RBE502 – Homework Set 1

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Problem 1

Equation (1) displays the differential equation for a point pendulum system. Here, m is the mass of the bob, l is the link length and u is the torque input applied at the base of the pendulum.

$$ml^2\ddot{q} = mgl\sin(q) + u \tag{1}$$

The equation for the control function, u, is defined by equation (2). Here

$$u = -mgl\sin(q) - k_1q - k_2\dot{q} \tag{2}$$

Equation (3) combines and simplifies equations (1) and (2).

$$ml^2\ddot{q} + k_2\dot{q} + k_1q = 0 \tag{3}$$

This is a second order linear differential equation. Therefore it can be solved by substituting an exponential function for$q$. Equations (4), (5) and (6) display this exponential function and its derivatives.

$$q = e^{nt} \tag{4}$$

$$\dot{q} = ne^{nt} \tag{5}$$

$$\ddot{q} = n^2e^{nt} \tag{6}$$

Equations (4), (5) and (6) can now be substituted into equation (3) to form equation (7).

$$ml^2n^2e^{nt} + k_2ne^{nt} + k_1e^{nt} = 0 \tag{7}$$

This equation can now be solved for n. Since all of the terms have the exponential it can be factored out, see equation (8)

$$e^{nt}(ml^2n^2 + k_2n + k_1) = 0 \tag{8}$$

Sincee^{nt}, will never equal 0, the expression inside of the parenthesis must be equal to 0, see equation (9).

$$ml^2n^2 + k_2n + k_1 = 0 \tag{9}$$

The equation to find n, equation (10), is obtained using the quadratic formula.

$$n = \frac{k_2 \pm \sqrt{{k_2}^2 - 4ml^2k_1}}{2ml^2} \tag{10}$$

Therefore two solutions for the differential equation are shown in equations (11) and (12)

$$q_1 = C_1 e^{\frac{k_2+\sqrt{{k_2}^2-4ml^2k_1}}{2ml^2}} \tag{11}$$

$$q_2 = C_2 e^{\frac{k_2-\sqrt{{k_2}^2-4ml^2k_1}}{2ml^2}} \tag{12}$$

The general solution for the system illustrated in equation (13).

$$q(t) = C_1 e^{\frac{k_2+\sqrt{{k_2}^2-4ml^2k_1}}{2ml^2}} + C_2 e^{\frac{k_2-\sqrt{{k_2}^2-4ml^2k_1}}{2ml^2}} \tag{13}$$

One of the cons of this system is that the real part of the eigenvector is positive, therefore the system is unstable. Another pro is that the angled components cancel each other out in the differential equations and therefore this problem becomes linear. This produces a very simple linear homogeneous solution that is simple to solve.

Problem 2

Equation (1) displays the differential equation for the 1D sliding cart system with the control function substituted. Here m is the mass of the cart, $\phi(t)$ is the disturbance force applied to the cart, s is the desired speed of the cart, k_p is the proportional gain, and k_d is the derivative gain.

$$(m + k_d)\dot{v} + k_p v = k_p s + \phi(t) \tag{1}$$

Equation (1) becomes equation (2) if rewritten in standard form.

$$\dot{v} + \frac{k_p}{(m+k_d)} v = k_p s + \phi(t) \tag{2}$$

Equation (2) can be rewritten so as to solve for$v(t)$, see equation (3).

$$v = \frac{1}{\mu} \int \mu(t) \left(\frac{k_p s + \phi(t)}{(m+k_d)} \right) dt + \frac{C}{\mu} \tag{3}$$

This is a first order differential equation, the integration factor method can be used to solve for$v(t)$. Equation for the integration factor is show in equation (4)

$$\mu = e^{\int \frac{k_p}{m+k_d} dt} = e^{\frac{k_p t}{m+k_d}} \tag{4}$$

The next part of this method involves multiplying both sides of the equation (3) by the integration factor, equation (4), this is seen in equation (5).

