

Point-Line Minimal Problems in Complete Multi-View Visibility

Kathlén Kohn

University of Oslo & ICERM

joint work with Timothy Duff (Georgia Tech),
Anton Leykin (Georgia Tech) & Tomas Pajdla (CTU in Prague)

Reconstruct 3D scenes and camera poses
from 2D images

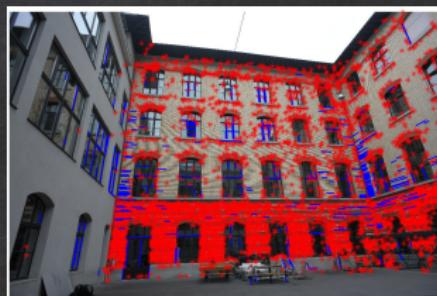
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- ◆ Step 1: Identify common points and lines on given images



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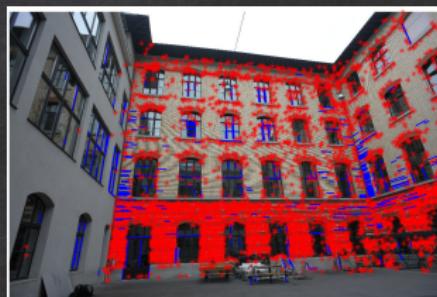
- ◆ Step 1: Identify common points and lines on given images



- ◆ Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

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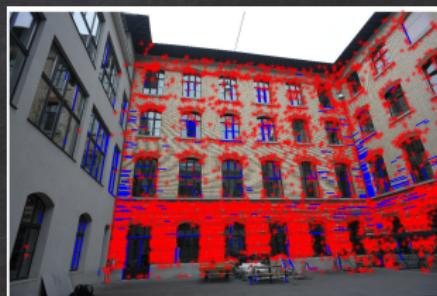
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We use calibrated perspective cameras:
each such camera is represented by a matrix
 $[R \mid t]$, where $R \in \text{SO}(3)$ and $t \in \mathbb{R}^3$

5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.



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This problem has 20 solutions over \mathbb{C} .

(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

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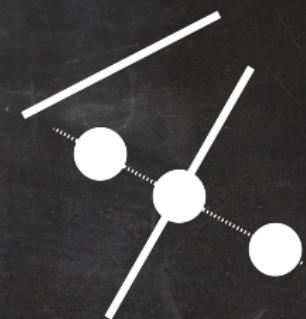
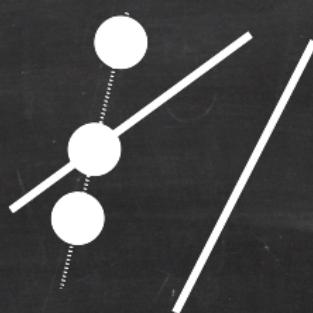
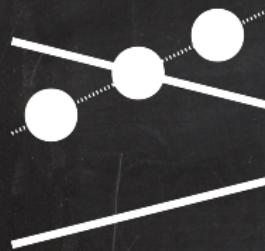
This problem has **20** solutions over \mathbb{C} .

(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

⇒ The 5-Point-Problem is a **minimal** problem!

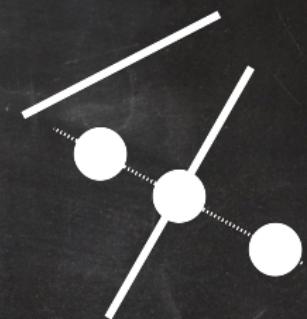
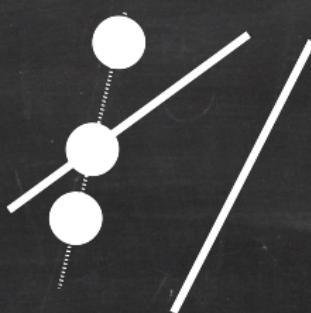
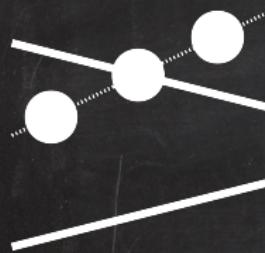
Another minimal problem

- Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses



Another minimal problem

- Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses



This problem has **40** solutions over \mathbb{C} .

(solution = 3 camera poses and 3D coordinates of points and lines)

⇒ It is a **minimal** problem!

