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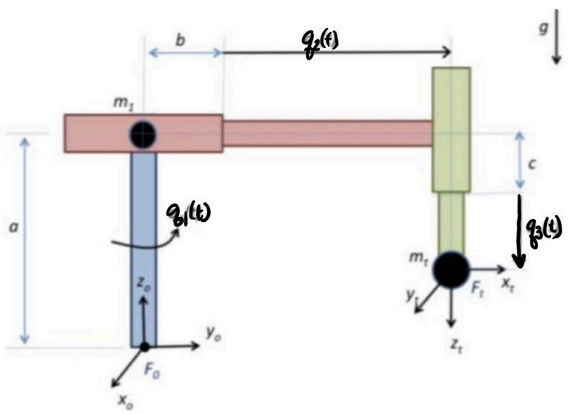
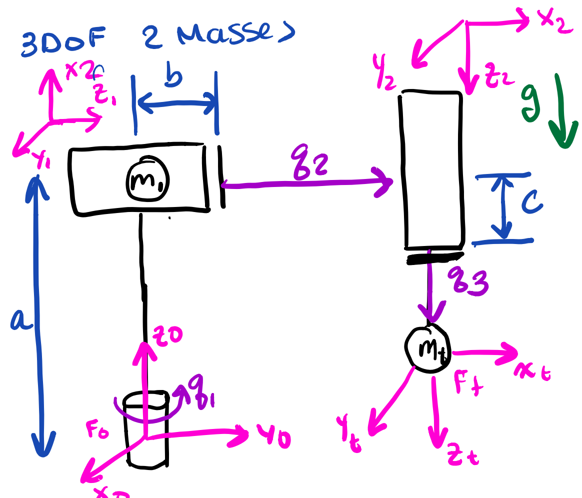
RBE 501 Final

Problem 1

Consider the three-DOF arm in the figure below. Consider the links to be massless, that is, consider only the point masses labeled m_1 and m_t below.



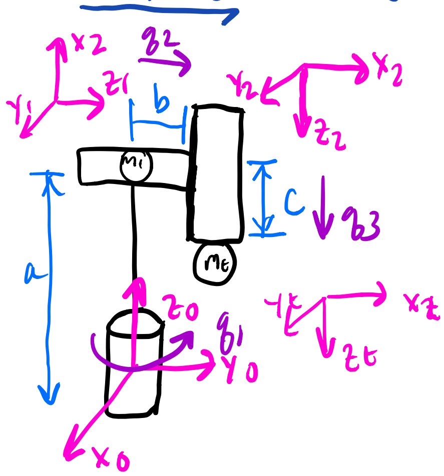


Can Consid 3 mass where $m_3 = m_t$ $m_2 = 0$ $I = 0$

a) Derive the dynamical model of the robot using Lagrange's approach and put it in the form of $\boldsymbol{\tau} = M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + g(\boldsymbol{q})$. **(30 points)**

at home

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$



Slist	ω	v
1	(0,0,1)	(0,0,0)
2	(0,0,0)	(0,1,0)
3	(0,0,0)	(0,0,1)

$$Slist = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{01} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

include b?

$$T_{01} = T\{1\} M_{01} = \begin{bmatrix} 0 & \cos(q_1) & -\sin(q_1) & 0 \\ 0 & \sin(q_1) & \cos(q_1) & 0 \\ 1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{02} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & 0 & -1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

include c?

$$T_{02} = T\{1\} T\{2\} M_{02} = \begin{bmatrix} -\sin(q_1) & \cos(q_1) & 0 & -\sin(q_1)*(b+q_2) \\ \cos(q_1) & \sin(q_1) & 0 & \cos(q_1)*(b+q_2) \\ 0 & 0 & -1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{03} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & 0 & -1 & a-c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{03} = T\{1\} T\{2\} T\{3\} M_{03} = \begin{bmatrix} -\sin(q_1) & \cos(q_1) & 0 & -\sin(q_1)*(b+q_2) \\ \cos(q_1) & \sin(q_1) & 0 & \cos(q_1)*(b+q_2) \\ 0 & 0 & -1 & a-c+q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Position vectors Position kinematics

