Minimal Problems in Computer Vision

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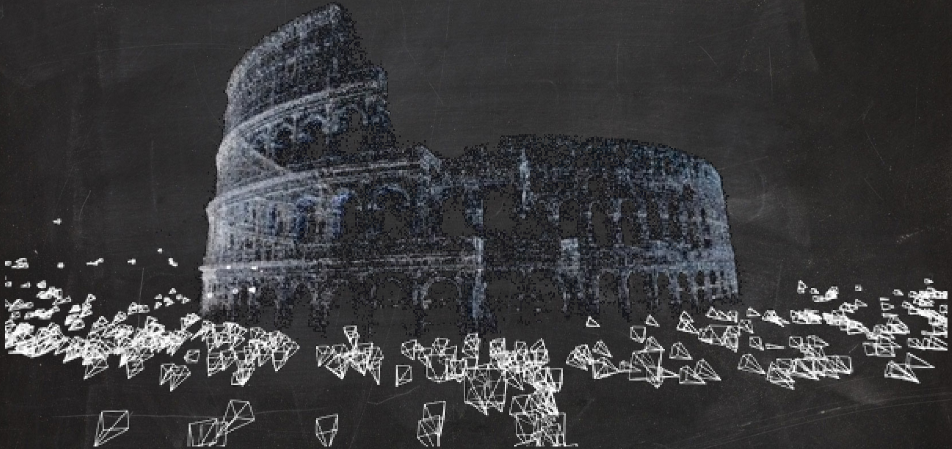
joint work with Timothy Duff (Georgia Tech),
Anton Leykin (Georgia Tech) & Tomas Pajdla (CTU in Prague)

Structure from Motion

Reconstruct 3D scenes and camera poses from 2D images



Rome in a Day: S. Agarwal, Y. Furukawa, N. Snavely, I. Simon, S. Seitz, R. Szeliski

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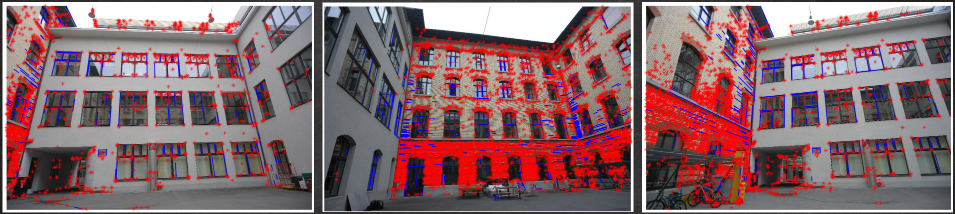
Reconstruct 3D scenes and camera poses from 2D images

- Step 1: Identify common points and lines on given images



- Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

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We use calibrated perspective cameras:
a camera is a matrix $C = [R \mid t]$, where $R \in \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$.

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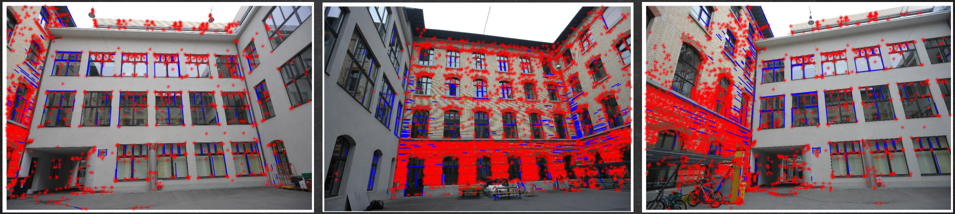


- **Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses**

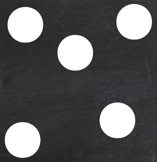
We use calibrated perspective cameras:

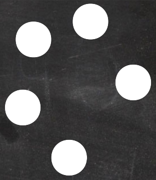
a camera is a matrix $C = [R \mid t]$, where $R \in \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$.

Taking a picture of a point $x \in \mathbb{P}^3$: $x \mapsto Cx$

5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.





5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.

This problem has 20 solutions over $\mathbb{C}$ on generic input images.
(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

5-Point-Problem

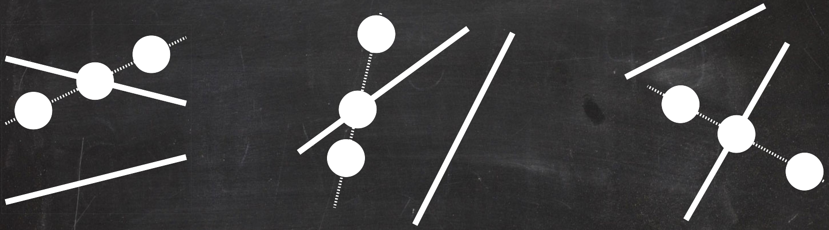
Given 2 images of 5 points, recover 5 points in 3D and both camera poses.

