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QUANTUM CORRELATIONS OF THE OUTPUT STATES GENERATED BY THE BROADCASTING OF ENTANGLEMENT

MASTER'S THESIS

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Contents

1 Introduction

The aim of this thesis is to describe the interesting and modern subject of quantum information theory, the study of quantum correlations obtained after broadcasting the entanglement. The field of quantum information theory is fascinating due to the fact that the strangeness of quantum mechanics is converted into physical applications. The study of quantum entanglement and quantum correlations respectively, have vast applications, for example quantum cryptography or increasing the computational power of quantum computers, scaling with the quantum efficiency. Considering entanglement as a physical resource, experimentalists focus their attention towards creating entangled states, distributing them over large distances and trying to manipulate them in the most coherent way possible, even though noisy environments affect them, producing decoherence effects [1].

The thesis is structured as follows:

We are dedicating Chapter 2 to the description of entanglement alongside the concept of separability. The Peres-Horodecki criterion is presented in order to find the separable states. A measure of entanglement convenient for us is the so called concurrence, which quantifies the inseparability of states. To check that indeed this measure is suitable, we are providing the isotropic state as an example.

Chapter 3 presents a special class of mixed states, the so called X states, of a system of two 1/2 spin particles. They are called X due to the structure of the associated density operator. We know that quantum correlations remain invariant under local unitary transformations, therefore we are bringing these states to their canonical form to help us ease our calculations.

Chapter 4 introduces another type of quantum correlations discovered by H. Olliver, W. Zurek and independently by L. Henderson and V. Vedral. They showed that the entanglement is not the only one responsible for all the non-classical correlations. Hence quantum discord appeared. In section 4.2 we are showing the method of evaluating the discord for the X states.

The approximate quantum cloning of the entangled states presented in Chapter 5 can be realized by the Pauli cloning machines. The fact that quantum entanglement can be broadcast at a distance by means of local optimal universal asymmetric cloning machines, is demonstrated in section 5.2.

Finally, Chapter 6 presents a new application, evaluating the concurrence and quantum discord for the broadcasting of the entangled X states, using the local asymmetric cloning machines.

Enjoy!

2 | Entanglement

"The only way to deal with an unfree world is to become so absolutely free that your very existence is an act of rebellion."

—Albert Camus

The story of entanglement starts in 1935 and 1936 when Schrödinger published a two part article [2, 3] in which he discussed and extended an argument proposed by Einstein-Podolsky-Rosen [4] (EPR) which is a critique to the Copenhagen interpretation of quantum mechanics, debating that the quantum theory is incomplete. Einstein emphasized on the existence of "hidden variables" within the measurements. He required certain conditions of separability and locality for composite systems. Each component within the system should separately be characterized by its own properties, and it should be impossible to alter the properties of a distant system instantaneously by acting on a local system.
After 30 years John Bell looked at entanglement in simpler systems than the EPR example [5]: matching correlations between two-valued dynamical quantities, such as polarization or spin, of two separated systems in an entangled state, showing that the statistical correlations between the measurement outcomes of different quantities on the two systems are inconsistent with an inequality derived from Einstein's separability and locality assumptions. This inequality is now known as Bell's inequality. An important feature of entanglement after Bell's investigation was that entanglement can persist over long distances.

But it was not until the 1980s that physicists, cryptologists and computer scientists began to consider the non-local correlations of entangled quantum states as a new kind of non-classical physical resource that could be exploited [6].

2.1 Definition of Entanglement

The basic unit in the field of quantum information is the qubit, i.e. quantum bit, which is a state in a two-dimensional space [7]:

$$a\,|0\rangle + b\,|1\rangle\,,$$

where the complex parameters satisfy $|a|^2 + |b|^2 = 1$ and $\{|0\rangle\,, |1\rangle\}$ is an orthonormal basis. For a d-dimensional space, the so called qudit state can be written as:

$$\boxed{|\psi\rangle = \sum_{i=0}^{d-1} a_i\,|i\rangle\,,} \tag{2.1}$$

whereas $\sum_{i=0}^{d-1} |a_i|^2 = 1$ and $\{|i\rangle\}$ is the set of orthonormal basis vectors.

Entanglement is a property of two or more quantum systems which exhibit correlations that cannot be explained by classical physics [8]. To gain a deeper understanding over this phenomena, let us evaluate a maximally entangled pure state from the Bell basis:

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \qquad \left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right),$$
$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right), \qquad \left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).$$

We try to write $|\Phi^+\rangle$ as a combination of two qubits $\alpha|0\rangle + \beta|1\rangle$ and $\gamma|0\rangle + \delta|1\rangle$. In order to find the values of these four coefficients we are expanding out and we obtain:

$$\begin{aligned}\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle + \beta|1\rangle\right)\left(\gamma|0\rangle + \delta|1\rangle\right)\\ &= \alpha\gamma|00\rangle + \beta\gamma|10\rangle + \alpha\delta|01\rangle + \beta\delta|11\rangle.\end{aligned}$$

We can see that in order to obtain the initial state, $\alpha\gamma = \beta\delta = 1$ and $\beta\gamma = \alpha\delta = 0$, which implies that either β or γ should be zero, and it cannot be the case. We may draw the conclusion that the Bell state cannot be written in terms of two separate qubit states. By these means we call it entangled.

Remark
A pure quantum state is entangled across two or more quantum systems when it cannot be expressed as a tensor product of states represented by the systems. Entanglement may exist between systems which are spatially separated by a great distance.

In order to describe a quantum system whose state is not completely known we are introducing the language of density operator. Assuming a quantum system is in a state $|\psi_i\rangle$, with the probability p_i, we might say $\{p_i, |\psi_i\rangle\}$ is an ensemble of pure states. The density operator for the system is described by

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|. \tag{2.2}$$

We call a *pure state* a quantum system whose state $|\psi\rangle$ is completely known, and the density operator will be written as $\rho = |\psi\rangle\langle\psi|$. Otherwise, ρ is a mixed state and it is said to be a mixture of pure states in the ensemble. An easy way to make the differentiation between a mixed and a pure state is to compute the trace of the density matrix squared, i.e. if $\mathsf{tr}(\rho^2) = 1$ then the state is pure, while for the mixed state $\mathsf{tr}(\rho^2) < 1$. To further detail, if a quantum system is prepared in a state ρ_i, with the probability p_i, from an ensemble $\{p_{ij}, |\psi_{ij}\rangle\}$ [1] of pure states, so

[1] Note that i is fixed.

the probability for being in the state $|\psi_{ij}\rangle$ is $p_i p_{ij}$ [9]. Thus, the density matrix for the system is

$$\begin{aligned} \rho &= \sum_{ij} p_i \underbrace{p_{ij} |\psi_{ij}\rangle \langle\psi_{ij}|}_{\rho_i} \\ &= \sum_i p_i \rho_i, \end{aligned} \tag{2.3}$$

and it is said that ρ is a mixture of the states ρ_i with probabilities p_i.

Definition 2.1 (Uncorrelated State [10])
A mixed state ρ of a composite system described in a Hilbert space $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, where the density matrix $\rho \in \mathcal{H}$ has the form

$$\rho = \rho^{(1)} \otimes \rho^{(2)}, \tag{2.4}$$

is said to be uncorrelated.

In this case, performing a measurement, described by two operators, each acting on a subsystem, the expectation value takes the product form

$$\mathsf{tr}\left(\rho A^{(1)} \otimes A^{(2)}\right) = \mathsf{tr}\left(\rho^{(1)} A^{(1)}\right) \mathsf{tr}\left(\rho^{(2)} A^{(2)}\right).$$

To extend the previous definition for our subject of interest, entanglement, we recall Eq. (2.3) and we might say:

Definition 2.2 (Separable State)
If we write the density operator as a convex sum of product states

$$\boxed{\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)},} \tag{2.5}$$

where $\sum_i p_i = 1$ and p_i are positive coefficients, then the mixed state of the composite system is called separable state.

The expectation value, in this case, of the two observables is the average of the product of the expectation values:

$$\mathsf{tr}\left(\rho A^{(1)} \otimes A^{(2)}\right) = \sum p_i \, \mathsf{tr}\left(\rho_i^{(1)} A^{(1)}\right) \mathsf{tr}\left(\rho_i^{(2)} A^{(2)}\right).$$

In the case that the density operator cannot be written as in Eq. (2.5), then the state is called entangled.

2.2 The Peres-Horodecki criterion of separability

The Peres-Horodecki criterion also known as the PPT (partial positive transposition) criterion is a necessary and sufficient condition for $\mathbb{C}^a \otimes \mathbb{C}^b$ (where $a \cdot b \leq 6$, i.e. $\mathcal{H}^2 \otimes \mathcal{H}^2, \mathcal{H}^3 \otimes \mathcal{H}^2$ and $\mathcal{H}^2 \otimes \mathcal{H}^3$), in order to determine the separability of a state.

Theorem 2.1 (Peres-Horodecki)
$' \Rightarrow'$ (Peres [11]): If a state $\rho \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ is separable, then the partial transpose of the density operator is positive:

$$\boxed{\sigma := \left(\mathbb{1}^{(1)} \otimes \mathsf{T}^{(2)}\right) \rho \geq 0,} \tag{2.6}$$

where σ represents the matrix obtained using the transposition on the second subsystem.
$' \Leftarrow'$ (Horodecki [12]): If the condition is fulfilled, then the state is separable. This condition is sufficient only for 2×2 and 2×3 systems.

Let us rewrite Eq. (2.5), to show that the partial transposition's eigenvalues are conserved (invariant under separate unitary transformations of the bases used by two observers [11])[2], in order to prove Theorem 2.1.

Proof
The density matrix becomes

$$\rho_{m\mu,n\nu} \overset{(2.5)}{=\!=} \sum_i p_i \left(\rho_i^{(1)}\right)_{mn} \otimes \left(\rho_i^{(2)}\right)_{\mu\nu},$$

where the Latin indices refer to the first subsystem and the Greek indices to the second one. Now, we are going to define a new matrix σ, obtained using the partial transpose of the initial matrix to the second subsystem[3]*:*

$$\begin{aligned}\sigma_{m\mu,n\nu} &\equiv \rho_{m\nu,n\mu} \\ &= \sum_i p_i \left(\rho_i^{(1)}\right)_{mn} \otimes \left(\rho_i^{(2)}\right)_{\nu\mu} \\ &= \sum_i p_i \left(\rho_i^{(1)}\right) \otimes \left(\rho_i^{(2)}\right)^T.\end{aligned}$$

If we perform the change of basis

$$\boxed{\rho \rightarrow \left(A^{(1)} \otimes A^{(2)}\right) \rho \left(A^{(1)} \otimes A^{(2)}\right)^\dagger}$$

[2] Meaning that the eigenvalues of $\sigma = \left(\mathbb{1}^{(1)} \otimes \mathsf{T}^{(2)}\right) \rho$ are equal to the eigenvalues of $\left(\mathbb{1}^{(1)} \otimes \mathsf{T}^{(2)}\right) \left(A^{(1)} \otimes A^{(2)}\right) \rho \left(A^{(1)} \otimes A^{(2)}\right)^\dagger$.

