Transcendence theorems for arithmetic Gevrey series

Mathematics
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Algebraic and Transcendental Numbers

An algebraic number α is the root of a polynomial with rational coefficients. The field of algebraic numbers is denoted $\overline{\mathbb{Q}}$. Examples of algebraic numbers include:

1. $\frac{3}{4} \in \overline{\mathbb{Q}}$ because $\frac{3}{4}$ is a root of $f(x) = 4x - 3$
2. $\sqrt{2} \in \overline{\mathbb{Q}}$ because $\sqrt{2}$ is a root of $f(x) = x^2 - 2$
3. $i \in \overline{\mathbb{Q}}$ because i is a root of $f(x) = x^2 + 1$

A transcendental number is a real or complex number that is not algebraic. In other words, a number β is algebraic if it is not the root of any polynomial with rational coefficients.

Examples of transcendental numbers include e, π, and $\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$.

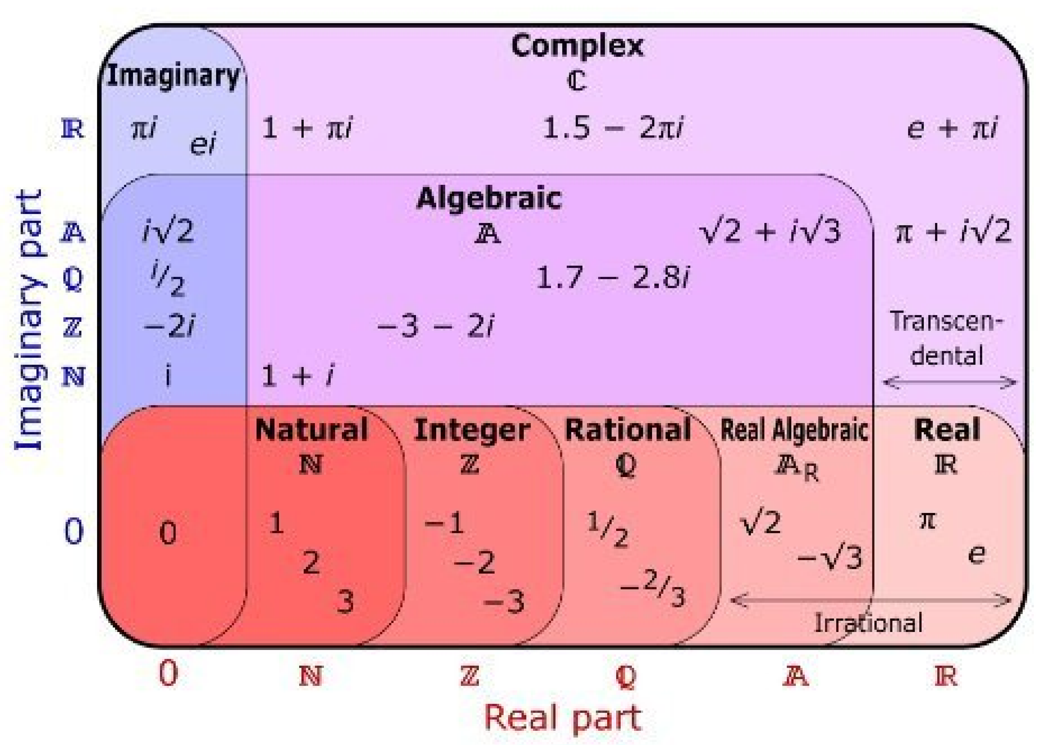

Figure 1. A classification of numbers within the field $\mathbb{C}$

The Lindemann-Weierstrass Theorem, transcendence of e and π

Theorem *(Lindemann-Weierstrass) If $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

This powerful theorem gives us many easy corollaries that resolve questions about transcendence of certain numbers.

Corollary 1. *If α is a nonzero algebraic number, then e^{α} is transcendental.* Indeed, since 0 and α are distinct algebraic numbers, $1 = e^0$ and e^{α} are linearly independent over $\overline{\mathbb{Q}}$, which says that there is no algebraic number s such that $s \cdot 1 = e^{\alpha}$, so e^{α} has to be transcendental.

Corollary 2. *e is transcendental.* This comes directly from corollary 1 using $\alpha = 1$.

Corollary 3. *π is transcendental.* Since i is algebraic, if π is algebraic then $i\pi$ would also be algebraic, implying $e^{i\pi} = -1$ is transcendental, which creates a contradiction.

Remark: The transcendence of e and π were proved much earlier in precedence to the Lindemann-Weierstrass theorem. The above corollaries were merely mentioned to demonstrate the power of the theorem, which goes way beyond e and π. In fact, it is not hard to show that $\cos(n), \sin(n)$ and e^n are indeed transcendental for nonzero algebraic numbers n, and $\ln(n)$ for n not equal to 0 or 1.

