Aerodynamics

Lecture 4:

Panel methods

G. Dimitriadis

Introduction

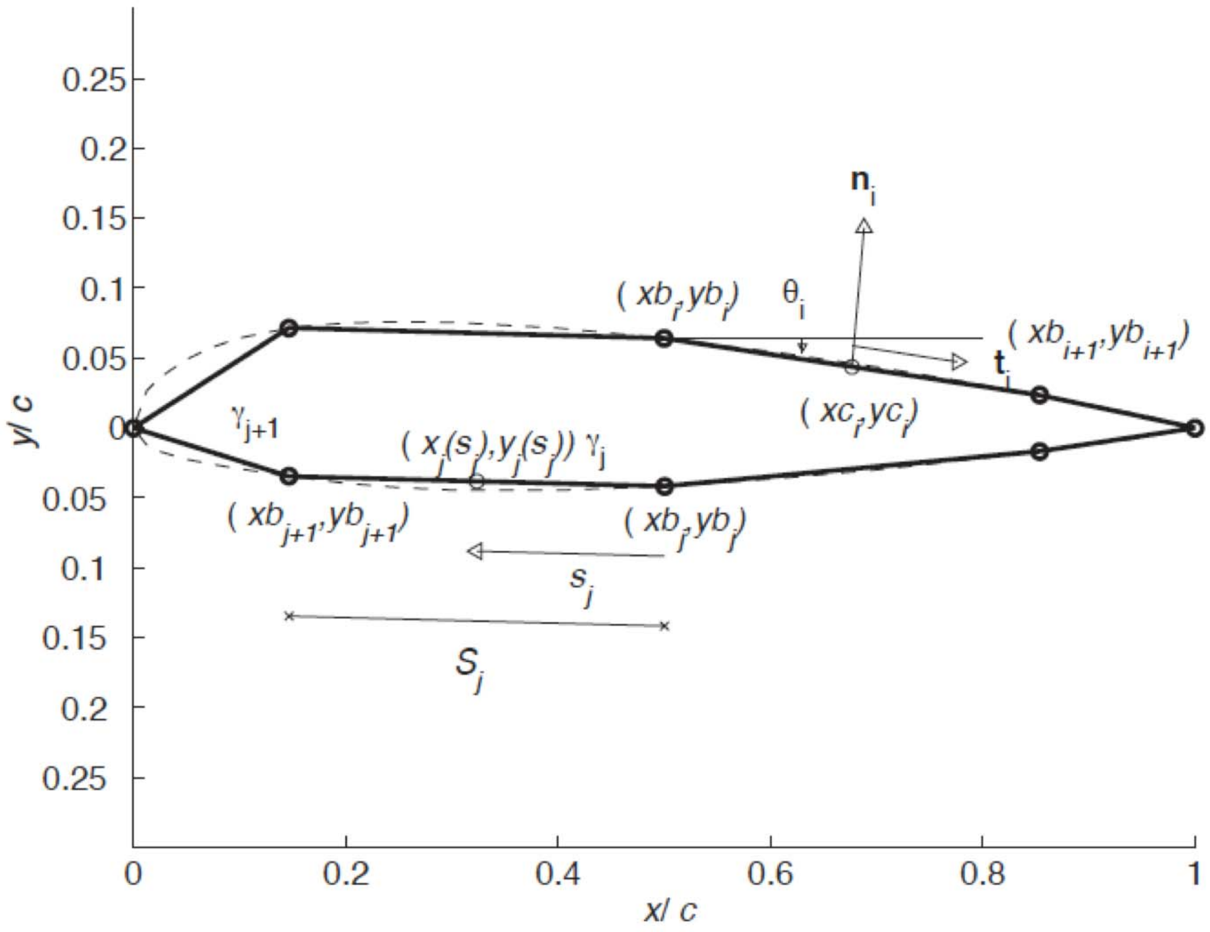
- Until now we’ve seen two methods for modelling wing sections in ideal flow:
  - Conformal mapping: Can model only a few classes of wing sections
  - Thin airfoil theory: Can model any wing section but ignores the thickness.
- Both methods are not general.
- A more general approach will be presented here.

2D Panel methods

- 2D Panel methods refers to numerical methods for calculating the flow around any wing section.
- They are based on the replacement of the wing section's geometry by singularity panels, such as source panels, doublet panels and vortex panels.
- The usual boundary conditions are imposed:
  - Impermeability
  - Kutta condition

Panel placement

- Eight panels of length S_j
- For each panel, $s_j=0\text{-}S_j$.
- Eight control (collocation) points (xc_j, yc_j) located in the middle of each panel.
- Nine boundary points (xb_j, yb_j)
- Normal and tangential unit vector on each panel, n_j, t_j
- Vorticity (or source strength) on each panel $\gamma_j(s)$ (or $\sigma_j(s)$)



Problem statement

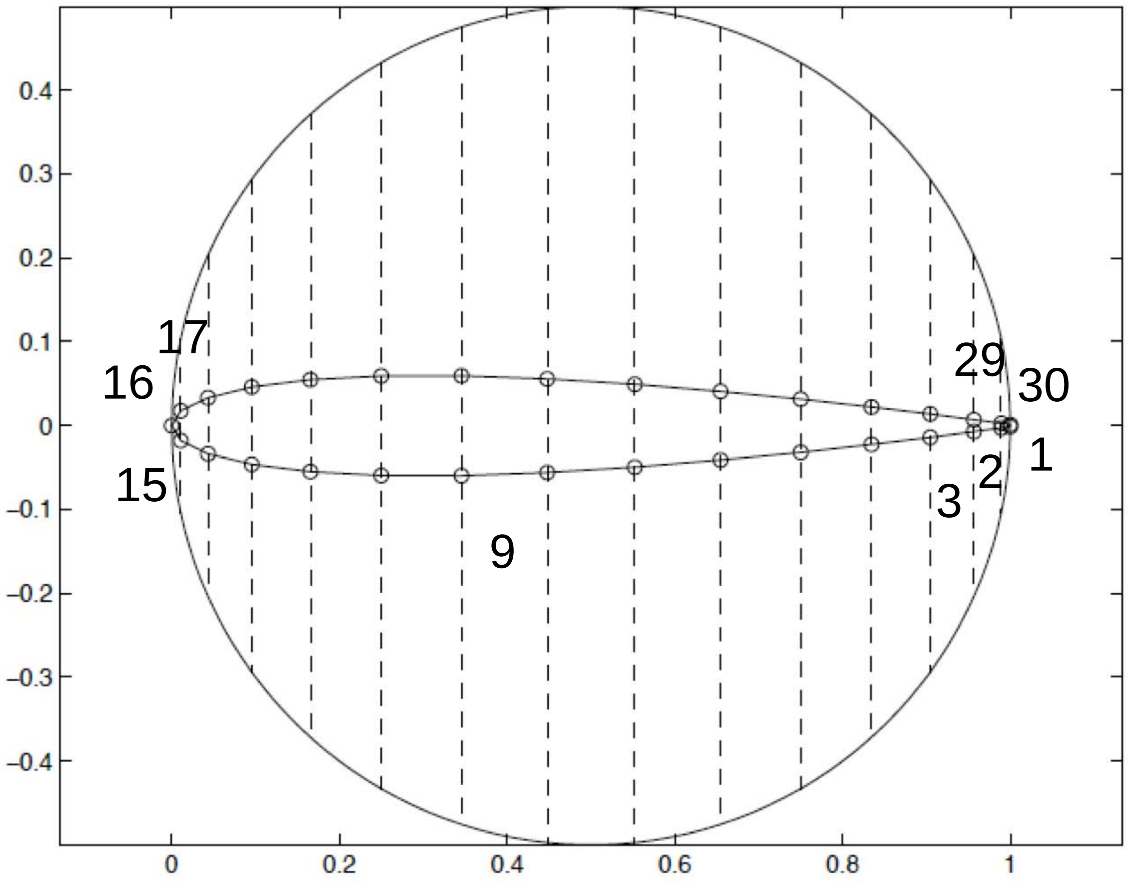
- Use linear panels
- Use constant singularity strength on each panel, $\gamma_j(s)=\gamma_j$ ($\sigma_j(s)=\sigma_j$).
- Add free stream U at angle α.
- Apply boundary conditions:
  - Far field: automatically satisfied if using source or vortex panels
  - Impermeability: Choose Neumann or Dirichlet.
- Apply Kutta condition.
- Find vortex and/or source strength distribution that will satisfy Boundary Conditions and Kutta condition.