[3] Note that the Latin indices of ρ are the same, while the Greek ones got transposed. Also note that it isn't a unitary transformation, but the matrix σ is still Hermitian.

then

$$\begin{aligned}\sigma &= \left(\mathbb{1}^{(1)} \otimes T^{(2)}\right)\left(A^{(1)} \otimes A^{(2)}\right)\rho\left(A^{(1)\dagger} \otimes A^{(2)\dagger}\right)\\ &= \left(\mathbb{1}^{(1)} \otimes T^{(2)}\right)\sum_i p_i A^{(1)}\rho_i^{(1)}A^{(1)\dagger} \otimes A^{(2)}\rho_i^{(2)}A^{(2)\dagger}\\ &= \sum_i p_i A^{(1)}\rho_i^{(1)}A^{(1)\dagger} \otimes \left(A^{(2)}\rho_i^{(2)}A^{(2)\dagger}\right)^T\\ &= \sum_i p_i A^{(1)}\rho_i^{(1)}A^{(1)\dagger} \otimes \left(A^{(2)\dagger}\right)^T \rho_i^{(2)T}A^{(2)T}\\ &= \sum_i p_i A^{(1)}\rho_i^{(1)}A^{(1)\dagger} \otimes A^{(2)*}\rho_i^{(2)T}\left(A^{(2)*}\right)^\dagger\\ &= \left(A^{(1)} \otimes A^{(2)*}\right)\rho_i^{(2)T}\left(A^{(1)} \otimes A^{(2)*}\right)^\dagger\end{aligned}$$

we have

$$\boxed{\sigma \rightarrow \left(A^{(1)} \otimes A^{(2)*}\right)\rho_i^{(2)T}\left(A^{(1)} \otimes A^{(2)*}\right)^\dagger}$$

which is also an unitary transformation, leaving the eigenvalues of σ *invariant. Therefore, it doesn't matter in which basis the observers view their basis.* □

It follows that

$$\sigma = \sum_i p_i \left(\rho_i^{(1)}\right) \otimes \left(\rho_i^{(2)}\right)^T, \tag{2.7}$$

since the transposed matrices $\left(\rho_i^{(2)}\right)^T$ are non-negative matrices with unit trace, they can also be separable density matrices, such that none of the eigenvalues are negative.

A short example proving this theorem is given by the isotropic state[4]

$$\rho_\alpha = \frac{\alpha}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right) + \frac{(1-\alpha)}{4}\mathbb{1} \otimes \mathbb{1}, \tag{2.8}$$

in the matrix form

$$\rho_\alpha = \begin{pmatrix} \frac{1+\alpha}{4} & 0 & 0 & \frac{\alpha}{2} \\ 0 & \frac{1-\alpha}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\alpha}{4} & 0 \\ \frac{\alpha}{2} & 0 & 0 & \frac{1+\alpha}{4} \end{pmatrix},$$

[4] An isotropic state is a d × d dimensional bipartite state, invariant under unitary transformation of the form

$$\rho = (U \otimes U^*)\,\rho\left(U^\dagger \otimes (U^*)^\dagger\right).$$

The isotropic states are a one-parameter family of states written as

$$\rho_\alpha = \alpha\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + \frac{(1-\alpha)\mathbb{1}\otimes\mathbb{1}}{d^2}$$

and $\alpha \in \mathbb{R}, \quad -\frac{1}{d^2-1} \leq \alpha \leq 1.$

where $\alpha \in \left(-\frac{1}{3}, 1\right]$ since the eigenvalues have to pe positive[5]. Therefore we can write the partial transpose of ρ_α with respect to the second subsystem as

$$\sigma_\alpha = \begin{pmatrix} \frac{1+\alpha}{4} & 0 & 0 & 0 \\ 0 & \frac{1-\alpha}{4} & \frac{\alpha}{2} & 0 \\ 0 & \frac{\alpha}{2} & \frac{1-\alpha}{4} & 0 \\ 0 & 0 & 0 & \frac{1+\alpha}{4} \end{pmatrix}.$$

The eigenvalues of σ_α are: $\lambda_{\overline{1,3}} = \frac{1+\alpha}{4}$ and $\lambda_4 = \frac{1-3\alpha}{4}$. For $\lambda_{\overline{1,3}}$ all α in the domain satisfy the fact that these are positive eigenvalues, while for λ_4 in order to be positive, $\alpha \leq \frac{1}{3}$. In accordance to the PPT criterion ρ_α is separable for $\alpha \in \left(-\frac{1}{3}, \frac{1}{3}\right]$ and entangled for $\alpha \in \left(\frac{1}{3}, 1\right]$[6].

2.3 The concurrence as a measure of entanglement of mixed states for bipartite systems

Entanglement measure quantifies how much entanglement is in a bipartite or multipartite systems.

For mixed states $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, a convenient entanglement measure for us is the entanglement of formation [13, 14].

Definition 2.3 (Entanglement of Formation)
For a mixed state ρ, the entanglement of formation is the average entanglement of the pure states of the decomposition, minimized over all decompositions of ρ:

$$\boxed{E(\rho) = \min \sum_i p_i E(\psi_i).} \tag{2.9}$$

Even though the definition of entanglement of formation is straightforward, the calculation is pretty challenging, often being difficult to determine all possible ensembles of the pure states within a given mixed state. For bipartite systems there is a method to calculate the entanglement of formation, by introducing another function of a 2-qubit density matrix ρ, the concurrence, as an entanglement measure[7]. Let us prepare the prerequisites before describing the concurrence.
The *spin − flip* operation on a pure state $|\psi\rangle$ is given by

$$|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y) |\psi^*\rangle, \tag{2.10}$$

[5] The eigenvalues of a denisty matrix must lie between 0 & 1.

[6] Thus, we may generalize that the isotropic state has the following properties:

$$-\frac{1}{d^2-1} \leq \alpha \leq \frac{1}{d+1} \Rightarrow \rho_\alpha^{(d)} \text{ separable,}$$
$$\frac{1}{d+1} < \alpha \leq 1 \Rightarrow \rho_\alpha^{(d)} \text{ entangled.}$$

[7] A concept nicely introduced by William Wootters [13].

where $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$. The Pauli operator σ_y[8] acts on the states $|0\rangle$ and $|1\rangle$ as $\sigma_y|0\rangle = i|1\rangle, \quad \sigma_y|1\rangle = -i|0\rangle$. The *spin − flip* operator on 2-qubit density matrices ρ takes the form

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\, \rho^* \, (\sigma_y \otimes \sigma_y)\,. \tag{2.11}$$

The *spin − flip* operation applied to a pure product state, takes the state of each qubit to the orthogonal state[9]. Therefore the concurrence of a pure product state is zero. On the other hand, a completely entangled state is left invariant (except possibly for a phase factor), in this case the concurrence takes the value one, the maximum value possible.

Theorem 2.2 (Entanglement of Formation as a Function of Concurrence)
The entanglement of formation of a bipartite state ρ is a function of concurrence:

$$E_F(\rho) = E_F(C(\rho)) = H\left(\frac{1+\sqrt{1-C^2}}{2}\right), \tag{2.12}$$

where H is the binary entropy function

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x). \tag{2.13}$$

Definition 2.4 (Concurrence [14])
The concurrence of a bipartite pure state $|\psi\rangle$ is defined to be

$$\boxed{C(|\psi\rangle) = |\langle\psi \mid \tilde{\psi}\rangle|,} \tag{2.14}$$

while for a mixed state ρ, it is

$$\boxed{C(\rho) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\},} \tag{2.15}$$

where $\sqrt{\lambda_i}$ are the square roots of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$ in decreasing order.

[8] In the matrix form $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

[9] The state diametrically opposite on the Bloch sphere.

In Fig. 2.1 we can see that the concurrence might be considered itself as an entanglement measure[10].

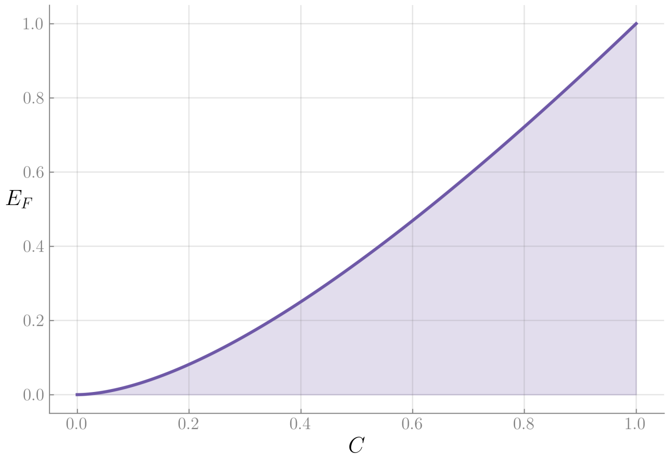


Figure 2.1: Entanglement of formation as a function of concurrence.

Let's take as example the previous isotropic state (Eq. 2.8). In order to obtain the concurrence (Eq. 2.15), firstly, the *spin − flipped* isotropic state $\tilde{\rho}$ will be computed:

$$\tilde{\rho} \overset{(2.8),(2.11)}{=\!=} \frac{\alpha}{2}\left(\sigma_y \otimes \sigma_y\right)\left[\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right) + \frac{(1-\alpha)}{4}\mathbb{1}\otimes\mathbb{1}\right]^*\left(\sigma_y \otimes \sigma_y\right), \tag{2.16}$$

in the matrix form

$$\tilde{\rho} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1+\alpha}{4} & 0 & 0 & \frac{\alpha}{2} \\ 0 & \frac{1-\alpha}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\alpha}{4} & 0 \\ \frac{\alpha}{2} & 0 & 0 & \frac{1+\alpha}{4} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

and we can see that

$$\tilde{\rho} = \rho_\alpha.$$

In the second step we are going to compute the square roots of the eigenvalues of $\rho_\alpha \cdot \tilde{\rho}$[11], which are actually the eigenvalues of the isotropic state themselves. Taking the eigenvalues in order: $\sqrt{\lambda_4} \geq \sqrt{\lambda_3} = \sqrt{\lambda_2} = \sqrt{\lambda_1}$ (i.e. $\sqrt{\lambda_4} = \frac{1+3\alpha}{4}$ and $\sqrt{\lambda_{1,3}} = \frac{1-\alpha}{4}$),

$$\sqrt{\lambda_4} - \sqrt{\lambda_3} - \sqrt{\lambda_2} - \sqrt{\lambda_1} = \frac{1}{2}(3\alpha - 1). \tag{2.17}$$

[10] As C increases from 0 to 1, E_F increases monotonically as well.

[11] In this case for ρ_α^2.

The concurrence is therefore:

$$C(\rho_\alpha) = 0 \quad \text{for} \quad -1/3 \leq \alpha \leq 1/3$$
$$C(\rho_\alpha) = \frac{3\alpha - 1}{2} \quad \text{for} \quad 1/3 < \alpha \leq 1.$$

We can also compute the entanglement of formation using Theorem 2.2, such that

$$E_F(\rho) = E_F(C(\rho)) = H\left(\frac{2 + \sqrt{-9\alpha^2 + 6\alpha + 1}}{4}\right). \tag{2.18}$$

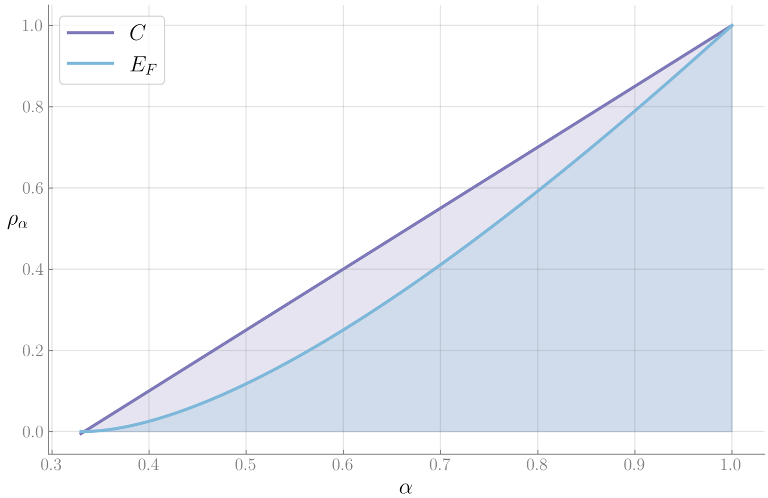


Figure 2.2: Entanglement of formation and concurrence of the isotropic state for values of α when the state is entangled.

