M328K: Homework 11

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Theorem 1. Let x_1, y_1 be the fundamental solution[1] of $x^2 - Dy^2 = 1$. Then every pair of integers x_n, y_n defined by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n, \quad n = 1, 2, 3, \ldots$$

is also a solution. In fact, every positive solution is given by such a pair x_n, y_n.

1. Prove that the integer pair x_n, y_n in Theorem 1 is a solution of $x^2 - Dy^2 = 1$ for all positive integers n.

   Proof. By definition,

   $$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

   Since the expressions are equal, we can multiply each side by its conjugate.

   $$(x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d}) = (x_1 + y_1\sqrt{d})^n(x_1 - y_1\sqrt{d})^n$$

   From the fundamental solution of $x_1 - dy_1^2 = 1$, we know

   $$(x_1 + y_1\sqrt{d})(x_1 - y_1\sqrt{d}) = 1$$

   Raising both sides to the nth power where $n = 1, 2, 3, \ldots$, we have

   $$(x_1 + y_1\sqrt{d})^n(x_1 - y_1\sqrt{d})^n = 1^n = 1$$

   Then by substitution and algebra,

   $$\begin{aligned}(x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d}) &= (x_1 + y_1\sqrt{d})^n(x_1 - y_1\sqrt{d})^n \\ (x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d}) &= 1 \\ x_n^2 + dy_n^2 &= 1\end{aligned}$$

   Thus the integer pair x_n, y_n is a solution of $x^2 - dy^2 = 1$ for all positive integers n. $\square$

2. (a) Calculate the continued fraction representation of $\sqrt{14}$ using the rationalizing denominators method. In particular, do not use the decimal representation of $\sqrt{14}$. Only use that the integer part of $\sqrt{14}$ is 3.

[1] Recall the *fundamental solution* of Pell's equation is the smallest positive integer solution.

Proof.

$$\begin{aligned}\sqrt{14} &= 3 + \sqrt{14} - 3 \\ &= 3 + \frac{1}{\frac{1}{\sqrt{14}-3}}\end{aligned}$$

$$\frac{1}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{14-9} = \frac{\sqrt{14}+3}{5}$$

The integer component of this fraction is 1 since the integer part of $\sqrt{14}$ is 3.

$$\frac{\sqrt{14}+3}{5} = 1 + \frac{\sqrt{14}+3}{5} - 1 = 1 + \frac{\sqrt{14}-2}{5} = 1 + \frac{1}{\frac{1}{\frac{\sqrt{14}-2}{5}}}$$

So,

$$\sqrt{14} = 3 + \frac{1}{1 + \frac{1}{\frac{1}{\frac{\sqrt{14}-2}{5}}}}$$

$$\frac{1}{\frac{\sqrt{14}-2}{5}} = \frac{5}{\sqrt{14}-2} = \frac{5\sqrt{14}+10}{10} = \frac{\sqrt{14}+2}{2}$$

The integer component is 2.

$$2 + \frac{\sqrt{14}+2}{2} - 2 = 2 + \frac{\sqrt{14}-2}{2} = 2 + \frac{1}{\frac{1}{\frac{\sqrt{14}-2}{2}}}$$

By substitution,

$$\sqrt{14} = 3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{1}{\frac{\sqrt{14}-2}{2}}}}}$$

$$\frac{1}{\frac{\sqrt{14}-2}{2}} = \frac{2}{\sqrt{14}-2} = \frac{2\sqrt{14}+4}{10} = \frac{\sqrt{14}+2}{5}$$

The integer component is 1.

$$\frac{\sqrt{14}+2}{5} = 1 + \frac{\sqrt{14}+2}{5} - 1 = 1 + \frac{\sqrt{14}-3}{5} = 1 + \frac{1}{\frac{1}{\frac{\sqrt{14}-3}{5}}}$$

By substitution,

$$\sqrt{14} = 3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{1}{\frac{\sqrt{14}-3}{5}}}}}}$$

$$\frac{1}{\frac{\sqrt{14}-3}{5}} = \frac{5}{\sqrt{14}-3} = \frac{5(\sqrt{14}+3)}{5} = \sqrt{14}+3$$

The integer component is 6.

$$6 + \sqrt{14} + 3 - 6 = 6 + \sqrt{14} - 3 = 6 + \frac{1}{\frac{1}{\sqrt{14}-3}}$$

By substitution,

$$\sqrt{14} = 3 + \cfrac{1}{1 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{\cfrac{1}{\sqrt{14}-3}}}}}}$$

We already know $\frac{1}{\sqrt{14}-3}$, so we know that the fractions will continue in the same pattern. Thus,

$$\sqrt{14} = 3 + \cfrac{1}{1 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\dots}}}}} = [3; \overline{1, 2, 1, 6}]$$

□

(b) Use the previous problem to find the fundamental solution of $x^2 - 14y^2 = 1$.

