Geometry of Morse Theory

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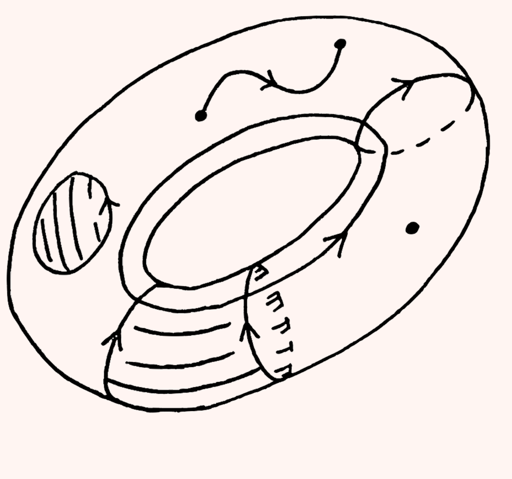
Motivation

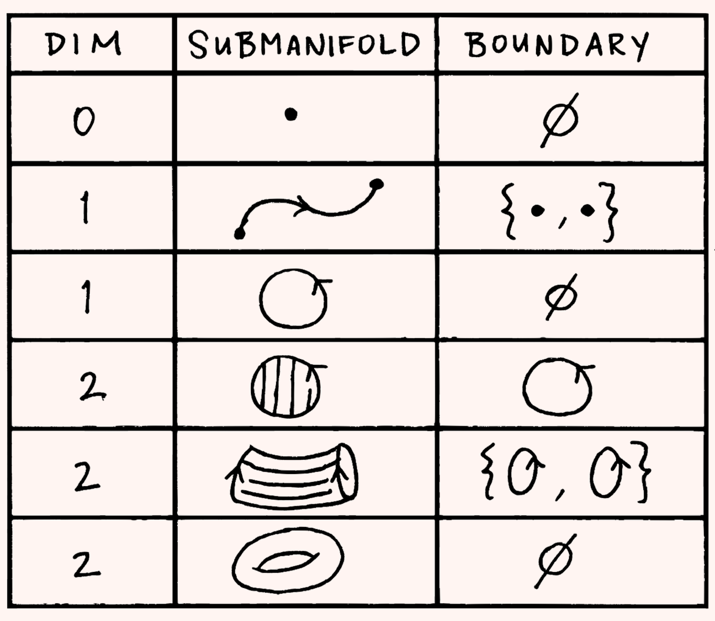
One goal of mathematical physics is to study *topological invariants of manifolds* like **Betti numbers** which count the **number of holes in each dimension** and the **Euler characteristic**.

Definition (Homology groups). For a given manifold M, the *n-th Homology group* is defined:

$$H_n(M) = \frac{\text{n-dim submanifolds without boundary}}{\text{n-dim boundaries to (n+1)-dim submanifolds}}$$

Torus: $H_0(T) = \mathbb{Z}$ $\quad H_1(T) = \mathbb{Z}^2$ $\quad H_2(T) = \mathbb{Z}$





Definition (Betti numbers). For a given manifold M, the *n-th Betti number*, denoted b_n, is the rank of the n-th Homology group.

Definition (Euler characteristic). The *Euler characteristic* for a manifold M is

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \, b_i$$

Torus: $b_0 = 1$ $\quad b_1 = 2$ $\quad b_2 = 1$

$$\chi(T) = 1 - 2 + 1 = 0$$

Hodge Theory

Hodge theory counts the number of holes in a manifold by studying **differential forms** and **cohomology classes**, the *dual* to Homology groups.

Definition (differential form). Given an n-dimensional manifold, a *(differential) p-form* $\omega \in \Omega^p(M)$ is a family of $(n-p)$-dimensional surfaces.

Definition (exterior derivative). The *exterior derivative* is a map that turns a p-form into a $(p+1)$-form. **(generalized gradient)**

