M328K: Homework 10

Katherine Ho

November 27, 2024

1. In this problem we will investigate an important arithmetic function that is *not* multiplicative. The *Mangoldt function* Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{, where } p \text{ is prime and } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that $\log n = \sum_{d|n} \Lambda(n)$. (Warning: In previous examples like this, it was sufficient to prove the equality for $n = p^k$ a prime power, but that is not enough here, since Λ is not multiplicative.)

Proof. Let n be a positive integer. The prime factorization of n is

$$n = p_1^{e_1} \dots p_k^{e_k}$$

Then taking the log of n we get

$$\log(n) = \log(p_1^{e_1} \dots p_k^{e_k})$$

By the properties of logarithms, we can rewrite this as

$$\log(n) = e_1 \log(p_1) + \cdots + e_k \log(p_k)$$

By the definition of the Mangoldt function, we know that

$$\Lambda(n) = \log(p) \quad \text{if } n = p^k \text{, where p is prime and } k \geq 1$$

In our equation, each term $e_k \log(p_k)$ divides n and contains a prime p_k raised to a positive exponent. So, each term $e_k \log(p_k) = \Lambda(n)$.
The sum of all such terms is

$$e_1 \log(p_1) + \cdots + e_k \log(p_k) = \sum_{d|n} \Lambda(n)$$

By substitution, we get

$$\log(n) = \sum_{d|n} \Lambda(n)$$

□

(b) Show that $\Lambda(n) = -\sum_{d|n} \mu(d)\log(d)$.

Proof. Using the Mobius inversion formula on $\log n = \sum_{d|n} \Lambda(n)$, we get

$$\Lambda(n) = \sum_{d|n} \mu(d)\log(\frac{n}{d})$$

By using algebra, we can rewrite this as

$$\begin{aligned}
\Lambda(n) &= \sum_{d|n} \mu(d)(\log(n) - \log(d)) \\
&= \sum_{d|n} \mu(d)\log(n) - \sum_{d|n} \mu(d)\log(d) \\
&= \log(n)\sum_{d|n} \mu(d) - \sum_{d|n} \mu(d)\log(d)
\end{aligned}$$

Consider the following property of the Mobius function:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

If $n = 1$, $\log(n)\sum_{d|n} \mu(d) = 0$ since $\log(1) = 0$.
So, $\log(n)\sum_{d|n} \mu(d) = 0$ in all cases.
This eliminates the term from the equation, leaving

$$\Lambda(n) = -\sum_{d|n} \mu(d)\log(d)$$

$\square$

2. Consider the continued fraction $[2; 5, 1, 3]$.

(a) Calculate the convergents C_0, C_1, C_2, C_3.

$$\begin{aligned}
C_0 &= 2 \\
C_1 &= 2 + \frac{1}{5} = \frac{11}{5} \\
C_2 &= 2 + \frac{1}{5 + \frac{1}{1}} = \frac{13}{6} \\
C_3 &= 2 + \frac{1}{5 + \frac{1}{1 + \frac{1}{3}}} = 2 + \frac{1}{5 + \frac{3}{5}} = 2 + \frac{1}{5 + \frac{3}{5}} = 2 + \frac{5}{28} = \frac{61}{28}
\end{aligned}$$

(b) If $C_k = p_k/q_k$, calculate the continued fraction expansions of p_k/p_{k-1} for $1 \leq k \leq 3$.

$$\begin{aligned}
p_0 &= a_0 = 2 \\
p_1 &= a_1 a_0 + 1 \\
&= 10 + 1 = 11
\end{aligned}$$

$\frac{p_1}{p_0} = \frac{11}{2}$

$$\begin{aligned} p_2 &= a_2 p_1 + p_0 \\ &= 1 \cdot 11 + 2 = 13 \end{aligned}$$

$\frac{p_2}{p_1} = \frac{13}{11}$

$$\begin{aligned} p_3 &= a_3 p_2 + p_1 \\ &= 3 \cdot 13 + 11 \\ &= 50 \end{aligned}$$

$\frac{p_3}{p_2} = \frac{50}{13}$

(c) Given a continued fraction $[a_0; a_1, \ldots, a_n]$ with $a_0 > 0$, form a conjecture about the continued fraction expansion of p_n/p_{n-1}. Prove it.

