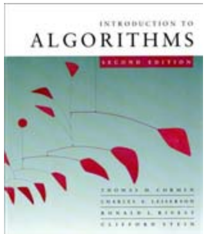


Divide-and-conquer algorithm

IDEA:

$n\times n$ matrix = 2×2 matrix of $(n/2)\times(n/2)$ submatrices:

$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \cdot \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]$$

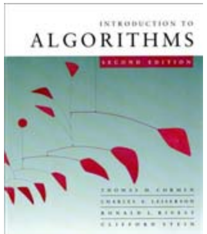
$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dh \\ u &= cf + dg \end{aligned}\right\}$$

recursive

8 mults of $(n/2)\times(n/2)$ submatrices

4 adds of $(n/2)\times(n/2)$ submatrices



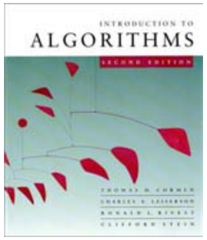
Analysis of D&C algorithm

$$T(n) = \underbrace{8}_{\#\ \textit{submatrices}}\, T(\underbrace{n/2}_{\textit{submatrix size}}) + \underbrace{\Theta(n^2)}_{\textit{work adding submatrices}}$$

 # submatrices

submatrix size

work adding submatrices



Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices

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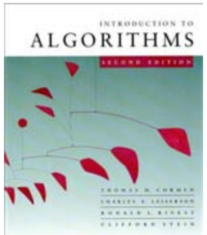
$$n^{\log_b a} = n^{\log_2 8} = n^3 \;\Rightarrow\; \textsc{Case 1} \;\Rightarrow\; T(n) = \Theta(n^3).$$



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submatrix size

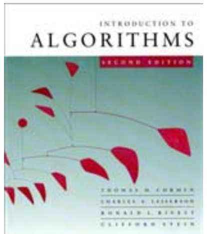
work adding submatrices

$$n^{\log_b a} = n^{\log_2 8} = n^3 \;\Rightarrow\; \text{Case 1} \;\Rightarrow\; T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.

Strassen's idea

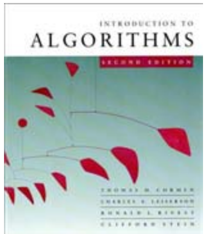
- Multiply 2×2 matrices with only 7 recursive mults.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$



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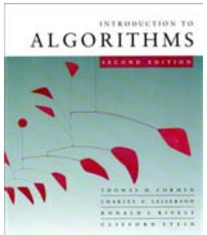
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$



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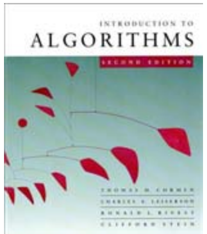
\end{aligned}
$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!



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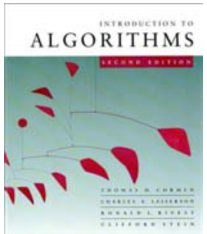
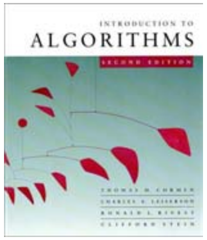
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$



Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.



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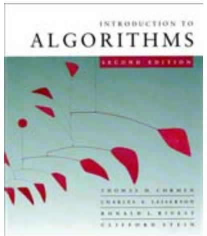
$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



Analysis of Strassen

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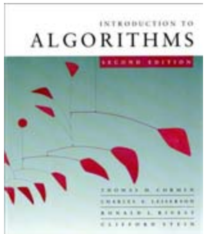
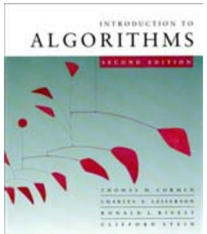


Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \;\Rightarrow\; \text{Case 1} \;\Rightarrow\; T(n) = \Theta(n^{\lg 7}).$$



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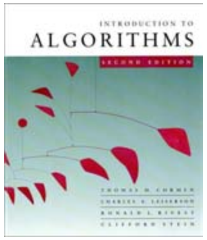
The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.