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

Definition (codifferential). The *codifferential* is a map that turns a p-form into a $(p-1)$-form. **(generalized divergence)**

$$\delta : \Omega^p(M) \to \Omega^{p-1}(M)$$

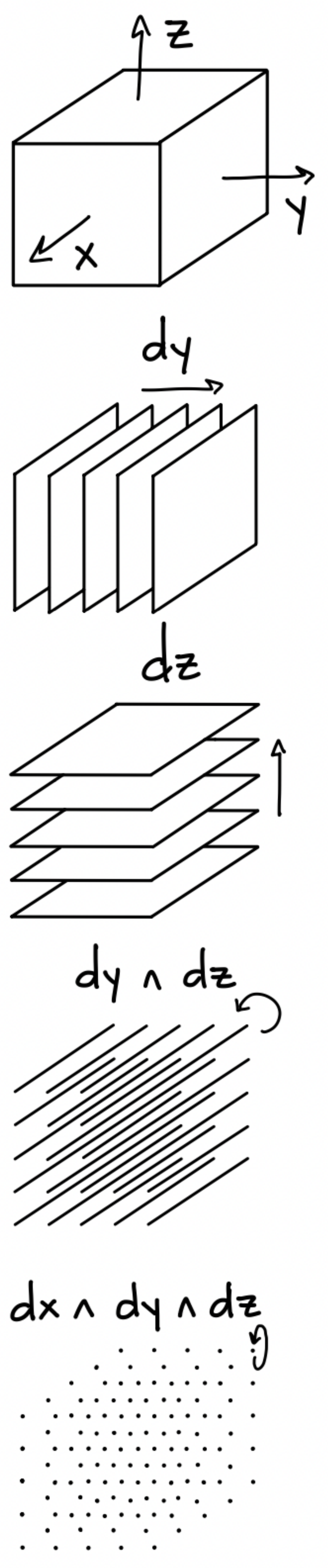
Definition (Laplacian). The *Laplacian* is a map from p-form to themselves.

$$\Delta = d\delta + \delta d$$

$$\Delta : \Omega^p(M) \to \Omega^p(M)$$

There are three types of differential forms:

1. closed forms $\quad (d\omega = 0)$
2. exact forms $\quad (\omega = d\alpha)$
3. harmonic forms $\quad (\Delta\omega = 0)$



Definition (de Rham cohomology classes). For a given manifold M, the *n-th cohomology class* is defined:

$$H^n(M) = \frac{\text{closed n-forms}}{\text{exact n-forms}} = [\text{harmonic forms}]$$

Morse Theory

Morse theory studies the **critical points** on a manifold and how to **reconstruct the manifold** given the critical points.

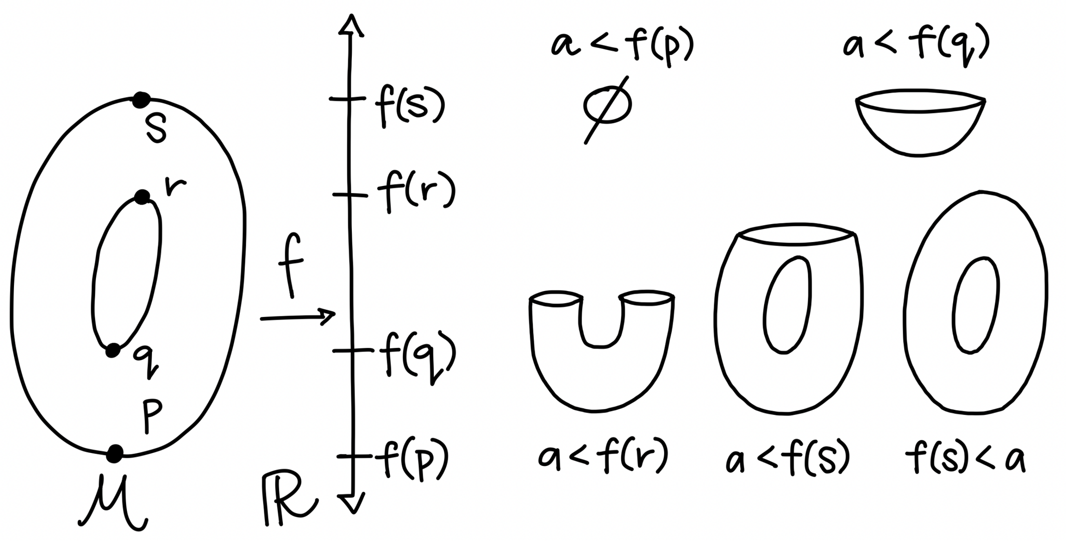
Definition (Morse function). Let $f : M \to \mathbb{R}$. We call f a *Morse function* if each critical point has a non-zero second derivative:

$$\begin{matrix}\textit{determinant of} \\ \textit{Hessian matrix}\end{matrix} = \det(H_f) = \det\left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}\right] \neq 0$$

