Documentation for anisotropic HW shell models

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1 Overview of equations

The GOY shell model is implemented for the Hasegawa Wakatani system of equations

$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right)\nabla^2\phi - C(\phi - n) = D_\phi$$

$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right)n + \kappa\frac{\partial\phi}{\partial y} - C(\phi - n) = D_n.$$

The Shell model was introduced to capture the richardson cascades in turbulent systems in a quick way. After representing the eqauation(s) in Fourier space,

$$\frac{\partial}{\partial t}\phi_k(t) = L[n_k, \phi_k] + \frac{1}{2}\sum_{k=k'+k''}\phi_{k'}(t)\phi_{k''}(t)$$

$$\frac{\partial}{\partial t}n_k(t) = L[n_k, \phi_k] + \frac{1}{2}\sum_{k=k'+k''}n_{k'}(t)\phi_{k''}(t)$$

where $L[\,]$ is the linear part of an equation and the nonlinear term is written as a sum over all possible fourier components.

The isotropized dynamical variable are $\Phi_n = \left[\frac{1}{k_n^2}\int_{k_n}^{k_{n+1}} \mathrm{d}k \int \mathrm{d}\alpha_k \left|\Phi_{\mathbf{k}}\right|^2 k^3\right]^{1/2}$ and $n_n = \left[\int_{k_n}^{k_{n+1}} \mathrm{d}k \int \mathrm{d}\alpha_k \left|n_{\mathbf{k}}\right|^2 k\right]^{1/2}$ over angles α_k for a fiven scale k and over all k vales between k_n and $k_n + dk$ with dk scaling geometrically, $dk = g^n$. However, Instead of integrating over all angles in Fourier Space the integral is over sectors in phase space resulting in the following dynamical variables for a given scale: Φ_n^X integrated from $\alpha_k = -\pi/8$ to $\pi/8$, Φ_n^0 from $\pi/8$ to $3\pi/8$, and Φ_n^Y from $3\pi/8$ to $5\pi/8$. We assume reflective symmetry with the x and y axese in Fourier space, therefore the counterpart of each Φ_n dynamical variable of the isotropic model is $2\Phi_n^X$'s, $4\Phi_n^0$'s, and $2\Phi_n^Y$'s in the anisotropic model. The various anisotropic shell variables still maintain a triad interaction where the shell with wave number k interacts with shells q, p such that $k = q + p$. In Fig. 1 we highlight which shells interact with each of the three anisotropic shells (Φ_n^X, Φ_n^0, Φ_n^Y) for a given n. The black sector is the n^{th} variable of concern (with k wave number) and the three other colors: blue for the $n-2,\ n-1$ interaction term, red for $n-1,\ n+1$ and red for $n+1,\ n+2$. Notice how the wave number add for the interactions such that $k = q + p$ where q, p are the wave numbers of the other two terms sharing the same color.

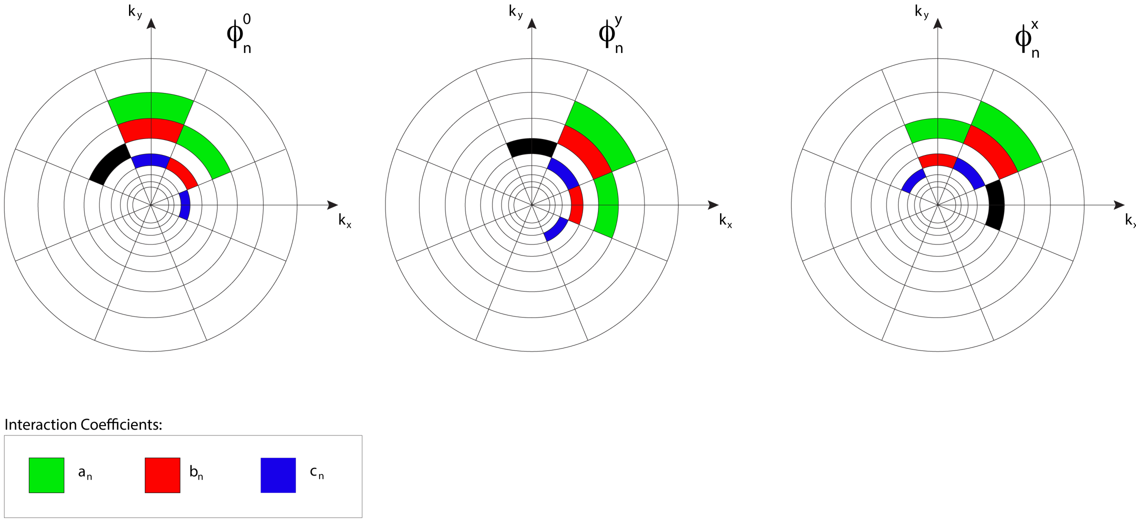

Figure 1: A schematic illustrating the interction of each anisotropic shell $\Phi_n^{X,0,Y}$ (black) with the $n-2,\ n-1$ shells (blue), $n-1,\ n+1$ shells (red), and $n+1,\ n+2$ shells (green).