Fig. 2.2 reveals that the entanglement of formation is always smaller or equal to the concurrence [15].

Summary

I	For a d-dimensional space, the playground of quantum information is the qudit, described by $\|\psi\rangle = \sum_{i=0}^{d-1} a_i \|i\rangle$.
II	According to the amount of information we have over a state, we can divide them in 2 categories: pure states, described by $\rho = \|\psi\rangle \langle\psi\|$, and mixed states, which are a mixture of pure states $\rho = \sum_i p_i \|\psi_i\rangle \langle\psi_i\|$.
III	Entanglement is a property of quantum mechanics without equivalent in classical physics. An interesting aspect of entanglement is that it may exist between spatially separated systems.
IV	To check if a state is entangled we analyze whether or not it can be written as a tensor product of states, represented by the system of correlation, i.e. if it cannot be written as $\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$.
V	A nice condition to determine the separability of a state is the Peres-Horodecki criterion. If a state is separable, the partial transpose of the density operator is positive: $\sigma := \left(\mathbb{1}^{(1)} \otimes \mathsf{tr}^{(2)}\right) \rho \geq 0$.
VI	In the last section we showed a measure of entanglement of a bipartite system, the entanglement of formation, as a function of concurrence.

3 | The X States

In his article *"Description of States in Quantum Mechanics by Density Matrix and Operator Techniques"* [16], Ugo Fano attempted to extend the treatment of quantum-mechanical problems by representing the states of physical systems in terms of parameters which are "logically consistent and also have a familiar, operational significance". He familiarizes the reader with the states of "less than maximum information", as density matrices in order to "reflect the features of physical phenomena more directly and in closer correspondence to macroscopic methods than is otherwise possible". Also when the states are represented by density matrices the calculations are way faster and one could avoid the introduction of unnecessary variables for the treatment of many-body problems.
In this chapter a similar approach will be considered for the parametrization of X states, firstly by introducing the two-qubit density matrix as Fano parametrization of the Bloch generalization of the density operator of a qubit [1] [17].

3.1 Definition. The Fano parametrization of an X state

Definition 3.1 (Density Matrix of a Bipartite System)
The Fano parametrization of the general expression of a two-qubit density operator, acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$[2]*, introduced by Horodecki in 1996 [18] is*

$$\rho = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + \sum_{n,m=1}^{3} t_{nm}\sigma_n\otimes\sigma_m\right), \tag{3.1}$$

where $\{\sigma_j\}_{j=\overline{1,3}}$, are the standard Pauli operators. Furthermore, $\mathbf{r},\mathbf{s}\in\mathbb{R}^3$ [3]*, and the coefficients $t_{nm} = \mathsf{tr}\,(\rho\sigma_n\otimes\sigma_m)$ form a real matrix denoted by T*[4]*.*

[1] The density matrix of a one-qubit system is

$$\rho = \frac{1}{2}\left(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}\right).$$

Where $\mathbb{1}$ stands for identity operator, $\mathbf{r}\in\mathbb{R}^3$ is called the Bloch vector and $\boldsymbol{\sigma}$ are the Pauli matrices. If $|\mathbf{r}| = 1$ then the state is said to be pure, whereas if $|\mathbf{r}| < 1$ the state is mixed.

[2] Let us rename the two quantum subsystems $(1)\to A$ and $(2)\to B$.

[3] Their expressions can be written as $r_j = \langle\sigma_j\otimes\mathbb{1}\rangle = \mathsf{tr}\,(\rho\,\sigma_j\otimes\mathbb{1})$ and $s_j = \langle\mathbb{1}\otimes\sigma_j\rangle = \mathsf{tr}\,(\rho\,\mathbb{1}\otimes\sigma_j)$.

[4] The T matrix is responsible for the correlations.

The reductions of the state ρ are determined by the local parameters $\mathbf{r}$ and $\mathbf{s}$, such that

$$\rho_A \equiv \mathrm{tr}_B\, \rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma}),$$
$$\rho_B \equiv \mathrm{tr}_A\, \rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{s} \cdot \boldsymbol{\sigma}).$$

It is possible to reduce the number of parameters by applying local unitary transformations, for more details check [18], but for our purpose we will just briefly show how the two-qubit density operator transforms under a local unitary operation. For any single-qubit unitary transformation U there is a unique rotation O such that

$$U\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} U^\dagger = (O\hat{\mathbf{n}}) \cdot \boldsymbol{\sigma}. \tag{3.2}$$

The state subjected to the local unitary transformation $U_A \otimes U_B$, has the form $\tilde{\rho} = U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger$, therefore, the parameters $\mathbf{r}, \mathbf{s}$, and T transform themselves as

$$\tilde{\mathbf{r}} = O_A \mathbf{r},$$
$$\tilde{\mathbf{s}} = O_B \mathbf{s},$$
$$\tilde{T} = O_A T O_B^T.$$

where O_A and O_B are related to U_A and U_B, through Eq. 3.2. We can see that given an arbitrary state, we can always choose such operators U_A & U_B with the corresponding rotations that will diagonalize its matrix T.

Definition 3.2 (X States [17])
A class of commonly occurring bipartite density matrices that only contain non-zero elements in an "X" formation along the main diagonal and anti-diagonal, are called "X states":

$$\rho_{\mathsf{X}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{3.3}$$

having the components $\{\rho_{jj}\}_{j=\overline{1,4}}$ real, while the off-diagonal terms are complex.

The general X state (Eq. 3.3) is written in the ordered computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Moreover we may write $\rho_{14} = |\rho_{14}|\, e^{i\varphi_{14}}$ and $\rho_{23} = |\rho_{23}|\, e^{i\varphi_{23}}$, also $\rho_{41} = \rho_{14}^*$ and $\rho_{32} = \rho_{23}^*$. The parametrization implies the unit trace condition, $\sum_{j=1}^4 \rho_{jj} = 1$, and the positivity condition

$$\sqrt{\rho_{11}\rho_{44}} \geq |\rho_{14}| \qquad \text{and} \qquad \sqrt{\rho_{22}\rho_{33}} \geq |\rho_{23}|\,. \tag{3.4}$$

3.2 The canonical form of the X states

Knowing that quantum correlations remain invariant under local unitary transformation, it is of a great importance to bring an arbitrary X state to its canonical form in order to compute different measures of quantum correlations.
To reach the canonical form of the Eq. 3.3, first we will write explicitly the general density operator from Eq. 3.1 in a matrix form to see the parameters used in an X state:

$$\rho = \frac{1}{4}\begin{pmatrix} 1+r_z+s_z+t_{33} & s_x-is_y+t_{31}-it_{32} & r_x-ir_y+t_{13}-it_{23} & t_{11}-t_{22}-i(t_{12}+t_{21}) \\ s_x+is_y+t_{31}+it_{32} & 1+r_z-s_z-t_{33} & t_{11}+t_{22}+i(t_{12}-t_{21}) & r_x-ir_y-t_{13}+it_{23} \\ r_x+ir_y+t_{13}+it_{22} & t_{11}+i(t_{21}-t_{12})+t_{22} & 1-r_z+s_z-t_{33} & s_x-is_y-t_{31}+it_{32} \\ t_{11}+i(t_{12}+t_{21})-t_{22} & r_x+ir_y-(t_{13}+it_{23}) & s_x+is_y-(t_{31}+it_{32}) & 1-r_z-s_z+t_{33} \end{pmatrix}. \tag{3.5}$$

Taking the diagonal and off-diagonal terms, we might see that

$$\begin{aligned} \rho_{11} &= \frac{1}{4}\left(1+r_z+s_z+t_{33}\right), & \rho_{14} &= \frac{1}{4}\left[t_{11}-t_{22}-i(t_{12}+t_{21})\right], \\ \rho_{22} &= \frac{1}{4}\left(1+r_z-s_z-t_{33}\right), & \rho_{23} &= \frac{1}{4}\left[t_{11}+t_{22}+i(t_{12}-t_{21})\right], \\ \rho_{33} &= \frac{1}{4}\left(1-r_z+s_z-t_{33}\right), & \rho_{32} &= \frac{1}{4}\left[t_{11}+t_{22}-i(t_{12}-t_{21})\right] = \rho_{23}^*, \\ \rho_{44} &= \frac{1}{4}\left(1-r_z-s_z+t_{33}\right), & \rho_{41} &= \frac{1}{4}\left[t_{11}-t_{22}+i(t_{12}+t_{21})\right] = \rho_{14}^*. \end{aligned}$$

So the Fano parametrization of an X state is given by[5]

$$r_z = \rho_{11}+\rho_{22}-\rho_{33}-\rho_{44}, \tag{3.7}$$
$$s_z = \rho_{11}-\rho_{22}+\rho_{33}-\rho_{44}, \tag{3.8}$$

$$t_{11} = 2\,\mathrm{Re}\left(\rho_{14}+\rho_{23}\right), \tag{3.9}$$
$$t_{22} = 2\,\mathrm{Re}\left(\rho_{23}-\rho_{14}\right), \tag{3.10}$$
$$t_{33} = \rho_{11}-\rho_{22}-\rho_{33}+\rho_{44}, \tag{3.11}$$
$$t_{12} = 2\,\mathrm{Im}\left(\rho_{23}-\rho_{14}\right), \tag{3.12}$$
$$t_{21} = -2\,\mathrm{Im}\left(\rho_{23}+\rho_{14}\right). \tag{3.13}$$

[5] Or more nicely written:

$$\begin{aligned} \mathbf{r}_\mathsf{X} :& \quad 0,0,r_z; \\ \mathbf{s}_\mathsf{X} :& \quad 0,0,s_z; \\ T_\mathsf{X} =& \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}. \end{aligned} \tag{3.6}$$

Now we are seeking to get rid of the phases of the off-diagonal elements by a local unitary transformation of the form $U_A \otimes U_B$ since it is irrelevant to carry unnecessary parameters[6]. By applying the local unitary operator [17, 19, 20]

$$\tilde{U}_A \otimes \tilde{U}_B = \mathsf{exp}\left[-\frac{i}{4}\left(\varphi_{14}+\varphi_{23}\right)\sigma_z\right] \otimes \mathsf{exp}\left[-\frac{i}{4}\left(\varphi_{14}-\varphi_{23}\right)\sigma_z\right], \tag{3.14}$$

we can now obtain the canonical form of the general X states:

$$\rho_{\mathsf{X}}^{\mathsf{can}} = \tilde{U}_A \otimes \tilde{U}_B\, \rho_{\mathsf{X}}\, \tilde{U}_A^{\dagger} \otimes \tilde{U}_B^{\dagger}, \tag{3.15}$$

and in the matrix form

$$\rho_{\mathsf{X}}^{\mathsf{can}} = \begin{pmatrix} \rho_{11} & 0 & 0 & |\rho_{14}| \\ 0 & \rho_{22} & |\rho_{23}| & 0 \\ 0 & |\rho_{32}| & \rho_{33} & 0 \\ |\rho_{41}| & 0 & 0 & \rho_{44} \end{pmatrix},$$

or equivalently

$$\rho_{\mathsf{X}}^{\mathsf{can}} = \frac{1}{4}\begin{pmatrix} 1+r_z+s_z+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1+r_z-s_z-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-r_z+s_z-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1-r_z-s_z+c_3 \end{pmatrix}. \tag{3.16}$$