Minimal Problems

A **Point-Line-Problem (PLP)** consists of

- ◆ a number m of cameras,
- ◆ a number p of points,
- ◆ a number ℓ of lines,
- ◆ a set \mathcal{I} of incidences between points and lines.

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Definition

A PLP $(m, p, \ell, \mathcal{I})$ is **minimal** if, given m generic 2D-arrangements each consisting of p points and ℓ lines satisfying the incidences \mathcal{I} , it has a positive and finite number of solutions over \mathbb{C} .

(solution = m camera poses and 3D coordinates of p points and ℓ lines satisfying the incidences \mathcal{I})

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Can we list **all** minimal PLPs?
How many solutions do they have?

Minimal PLPs

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^d l^f l_\alpha^a$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
(p, l, \mathcal{I})													
Minimal Degree	Y $> 450k^*$	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^f p^d l^f l_\alpha^a$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
(p, l, \mathcal{I})													
Minimal Degree	Y 544*	Y 360	Y 552	Y 480	N	N	Y 264	Y 432	Y 328	Y 480	Y 240	Y 64	Y 216
m views	3	3	3	3	3	3	3	2	2	2	2	2	2
$p^f p^d l^f l_\alpha^a$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
(p, l, \mathcal{I})													
Minimal Degree	Y 312	Y 224	Y 40	Y 144	Y 144	Y 144	Y 64	N	Y 20	Y 16	Y 12	N	N

Joint camera map

(3D-arrangement , $\text{cam}_1, \dots, \text{cam}_m$)
of p points and ℓ lines
satisfying incidences \mathcal{I}

Joint camera map

(3D-arrangement , $\text{cam}_1, \dots, \text{cam}_m$) \longmapsto (2D-arr₁, ..., 2D-arr_m)
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Joint camera map

$$\begin{array}{ccc} \mathcal{X} & \times & \mathcal{C} \\ (\text{3D-arrangement} & , & \text{cam}_1, \dots, \text{cam}_m) \\ \text{of } p \text{ points and } \ell \text{ lines} & & \longmapsto \\ \text{satisfying incidences } \mathcal{I} & & (\text{2D-arr}_1, \dots, \text{2D-arr}_m) \end{array}$$

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- ◆ $\mathbb{P}^n = n$ -dimensional projective space
- ◆ $\mathbb{G}_{1,n} = \{\text{lines in } \mathbb{P}^n\} = \text{Grassmannian of lines in } \mathbb{P}^n$
- ◆ $\mathcal{X} = \{(X_1, \dots, X_p, L_1, \dots, L_\ell) \in (\mathbb{P}^3)^p \times (\mathbb{G}_{1,3})^\ell \mid \forall (i,j) \in \mathcal{I} : X_i \in L_j\}$

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- ◆ $\mathcal{Y} = \left\{ \begin{array}{c} (x_{1,1}, \dots, x_{m,p}, l_{1,1}, \dots, l_{m,\ell}) \\ \in (\mathbb{P}^2)^{mp} \times (\mathbb{G}_{1,2})^{m\ell} \end{array} \middle| \begin{array}{l} \forall k = 1, \dots, m \\ \forall (i,j) \in \mathcal{I} : x_{k,i} \in l_{k,j} \end{array} \right\}$

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\mathcal{X} \times \mathcal{C} \longrightarrow \mathcal{Y}
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- ◆ $\mathcal{C} = \left\{ ([R_1|t_1], \dots [R_m|t_m]) \middle| \begin{array}{l} \forall i = 1, \dots, m : R_i \in \text{SO}(3), t_i \in \mathbb{R}^3, \\ R_1 = I_3, t_1 = 0, t_{2,1} = 1 \end{array} \right\}$

Joint camera map

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 \mathcal{X} & \times & \mathcal{C} \\
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Lemma

If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$.

Algebraic varieties

Definition

A **variety** is the common zero set of a system of polynomial equations.

A variety looks like a manifold **almost everywhere**:



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\mathcal{X} , \mathcal{C} and \mathcal{Y} are irreducible varieties!

