Transcendence theorems for arithmetic Gevrey series

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Algebraic and transcendental numbers

An algebraic number α is the root of a polynomial with rational coefficients. The field of algebraic numbers is denoted $\overline{\mathbb{Q}}$. Examples of algebraic numbers include:

1. $\frac{3}{4} \in \overline{\mathbb{Q}}$ because $\frac{3}{4}$ is a root of $f(x) = 4x - 3$
2. $\sqrt{2} \in \overline{\mathbb{Q}}$ because $\sqrt{2}$ is a root of $f(x) = x^2 - 2$
3. $i \in \overline{\mathbb{Q}}$ because i is a root of $f(x) = x^2 + 1$

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A transcendental number is a real or complex number that is not algebraic. In other words, a number β is transcendental if it is not the root of any polynomial with rational coefficients.

Examples of transcendental numbers include e, π, and $\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$.

Transcendence theorem on exp: Lindemann-Weierstrass

Theorem: (Lindemann-Weierstrass) *If $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

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This powerful theorem gives us many easy corollaries that resolve questions about transcendence of certain numbers.
Corollary 1. *If α is a nonzero algebraic number, then e^{α} is transcendental.* Indeed, since 0 and α are distinct algebraic numbers, $1 = e^0$ and e^{α} are linearly independent over $\overline{\mathbb{Q}}$, which says that there is no algebraic number s such that $s \cdot 1 = e^{\alpha}$, so e^{α} has to be transcendental.

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Remark: In fact, this theorem shows that $e^\alpha, \cos\alpha, \sin\alpha$ are all transcendental for a nonzero algebraic number α.

Arithmetic Gevrey Series

Let $k \in$. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n!)^k} z^n.$$

$f(z)$ is called an *arithmetic Gevrey series of order* $-k$ if and only if

1. $a_0, \ldots, a_n \in \overline{\mathbb{Q}}$
2. $f(z)$ is a solution to a differential equation with coefficients in $\overline{\mathbb{Q}}[z]$
3. There exists $C > 0$ such that for all n :

$$H(a_n) := \max_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n$$

   and $\text{den}(a_0, \ldots, a_n) < C^n$. Where $\text{den}(a_0, \ldots, a_n)$ denotes the common denominator of $(a_0, \ldots, a_n)$.

E-functions and G-functions

In particular, $f(z)$ is called an **E-function** if it's an arithmetic Gevery series of order -1. Namely, it is a power series with denominator bounds of the same order as the exponential, of the form

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An arithmetic Gevrey series of order 0 is called a G-function. The G in the name comes from the fact that geometric series fall under this category.

The Nesterenko-Shidlovskii theorem

Theorem: (Nesterenko-Shidlovskii, Beukers) Let $f_1, \ldots, f_n$ be E-functions and $\vec{y} = (f_1, \ldots, f_n)^t$. If there exists an $n \times n$ matrix A with entries in $\overline{\mathbb{Q}}(z)$, such that $\vec{y}$ satisfies the differential equation $Y' = AY$, with the common denominator of entries in A denoted by $T(z)$, then for all $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$, the following holds: For any homogeneous polynomial relation $P(f_1(\xi), \ldots, f_n(\xi)) = 0$ with $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_n]$, homogenous in X_i, such that $Q(z, f_1(z), \ldots, f_n(z)) \equiv 0$ and $P(X_1, \ldots, X_n) = Q(\xi, X_1, \ldots, X_n)$.

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Theorem: The Nesterenko-Shidlovskii theorem also holds when "E-functions" are replaced with Arithmetic Gevery Series of order $-k_1, -k_2, ..., -k_n$, where each k_i is a positive integer.

An Example for the N-S theorem

- $\cos(1) \approx 0.5403023058...$
- $\sin(1) \approx 0.8414709848...$

One notices that $\cos(1)^2 + \sin(1)^2 - 1 = 0$, and the Nesterenko-Shidlovskii theorem says that one cannot have such a homogeneous polynomial relationship between the two transcendental numbers unless there is a homogeneous polynomial relationship between the variables $\cos(z), \sin(z), z$ whose specialization at the point $z = 1$ gives such a relationship.

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Indeed, $\cos^2(z) + \sin^2(z) - 1 = 0$ is the usual trigonometric identity. In the case that we don't konw if such a relationship between the functions exist, the N-S theorem garantees them to exist and agree with the relationship of the values of the functions at a point.

Generalizing L-W to arithmetic Gevrey series

Definition: For f any Gevery Series of order $-k$, let $\mathfrak{S}(f)$ denote the set of singularities (in the sense of meromorphic continuation) of finite distance of the corresponding G-function, obtained through formal transformation $\psi_k(\sum_{i=0}^{\infty} \frac{a_n}{n!^k} z^n) = \sum_{i=0}^{\infty} a_n z^n$.

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Theorem: (Delaygue) Let $f_1, \ldots, f_n$ be E-functions with pairwise disjoint sets $\mathfrak{S}(f_i)$ and let $\alpha \in \overline{\mathbb{Q}}$. Then the numbers $1, f_1(\alpha), \ldots, f_n(\alpha)$ are linearly independent over $\overline{\mathbb{Q}}$, unless two of them are algebraic.

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Corollary:(L-W for E-functions, Delaygue) Let f be an arithmetic Gevrey series of order $-k$. If $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers such that $\frac{\alpha_i}{\alpha_j} \neq \frac{\rho_1}{\rho_2}$ for all $\rho_1, \rho_2 \in \mathfrak{S}(f)$, and all $i \neq j$. Then the numbers $1, f(\alpha_1), \ldots, f(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$, unless two of them are algebraic.

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Theorem: The above theorem also holds for arithmetic Gevrey series of order $-k$.

Method of proof

Theorem:(André) Let f be an E-function and L be its minimal differential operator, then the equation $Ly = 0$ only has non-trivial singularities at 0 and ∞.

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Lemma: If $f(z)$ is an arithmetic Gevrey series of order $-k$, then $f(z^k)$ is an E-function.

Next steps

It is possible to proceed and prove a Nesterenko-Shidlovshii theorem for all arithmetic Gevrey series of negative rational order.

It is possible to loosen the conditions for the Lindemann-Weierstrass theorem for E-functions so that the sets $\mathfrak{S}(f_i)$ are allowed to have intersections at common poles. The theorem should also be extendable to arithmetic Gevrey series of negative integer order.

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