Using Monte Carlo Renormalization Group (MCRG) to study the Ising model on a two dimensional square lattice

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1 Introduction

Monte Carlo Renormalization Group (MCRG) is a computational method for investigating critical phenomena in thermodynamical systems by combining standard Monte Carlo simulations with real space renormalization group analysis. In this work, I follow a description of MCRG by Swendsen [1] and Binder [2]. I apply MCRG to extract critical exponents of the Ising Model on a two dimensional square lattice. We know the exact critical exponents for the Ising Model in two dimensions [3], so this is a good check for the validity of the MCRG method.

MORE THEORY : MC MORE THEORY: RG

2 Simulation Method

Consider a system Ω of Ising spins $\sigma_i = \pm 1$ situated on a two dimensional ($d = 2$) square lattice with linear dimension L and lattice spacing 1. Then the total number of lattice sites is $N_s = L^2$. We use priodic boundary conditions. For simplicity, we start our analysis with a microscopic Hamiltonian with only nearest neighbor interactions:

$$\mathcal{H} = -\beta H_\Omega = K \sum_{\langle i,j \rangle} \sigma_i \sigma_j + h \sum_{i=0}^{N_s} \sigma_i$$

and set coupling constants to the known critical values ($K = K_c, h = 0$). As discussed previously, because we have started on the critical manifold, as we repeatedly apply renormalization group transformations, the system will be moved towards the critical fixed point $\mathbf{K}^*$. (With finite precision numbers, we are not *exactly* starting *on* the critical manifold, but we still expect the system to move towards the fixed point in the few RG iterations that we apply before diverging along one of the relevant directions.)

For the standard Monte Carlo part, I choose to use the Metropolis-Hastings method to generate spin configurations. The Monte Carlo simulation settings for a few different lattice sizes are listed in Table 1.

Lattice linear dimension L	64	32	16	8
# of burn-in steps N_{warm}	1×10^4	1×10^4	1×10^4	1×10^4
# of measurement steps N_{meas}	1×10^4	1×10^4	1×10^4	1×10^4
# of MC steps between measurements Δ_N	10	10	10	10
# of samples $N_{data} = N_{meas}/\Delta_N$	1×10^4	1×10^4	1×10^4	1×10^4

Table 1: Monte Carlo simulation settings

For the renormalization group analysis part, I use a simple block-spin transformation with scale factor $b = 2$. The renormalized block-spin value is determined by majority rule, with ties broken by random assignments of $+1$ and -1.

//TODO: DIAGRAM

(CHECK FACT)Due to a special symmetry of the two dimensional square lattice, we can analyze the even and odd coupling constants in the Hamiltonian separately. In other words, we can suppose that renormalization group transformations do not mix even and odd coupling constant spaces. For our purposes, the largest (in magnitude) eigenvalue λ_e of the linearized RG transformation matrix for even coupling constants produce the thermal exponent y_T via:

$$y_T = \frac{\log \lambda_e}{\log b}$$

The largest (in magnitude) eigenvalue λ_o of the linearized RG transformation matrix for odd coupling constants produce the megnetization exponent y_H via:

$$y_H = \frac{\log \lambda_o}{\log b}$$

From Onsager's exact solution [3] we know the exact critical exponents of the Ising model in two dimensions are $\nu = 1,\ \eta = 1/4$. So we expect to find

$$y_T = 1/\nu = 1 \qquad y_H = d - \frac{d-2+\eta}{2} = \frac{15}{8}$$

The even coupling constants that are considered in the RG analysis are given in Table. 2.

Even couplings	
Notation	Meaning
K_1	nearest neighbor $(0,0)$ - $(1,0)$
K_2	next-nearest neighbor $(0,0)$ - $(1,1)$
K_3	third nearest neighbor $(0,0)$ - $(2,0)$
K_4	fourth nearest neighbor $(0,0)$ - $(2,1)$
K_5	fifth nearest neighbor $(0,0)$ - $(2,2)$
K_6	four spins on a plaquette $(1,0)$ - $(1,1)$ - $(0,1)$ - $(0,0)$
K_7	four spins on a sublattice plaquette $(2,0)$ - $(0,2)$ - $(-2,0)$ - $(0,-2)$

Table 2: First few even short range coupling constants that may be used in the RG analysis to find y_T

The odd coupling constants that are considered in the RG analysis are given in Table. 3.

Odd couplings	
Notation	Meaning
K_1	Magnetization $(0,0)$
K_2	Three spins on a plaquette $(0,0)$ - $(1,0)$ - $(1,1)$
K_3	Three spins in a row $(0,0)$ - $(1,0)$ - $(2,0)$
K_4	Three spins at an angle $(0,0)$ - $(1,0)$ - $(2,1)$

Table 3: First few odd short range coupling constants that may be used in the RG analysis to find y_H

3 Thermal Eigenvalue Results

	L	64	32	16	8	4
Nc		1×10^4	1×10^4	1×10^4	1×10^4	1×10^4
# of measurement steps N_{meas}		1×10^4	1×10^4	1×10^4	1×10^4	1×10^4
# of MC steps between measurements Δ_N		10				
# of samples $N_{data} = N_{meas}/\Delta_N$		1×10^4	1×10^4	1×10^4	1×10^4	1×10^4

Table 4: thermal eigenvalue exponent y_T as a function of the number of RG iterations N_r, the number of coupling constants in the RG analysis N_c

4 Magnetic Eigenvalue Results

	L	64	32	16	8	4
Nc		1×10^4	1×10^4	1×10^4	1×10^4	1×10^4
# of measurement steps N_{meas}		1×10^4	1×10^4	1×10^4	1×10^4	1×10^4
# of MC steps between measurements Δ_N		10				
# of samples $N_{data} = N_{meas}/\Delta_N$		1×10^4	1×10^4	1×10^4	1×10^4	1×10^4

Table 5: thermal eigenvalue exponent y_T as a function of the number of RG iterations N_r, the number of coupling constants in the RG analysis N_c

References

[1] R. H. Swendsen, *Monte Carlo Renormalization*, pp. 57–84. Topics in current physics, Springer-Verlag, 1982.

[2] D. P. Landau and K. Binder, *Monte Carlo renormalization group methods*, p. 364–377. Cambridge University Press, 4 ed., 2014.

[3] L. Onsager, "Crystal statistics. i. a two-dimensional model with an order-disorder transition," *Phys. Rev.*, vol. 65, pp. 117–149, Feb 1944.