Panel choice

- It is best to chose small panels near the leading and trailing edge and large panels in the middle:

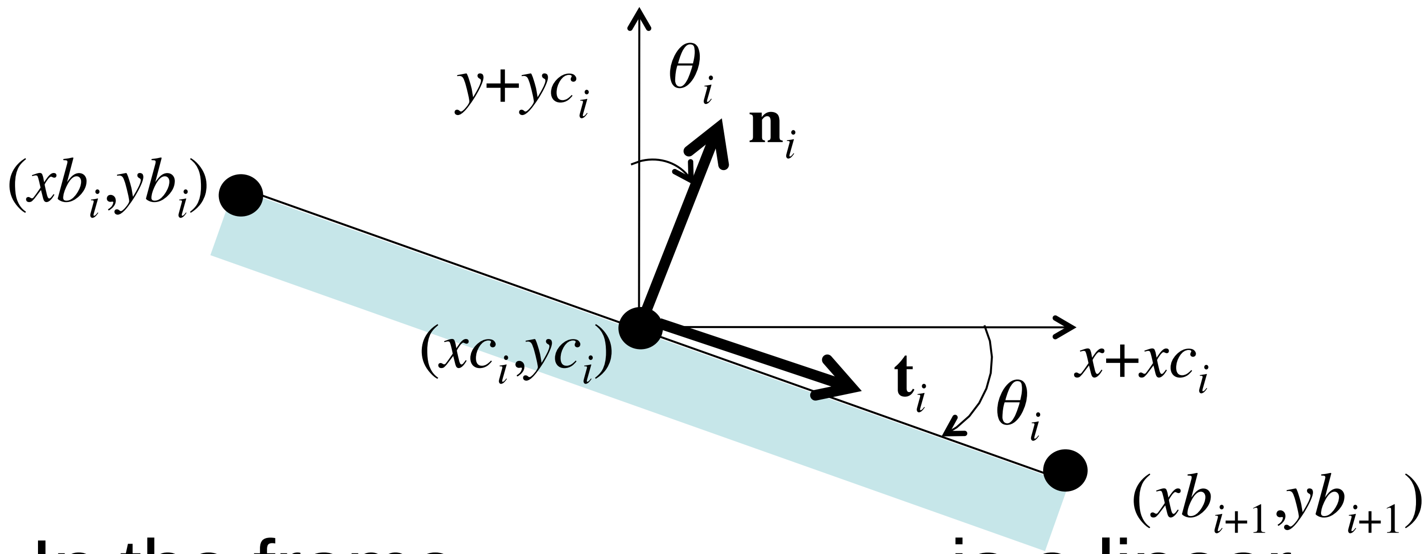
$$\frac{x}{c} = \frac{1}{2}\left(\cos\zeta + 1\right), \quad \zeta = 0 - 2\pi$$

Notice that now x/c begins at 1, passes through 0 and then goes back to 1.
The usual numbering scheme is: lower trailing edge to lower leading edge, upper leading edge to upper trailing edge.



Panel normal and tangent

- Consider a source panel on an airfoil's upper surface, near the trailing edge.



- In the frame $x+xc_i, y+yc_i$, n is a linear function:

$$n = \frac{\partial x}{\partial n} x, \quad n = \frac{\partial y}{\partial n} y$$

- So that $\dfrac{\partial x}{\partial n} = \cos\left(\dfrac{\pi}{2} - \theta_i\right) = \sin\theta_i, \quad \dfrac{\partial y}{\partial n} = -\cos\theta_i$

NACA four digit series airfoils

- The NACA 4-digit series is defined by four digits, e.g. NACA 2412, m=2%, p=40%, t=12%
- The equations are:

$$y_t = \frac{t}{0.2}\left(0.2969\sqrt{x} - 0.126x - 0.35160x^2 + 0.2843x^3 - 0.1015x^4\right)$$

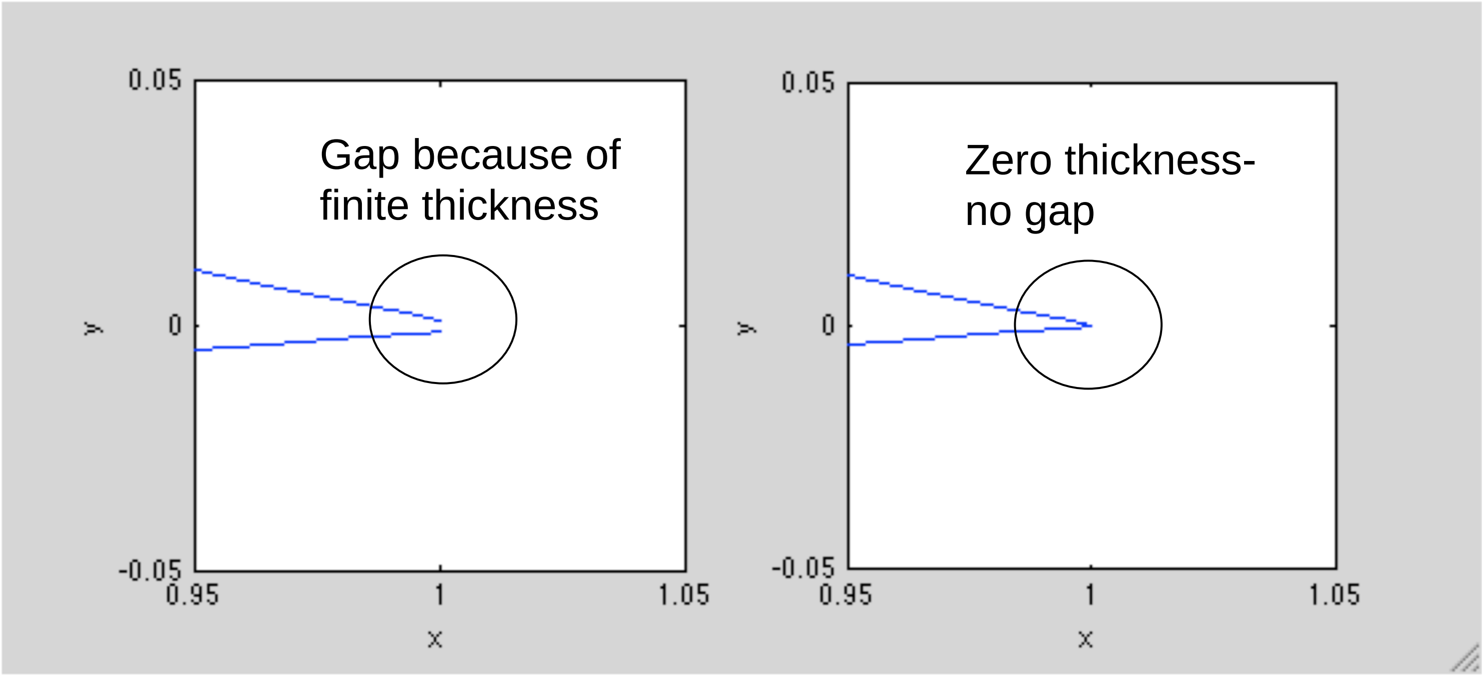
$$y_c = \begin{cases} \dfrac{m}{p^2}\left(2px - x^2\right) & \text{for } x < p \\ \dfrac{m}{\left(1 - p^2\right)}\left(\left(1 - 2p\right) + 2px - x^2\right) & \text{for } x > p \end{cases}$$

- Where t is the maximum thickness as a percentage of the chord, m is the maximum camber as a percentage of the chord, p is the chordwise position of the maximum camber as a tenth of the chord.
- The complete geometry is given by $y=y_c+y_t$.

NACA 4-digit trailing edge

- It should be stressed that the NACA 4-digit thickness equation specifies a trailing edge with a finite thickness:

The equation can be modified so that $y_t(1)=0$



Actual trailing edge

Modified trailing edge

NACA 4-digit with thin airfoil theory

- Thin airfoil theory solutions for NACA four-digit series airfoils can be readily obtained.
- The camber slope is obtained by differentiating the camber line and substituting for $\theta=\cos^{-1}(1-2x/c)$:

$$\frac{dz}{dx}=\frac{m}{p^2}(2p-1+\cos\theta) \text{ for } \theta\le\theta_p$$

$$\frac{dz}{dx}=\frac{m}{(1-p)^2}(2p-1+\cos\theta) \text{ for } \theta\ge\theta_p$$

Substitute these in:

$$A_0=\alpha-\frac{1}{\pi}\int_0^{\pi}\frac{\mathrm{d}z}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_n=\frac{2}{\pi}\int_0^{\pi}\frac{\mathrm{d}z}{\mathrm{d}x}\cos n\theta\,\mathrm{d}\theta$$

NACA 4-digit with thin airfoil theory

- To obtain:

$$A_0 = \alpha - \left[\frac{m}{\pi p^2} \left((2p-1)\theta_p + \sin\theta_p \right) + \frac{m}{\pi(1-p)^2} \left((2p-1)(\pi - \theta_p) - \sin\theta_p \right) \right]$$

$$A_1 = \frac{2m}{\pi p^2} \left((2p-1)\sin\theta_p + \frac{1}{4}\sin 2\theta_p + \frac{\theta_p}{2} \right)$$

$$- \frac{2m}{\pi(1-p)^2} \left((2p-1)\sin\theta_p + \frac{1}{4}\sin 2\theta_p - \frac{1}{2}(\pi - \theta_p) \right)$$

- Which are easily substituted into:

$$c_l = 2\pi A_0 + \pi A_1$$

Source panel airfoils

- Consider an airfoil idealized as m linear source panels with constant strength.
- The potential induced at any point (x,y) in the flowfield by the jth panel is:

$$\phi_j(x,y) = \frac{\sigma_j}{2\pi} \int_0^{S_j} \ln \sqrt{\left(x - x_j(s_j)\right)^2 + \left(y - y_j(s_j)\right)^2}\, \mathrm{d}s_j$$

- Including the free stream and summing the contributions of all the panels, the total potential at point (x,y) is:

$$\phi(x,y) = U(x\cos\alpha + y\sin\alpha) + \sum_{j=1}^{m} \frac{\sigma_j}{2\pi} \int_0^{S_j} \ln \sqrt{\left(x - x_j(s_j)\right)^2 + \left(y - y_j(s_j)\right)^2}\, \mathrm{d}s_j$$

