Homework 1

1. i. $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$

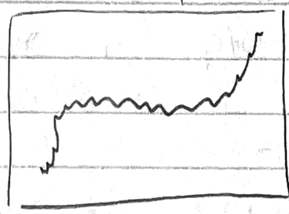
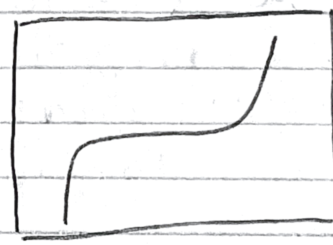
i. plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ via coeffs.

plot created in back

ii. plot $p(x)$ for $x = [1.920 : 0.001 ; 2.080]$ via $(x-2)^9$

plot created in back

iii. the main differences in the two plots is that the 1^{st} plot using coefficients / the expanded polynomial is very rough, where as when we look at the second plot using the closed expression it is much smoother.

Ex  ① vs.  ②

the descrepency comes from the fact that the 1^{st} plot has 10 operations to take into account where the 2^{nd} plot only has 1. this causes errors in the estimations which is reflected in the roughness of the graph.

we would want to use / the correct plot is plot 2. this is going to give us the least amount of errors in our estimation and is going to be smoother / more stable to our $P(x)$.

avoiding cancelation for:

2\. i. $\sqrt{x+1}-1$ for $x \simeq 0$

→ multiply by the recipricol:

$$\sqrt{x+1}-1 \cdot \left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) = \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{\sqrt{x+1}+1} = \frac{x+1+\sqrt{x+1}-\sqrt{x+1}-1}{\sqrt{x+1}+1}$$

$$= \left.\frac{x}{\sqrt{x+1}+1}\right|_{x\simeq 0} = 0 \checkmark$$

ii. $\sin(x)-\sin(y)$ for $x \simeq y$

assumption: the problem lies in the subtraction, to get rid of this we can again multiply by the recipricol

$$\sin(x)-\sin(y) \cdot \left(\frac{\sin(x)+\sin(y)}{\sin(x)+\sin(y)}\right) = \frac{(\sin(x)-\sin(y))\cdot(\sin(x)+\sin(y))}{\sin(x)+\sin(y)}$$

$$= \frac{\sin^2(x)-\sin^2(y)}{\sin(x)+\sin(y)}$$

using the fact that $\sin(a+b)\cdot\sin(a-b) = \sin^2(x)-\sin^2(x)$

$$\Rightarrow = \left.\frac{\sin(x+y)\sin(x-y)}{\sin(x)+\sin(y)}\right|_{x\simeq y}$$

$$= \frac{\sin(2x)\sin(0)}{2\sin(x)} = 0 \checkmark$$

iii. $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$

assumption: problem lies in the numerator, we can (again) multiply by the recipricol to eliminate this:

$$\frac{1-\cos(x)}{\sin(x)} \cdot \left(\frac{1+\cos(x)}{1+\cos(x)}\right) = \frac{(1-\cos x)(1+\cos x)}{\sin(x)+\sin(x)\cos(x)} = \frac{1-\cos^2(x)}{\sin(x)+\sin(x)\cos(x)}$$

using the trig identity $\sin^2(x)+\cos^2(x)=1$

$$= \frac{\sin^2(x)}{\sin(x)(1+\cos(x))}$$

$$= \left.\frac{\sin(x)}{1+\cos(x)}\right|_{x=0}$$

$$= \frac{0}{2} = 0 \checkmark$$

3.a. 2^{nd} degree Taylor Polynomial $P_2(x)$, $f(x) = (1+x+x^3)\cos(x)$

about $x_0 = 0$

$P_2(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + f''(x_0)\cdot(x\cdot x_0)^2 \cdot \frac{1}{2!}$

$= f(0)$

$+ [-x\sin(x) - x^3\sin(x) + 3x^2\cos(x) - \sin(x) + \cos(x)](0)\cdot x$

$+ [5x\cos(x) - x^3\cos(x) - 6x^2\sin(x) - \cos(x) - 2\sin(x)](0)\cdot \frac{x^2}{2}$

$\boxed{P_2(x) = 1 + x - \frac{x^2}{2}}$

a. $P_2(0.5) = 1 + \frac{1}{2} - \frac{(1/2)^2}{2} = \frac{11}{8}$ $\boxed{P_2(0.5) = \frac{11}{8}}$ $\approx f(0.5)$

Absolute error:

