Taylor Series

A few days ago we learned to find a Taylor polynomial for a function f. Today we will extend our knowledge of Taylor polynomials to find a **Taylor** **series** for a function f.

Taylor Series for f centered at $x = c$: $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c = 0$, the series is called a Maclaurin series.

There are three special Maclaurin series you must **know**. These are the series for e^x, $\sin x$, and $\cos x$.

To derive a series for e^x:

n	$f^n(x)$	$f^n(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1
5	e^x	1

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n} \right| = 0$$

$0 < 1 \quad \therefore \text{RoC} = \infty$

For what values of x does this series converge?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for } (-\infty, \infty)$$

To derive a series for $\sin x$:

n	$f^n(x)$	$f^n(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1

$$0 + x + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!} + \cdots \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}(-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(-1)\,x^2}{(2n+3)(2n+2)} \right| = 0$$

$0 < 1 \quad \therefore \text{RoC} = \infty$

For what values of x does this series converge?

$$\sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for } (-\infty, \infty)$$

To derive a series for $\cos x$:

n	$f^n(x)$	$f^n(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
5	$-\sin x$	0
6	$-\cos x$	-1

$$1 + 0x - \frac{x^2}{2!} + \frac{0x^3}{3!} + \frac{x^4}{4!} + \frac{0x^5}{5!} - \frac{x^6}{6!} + \cdots \frac{x^{2n}}{(2n)!}(-1)^n$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}\cdot(-1)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}\cdot(-1)^n}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$$

$0 < 1 \quad \therefore \text{ROC} = \infty$

$$\boxed{\cos x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(-1)^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for } (-\infty, \infty)}$$

We can manipulate these three special series (or any series we are given) to find other series by using the following techniques:

1) Substitute into the series
2) Multiply or divide the series by a constant and/or a variable
3) Add or subtract two series
4) Differentiate or integrate a series
5) Recognizing it as the sum of a geometric power series

Ex. Find a Maclaurin series for $f(x) = \sin(x^2)$. Find the first four nonzero terms and the general term.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

$$* \ \sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \frac{x^{4n+2}}{(2n+1)!}(-1)^n$$

Ex. Find a Maclaurin series for $f(x) = x\cos x$. Find the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \frac{x^{2n}}{(2n)!}(-1)^n$$

$$* \ x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots \frac{x^{2n+1}}{(2n)!}(-1)^n$$

Ex. Find a Maclaurin series for $h(x) = \dfrac{e^x + e^{-x}}{2}$. Find the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \frac{x^n}{n!}(-1)^n$$

$$* \ \frac{e^x + e^{-x}}{2} = \frac{2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \frac{x^{2n}}{(2n)!}$$

Finding the sum of a series:
Geometric:

$$\sum_{n=0}^{\infty} a_1 r^n = \frac{a_1}{1-r} \quad if \ |r|<1$$

What if $\sum_{n=0}^{\infty}(x)^n$? $= 1 + x + x^2 + x^3 + x^4 + \dots$

geometric $r = x$

$|r| < 1$ when $|x| < 1$

$-1 < x < 1$

What will the series converge to? On what interval?

$$S = \frac{1}{1-x} \quad \text{when} \quad -1 < x < 1$$

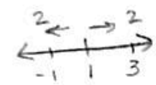
* Endpts always diverge when geo.

What is the sum of : $\frac{x-1}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{8} - \frac{(x-1)^4}{16} + \dots$

geometric

$r = \frac{x-1}{2}$ $\quad \left|\frac{x-1}{2}\right| < 1$ when

$|x-1| < 2$



$$S = \frac{\frac{x-1}{2}}{1 - \frac{x-1}{2}} = \frac{(x-1)}{2-x+1} = \frac{x-1}{-x+1} \quad \text{when} \quad -1 < x < 3.$$

If we are given the function, and want the power series, we must rewrite the function in the form $f(x) = \frac{a_1}{1-r}$

Example:

Find the power series for $f(x) = \frac{1}{1-x}$, centered at x=-1. $\rightarrow (x+1)$

$$f(x) = \frac{1}{1-(x+1)+1} = \frac{1/2}{\frac{2-(x+1)}{2}} = \frac{\frac{1}{2}}{1 - \frac{(x+1)}{2}}$$

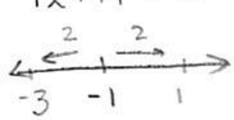
$a_1 = \frac{1}{2}$

$r = \frac{x+1}{2}$

The series will converge

when $\left|\frac{x+1}{2}\right| < 1$

$|x+1| < 2$



$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n = \frac{1}{1-x}$$

when $-3 < x < 1$

Example: Find the geometric power series centered at $\underline{x = 0}$ for $f(x) = \frac{4}{x+2}$.

$$f(x) = \frac{4/2}{\frac{2}{2} - \frac{(-x)}{2}} = \frac{2}{1 - (-\frac{x}{2})}$$

$a_1 = 2$

$r = -\frac{x}{2}$

The series will converge when $\left| -\frac{x}{2} \right| < 1$

$|x| < 2$

$$\therefore \boxed{\sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n = \frac{4}{x+2} \quad \text{when} \quad -2 < x < 2}$$

Example: Find a power series for $f(x) = \frac{1}{x}$, centered at $\underline{x = 1}$.

$$f(x) = \frac{1}{1 - [-(x-1)]}$$

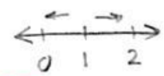
$a_1 = 1$

$r = -(x-1)$

The series will converge when

$|-(x-1)| < 1$

$|x-1| < 1$



$$\boxed{\sum_{n=0}^{\infty} (1-x)^n = \frac{1}{x} \quad \text{when} \quad 0 < x < 2}$$