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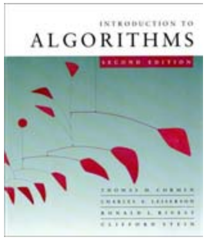
Best to date (of theoretical interest only): $\Theta(n^{2.376\cdots})$.

VLSI layout

Problem: Embed a complete binary tree with n leaves in a grid using minimal area.



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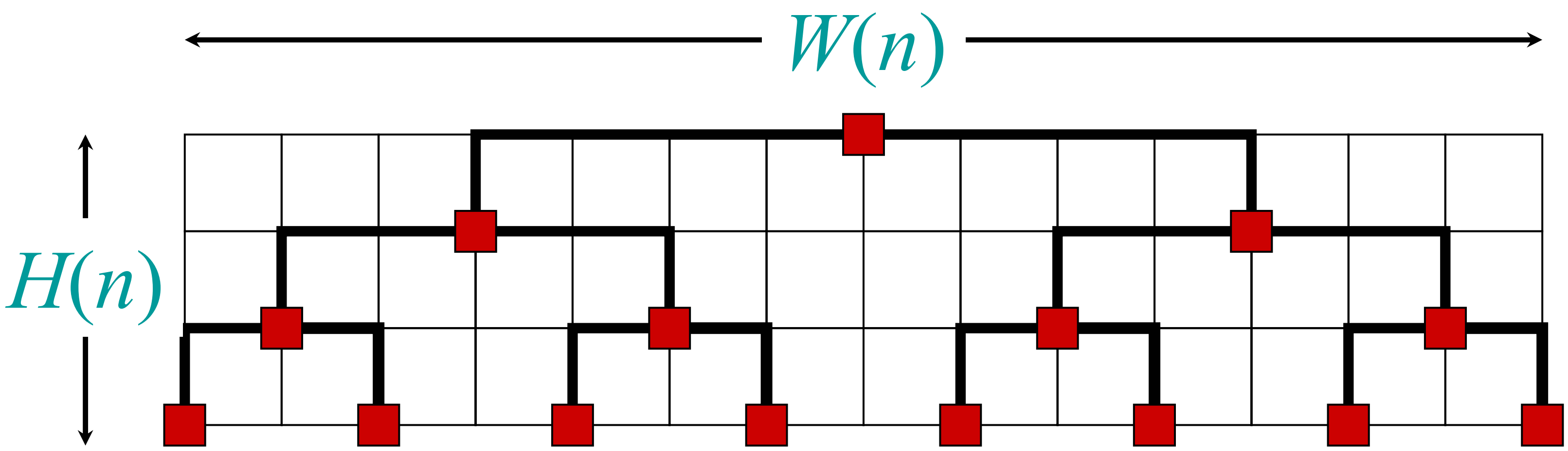