$$O_1 = T_{01}(1:3,4) = \begin{bmatrix} 0 \\ 0 \\ a \\ 1 \end{bmatrix}$$

$$O_2 = T_{02}(1:3,4) = \begin{bmatrix} -\sin(q_1)*(b+q_2) \\ \cos(q_1)*(b+q_2) \\ a \\ 1 \end{bmatrix}$$

$$O_3 = T_{03}(1:3,4) = \begin{bmatrix} -\sin(q_1)*(b+q_2) \\ \cos(q_1)*(b+q_2) \\ a-c+q_3 \\ 1 \end{bmatrix}$$

Now I'll do the velocity kinematics

For m_1:

$$v_1 = \frac{d}{dt}\theta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For m_2: I'm assuming $m_2 = 0$

$$v_2 = \frac{d}{dt}\theta_2 = \begin{bmatrix} -\dot{q}_2 \sin(q_1) - \dot{q}_1 \cos(q_1) * (b+q_2) \\ \dot{q}_2 \cos(q_1) - \dot{q}_1 \sin(q_1) * (b+q_2) \\ 0 \end{bmatrix}$$

For m_3: $m_3 = m_t$

$$v_3 = \frac{d}{dt}\theta_3 = \begin{bmatrix} -\dot{q}_2 \sin(q_1) - \dot{q}_1 \cos(q_1) * (b+q_2) \\ \dot{q}_2 \cos(q_1) - \dot{q}_1 \sin(q_1) * (b+q_2) \\ \dot{q}_3 \end{bmatrix}$$

$\boxed{\text{Step 1}}$ Find lagrangian

$L = K - P$

Find kinetic energy: $K = \frac{1}{2}Mv^2$

$K_1 = \frac{1}{2} m_1 v_1^T v_1 = 0$

$K_2 = \frac{1}{2} m_2 v_2^T v_2$

$= \frac{1}{2} m_2 \left[\dot{q}_2 \sin(q_1) + \dot{q}_1 \cos(q_1)(b+q_2)\right]^2 + \frac{1}{2} m_2 \left[\dot{q}_2 \cos(q_1) - \dot{q}_1 \sin(q_1)(b+q_2)\right]^2$

$K_3 = \frac{1}{2} m_3 v_3^T v_3$

$= \frac{1}{2} m_3 \left[\dot{q}_2 \sin(q_1) + \dot{q}_1 \cos(q_1)(b+q_2)\right]^2 + \frac{1}{2} m_3 \left[\dot{q}_2 \cos(q_1) - \dot{q}_1 \sin(q_1)(b+q_2)\right]^2$

$+ \frac{1}{2} m_3 \dot{q}_3$

Total kinetic energy

$K = K_1 + K_2 + K_3$

Find Potential Energy: $P = mg\,O(3)$ Position in z dir

$P_1 = m_1 g a$

$P_2 = m_2 g a$

$P_3 = m_3 g (a - c + q_3)$

Total Potential energy

$P = P_1 + P_2 + P_3 = m_3 g(a - c + q_3) + m_2 g a + m_1 g a$

$L = K - P$

$\boxed{\text{Step 2}}$ lagrangian equation

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1}$$

$$= \ddot{q}_1 (b + q_2)^2 (m_2 + m_3) + 2\dot{q}_1 \dot{q}_2 (b + q_2)(m_2 + m_3)$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2}$$

$$= \ddot{q}_2 (m_2 + m_3) + m_2 \dot{q}_1 \sin(q_1)\left[\dot{q}_2 \cos(q_1) - \dot{q}_1 \sin(q_1)(b + q_2)\right]$$

$$+ m_3 \dot{q}_1 \sin(q_1)\left[\dot{q}_2 \cos(q_1) - \dot{q}_1 \sin(q_1)(b + q_2)\right]$$

$$- m_2 \dot{q}_1 \cos(q_1)\left[\dot{q}_2 \sin(q_1) + \dot{q}_1 \cos(q_1)(b + q_2)\right]$$

$$- m_3 \dot{q}_1 \cos(q_1)\left[\dot{q}_2 \sin(q_1) + \dot{q}_1 \cos(q_1)(b + q_2)\right]$$

$$\tau_3 = m_3 g + m_3 \ddot{q}_3$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$M = \begin{bmatrix} (b+q_2)^2(m_2+m_3) & 0 & 0 \\ 0 & m_2+m_3 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ g m_3 \end{bmatrix}$$

$$C = \begin{bmatrix} 2\dot{q}_1 \dot{q}_2 (b+q_2)(m_2+m_3) \\ -\dot{q}_1^2 (b+q_2)(m_2+m_3) \\ 0 \end{bmatrix}$$

