Transcendental Theorems for Arithmetic Gevrey Series

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Algebraic and Transcendental Numbers

An **algebraic number** is the root of a polynomial with rational coefficients. We denote the field of algebraic numbers as $\overline{\mathbb{Q}}$.

Example: $\dfrac{3}{\sqrt{7}}$ is an algebraic number.

$$x = \frac{3}{\sqrt{7}} \qquad x^2 = \frac{9}{7} \qquad 7x^2 = 9 \qquad 7x^2 - 9 = 0$$

So, $\dfrac{3}{\sqrt{7}}$ is an algebraic number since it is the root of $7x^2 - 9 = 0$.

Other examples of algebraic numbers:

1. all fractions $\left(\frac{a}{b}\right)$
2. all n-th roots ($\sqrt[n]{a}$)
3. the imaginary unit i ($\sqrt{-1}$)

A **transcendental number** is a number that is not algebraic (cannot be expressed as a polynomial with rational coefficients). Proving a specific number is transcendental is difficult and uses some clever constructions.

Some known transcendental numbers:

$$e, \pi, \ln 2, i^i, e^\pi$$

Unknown if transcendental:

$$e^e, e+\pi, e\pi$$

The Lindemann-Weierstrass Theorem

Theorem (**L-W**). *If $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

This is to say, for any non-zero algebraic number a, e^a is transcendental. All of the following are transcendental numbers:

$$e^2, e^{\sqrt{3}}, e^{5i+2}, e^{22/7}$$

Because of this, we say that $f(x) = e^x$ is a **purely transcendental function**. In fact, $f(x) = e^x$ belongs to the family of E-functions.

Arithmetic Gevrey Series

In order to study functions, we categorized them into different types based on the function's **formal power series expansion**. Let's define three types of functions:

$$e(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n \qquad g(x) = \sum_{n=0}^{\infty} a_n x^n \qquad a_k(x) = \sum_{n=0}^{\infty} (n!)^k a_n x^n$$

E-function G-function arithmetic Gevrey series

E-functions are arithmetic Gevrey series of order -1:

$$e(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n = \sum_{n=0}^{\infty} (n!)^{-1} a_n x^n = a_{-1}(x)$$

G-functions are arithmetic Gevrey series of order 0:

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n!)^{0} a_n x^n = a_0(x)$$

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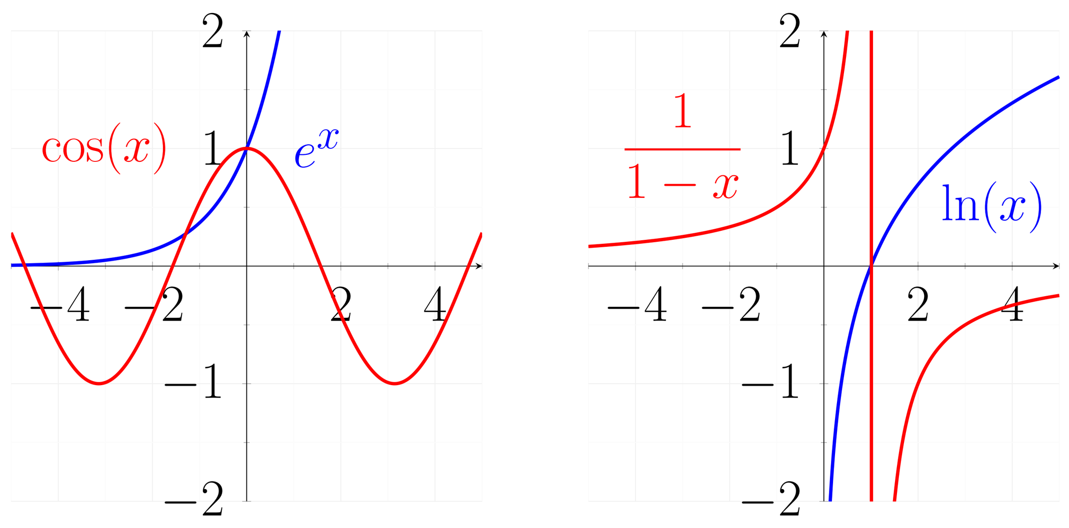

E-function examples:

$$e_1(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \qquad e_2(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

G-function examples:

$$g_1(x) = \ln(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n+1} \qquad g_2(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Motivation and Goal

Generalizing theorems is an important part of mathematics research, and this project attempts generalize transcendental theorems to a larger category of functions.

Transcendental theorems are well known for E-functions. As it turns out, all E-functions are arithmetic Gevrey series of order -1. The goal of this project is to extend transcendental theorems that have been proven for E-functions to arithmetic Gevrey series of order $-k$ where $k \in \mathbb{Z}^+$.

The Nesterenko-Shidlovskii Theorem

Theorem (**N-S**). *Let $f_1, \ldots, f_n$ be arithmetic Gevery series of order $-k$ with $k \in \mathbb{Z}^+$ and $y = (f_1, \ldots, f_n)^t$. If there exists an $n \times n$ matrix A with entries in $\overline{\mathbb{Q}}(z)$, such that $y' = Ay$ and the common denominator of entries in A are denoted by $T(z)$, then for all $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$, the following holds: For any homogeneous polynomial relation $P(f_1(\xi), \ldots, f_n(\xi)) = 0$ with $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_n]$, homogenous in X_i, such that $Q(z, f_1(z), \ldots, f_n(z)) \equiv 0$ and $P(X_1, \ldots, X_n) = Q(\xi, X_1, \ldots, X_n)$.*

This theorem allows us to derive transcendence of numbers from the transcendence of functions.

Example: $f_1(x) = \cos(x), \quad f_2(x) = \sin(x), \quad f_3(x) = x, \quad \xi = 1$

The relation $\cos(1) + \sin(1) = 1$ can be written as an algebraic relation $P(f_1(\xi), f_2(\xi), f_3(\xi)) = 0$, where

$$P(a,b,c) = a^2 + b^2 - c^2 \in \overline{\mathbb{Q}}[a,b,c]$$

is a homogeneous polynomial in a, b, c, with

$$P(f_1(\xi), f_2(\xi), f_3(\xi)) = \cos^2(1) + \sin^2(1) - 1 = 0$$

Indeed, this comes from an algebraic relation between the functions $f_1(x), f_2(x), f_3(x)$:

$$Q(z,a,b,c) = z^2a^2 + z^2b^2 - c^2 \in \overline{\mathbb{Q}}[z,a,b,c]$$

$$\begin{aligned} Q(z, f_1(z), f_2(z), f_3(z)) &= z^2\cos^2(z) + z^2\sin^2(z) - z^2 \\ &= z^2(\cos^2(z) + \sin^2(z)) - z^2 \\ &\equiv 0 \end{aligned}$$

and it also satisfies

$$Q(\xi, a, b, c) = a^2 + b^2 - c^2 = P(a,b,c)$$

$\cos(x)$ and $\sin(x)$ are both E-functions. This algebraic relation exists between them because the purely transcendental function of $\cos(x)$ and $\sin(x)$ is e^x for both.

Further Research

Although we have proved some important theorems in transcendental number theory for arithmetic Gevrey series of negative integer order, there is still work to be done. We are currently working to prove the following for AGS$\{-k\}$:

1. a structure theorem (canonical decomposition)
2. relaxing the conditions for the extended Lindemann-Weierstrass theorem
3. computing approximations for upper and lower bounds
4. potentially extending our research into arithmetic Gevrey series of negative *rational* order.

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