Problem of the Week Teacher Packet

Broken Pottery

A "sherd" is part of a piece of pottery that one might dig up at an archaeological site where pottery-making people once lived. Archaeologists usually want to figure out how big the original piece of pottery was, as that can tell them something about who might have made the piece and when it was made.

Using the sherd shown, devise a method for determining the diameter of the original plate.

Extra: Can you come up with another method?

Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually get the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

There are a lot of ways to tackle this task. Most (but not all) involve chords and radii or even tangents.

Think you've got a good method? Here are some things to think about:

- Would your method enable you to find the diameter to at least the nearest quarter inch?
- Would your method work on different sized "sherds"? (You can try this by drawing various arcs on a piece of paper, where you know the actual radius of the starting circle.)
- Can you think of things that might make your method more or less accurate?
- Did you try to come up with more than one method?

If any of those ideas help you, you might revise your answer, and then leave a comment that tells us what you did.

Our Solutions

Method 1: Solve a Simpler Problem (Chord Bisectors)

When I first started looking at this problem, I felt really stuck. All I knew was that the problem was about circles. I didn't know where the center was, what the radius was, anything. I decided to start by listing what I did know or could measure:
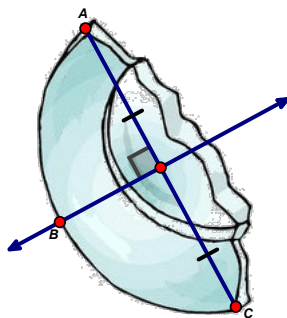
- The length of the arc
- The straight distance from one edge of the arc to the other (the longest chord in the circle)

One way to make a problem simpler is to try to *connect it to a problem I've solved before*. I could think of a few things I already know about arcs, chords, and circles.

- The perpendicular bisector of a chord goes through the center of the circle
- The arc length is a certain percentage of the circumference of the circle, the same percentage as the arc angle is of 360°
- Half of the chord, the perpendicular bisector of the chord, and the radius form a right triangle
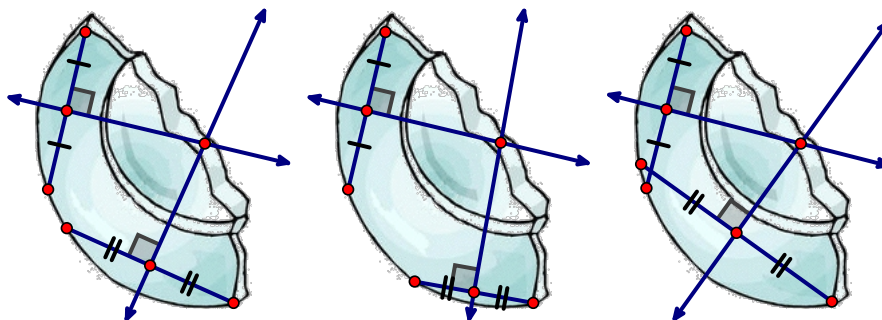
Now I had some ideas I could relate the problem to, but I was still finding it hard, so I asked myself, "What makes this problem hard?" The answer: I don't know where the center of the circle is!

So I thought about the fact that the perpendicular bisector of a chord goes through the center. I know one chord in the circle, so I can make a line that the center is on:

Now what's hard is that I have a line the center is on, but I need another point. If I could make the problem easier, I'd find another sherd from the same circle. I could put them together and make another chord and draw a second perpendicular bisector. The intersection would be the center!

Even though that's wishful thinking, it's actually useful! Why can't I think of this sherd in two pieces? Then I could draw two lines, and the intersection would give me the center of the circle! I did it a few different ways to be sure the choice of chords didn't change the answer.
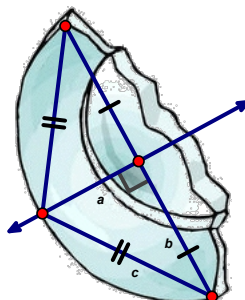

Here is the summary of my method: Pick four points on the edge of the plate. Draw chords connecting two pairs of points. Construct perpendicular bisectors of each chord. The intersection of the perpendicular bisectors of the chords is at the center of the circle. Now that you know where the center is, measure along any of the perpendicular bisectors from the center to the edge of the sherd to find the radius. Of course, you need to double that to find the diameter.