1.1 Shell model for anisotropic shell model

We present the detailed equation for the evolution of the anisotropic shells as illustrated in Fig. 1. For berivty we only present the nonlinear terms keeping in mind that the evolution of each variable also includes a $L[\phi_n^S]$ linear term.

$$\frac{\partial \Phi_n^{(x)}}{\partial t} = a_n^{(x)} \left(\Phi_{n-2}^{(0)} \Phi_{n-1}^{(0)}\right) + b_n^{(x)} \left(\Phi_{n-1}^{(y)} \Phi_{n+1}^{(0)}\right) + c_n^{(x)} \left(\Phi_{n+1}^{(y)} \Phi_{n+2}^{(0)}\right) \tag{1a}$$

$$\frac{\partial \Phi_n^{(y)}}{\partial t} = a_n^{(y)} \left(\Phi_{n-2}^{(0)} \Phi_{n-1}^{(0)}\right) + b_n^{(y)} \left(\Phi_{n-1}^{(x)} \Phi_{n+1}^{(0)}\right) + c_n^{(y)} \left(\Phi_{n+1}^{(x)} \Phi_{n+2}^{(0)}\right) \tag{1b}$$

$$\begin{aligned}\frac{\partial \Phi_n^{(0)}}{\partial t} =& a_n^{(0)} \left(\Phi_{n-2}^{(x)} \Phi_{n-1}^{(y)} + \Phi_{n-1}^{(x)} \Phi_{n-2}^{(y)}\right) + b_n^{(0,x)} \left(\Phi_{n-1}^{(0)} \Phi_{n+1}^{(x)}\right) \\ &+ b_n^{(0,y)} \left(\Phi_{n-1}^{(0)} \Phi_{n+1}^{(y)}\right) + c_n^{(0,x)} \left(\Phi_{n+1}^{(0)} \Phi_{n+2}^{(x)}\right) + c_n^{(0,y)} \left(\Phi_{n+1}^{(0)} \Phi_{n+2}^{(y)}\right)\end{aligned} \tag{1c}$$

where a_n, b_n and $c_n$are calculated for a given system using conservation laws. For the Hasegawa Wakatani system there is also the equations for the evoution of denisty which are written in shell variables as

$$\frac{\partial n_n^{(x)}}{\partial t} = a_n^{(x)\prime} \left(n_{n-2}^{(0)} \Phi_{n-1}^{(0)} - \Phi_{n-2}^{(0)} n_{n-1}^{(0)}\right) + b_n^{(x)\prime} \left(n_{n-1}^{(y)} \Phi_{n+1}^{(0)} - \Phi_{n-1}^{(y)} n_{n+1}^{(0)}\right) + c_n^{(x)\prime} \left(n_{n+1}^{(y)} \Phi_{n+2}^{(0)} - \Phi_{n+1}^{(y)} n_{n+2}^{(0)}\right) \tag{2a}$$

$$\frac{\partial n_n^{(y)}}{\partial t} = a_n^{(y)\prime} \left(n_{n-2}^{(0)} \Phi_{n-1}^{(0)} - \Phi_{n-2}^{(0)} n_{n-1}^{(0)}\right) + b_n^{(y)\prime} \left(n_{n-1}^{(x)} \Phi_{n+1}^{(0)} - \Phi_{n-1}^{(x)} n_{n+1}^{(0)}\right) + c_n^{(y)\prime} \left(n_{n+1}^{(x)} \Phi_{n+2}^{(0)} - \Phi_{n+1}^{(x)} n_{n+2}^{(0)}\right) \tag{2b}$$