$$v = e^{\frac{-k_p t}{m+k_d}} \left(\int e^{\frac{k_p t}{m+k_d}} \left(\frac{k_p s + \phi(t)}{(m+k_d)} \right) dt \right) + C e^{\frac{-k_p t}{m+k_d}} \tag{5}$$

Equation (5) can be further simplified to equation (6)

$$v = e^{\frac{-k_p t}{m+k_d}}\left(\int \frac{k_p s e^{\frac{k_p t}{m+k_d}}}{m+k_d}dt + \int \frac{\phi(t) e^{\frac{k_p t}{m+k_d}}}{m+k_d}dt\right) + Ce^{\frac{-k_p t}{m+k_d}} \tag{6}$$

By solving the two integral expressions in equation (6) you get equation (7).

$$v = e^{\frac{-k_p t}{m+k_d}}\left(se^{\frac{k_p t}{m+k_d}} + \frac{\phi(t) e^{\frac{k_p t}{m+k_d}}}{k_p} - \frac{1}{k_p}\int e^{\frac{k_p t}{m+k_d}}\dot{\phi(t)}\, dt\right) + Ce^{\frac{-k_p t}{m+k_d}} \tag{7}$$

Then by crossing out like terms you get equation (8)

$$v = s + \frac{\phi(t)}{k_p} - \frac{e^{\frac{-k_p t}{m+k_d}}}{kp}\int e^{\frac{k_p t}{m+k_d}}\,\dot{\phi(t)}dt + Ce^{\frac{-k_p t}{m+k_d}} \tag{8}$$

In order to find the upper bound of $\lceil v(t) \rceil$, the limit as $t \to \infty$ is taken of equation (8), shown in equation (9).

$$\lim_{t\to\infty}\lceil v(t) \rceil = s + \frac{\phi(t)}{k_p} - \frac{e^{-\infty}}{kp}\int e^{\infty}\,\dot{\phi(t)}dt + Ce^{-\infty} \tag{8}$$

Since $e^{\infty} = 0$, the equation reduces to equation (9):

$$\lim_{t\to\infty}\lceil v(t) \rceil = s + \frac{\phi(t)}{k_p} \tag{9}$$

Since $\phi(t) < a$, the upper bound of the velocity is shown in equation (10)

$$\lceil v(t) \rceil \leq s + a \tag{10}$$

Problem 3

Part 1

Figure 1 illustrates the response of the system for ϕ (e) = e and r = 1 for the parameters listed in Table 1.

Simulation time	$t_0 = 0, t_f = 10$
Initial conditions	$y(t_0) = \dot{y}(t_0) = 0$
System parameters	$m = 1, b = 2,$ and $k = 4$
Reference Input	$r = 1$

Table 1: System parameters used in the simulation

Based on the simulation results, we have $e(10)$ ≈0.20, this means that difference between the desired position value, *r*, and the actual position value, *y (t)*, is a proximately 0.20. Ideally we would like for $e(t)$ ≈0 so that *r = y (t)*. With the controller in this problem, this would not be the case and therefore the mass will not stay in the desired positon. In quiz 1, that the limit as $e(t)$ is the expression shown in equation (1)

$$\lim_{t\to\infty} e(t) = \frac{kr}{k+1} \tag{1}$$

From this we know that the error will not equal zero and that the mass will not stay at the desired position. The answer of $e(10)$ ≈0.20 corroborates this finding, as time progresses $e(t)$ is not zero, therefore the mass is not staying at the desired position.

Plugging in the values *k = 4* and *r =1* into equation (1), you get equation (2)

$$\lim_{t\to\infty} e(t) = \frac{kr}{k+1} = \frac{4\cdot 1}{4+1} = \frac{4}{5} = 0.80$$

Since *y(t) = r - e(t)*, you get equation (3)

$$y = r - e = 1 - 0.8 = 0.20$$

The result of equation (3) agrees with the answer that was obtained in the quiz. The value of obtained for y agrees with what'd we'd expect by using equation 1 from the quiz.