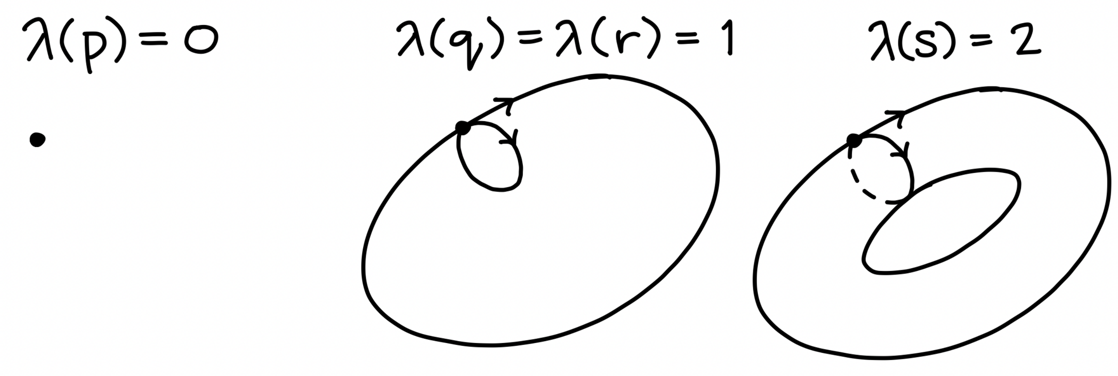
Definition (index). The *(Morse) index* of a Morse function f is the number of negative eigenvalues of the function's Hessian matrix.

Suppose that M is a compact manifold and $f : M \to \mathbb{R}$ is a Morse function. Define *compact submanifolds* of M:

$$M^a = \{x \in M \mid f(x) \leq a\}$$



When f passes through a critical point, the *homology* of the compact submanifolds change. In fact, when M^a passes through a critical point p and "becomes" M^b **one adds a λ-cell to M^a** where λ is the index of p (ie. $M^b \simeq M^a \cup e^\lambda$).



Theorem (Morse Inequalities). Let c_λ denote the number of critical points of index λ in M. Then,

$$b_\lambda \leq c_\lambda$$

$$\sum_{\lambda=0}^{n} (-1)^\lambda \, c_\lambda = \chi(M)$$

Torus: $c_0 = 1$ $\quad c_1 = 2$ $\quad c_2 = 1$

$$\chi(T) = 1 - 2 + 1 = 0$$

Witten's Insights

Witten used *supersymmetry* and *Hodge theory* in order to **connect Morse theory to cohomology classes**.

Let $h : M \to \mathbb{R}$ and $t \in \mathbb{R}$.

$$d_t = e^{-ht} \, d \, e^{ht} \qquad \delta_t = e^{ht} \, \delta \, e^{-ht}$$

$$\Omega^0, \Omega^2, \ldots, \Omega^n \qquad \Omega^1, \Omega^3, \ldots, \Omega^{n-1}$$

bosonic states $\qquad$ *fermionic states*

Definition (Witten symmetry operators). A supersymmetry theory is one with *(Hermitian) symmetry operators*.

$$Q_{1t} = d_t + \delta_t \qquad Q_{2t} = i(d_t - \delta_t) \qquad H_t = d_t\delta_t + \delta_t d_t$$

Proof (of First Morse Inequality).

1. e^{ht} is invertible $\implies$ statements hold for all $t \in \mathbb{R}$
2. number of zero eigenvalues of H_t acting on $\Omega^p = b_p(t) = b_p$
3. $(t \to \infty)$ eigenvalues of H_t can be calculated in terms of the local data at critical points
4. around A (critical point of index p), H_t has **one** zero eigenvalue which is a p-form, denoted $|a\rangle$

$\implies$ *at most* the number of zero energy p-forms equals the number of critical points with index p $\implies$ $b_p \leq c_p$ □

Proof (of Second Morse Inequality). *(goal: define a coboundary operator using critical points)*

1. potential energy has minimum at each critical point $\implies$ tunneling from one critical point A to another B
2. tunneling paths $\Lambda : A \to B$ and $\Gamma : B \to A$ are of steepest descent $\implies$ A and B have index difference of 1
3. paths induce orientation $\implies$ if Γ and Λ agree then $n_\Gamma = +1$, and if not then $n_\Gamma = -1$ $\implies$ define $n(a,b) = \sum_\Gamma n_\Gamma$

$\implies$ define a coboundary operator such that:

$$\partial \, |a\rangle = \sum_b n(a,b) \, |b\rangle$$

$\implies$ Betti numbers of ∂ are the same Betti numbers of M □

Acknowledgements

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