This problem has 20 solutions over $\mathbb{C}$ on generic input images.
(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

$\Rightarrow$ **Since $0 < 20 < \infty$, the 5-Point-Problem is a minimal problem!**

Another minimal problem

- Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses



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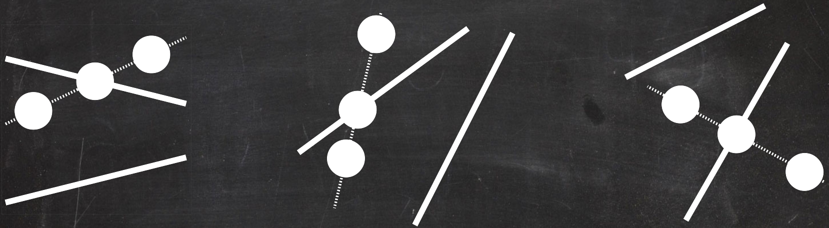

This problem has 40 solutions over $\mathbb{C}$ on generic input images.
(solution = 3 camera poses and 3D coordinates of points and lines)

$\Rightarrow$ **It is a minimal problem!**

Minimal Problems

A **Point-Line-Problem (PLP)** consists of

- a number m of cameras,
- a number p of points,
- a number ℓ of lines,
- a set $\mathcal{I}$ of incidences between points and lines.

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Definition

A PLP $(m, p, \ell, \mathcal{I})$ is **minimal** if, given m generic 2D-arrangements each consisting of p points and ℓ lines satisfying the incidences $\mathcal{I}$, it has a positive and finite number of solutions over $\mathbb{C}$.

(solution $=$ m camera poses and 3D coordinates of p points and ℓ lines satisfying the incidences $\mathcal{I}$)

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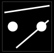
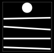
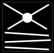
Can we list all minimal PLPs?
How many solutions do they have?

Minimal PLPs

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p,l,\mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p,l,\mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

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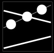
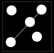
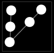
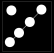
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p,l,\mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

Joint camera map

(3D-arrangement , $\text{cam}_1, \ldots, \text{cam}_m$)

of p points and ℓ lines

satisfying incidences $\mathcal{I}$

Joint camera map

$$\begin{array}{ccccc} (\text{3D-arrangement} & , & \mathrm{cam}_1, \ldots, \mathrm{cam}_m) & \longmapsto & (\text{2D-arr}_1, \ldots, \text{2D-arr}_m) \\ \text{of } p \text{ points and } \ell \text{ lines} & & & & \\ \text{satisfying incidences } \mathcal{I} & & & & \end{array}$$

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- $\mathbb{P}^n$ = n-dimensional projective space
- $\mathbb{G}_{1,n} = \{\text{lines in } \mathbb{P}^n\}$ = **Grassmannian of lines in $\mathbb{P}^n$**
- $\mathcal{X} = \{(X_1, \ldots, X_p, L_1, \ldots, L_\ell) \in (\mathbb{P}^3)^p \times (\mathbb{G}_{1,3})^\ell \mid \forall (i,j) \in \mathcal{I} : X_i \in L_j\}$

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Lemma

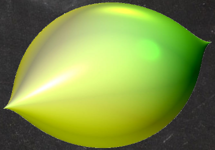
If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$.

Algebraic varieties

Definition
A **variety** is the common zero set of a system of polynomial equations.

A variety looks like a manifold **almost everywhere**:





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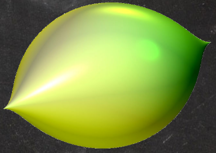

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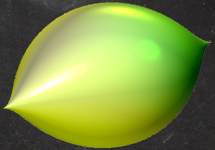

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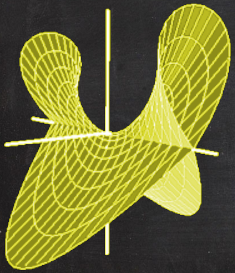

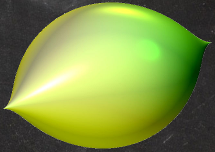

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The **dimension** of an irreducible variety is its local dimension as a manifold.

$\mathcal{X}$, $\mathcal{C}$ and $\mathcal{Y}$ are irreducible varieties!