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} (b+q_2)^2(m_2+m_3) & 0 & 0 \\ 0 & m_2+m_3 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 2\dot{q}_1 \dot{q}_2 (b+q_2)(m_2+m_3) \\ -\dot{q}_1^2 (b+q_2)(m_2+m_3) \\ 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

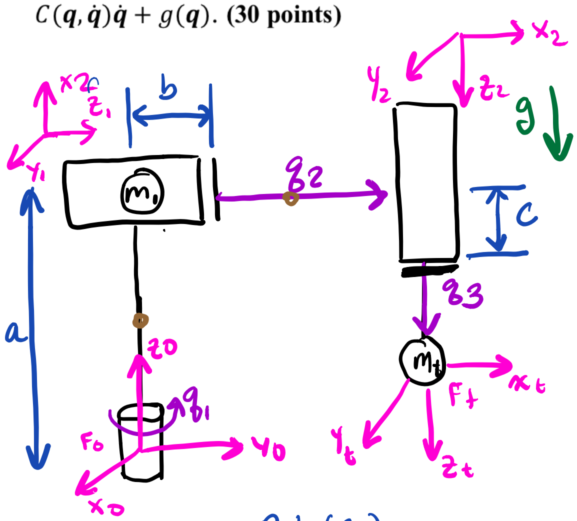
$$+ \begin{bmatrix} 0 \\ 0 \\ g m_3 \end{bmatrix}$$

<u>Where $m_2 = 0$</u>

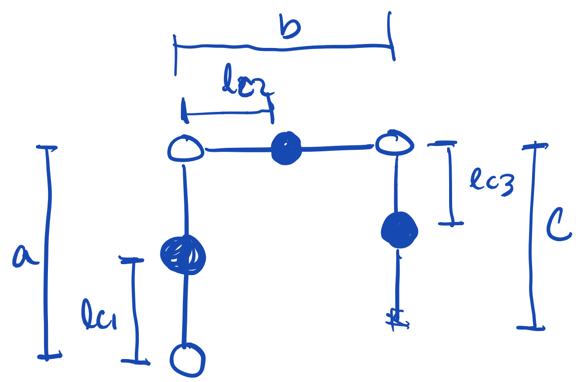
$$
\boxed{\begin{aligned}
\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} &= \begin{bmatrix} (b+q_2)^2(m_3) & 0 & 0 \\ 0 & m_3 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} \\
&+ \begin{bmatrix} 2\dot{q}_1\dot{q}_2(b+q_2)(m_3) \\ -\dot{q}_1^{2}(b+q_2) \quad (m_3) \\ 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ g m_3 \end{bmatrix}
\end{aligned}}
$$

$$
\tau = \begin{bmatrix} m_3(b+q_2)(2\dot{q}_2\dot{q}_1^{2} + b\ddot{q}_1 + q_2\ddot{q}_1) \\ -m_3(b\dot{q}_1^{2}\dot{q}_2 - \ddot{q}_2 + q_2\dot{q}_1^{2}\dot{q}_2) \\ m_3(g + \ddot{q}_3) \end{bmatrix}
$$

b) Derive the dynamical model using Newton's approach and put it in the form of $\boldsymbol{\tau} = M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + g(\boldsymbol{q})$. **(30 points)**



$$\omega_0 = \alpha_0 = a_{c0} = a_{ce} = 0$$



$$R_{01} = \overset{Rot_z(q_1)}{\begin{bmatrix} \cos(q_1) & -\sin(q_1) & 0 \\ \sin(q_1) & \cos(q_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$l_{c1} = a$