Proof. First, find the convergents of the continued fraction of $\sqrt{14}$, where $C_n = \frac{p_n}{q_n}$.

$$C_0 = \frac{3}{1}$$
$$C_1 = 3 + \frac{1}{1} = \frac{4}{1}$$
$$C_2 = 3 + \cfrac{1}{1 + \frac{1}{2}} = \frac{11}{3}$$
$$C_3 = 3 + \cfrac{1}{1 + \cfrac{1}{2+\frac{1}{1}}} = \frac{15}{4}$$
$$C_4 = 3 + \cfrac{1}{1 + \cfrac{1}{2+\cfrac{1}{1+\frac{1}{6}}}} = \frac{41}{11}$$

The fundamental solution satisfies $p_n^2 - 14q_n^2 = 1$. So we use the convergents to find the solution pair p_n, q_n with the smallest n.

$$n = 0: \quad 3^2 - 14 \cdot 1^2 = -5$$
$$n = 1: \quad 4^2 - 14 \cdot 1^2 = 2$$
$$n = 2: \quad 11^2 - 14 \cdot 3^2 = -5$$
$$n = 3: \quad 15^2 - 14 \cdot 4^2 = 1$$

Thus the fundamental solution of $x^2 - 14y^2 = 1$ is $(x, y) = (15, 4)$. □

(c) Use Theorem 1 to calculate two more distinct positive solutions of $x^2 - 14y^2 = 1$.

Proof. By Theorem 1, the solutions (x_n, y_n) for positive integers n are

$$x_n + y_n\sqrt{14} = (15 + 4\sqrt{14})^n$$

For $n = 2$: $(15 + 4\sqrt{14})^2 = 225 + 120\sqrt{14} + 224 = 449 + 120\sqrt{14}$
So, one solution is $(x_2, y_2) = (449, 120)$

For $n = 3$:

$$
\begin{aligned}
(15 + 4\sqrt{14})^3 &= (449 + 120\sqrt{14})(15 + 4\sqrt{14}) \\
&= 6735 + 3596\sqrt{14} + 6720 \\
&= 13455 + 3596\sqrt{14}
\end{aligned}
$$

So, another solution is $(x_3, y_3) = (13455, 3596)$ □

3. Let x be irrational, and let $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ be two consecutive convergents of x. Show that at least one of the convergents satisfies the inequality

$$\left| x - \frac{p_i}{q_i} \right| < \frac{1}{2q_i^2}.$$

Hint: Since x lies between the two convergents, we have

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right|.$$

Now argue by contradiction.

Proof. First, a property of consecutive convergents is

$$C_{n+1} - C_n = \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^n}{q_{n+1}q_n}$$

Taking the absolute value,

$$|C_{n+1} - C_n| = \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_{n+1}q_n}$$

Now suppose that neither convergent satisfies the aforementioned inequality. That is, ‘

$$\left| x - \frac{p_n}{q_n} \right| \geq \frac{1}{2q_n^2} \quad \text{and} \quad \left| x - \frac{p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{2q_{n+1}^2}$$

Adding these inequalities, we get

$$\left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

And by substitution,

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

Since $\frac{1}{q_n^2} + \frac{1}{q_{n+1}^2} \geq \frac{2}{q_n q_{n+1}}$, then we can conclude

$$\frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} \geq \frac{1}{q_n q_{n+1}}$$

Thus, we have

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{q_n q_{n+1}}$$

However, this contradicts the property from the beginning of the proof. Thus at least one convergent satisfies the inequality

$$\left| x - \frac{p_i}{q_i} \right| < \frac{1}{2q_i^2}.$$

□