I noticed that since the sherd was not a perfect circle, where I put the chords could change where the center was slightly. Realistically, a pottery sherd won't be a perfect circle either, so I would recommend that the archaeologists use multiple pairs of chords and find an approximate center.
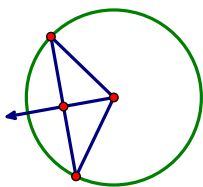
Method 2: Getting Unstuck (Pythagorean Theorem)

This problem reminded me of problems I've done before in which you know the length of a chord and you draw its perpendicular bisector and use algebra and the Pythagorean theorem to find the length of the radius. I right away drew this picture and labeled variables:
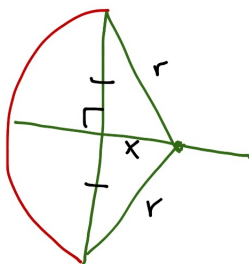
But I could measure all of those lengths and I don't see how they relate to the radius or diameter.

I decided to go back to the problem I had solved that this reminded me of. Here was the diagram from that problem:
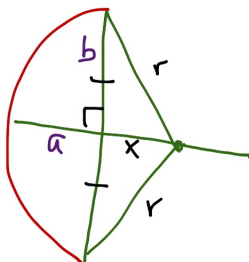
I needed to include the center in my diagram, even though I don't know where it is. Maybe a rough sketch will help...

I can measure the two halves of the chord, so those are known quantities. But I have two unknown quantities and no other relationship between them. I'm stuck again!

I asked myself, "What am I missing?" I read through my problem solving and realized that there were unlabeled lines in my diagram whose length I could measure (labeled *a* and *b* below).

I can measure *a* and *b* and then I can use this relationship to write *x* in terms of *r*: $x = r - a$.

Now I can use the Pythagorean theorem with *x*, *r*, and *b*.

$$r^2 = x^2 + b^2$$

$$r^2 = (r - a)^2 + b^2$$

$$r^2 = r^2 - 2ar + a^2 + b^2$$

$$2ar = a^2 + b^2$$

$$r = \frac{a^2 + b^2}{2a}$$

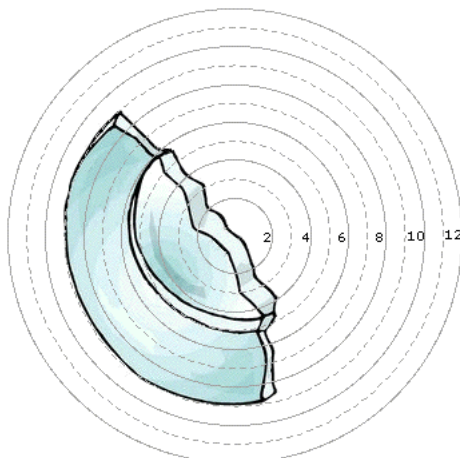$$d = \frac{a^2 + b^2}{a}$$

Voila! Now I have a formula for the archaeologists. Draw the longest chord you can in the circle. Measure the distance from its midpoint to the closest point on the circle (on its perpendicular bisector) and the farthest point (the end of the chord). Call the shorter measurement *a* and the longer *b*. Use

$$d = \frac{a^2 + b^2}{a}$$

to find the length of the diameter, *d*.

Method 3: Using a Template (What the Archaeologists Do)

I used a compass to construct circles with different radii. Then I printed out and cut out my sherd and lined it up with different circles until I found the best match:

Archaeologists could have a template like this, the way my dad has one for rolling out different diameter piecrusts. In fact, that's what gave me the idea for this strategy.

If they didn't have a template, they could just use a compass and keep adjusting the angle of the compass until the arc of the circle they drew best matched the sherd. Then they could measure the diameter of that circle.

Standards

If your state has adopted the [Common Core State Standards](), you might find the following alignments helpful.

