M328K: Homework 10

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1. In this problem we will investigate an important arithmetic function that is *not* multiplicative. The *Mangoldt function* Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \text{ where } p \text{ is prime and } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that $\log n = \sum_{d|n} \Lambda(n)$. (Warning: In previous examples like this, it was sufficient to prove the equality for $n = p^k$ a prime power, but that is not enough here, since Λ is not multiplicative.)

   (b) Show that $\Lambda(n) = -\sum_{d|n} \mu(d) \log(d)$.

2. Consider the continued fraction $[2; 5, 1, 3]$.

   (a) Calculate the convergents C_0, C_1, C_2, C_3.

   (b) If $C_k = p_k/q_k$, calculate the continued fraction expansions of p_k/p_{k-1} for $1 \leq k \leq 3$.

   (c) Given a continued fraction $[a_0; a_1, \ldots, a_n]$ with $a_0 > 0$, form a conjecture about the continued fraction expansion of p_n/p_{n-1}. Prove it.

3. Compute continued fraction expansions of the following:

   (a) $\sqrt{5}$

   (b) $\dfrac{5 + \sqrt{37}}{2}$

   (c) $\sqrt{n^2 + 1}$ for any $n > 0$.

4. Using the continued fraction of $\sqrt{5}$ from the previous problem, find the first convergent that gives a rational approximation of $\sqrt{5}$ accurate to four decimal places.