$|f(0.5) - P_2(0.5)| < f'''(z)\cdot(x-x_0)^3 \cdot \frac{1}{3!}$ $| x_0 = 0$

$= [6\cos(z) - 6z\sin(z) - 12z\sin(z) - 2(1+3z^2)\cos(z) -$

$\ldots - (1+3z^2)\cos(z) + (+z-z^3)\sin(z)] \cdot x^3 \cdot \frac{1}{3!}$

$|f(0.5) - P_2(0.5)| < f'''(0)\cdot x^3 \cdot \frac{1}{6}$

$< [6 - 0 - 0 - 2 - 1 + 0]\cdot(0.5)^3 \cdot \frac{1}{6}$

$< [3]\cdot\frac{1}{8}\cdot\frac{1}{6}$

$< \frac{1}{16}$

$\boxed{|f(0.5) - P_2(0.5)| < \frac{1}{16}}$

actual error $= f(0.5) - P_2(0.5) = 0.0511$

$1/16 = 0.0625$ $>$ our upper bound holds true

b. error bound for $|f(x) - P_2(x)|$ w/ $P_2(x)$ approx. $f(x)$

$|f(x) - P_2(x)| < \left| P_2'''(x)\cdot x^3 \cdot \frac{1}{3!} \right|$

$< \left| -x \cdot x^3 \cdot \frac{1}{6} \right| = \left| \frac{-x^4}{6} \right| = \frac{x^4}{6}$

$\boxed{|f(x) - P_2(x)| < \frac{x^4}{6}}$

c. approx. $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$

$$\int_0^1 P_2(x)\,dx = \int_0^1 1 + x - x^2/2\,dx$$
$$= \left[x + \frac{x^2}{2} - \frac{x^3}{6}\right]_0^1$$
$$= \left(1 + \frac{1}{2} - \frac{1}{6}\right) - (0 + 0 - 0) = \boxed{\frac{4}{3}} \simeq \int_0^1 f(x)\,dx$$

d. estimated error in $\int_0^1 P_2(x)\,dx \simeq \int_0^1 f(x)\,dx$

using error formula created in (b): $|f(x) - P_2(x)| < \frac{x^4}{6}$

$$\Rightarrow \int_0^1 \frac{x^4}{6}\,dx$$
$$= x^5/30 \Big|_0^1$$
$$= \frac{1}{30} - 0 = \frac{1}{30}$$

Note: $\int_0^1 f(x)\,dx = \int_0^1 (1 + x + x^3)\cos(x)\,dx \doteq 1.3950$

$\left|\int_0^1 f(x)\,dx - \int_0^1 P_2(x)\,dx\right| \doteq 0.0616$, close to our estimate

estimated error in $\int_0^1 P_2(x)\,dx \simeq \int_0^1 f(x)\,dx = \boxed{\frac{1}{30}}$

4.a. Quadratic equation: $ax^2+bx+c=0$, $a=1$, $b=-56$, $c=1$

this gives us: $x^2-56x+1=0$

compute roots correct w/ 3 decimal places:

$$r_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\Bigg|_{a=1,\,b=-56,\,c=1}$$

$$=\frac{56\pm\sqrt{(-56)^2-(4\cdot1\cdot1)}}{2\cdot1}=\frac{56\pm\sqrt{3132}}{2}$$

$\oplus$: $\frac{56+\sqrt{3132}}{2}=\frac{56}{2}+\frac{\sqrt{3132}}{2}=28+3\sqrt{87}=55.981$

$\ominus$: $\frac{56-\sqrt{3132}}{2}=28-3\sqrt{87}=0.019$

NOTE:

$\sqrt{87}\simeq 9.327$ using 3 decimal correction

when solving for the roots w/ coding we get: $\oplus=55.9821\ldots$; $\ominus=0.0179\ldots$ (used full #s in calculations)

relative error: $r_+=\frac{|55.981-55.9821|}{55.9821}=2.031\cdot10^{-5}$

$r_-=\frac{|0.19-0.0179|}{0.0179}=0.063$

$\boxed{r_+=2.031\cdot10^{-5},\ r_-=0.063}$ r_- is the bad root

b. "bad" root approx. is better when we manipulate $(x-r_1)(x-r_2)=0$ so roots are related to a,b,c