High Schools: Geometry: Circles

2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

High School: Geometry: Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

Teaching Suggestions

Problems that come from real-life scenarios like this one have great possibilities, and they can be extra challenging. In this problem, for example, drawing and measuring are viable solutions because they are solutions archaeologists would use in the field. They're accurate enough for archaeological purposes. Students might be used to trying to solve problems geometrically or with algebra and if they're stuck, encourage them to think about what kinds of measurement and drawing tools actual archaeologists could use. (Though, in truth, archaeologists do use the method suggested in Method 3, above—they have a template they carry with them.)

I think "getting your hands dirty" is pretty important to solving this problem (no archaeological puns intended). Most of the students who gave us inaccurate methods seemed to be solving the problem in theoretically, saying things like, "measure from the center to the end of the sherd." Printing out the paper, cutting out the sherd, and testing your method will make it a lot harder to blithely say, "just measure from the center."

Students who do use geometric or algebraic methods need to figure out how to apply them to this situation that has no numbers, no helpful diagram, etc. They need to figure out what geometry has to do with this

situation. One really helpful strategy in this case is "Solve a Simpler Problem" in which you try to relate this situation to others you're more familiar with, change the situation by mathematizing it (drawing extra lines and triangles, etc.) and ask yourself, "What makes this problem hard? How can I make it easier?"

Sample Student Solutions – Focus on Strategy

In the solutions below, we've focused on students' "strategy". Generally speaking, this reflects whether the student has chosen a mathematically sound method for solving the problem that doesn't rely on luck in any way. My comments focus on what I feel is the area in which they need the most improvement.

Novice	Apprentice	Practitioner	Expert
Has no ideas that will lead them toward a successful solution. Shows no evidence of their strategy.	Uses a method that really relies on luck – their method need not be 100% accurate, since it's a rough piece of pottery. Might be based on drawing a very rough picture without any real math idea.	Picks a sound strategy—success achieved through skill, not luck. Could be a Practitioner even if they assume it's a semicircle – that's reflected in Interpretation. Must have some mathematical basis.	Might talk about which strategies will do a better job, or how they'll be more accurate if the piece is bigger.

Samantha, age 15, Novice

Drawing

Draw the rest of the plate on a peice of paper, using tools such as the ruler, protracter, ect..

Samantha hasn't told us enough about her drawing strategy to know if she will get an accurate sketch or not. I wonder how the protractor will help her with her drawing. I would ask her to tell me more about how she can guarantee an accurate drawing.

K C, age 14, Novice

This would only work is the piece of pottery is suppose to make a circular plate. Here is how I find out its diameter: (you might want to draw down what i said to have a better understanding of it.)

First, I looked at the plate piece as a 2D piece of a circular plate. i know by looking at the piece that the curved side is part of a the circumference of the whole plate. I connect the two end points of the circumference with a straight line.

Then i measured the line and made a mid-point on it. Then mark the line #1.

I made another line that cut though the mid-point and creates 90 degree angles between the two line i made. Then mark the line #2

I will extend the second line to the circumference and also triple the length of the line on the other side of the mid-point.

K C has a lot of ideas but I'm not sure what the mathematical reasoning behind them is. I tried following K C's method but the intersection point I created wasn't the center of the circle. I can't think of a method using an equilateral triangle so I would ask K C what this problem reminded them of and what they had noticed about the problem as they chose their strategy. Maybe K C had some other ideas they had thought of that we could build on.

At one end of the first line I drew, I will draw another line towards the second line(not the circumference's direction) but make sure the angle between this line and the first line is 60 degree.

Do the same thing with the other end point from the first line.

Now, you created a equilateral triangle!

Find the mid-points of every side of the triangle and connect each one to the triangle's corner directly across from it.

Find the point where all three lines inside the triangle intercept. That's also the center point of the whole plate.

now, measure from the center point to the circumference and you will find the radius. Double that and you will find the diameter!

Sean, age 17, Apprentice

Using a geometrical method that includes a "Chord" instead of Diameters, to find the Diameter.

Line Segment AB is a Chord, Consisting of Vertex A and B as shown in the Picture given. To find the Diameter, Take Chord AB, and duplicate it, but have it perpendicular to one of the points. Do the same procedure two times more. You will form a Square from the Chords. At that point, you can create and measure the Hypotenuse between the angles. The hypotenuse is the Diameter. Divide the Diameter in half to create the Radius, and then you can find your center point.

