Hybrid Tree-Shell Model for Turbulence Cascades

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1 Hybrid Tree-Goy model

The tree model is an extension of the shell model to account for inhomogeneity in space. Akin to the wavelet analysis, the dynamical variables $\Phi_{n,N}$ is defined for a given space and scale. This model is proposed by Aurel (1994) and applied in (1997) on 2D fluid turbulence. We propose a hybrid tree-shell model that incorporates inhomogeneities up to a certain sclae beyond which the homogeneity is assumed. This results in a tree structure that tranistions from a hierarchical tree to a set of shell models at different spatial points. In addition to the hybrid aspet of the model, we introduce spatial diffusion.

2 Equation for Hybrid Tree-Goy shell model

Instead of having one fourier value Φ_n to quantify the strength at that scale, $\Phi_{n,i}$ is used to specify the strength of the fourier component at a scale n in the ith spatial location. Figure shows the division and component representation in 2d turbulence, while fig. shows the

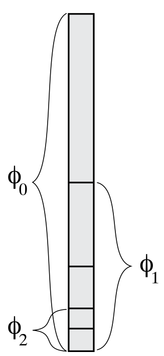

Figure 1: Representation of $\Phi_{n,i}$ in 1d.

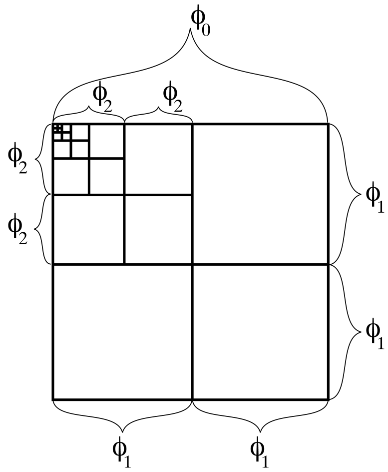

Figure 2: Representation of $\Phi_{n,i}$ in 2d.

Each compoenent interacts with the nearest components best represented in a hierarchical tree os degree $D = 2^d$ where d is the spatial dimensions of tubrulence considered. Akin to GOY shell model the interaction is

$$\begin{aligned}\frac{\Phi_{n,i}}{dt} &= a_n\Phi_{n-1,ceil(i/D)}\Phi_{n-2,ceil(i/D^2)} + b_n\Phi_{n-1,ceil(i/D)}\frac{\sum_k \Phi_{n+1,k}}{D} + c_n \\ &+c_n\frac{\sum_{k=(i-1)*D:i*D}\Phi_{n+1,k}}{D}\frac{\sum_{k=(i-1)*D^2:i*D^2}\Phi_{n+2,k}}{D^2}\end{aligned} \tag{1}$$

For a given node in the part of the tree structure, the node interacts with its parent, grandparent, children and all its grandchildren accoding to Eq. 1. We represent the interactions graphically in figs. 3 and 4 for a binary tree modeling 1d turbulence. The node in solid red and all the nodes it interacts with with solid borders both when the node and all its nearest neighbors are completely in the tree structure (fig. 3) and when the node is near the interface (fig. 4) where it has neighbors both within the tree and the peripheral parts that are a shell model.

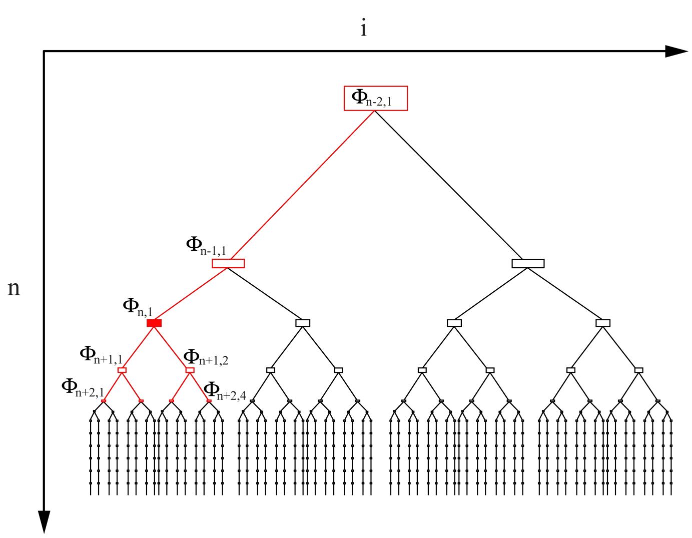

Figure 3: interactions for nodes within the hierarchical tree of a hybrid hierarchical tree