Conjecture: $\frac{p_n}{p_{n-1}}$ is the reversal of the continued fraction $[a_0; a_1, \ldots, a_n]$. That is,

$$\frac{p_n}{p_{n-1}} = [a_n; a_{n-1}, a_{n-2}, \ldots, a_0]$$

Proof. We know that $p_n = a_n p_{n-1} + p_{n-2}$, so we can write:

$$\begin{aligned} \frac{p_n}{p_{n-1}} &= \frac{a_n p_{n-1} + p_{n-2}}{p_{n-1}} \\ &= a_n + \frac{p_{n-2}}{p_{n-1}} \\ &= a_n + \frac{1}{\frac{1}{\frac{p_{n-2}}{p_{n-1}}}} \end{aligned}$$

Similarly, we know that $p_{n-1} = a_{n-1} p_{n-2} + p_{n-3}$, so

$$\frac{p_{n-1}}{p_{n-2}} = a_{n-1} + \frac{p_{n-3}}{p_{n-2}}$$

and

$$\frac{p_{n-2}}{p_{n-1}} = \frac{1}{a_{n-1} + \frac{p_{n-3}}{p_{n-2}}}$$

So,

$$\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1} + \frac{1}{\frac{p_{n-3}}{p_{n-2}}}}$$

We can see that if we continue to transform $\frac{1}{\frac{p_{n-3}}{p_{n-2}}}$ and so on, we will see the following pattern:

$$\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1} + \frac{1}{a_{n-2} + \cdots + \frac{1}{a_0}}}$$

This continued fraction is $[a_n; a_{n-1}, a_{n-2}, \ldots, a_0]$, thus proving the conjecture.

□

3. Compute continued fraction expansions of the following:

(a) $\sqrt{5}$

Proof. Since $\sqrt{4} = 2 < \sqrt{5} < \sqrt{9} = 3$, then the integer component of $\sqrt{5}$ is 2. We can rewrite $\sqrt{5}$ as

$$\begin{aligned}\sqrt{5} &= 2 + (\sqrt{5} - 2)\\ &= 2 + \frac{1}{\frac{1}{\sqrt{5}-2}}\end{aligned}$$

The fraction $\frac{1}{\sqrt{5}-2} = \frac{\sqrt{5}+2}{(\sqrt{5}-2)(\sqrt{5}+2)} = \frac{\sqrt{5}+2}{1} = 4 + \sqrt{5} - 2 = 4 + \frac{1}{\frac{1}{\sqrt{5}-2}}$.
So by substitution,

$$\sqrt{5} = 2 + \frac{1}{4 + \frac{1}{\frac{1}{\sqrt{5}-2}}}$$

Since we know the value of $\frac{1}{\sqrt{5}-2}$, then we also know that this expression will continue to generate $4 + \frac{1}{\frac{1}{\sqrt{5}-2}}$.
So, the continued fraction expression is

$$\sqrt{5} = 2 + \frac{1}{4 + \frac{1}{4+\frac{1}{4+\frac{1}{\cdots}}}} = [2; \overline{4}]$$

□

(b) $\dfrac{5 + \sqrt{37}}{2}$

Proof. First we know that $\sqrt{36} = 6 < \sqrt{37} < \sqrt{49} = 7$, so

$$\begin{aligned}\frac{5+6}{2} &< \frac{5+\sqrt{37}}{2} < \frac{5+7}{2}\\ 5.5 &< \frac{5+\sqrt{37}}{2} < 6\end{aligned}$$

So, the integer component is 5.

$$\begin{aligned}\frac{5+\sqrt{37}}{2} &= 5 + \frac{5+\sqrt{37}}{2} - 5\\ &= 5 + \frac{1}{\frac{1}{\frac{5+\sqrt{37}}{2}-5}}\end{aligned}$$

The expression $\frac{5+\sqrt{37}}{2} - 5 = \frac{5+\sqrt{37}}{2} - \frac{10}{2} = \frac{\sqrt{37}-5}{2}$
The inverse is $\frac{1}{\frac{\sqrt{37}-5}{2}} = \frac{2}{\sqrt{37}-5} = \frac{2(\sqrt{37}+5)}{12} = \frac{\sqrt{37}+5}{6}$

$$\begin{aligned}\frac{6+5}{6} &< \frac{\sqrt{37}+5}{6} < \frac{7+5}{6}\\ 1\frac{5}{6} &< \frac{\sqrt{37}+5}{6} < 2\end{aligned}$$

So the integer component of this fraction is 1. Then,

$$\begin{aligned}\frac{\sqrt{37}+5}{6} &= 1 + \frac{\sqrt{37}+5}{6} - 1 \\ &= 1 + \frac{\sqrt{37}-1}{6}\end{aligned}$$

and

$$\frac{5+\sqrt{37}}{2} = 5 + \frac{1}{1+\frac{\sqrt{37}-1}{6}} = 5 + \frac{1}{1+\frac{1}{\frac{1}{\frac{\sqrt{37}-1}{6}}}}$$