The Fano parametrization of the canonical form of the X state is given by $T = \mathsf{diag}(c_1, c_2, c_3)$:

$$\boxed{\begin{aligned} r^{\mathsf{can}} &= r_z = \rho_{11}+\rho_{22}-\rho_{33}-\rho_{44}, & (3.17)\\ s^{\mathsf{can}} &= s_z = \rho_{11}-\rho_{22}+\rho_{33}-\rho_{44}, & (3.18)\end{aligned}}$$

$$\boxed{\begin{aligned} c_1 &= t_{11}^{\mathsf{can}} = 2\left(|\rho_{23}|+|\rho_{14}|\right), & (3.19)\\ c_2 &= t_{22}^{\mathsf{can}} = 2\left(|\rho_{23}|-|\rho_{14}|\right), & (3.20)\\ c_3 &= t_{33}^{\mathsf{can}} = t_{33} = \rho_{11}-\rho_{22}-\rho_{33}+\rho_{44}. & (3.21)\end{aligned}}$$

Finally, the Bloch form of the canonical density operator of an X state given by the Fano parametrization is

$$\rho_{\mathsf{X}}^{\mathsf{can}} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + r_z\sigma_z\otimes\mathbb{1} + s_z\mathbb{1}\otimes\sigma_z + \sum_{j=1}^{3} c_j\sigma_j\otimes\sigma_j\right). \tag{3.22}$$

The expressions of the coefficients of the density matrix might be expressed as: $r_z = \mathsf{tr}\left(\rho_{\mathsf{X}}^{\mathsf{can}}\,\sigma_z\otimes\mathbb{1}\right)$, $s_z = \mathsf{tr}\left(\rho_{\mathsf{X}}^{\mathsf{can}}\,\mathbb{1}\otimes\sigma_z\right)$ and $c_j = \mathsf{tr}\left(\rho_{\mathsf{X}}^{\mathsf{can}}\sigma_j\otimes\sigma_j\right)$.

[6] i.e. the matrix T can be diagonalized by the associated rotations O_A and O_B along the z axis, of the operation in Eq. 3.14.

3.3 The concurrence of an arbitrary X state

As done by *Wooters* [13] we have defined the concurrence in Eq. 2.14. In order to compute it for the canonical shape of the X states, we are going to follow the same procedure as in the previous example for the isotropic state. Therefore, we will compute the *spin-flipped* X state $\tilde{\rho}_{\mathsf{X}}^{\mathsf{can}}$:

$$\begin{aligned}\tilde{\rho}_{\mathsf{X}}^{\mathsf{can}} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} & 0 & 0 & |\rho_{14}^*| \\ 0 & \rho_{22} & |\rho_{23}^*| & 0 \\ 0 & |\rho_{32}^*| & \rho_{33} & 0 \\ |\rho_{41}^*| & 0 & 0 & \rho_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \rho_{44} & 0 & 0 & |\rho_{41}^*| \\ 0 & \rho_{33} & |\rho_{32}^*| & 0 \\ 0 & |\rho_{23}^*| & \rho_{22} & 0 \\ |\rho_{14}^*| & 0 & 0 & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{44} & 0 & 0 & |\rho_{14}| \\ 0 & \rho_{33} & |\rho_{23}| & 0 \\ 0 & |\rho_{23}^*| & \rho_{22} & 0 \\ |\rho_{14}^*| & 0 & 0 & \rho_{11} \end{pmatrix}, \end{aligned} \tag{3.23}$$

following the calculation for obtaining the eigenvalues of the matrix product $\rho_{\mathsf{X}}^{\mathsf{can}} \cdot \tilde{\rho}_{\mathsf{X}}^{\mathsf{can}}$

$$\rho_{\mathsf{X}}^{\mathsf{can}} \cdot \tilde{\rho}_{\mathsf{X}}^{\mathsf{can}} = \begin{pmatrix} \rho_{11}\rho_{44} + |\rho_{14}|^2 & 0 & 0 & 2\rho_{11}\,|\rho_{14}| \\ 0 & \rho_{22}\rho_{33} + |\rho_{23}|^2 & 2\rho_{22}\,|\rho_{23}| & 0 \\ 0 & 2\rho_{33}\,|\rho_{23}| & \rho_{22}\rho_{33} + |\rho_{23}|^2 & 0 \\ 2\rho_{44}\,|\rho_{14}| & 0 & 0 & \rho_{11}\rho_{44} + |\rho_{14}|^2 \end{pmatrix}. \tag{3.24}$$

Therefore, the eigenvalues are:

$$\lambda_1 = -2\sqrt{\rho_{11}\rho_{44}}|\rho_{14}| + \rho_{11}\rho_{44} + |\rho_{14}|^2 = \left(|\rho_{14}| - \sqrt{\rho_{11}\rho_{44}}\right)^2, \tag{3.25}$$
$$\lambda_2 = 2\sqrt{\rho_{11}\rho_{44}}|\rho_{14}| + \rho_{11}\rho_{44} + |\rho_{14}|^2 = \left(|\rho_{14}| + \sqrt{\rho_{11}\rho_{44}}\right)^2, \tag{3.26}$$
$$\lambda_3 = -2\sqrt{\rho_{22}\rho_{33}}|\rho_{23}| + \rho_{22}\rho_{33} + |\rho_{23}|^2 = \left(|\rho_{23}| - \sqrt{\rho_{22}\rho_{33}}\right)^2, \tag{3.27}$$
$$\lambda_4 = 2\sqrt{\rho_{22}\rho_{33}}|\rho_{23}| + \rho_{22}\rho_{33} + |\rho_{23}|^2 = \left(|\rho_{23}| + \sqrt{\rho_{22}\rho_{33}}\right)^2. \tag{3.28}$$

Let us remember the positivity condition, $\sqrt{\rho_{11}\rho_{44}} \geq |\rho_{14}|$ & $\sqrt{\rho_{22}\rho_{33}} \geq |\rho_{23}|$, given for the parametrization of the X state in Eq. 3.4. Now, allow us to rewrite the eigenvalues in the convenient form:

$$\sqrt{\lambda_1} = \left||\rho_{14}| - \sqrt{\rho_{11}\rho_{44}}\right| = \sqrt{\rho_{11}\rho_{44}} - |\rho_{14}|, \tag{3.29}$$
$$\sqrt{\lambda_2} = \left||\rho_{14}| + \sqrt{\rho_{11}\rho_{44}}\right| = \sqrt{\rho_{11}\rho_{44}} + |\rho_{14}|, \tag{3.30}$$
$$\sqrt{\lambda_3} = \left||\rho_{23}| - \sqrt{\rho_{22}\rho_{33}}\right| = \sqrt{\rho_{22}\rho_{33}} - |\rho_{23}|, \tag{3.31}$$
$$\sqrt{\lambda_4} = \left||\rho_{23}| + \sqrt{\rho_{22}\rho_{33}}\right| = \sqrt{\rho_{22}\rho_{33}} + |\rho_{23}|. \tag{3.32}$$

To obtain the concurrence we observe that two cases may be available[7]:

I when $\sqrt{\lambda_2} > \sqrt{\lambda_4}$:

$$\sqrt{\lambda_2} - \sqrt{\lambda_1} - \sqrt{\lambda_3} - \sqrt{\lambda_4} = 2|\rho_{14}| - 2\sqrt{\rho_{22}\rho_{33}}. \tag{3.33}$$

II when $\sqrt{\lambda_4} > \sqrt{\lambda_2}$:

$$\sqrt{\lambda_4} - \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} = 2|\rho_{23}| - 2\sqrt{\rho_{11}\rho_{44}}. \tag{3.34}$$

Finally we can see that the concurrence of an arbitrary X state is given by [21]

$$\boxed{C\left(\rho_{\mathsf{X}}^{\mathsf{can}}\right) = 2\,\mathsf{max}\left\{0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\;,\; |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\right\}.} \tag{3.35}$$

Summary

I The general expression of the two-qubit density operator is

$$\rho = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + \sum_{n,m=1}^{3} t_{nm}\sigma_n\otimes\sigma_m\right).$$

II We defined the X state as a class of commonly occurring bipartite density matrices with the form

$$\rho_{\mathsf{X}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}.$$

III The Fano parametrization of the X state is given by

$$\begin{aligned} \mathbf{r}_{\mathsf{X}} : &\quad 0, 0, r_z; \\ \mathbf{s}_{\mathsf{X}} : &\quad 0, 0, s_z; \\ T_{\mathsf{X}} = &\begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}. \end{aligned}$$

IV In order to reach the canonical form we apply a pair of local unitary transformations to make all the coefficients positive. Because these

[7] Keeping in mind that the eigenvalues are taken in decreasing order.

unitary transformations do not change the correlations, we will remain just with the absolute values of the correlations that will be dependent on coherence but not on the phases. Therefore we obtain the canonical form of an X state

$$\rho_{\mathsf{X}}^{\mathsf{can}} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + r_z\sigma_z\otimes\mathbb{1} + s_z\mathbb{1}\otimes\sigma_z + \sum_{j=1}^{3} c_j\sigma_j\otimes\sigma_j\right).$$

V We found the relation describing the concurrence of a genreal X state as

$$C\left(\rho_{\mathsf{X}}\right) = 2\,\mathsf{max}\left\{0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\,,\ |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\right\}.$$

4 Quantum Discord

"You must have chaos within you to give birth to a dancing star."
— Friedrich Nietzsche

The early days of quantum information viewed quantum entanglement as the main feature that gives quantum computers an advantage over the classical ones. Without entanglement, superposition was seen insufficient, given the fact that it exists in classical physics as well [22]. It was shown that in the case of qubits in a maximally mixed states, exept one qubit, the quantum computation can achieve an exponential improvement in efficiency over classical computers for a limited set of tasks [23]. In 2001 while analyzing different measures of information in quantum theory, Harold Ollivier and Wojciech H. Zurek [24] and, independently, Leah Henderson and Vlatko Vedral [25] concluded that entanglement is not responsible for all non-classical correlations and that even separable states may contain correlations that are not entirely classical, hence the notion of Quantum Discord was introduced. Further studies [26] showed that discord scales with the quantum efficiency, unlike entanglement that remains vanishingly small throughout the computation[1].

4.1 Definition

Two subsystems are correlated if together they contain more information than when they are taken separately.

Theorem 4.1 (Mutual Information, the Classical Case)
The mutual information measure, quantifies the gain of information of one subsystem as a result of a measurement on the other one:

$$\boxed{\begin{aligned} I(A:B) &\equiv H(A) - H(A|B) \\ &\equiv H(B) - H(B|A) \\ &\equiv H(A) + H(B) - H(A,B), \end{aligned}} \tag{4.1}$$

where H is the Shannon entropy. $H(A)$ & $H(B)$ are individual entropies, $H(A|B)$ & $H(B|A)$ are the conditional entropies and $H(A,B)$ is the joint entropy (Fig. 4.1).[2]

[1] Introduction inspired from *Kavan Modi et al.* [22].

[2] For further use, we will make the difference between the two formulas (which coincide in the classical case) of the mutual information [24]:

$$J(A:B) = H(A) - H(A|B), \tag{4.2}$$
$$I(A:B) = H(A) + H(B) - H(A,B). \tag{4.3}$$

We could also say that the mutual information content of A and B measures how much information the variables have in common. It has been obtained by adding the information content of A to the information content of B, and it may be seen that the information that is common between the two, is counted two times, while the uncommon information is counted once. The final form is taken when the joint information of (A, B) is subtracted[3] [9].