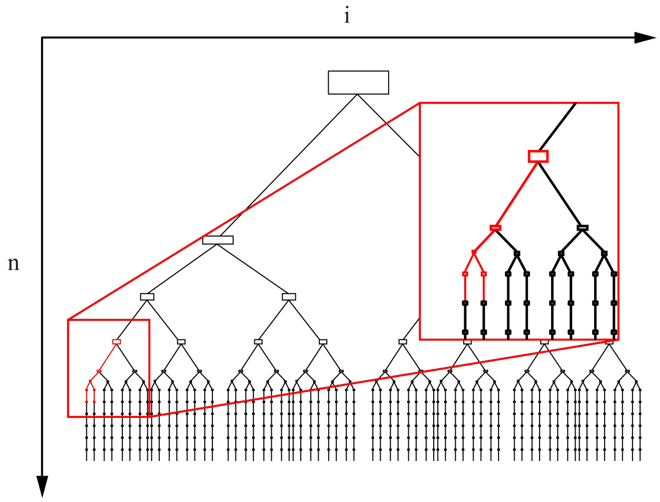

Figure 4: interactions for nodes near the interface of the tree and shell parts of a hybrid hierarchical tree

2.1 2D fluid turbulence

2.1.1 Coefficients

$$a_n = \alpha k_n^4 \frac{g^2 - 1}{g^7(k_n + 1)}$$

$$b_n = -\alpha k_n^4 \frac{g^4 - 1}{g^4(k_n + 1)}$$

$$c_n = \alpha k_n^4 \frac{g^2 - 1}{g^7(k_n + 1)}$$

3 Benchmarking

To benchmark the hierarchical tree model, we compare the results for homogeneous 2D turbulence resulting from tree model. Below is the result for choosing 10 levels in teh tree and 19 levels for the shell after that for a total of 29 scales. The system is forced at the 8th scale in a homogeous way (all shells at the 8th scale are forced) and there is a dissipation at the small and large scales. The hierarchical tree structure captures the inverse cascade of enstropy better then the convensional shell model.

4 Diffusion in Wavelet form

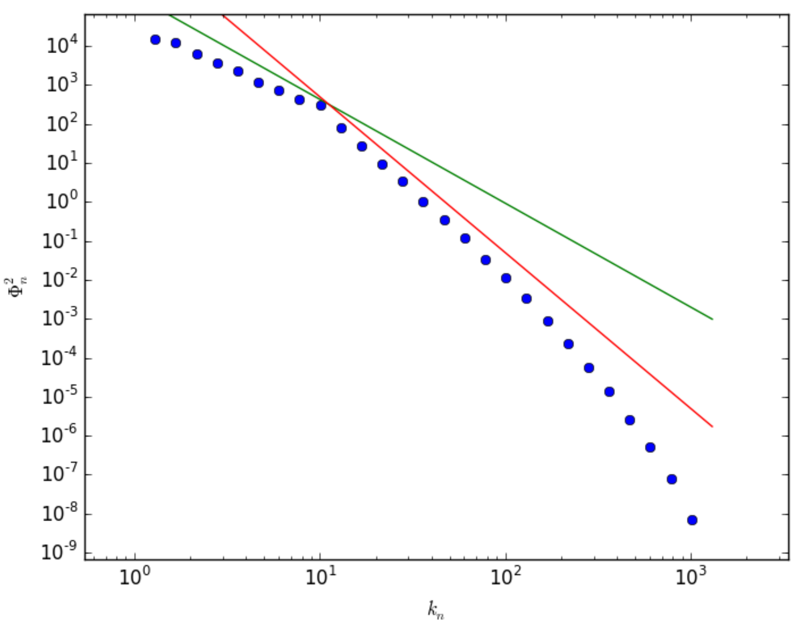

Figure 5: Benchmarking the hybrid tree-shell model in teh case of homogeneous 2d turbulence. The blue does are the averaged $\Phi^2_{n,i}$ for all i at a given scale n over 1000 runs, the green line is $k^{-8/3}$ and the red line is k^{-4}.