E-Functions and Arithmetic Gevrey Series

Let k be an positive integer. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n!)^k} z^n.$$

$f(z)$ is called an *arithmetic Gevrey series of order $-k$* if and only if

1. $a_0, \ldots, a_n \in \overline{\mathbb{Q}}$
2. $f(z)$ is a solution to a differential equation with coefficients in $\overline{\mathbb{Q}}[z]$
3. There exists $C > 0$ such that for all n :

$$H(a_n) := \max_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(a_n)| < C^n$$

and $\text{den}(a_0, \ldots, a_n) < C^n$. Where $\text{den}(a_0, \ldots, a_n)$ denotes the common denominator of $(a_0, \ldots, a_n)$.

In particular, $f(z)$ is called an **E-function** if it's an arithmetic Gevery series of order -1. Examples of E-functions include $\exp$, $\sin$ and $\cos$, of which Arithmetic Gevrey Series are a generalization.

Main Result

Less formally, the Nesterenko-Shidlovskii Theorem states that any algebraic relation of values of a specific family of E-functions at some point would imply an algebraic relation between the functions. An example of this:

We know that $\sin(z)$ and $\cos(z)$ are E-functions, and they satisfy a well-known identity: $\sin^2(z) + \cos^2(z) = 1$ and the theorem would imply that any algebraic relations between $\sin(z_0)$ and $\cos(z_0)$ for some $z_0 \in \overline{\mathbb{Q}}$ comes from algebraic relations of functions like $\sin^2(z) + \cos^2(z) = 1$. Further more, the Lindemann-Weierstrass theorem is a corollary of the Nesterenko-Shidlovskii theorem.

Our result is generalization of the Nesternko-Shidlovskii theorem to arithmetic Gevrey sereis:

Let $f_1, \ldots, f_n$ be arithmetic Gevery series of order $-k$ with $k \in \mathbb{Z}^{>0}$ and $y = (f_1, \ldots, f_n)^t$. If there exists an $n \times n$ matrix A with entries in $\overline{\mathbb{Q}}(z)$, such that $y' = Ay$ and the common denominator of entries in A are denoted by $T(z)$, then for all $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$, the following holds: For any homogeneous polynomial relation $P(f_1(\xi), \ldots, f_n(\xi)) = 0$ with $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_n]$, homogenous in X_i, such that $Q(z, f_1(z), \ldots, f_n(z)) \equiv 0$ and $P(X_1, \ldots, X_n) = Q(\xi, X_1, \ldots, X_n)$.

From which we can derive a Lindemann-Weierstrass theorem:

Let f be an arithmetic Gevrey series of order $-k$ and let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers such that $\frac{\alpha_i}{\alpha_j} \neq \frac{\rho_1}{\rho_2}$ for all $\rho_1, \rho_2 \in \mathfrak{S}(f)$ and all $i \neq j$. Then the numbers $1, f(\alpha_1), \ldots, f(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$, unless two of them are algebraic.

the arithmetic condition on the algebraic numbers is imposed in view of certain linear properties of the function at different values, such as $\cos(z) = -\cos(-z)$.

Research Focus: Generalizing Lindemann-Weierstrass to Arithmetic Gevrey Series

E-Functions, and arithmetic Gevrey series more generally, represent an important class of functions with regards to transcendence. Recent research has developed transcendence results for E-functions, such as the Nesternko-Shidlovskii and Lindemann-Weierstrass Theorems for E-Functions. Since E-functions are arithmetic Gevery Series of order -1, we seek to generalize transcendence results for E-functions to similar results for arithmetic Gevery Series of order $-k$ for all positive integers k.

Method of proof

In generalizing Nesterenko-Shidlovsky, we use a transformation $g(z) = f(z^k)$ that takes Gevery series of order -k to E-functions and use the results we already have to obtain the results.

Whereas for Lindemann-Weierstrass we adopt a generalised Laplace transformation $\psi_k(\sum_{i=0}^{\infty} \frac{a_n}{n!^k} z^n) = \sum_{i=0}^{\infty} a_n z^n$. That doesn't add any singularities under the differential operators $z\frac{d}{dz}$ and z, and since we can write any differential operators in terms of $z\frac{d}{dz}$ instead of $\frac{d}{dz}$, the same proof in [4] would work for our generalization.

Next Steps and Further Research

In addition to generalizing the Nesterenko-Shidlovskii Theorem and the Lindemann-Weierstrass Theorem to Arithmetic Gevrey series of order $-k$, we seek to provide a structure theorem for arithmetic Gevrey series similar to existing theorems for E-Functions. moreover, we intend to formulate an effective version of the Nesternko-Shidlovskii Theorem for arithmetic Gevrey series by providing a quantitative measure like that developed by Fischler and Rivoal for E-Functions.

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