Witten and Morse Theory

Kate Mekechuk

Columbia Undergraduate
Mathematics Society

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Topological Invariants

Definition (Homology groups)

For a given manifold M, the *n-th Homology group* is defined:

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Definition (Euler characteristic)

The *Euler characteristic* for a manifold M is

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \, b_i$$

Homology of the Torus

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DIM	SUBMFLD	BOUNDARY
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1		−
1		∅
2		
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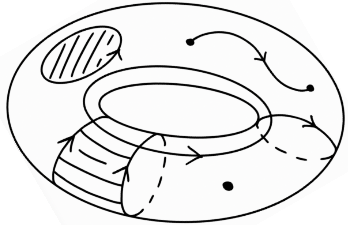

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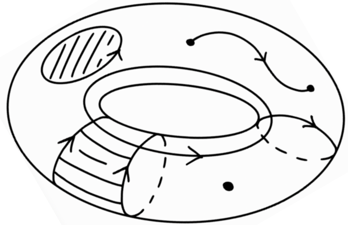

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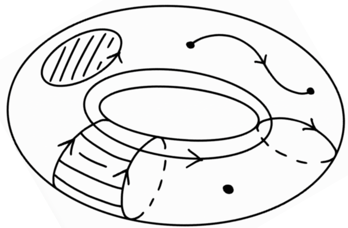

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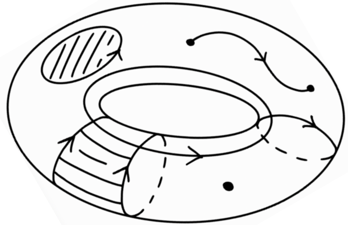

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Let $f : M \to \mathbb{R}$. We call a smooth real valued function f a *Morse function* if every critical point of f is non-degenerate (has a non-singular Hessian matrix).

$$\frac{\textit{determinant of}}{\textit{Hessian matrix}} = \det(H_f) = \det\left[\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right] \neq 0$$

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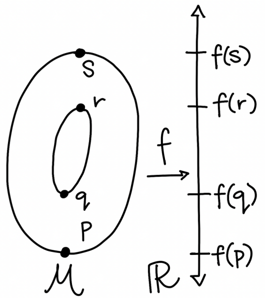
Definition (index)

The *(Morse) index* of a Morse function f is the maximal dimension of the subspace where H_f is negative definite (the number of negative eigenvalues of H_f).

Morse Theory and CW Complexes

Theorem

A Morse function on M admits a CW complex such that each critical point of index λ is a λ cell in the complex.



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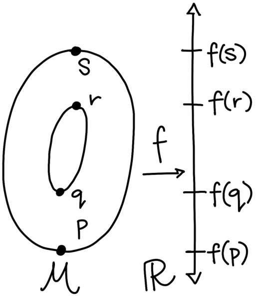

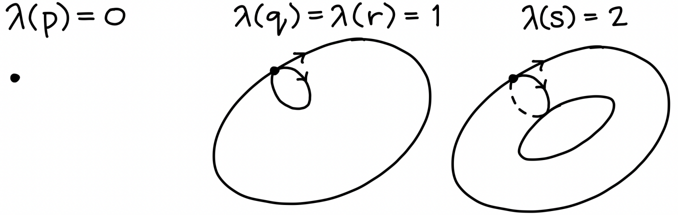

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Torus: $c_0 = 1 \qquad c_1 = 2 \qquad c_2 = 1$

$$\chi(T) = 1 - 2 + 1 = 0$$

Differential Forms

Let $N = \{1, \ldots, n\}$ be a set of numbers. We define $I = (i_1 < \ldots < i_p)$ to be a strictly ascending multi-index where $p \leq n$. We denote

$$\mathcal{J}_{p,n} = \{I = (i_1, \ldots, i_p) \mid 1 \leq i_1 < \cdots < i_p \leq n\}$$

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Example

$$f = (3x + 2y^2)dx \wedge dy + (7z^2 + 4)dx \wedge dz + (e^x + \sqrt{z})dy \wedge dz$$

Operators on Forms

Definition (Exterior Derivative)

The *exterior derivative* is a unique map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ of deg 1 such that $d^2 = 0$ and $d|_{\Omega^0(M)} = d|_{C^\infty(M)}$ is the usual differential.

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Definition (Laplacian Operator)

The *(Hodge) Laplacian operator* is a map $\Delta : \Omega^p(M) \to \Omega^p(M)$ such that

$$\Delta = d\delta + \delta d$$

De Rham Cohomology

Any form is decomposed into $\omega = d\alpha + \delta\beta + \gamma$ where $\Delta\gamma = 0$

1. closed forms $(d\omega = 0)$
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We conclude that the space of harmonic forms as a vector space is isomorphic to the space of the cohomology class.

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$$H^0(T) = \mathbb{R}/0 = \mathbb{R}$$

$$\dim(\mathbb{R}) = \textit{rank}\,(\mathbb{Z}) = b_0 = 1$$

Supersymmetric Theory

A supersymmetric quantum theory has a decomposition of the Hilbert Space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+$ is the space of bosonic states and $\mathcal{H}^-$ is the space of fermionic states.

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bosonic states | *fermionic states*

Witten defined the following supersymmetric operators on p-forms

$$Q_{1t} = d_t + \delta_t \qquad Q_{2t} = i(d_t - \delta_t) \qquad H_t = d_t\delta_t + \delta_t d_t = \Delta_t$$

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$\implies$ *at most* the number of zero energy p-forms equals the number of critical points with index p $\implies$ $b_p \leq c_p$ $\Box$

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$\implies$ Betti numbers of ∂ are the same Betti numbers of M $\square$