the README file.

If the equations to be solved are different, then the coeffients need to be updated manually in the file HW_iso.cpp. We discuss briefly the files and how to modify things if any to make it work for a different set of equations.

2.1 work.cpp

This is where the main function is. It calls functions from HW_iso.cpp to integrate the set of ODEs using GSL solver. Most of the parameters in this file can be specified in the INPUT file as explained in the README file. However, the gsl tolerence can be modified (lines 93 and 94) as well as the format of the output files can be modified (lines 183 and 185).

2.2 HW_iso.cpp

This program is written specifically to solve for the Hasegawa Wakatani system in Eqs. 2, but it really can solve the set of Eqs. 1. To modify the coefficients one needs to modify both functions set_alph_HW on line 62 and setCoef_HW on line 71 of the file HW_iso.cpp. The coeffiecient are such that $a_n = phi\_an * an$ for example where an is specified in set_alpha_HW and phi_an is specified in setCoef_HW. alph and alph_n which are the input parameters of set_alph_HW are hard coded as $g^2/2$ in the main file lines 68 to 70. Getting rid of the linear terms is as easy as setting C and kappa to 0 in the INPUT file.