Example: Find a power series for $f(x) = \frac{12}{4+x}$, centered at $\underline{x = 0}$. for what values of x does your series converge to $f(x)$?

$$f(x) = \frac{12/4}{\frac{4}{4} - \left(\frac{-x}{4}\right)} = \frac{3}{1 - (-\frac{x}{4})}$$

$a_1 = 3$

$r = -\frac{x}{4}$

The series will converge when $\left| \frac{-x}{4} \right| < 1$

$|x| < 4$

$$\boxed{\sum_{n=0}^{\infty} 3\left(-\frac{x}{4}\right)^n = \frac{12}{4+x} \quad \text{when } -4 < x < 4.}$$

Finding the sum of a Taylor Series: ① Either geometric, e^x, $\sin x$, $\cos x$

Ex. Find the sum of $1+\frac{3}{1!}+\frac{9}{2!}+\frac{27}{3!}+...+\frac{3^n}{n!}+...$

$1+\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\ldots$ looks like e^x! $\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)$

$x=3$

$\therefore \sum_{n=0}^{\infty}\frac{3^n}{n!}=e^3$

Ex. Find the sum of $2-\frac{8}{3!}+\frac{32}{5!}-\frac{128}{7!}+...+\frac{(-1)^n 2^{2n+1}}{(2n+1)!}+...$

$2^1-\frac{2^3}{3!}+\frac{2^5}{5!}-\frac{2^7}{7!}+\cdots$ looks like $\sin x$

$x=2$

$\therefore \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n+1}}{(2n+1)!}=\sin 2$

Ex. Find the sum of $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+...\left(\frac{-2}{3}\right)^n+...$ Geometric!

$a_1=1$

$r=-\frac{2}{3}$

$$\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n=\frac{1}{1+2/3}=\frac{3}{5}$$

Theorem: If the function given by $f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+...=\sum_{n=0}^{\infty}a_n(x-c)^n$ has a radius of converegence of R>0, then on the interval (c-R,c+r), $f(x)$ is differentiable (and therefore continuous). Moreover, the derivative and antiderative of $f(x)$ are as follows:

$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+...=\sum_{n=1}^{\infty}na_n(x-c)^{n-1}$$

$$\int f(x)\,dx=C+a_0(x-c)+\frac{a_1(x-c)^2}{2}+...=C+\sum_{n=0}^{\infty}\frac{a_n(x-c)^{n+1}}{n+1}$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of converence, however, may differ as a result of the behavior at the endpoints.

Ex. The function $f(x)$ is defined by $f(x)=\frac{1}{1-x}$. ← geometric $a_1=1$ $r=x$

(a) Write the Maclaurin series for $f(x)$. Give the first four nonzero terms and the general term. For what values of x does the series converge?

* $\frac{1}{1-x}=1+x+x^2+x^3+x^4+\dots x^n=\sum_{n=0}^{\infty}x^n$ when $-1<x<1$

(b) Use your answer to find the MacLaurin series of $f'(x)$. Give the first four nonzero terms and the general term. For what values of x does the series converge?

$f'(x)=1+2x+3x^2+4x^3+\dots=\sum_{n=0}^{\infty}(n+1)\cdot x^n$ when $-1<x<1$

RoC = 1 but we must check endpoints! (no longer geometric)

Test $x=-1$

$\sum_{n=0}^{\infty}(n+1)(-1)^n$ - diverges by n^{th} term $\lim_{n\to\infty}(n+1)(-1)^n\neq 0$

Test $x=1$

$\sum_{n=0}^{\infty}(n+1)$ - diverges by n^{th} term $\lim_{n\to\infty}(n+1)\neq 0$

(c) Use your answers to part (b) to find the sum of the infinite series $1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\dots\frac{n}{3^{n-1}}+\dots$

$1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\dots$ is $f'(\frac{1}{3})$ $\quad -1<\frac{1}{3}<1$

$f(x)=\frac{1}{1-x}$ $\quad\therefore\quad$ $f'(\frac{1}{3})=\frac{-1}{(1-\frac{1}{3})^2}=\frac{1}{\frac{4}{9}}=\frac{9}{4}$

$f'(x)=\frac{-1}{(1-x)^2}$

(d) Use your answer to part (a) for find the Maclaurin series for $\int_0^x f(t)dt$. Give the first four non zero terms and the general term. For what values of x does the series converge?

$\int_0^x \frac{1}{1-t}\,dt=\int_0^x(1+t+t^2+t^3+\quad)\,dt=t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}+\dots\Big|_0^x$

$=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots=\sum_{n=1}^{\infty}\frac{x^n}{n}$ when $-1\le x<1$.

RoC is still 1

Test $x=-1$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges, Alt. Harmonic

Test $x=1$

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, Harmonic

e) Use your answer to part (d) to find the sum of the infinite series

$\frac{1}{3}+\frac{1}{3^2(2)}+\frac{1}{3^3(3)}+\frac{1}{3^4(4)}+\dots+\frac{1}{3^n(n)}+\dots$

$x=\frac{1}{3}$ $\quad -1\le\frac{1}{3}<1$

$\int_0^{\frac{1}{3}}\frac{1}{1-t}\,dt=-\ln|1-t|\Big|_0^{\frac{1}{3}}=-\ln(\frac{2}{3})+\ln 1$

$=\ln(\frac{3}{2})$