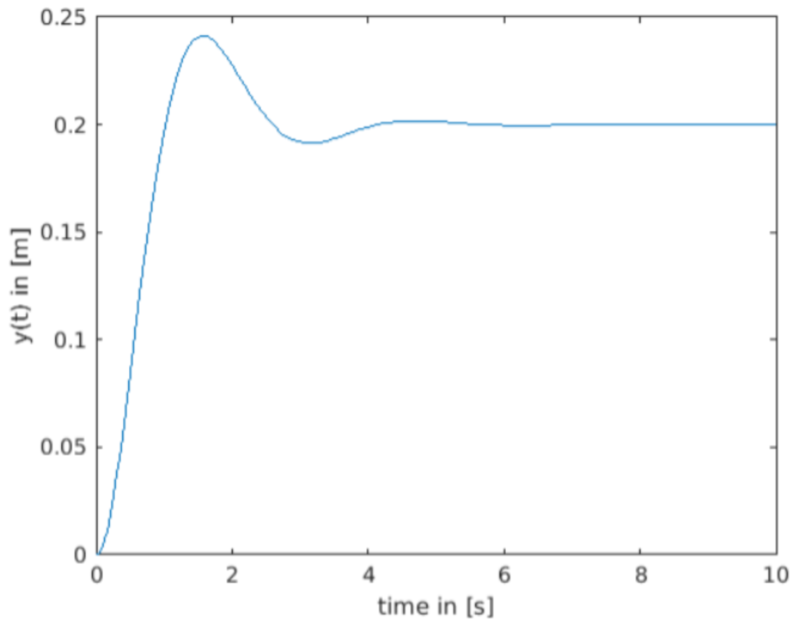


Figure 1: System response for $\phi(t)(e) = e$ and $r = 1$

Part 2

Figure 2 illustrates the response of the system to PID controller

$$u(t) = k_p e(t) + k_i \int_{t_0}^{t} e(\tau)d\tau + k_d \dot{e} \qquad (1)$$

For the 3 case studies as listed in Table 2.

Case Study	k_p	k_i	k_d	$e(t_f)$	J	$u(t_f)$
I	5	0	0	0.44	4.57	5.57
	10	0	0	0.29	2.96	3.96
	20	0	0	1.90	1.90	2.74
II	10	2	0	0.030	1.04	1.79
	10	5	0	-0.0032	0.97	0.81
	10	10	0	-0.0021	1.64	0.40

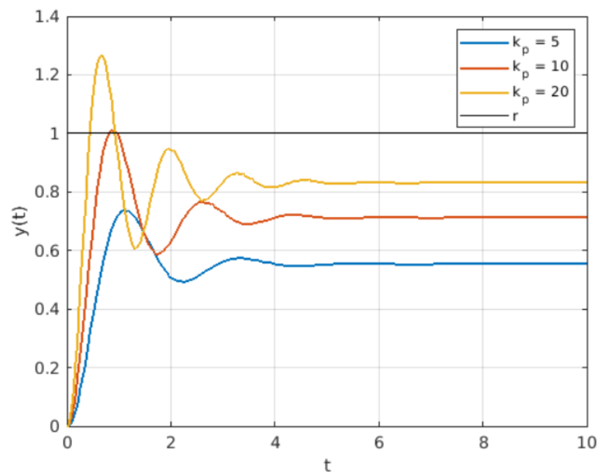
III	10	0	2	0.29	3.06	4.06
	10	0	5	0.29	3.21	4.21
	10	0	10	0.29	3.47	4.47

