Transcendence theorems for arithmetic Gevrey series

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Algebraic and Transcendental Numbers

An algebraic number α is the root of a polynomial with rational coefficients. The field of algebraic numbers is denoted $\overline{\mathbb{Q}}$. Examples of algebraic numbers include:

1. $\frac{3}{4} \in \overline{\mathbb{Q}}$ because $\frac{3}{4}$ is a root of $f(x) = 4x - 3$
2. $\sqrt{2} \in \overline{\mathbb{Q}}$ because $\sqrt{2}$ is a root of $f(x) = x^2 - 2$
3. $i \in \overline{\mathbb{Q}}$ because i is a root of $f(x) = x^2 + 1$

A transcendental number is a real or complex number that is not algebraic. In other words, a number β is algebraic if it is not the root of any polynomial with rational coefficients.

Examples of transcendental numbers include e, π, and $\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$.

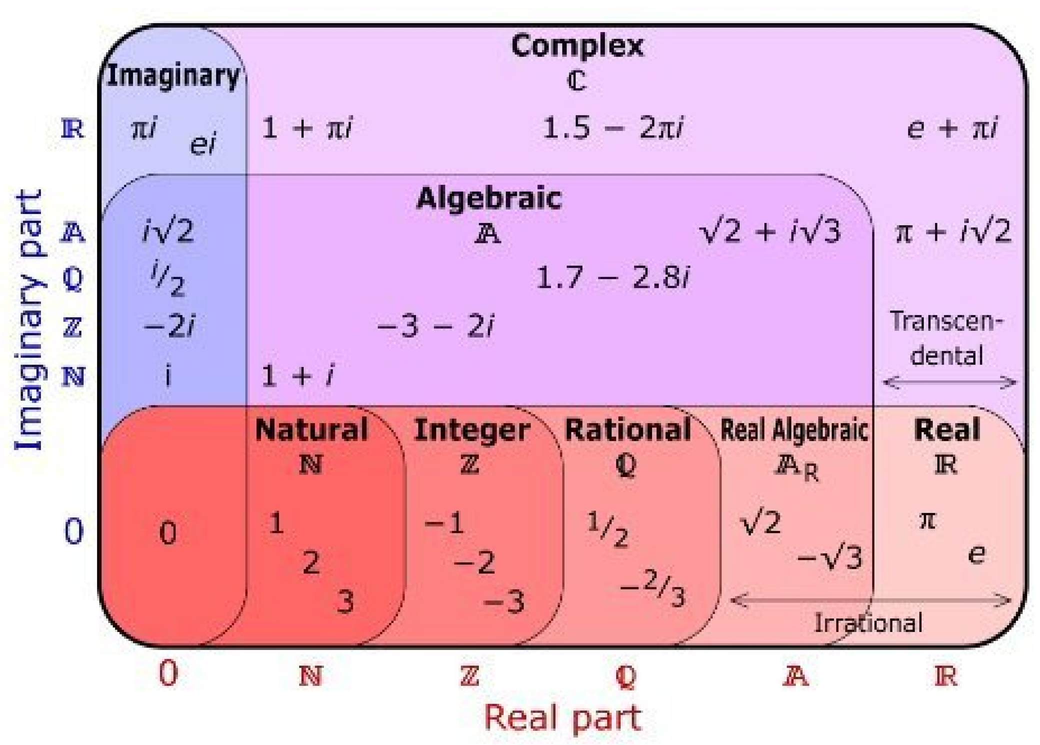

Figure 1. A classification of numbers within the field $\mathbb{C}$

The Lindemann-Weierstrass Theorem, transcendence of e and π

Theorem *(Lindemann-Weierstrass) If $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

This powerful theorem gives us many easy corollaries that resolve questions about transcendence of certain numbers.

Corollary 1. *If α is a nonzero algebraic number, then e^{α} is transcendental.* Indeed, since 0 and α are distinct algebraic numbers, $1 = e^0$ and e^{α} are linearly independent over $\overline{\mathbb{Q}}$, which says that there is no algebraic number s such that $s \cdot 1 = e^{\alpha}$, so e^{α} has to be transcendental.

Corollary 2. *e is transcendental.* This comes directly from corollary 1 using $\alpha = 1$.

Corollary 3. *π is transcendental.* Since i is algebraic, if π is algebraic then $i\pi$ would also be algebraic, implying $e^{i\pi} = -1$ is transcendental, which creates a contradiction.

Remark: The transcendence of e and π were proved much earlier in precedence to the Lindemann-Weierstrass theorem. The above corollaries were merely mentioned to demonstrate the power of the theorem, which goes way beyond e and π. In fact, it is not hard to show that $\cos(n), \sin(n)$ and e^n are indeed transcendental for nonzero algebraic numbers n, and $\ln(n)$ for n not equal to 0 or 1.