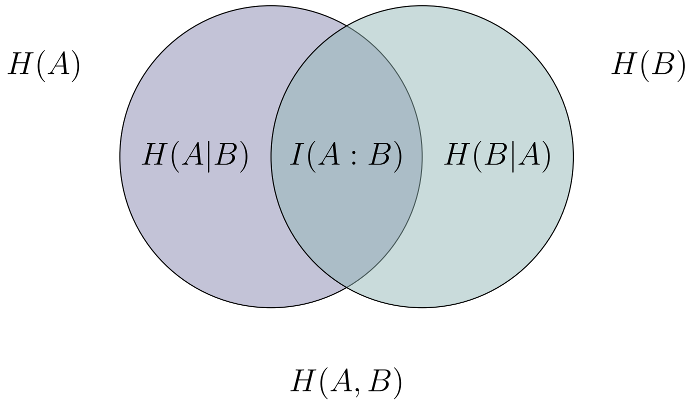


Figure 4.1: Representative diagram of the relationship between different information measures.

For classical variables the conditional probability is defined by Bayes' rule as $P_{a|b} = p_{ab}/p_b$ [4]. The conditional entropy $H(A|B)$ is the measure of ignorance (lack of knowledge) of A, given some knowledge over the state B, i.e. when the state B is known to be in the b^{th} state, being weighted by the probability of b^{th} outcome[5]:

$$\begin{aligned} H(A \mid B) &= \sum_b p_b H\left(P_{a|b}\right) \\ &= -\sum_b p_b \sum_a \frac{p_{ab}}{p_b} \log\left(\frac{p_{ab}}{p_b}\right) \\ &= -\sum_{a,b} p_{ab} \log\left(p_{ab}\right) + \sum_{a,b} p_{ab} \log\left(p_b\right) \\ &= H(A,B) - H(B). \end{aligned}$$

So far, we know that a bipartite quantum system contains both classical and quantum correlations. To quantify the total correlations we will use an equivalent form of Eqs. 4.2 & 4.3, by replacing the classical probability distributions with density operators and the Shannon entropy with von Neumann entropy[6].

[3] Note that Eq. 4.2 is depicting the conditional entropy and mutual information.

[4] $P_{ab} = \{p_{ab}\}$ is the joint probability distribution. $P_a = \{\sum_b p_{ab}\}$ and $P_b = \{\sum_a p_{ab}\}$ represent the marginal probability distribution obtained by summing over one of the indeces.

[5] Since $H(A,B) = H\left(P_{ab}\right) = -\sum_{ab} p_{ab} \log\left(p_{ab}\right)$.

[6] The properties of both entropies are shown in Appendix A.

Theorem 4.2 (Quantum Mutual Information [24])
Considering a bipartite system with the density operator ρ_{AB} and $\rho_A = \mathrm{tr}_B(\rho_{AB})$ is the reduced density operator over the subsystem B, then the **quantum mutual information** *is defined as*

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{4.4}$$

where $S(\rho_{AB}) = -\mathrm{tr}(\rho_{AB}\log\rho_{AB})$[7] is the **von Neumann entropy***.*

In quantum mechanics we know that the measurement disturbs the quantum state[8], however, a measurement over one subsystem may be described by a positive-operator-valued measure (POVM). In [24], in order to generalize the classical mutual information to the quantum state, the taken POVM consists of the set of one-dimensional projectors $\{\Pi_k^B\}_k$ performed on subsystem B. The conditional density operator $\rho_{A|\Pi_k^B}$ associated with the measurement outcome k is [17]

$$\boxed{\rho_{A|\Pi_k^B} = \frac{1}{p_k}\mathrm{Tr}_B\left(\mathbb{1}\otimes\Pi_k^B\,\rho_{AB}\,\mathbb{1}\otimes\Pi_k^B\right),} \tag{4.5}$$

where the probability $p_k = \mathrm{tr}\left(\rho_{AB}\,\mathbb{1}\otimes\Pi_k^B\right)$ and $\mathbb{1}$ is the identity operator acting on the subsystem A. The quantum conditional entropy with respect to the measurement $\{\Pi_k^B\}_k$ on B is obtained by considering all the possible outcomes

$$S\left(\rho_{A|\{\Pi_k^B\}}\right) = \sum_k p_k S\left(\rho_{A|\Pi_k^B}\right). \tag{4.6}$$

Therefore the associated quantum mutual information, generalizing Eq. 4.2, is

$$\mathcal{J}(\rho_{AB})|_{\{\Pi_k^B\}} = S(\rho_A) - S\left(\rho_{A|\{\Pi_k^B\}}\right), \tag{4.7}$$

representing the information gained about the system A as a result of the measurement $\{\Pi_k^B\}_k$.

To give a definition for the quantum discord we are going to introduce the classical correlation by considering the supremum over all the possible von Neumann measurements$\{\Pi_k^B\}_k$ [17]:

$$\mathcal{C}(\rho_{AB}) = \sup_{\{\Pi_k^B\}} \mathcal{J}(\rho_{AB})\,|_{\{\Pi_k^B\}}. \tag{4.8}$$

Definition 4.1 (Quantum Discord)
The **quantum A-discord** *is a measure of quantum correlations of a bipartite quantum state*

$$\mathcal{D}_A(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{C}(\rho_{AB}), \tag{4.9}$$

given by the difference between the mutual information $\mathcal{I}(\rho_{AB})$ and the classical correlation $\mathcal{C}(\rho_{AB})$ [17, 24].

---

[7] Note that throughout this thesis, the convention for the logarithm, log, is taken to base two.

[8] Measurements in quantum theory are basis dependent and they are changing the state of the system.

We need to observe the fact that there is a second definition for the quantum discord, the quantum B-discord, considering the von Neumann measurements performed on the first system. By this definition, the obvious consequence of asymmetry under the change $A \leftrightarrow B$ [9] is validated. For further use, the notation $\mathcal{D}(\rho) = \mathcal{D}_A(\rho)$ will be considered.

Remark
Quantum discord is a different type of correlation than entanglement, that is because separable states can have non-vanishing quantum discord.
It is a measure of non-classical correlations that may include entanglement, however it is an independent measure.

4.2 Quantum Discord for an arbitrary X state

In this part of the chapter, the algorithm found by Li et al. [27] is used to evaluate the quantum discord of an X state in its canonical form[10].

Recalling the canonical form of the X state, given in Eq. 3.16, it's eigenvalues are given by

$$\begin{aligned} \mu_{1,2} &= \frac{1}{4}\left[1 - c_3 \pm \sqrt{(r_z - s_z)^2 + (c_1 + c_2)^2}\right], \\ \mu_{3,4} &= \frac{1}{4}\left[1 + c_3 \pm \sqrt{(r_z + s_z)^2 + (c_1 - c_2)^2}\right]. \end{aligned} \tag{4.10}$$

To give the expression of the quantum mutual information, the following monotonically decreasing function will be introduced for convenience:

$$\boxed{f(x) = -\frac{1-x}{2}\log(1-x) - \frac{1+x}{2}\log(1+x),} \tag{4.11}$$

where $x \in [0, 1]$.
According to Eq. 4.4 the quantum mutual information is given by

$$\mathcal{I}\left(\rho_{AB}\right) = S\left(\rho_A\right) + S\left(\rho_B\right) + \sum_{j=1}^{4} \mu_j \log \mu_j, \tag{4.12}$$

however, the von Neumann entropy of the reduced density matrices is:
$S\left(\rho_A\right) = 1 + f(r_z)$ and $S\left(\rho_B\right) = 1 + f(s_z)$.

[9] This means that the quantum discord depends on which subsystem the measurement is performed on.

[10] Due to the fact that analytical expressions for quantum correlations and quantum discord have been accessible for a bipartite Bell-diagonal state and a seven-parameter family of two-qubit X state.

The classical correlation is computed in detail in [27]. We will make use of the functions

$$g_1 = -\frac{1+r_z+s_z+c_3}{4}\log\frac{1+r_z+s_z+c_3}{2(1+s_z)} - \frac{1-r_z+s_z-c_3}{4}\log\frac{1-r_z+s_z-c_3}{2(1+s_z)} - \\ - \frac{1+r_z-s_z-c_3}{4}\log\frac{1+r_z-s_z-c_3}{2(1-s_z)} - \frac{1-r_z-s_z+c_3}{4}\log\frac{1-r_z-s_z+c_3}{2(1-s_z)},$$

$$g_2 = 1 + f\left(\sqrt{r_z^2+c_1^2}\right),$$

$$g_3 = 1 + f\left(\sqrt{r_z^2+c_2^2}\right).$$

to give its expression.

Theorem 4.3 (Classical Correlation of an X state)
For any state ρ_X^{can} provided by the Eq. 3.16, the classical correlation is given by

$$\mathcal{C}(\rho) = S(\rho_A) - \min\{g_1, g_2, g_3\}. \tag{4.13}$$

Given this theorem, and recalling Eq. 4.9 the quantum discord for an X state will have the form

$$\boxed{\mathcal{D} \overset{(4.9).(4.12),(4.13)}{=\!=} 1 + f(s_z) + \sum_{j=1}^{4}\mu_j\log\mu_j + \min\{g_1, g_2, g_3\}.} \tag{4.14}$$

Summary

I Quantum Discord was introduced as a result of the discovery that entanglement is not responsible for all non-classical correlations and that even separable states may contain quantum correlations.

II To obtain the definition of Quantum Discord the first steps taken in this chapter were to define Classical Mutual Information and then shift it to the Quantum Mutual Information in terms of density operators, given by

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

III The Classical Correlation is also defined, and mathematically it represents the supremum over all the possible von Neumann measurements $\{\Pi_k^B\}_k$:

$$\mathcal{C}(\rho_{AB}) = \sup_{\{\Pi_k^B\}} \mathcal{J}(\rho_{AB})\,|_{\{\Pi_k^B\}}.$$

IV Hence, the quantum discord is a measure of quantum correlations of a bipartite quantum state given by the difference between the mutual information and the classical correlations

$$\mathcal{D}_A\left(\rho_{AB}\right)=\mathcal{I}\left(\rho_{AB}\right)-\mathcal{C}\left(\rho_{AB}\right).$$

V Further, this chapter presents the main steps towards the construction of Quantum Discord for the case of X states.

5 | Broadcasting of Entanglement

In contrast with the classical information, quantum information has a fundamental property that it cannot be copied. There exist no physical process reproducing a perfect copy of a system that is initially in an unknown state[1]. This is named no-cloning theorem [28, 29]. Although quantum perfect cloning is not allowed, it is possible to construct a cloning machine that yields imperfect copies of the original quantum state, meaning that the cloning process presents errors.

Assuming a quantum machine duplicating a two-state system, initially in an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where the amplitudes α and β are unknown, it is possible to construct a cloning machine yielding two imperfect copies of a single qubit in the state $|\psi\rangle$. An universal cloning machine (UCM) was implemented [30] that creates two identical copies of $|\psi\rangle$, each characterized by the same density operator $\rho = (2/3)|\psi\rangle\langle\psi| + \mathbb{1}/6$. It is called universal because it produces copies that are not dependent on the input state.[2]

5.1 Local optimal universal asymmetric cloning machines of qubits

5.1.1 Characteristics of a Pauli channel using Bell states

A Pauli channel is defined with the help of four error operators[3] group, acting on a qubit in an arbitrary pure state $|\psi\rangle$ by rotating it or leaving it unchanged[4]. However, Pauli channel can be described also in another manner. Consider an input qubit X initially entangled with a reference qubit R, in the Bell state $|\phi^+\rangle$. If the reference qubit R remains unchanged while X is processed by the channel, their joint state is

$$|\psi\rangle_{RX} = \left|\Phi^+\right\rangle, \tag{5.1}$$

then the joint state of the qubit R and the output Y is a mixture of the four Bell states [5]

$$\rho^{RY} = (1-p)\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + p_z\left|\Phi^-\right\rangle\left\langle\Phi^-\right| + p_x\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + p_y\left|\Psi^-\right\rangle\left\langle\Psi^-\right|, \tag{5.2}$$

where $p = p_x + p_y + p_z$, the weights of the Bell states are simply the associated probabilities with the four error operators [32].