$\Rightarrow x^2-xr_2-xr_1+r_1r_2=0$

$x^2-x(r_1+r_2)+r_1r_2=0$ compared to: $x^2-56x+1=0$

so: $(r_1+r_2)=56$ and $r_1\cdot r_2=1$

$r_2=\frac{1}{r_1}\Big|$ approx. for r_1

$=0.178632\ldots$

new rel. error $=2.0313\cdot10^{-5}$

<u>MUCH</u> smaller (factor of 10^3)

5.a. $y = x_1 - x_2$, w/ $\hat{x}_1 = x_1 + \Delta x_1$, $\hat{x}_2 = x_2 + \Delta x_2$

operation $x_1 - x_2$ carried out gives us $\tilde{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$

upper bound on absolute error:

$$\tilde{y} = x_1 - x_2 + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y} \Rightarrow |\Delta y| = |\Delta x_1 - \Delta x_2|$$

by triangle inequality:

$$|\Delta y| = |\Delta x_1 - \Delta x_2| \leq |\Delta x_1| - |\Delta x_2|$$

upper bound on relative error:

$$\frac{|\Delta y|}{|y|} = \frac{|\Delta x_1| - |\Delta x_2|}{|x_1 - x_2|} \leftarrow \text{largest when } |x_1 - x_2| \to 0$$

so when $x_1 \simeq x_2$

$$\boxed{|\Delta y| = |\Delta x_1| - |\Delta x_2|, \quad \frac{|\Delta y|}{|y|} = \frac{|\Delta x_1| - |\Delta x_2|}{|x_1 - x_2|}, \text{ largest when } x_1 \simeq x_2}$$

b. $\cos(x+\delta) - \cos(x)$ into expression w/out subtraction

note: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B$

$\cos(x+\delta) - \cos(x)$

$x = A - B$, $\cos(A - B + \delta) = \cos(A+B)$, $\delta = 2B$

$A = x + \frac{\delta}{2}$ $\quad B = \frac{\delta}{2}$

So: $\boxed{\cos(x+\delta) - \cos(x) = -2 \cdot \sin(x + \frac{\delta}{2}) \cdot \sin(\frac{\delta}{2})}$

plotting for difference of: $-2\sin(x+\frac{\delta}{2})\sin(\frac{\delta}{2}) + \cos(x+\delta) - \cos(x)$

we find that our $\boxed{\text{new expression is more stable}}$

we know this even w/out plotting because we do not want to divide by 0 or subtract close to 0. Since our original expression has the later of the 2 problems, we will have a higher stability w/ our 1* expression, especially w/ smaller deltas.

5.c. $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi), \quad \xi \in [x, x+\delta]$

$f(x+\delta) = \cos(x+\delta)$

$f(x) = \cos(x)$

$\Rightarrow f'(x) = -\sin(x)$

$f''(\xi) = -\cos(\xi)$

$\Rightarrow f(x+\delta) - f(x) = -\delta \sin(x) - \frac{\delta^2}{2!} \cos(\xi)$

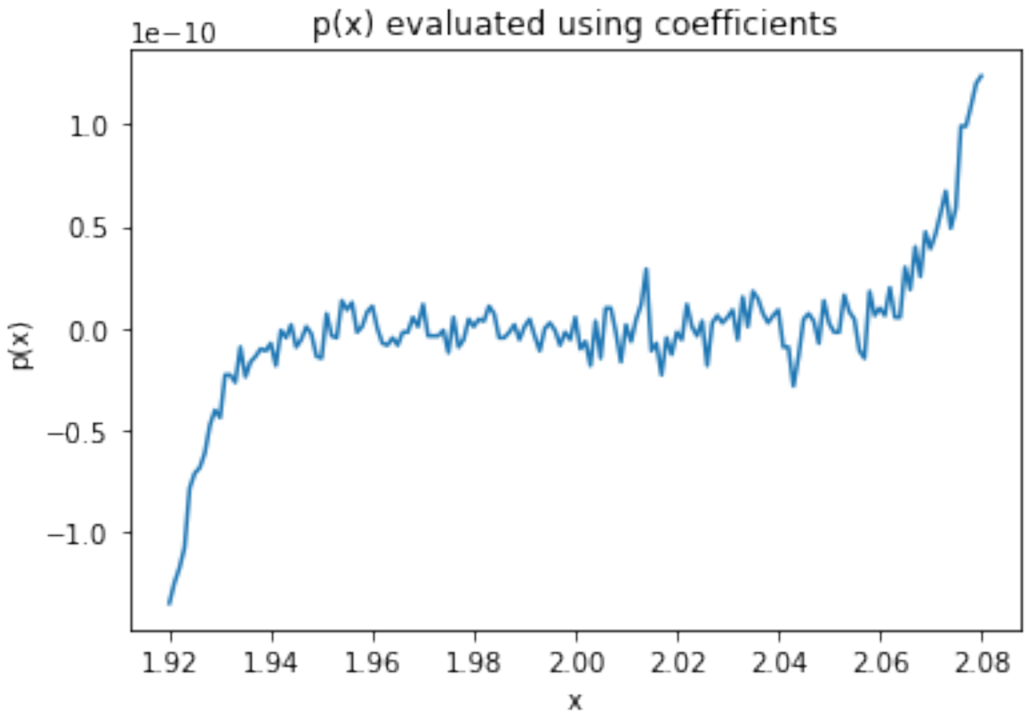
$$\boxed{\text{algorithm: } -\delta \sin(x) - \frac{\delta^2}{2!} \cos(\xi)}$$

