The Adjoint of a Polytope

joint works with Kristian Ranestad (Universitetet i Oslo) /
Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig, UC Berkeley)

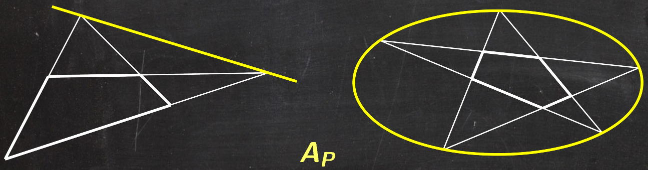
April 13, 2019

The Adjoint of a Polygon

Wachspress (1975)

Definition

The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.

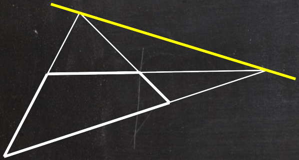


$(\deg A_P = |V(P)| - 3)$

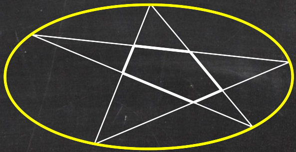
The Adjoint of a Polygon

Wachspress (1975)

Definition
The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



A_P



$(\deg A_P = |V(P)| - 3)$

Generalization to higher-dimensional polytopes?

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition
$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$. *So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

I $\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$. *So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

II *If P is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.

(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ dual polytope of P)

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition
$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

I $\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$. *So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

II *If* P *is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.

(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of* P*)*

Geometric definition using a vanishing condition à la Wachspress?

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P

The Adjoint of a Polytope

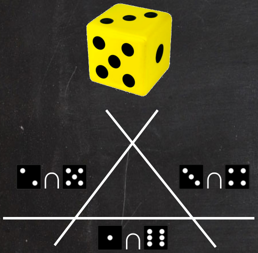
- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

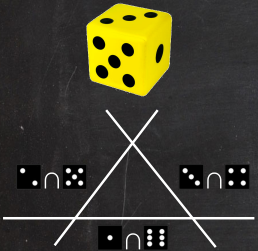
The Adjoint of a Polytope

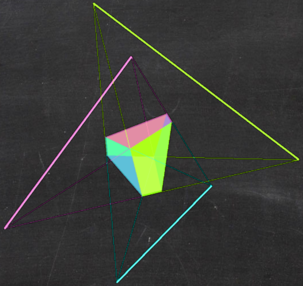
- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P



The Adjoint of a Polytope

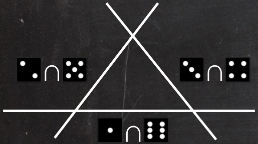
- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