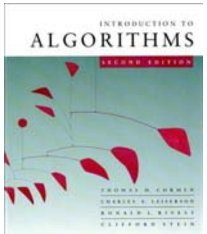

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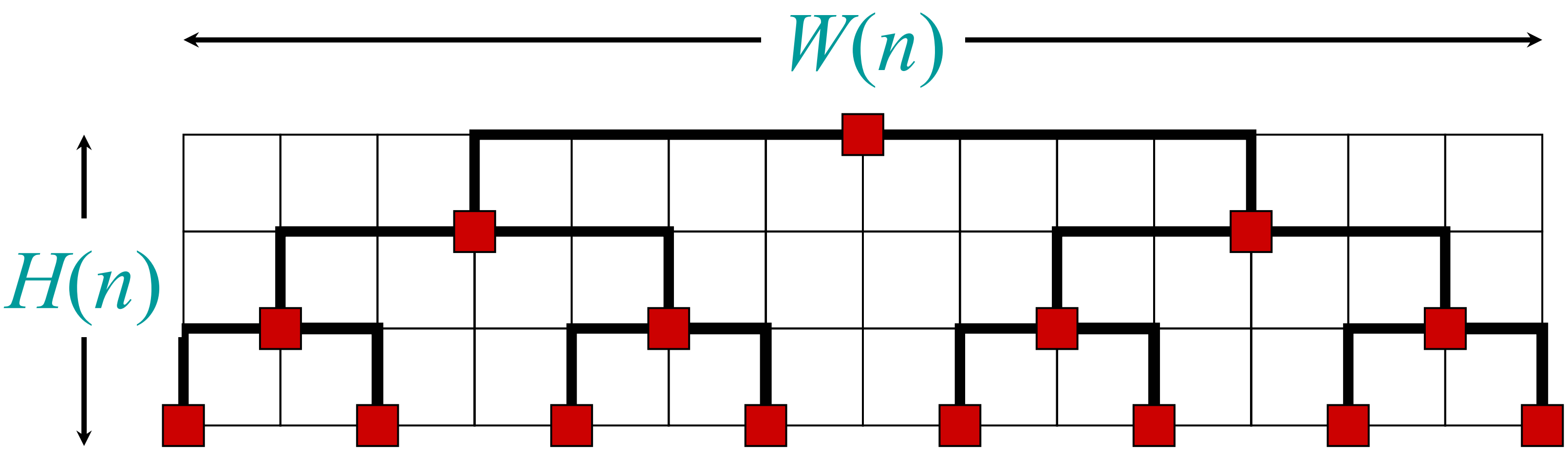

$$
\begin{aligned}
H(n) &= H(n/2) + \Theta(1) \\
&= \Theta(\lg n)
\end{aligned}
$$

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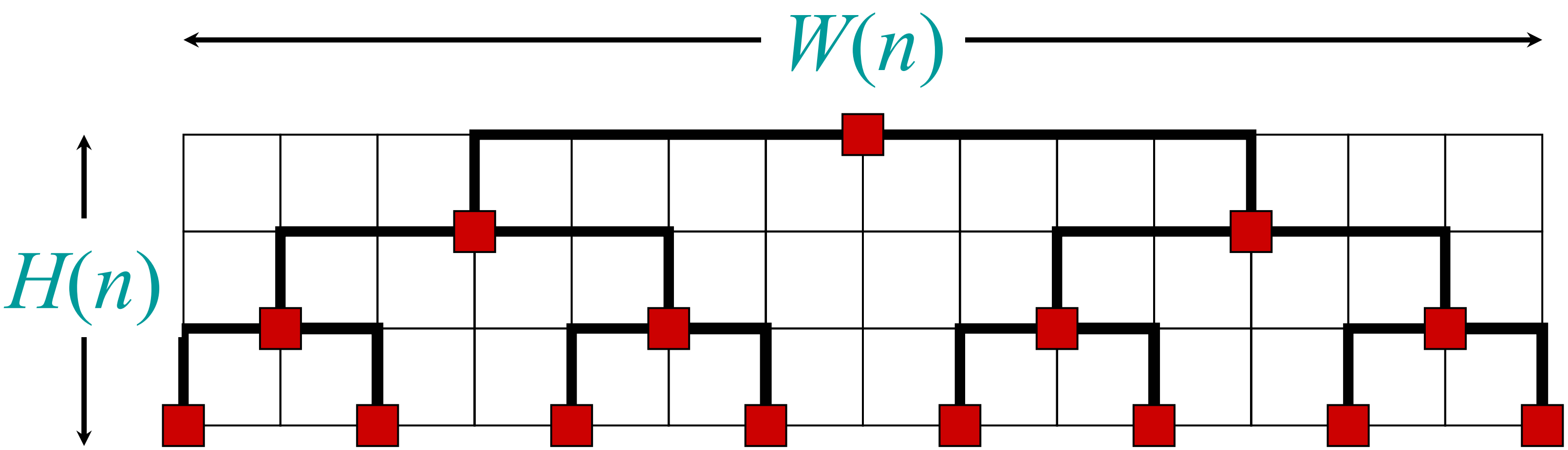
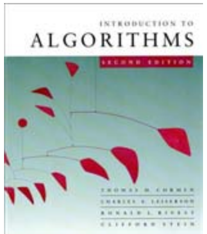