$l_{c2} = 0$

$l_{c2} = q_3$

$$R_{12} = \overset{M_1^2}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \overset{Rot_z(q_2)}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_2^0 = R_1^0 R_2^1 = \begin{bmatrix} 0 & \cos(q_1) & -\sin(q_1) \\ 0 & \sin(q_1) & \cos(q_1) \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{23} = \overset{M_2^1}{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \overset{Rot_z(q_3)}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_3^0 = R_2^0 R_3^2 = \begin{bmatrix} -\sin\theta_1 & \cos\theta_1 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -\sin(q_1) \\ \cos(q_1) \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Forward Recurssion

$$\omega_i = (R_i^{i-1})^T \omega_{i-1} + b_i \dot{q}_i$$

$$b_i = (R_i^0)^T z_{i-1}$$

$$b_1 = (R_1^0)^T z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \omega_1 = (R_1^0)^T \omega_0 + b_1 \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

$$b_2 = (R_2^0)^T z_1 = \begin{bmatrix} 0 \\ 0 \\ \cos^2(q_1) + \sin^2(q_1) \end{bmatrix} \qquad \omega_2 = (R_2^1)^T \omega_1 + b_2 \dot{q}_2 = \begin{bmatrix} \dot{q}_1 \\ 0 \\ \dot{q}_2(\cos^2(q_1) + \sin^2(q_1)) \end{bmatrix}$$

$$b_3 = (R_3^0)^T z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \omega_3 = (R_3^2)^T \omega_2 + b_3 \dot{q}_3 = \begin{bmatrix} \dot{q}_2(\cos^2(q_1) + \sin^2(q_1)) \\ 0 \\ \dot{q}_3 - \dot{q}_1 \end{bmatrix}$$

$$\alpha_i = (R_i^{i-1})^T \alpha_{i-1} + b_i \ddot{q}_i + \omega_i \times b_i \dot{q}_i$$

$$\alpha_1 = (R_1^0)^T \alpha_0 + b_1 \ddot{q}_1 + \omega_1 \times b_1 \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 \end{bmatrix}$$

$$\alpha_2 = (R_2^1)^T \alpha_1 + b_2 \ddot{q}_2 + \omega_2 \times b_2 \dot{q}_2 = \begin{bmatrix} \ddot{q}_1 \\ -\dot{q}_1 \dot{q}_2(\cos^2(q_1) + \sin^2(q_1)) \\ \ddot{q}_2(\cos^2(q_1) + \sin^2(q_1)) \end{bmatrix}$$

$$\alpha_3 = (R_3^2)^T \alpha_2 + b_3 \ddot{q}_3 + \omega_3 \times b_3 \dot{q}_3 = \begin{bmatrix} \ddot{q}_2(\cos^2(q_1) + \sin(q_1)^2) \\ -\dot{q}_1 \dot{q}_2(\cos^2(q_1) + \sin^2(q_1)) - \dot{q}_2 \dot{q}_3(\cos^2(q_1) + \sin^2(q_1)) \\ \ddot{q}_3 - \ddot{q}_1 \end{bmatrix}$$

c) Consider the following physical parameters for the robot: $a = 0.3\ m$, $b = c = 0.1\ m$, $m_1 = m_2 = 0.5\ kg$, $g = 9.8$. Substitute these parameters in the models derived in parts a and b and compare the results. **(10 points)**

(See matlab)

d) Plan a trajectory such that the end-effector starts at $t = 0s$ with the initial position of $x_{ti} = (0, 200, 150)\ mm$ with zero velocity, and while following a straight line, ends at $t = 10s$ with the final position of $x_{tf} = (200, 0, 200)\ mm$ with zero velocity. Plot the time-position, time-velocity, and time-acceleration trajectory graphs for all the joints from $t = 0$ to $t = 10$ seconds (joint space). Also, plot the resultant end-effector path/trajectory using the forward kinematics i.e. track the position of the end-effector (x-y-z) as a result of your solution (task space) within the motion. Show both your solution and the desired trajectory together in a single graph (3D) for comparison. **(30 points)**