$$\begin{aligned}\frac{\partial n_n^{(0)}}{\partial t} =& a_n^{(0)\prime} \left(n_{n-2}^{(x)} \Phi_{n-1}^{(y)} - \Phi_{n-2}^{(x)} n_{n-1}^{(y)} + n_{n-2}^{(y)} \Phi_{n-1}^{(x)} - \Phi_{n-2}^{(y)} n_{n-1}^{(x)}\right) + b_n^{(0,x)\prime} \left(n_{n-1}^{(0)} \Phi_{n+1}^{(x)} - \Phi_{n-1}^{(0)} n_{n+1}^{(x)}\right) \\ &+ b_n^{(0,y)\prime} \left(n_{n-1}^{(0)} \Phi_{n+1}^{(y)} - \Phi_{n-1}^{(0)} n_{n+1}^{(y)}\right) + c_n^{(0,x)\prime} \left(n_{n+1}^{(0)} \Phi_{n+2}^{(x)} - \Phi_{n+1}^{(0)} n_{n+2}^{(x)}\right) + c_n^{(0,y)\prime} \left(n_{n+1}^{(0)} \Phi_{n+2}^{(y)} - \Phi_{n+1}^{(0)} n_{n+2}^{(y)}\right)\end{aligned} \tag{2c}$$

Using the conservation laws for Kinetic Energy

$$E_K = \sum_n k_n^2 \Phi_n^2 = \sum_n \left[k_n^2 \Phi_n^{(x)2} + k_n^2 \Phi_n^{(y)2} + 2k_n^2 \Phi_n^{(0)2}\right]$$

for Enstopy

$$W = \sum_n k_n^4 \Phi_n^2 = \sum_n \left[k_n^4 \Phi_n^{(x)2} + k_n^4 \Phi_n^{(y)2} + 2k_n^4 \Phi_n^{(0)2}\right]$$

and Internal energy

$$E_I = \sum_n n_n^2 = \sum_n \left[n_n^{(x)2} + n_n^{(y)2} + 2n_n^{(0)2} \right]$$

and Cross Helicity

$$H = \sum_n k_n^2 \Phi_n n_n = \sum_n k_n^2 \left[n_n^{(x)} \Phi_n^{(x)} + n_n^{(y)} \Phi_n^{(y)} + 2n_n^{(0)} \Phi_n^{(0)} \right]$$

We find the relations for the coefficients and the anisotropic shell equations for Hasegawa Wakatani take the form:

$$\frac{\partial \Phi_n^{(x)}}{\partial t} = k_n^2 \left(g^2 - 1\right) \left[\alpha^{(x)} g^{-7} \left(\Phi_{n-2}^{(0)} \Phi_{n-1}^{(0)} \right) - 2\alpha^{(0)} g^{-3} \left(g^2 + 1\right) \left(\Phi_{n-1}^{(y)} \Phi_{n+1}^{(0)} \right) + 2\alpha^{(0)} g^3 \left(\Phi_{n+1}^{(y)} \Phi_{n+2}^{(0)} \right) \right] \tag{3a}$$

$$\frac{\partial \Phi_n^{(y)}}{\partial t} = k_n^2 \left(g^2 - 1\right) \left[\alpha^{(y)} g^{-7} \left(\Phi_{n-2}^{(0)} \Phi_{n-1}^{(0)} \right) - 2\alpha^{(0)} g^{-3} \left(g^2 + 1\right) \left(\Phi_{n-1}^{(x)} \Phi_{n+1}^{(0)} \right) + 2\alpha^{(0)} g^3 \left(\Phi_{n+1}^{(x)} \Phi_{n+2}^{(0)} \right) \right] \tag{3b}$$

$$\begin{aligned} \frac{\partial \Phi_n^{(0)}}{\partial t} =& \frac{1}{2} k_n^2 \left(g^2 - 1\right) \Big\{ 2\alpha^{(0)} g^{-7} \left(\Phi_{n-2}^{(x)} \Phi_{n-1}^{(y)} + \Phi_{n-1}^{(x)} \Phi_{n-2}^{(y)} \right) - \alpha^{(x)} \left(g^2 + 1\right) g^{-3} \left(\Phi_{n-1}^{(0)} \Phi_{n+1}^{(x)} \right) \\ & - \alpha^{(y)} \left(g^2 + 1\right) g^{-3} \left(\Phi_{n-1}^{(0)} \Phi_{n+1}^{(y)} \right) + \alpha^{(x)} g^3 \left(\Phi_{n+1}^{(0)} \Phi_{n+2}^{(x)} \right) + \alpha^{(y)} g^3 \left(\Phi_{n+1}^{(0)} \Phi_{n+2}^{(y)} \right) \Big\} \end{aligned} \tag{3c}$$