$$H(n) = H(n/2) + \Theta(1) = \Theta(\lg n)$$

$$W(n) = 2\,W(n/2) + \Theta(1) = \Theta(n)$$



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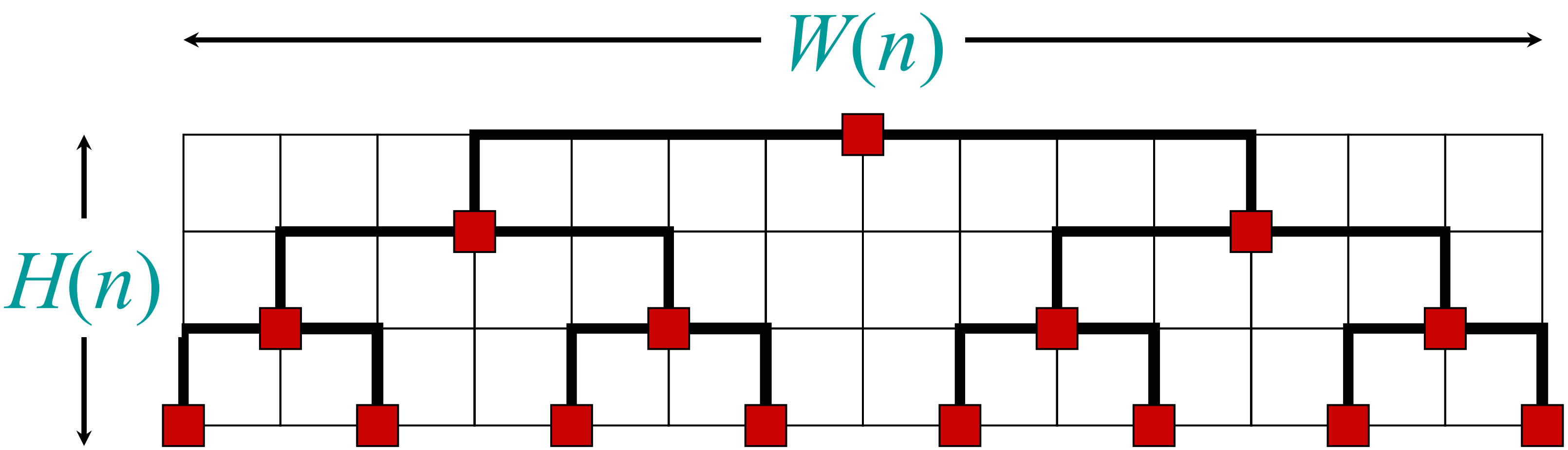
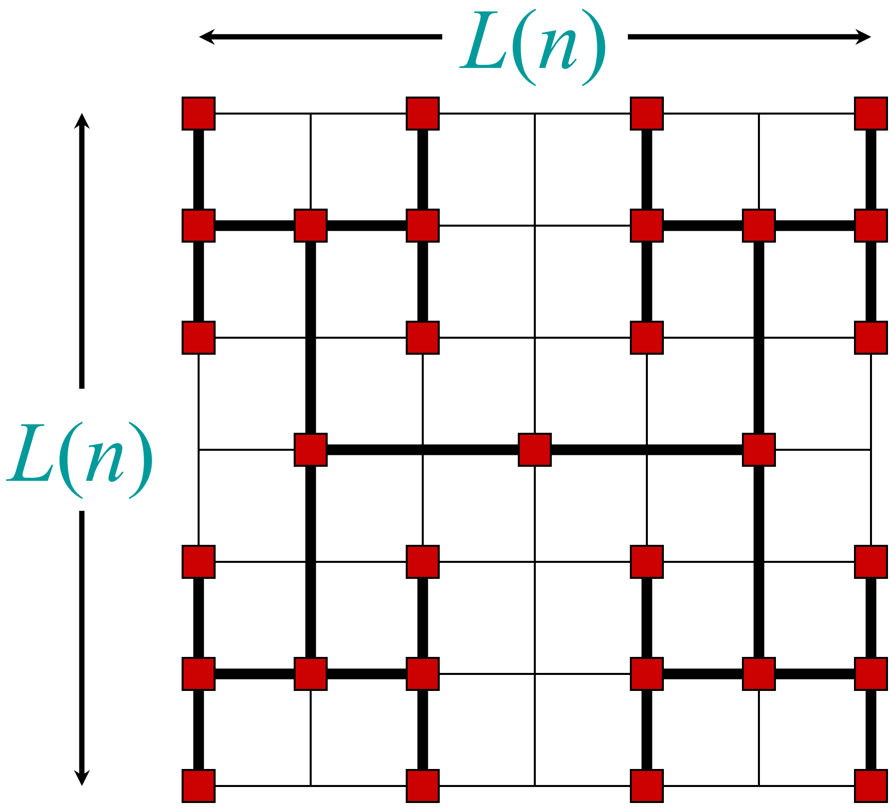