The expression $\frac{1}{\frac{\sqrt{37}-1}{6}} = \frac{6}{\sqrt{37}-1} = \frac{6(\sqrt{37}+1)}{36} = \frac{\sqrt{37}+1}{6}$

$$\begin{aligned}\frac{6+1}{6} < &\frac{\sqrt{37}+1}{6} < \frac{7+1}{6} \\ 1\frac{1}{6} < &\frac{\sqrt{37}+1}{6} < 1\frac{1}{3}\end{aligned}$$

The integer component is 1. So,

$$\frac{\sqrt{37}+1}{6} = 1 + \frac{\sqrt{37}+1}{6} - 1 = 1 + \frac{\sqrt{37}-5}{6}$$

$$\frac{5+\sqrt{37}}{2} = 5 + \frac{1}{1+\frac{1}{1+\frac{\sqrt{37}-5}{6}}} = 5 + \frac{1}{1+\frac{1}{1+\frac{1}{\frac{1}{\frac{\sqrt{37}-5}{6}}}}}$$

The expression $\frac{1}{\frac{\sqrt{37}-5}{6}} = \frac{6}{\sqrt{37}+5} = \frac{6(\sqrt{37}+5)}{12} = \frac{\sqrt{37}+5}{2}$

From an earlier step, we already know

$$\frac{5+\sqrt{37}}{2} = 5 + \frac{1}{\frac{1}{\frac{5+\sqrt{37}}{2}-5}}$$

so by substitution,

$$\frac{5+\sqrt{37}}{2} = 5 + \frac{1}{1+\frac{1}{1+\frac{1}{\frac{1}{5+\frac{1}{\frac{1}{\frac{5+\sqrt{37}}{2}-5}}}}}}$$

Now we can see that the pattern will continue.
Thus, the continued fraction expression is

$$\frac{5+\sqrt{37}}{2} = 5 + \frac{1}{1+\frac{1}{1+\frac{1}{\frac{1}{5+\frac{1}{1+\dots}}}}} = [5; \overline{1,1,5}]$$

$\square$

(c) $\sqrt{n^2+1}$ for any $n > 0$.

Proof. First, the integer component is n and

$$\begin{aligned}\sqrt{n^2+1} &= n + \sqrt{n^2+1} - n \\ &= n + \frac{1}{\frac{1}{\sqrt{n^2+1}-n}}\end{aligned}$$

Then, $\frac{1}{\sqrt{n^2+1}-n} = \frac{\sqrt{n^2+1}+n}{n^2+1-n^2} = \sqrt{n^2+1} + n$.
The integer component is $2n$, so

$$\begin{aligned}\sqrt{n^2+1} + n &= 2n + \sqrt{n^2+1} + n - 2n \\ &= 2n + \sqrt{n^2+1} - n \\ &= 2n + \frac{1}{\frac{1}{\sqrt{n^2+1}-n}}\end{aligned}$$

By substitution,

$$\sqrt{n^2+1} = n + \frac{1}{2n + \frac{1}{\frac{1}{\sqrt{n^2+1}-n}}}$$

Since we already know the value of $\frac{1}{\sqrt{n^2+1}-n}$, we know that the fraction will infinitely continue in this pattern. So, the continued fraction expression is

$$\sqrt{n^2+1} = n + \frac{1}{2n + \frac{1}{2n + \frac{1}{2n+\dots}}} = [n; \overline{2n}]$$

□

4. Using the continued fraction of $\sqrt{5}$ from the previous problem, find the first convergent that gives a rational approximation of $\sqrt{5}$ accurate to four decimal places.

Proof. The actual value of $\sqrt{5} = 2.2360\dots$.

$$\begin{aligned}C_0 &= 2 \\ C_1 &= 2 + \frac{1}{4} = 2.25 \\ C_2 &= 2 + \frac{1}{4+\frac{1}{4}} = 2 + \frac{1}{\frac{17}{4}} = 2 + \frac{4}{17} = \frac{38}{17} = 2.2352\dots \\ C_3 &= 2 + \frac{1}{4+\frac{1}{4+\frac{1}{4}}} = 2 + \frac{1}{4+\frac{4}{17}} = 2 + \frac{17}{72} = \frac{161}{72} = 2.2361\dots \\ C_4 &= 2 + \frac{1}{4+\frac{1}{4+\frac{1}{4+\frac{1}{4}}}} = 2 + \frac{1}{4+\frac{17}{72}} = 2 + \frac{72}{305} = \frac{682}{305} = 2.2360\dots\end{aligned}$$

Thus the first convergent that gives a rational approximation of $\sqrt{5}$ is

$$C_4 = \frac{682}{305}$$

□