Sean's strategy is valid for the case when the arc is 1/4 of the circle. There are methods to calculate the arc angle and Sean could refine his method to account for different arc angles. I'm not sure what Sean knows about calculating arc angles, though, so I might just ask him, "What would happen if the sherd were 1/3 of the circle? Would your method still work? If not, how could you find the diameter for a sherd that's an unknown fraction of the circle?"

Nick, age 14, Apprentice

My way to find the diameter is by measuring from the center to the outside of the plate and then multiply the measurement by 2.

The way I got that is by taking what I knew about diameter. Which is that if you multiply the radius by 2 you get diameter. That is how I got my answer.

Most of Nick's strategy is perfect. The only trouble is, how does he know where the center of the circle is? That's exactly what I would ask him, along with the question of what tools could he use (e.g. compass, ruler, etc.)

Hayley, age 15, Practitioner

You can trace the curve of the sherd, and move it around until you get a circle.

If you place the sherd on a piece of paper you can use a pencil to trace the curve edge. Once you have traced the entire curved edge, simply adjust the curve so part of it is still lined up with the curve you've drawn, but now its position lets you trace more of the circle. This way, you are keeping the same curve, but can complete the circle. Continue tracing and adjusting the sherd until a full circle has been made. Then, measure the distance from one point on the circle's edge to the point on the opposite edge to find the diameter.

Hayley's tracing strategy is sound – this type of real-world problem is one where careful drawing is a great strategy. I would ask Hayley how accurate they think this method is, and what could affect that accuracy.

James, age 14, Practitioner

One way you can find the diameter of an old plate or something like it, is to have a compass and you trase around the plate for an aprroximate size

First, you need a pencil, compass, paper, and the art work. You then place the art on the paper so you can trace around it. You then find a spot where the compass will follow the art all the way around. You then trace so that part of the plat or something is in the circle and fits the curve the compass made. You can then measure the diameter to find the approximate size of the art work.

James has done a nice job making use of an appropriate tool: the compass. I would ask James about accuracy as well.

Jayoung, age 15, Practitioner

There are a lot of ways to tackle this task. Most (but not all) involve chords and radii or even tangents.

To figure out this problem, you need to assume that the plate/potter peice has been broken around or near the center. Thus from the edge of the break to the edge of the plate/pttery peice would be the radius so after that use the formula: (pi) r squared and u should get the answer instering whatever it is the radius would be.

Jayoung's Intepretation of the problem has several errors: finding area rather than diameter, assuming the plate has to be broken near the center. But the strategy based on this interpretation is sound. I would ask Jayoung what they would do if the archaeologist found a small sherd that wasn't broken near the center.
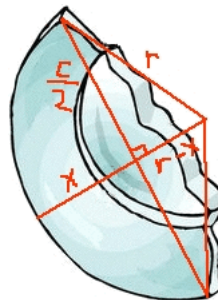
Tayshaun, age 13, Practitioner

I used the Pythagorean theorem.

I drew a chord across the sherd from the furthest points. Then I drew a perpendicular bisector of that chord. I know that bisector goes through the center of the circle because that is a theorem we learned. That means the center has to be on it somewhere.

I drew the picture below. I made the whole chord c. The longest distance from the chord to the circle is x. The radius is r. The part between the chord and the center of the circle is r - x. I used the Pythagorean Theorem.

$(c/2)^2 + (r - x)^2 = r^2$

When you find the sherd, you can measure the chord and measure x, and then put them in the formula.

Tayshaun has a sound strategy that would be pretty accurate if the measurements are accurate. He's also explained his strategy pretty clearly and completely. I'd ask him to attempt the Extra by trying to find a second method… Maybe one that the archaeologists wouldn't need a calculator to use?

Paul, age 17, Practitioner

The diameter of the original plate is 2 and 3/4 inches.