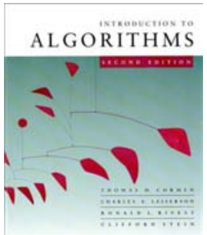
$$H(n) = H(n/2) + \Theta(1) = \Theta(\lg n)$$

$$W(n) = 2\,W(n/2) + \Theta(1) = \Theta(n)$$

$$\text{Area} = \Theta(n \lg n)$$

H-tree embedding





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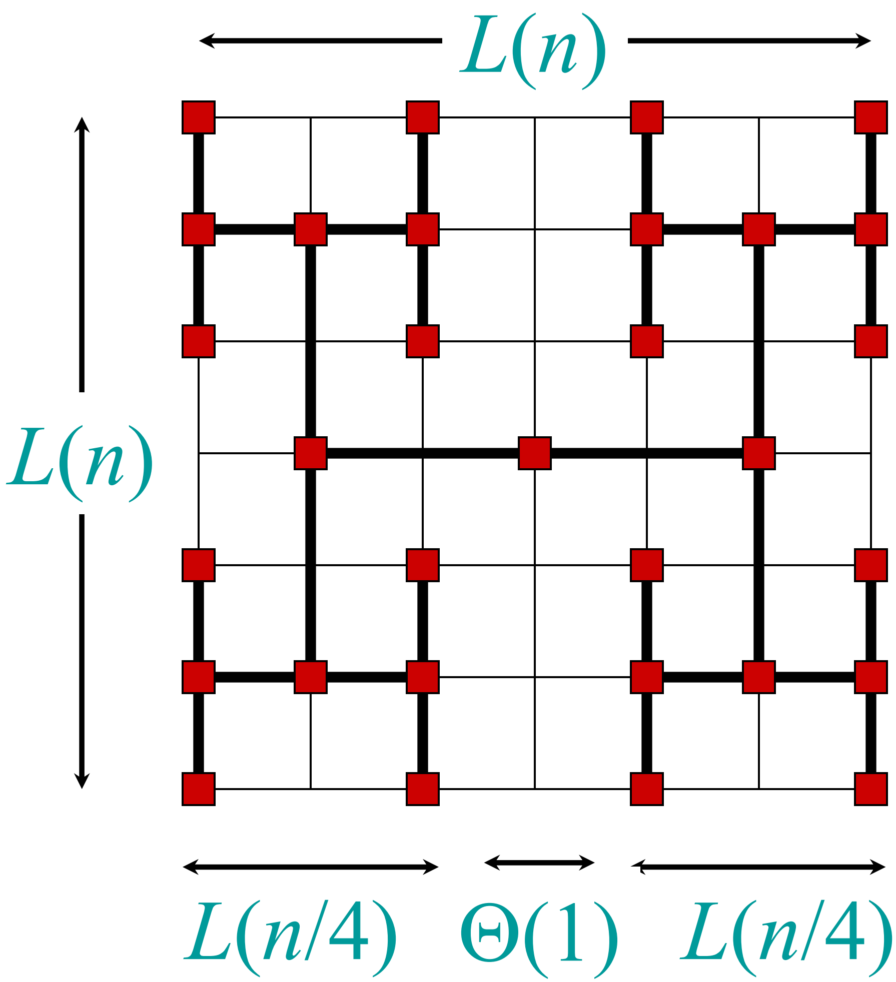