[1] If this would've been allowed Heisenberg uncertainty principle would be violated when conjugate observables were measured.

[2] Introduction inspired from *N. Cerf* [31].

[3] The three Pauli matrices and the identity.

[4] The input qubit undergoes a phase flip by σ_z, a bit flip σ_x or their combination $\sigma_x\sigma_z = -i\sigma_y$, with the probabilities p_z, p_x, p_y . If the qubit remains unchanged $p = 1 - p_x - p_y - p_z$.

[5] Since whenever a local action of the error operators is applied on one of the Bell states the other three states result: $(\mathbb{1}\otimes\sigma_z)|\Phi^+\rangle = |\Phi^-\rangle$, $(\mathbb{1}\otimes\sigma_x)|\Phi^+\rangle = |\Psi^+\rangle$, and $(\mathbb{1}\otimes\sigma_x\sigma_z)|\Phi^+\rangle = |\Psi^-\rangle$.

5.1.2 Asymmetric Pauli cloning machines

Definition 5.1 (PCM)
A Pauli cloning machine is a unitary transformation acting on a input qubit along with two auxiliary qubits (a blank copy and an ancilla). The asymmetric machine has two outputs emerging from distinct Pauli channels.

To emphasize a little bit more on the previous definition, in Fig. 5.1, we can see that the operation of PCM is described by considering a 4-qubit system consisting of the input X fully entangled with a reference R, i.e. $|\psi\rangle_{RX} = |\Phi^+\rangle$ and two auxiliary qubits in the state $|0\rangle$. The machine admits two output states A and B emerging from distinct Pauli channels, whereas the density operators ρ^{RA} and ρ^{RB} must be mixtures of Bell states.

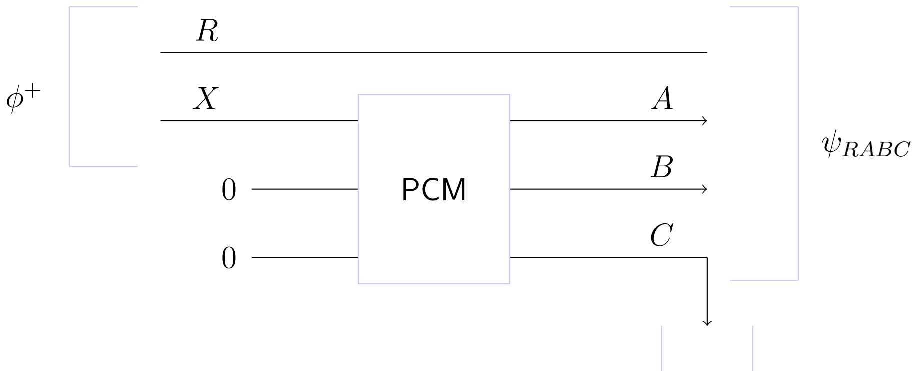


Figure 5.1: Pauli cloning machine of input X entangled initially with a reference R having the output A and B. The additional output C refers to the ancilla or the cloning machine. This design is inspired from [31, 32].

The additional output, the ancilla or the cloning machine itself, is used because if the first output ρ^{RA} is assumed to result from the partial trace of a pure state in an 4-dimensional Hilbert space. By Schmidt decomposition, to accommodate it's eigenvalues, a 4-dimensional additional space is necessary, such that the second qubit B (with a 2-dimensional space) is insufficient [32]. We will restrict to the minimal case of a 2-dimensional ancilla C [30].

After cloning took place, the qubits R, A, B and C are in a pure state $|\psi\rangle_{RABC}$, with ρ^{RA} and ρ^{RB} mixtures of Bell states[6], as well, ρ^{RC} is also a Bell mixture, so that C may be viewed as a third output emerging from another Pauli channel.

[6] A and B emerge from a Pauli channel.

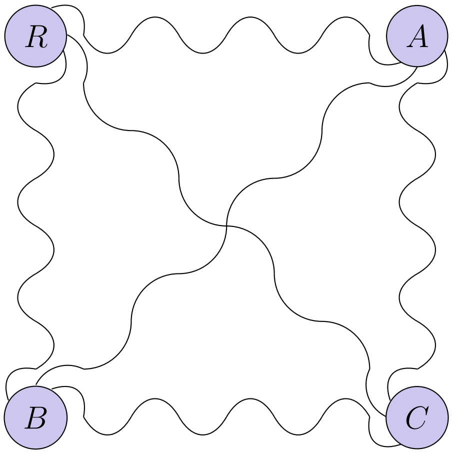


Figure 5.2: Representation of the entangled wavefunction $|\psi\rangle_{RABC}$.

The way to characterize the PCM is by the wavefunction[7] of the entangled state $|\psi\rangle_{RABC}$. To satisfy the requirement that the state of every qubit pair is a mixture of the four Bell states, we shall find the 4-qubit wave function $|\psi\rangle_{RABC}$, making use of the Schmidt decomposition for the bipartite partition RA vs BC as a superposition of double Bell states [32]

$$|\Psi\rangle_{RA;BC} = \left\{v\left|\Phi^+\right\rangle\left|\Phi^+\right\rangle + z\left|\Phi^-\right\rangle\left|\Phi^-\right\rangle + x\left|\Psi^+\right\rangle\left|\Psi^+\right\rangle + y\left|\Psi^-\right\rangle\left|\Psi^-\right\rangle\right\}_{RA;BC}, \tag{5.3}$$

where $x, y, z,$ and v are complex amplitudes with $|x|^2 + |y|^2 + |z|^2 + |v|^2 = 1$. We shall see that the requirement of the pairs RA and BC to be Bell mixtures is thus satisfied, thus, $\rho^{RA} = \rho^{BC}$ is of the form 5.2 when $p_x = |x|^2, p_y = |y|^2, p_z = |z|^2$ and $1-p = |v|^2$. A feature of these double Bell states for the partition RA vs BC is that they transform into superpositions of double Bell states for the other two possible partitions RB vs AC, RC vs AB). E.g., the transformation associated with the partition RB vs AC is

$$\left|\Phi^+\right\rangle_{RA}\left|\Phi^+\right\rangle_{BC} = \frac{1}{2}\left\{\left|\Phi^+\right\rangle\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\left|\Phi^-\right\rangle + \left|\Psi^+\right\rangle\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\left|\Psi^-\right\rangle\right\}_{RB;AC}. \tag{5.4}$$

This implies that $|\Psi\rangle_{RABC}$ is also a superposition of double Bell states (having different amplitudes) for these two other partitions. Hence, when tracing over half of the system, mixtures of Bell states are obtained . E.g, for the partition RB vs AC

$$|\Psi\rangle_{RB;AC} = \left\{v'\left|\Phi^+\right\rangle\left|\Phi^+\right\rangle + z'\left|\Phi^-\right\rangle\left|\Phi^-\right\rangle + x'\left|\Psi^+\right\rangle\left|\Psi^+\right\rangle + y'\left|\Psi^-\right\rangle\left|\Psi^-\right\rangle\right\}_{RB;AC}, \tag{5.5}$$

having

[7] Instead of specifying it by particular unitary transformations acting on the input state.

$$
\begin{aligned}
v' &= \frac{1}{2}(v+z+x+y),\\
z' &= \frac{1}{2}(v+z-x-y),\\
x' &= \frac{1}{2}(v-z+x-y),\\
y' &= \frac{1}{2}(v-z-x+y),
\end{aligned}
\tag{5.6}
$$

therefore the second output B comes out from a Pauli channel with probabilities $p'_x = |x'|^2, p'_y = |y'|^2$, and $p'_z = |z'|^2$. Likewise, the third output C is described by considering the partition RC vs AB,

$$
|\Psi\rangle_{RC;AB} = \left\{ v'' \left|\Phi^+\right\rangle \left|\Phi^+\right\rangle + z'' \left|\Phi^-\right\rangle \left|\Phi^-\right\rangle + x'' \left|\Psi^+\right\rangle \left|\Psi^+\right\rangle + y'' \left|\Psi^-\right\rangle \left|\Psi^-\right\rangle \right\}_{RC;AB}
\tag{5.7}
$$

having

$$
\begin{aligned}
v'' &= \frac{1}{2}(v+z+x-y),\\
z'' &= \frac{1}{2}(v+z-x+y),\\
x'' &= \frac{1}{2}(v-z+x+y),\\
y'' &= \frac{1}{2}(v-z-x-y),
\end{aligned}
\tag{5.8}
$$

We can see that the related amplitudes (related by Eqs. 5.6 & 5.8) of the double Bell states, for the three possible partitions of the four qubits into two pairs, are specifying the entire set of asymmetric PCM [31].

Let us rewrite the amplitudes of $|\psi\rangle_{RA:BC}$ as a two-dimensional discrete function $\alpha_{m,n}$ with $m, n = 0, 1$:

$$
\begin{aligned}
\alpha_{0,0} &= v,\\
\alpha_{0,1} &= z,\\
\alpha_{1,0} &= x,\\
\alpha_{1,1} &= y,
\end{aligned}
\tag{5.9}
$$

The A output emerges from a Pauli channel with the probability distribution $p_{m,n} = |\alpha_{m,n}|^2$, where $p_{0,1} = p_z, p_{1,0} = p_x, p_{1,1} = p_y$, and $p_{0,0}$ is the probability of the qubit that remains unchanged. In the same manner, output B can be characterized by a two-dimensional function $\beta_{m,n}$ defined as

$$\begin{aligned}\beta_{0,0} &= v', \\ \beta_{0,1} &= z', \\ \beta_{1,0} &= x', \\ \beta_{1,1} &= y',\end{aligned} \tag{5.10}$$

with the probability distribution $p'_{m,n} = |\beta_{m,n}|^2$.
Using this notation, we can see that Eq. 5.6 is a two-dimensional discrete Fourier transform [31]

$$\beta_{m,n} = \frac{1}{2}\sum_{x,y=0}^{1}(-1)^{nx+my}\alpha_{x,y}. \tag{5.11}$$

If output A is close to perfect then output B is very noisy, and conversely. The probability distributions $p_{m,n} = |\alpha_{m,n}|^2$ and $p'_{m,n} = |\beta_{m,n}|^2$ characterizing the channels leading to outputs A and B cannot have a variance simultaneously tending to zero. This gives rise to the uncertainty principle that governs the exchange between the quality of the copies [31].

Thereafter, we can describe the cloner in a more advanced way[8]

$$\begin{aligned}|\psi\rangle_{RBCD} &:= \sum_{m,n=0}^{1} \alpha_{m,n}\,|\Phi_{m,n}\rangle_{RA}\,|\Phi_{m,n}\rangle_{BC} \\ &= \sum_{m,n=0}^{1} \beta_{m,n}\,|\Phi_{m,n}\rangle_{RB}\,|\Phi_{m,n}\rangle_{AC}\,.\end{aligned} \tag{5.12}$$

Hence, the action of the cloner on the input state is given by [33]

$$|\phi_j\rangle := U|j\rangle_B|00\rangle_{CD} = \sum_{m,s=0}^{d-1} b_{m,s}^{(j)}|\overline{j+m}\rangle_B|s\rangle_C|\overline{s+m}\rangle_D \tag{5.13}$$

where $\{|j\rangle\}_{j=0,1}$ is the computational basis and $\overline{j+m} = j+m$ modulo d, $\overline{s+m}$ respectively .