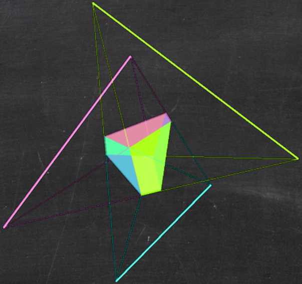


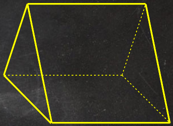


The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

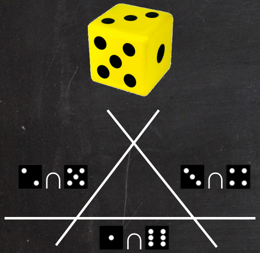


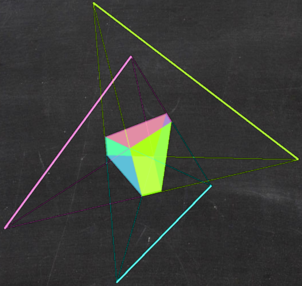


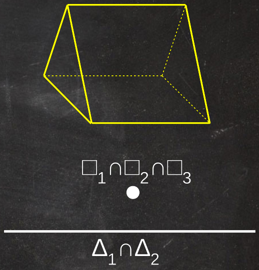


The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

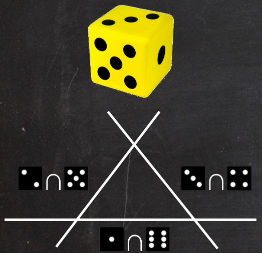


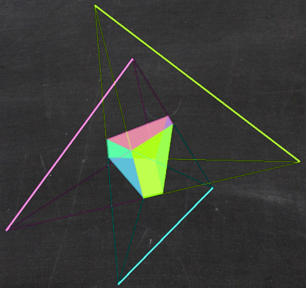


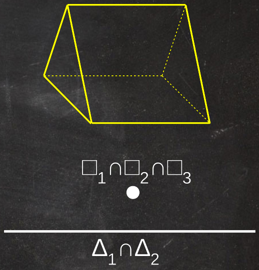


The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





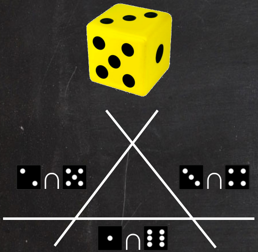


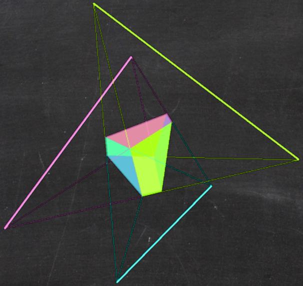
Theorem (K., Ranestad)

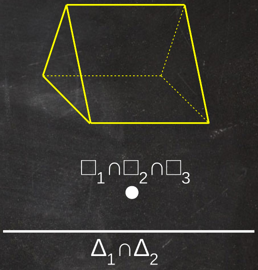
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





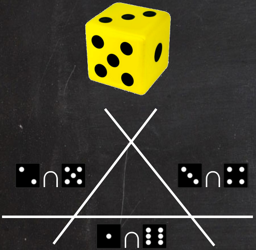


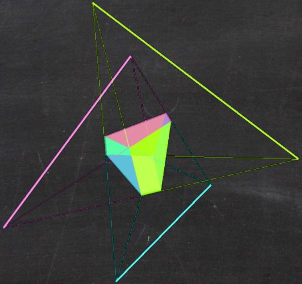
Theorem (K., Ranestad)

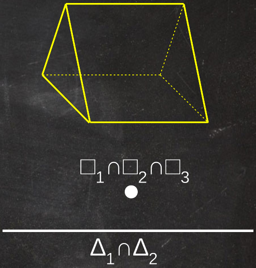
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





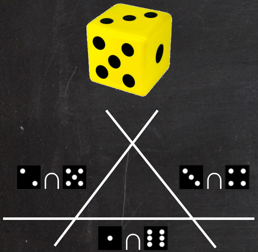


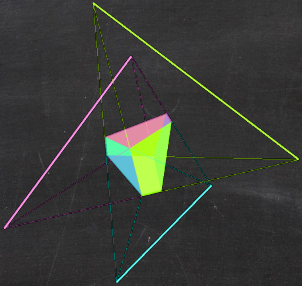
Theorem (K., Ranestad)

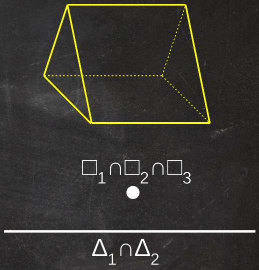
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the ***adjoint*** *of P.*

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P







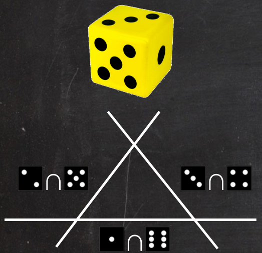
adjoint plane

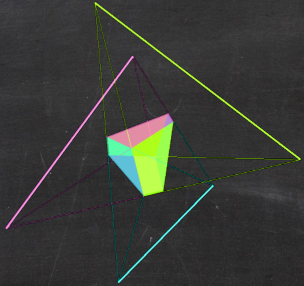
Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

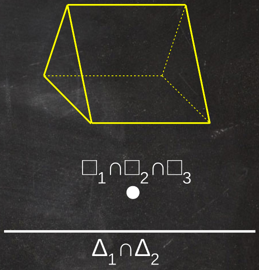
The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





adjoint quadric surface



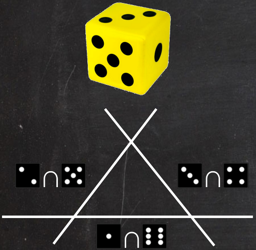
adjoint plane

Theorem (K., Ranestad)

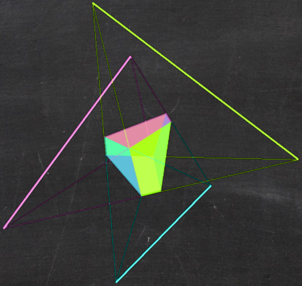
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