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Deriving the big table

$$\begin{array}{ccc} \mathcal{X} & \times & \mathcal{C} \\ \text{(3D-arrangement} & , & \text{cam}_1, \dots, \text{cam}_m) \\ \text{of } p \text{ points and } \ell \text{ lines} & & \longmapsto \\ \text{with incidences } \mathcal{I} & & \end{array} \quad \begin{array}{c} \mathcal{Y} \\ (2\text{D-arr}_1, \dots, 2\text{D-arr}_m) \end{array}$$

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If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$.

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If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$.

Theorem

- ◆ If $m > 6$, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) \neq \dim(\mathcal{Y})$.

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If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$.

Theorem

- ♦ If $m > 6$, then $\dim(\mathcal{X}) + \dim(\mathcal{C}) \neq \dim(\mathcal{Y})$.
- ♦ There are exactly 39 PLPs with $\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})$:

m views	6	6	6	5	5	5	4	4	4	4	4	4
$p^f p^d f^f l^a_\alpha$	1021 ₁	1013 ₃	1005 ₅	2011 ₁	2003 ₂	2003 ₃	1030 ₀	1022 ₂	1014 ₄	1006 ₆	3001 ₁	2110 ₀
(p, l, \mathcal{I})	•—	—*	*—	—*—	—*—	—*—	•—	—*	—*	—*	*	—*
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y
Degree	> 450 k^*			11306*	26240*	11008*	3040*	4524*			1728*	32*
m views	4	3	3	3	3	3	3	3	3	3	3	3
$p^f p^d f^f l^a_\alpha$	2102 ₂	1040 ₀	1032 ₂	1024 ₄	1016 ₆	1008 ₈	2021 ₁	2013 ₂	2013 ₃	2005 ₄	2005 ₅	3010 ₀
(p, l, \mathcal{I})	—*	•—	—*	*—	*—	*—	—*	—*	—*	—*	—*	—*
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y
Degree	544*	360	552	480			264	432	328	480	240	64
m views	3	3	3	3	3	3	3	3	2	2	2	2
$p^f p^d f^f l^a_\alpha$	3002 ₁	3002 ₂	2111 ₁	2103 ₁	2103 ₂	2103 ₃	3100 ₀	2201 ₁	5000 ₂	4110 ₃	3200 ₃	3200 ₄
(p, l, \mathcal{I})	—*	—*	—*	—*	—*	—*	—*	—*	—*	—*	—*	—*
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N
Degree	312	224	40	144	144	144	64		20	16	12	

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Lemma

A PLP with $\dim(\mathcal{X}) + \dim(\mathcal{C}) \neq \dim(\mathcal{Y})$ is minimal if and only if its joint camera map $\mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y}$ is dominant.

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Definition

A map $\varphi : A \rightarrow B$ is **surjective** if for every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

Definition

A map $\varphi : A \rightarrow B$ is **dominant** if for almost every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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A map $\varphi : A \rightarrow B$ is **dominant** if for almost every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

Fact A map $\varphi : A \rightarrow B$ between irreducible varieties A and B is dominant if and only if

for almost every $a \in A$ the differential $D_a\varphi : T_a A \rightarrow T_{\varphi(a)} B$ is surjective.

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Can check this computationally! It is only linear algebra!

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^{dl} l^f l_\alpha^a$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
(p, l, \mathcal{I})													
Minimal Degree	Y $> 450k^*$	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
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Minimal Degree	Y 544^*	Y 360	Y 552	Y 480	N	N	Y 264	Y 432	Y 328	Y 480	Y 240	Y 64	Y 216
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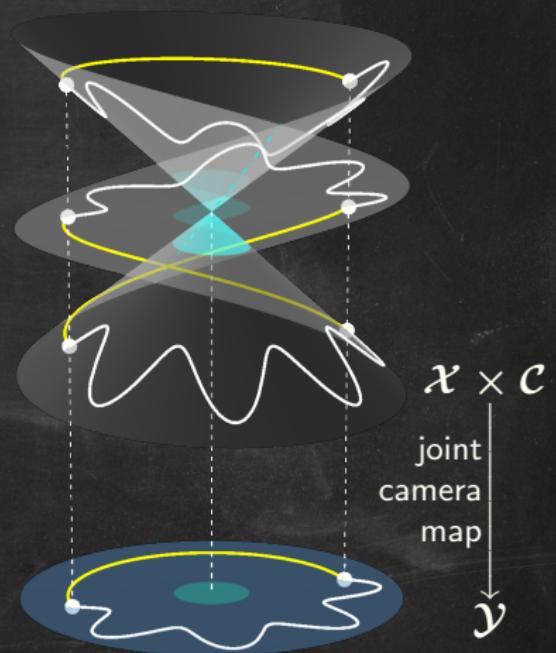
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m views	3	3	3	3	3	3	3	3	2	2	2	2
$p^f p^d l^f l_\alpha^a$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4
(p, l, \mathcal{I})												
Minimal Degree	Y 312	Y 224	Y 40	Y 144	Y 144	Y 144	Y 64	N	Y 20	Y 16	Y 12	N N