Source panel airfoils

- As the panels are linear, then

$$x_j(s_j) = \frac{xb_{j+1} - xb_j}{S_j} s_j + xb_j = \cos\theta_j s_j + xb_j$$

$$y_j(s_j) = \frac{yb_{j+1} - yb_j}{S_j} s_j + yb_j = \sin\theta_j s_j + yb_j$$

- So that the total potential becomes:

$$\phi(x,y) = U(x\cos\alpha + y\sin\alpha) + \sum_{j=1}^{m} \frac{\sigma_j}{2\pi} \int_0^{S_j} \ln\sqrt{\left(x - xb_j - \cos\theta_j s_j\right)^2 + \left(y - yb_j - \sin\theta_j s_j\right)^2}\,\mathrm{d}s_j$$

- There is no obvious expression for this integral…

Boundary condition

- Try the Neumann boundary condition:

$$\frac{\partial \phi}{\partial n} = 0$$

- This condition is applied on the control point of each panel so that:

$$\frac{\partial \phi(xc_i, yc_i)}{\partial n_i} = U\left(\frac{\partial x}{\partial n_i}\cos\alpha + \frac{\partial y}{\partial n_i}\sin\alpha\right) + \sum_{j=1}^{m}\frac{\sigma_j}{2\pi}\int_0^{S_j}\frac{\partial}{\partial n_i}\left(\ln\sqrt{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2}\right)\mathrm{d}s_j$$

- Notice that: $\frac{\partial x}{\partial n_i} = \sin\theta_i, \frac{\partial y}{\partial n_i} = -\cos\theta_i$ and then

$$\sum_{j=1}^{m}\frac{\sigma_j}{2\pi}\int_0^{S_j}\frac{\partial}{\partial n_i}\left(\ln\sqrt{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2}\right)\mathrm{d}s_j = U\sin(\theta_i - \alpha)$$

Differentiation

- Carrying out the differentiation in the integral:

$$\frac{\partial}{\partial n_i}\left(\ln\sqrt{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2}\right) =$$

$$\frac{1}{2}\frac{\frac{\partial}{\partial n_i}\left(\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2\right)}{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2} =$$

$$\frac{2\left(xc_i - xb_j - \cos\theta_j s_j\right)\frac{\partial x}{\partial n_i} + 2\left(yc_i - yb_j - \sin\theta_j s_j\right)\frac{\partial y}{\partial n_i}}{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2} =$$

$$\frac{\left(xc_i - xb_j - \cos\theta_j s_j\right)\cos\theta_i - \left(yc_i - yb_j - \sin\theta_j s_j\right)\sin\theta_i}{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2}$$

Integration

- After this differentiation, it is now possible to evaluate the integral.
- The boundary condition becomes:

$$\sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left(-\frac{C_{ij}F_{ij}}{2} + D_{ij}G_{ij}\right) = U\sin(\theta_i - \alpha)$$

Where:

$$A_{ij} = -(xc_i - xb_j)\cos\theta_j - (yc_i - yb_j)\sin\theta_j$$

$$B_{ij} = (xc_i - xb_j)^2 + (yc_i - yb_j)^2$$

$$C_{ij} = \sin(\theta_i - \theta_j),\ \ D_{ij} = \cos(\theta_i - \theta_j),\ F = \ln\left(1 + \frac{S_j^2 + 2A_{ij}S_j}{B_{ij}}\right)$$

$$E_{ij} = (xc_i - xb_j)\sin\theta_j - (yc_i - yb_j)\cos\theta_j,\ \ G_{ij} = \tan^{-1}\left(\frac{E_{ij}S_j}{A_{ij}S_j + B_{ij}}\right)$$

$$-\frac{C_{ii}F_{ii}}{2} + D_{ii}G_{ii} = -\pi$$

System of equations

- Therefore, the problem of choosing the correct source strengths to enforce impermeability has been reduced to:

$$\sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left(-\frac{C_{ij}F_{ij}}{2} + D_{ij}G_{ij}\right) = U \sin(\theta_i - \alpha)$$

- Or, in matrix notation, $\mathbf{D}_n\boldsymbol{\sigma}=U\sin(\alpha-\boldsymbol{\theta})$
- Where $\mathbf{D}_n=(-\mathbf{CF}/2+\mathbf{DG})/2\pi$
- Which can be solved directly for the unknown $\boldsymbol{\sigma}$.

Tangential Velocities

- The velocities tangential to the panels are given by:

$$\frac{\partial\phi(xc_i, yc_i)}{\partial t_i} = U\left(\frac{\partial x}{\partial t_i}\cos\alpha + \frac{\partial y}{\partial t_i}\sin\alpha\right)$$

$$+\sum_{j=1}^{m}\frac{\sigma_j}{2\pi}\int_0^{S_j}\frac{\partial}{\partial t_i}\left(\ln\sqrt{\left(xc_i - xb_j - \cos\theta_j s_j\right)^2 + \left(yc_i - yb_j - \sin\theta_j s_j\right)^2}\right)\mathrm{d}s_j$$

- So that:

$$v_{t_i} = U\cos(\theta_i - \alpha) - \sum_{j=1}^{m}\frac{\sigma_j}{2\pi}\left(\frac{D_{ij}F_{ij}}{2} + C_{ij}G_{ij}\right)$$

- As usual: $c_{p_i} = 1 - \left(\frac{v_{t_i}}{U}\right)^2$

Cartesian velocities

- For plotting the velocity field around the airfoil, the cartesian velocities, u, v are needed.
- These can be obtained from the normal and tangential expressions:

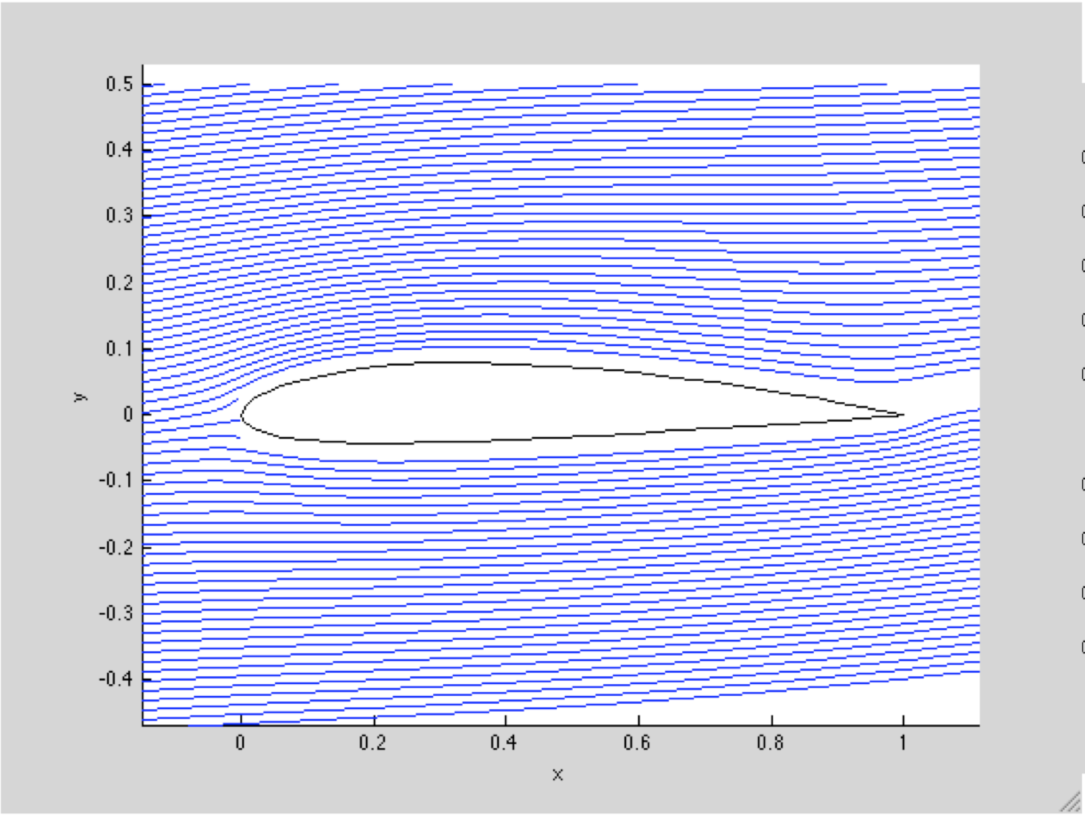
$$u = U\cos(\theta_i - \alpha) - \sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left(\frac{D_{ij}F_{ij}}{2} + C_{ij}G_{ij}\right)$$

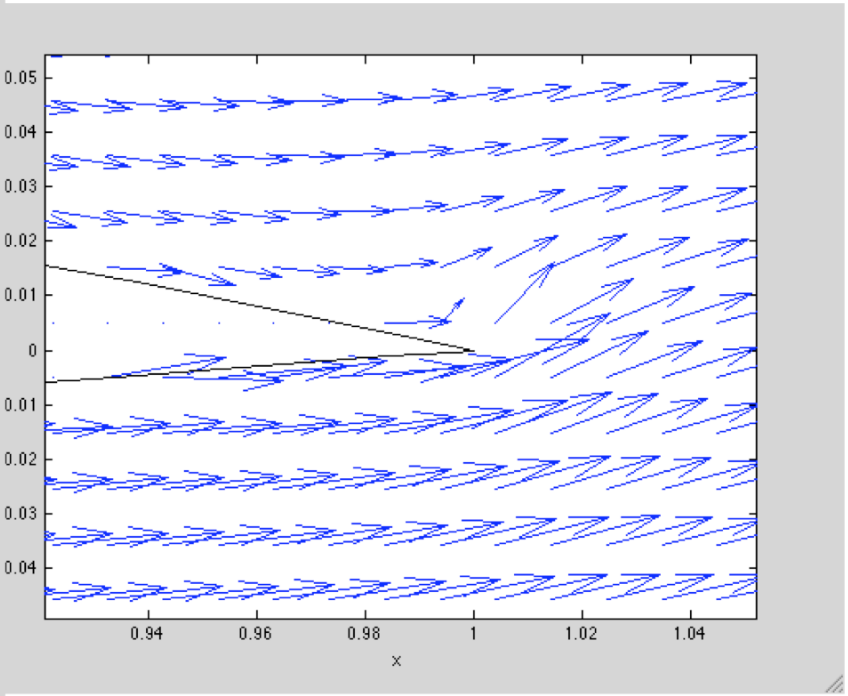
$$v = -U\sin(\theta_i - \alpha) + \sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left(-\frac{C_{ij}F_{ij}}{2} + D_{ij}G_{ij}\right)$$

- for θ_i=0.