First, I drew a circle with a compass. Then, I drew more circles around that circle. Then, I cut out the sherd from the problem page, and placed it on the paper with the circles and matched up the sherd to one of the circles. Then, I used the compass to draw the rest of the sherd, which was circular. I placed a ruler in a straight line from left to right across the dot made by the compass in the center of the pottery and got the diameter of 2 and 3/4 inches.

Another way you could do this is to put the paper with the circles under the paper with the sherd, and shine a bright light on the top page. Then you can match up the sherd with one of the circles, draw the rest of the sherd, and use a ruler to get the diameter.

Paul used the same method archaeologists used! I'm not sure his second method is really mathematically different from his first, so I would give him the challenge to calculate the diameter without drawing a circle.

Becca, age 17, Practitioner

I would find the diameter by using perpendicular lines and the tangent points.

First you would have to find the radius. You do this by:

1. Draw two lines that are perpendicular to each other against the outside of the sherd.

2. Find the tangent point of each line where it touches the plate.

3. From the tangent point draw a line perpendicular to it. (Do this to both tangent points)

4. Where the two new lines meet is where the middle of the plate should be. This would be the radius.

5. Once you have the radius then you can find the diameter by doubleing the radius.

Becca used tangent lines to find the center of the circle, another sound strategy. I'm not convinced that the lines have to be perpendicular to each other (her step 1) and wonder if she meant two lines that are perpendicular to the radii of the sherd?

Aman, age 13, Expert

1. The method is simply to construct a tangent line to the inner circle with endpoints on the circumference of the outer circle. This creates a right triangle after which you can use the Pythagorean Theorem. EXTRA: Another method is to use the c

Before the plate was broken, it obviously consisted of two concentric circles (two circles, where the distance between their circumferences are constant around the circumference). Let their center be C, although you cannot see the center. Now, draw two points on the circumference of the outer circle, A and B, where the line AB is tangent to the inner circle at point D. So now, you have what looks like a line beginning and ending on the circumference of the outer circle and touching the inner circle. Next, draw ray CD and let the intersection of this ray and the outer circle be at point E. The distance DE is therefore, the distance between the two circles. Observe how the triangle DA AC CD is a right triangle. DA is a constant since you can measure it and let's use x to represent it. AC is the radius of the outer circle and CD is the radius of the inner circle. So, AC = CD + DE. DE is a measurable constant as well and let's use y to represent it. let r = CD. So AC = y + r. According to the Pythagorean Theorem,

$DA^2 + CD^2 = AC^2$

$x^2 + r^2 = (y + r)^2$

$x^2 + r^2 = y^2 + 2yr + r^2$

$x^2 = y^2 + 2yr$

$x^2 - y^2 = 2yr$

$(x^2 - y^2)/2y = r$

As AC was the original radius, I'm not done yet.

$AC = y + r$

$AC = y + (x^2 - y^2)/2y$

$AC = 2y^2/2y + (x^2 - y^2)/2y$

$AC = (x^2 + y^2)/2y$

$2AC = (x^2 + y^2)/y$

Aman generated two pretty different methods, using known measurements to find unknown ones. His explanation was clear and complete as well. Both methods relied on the inner and outer circles of the sherd. I'd ask Aman if he would take on the challenge of what to do if the sherd didn't have an obvious inner circle.

That is the diameter. Just substitute values, which you have to measure for x and y.

EXTRA: First, let me define all of my variables. let a = angle(the central angle at which the plate was broken) let x = radius of the smaller circle let y = distance between the two circumferences of the two circles let c = arc of inner circle that is showing let d = arc of outer circle that is showing So, with my variables I can set up a simple circumference equation in terms of the central angle and the radius. Similarly, for the outer circle, the radius will be the radius of the inner circle plus y. So,

(a/360)(2xpi) = c

(a/360)(2xpi + 2ypi) = d

After simplifying and solving for a, I get

a = (180d - 180c)/ypi

After substituting this into the former equation and simplifying, I arrive at

x = cy/(d - c)

Of course, c, d, and y and constants that can be measured.

Scoring Rubric

A **problem-specific rubric** can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work.

We hope these packets are useful in helping you make the most of Geometry Problems of the Week. Please let me know if you have ideas for making them more useful.

https://www.nctm.org/contact-us/