Deriving the big table

$$\begin{array}{ccccccc} \mathcal{X} & \times & \mathcal{C} & \longrightarrow & \mathcal{Y} \\ (\text{3D-arrangement} & , & \text{cam}_1, \ldots, \text{cam}_m) & \longmapsto & (\text{2D-arr}_1, \ldots, \text{2D-arr}_m) \\ \text{of } p \text{ points and } \ell \text{ lines} & & & & \\ \text{with incidences } \mathcal{I} & & & & \end{array}$$

Lemma

If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$.

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Theorem

- *If $m > 6$, then* $\dim(\mathcal{X}) + \dim(\mathcal{C}) \neq \dim(\mathcal{Y})$.

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$\mathcal{X}$	$\times$	$\mathcal{C}$	$\longrightarrow$	$\mathcal{Y}$
(3D-arrangement of p points and ℓ lines with incidences $\mathcal{I}$	,	$\text{cam}_1, \dots, \text{cam}_m)$	$\longmapsto$	$(\text{2D-arr}_1, \dots, \text{2D-arr}_m)$

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Theorem

- *If* $m > 6$*, then* $\dim(\mathcal{X}) + \dim(\mathcal{C}) \neq \dim(\mathcal{Y})$.
- *There are exactly 39 PLPs with* $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$:

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
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Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
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Lemma

A PLP with $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$ *is minimal if and only if its joint camera map* $\mathcal{X} \times \mathcal{C} \to \mathcal{Y}$ *is dominant.*

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A map $\varphi : A \to B$ is **surjective** if for every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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A map $\varphi : A \to B$ is **dominant** if for almost every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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Fact A map $\varphi : A \to B$ between irreducible varieties A and B is dominant
if and only if
for almost every $a \in A$ the differential $D_a\varphi : T_aA \to T_{\varphi(a)}B$ is surjective.

Deriving the big table

$$\begin{array}{ccccc} \mathcal{X} & \times & \mathcal{C} & \longrightarrow & \mathcal{Y} \\ (\text{3D-arrangement} & , & \text{cam}_1, \ldots, \text{cam}_m) & \longmapsto & (\text{2D-arr}_1, \ldots, \text{2D-arr}_m) \end{array}$$

of p points and ℓ lines
satisfying incidences $\mathcal{I}$

Lemma
A PLP with $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$ *is minimal if and only if its joint camera map* $\mathcal{X} \times \mathcal{C} \to \mathcal{Y}$ *is dominant.*

Definition
A map $\varphi : A \to B$ is **surjective** if for every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

Definition
A map $\varphi : A \to B$ is **dominant** if for almost every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

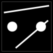
Fact A map $\varphi : A \to B$ between irreducible varieties A and B is dominant
if and only if
for almost every $a \in A$ the differential $D_a\varphi : T_aA \to T_{\varphi(a)}B$ is surjective.

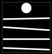
Can check this computationally! It is only linear algebra!

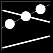
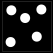
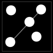
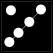
m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

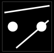
m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

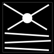
m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\text{f}}p^{\text{d}}l^{\text{f}}l^{\text{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

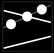
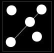
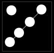
m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\text{f}}p^{\text{d}}l^{\text{f}}l^{\text{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\text{f}}p^{\text{d}}l^{\text{f}}l^{\text{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

- For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)

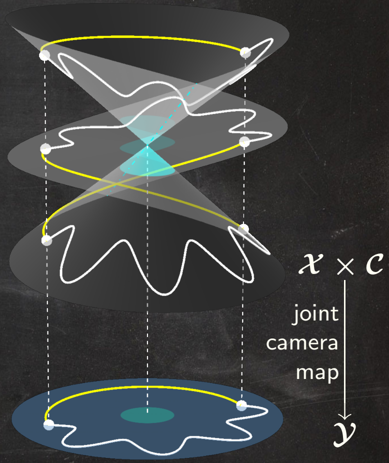
m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

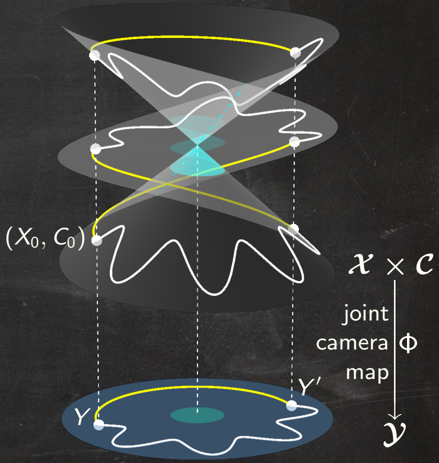
- For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)
- For $m \in \{4, 5, 6\}$: compute number of solutions with **homotopy continuation** and **monodromy** (state-of-the-art method in numerical algebraic geometry)

