Given a joint probability $p(x, y)$, where $x \in \{0,1\}$ and $y \in \{0,1\}$:

$$
\left.
\begin{aligned}
p(x = 0, y = 0) &= a \\
p(x = 0, y = 1) &= b \\
p(x = 1, y = 0) &= c \\
p(x = 1, y = 1) &= d
\end{aligned}
\right\}
$$

For $p(x, y)$ to be a valid probability:
$a + b + c + d = 1$

| $p(x, y)$ | x=0 | x=1 |
|---|---|---|
| y=0 | a | c |
| y=1 | b | d |

Find the marginal probability $p(x)$:

$p(x) = \sum_y p(x, y) = p(x, y = 0) + p(x, y = 1)$, i.e.

$$p(x = 0) = p(x = 0, y = 0) + p(x = 0, y = 1) = a + b,$$
$$p(x = 1) = p(x = 1, y = 0) + p(x = 1, y = 1) = c + d$$

Check that $p(x)$ is a valid probability: $(a + b) + (c + d) = 1$

| $p(x)$ | x=0 | x=1 |
|---|---|---|
| | a + b | c + d |

Find the conditional probability $p(y \mid x)$:

$$p(y \mid x) = \frac{p(x, y)}{p(x)}$$

$$p(y = 0 \mid x = 0) = \frac{p(x = 0, y = 0)}{p(x = 0)} = \frac{a}{a + b}$$

$$p(y = 1 \mid x = 0) = \frac{p(x = 0, y = 1)}{p(x = 0)} = \frac{b}{a + b}$$

For $p(y \mid x = 0)$to be a valid probability: $\frac{a}{a+b} + \frac{b}{a+b} = 1$

$$p(y = 0 \mid x = 1) = \frac{p(x = 1, y = 0)}{p(x = 1)} = \frac{c}{c + d}$$

$$p(y = 1 \mid x = 1) = \frac{p(x = 1, y = 1)}{p(x = 1)} = \frac{d}{c + d}$$

For $p(y \mid x = 1)$to be a valid probability: $\frac{c}{c+d} + \frac{d}{c+d} = 1$

| $p(y \mid x)$ | x=0 | x=1 |
|---|---|---|
| y=0 | $\frac{a}{a+b}$ | $\frac{c}{c+d}$ |
| y=1 | $\frac{b}{a+b}$ | $\frac{d}{c+d}$ |

Find the $p(x \mid y)$ using Bayes' rule:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{\sum_x p(y \mid x)p(x)}$$

$$p(x = 0 \mid y = 0) = \frac{p(y = 0 \mid x = 0)p(x = 0)}{p(y = 0 \mid x = 0)p(x = 0) + p(y = 0 \mid x = 1)p(x = 1)}$$

$$p(x = 1 \mid y = 0) = \frac{p(y = 0 \mid x = 1)p(x = 1)}{p(y = 0 \mid x = 0)p(x = 0) + p(y = 0 \mid x = 1)p(x = 1)}$$

$$p(x = 0 \mid y = 1) = \frac{p(y = 1 \mid x = 0)p(x = 0)}{p(y = 0 \mid x = 0)p(x = 0) + p(y = 0 \mid x = 1)p(x = 1)}$$

$$p(x = 1 \mid y = 1) = \frac{p(y = 1 \mid x = 1)p(x = 1)}{p(y = 0 \mid x = 0)p(x = 0) + p(y = 0 \mid x = 1)p(x = 1)}$$

$$p(x = 0 \mid y = 0) = \frac{p(y = 0 \mid x = 0)p(x = 0)}{p(y = 0 \mid x = 0)p(x = 0) + p(y = 0 \mid x = 1)p(x = 1)}$$

$$= \frac{\left(\frac{a}{a + b}\right)(\mathrm{a{+}b})}{\left(\frac{a}{a + b}\right)(\mathrm{a{+}b}) + \left(\frac{c}{c + d}\right)(c + d)} = \frac{a}{a + c}$$

$$p(x = 1 \mid y = 0) = \frac{p(y = 0 \mid x = 1)p(x = 1)}{p(y = 0 \mid x = 0)p(x = 0) + p(y = 0 \mid x = 1)p(x = 1)}$$

$$= \frac{(\frac{c}{c + d})(c + d)}{\left(\frac{a}{a + b}\right)(\mathrm{a{+}b}) + \left(\frac{c}{c + d}\right)(c + d)} = \frac{c}{a + c}$$

$$p(x=0 \mid y=1) = \frac{p(y=1 \mid x=0)p(x=0)}{p(y=1 \mid x=0)p(x=0) + p(y=1 \mid x=1)p(x=1)}$$

$$= \frac{\left(\frac{b}{a+b}\right)(\mathrm{a+b})}{\left(\frac{b}{a+b}\right)(\mathrm{a+b}) + \left(\frac{d}{c+d}\right)(c+d)} = \frac{b}{b+d}$$

$$p(x=1 \mid y=1) = \frac{p(y=1 \mid x=1)p(x=1)}{p(y=0 \mid x=0)p(x=0) + p(y=0 \mid x=1)p(x=1)}$$

$$= \frac{\left(\frac{d}{c+d}\right)(c+d)}{\left(\frac{b}{a+b}\right)(\mathrm{a+b}) + \left(\frac{d}{c+d}\right)(c+d)} = \frac{d}{b+d}$$

We have $p(x \mid y) = \frac{p(y|x)p(x)}{\sum_x p(y|x)p(x)}$ in the following table:

| $p(x \mid y)$ | x=0 | x=1 |
|---|---|---|
| y=0 | $\frac{a}{a+c}$ | $\frac{c}{a+c}$ |
| y=1 | $\frac{b}{b+d}$ | $\frac{d}{b+d}$ |

Check that $\sum_x p(x \mid y = 0) = 1$:

$$\sum_x p(x \mid y = 0) = p(x = 0 \mid y = 0) + p(x = 1 \mid y = 0)$$

$$= \frac{a}{a+c} + \frac{c}{a+c} = 1. \qquad \square$$

Check that $\sum_x p(x \mid y = 1) = 1$:

$$\sum_x p(x \mid y = 1) = p(x = 0 \mid y = 1) + p(x = 1 \mid y = 1)$$

$$= \frac{b}{b+d} + \frac{d}{b+d} = 1. \qquad \square$$