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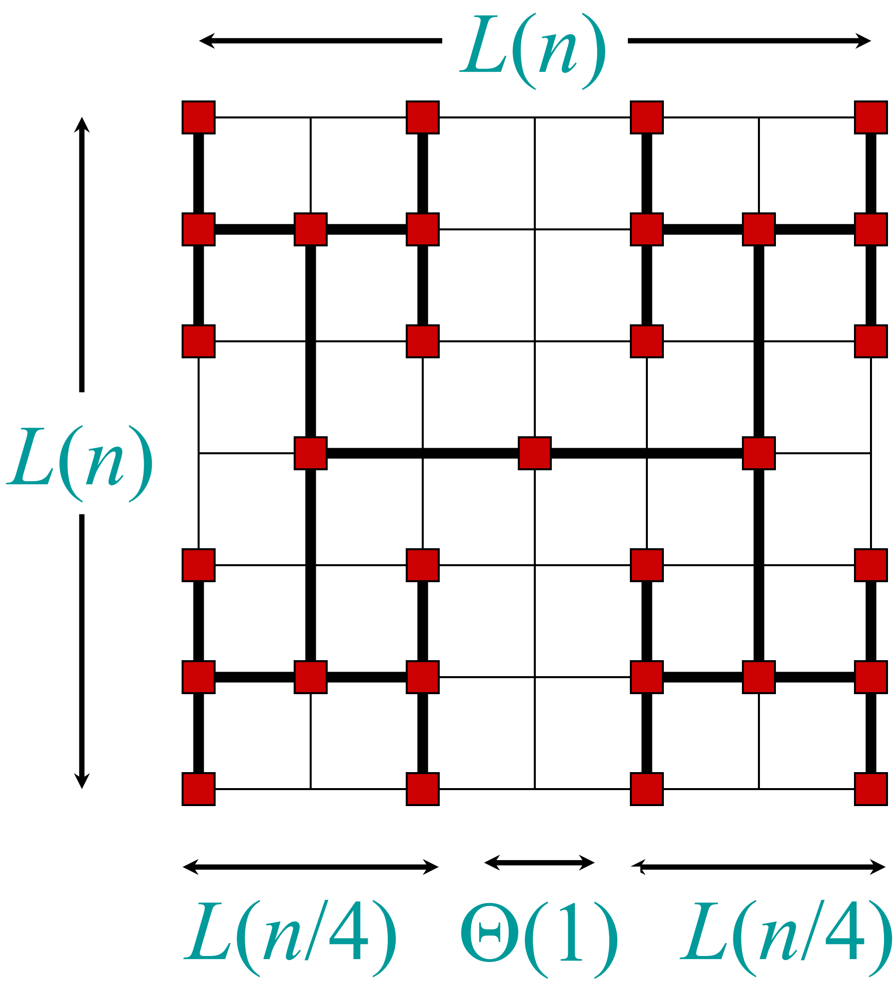
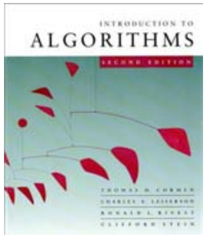


$$L(n) = 2L(n/4) + \Theta(1)$$
$$= \Theta(\sqrt{n})$$

Area $= \Theta(n)$



Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- The divide-and-conquer strategy often leads to efficient algorithms.