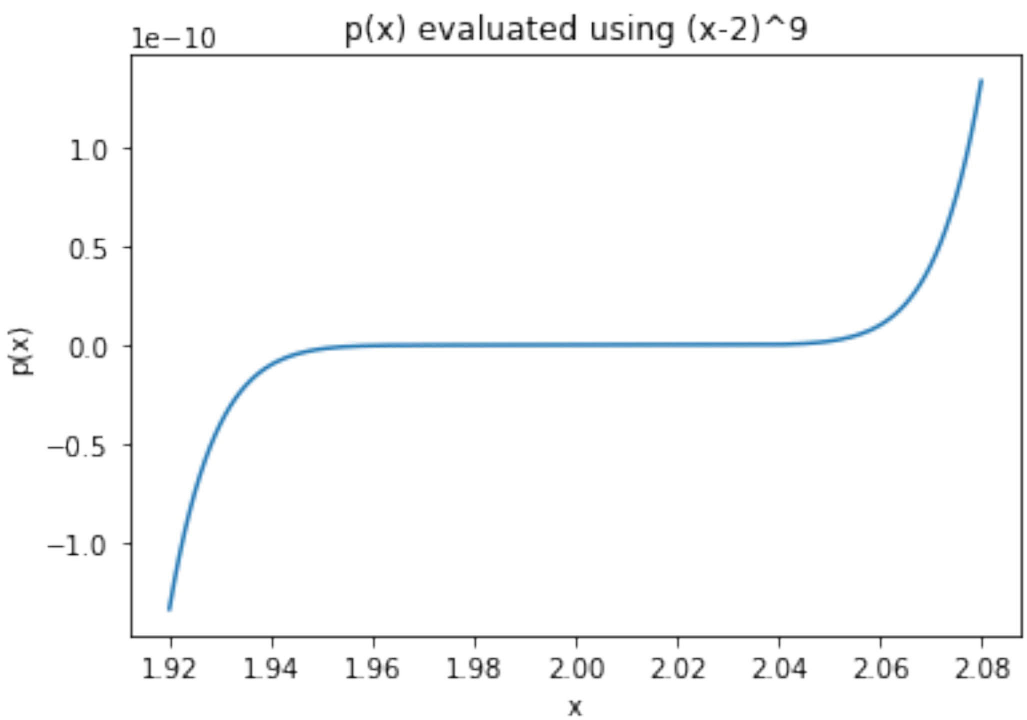
Note: We could choose to only use $-\delta \sin(x)$, since we are using a very small δ, the second term with δ^2 is practically negligible to the approximation of $\cos(x+\delta) - \cos(x)$, but for the sake of comparing the 2:

the best representation of our $\cos(x+\delta) - \cos(x)$ function is our part (b) simplification, $-2\sin(x + \frac{\delta}{2}) \cdot \sin(\frac{\delta}{2})$.

this is the best representation for our machine to plot because it minimizes the error in subtraction and noise in multiple terms like the original function and taylor expansion have.

Additionally, since we are only using the first 2 approximations to the taylor expansion, this will yield additional errors in the representation of the original function.


p(x) evaluated using coefficients
1e−10
1.0
0.5
0.0
−0.5
−1.0
p(x)
1.92
1.94
1.96
1.98
2.00
2.02
2.04
2.06
2.08
x


p(x) evaluated using (x-2)^9
1e−10
1.0
0.5
0.0
−0.5
−1.0
p(x)
1.92
1.94
1.96
1.98
2.00
2.02
2.04
2.06
2.08
x