Example:

- NACA 2412 airfoil at 5^{o} angle of attack
- 50 panels





Full flowfield

Near trailing edge

Discussion

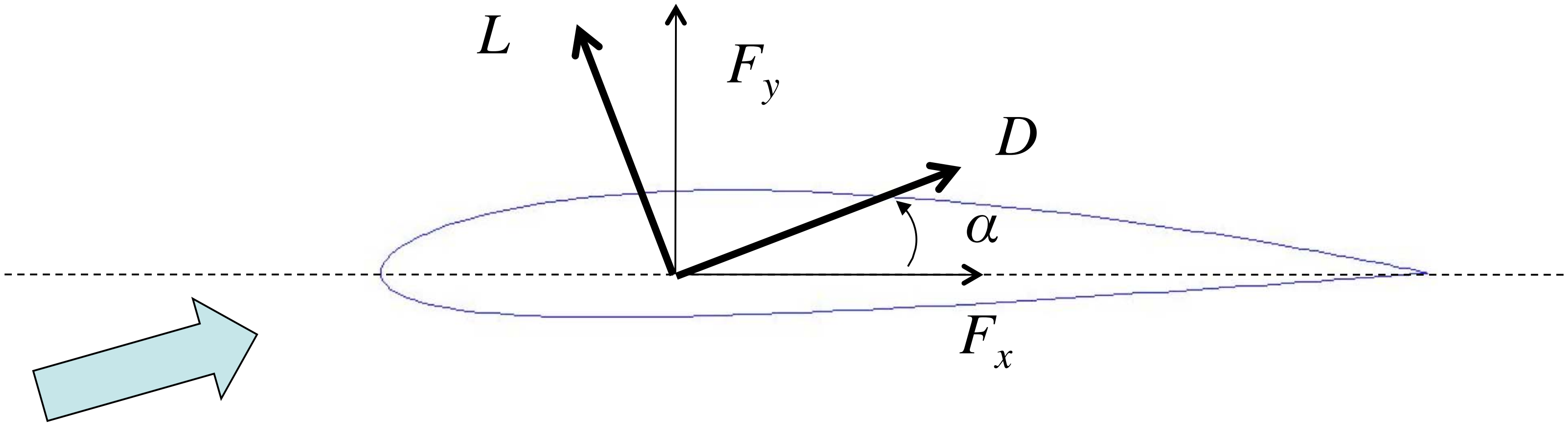
- The usual problem: the Kutta condition was not enforced. The flow separates on the airfoil's upper surface.
- Additionally, the lift must be equal to zero, since there is no ciurculation in the flow. But is it?
- Calculate