The Adjoint of a Polytope

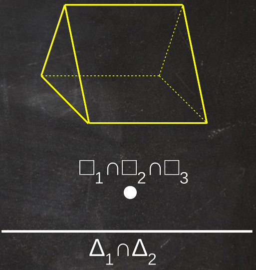
- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P



adjoint double plane



adjoint quadric surface



adjoint plane

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the ***adjoint*** *of P.*

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

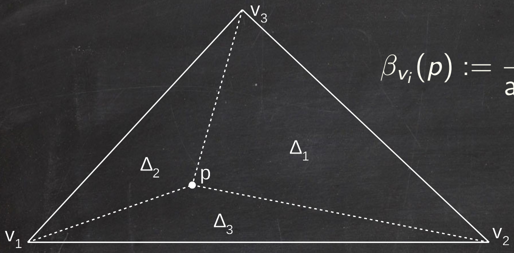
Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

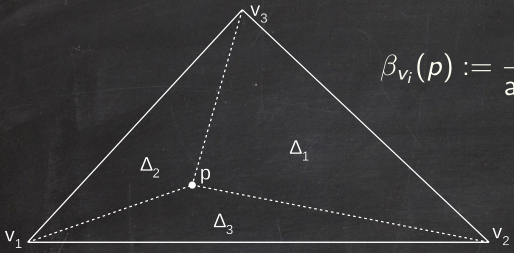
Application 1: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

Application 1: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

Definition

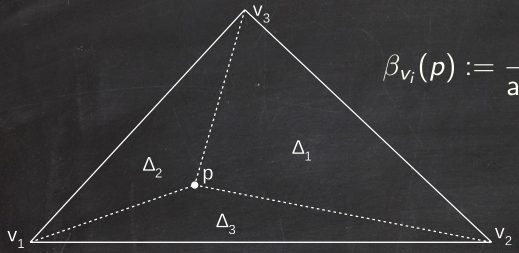
Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

(i) $\forall u \in V(P) : \beta_u(p) > 0$,

(ii) $\sum_{u \in V(P)} \beta_u(p) = 1$, and

(iii) $\sum_{u \in V(P)} \beta_u(p) u = p$.

Application 1: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

Definition

Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

(i) $\forall u \in V(P) : \beta_u(p) > 0$,

(ii) $\sum_{u \in V(P)} \beta_u(p) = 1$, and

(iii) $\sum_{u \in V(P)} \beta_u(p) u = p$.

Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

Application 1: Barycentric Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

Application 1: Barycentric Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$V(P) \stackrel{1:1}{\longleftrightarrow} \mathcal{F}(P^*)$$
$$v \longmapsto F_v$$

Application 1: Barycentric Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$V(P) \overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*) \qquad\qquad \mathcal{F}(P) \overset{1:1}{\longleftrightarrow} V(P^*)$$

$$v \longmapsto F_v \qquad\qquad F \longmapsto v_F$$

Application 1: Barycentric Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$\begin{aligned} V(P) &\overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*) \\ v &\longmapsto F_v \end{aligned} \qquad\qquad \begin{aligned} \mathcal{F}(P) &\overset{1:1}{\longleftrightarrow} V(P^*) \\ F &\longmapsto v_F \end{aligned}$$

Proposition (Warren)

The **Wachspress coordinates**

$$\beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}$$

for $u \in V(P)$

are generalized barycentric coordinates for P.

Application 1: Barycentric Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$V(P) \overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*) \qquad v \longmapsto F_v$$

$$\mathcal{F}(P) \overset{1:1}{\longleftrightarrow} V(P^*) \qquad F \longmapsto v_F$$

Proposition (Warren)

The **Wachspress coordinates**

$$\beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}$$

for $u \in V(P)$

are generalized barycentric coordinates for P.

For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM): [Floater: Generalized barycentric coordinates and applications, Acta Numerica 24 (2015)]

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- P: convex polytope in $\mathbb{R}^n$
- μ_P: uniform probability distribution on P

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- P: convex polytope in $\mathbb{R}^n$
- μ_P: uniform probability distribution on P
- moments

$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \dots w_n^{i_n} \, d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- P: convex polytope in $\mathbb{R}^n$
- μ_P: uniform probability distribution on P
- moments

$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \dots w_n^{i_n} d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}}\, m_{\mathcal{I}}(P)\, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

where $c_{\mathcal{I}} := \binom{i_1+i_2+\dots+i_n+n}{i_1, i_2, \dots, i_n, n}$.