Following the computations made in [33] which are based on d-level systems, I will brutally extract the general expression for the three-particle system after applying the cloning machine, keeping in mind that we are playing with 2-dimensional system:

[8] We replace the notation for the Bell states in the following way:

$$\begin{aligned}|\Phi_{0,0}\rangle &= |\Phi^+\rangle\,, \\ |\Phi_{0,1}\rangle &= |\Phi^-\rangle\,, \\ |\Phi_{1,0}\rangle &= |\Psi^+\rangle\,, \\ |\Phi_{1,1}\rangle &= |\Psi^-\rangle\,.\end{aligned}$$

$$U|j\rangle|00\rangle = \frac{1}{\sqrt{1+(d-1)\left(p^2+q^2\right)}}\left(|j\rangle|j\rangle|j\rangle + p\sum_{r=1}^{d-1}|j\rangle|\overline{j+r}\rangle|\overline{j+r}\rangle \right. \\ \left. +q\sum_{r=1}^{d-1}|\overline{j+r}\rangle|j\rangle|\overline{j+r}\rangle\right). \tag{5.14}$$

Assuming we want to clone the state $|\psi\rangle = \sum_{j=0}^{d-1}\alpha_j|j\rangle$. Then the total state of the two clones and ancilla, after cloning, is

$$\boxed{|\Pi\rangle = \frac{1}{\sqrt{1+(d-1)\left(p^2+q^2\right)}}\sum_{j=0}^{d-1}\alpha_j\left(|j\rangle|j\rangle|j\rangle + p\sum_{r=1}^{d-1}|j\rangle|\overline{j+r}\rangle|\overline{j+r}\rangle + +q\sum_{r=1}^{d-1}|\overline{j+r}\rangle|j\rangle|\overline{j+r}\rangle\right),} \tag{5.15}$$

having $p+q=1$.
The states of the clones therefore are

$$\boxed{\rho^A = \frac{1}{1+(d-1)\left(p^2+q^2\right)}\left\{\left[1-q^2+(d-1)p^2\right]|\psi\rangle\langle\psi| + q^2 I\right\},} \tag{5.16}$$

and

$$\boxed{\rho^B = \frac{1}{1+(d-1)\left(p^2+q^2\right)}\left\{\left[1-p^2+(d-1)q^2\right]|\psi\rangle\langle\psi| + p^2 I\right\}.} \tag{5.17}$$

5.2 Broadcasting of entanglement using local optimal universal asymmetric cloners

Before we are going through calculations I would like to say that this section is fully inspired from the work done by *I. Ghiu* in [33].

We are going to investigate the case when two distant observers (Alice and Bob) apply locally an asymmetric cloning machine. The initial entanglement shared between Alice and Bob is

$$|\psi\rangle_{12} = \alpha|00\rangle + \beta|11\rangle, \tag{5.18}$$

where we assume that α, $\beta \in \mathbb{R}$.

Alice and Bob use the optimal universal asymmetric Pauli cloning machine

$$U|0\rangle|00\rangle = \frac{1}{\sqrt{1+p^2+q^2}}(|000\rangle + p|011\rangle + q|101\rangle), \tag{5.19}$$

$$U|1\rangle|00\rangle = \frac{1}{\sqrt{1+p^2+q^2}}(|111\rangle + p|100\rangle + q|010\rangle), \tag{5.20}$$

with $p+q=1$. Therefore, the state of the total system, consisting of the two particles 1 and 2 , and another four particles: the blank states 3 and 4 , and the ancillas 5 and 6, (Fig. 5.3), after applying the cloning transformation (Eqs. 5.19 & 5.20) by Alice and Bob is [33] .

$$\begin{aligned}
|\Pi'\rangle =& U \otimes U |\psi\rangle_{12}|00\rangle_{35}|00\rangle_{46} \\
=& \frac{1}{\sqrt{1+p^2+q^2}} \Big\{ |00\rangle_{56} \big[\alpha|00\rangle_{13}|00\rangle_{24} + \beta p^2|10\rangle_{13}|10\rangle_{24} + \beta pq|10\rangle_{13}|01\rangle_{24} + \\
& + \beta pq|01\rangle_{13}|10\rangle_{24} + \beta q^2|01\rangle_{13}|01\rangle_{24}\big] + \\
& + |01\rangle_{56} [\alpha p|00\rangle_{13}|01\rangle_{24} + \alpha q|00\rangle_{13}|10\rangle_{24} + \beta p|10\rangle_{13}|11\rangle_{24} + \beta q|01\rangle_{13}|11\rangle_{24}] + \\
& + |10\rangle_{56} [\alpha q|10\rangle_{13}|00\rangle_{24} + \alpha p|01\rangle_{13}|00\rangle_{24} + \beta p|11\rangle_{13}|10\rangle_{24} + \beta q|11\rangle_{13}|01\rangle_{24}] + \\
& + |11\rangle_{56} \big[\alpha p^2|01\rangle_{13}|01\rangle_{24} + \alpha pq|01\rangle_{13}|10\rangle_{24} + \alpha pq|10\rangle_{13}|01\rangle_{24} + \\
& + \alpha q^2|10\rangle_{13}|10\rangle_{24} + \beta|11\rangle_{13}|11\rangle_{24}\big]\Big\} \\
=& \alpha\,|\phi_0\rangle + \beta\,|\phi_1\rangle\,.
\end{aligned} \tag{5.21}$$

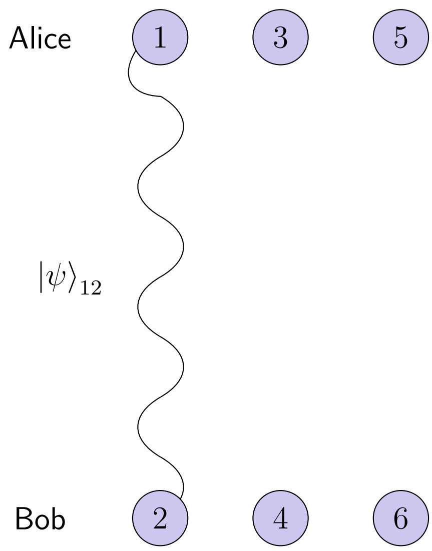


Figure 5.3: The initial state of the total system.

Two necessary conditions need to be satisfied in order to say that the input state $|\psi\rangle_{12}$ has been broadcast:

- the local reduced density operators ρ_{13} and ρ_{24} are separable,
- the nonlocal states ρ_{14} and ρ_{23} are inseparable.

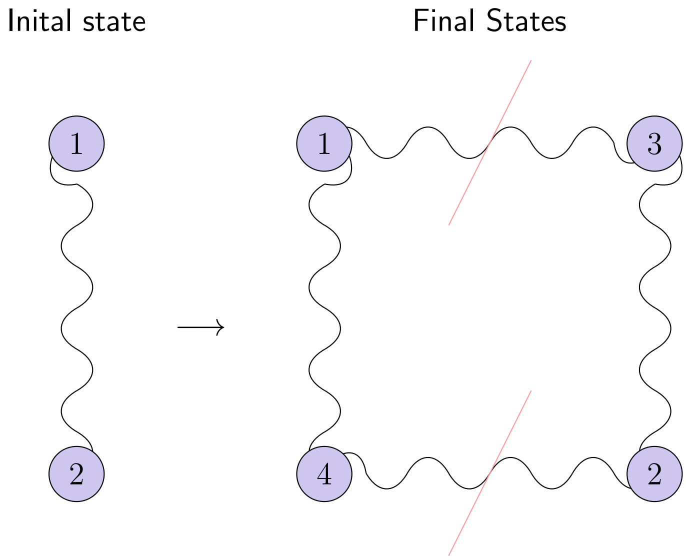


Figure 5.4: The broadcasting process implies that having initially a pure state $|\psi\rangle_{12}$, we will end up with two entangled mixed states and two separable states.

The reduced density operators of the local states are therefore calculated

$$\boxed{\begin{aligned}\rho^{13} = \rho^{24} = &\frac{1}{(1+p^2+q^2)^2}\left[\alpha^2\left(1+p^2+q^2\right)|00\rangle\langle 00\,|+\right.\\&+\ \beta^2\left(1+p^2+q^2\right)|\,11\rangle\langle 11|+\\&+\left(p^2q^2+\beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2\right)|01\rangle\langle 01|+\\&+\left(p^2q^2+\beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2\right)|10\rangle\langle 10|+\\&\left.+\left(pq+p^3q+pq^3\right)(|01\rangle\langle 10|+|10\rangle\langle 01|)\right].\end{aligned}} \tag{5.22}$$

The condition for the separability of the local states

$$\alpha^2\beta^2 - p^2q^2 \geqslant 0$$

or equivalently

$$\frac{1}{2}\left[1-\sqrt{1-4p^2(1-p)^2}\right] \leqslant \alpha^2 \leqslant \frac{1}{2}\left[1+\sqrt{1-4p^2(1-p)^2}\right]. \tag{5.23}$$

is obtained by applying the Peres-Horodecki (2.6) theorem.

The reduced density operators of the nonlocal pairs of particles are:

$$\boxed{\begin{aligned}\rho^{14} =& \frac{1}{\left(1+p^2+q^2\right)^2}\left\{\left[p^2q^2+\alpha^2\left(1+p^2+q^2\right)\right]|00\rangle\langle 00| + \left[p^2q^2 + \right.\right. \\ & \left. +\beta^2\left(1+p^2+q^2\right)\right]|11\rangle\langle 11| + 4pq\alpha\beta(|00\rangle\langle 11| + |11\rangle\langle 00|) + \\ & +\left(\beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2\right)|01\rangle\langle 01| + \\ & \left. +\left(\beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2\right)|10\rangle\langle 10|\right\}\end{aligned}}, \tag{5.24}$$

and

$$\boxed{\begin{aligned}\rho^{23} =& \frac{1}{\left(1+p^2+q^2\right)^2}\left\{\left[p^2q^2+\alpha^2\left(1+p^2+q^2\right)\right]|00\rangle\langle 00| + \left[p^2q^2 + \right.\right. \\ & \left. +\beta^2\left(1+p^2+q^2\right)\right]|11\rangle\langle 11| + 4pq\alpha\beta(|00\rangle\langle 11| + |11\rangle\langle 00|) + \\ & +\left(\beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2\right)|01\rangle\langle 01| + \\ & \left. +\left(\beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2\right)|10\rangle\langle 10|\right\}.\end{aligned}} \tag{5.25}$$

The two nonlocal states are inseparable if

$$\left(\beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2\right)\left(\beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2\right) - 16\alpha^2\beta^2p^2q^2 \leqslant 0. \tag{5.26}$$

or equivalently

$$\frac{1}{2}(1-\sqrt{1-4\lambda}) \leqslant \alpha^2 \leqslant \frac{1}{2}(1+\sqrt{1-4\lambda}). \tag{5.27}$$

where

$$\lambda = \frac{p^4q^4+p^2q^4+p^4q^2+p^2q^2}{2p^4q^4+2p^4q^2+2p^2q^4-q^8-2q^6-q^4-p^8-2p^6-p^4+18p^2q^2} \tag{5.28}$$

If $1-4\lambda$ is positive and the local states are separable when the nonlocal ones are inseparable, leads to [33]

$$\frac{1}{2}(1-\sqrt{-9+2\sqrt{21}}) \leqslant p \leqslant \frac{1}{2}(1+\sqrt{-9+2\sqrt{21}}). \tag{5.29}$$

Summary

I In this chapter the Pauli cloning machine (PCM) was introduced. It produces generally two non-identical output qubits emerging from a Pauli channel.