$$c_x = \oint c_p \mathrm{d}x$$

$$c_y = \oint c_p \mathrm{d}y$$

Lift definition

- Lift is the force perpendicular to the free stream



Therefore: $L = F_y \cos\alpha - F_x \sin\alpha$

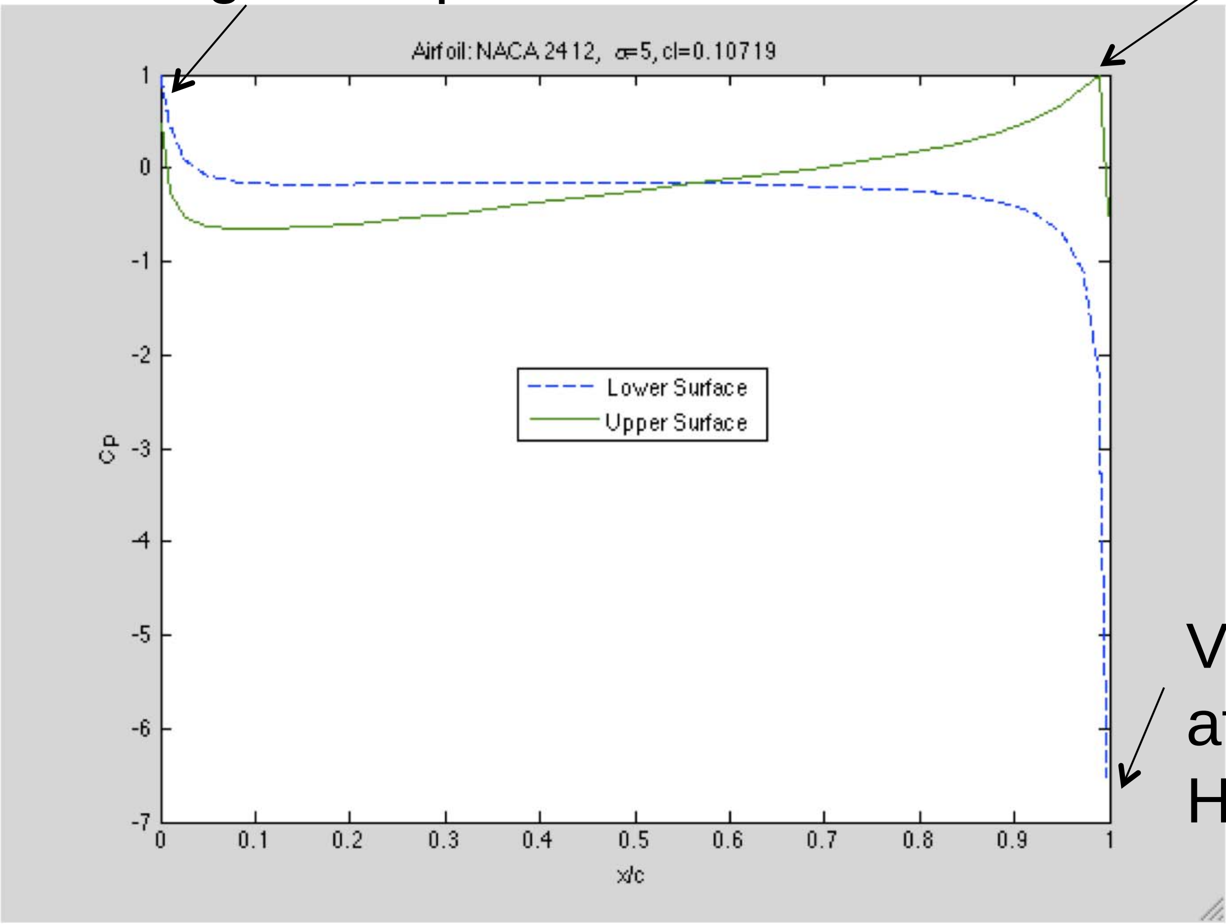
$D = F_y \sin\alpha + F_x \cos\alpha$

Or: $c_l = c_y \cos\alpha - c_x \sin\alpha$

$c_d = c_y \sin\alpha + c_x \cos\alpha$

Pressure distribution

Stagnation point



Stagnation point

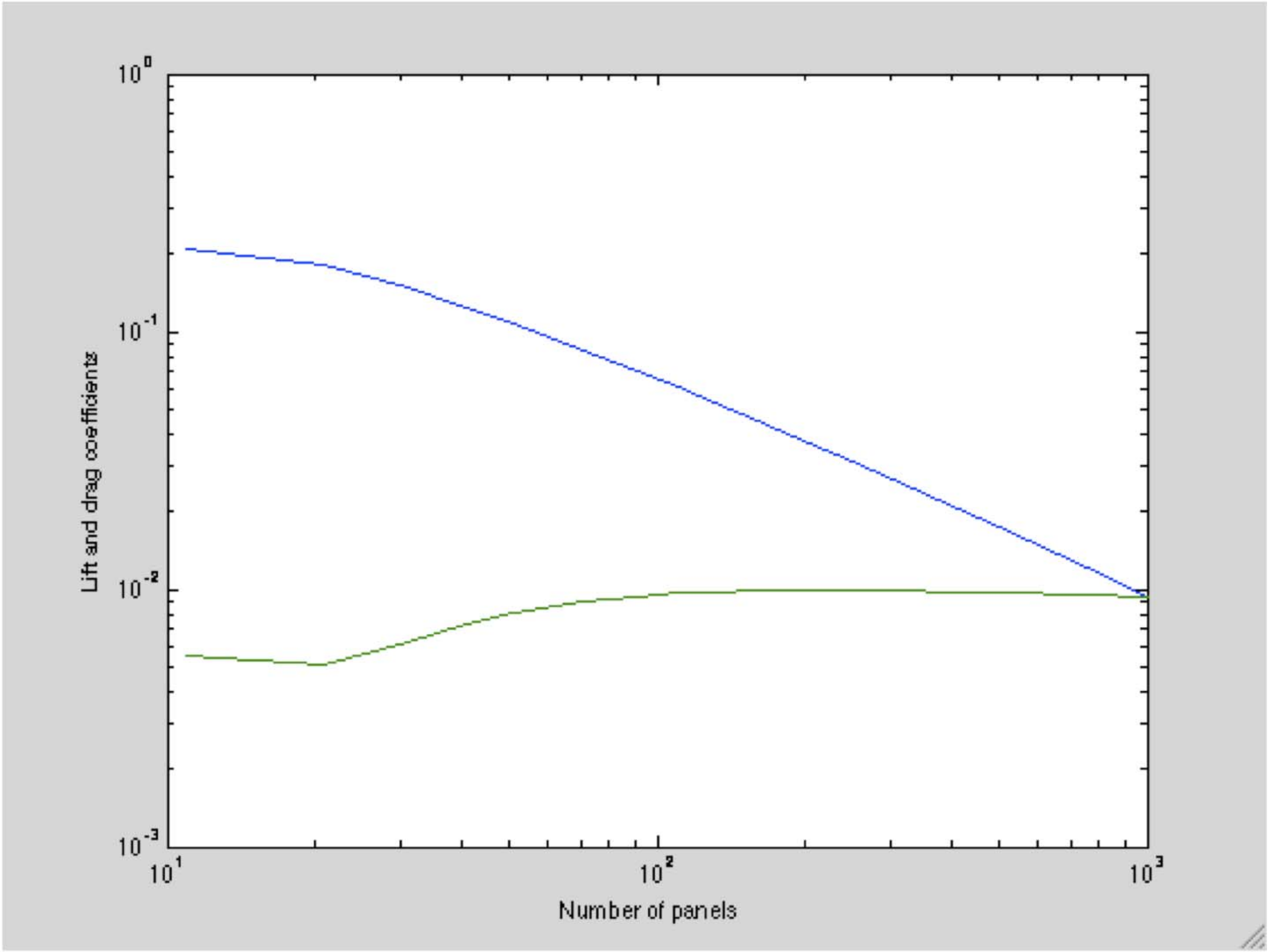
c_l=0.1072
c_d=-0.0082

The aerodynamic forces are not zero.

Very high velocity at trailing edge. Hence very low pressure.

Increasing number of panels

- Increasing the number of panels also increases the accuracy.
- The forces move very slowly towards 0.
- The problem is the infinite velocity at the trailing edge.



Enforcing the Kutta condition

- The number of equations was equal to the number of unknowns
- Therefore, the Kutta condition could not be enforced anyway, it would have been an additional equation.
- More equations than unknowns means a least squares solution.
- Conclusion: we need an additional equation (Kutta condition) and an additional unknown.

Vortex panels

- Vortex panels with exactly the same geometry as the source panels are added.
- If there are m source panels, there will now be additionally m vortex panels.
- The vorticity on all the panels is equal. Only one new unknown is introduced, γ.
- The potential equation becomes:

$$\phi(x,y) = U(x\cos\alpha + y\sin\alpha) + \sum_{j=1}^{m} \frac{\sigma_j}{2\pi} \int_0^{S_j} \ln\sqrt{\left(x - xb_j - \cos\theta_j s_j\right)^2 + \left(y - yb_j - \sin\theta_j s_j\right)^2}\,\mathrm{d}s_j$$

$$- \sum_{j=1}^{m} \frac{\gamma}{2\pi} \int_0^{S_j} \tan^{-1}\left(\frac{y - yb_j - \sin\theta_j s_j}{x - xb_j - \cos\theta_j s_j}\right)\mathrm{d}s_j$$

Boundary condition

- The Neumann impermeability boundary condition is still:

$$\frac{\partial \phi}{\partial n} = 0$$

- So that, now:

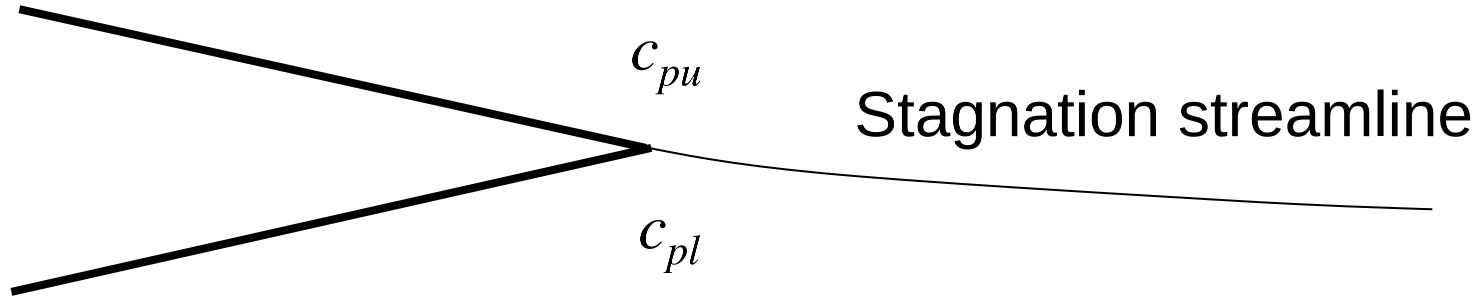
$$\sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left(-\frac{C_{ij}F_{ij}}{2} + D_{ij}G_{ij}\right) - \frac{\gamma}{2\pi}\sum_{j=1}^{m}\left(\frac{D_{ij}F_{ij}}{2} + C_{ij}G_{ij}\right) = U\sin(\theta_i - a)$$

- The tangential velocity is:

$$v_{t_i} = U\cos(\theta_i - \alpha) - \sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left(\frac{D_{ij}F_{ij}}{2} + C_{ij}G_{ij}\right) + \frac{\gamma}{2\pi}\sum_{j=1}^{m}\left(\frac{C_{ij}F_{ij}}{2} - D_{ij}G_{ij}\right)$$

Kutta condition

- The Kutta condition can be applied to this flow by enforcing that the pressures just above and just below the trailing edge must be equal



If the two pressures are not equal, then the stagnation Streamline will wrap itself around the trailing edge.

Kutta condition (2)

- Therefore, $c_p(m) = c_p(1)$
- And $v_t(m) = -v_t(1)$
- So that:

$$\sum_{j=1}^{m} \frac{\sigma_j}{2\pi}\left[\left(\frac{D_{mj}F_{mj}}{2} + C_{mj}G_{mj}\right) + \left(\frac{D_{1j}F_{1j}}{2} + C_{1j}G_{1j}\right)\right]$$

$$-\frac{\gamma}{2\pi}\sum_{j=1}^{m}\left[\left(\frac{C_{mj}F_{mj}}{2} - D_{mj}G_{mj}\right) + \left(\frac{C_{1j}F_{1j}}{2} - D_{1j}G_{1j}\right)\right]$$

$$= U\left(\cos(\theta_1 - \alpha) + \cos(\theta_m - \alpha)\right)$$

System of Equations

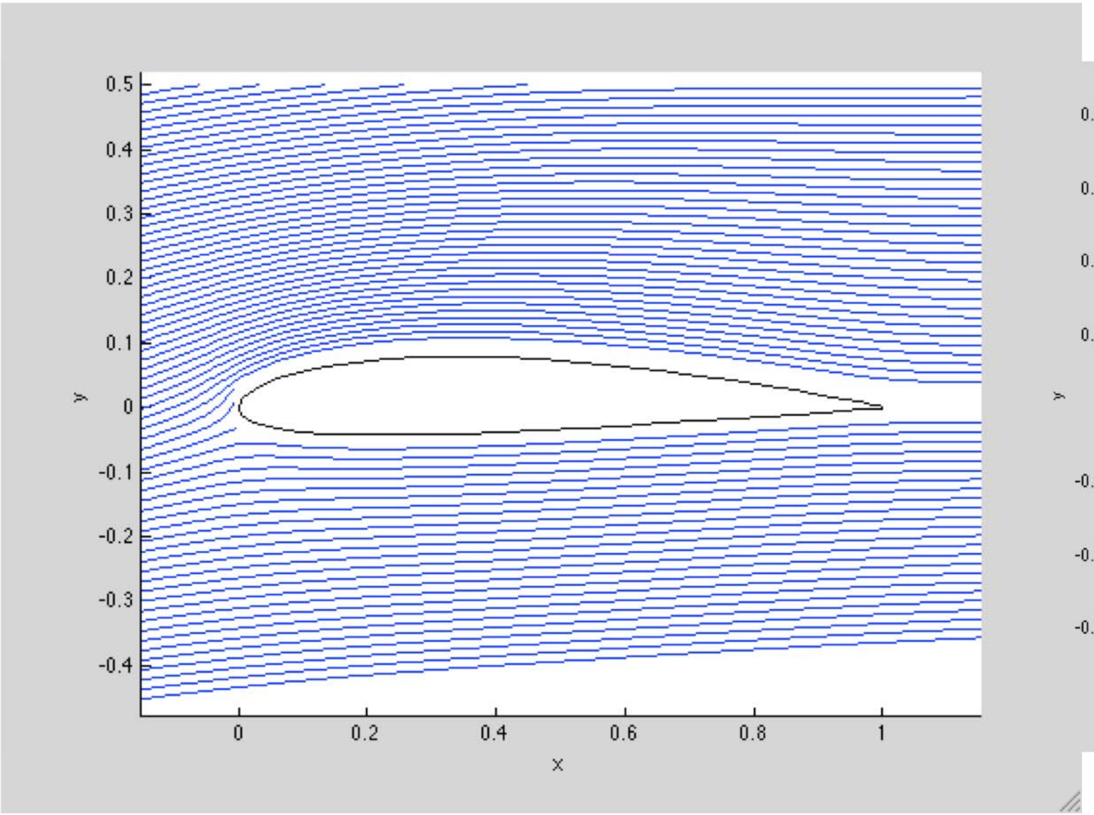
- The impermeability boundary conditions on the panels and the Kutta condition make up $m+1$ equations with $m+1$ unknowns (m source strengths and 1 vorticity).
- The complete system of equations becomes: $\mathbf{A}_n\mathbf{q}=\mathbf{R}$
- where:

$$\mathbf{A}_n = \frac{1}{2\pi}\begin{bmatrix} \left(-\dfrac{C_{ij}F_{ij}}{2} + D_{ij}G_{ij}\right) & -\displaystyle\sum_{j=1}^{m}\left(\dfrac{D_{ij}F_{ij}}{2} + C_{ij}G_{ij}\right) \\ \left[\left(\dfrac{D_{mj}F_{mj}}{2} + C_{mj}G_{mj}\right) + \left(\dfrac{D_{1j}F_{1j}}{2} + C_{1j}G_{1j}\right)\right] & -\displaystyle\sum_{j=1}^{m}\left[\left(\dfrac{C_{mj}F_{mj}}{2} - D_{mj}G_{mj}\right) + \left(\dfrac{C_{1j}F_{1j}}{2} - D_{1j}G_{1j}\right)\right] \end{bmatrix}$$

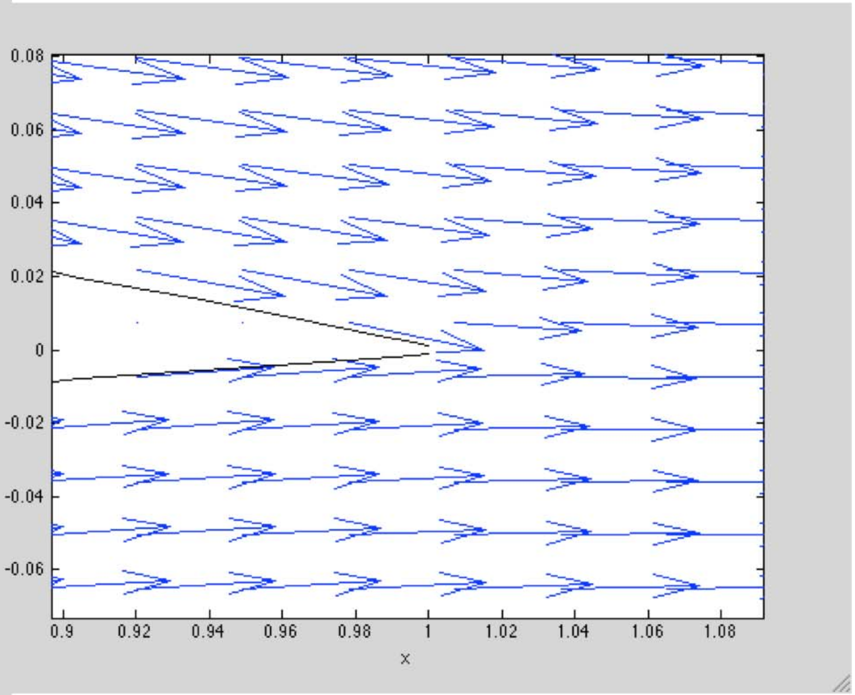
$$\mathbf{R} = \begin{bmatrix} U\sin(\theta_i - \alpha) \\ U\left(\cos(\theta_1 - \alpha) + \cos(\theta_m - \alpha)\right) \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \sigma_i \\ \gamma \end{bmatrix}$$

Example:

- NACA 2412 airfoil at 5^o angle of attack
- 50 panels

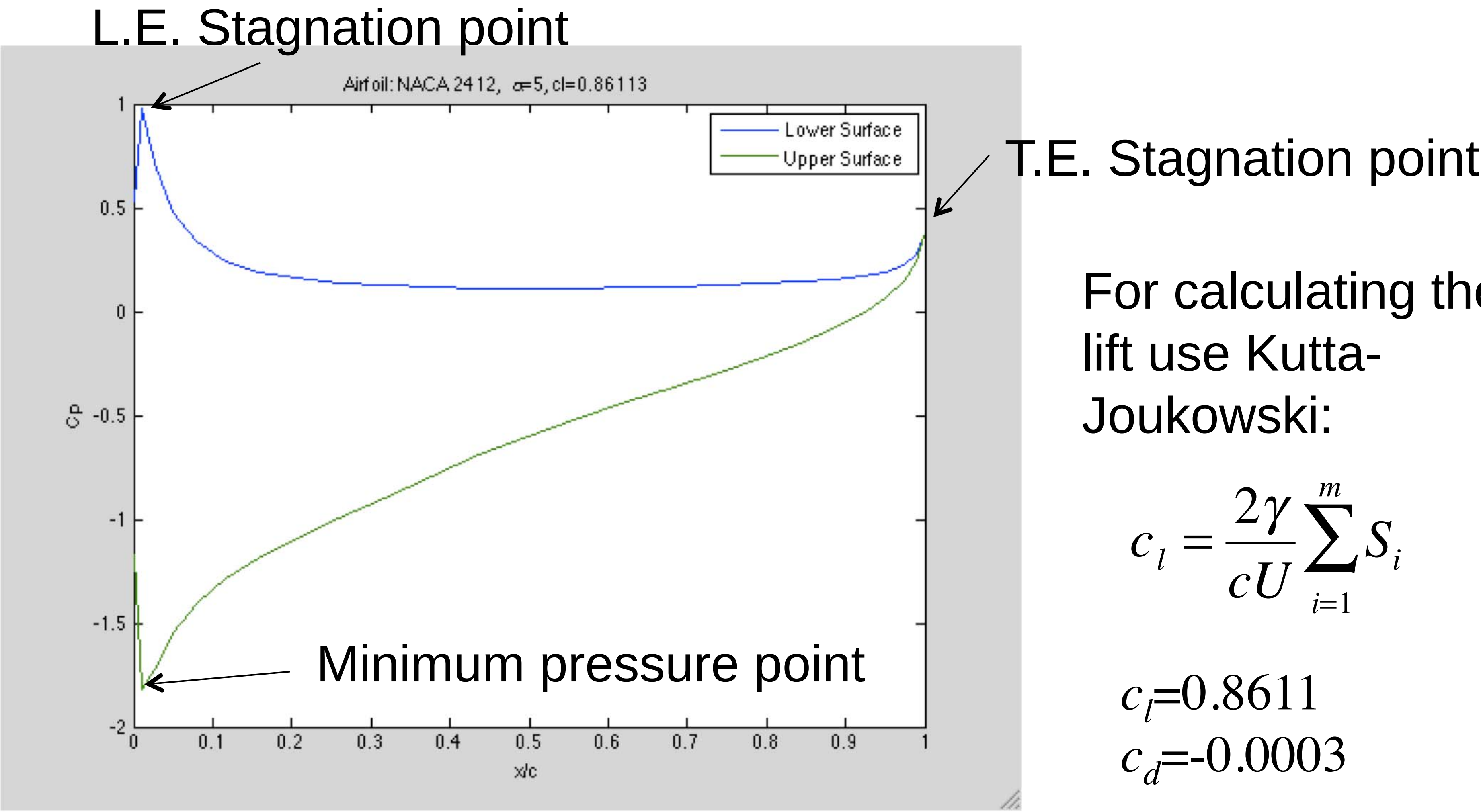


Full flowfield



Near trailing edge

Pressure distribution



For calculating the lift use Kutta-Joukowski:

