Quantitative Genomics and Genetics - Spring 2019
BTRY 4830/6830; PBSB 5201.01

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Problem 1 (Easy)

Consider an experiment of one coin flip.

a. What is the sample space Ω of this experiment?

   $\Omega = \{H, T\}$

b. What is the Sigma-algebra (containing all events) for this sample space?

   $\mathcal{F} = \emptyset, \{H\}, \{T\}, \{H, T\}$

c. Define a probability model on Ω (i.e. assign specific probabilities to each outcome) such that $Pr(H) = Pr(T) = 0.5$. What is the probability of each event of the Sigma-algebra that you defined in part [b]?

   $Pr(\emptyset) = 0$ $Pr(H) = 0.5$ $Pr(T) = 0.5$ $Pr(H \cup T) = 1$

d. Considering the probability model in part [c] calculate $Pr(H \cap T)$ and explain why this demonstrates these events are not independent.

$Pr(H \cap T) = Pr(\emptyset) = 0$

For events to be independent, they must follow $Pr(A \cap B) = Pr(A)Pr(B)$ $Pr(H)Pr(T). = 0.5 * 0.5 = 0.25 \neq 0$ This also makes logical sense because we are assuming a fair coin flip based on the probability model which suggests if you flip a head you could not have flipped a tail.

Problem 2 (Medium)

Consider an experiment of two coin flips.

a. What is the sample space Ω of this experiment and what is the Sigma-algebra (containing all events) for this sample space? $\Omega = \{HH, TH, HT, TT\}$ $\mathcal{F} = \emptyset, \{HH\}, \{TT\}, \{HT\}, \{TH\}$

$\{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}, \{HH, HT, TH, TT\}$

b. Which of these events is 'neither flip is tails' (i.e., the event $\{\neg TT\}$ with $\neg$ indicating 'not' where you may symbolize this event as {not(TT)})? Which of these events is the event 'the second flip is heads' (i.e., $\{H_{2nd}\}$)?

   $\{\neg TT\} = \{HH\}$ $\{H_{2nd}\} = \{HH, TH\}$

c. Define a probability model on Ω (i.e. assign specific probabilities to each outcome) such that $Pr(HH) = 0.2, Pr(HT) = 0.3, Pr(TH) = 0.3, Pr(TT) = 0.2$. What is $Pr(H_{2nd}|\neg TT)$?

   $Pr(\emptyset) = 0$ $Pr(HH) = 0.2$ $Pr(HT) = 0.3$ $Pr(TH) = 0.3$ $Pr(TT) = 0.2$ $Pr(HH \cup HT) = 0.5$ $Pr(HH \cup TH) = 0.5$ $Pr(HH \cup TT) = 0.4$ $Pr(HT \cup TH) = 0.6$ $Pr(HT \cup TT) = 0.5$ $Pr(TH \cup TT) = 0.5$ $Pr(HH \cup HT \cup TH) = 0.8$ $Pr(HH \cup HT\ cupTT) = 0.7$ $Pr(HH \cup TH \cup TT) = 0.7$ $Pr(HT \cup TH \cup TT) = 0.8$ $Pr(HH \cup HT \cup TH \cup TT) = 1$ $Pr(H_{2nd}|\neg TT) = \frac{Pr(H_{2nd} \cap \neg TT)}{Pr(\neg TT)} = Pr(HH)/Pr(HH) = 0.2/0.2 = 1$

d. Could the probability model defined in part [c] be the correct probability model for a two coin flip experiment (explain your answer)? If so, provide a description of a system where this probability model would be correct.
   Yes, the probability function defined for the Sigma Field obeys the three axioms of probability. This model could be correct if the coin is not fair and for whatever reason the outcomes HT, TH are more likely than HH, TT.

e. Define the random variable X_1 that 'takes the value one if the two flips are the same and zero if the two flips are different' and considering the probability model in part [c], show the values for this random variable for each possible outcome and the probabilities for each possible outcome. Define the random variable X_2 that is 'number of heads on the first flip' and considering the probability model in part [c], show the values for this random variable for each possible outcome and the probabilities for each possible outcome. Are the random variables X_1 and X_2 independent? Demonstrate that this is the case.

Outcome	X_1	X_2
$\emptyset$	0	0
HH	1	1
HT	0	1
TH	0	0
TT	1	0

   $Pr(X_1 = 1) = Pr(HH \cup TT) = 0.4$ $Pr(X_1 = 0) = Pr(HT \cup TH) = 0.6$ $Pr(X_2 = 1) = Pr(HH \cup HT) = 0.5$ $Pr(X_2 = 0) = Pr(TH \cup TT) = 0.5$

   By the definition of independence we would expect, $Pr(A_i \cap A_j) = Pr(A_i)Pr(A_j) = 0.4*0.5 = 0.2$ And given the probability model $Pr(X_1 = 1 \cap X_2 = 1) = Pr((HH \cup TT) \cap (HH \cup HT)) = Pr(HH) = 0.2$ Both give 0.2 so the events are independent.

Problem 3 (Difficult)

Consider the sample space and Sigma-algebra in question [2]. This is not the only Sigma-algebra that can be defined for this sample space. Write down another Sigma-algebra for this sample space and show that its sets satisfy the three properties of a Sigma Algebra. Also provide an intuitive explanation as to why this alternative Sigma-algebra you have presented cannot be used to define a useful probability function (model) for the experiment.

$\mathcal{F} = \emptyset, \{HH\}, \{HT, TH, TT\}, \{HH, HT, TH, TT\}$ This set obeys the properties of a Sigma Algebra: 1. It contains the empty set 2. It contains the event $\{HH\}$ and its complement $\{HT, TH, TT\}$ 3. It contains the union of all the sets $\{HT, TH, HH, TT\}$

We dont use this Sigma Algebra to define a probability function because its not as useful because it doesnt allow us to assign probabilities to all possible combinations of outcomes from Ω.