E-Functions and Arithmetic Gevrey Series

Let k be an positive integer. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n!)^k} z^n.$$

$f(z)$ is called an *arithmetic Gevrey series of order $-k$* if and only if

1. $a_0, \ldots, a_n \in \overline{\mathbb{Q}}$
2. $f(z)$ is a solution to a differential equation with coefficients in $\overline{\mathbb{Q}}[z]$
3. There exists $C > 0$ such that for all n :

$$H(a_n) := \max_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n$$

and $\text{den}(a_0, \ldots, a_n) < C^n$. Where $\text{den}(a_0, \ldots, a_n)$ denotes the common denominator of $(a_0, \ldots, a_n)$.

In particular, $f(z)$ is called an **E-function** if it's an arithmetic Gevery series of order -1. Examples of E-functions include $\exp$, $\sin$ and $\cos$, of which Arithmetic Gevrey Series are a generalization.

Main Result

Less formally, the Nesterenko-Shidlovskii Theorem states that any algebraic relation of values of a specific family of E-functions at some point would imply an algebraic relation between the functions. An example of this:

We know that $\sin(z)$ and $\cos(z)$ are E-functions, and they satisfy a well-known identity: $\sin^2(z) + \cos^2(z) = 1$ and the theorem would imply that any algebraic relations between $\sin(z_0)$ and $\cos(z_0)$ for some $z_0 \in \overline{\mathbb{Q}}$ comes from algebraic relations of functions like $\sin^2(z) + \cos^2(z) = 1$. Further more, the Lindemann-Weierstrass theorem is a corollary of the Nesterenko-Shidlovskii theorem.

Our result is generalization of the Nesternko-Shidlovskii theorem to arithmetic Gevrey sereis:

Let $f_1, \ldots, f_n$ be arithmetic Gevery series of order $-k$ with $k \in \mathbb{Z}^{>0}$ and $y = (f_1, \ldots, f_n)^t$. If there exists an $n \times n$ matrix A with entries in $\overline{\mathbb{Q}}(z)$, such that $y' = Ay$ and the common denominator of entries in A are denoted by $T(z)$, then for all $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$, the following holds: For any homogeneous polynomial relation $P(f_1(\xi), \ldots, f_n(\xi)) = 0$ with $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_n]$, homogenous in X_i, such that $Q(z, f_1(z), \ldots, f_n(z)) \equiv 0$ and $P(X_1, \ldots, X_n) = Q(\xi, X_1, \ldots, X_n)$.

From which we can derive a Lindemann-Weierstrass theorem:

Let f be an arithmetic Gevrey series of order $-k$ and let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers such that $\frac{\alpha_i}{\alpha_j} \neq \frac{\rho_1}{\rho_2}$ for all $\rho_1, \rho_2 \in \mathfrak{S}(f)$ and all $i \neq j$. Then the numbers $1, f(\alpha_1), \ldots, f(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$, unless two of them are algebraic.

The arithmetic condition on the algebraic numbers is imposed in view of certain linear properties of the function at different values. An example of the theorem is on the E-function $\cos$. Calculation shows that $\mathfrak{S}(\cos(z)) = \{1, -1\}$, so for any set of distinct numbers $\alpha_1, \ldots, \alpha_n, \cos\alpha_1, \ldots, \cos\alpha_n$ are linearly independent as long as no $\alpha_i = \pm\alpha_j$. This is in alignment with the fact that $\cos(z) = -\cos(-z)$.

Research Focus: Generalizing Lindemann-Weierstrass to Arithmetic Gevrey Series

E-Functions, and arithmetic Gevrey series more generally, represent an important class of functions with regards to transcendence. Recent research has developed transcendence results for E-functions, such as the Nesternko-Shidlovskii and Lindemann-Weierstrass Theorems for E-Functions. Since E-functions are arithmetic Gevery Series of order -1, we seek to generalize transcendence results for E-functions to similar results for arithmetic Gevery Series of order $-k$ for all positive integers k.

Method of proof

In generalizing Nesterenko-Shidlovsky, we use a transformation $g(z) = f(z^k)$ that takes Gevery series of order -k to E-functions and use the results we already have to obtain the results.

Whereas for Lindemann-Weierstrass we adopt a generalised Laplace transformation $\psi_k(\sum_{i=0}^{\infty} \frac{a_n}{n!^k} z^n) = \sum_{i=0}^{\infty} a_n z^n$. That doesn't add any singularities under the differential operators $z\frac{d}{dz}$ and z, and since we can write any differential operators in terms of $z\frac{d}{dz}$ instead of $\frac{d}{dz}$, the same proof in [4] would work for our generalization.

Next Steps and Further Research

In addition to generalizing the Nesterenko-Shidlovskii Theorem and the Lindemann-Weierstrass Theorem to Arithmetic Gevrey series of order $-k$, we seek to provide a structure theorem for arithmetic Gevrey series similar to existing theorems for E-Functions. moreover, we intend to formulate an effective version of the Nesternko-Shidlovskii Theorem for arithmetic Gevrey series by providing a quantitative measure like that developed by Fischler and Rivoal for E-Functions.

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