$$\frac{\partial n_n^{(x)}}{\partial t} = k_n^2 \left[\alpha'^{(x)} g^{-3} \left(n_{n-2}^{(0)} \Phi_{n-1}^{(0)} - \Phi_{n-2}^{(0)} n_{n-1}^{(0)} \right) - 2\alpha'^{(0)} g^{-1} \left(n_{n-1}^{(y)} \Phi_{n+1}^{(0)} - \Phi_{n-1}^{(y)} n_{n+1}^{(0)} \right) + 2\alpha'^{(0)} g \left(n_{n+1}^{(y)} \Phi_{n+2}^{(0)} - \Phi_{n+1}^{(y)} n_{n+2}^{(0)} \right) \right] \tag{3d}$$

$$\frac{\partial n_n^{(y)}}{\partial t} = k_n^2 \left[\alpha'^{(y)} g^{-3} \left(n_{n-2}^{(0)} \Phi_{n-1}^{(0)} - \Phi_{n-2}^{(0)} n_{n-1}^{(0)} \right) - 2\alpha'^{(0)} g^{-1} \left(n_{n-1}^{(x)} \Phi_{n+1}^{(0)} - \Phi_{n-1}^{(x)} n_{n+1}^{(0)} \right) + 2\alpha'^{(0)} g \left(n_{n+1}^{(x)} \Phi_{n+2}^{(0)} - \Phi_{n+1}^{(x)} n_{n+2}^{(0)} \right) \right] \tag{3e}$$

$$\begin{aligned} \frac{\partial n_n^{(0)}}{\partial t} =& \frac{k_n^2}{2} \Big\{ 2\alpha'^{(0)} g^{-3} \left(n_{n-2}^{(x)} \Phi_{n-1}^{(y)} - \Phi_{n-2}^{(x)} n_{n-1}^{(y)} + n_{n-2}^{(y)} \Phi_{n-1}^{(x)} - \Phi_{n-2}^{(y)} n_{n-1}^{(x)} \right) - \alpha'^{(x)} g^{-1} \left(n_{n-1}^{(0)} \Phi_{n+1}^{(x)} - \Phi_{n-1}^{(0)} n_{n+1}^{(x)} \right) \\ & - \alpha'^{(y)} g^{-1} \left(n_{n-1}^{(0)} \Phi_{n+1}^{(y)} - \Phi_{n-1}^{(0)} n_{n+1}^{(y)} \right) + \alpha'^{(x)} g \left(n_{n+1}^{(0)} \Phi_{n+2}^{(x)} - \Phi_{n+1}^{(0)} n_{n+2}^{(x)} \right) + \alpha'^{(y)} g \left(n_{n+1}^{(0)} \Phi_{n+2}^{(y)} - \Phi_{n+1}^{(0)} n_{n+2}^{(y)} \right) \Big\} \end{aligned} \tag{3f}$$

2 Specifics of the code

The working file work.cpp, the functions in HW_iso.cpp and the header file HW_iso.h are compiled with the GSL library and openMP to create the executive shell_iso_HW. This executive solves the set of ODE equations for the Hasegawa Wakatani specifically where the paramters specific to the HW system can be specifed in the input file INPUT as discussed in the README file.

If the equations to be solved are different, then the coeffients need to be updated manually in the file HW_iso.cpp. We discuss briefly the files and how to modify things if any to make it work for a different set of equations.

2.1 work.cpp

This is where the main function is. It calls functions from HW_iso.cpp to integrate the set of ODEs using GSL solver. Most of the parameters in this file can be specified in the INPUT file as explained in the README file. However, the gsl tolerence can be modified (lines 92 and 93) as well as the format of the output files can be modified (lines 178 and 179).

2.2 HW_iso.cpp

This program is written specifically to solve for the Hasegawa Wakatani system in Eqs. 3, but it really can solve the set of Eqs. 2. To modify the coefficients one needs to modify both functions set_alph_HW on line 62 and setCoef_HW on line 71 of the file HW_iso.cpp. The coeffiecient are such that $a_n^y = phi\_an * any$ for example where an is specified in set_alpha_HW and phi_an is specified in setCoef_HW. alph and alph_n which are the input parameters of set_alph_HW are hard coded as $g^2/2$ in the main file lines 67 to 71. Getting rid of the linear terms is as easy as setting C and kappa to 0 in the INPUT file.