Monodromy



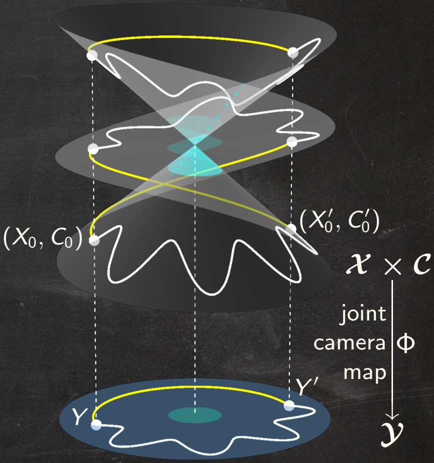
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$



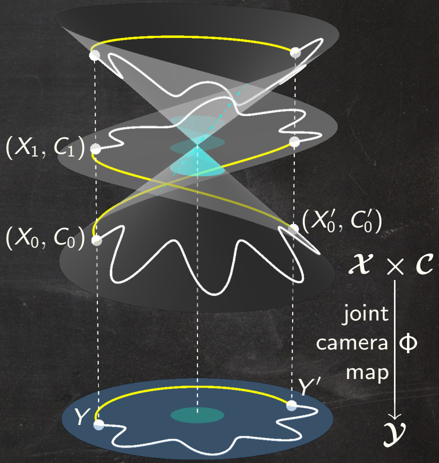
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**



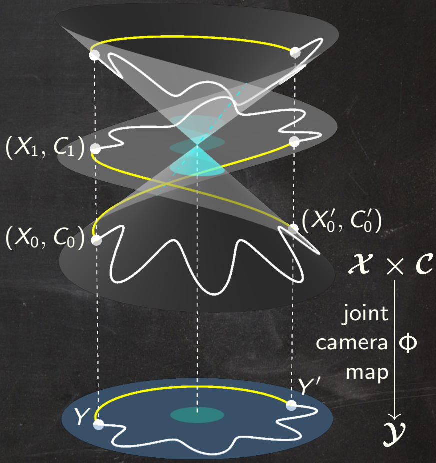
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**



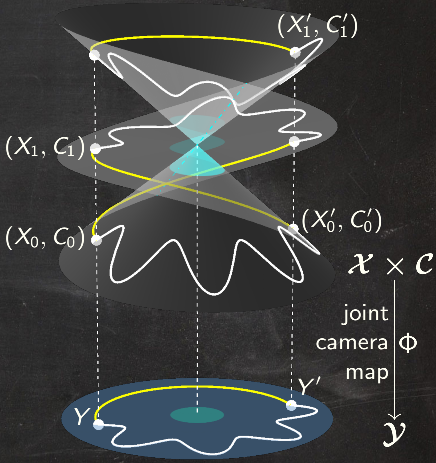
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- Keep on circulating between Y and Y' until no more solutions for Y are found



Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- Keep on circulating between Y and Y' until no more solutions for Y are found



Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
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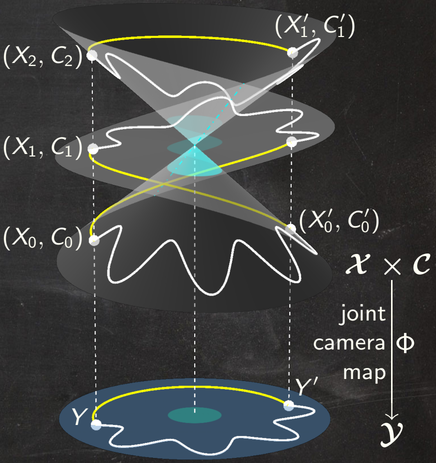
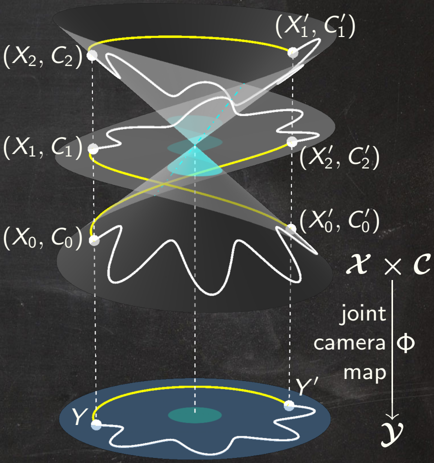
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- Keep on circulating between Y and Y' until no more solutions for Y are found

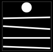


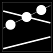
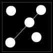
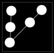
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- Keep on circulating between Y and Y' until no more solutions for Y are found



m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

Thanks for your attention!