模式识别2020 作业1

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1.第一章讲义习题

(a) $\frac{8a-1}{3} \geq 0$,即$a \geq \frac{1}{8}$

(b)$2\sqrt[3]{a} = 1$

(c)I think another peculiar value for a is $\frac{1}{2}$, so that $\sqrt{\frac{8a-1}{3}} = 1$. So that Equation 1's value is $\sqrt[3]{\frac{1}{2}+\frac{1}{2}} + \sqrt[3]{\frac{1}{2}-\frac{1}{2}} = 1$. Given these observations in question (b) and (c), I think the value of Equation 1 may be a constant, and the result is 1.

(d)the value this command returns is $1.2182 + 0.1260i$.

(e) It is because the second part $a - \frac{a+1}{3}\sqrt{\frac{8a-1}{3}} < 0$, and when calculating $\sqrt[3]{}$, the program will give the first result it comes up with ,which is a complex number. The correct value is 1.0000. The revised function is as below.

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>> f = ( a + (a+1)/3*sqrt((8*a-1)/3) )^(1/3) + ...
       sign( a - (a+1)/3*sqrt((8*a-1)/3))*abs( a - (a+1)/3*sqrt((8*a-1)/3) )^(1/3)

f =

    1.0000
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I have tried different a values, and the code can support my idea. The Equation 1's result is 1.

(f)mathematical statement: define $\equiv$ identically equal to, so

$$\sqrt[3]{a+\frac{a+1}{3}\sqrt{\frac{8a-1}{3}}} + \sqrt[3]{a-\frac{a+1}{3}\sqrt{\frac{8a-1}{3}}} \equiv 1(for \forall a \in R, a \geq \frac{1}{8})$$

Rigorous proof:

$$let\, x = \sqrt[3]{a + \frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}, and\, let\, y = \sqrt[3]{a - \frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (x+y)((x+y)^2 - 3xy)\, while\ x+y\ is\ the\ required\ result$$

$$x^3 + y^3 = a + \frac{a+1}{3}\sqrt{\frac{8a-1}{3}} + a - \frac{a+1}{3}\sqrt{\frac{8a-1}{3}} = 2a$$

$$3xy = 3\sqrt[3]{(a + \frac{a+1}{3}\sqrt{\frac{8a-1}{3}})(a - \frac{a+1}{3}\sqrt{\frac{8a-1}{3}})}$$

$$= 3\sqrt[3]{a^2 - \frac{(a+1)^2}{9}\frac{8a-1}{3}} = 3\sqrt[3]{\frac{-8a^3 + 12a^2 - 6a + 1}{27}}$$

$$= 3\sqrt[3]{(\frac{-2a+1}{3})^3} = 3 * \frac{-2a+1}{3} = -2a + 1$$

$$let\ the\ result\ be\ t\ ,so\ that\ \ 2a = t(t^2 - (1-2a))$$

$$2a(t-1) = t(1-t)(1+t) always\ stands\ up(for\ all\ a \geq \frac{1}{8})$$

$$so\, t - 1 = 0, i.e. t = 1 \qquad QED.$$

(g) The expression is another peculiar case of Equation 1 when $a = 2$. Since $2 > \frac{1}{8}$, so the result of the expression is 1.

(h) The Cardano's method is that given the equation $z^3 + pz + q = 0$, the result $z = \alpha + \beta$.

$\alpha = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$, and $\beta = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$. According to Equation 1, $a = -\frac{q}{2}$, so $q = -2a$. And $\frac{p^3}{27} + \frac{q^2}{4} = \frac{(a+1)^2}{9}\frac{8a-1}{3}$,while $a = -\frac{q}{2}$, so we can have $p^3 = -(1 + 3q + 3q^2 + q^3) = -(1+q)^3$,so$p = -(1+q)$.

So that the original equation becomes$z^3 - (1+q)z + q = 0$,i.e.$z(z+1)(z-1) = q(z-1)$, so the result is $z = 1$,which means $\alpha + \beta = 1$ and proves the correctness of Equation 1.