Table 2: Results of the three case study for part 2

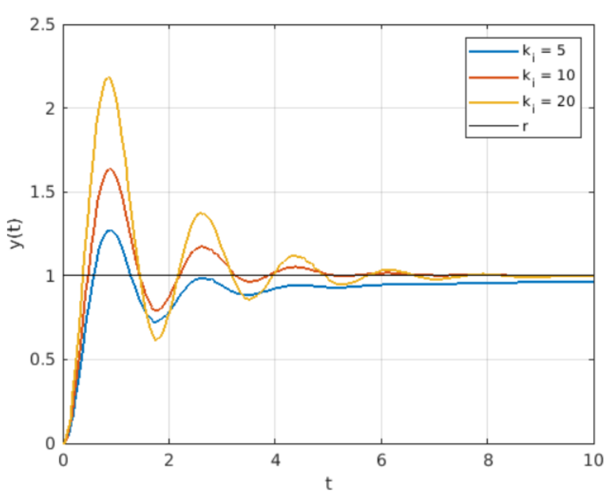
The performance index $J \in \mathrm{R}^+$ defined as

$$J = \int_{t_0}^{t_f} |e(t)|dt = \int_{t_0}^{t_f} |r(t) - y(t)|dt \tag{2}$$

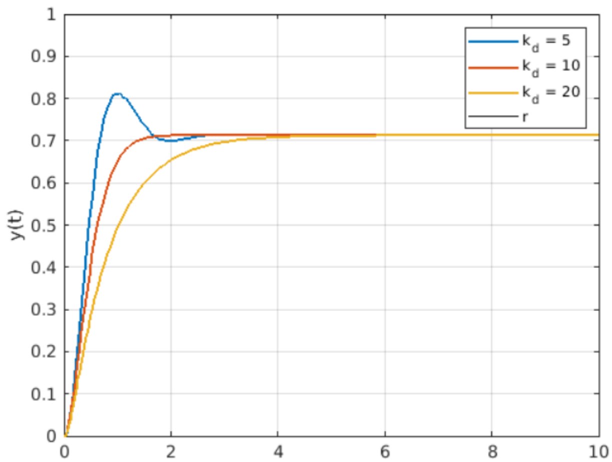
The performance index, *J*, is a quantitative measure of the performance of a system and it is chosen to give emphasis on the desired outcome. The performance index is the integral of the error function from the start time to the stop time. It is the sum of all error over that time. Ideally we would like to have the value of *J* to be small because it means we are closer to obtaining the desired value on average, than we would if *J* were larger.



(a)Case study I



(b) Case study II



(c) Case study III

Figure 2: System response to PID controller for the three case studies of Part 2

When looking at the three studies depicted in Figure 2, it can be seen that Case study II seems to be the most ideal. The value for *y(t)* seems to converge to 1, which is the desired value *r*. In the second question of the quiz, a value *h* that could reduce the error to zero in time was found, see equation (3) and (4). The equation for the control function u, is shown in equation (5).

$$e = r - \frac{r+h}{k+1} \tag{3}$$

$$h = kr \tag{4}$$

$$u = e + h \tag{5}$$

If we substitute the values corresponding to this system of *k = 4* and *r = 1* into equations (3),(4) and (5), we can verify if the *h* really does reduce the error to zero with time and the control function reduces to *h*.

$$h = 4 \cdot 1 = 4$$

$$e = 1 - \frac{1+4}{4+1} = 1 - 1 = 0$$

$$u = 0 + 4 = 4$$

Importing system's values into the equation derived in problem 2 of the quiz, it is evident that the equation is valid and that incorporating the *h* value into the error calculation does indeed reduce the value of the error to zero. Also, the control function, u , does indeed equal *h* which is obtained through equation (4).The values of for $u(t_f)$ obtained for case study II represent *h*, a value that reduces the error value of the system to zero as time progresses.

Comparing plots (a), (b) and (c) of Figure 2, one can compare the effects of the three different gains on the system response. Increasing the proportional gain will increase the value of y that the system settles on as time goes to infinity it also increased the performance index value. Therefore the more you increase the proportional gain the worse performance you get because the response is deviating more and more from the desired valued. Increasing the integral gain increases the error early on in the response but then settles down and converges to roughly the same value as if the integral gain was less. Increasing the integral gain, also increases the overall performance index but not as much as increasing the proportional gain would. Increasing the derivative gain decreases the performance index, which is good, but it is still higher than obtained when increasing the integral gain by similar amounts. Increasing the derivative gain makes the response oscillate less.