$$c_l = \frac{2\gamma}{cU} \sum_{i=1}^{m} S_i$$

c_l=0.8611

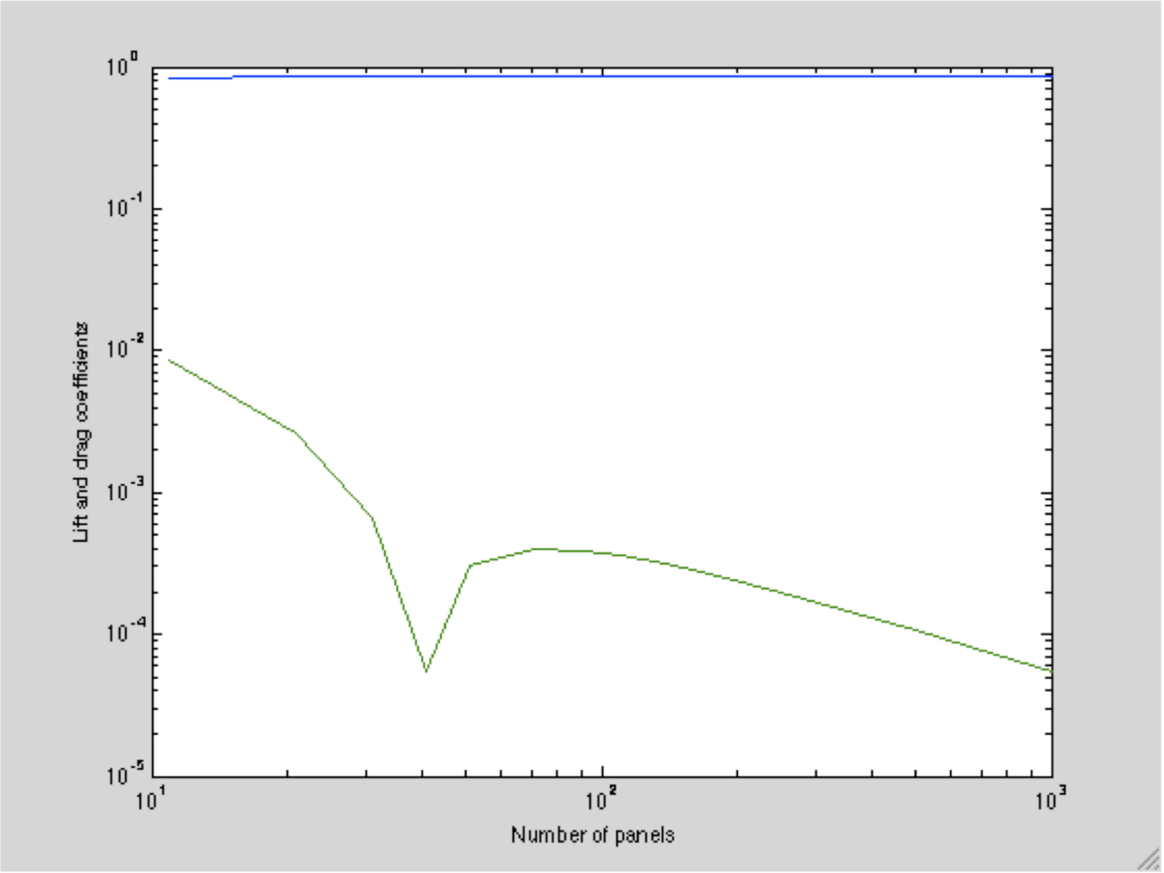
c_d=-0.0003

Discussion

- This is a typical pressure distribution for attached flow over a 2D airfoil.
- The c_p values at the two stagnation points are not exactly 1.
  - The leading edge stagnation point is somewhere on the bottom surface, not necessarily on a control point
  - The trailing edge stagnation point is on the trailing edge, certainly not on a control point.
- The drag is still not exactly zero.

Increasing number of panels

- This is a nice situation:
  - The lift is almost constant with number of panels
  - The drag is high for few panels but drops a lot for many panels
  - It remains to decide which number of panels is acceptable



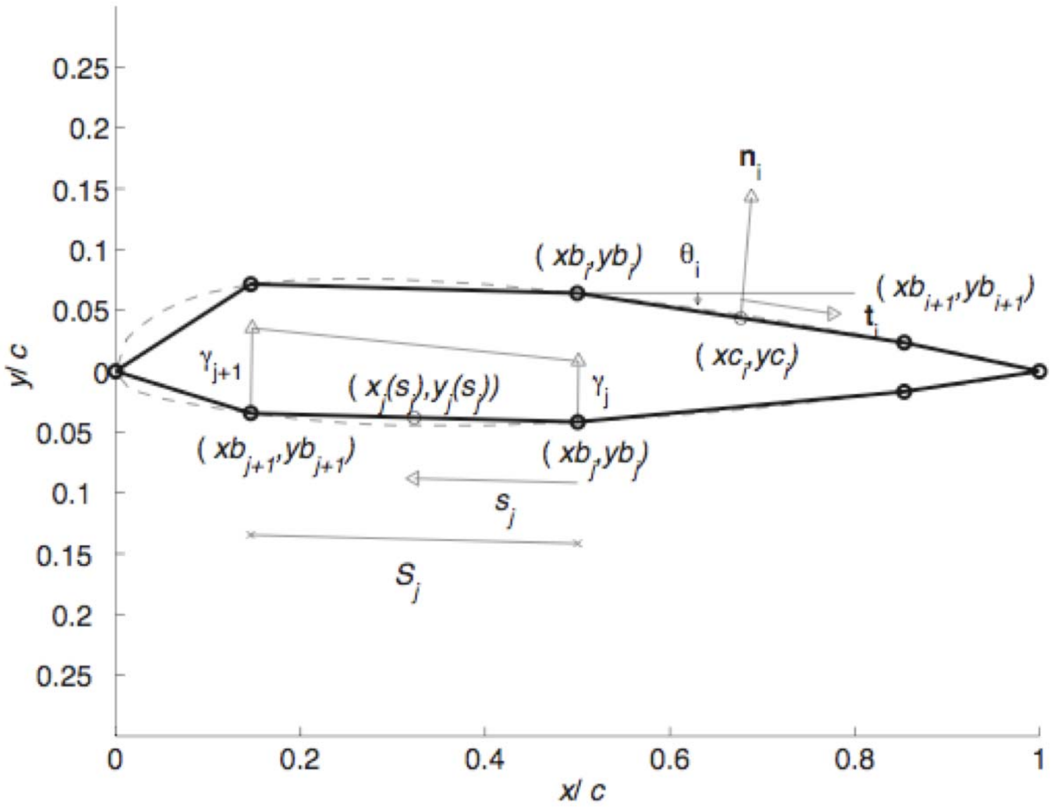
Higher order accuracy

- The panel methods shown here have a constant strength (source or vortex) on every panel.
- Higher orders of accuracy can be obtained if the singularity strength is allowed to vary.
- For example, an airfoil can be modeled using vortex panels only with linearly varying vorticity.

Linearly varying vortex panels

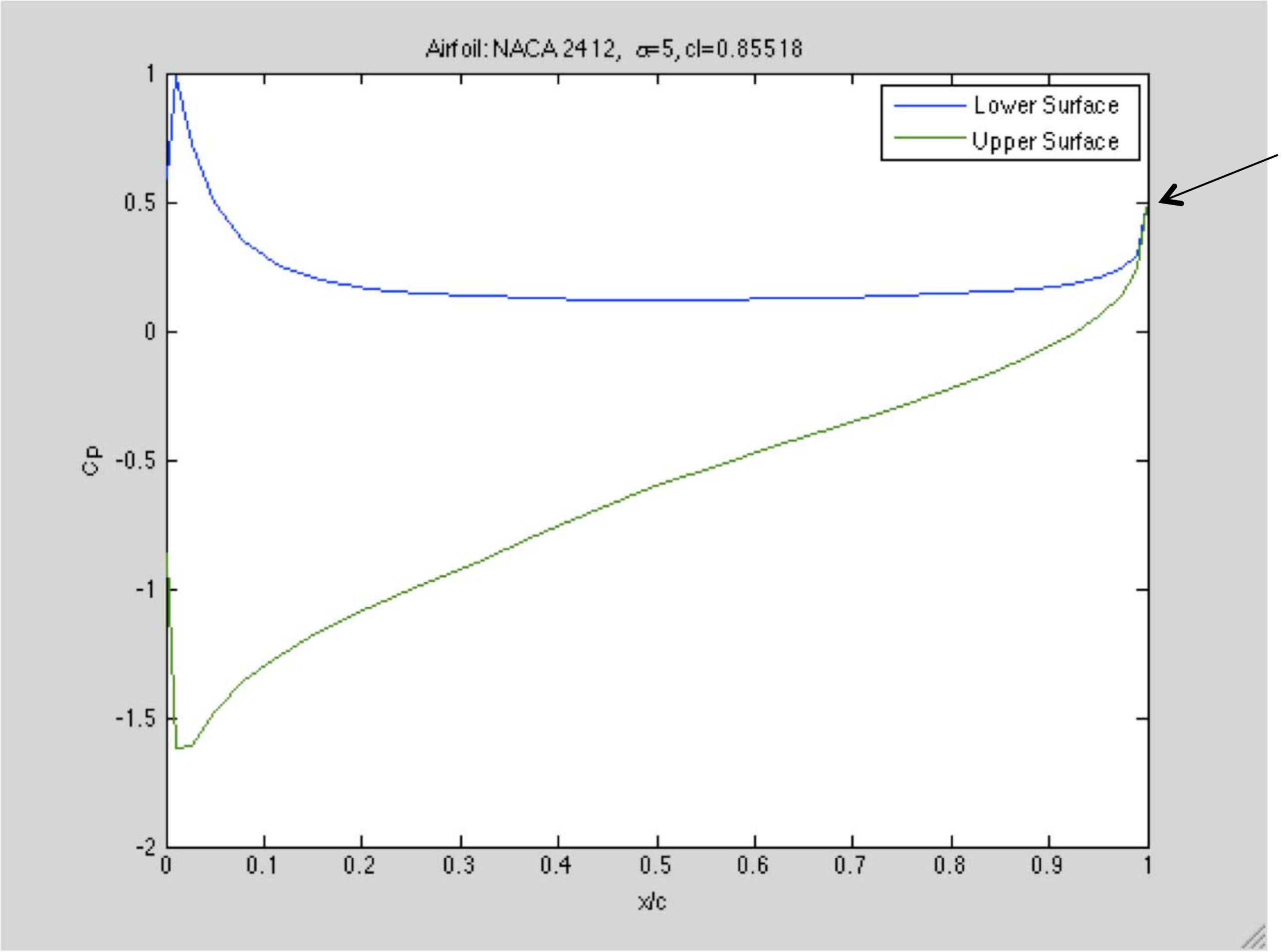
Now there are m+1 unknowns, γ_i, with m boundary conitions and one Kutta condition.