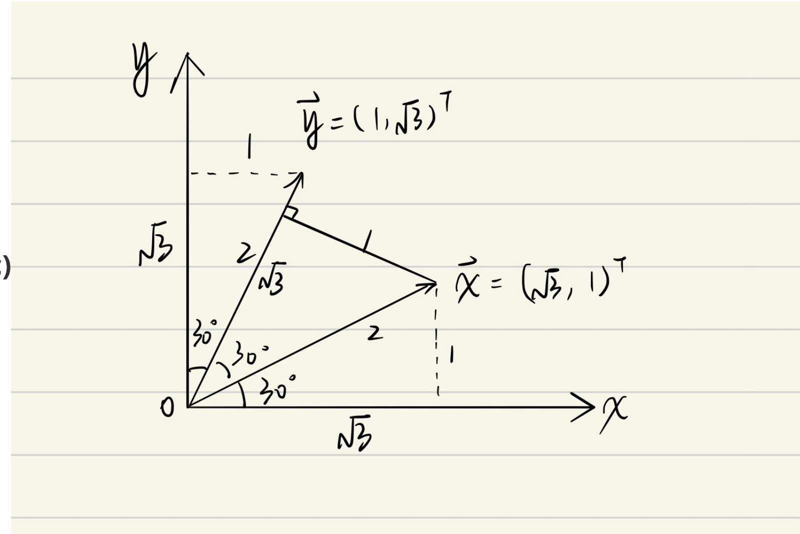
2.第二章讲义习题1

(a)the value of $x_\perp = ||x||cos\theta = ||x||\frac{x^Ty}{||x||||y||} = \frac{x^Ty}{||y||} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

(b)the direction of $x_\perp$ is $\frac{y}{||y||} = (\frac{1}{2}, \frac{\sqrt{3}}{2})^T$, so $x_\perp = \sqrt{3} * (\frac{1}{2}, \frac{\sqrt{3}}{2})^T = (\frac{\sqrt{3}}{2}, \frac{3}{2})^T$.

so that $x - x_\perp = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$, so that $y^T(x - x_\perp) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 0$, so that $y \perp (x - x_\perp)$. QED.

(c)



(d)proof: according to the Hint that the geometry between these vectors suggests that $||x - x_{\perp}||^2 + ||x_{\perp} - \lambda y||^2 = ||x - \lambda y||^2$,because we have $||x_{\perp} - \lambda y||^2 \geq 0$, so that $||x - \lambda y||^2 - ||x - x_{\perp}||^2 \geq 0$, so that $||x - x_{\perp}||^2 \leq ||x - \lambda y||^2$, so $||x - x_{\perp}|| \leq ||x - \lambda y||$. QED.

3.第二章讲义习题2

(a)$x > 0$

(b)because $det(X) = \prod_{i=1}^{n} \lambda_i$, so $72 = 12x$, i.e. $x = 6$.

4.第二章讲义习题6

(a)

$$E[X] = \int_x p(x)xdx = \int_0^{+\infty} \beta e^{-\beta x}xdx = -\int_0^{+\infty} xd(e^{-\beta x})$$
$$= -(xe^{-\beta x}|_0^{+\infty} - \int_0^{+\infty} e^{-\beta x}dx) = \frac{1}{\beta}\int_{+\infty}^{0} d(e^{-\beta x}) = \frac{1}{\beta}$$
$$E[X^2] = \int_x p(x)x^2dx = \int_0^{+\infty} \beta e^{-\beta x}x^2dx = -\int_0^{+\infty} x^2d(e^{-\beta x})$$
$$= -(x^2e^{-\beta x}|_0^{+\infty} - \int_0^{+\infty} 2xe^{-\beta x}dx) = \frac{2}{\beta}E[x] = \frac{2}{\beta^2}$$
$$Var[X] = E[X^2] - (E[X])^2 = \frac{2}{\beta^2} - \frac{1}{\beta^2} = \frac{1}{\beta^2}$$