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P

Idea:

$$P \rightsquigarrow \underbrace{\mathcal{H}_P}_{\substack{\text{hypersurface} \\ \text{of degree } d}}$$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$P \rightsquigarrow \mathcal{H}_P \rightsquigarrow D$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\begin{tikzcd}
 & & \mathbb{P}^n & X \arrow[l, "\text{blowup } \pi"'] & \text{smooth} \\
P \arrow[r, rightsquigarrow] & \mathcal{H}_P \arrow[r, rightsquigarrow] & D \arrow[u, hook] \arrow[r, rightsquigarrow] & \tilde{D} \arrow[u, hook] & \text{smooth}
\end{tikzcd}$$

hypersurface
of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

Why “Adjoint”?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\begin{tikzcd}
 & & \mathbb{P}^n & X \text{ smooth} \arrow[l, "\text{blowup } \pi"'] \\
P \arrow[r, squiggly] & \mathcal{H}_P \arrow[r, squiggly] & D \arrow[u, hook] \arrow[r, squiggly] & \tilde{D} \text{ smooth} \arrow[u, hook]
\end{tikzcd}$$

hypersurface
of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\begin{tikzcd}
 & & \mathbb{P}^n & X \arrow[l, "\text{blowup } \pi"'] \quad \text{smooth} \\
P \arrow[r, rightsquigarrow] & \mathcal{H}_P \arrow[r, rightsquigarrow] & D \arrow[u, hook] \arrow[r, rightsquigarrow] & \tilde{D} \arrow[u, hook] \quad \text{smooth}
\end{tikzcd}$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Def.: An **adjoint to $\tilde{D}$ in X** is a hypersurface A in X s.t. $[A] = K_X + [\tilde{D}]$.

Why “Adjoint”?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_\mathcal{P}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\begin{tikzcd} & & \mathbb{P}^n & X \text{ smooth} \arrow[l, "\text{blowup } \pi"'] \\ P \arrow[r, rightsquigarrow] & \mathcal{H}_P \arrow[r, rightsquigarrow] & D \arrow[u, hook] \arrow[r, rightsquigarrow] & \tilde{D} \text{ smooth} \arrow[u, hook] \end{tikzcd}$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Def.: An **adjoint to $\tilde{D}$ in X** is a hypersurface A in X s.t. $[A] = K_X + [\tilde{D}]$.

Proposition (K., Ranestad)

$\tilde{D}$ has a unique adjoint A in X, and thus a unique canonical divisor: $A \cap \tilde{D}$. Moreover, $\pi(A) = A_P$.

Why “Adjoint”?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\begin{tikzcd}
 & & \mathbb{P}^n & & X \arrow[ll, "\text{blowup } \pi"'] & \text{smooth} \\
P \arrow[r, squiggly] & \mathcal{H}_P \arrow[r, squiggly] & D \arrow[u, hook] \arrow[rr, squiggly] & & \tilde{D} \arrow[u, hook] & \text{smooth}
\end{tikzcd}$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

When can we find a polytopal hypersurface D?

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Def.: An **adjoint to $\tilde{D}$ in X** is a hypersurface A in X s.t. $[A] = K_X + [\tilde{D}]$.

Proposition (K., Ranestad)

$\tilde{D}$ has a unique adjoint A in X, and thus a unique canonical divisor: $A \cap \tilde{D}$. Moreover, $\pi(A) = A_P$.

Polytopal Hypersurfaces

Proposition (K., Ranestad)

Let P be a general d-gon in $\mathbb{P}^2$. There is a polygonal curve D iff $d \leq 6$. In that case, D is an elliptic curve.

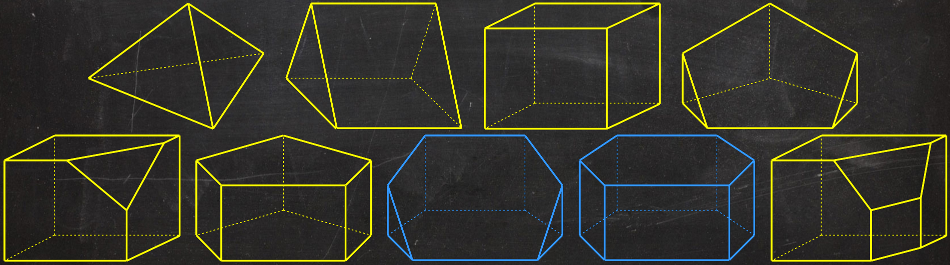
Polytopal Hypersurfaces

Proposition (K., Ranestad)

Let P be a general d-gon in $\mathbb{P}^2$. There is a polygonal curve D iff $d \leq 6$. In that case, D is an elliptic curve.

Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



In that case, the general D is either an elliptic surface or a K3-surface.