- ◆ For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^d l^f l_\alpha^a$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
(p, l, \mathcal{I})													
Minimal Degree	Y $> 450k^*$	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^f p^d l^f l_\alpha^a$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
(p, l, \mathcal{I})													
Minimal Degree	Y 544^*	Y 360	Y 552	Y 480	N	N	Y 264	Y 432	Y 328	Y 480	Y 240	Y 64	Y 216
m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^f p^d l^f l_\alpha^a$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
(p, l, \mathcal{I})													
Minimal Degree	Y 312	Y 224	Y 40	Y 144	Y 144	Y 144	Y 64	N	Y 20	Y 16	Y 12	N	N

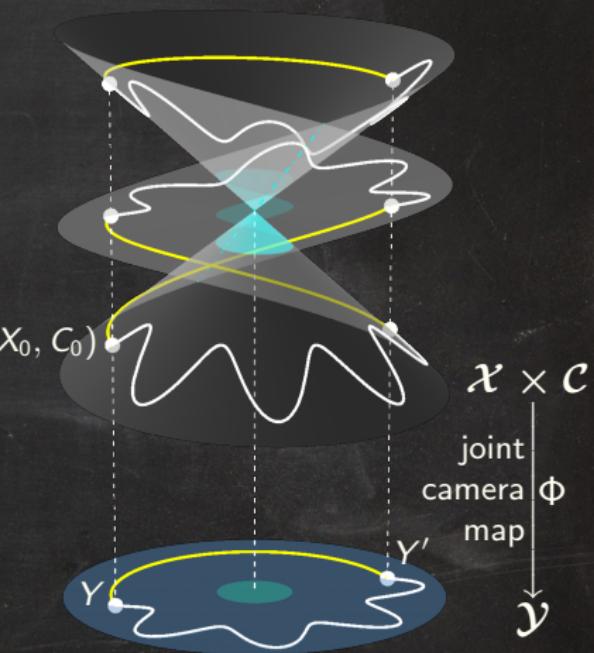
- ◆ For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)
- ◆ For $m \in \{4, 5, 6\}$: compute number of solutions with **homotopy continuation** and **monodromy** (state-of-the-art method in numerical algebraic geometry)

Monodromy



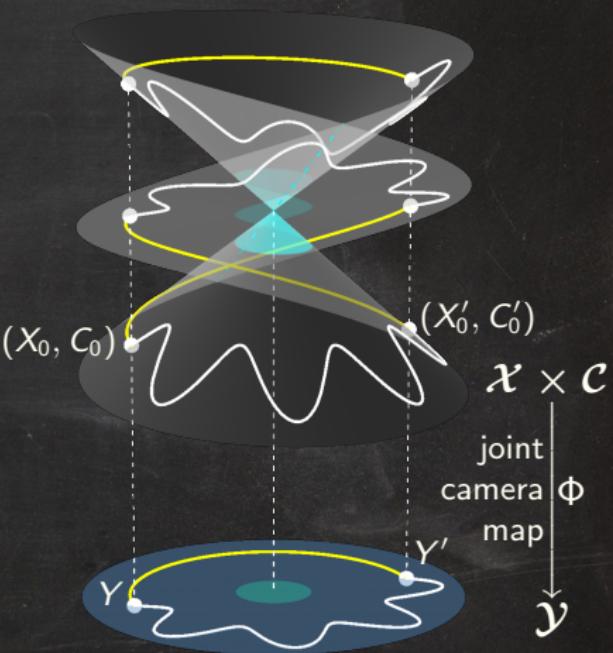
Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$