(b)

$$F(x) = Pr(X \leq x) = \int_{-\infty}^{+\infty} p(x)dx$$
$$while\ x < 0, F(x) = \int_{-\infty}^{x} 0dx = 0$$
$$while\ x \geq 0, F(x) = \int_0^{x} \beta e^{-\beta x}dx = e^{-\beta x}|_x^0 = 1 - e^{-\beta x}$$
$$so\ F(x) = \begin{cases} 0, x < 0 \\ 1 - e^{-\beta x}, x \geq 0 \end{cases}$$

(c)

$$while\ x > 0, Pr(X \geq x) = \int_x^{+\infty} p(x)dx = F(x)|_x^{+\infty} = 1 - (1 - e^{-\beta x}) = e^{-\beta x}$$
$$Pr(X \geq a+b|X \geq a) = \frac{Pr(X \geq a+b)}{Pr(X \geq a)} = \frac{e^{-\beta(a+b)}}{e^{-\beta a}} = e^{-\beta b} = Pr(X \geq b)$$

(d) The expected lifetime is $E[x] = \frac{1}{\beta} = 1000(hours)$. According to question (c), we know that the exponential distribution has memoryless property. So Pr(the light bulb can work b hours more after it has worked a hours)=Pr(the bulb can work b hours from the beginning), which means the bulb has no memory for how many hours it has worked. In this case, the answer for the remaining lifetime is still 1000 hours.

5.第二章讲义习题10

(a)

$$to\ prove\ f(x) = e^{ax} is\ a\ convex\ function$$
$$is\ to\ prove\ that f(x-d) + f(x+d) \geq 2f(x)\ for\ \forall d > 0, x \in R$$
$$that\ is\ to\ prove\ e^{a(x+d)} + e^{a(x-d)} \geq 2e^{ax}$$
$$that\ is\ to\ prove\ e^{ad} + \frac{1}{e^{ad}} \geq 2, which\ is\ obviously\ right\ for\ that\ e^{ad} > 0.\ Q.E.D.$$

(b)

$$in\ order\ to\ prove\ g(x) = ln(x)\ is\ a\ concave\ function$$
$$is\ to\ prove\ that\ g(x-d) + g(x+d) \leq 2g(x)\ for\ \forall 0 < d < x$$
$$is\ to\ prove\ that\ ln(x-d) + ln(x+d) \leq 2ln(x)$$
$$is\ to\ prove\ that\ (x-d)(x+d) \leq x^2, i.e. x^2 - d^2 \leq x^2$$
$$which\ is\ obviously\ right\ since\ d > 0,\ Q.E.D.$$

(c)$h'(x) = ln(x) + 1$ exists, and $h''(x) = \frac{1}{x} > 0\ for\ x > 0$. So $h(x)$ is a convex function on $\{x|x \geq 0\}$.

(d)

$$p_1 + p_2 + \ldots + p_n = 1, so\ let\ \varphi(p_1, p_2, \ldots, p_n) = p_1 + p_2 + \ldots + p_n - 1 = 0$$
$$L(p_1, p_2, \ldots, p_n, \lambda) = H + \lambda\varphi = -\sum_{i=1}^n p_i log_2 p_i + \lambda(\sum_{i=1}^n p_i - 1) = \sum_{i=1}^n p_i(\lambda - log_2 p_i) - \lambda$$
$$\frac{\partial L}{\partial p_i} = \lambda - log_2 p_i + p_i(-\frac{1}{ln2 * p_i}) = \lambda - log_2 p_i - \frac{1}{ln2}$$
$$let\ \frac{\partial L}{\partial p_i} = 0, so\ that\ p_i = 2^{\lambda - \frac{1}{ln2}} = \frac{2^\lambda}{e}$$
$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^n p_i - 1 = 0, beacuse\ p_i = const, so\ that\ n * p_i = 1, p_i = \frac{1}{n}.$$
$$so\ \lambda = \frac{1}{ln2} - log_2 n = log_2 \frac{e}{n}.$$
$$so\ that\ when\ pi = const\ \frac{1}{n}, the\ entropy\ reaches\ maximum\ log_2 n.$$