II A Pauli channel is a special case of Heisenberg channel, defined by the four element group of error operators for qubits (generated by the bit/phase flip errors).

III The family of Pauli cloning machines relies on a 4-qubit wave functions parametrization in which all qubit pairs are in a mixture of Bell states.

IV We investigated the case when two distant observers apply a PCM. The output states after broadcasting the entangled input state $|\psi\rangle_{12}$ are calculated.

6 Quantum correlations of the output states generated by the broadcasting of entanglement using local optimal universal asymmetric cloners

In this section, we present a comparison between the concurrence and quantum discord of the output states generated in the process of broadcasting of entanglement.

6.1 The concurrence of the output states

The values for the concurrence that are of interest for us, are for the nonlocal pairs from Eq. 5.24 & 5.25,[1] given by

$$C(\rho^{14}) = 2\,\mathsf{max}\left\{0,\ \frac{1}{(1+p^2+q^2)^2}\left[4pq\alpha\beta - \right.\right.$$
$$\left.\left. -\sqrt{(\beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2)(\beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2)}\right]\right\}. \tag{6.3}$$

$$C(\rho^{23}) = 2\,\mathsf{max}\left\{0,\ \frac{1}{(1+p^2+q^2)^2}\left[4pq\alpha\beta - \right.\right.$$
$$\left.\left. -\sqrt{(\beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2)(\beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2)}\right]\right\}. \tag{6.4}$$

We can see that the values for the concurrence of the two nonlocal pairs ρ^{14} and ρ^{23} are equal.

[1] In order to emphasize that the relations (5.24 & 5.25) are X states, we are writing them in a matrix form, as following

$$\rho^{14} = \frac{1}{(1+p^2+q^2)^2}\begin{pmatrix} p^2q^2+\alpha^2(1+p^2+q^2) & 0 & 0 & 4pq\alpha\beta \\ 0 & \beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2 & 0 & 0 \\ 0 & 0 & \beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2 & 0 \\ 4pq\alpha\beta & 0 & 0 & p^2q^2+\beta^2(1+p^2+q^2) \end{pmatrix}, \tag{6.1}$$

$$\rho^{23} = \frac{1}{(1+p^2+q^2)^2}\begin{pmatrix} p^2q^2+\alpha^2(1+p^2+q^2) & 0 & 0 & 4pq\alpha\beta \\ 0 & \beta^2p^4+\beta^2p^2+\alpha^2q^4+\alpha^2q^2 & 0 & 0 \\ 0 & 0 & \beta^2q^4+\beta^2q^2+\alpha^2p^4+\alpha^2p^2 & 0 \\ 4pq\alpha\beta & 0 & 0 & p^2q^2+\beta^2(1+p^2+q^2) \end{pmatrix}. \tag{6.2}$$

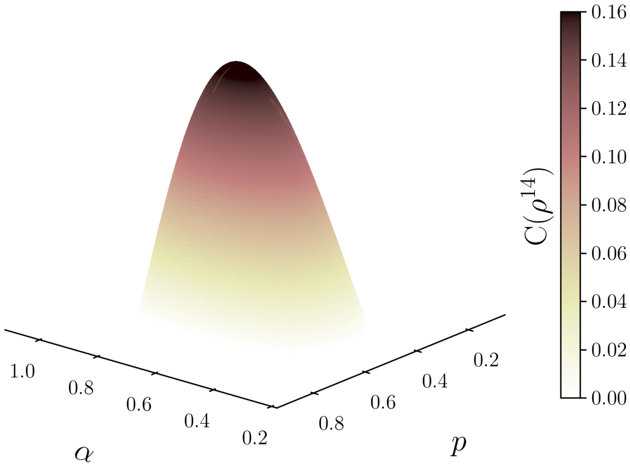
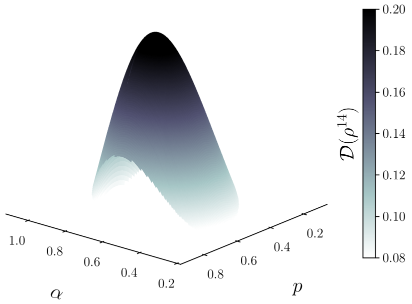


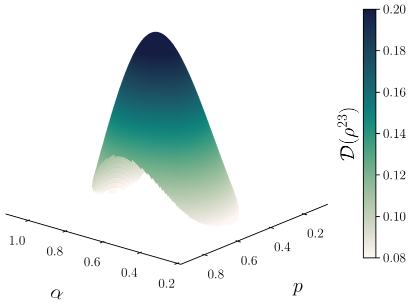
Figure 6.1: Concurrence of the output states ρ^{14} & ρ^{23} generated by broadcasting the entanglement.

6.2 The quantum discord of the output states

The quantum discord of the two nonlocal states ρ^{14} and ρ^{23} (Eq. 5.24 & 5.25), using the complete algorithm from the section 4.2. of the X states, is plotted in Fig. 6.2.



(a) QD ρ^{14}.

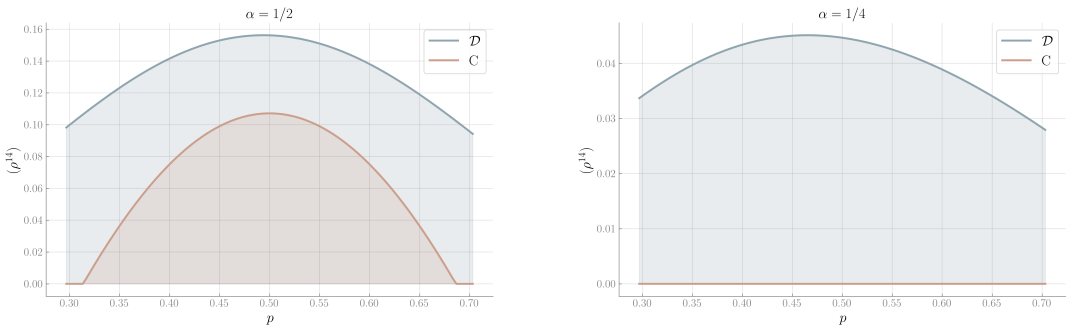


(b) QD ρ^{23}.

Figure 6.2: Quantum Discord of the output states ρ^{14} & ρ^{23} generated by broadcasting the entanglement.

6.3 A comparison between the concurrence and the quantum discord of the output states

In Fig. 6.3 (a) we plot both the concurrence and discord of the state ρ^{14}, when the parameter α, which characterizes the initial entangled state (see Eq. 5.18), is equal to $1/2$.We remark that the discord is larger than the concurrence.
In addition, in the case when $\alpha = 1/4$, we notice in Fig. 6.3 (b) that the state ρ^{14} is separable (C=0) and, at the same time, its quantum discord is non-zero, this being an example of separable state characterized by another kind of correlation - qauntum discord.

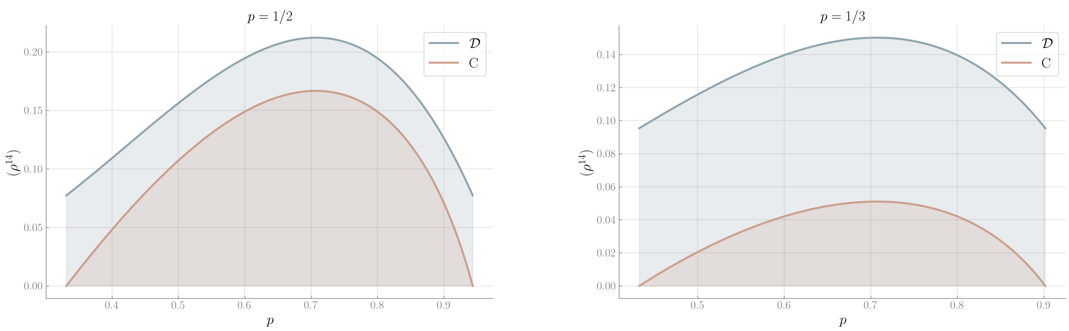


(a) $\alpha = 1/2$

(b) $\alpha = 1/4$

Figure 6.3: Concurrence vs Discord for ρ^{14}.

On the other hand, we present the comparison between concurrence and discord for some fixed values of the parameter p, which describes the cloning machine (see Eqs. 5.19 and 5.20). The case of optimal universal symmetric Pauli cloning machine corresponds to $p = 1/2$ and this is shown in Fig. 6.4 (a), where we see that the discord is larger than the concurrence. Finally, we plot the concurrence and discord in Fig. 6.4 (b) in the case when asymmetric cloning machine is applied for $p = 1/3$. Again the discord is larger than concurrence.



(a) $p = 1/2$

(b) $p = 1/3$

Figure 6.4: Concurrence vs Discord for ρ^{14}.

Conclusions

The results of this thesis offer a new insight in the world of quantum information. We have provided the tools for broadcasting the entangled state presented in Chapter 5, and analyze the output of that initial state after applying an optimal universal asymmetric Pauli cloning machine. Further, by the means of concurrence and quantum discord of an arbitrary X state, provided in Chapter 3 & Chapter 4, we plot them for the non-local states ρ^{14} and ρ^{23}, giving a comparison between concurrence and discord by fixating firstly the parameter characterizing the initial entangled state, then the parameter describing the cloning machine. In the end we have seen an example of separable state characterized by another type of correlation -quantum discord.

Studying quantum correlations, other than quantum entanglement, may give new sorts of approaches into increasing the efficiency of quantum algorithms on large quantum systems. Developing future applications and advancing our comprehension about the rule played by this correlations, like quantum communications, may provide new insights within the realm of physics.

A | Properties of Shannon and von Neumann Entropy

Properties of the Shannon entropy [9]:

- $H(A, B) = H(B, A)$, $I(X : B) = I(B : A)$.
- $H(A|B) \geq 0$ thus $H(A : B) \leq H(B) \Leftrightarrow B = f(A)$.
- $H(A) \leq H(A, B)$, where $H(A) = H(A, B) \Leftrightarrow B = f(A)$.
- Subadditivity: $H(A, B) \leq H(A) + H(B)$, where $H(A, B) = H(A) + H(B) \Leftrightarrow$ A and B are independent random variables.
- $H(B|A) \geq H(B)$ and thus $I(A : B) \geq 0$, with equality in each $\Leftrightarrow$ A and B are independent random variables.
- Strong subadditivity: $H(A, B, C) + H(B) \leq H(A, B) + H(B, C)$, with equality $\Leftrightarrow$ $A \rightarrow B \rightarrow C$ forms a Markov chain.
- Conditioning reduces entropy: $H(A|B, C) \leq H(A|B)$.

Properties of von Neumann entropy [9]:

- The entropy is non-negative. $S(\rho) = 0 \Leftrightarrow \rho$ is a pure state.
- In a d-dimensional Hilbert space the entropy is at most $\log d$. If the system is in a maximally mixed state the entropy is $\log d$.
- If a composite system AB is in a pure state then $S(A) = S(B)$.
- If p_i are probabilities, and the states ρ_i are orthogonal, then

$$S\left(\sum_i p_i \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i).$$

- Joint entropy theorem: If p_i are probabilities, $|i\rangle$ are orthogonal states for a system A, and ρ_i is any set of density operators for a system B, then

$$S\left(\sum_i p_i |i\rangle \langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i).$$

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