The Kutta condition states that the vorticity at the trailing edge must be zero, i.e. $\gamma_{m+1}+\gamma_1=0$.



Example

- NACA 2412 airfoil at 5º angle of attack
- 50 panels



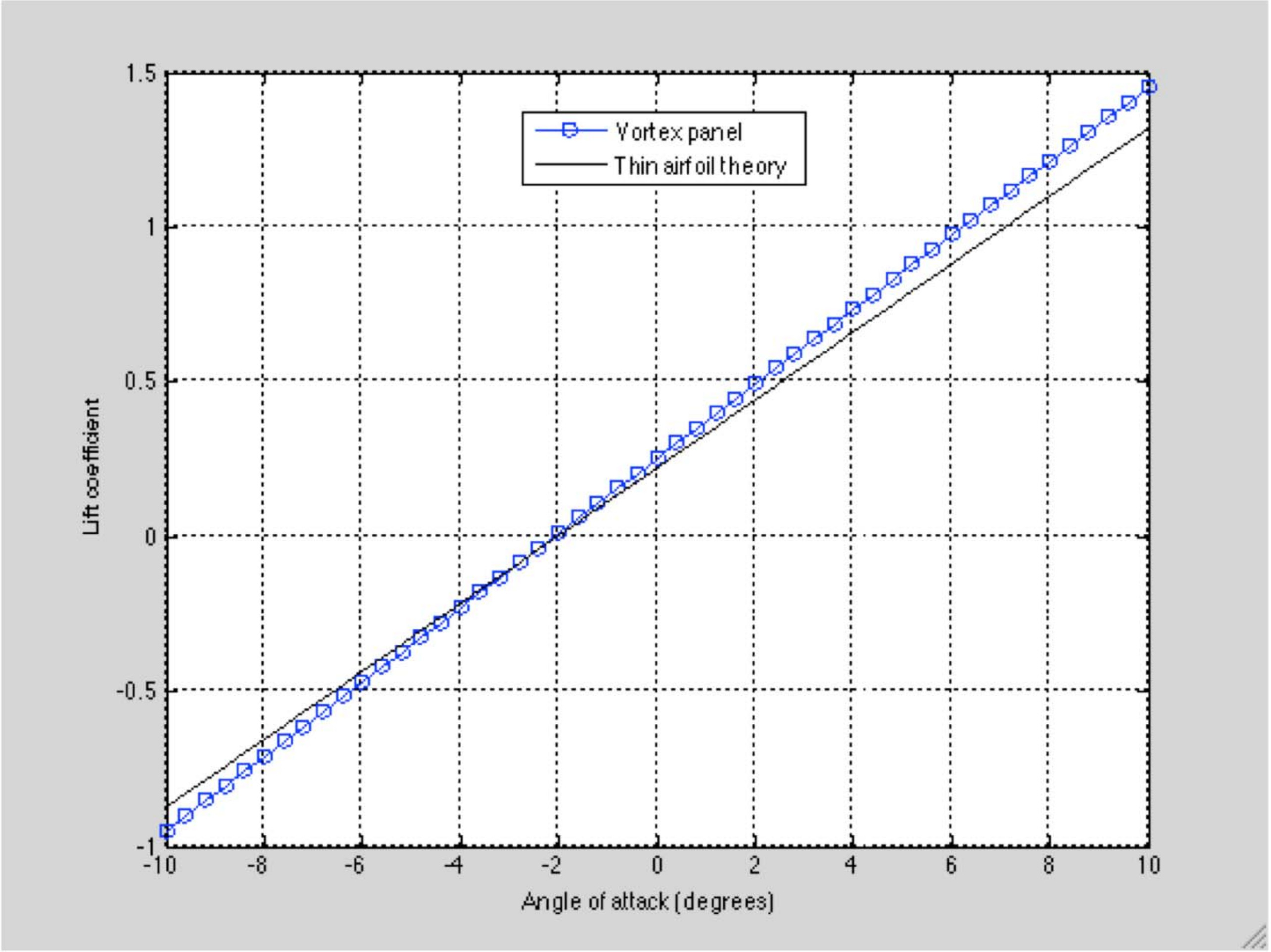
Trailing edge control points. The pressure coefficient is closer to 1

Observations

- Panel methods allow the modeling of any airfoil shape, as long as the coordinates of the airfoil are known.
- As they are numerical methods, their results depend on parameters, such as the number, order and choice of panels.
- Second order panels, i.e. panels with quadratically varying singularity strength are even more accurate.
- Panel methods are supposed to be fast and easy to implement:
  - Increasing the order and increasing the number of panels too much will render these methods so computationally expensive that their main advantage, speed, is lost.

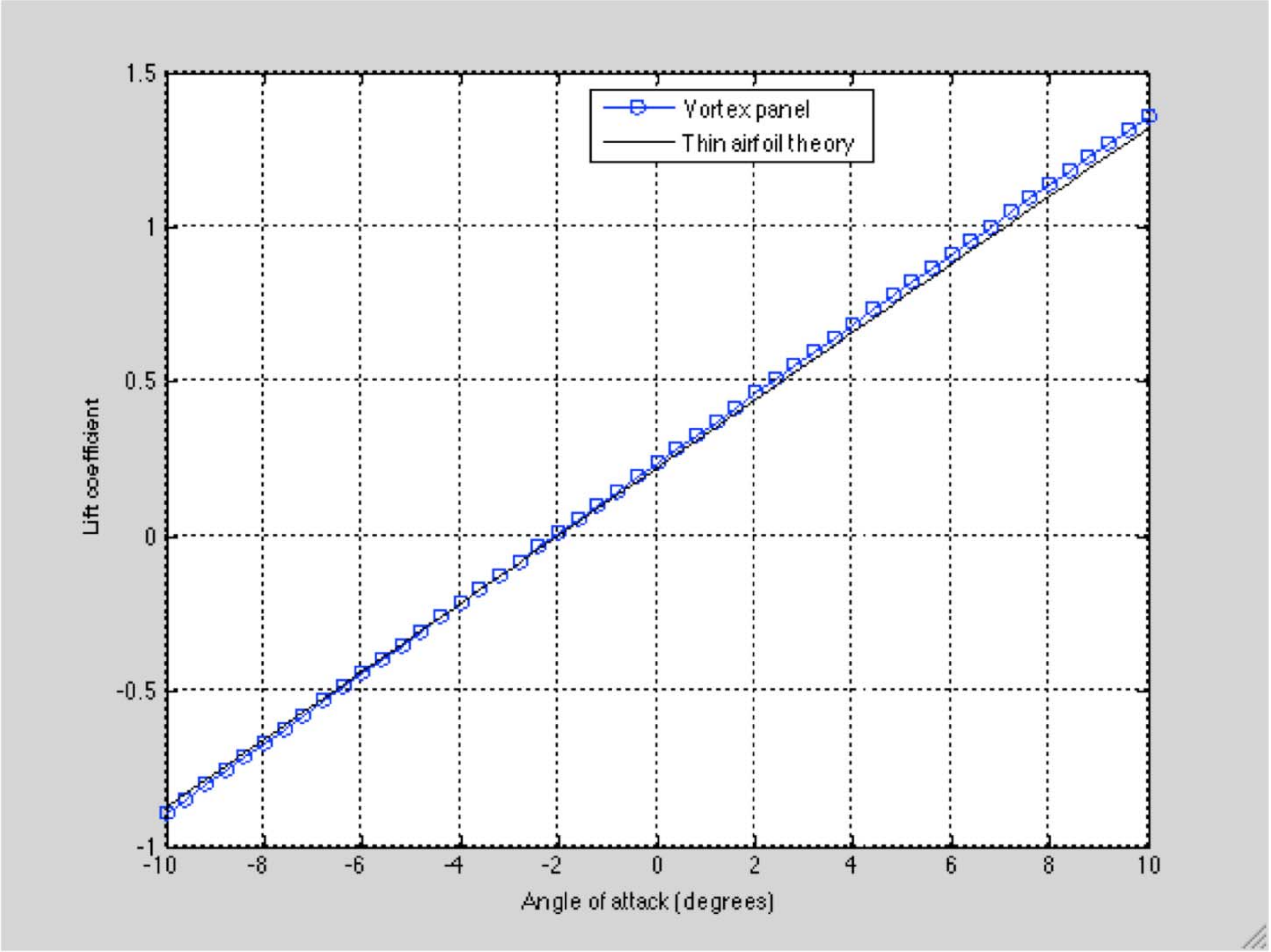
Comparison with thin airfoil theory – NACA 2412

- The zero-lift angles are identical
- The lift-curve slopes are different
- Thin airfoil theory cannot account for thickness effects



Comparison with thin airfoil theory – NACA 2404

- The lift-curve slopes are much more similar
- Clearly, 12% thickness is too much for thin airfoil theory.
- At 4% thickness, the thin airfoil theory is much more representative



XFOIL

- XFOIL is a free panel method software developed by Mark Drela at MIT.
- Website: http://web.mit.edu/drela/Public/web/xfoil/
- It can model the flow around any 2D airfoil using panel methods. It can also:
  - Perform corrections for viscosity
  - Perform corrections for compressibility
  - Design an airfoil given specifications

NACA 2412 at 5º