$t_0 = 0s$ $\qquad$ $t_f = 10s$

$x_{ti} = (0, 200, 150)$ $\qquad$ $x_{tf} = (200, 0, 200)$

$V_i = 0$ $\qquad$ $V_f = 0$

$$x_{ti} = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3$$

$$\dot{x}_{ti} = a_1 + 2a_2 t_0 + 3a_3 t_0^2 = 0$$

$$x_{tf} = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$$

$$\dot{x}_{tf} = a_1 + 2a_2 t_f + 3a_3 t_f^2 = 0$$

$t_0 = 0$, $x_{ti} = (0, 200, 150)$, $\dot{x}_{ti} = 0$

$$\begin{bmatrix} 0 \\ 200 \\ 150 \end{bmatrix} = a_0 + \cancel{a_1 t_0} + \cancel{a_2 t_0^2} + \cancel{a_3 t_0^3} \Rightarrow \boxed{a_0 = \begin{bmatrix} 0 \\ 200 \\ 150 \end{bmatrix}}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a_1 + \cancel{2a_2 t_0} + \cancel{3a_3 t_0^2} = 0 \Rightarrow \boxed{a_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$t_f = 10$, $x_{tf} = (200, 0, 200)$, $\dot{x}_{tf} = 0$

$$\begin{bmatrix} 200 \\ 0 \\ 200 \end{bmatrix} = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$$

$$= \begin{bmatrix} 0 \\ 200 \\ 150 \end{bmatrix} + \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}(10)} + a_2(10^2) + a_3(10)^3$$

$$\begin{bmatrix} 200 \\ -200 \\ 50 \end{bmatrix} = 100 a_2 + 1000 a_3$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \cancel{a_1} + 2a_2 t_f + 3a_3 t_f^2$$

$$\left[\begin{bmatrix}0\\0\\0\end{bmatrix} = 20a_2 + 300a_3\right] \times 5 \Rightarrow \begin{bmatrix}0\\0\\0\end{bmatrix} = 100a_2 + 1500a_3$$

$$\left[\begin{bmatrix}200\\-200\\50\end{bmatrix} = \cancel{100a_2} + 1000a_3\right]$$

$$-\left[\begin{bmatrix}0\\0\\0\end{bmatrix} = \cancel{100a_2} + 1500a_3\right]$$

$$\begin{bmatrix}200\\-200\\50\end{bmatrix} = -500a_3 \Rightarrow a_3 = \begin{bmatrix}-2/5\\2/5\\-\frac{1}{10}\end{bmatrix}$$

$$\begin{bmatrix}20\cancel{0}\\-20\cancel{0}\\5\cancel{0}\end{bmatrix} = 10\cancel{0}a_2 + 100\cancel{0}\begin{bmatrix}-2/5\\2/5\\-1/10\end{bmatrix}$$

$$\boxed{a_3 = \begin{bmatrix}-0.4\\0.4\\-0.1\end{bmatrix}}$$

$$\begin{bmatrix}20\\-20\\5\end{bmatrix} = 10a_2 + \begin{bmatrix}-40\\40\\-10\end{bmatrix} \Rightarrow \begin{bmatrix}60\\-60\\15\end{bmatrix} = 10a_2$$

$$\Rightarrow \boxed{a_2 = \begin{bmatrix}6\\-6\\1.5\end{bmatrix}}$$

$$a_0 = \begin{bmatrix} 0 \\ 200 \\ 150 \end{bmatrix} \quad a_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 6 \\ -6 \\ 1.5 \end{bmatrix} \quad a_3 = \begin{bmatrix} -0.4 \\ 0.4 \\ -0.1 \end{bmatrix}$$

$$P(t) = a_0 + \cancelto{0}{a_1 t} + a_2 t^2 + a_3 t^3$$

$$P(t) = \begin{bmatrix} 0 \\ 200 \\ 150 \end{bmatrix} + \begin{bmatrix} 6 \\ -6 \\ 1.5 \end{bmatrix} t^2 + \begin{bmatrix} -0.4 \\ 0.4 \\ -0.1 \end{bmatrix} t^3$$