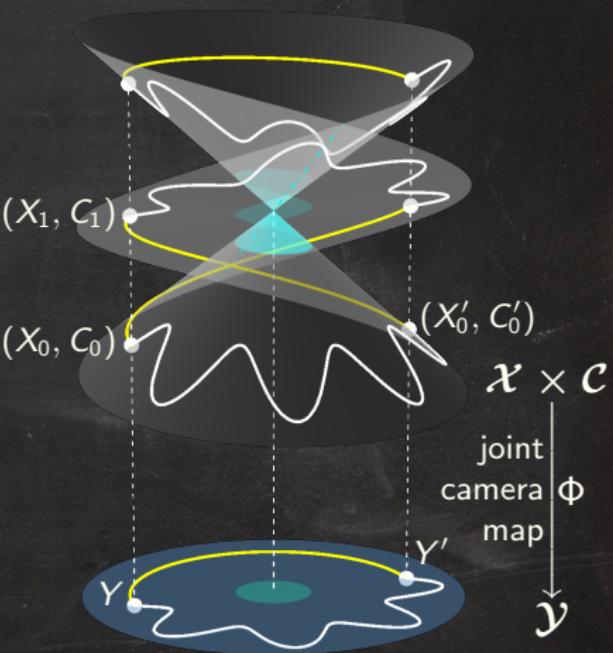
Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$
- ◆ Along a random path from Y to Y'
track the solution (X_0, C_0) for Y
to a solution (X'_0, C'_0) for Y'
via **homotopy continuation**



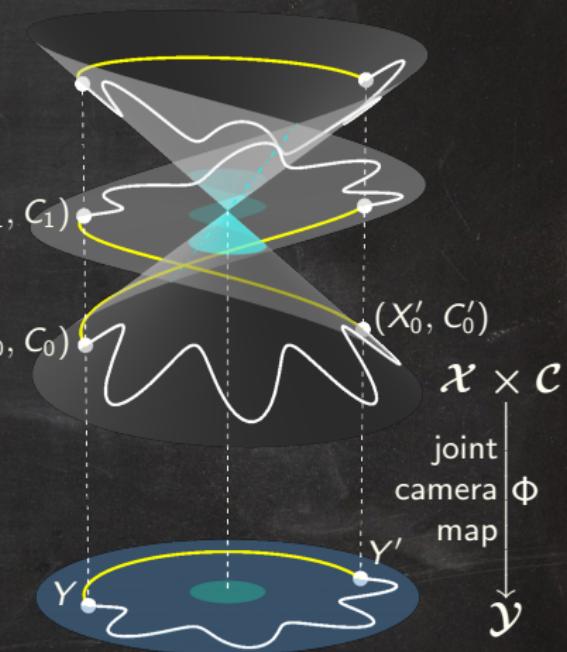
Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$
- ◆ Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X'_0, C'_0) for Y' via **homotopy continuation**
- ◆ Along a random path from Y' to Y track the solution (X'_0, C'_0) for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**



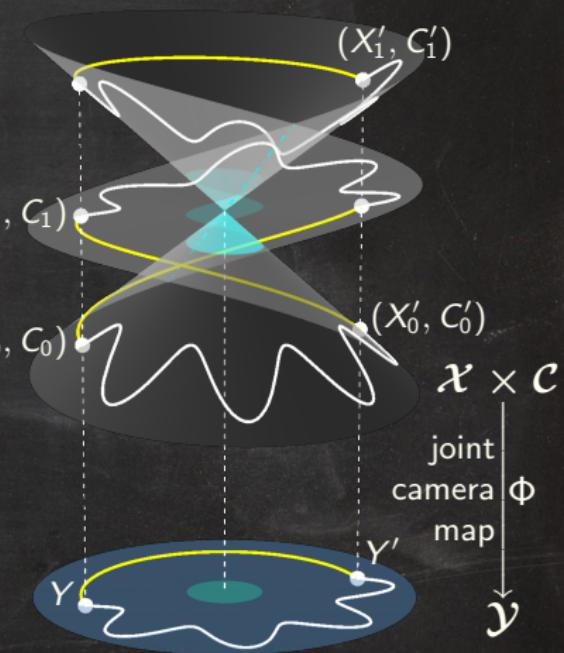
Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$
- ◆ Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X'_0, C'_0) for Y' via **homotopy continuation**
- ◆ Along a random path from Y' to Y track the solution (X'_0, C'_0) for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- ◆ Keep on circulating between Y and Y' until no more solutions for Y are found



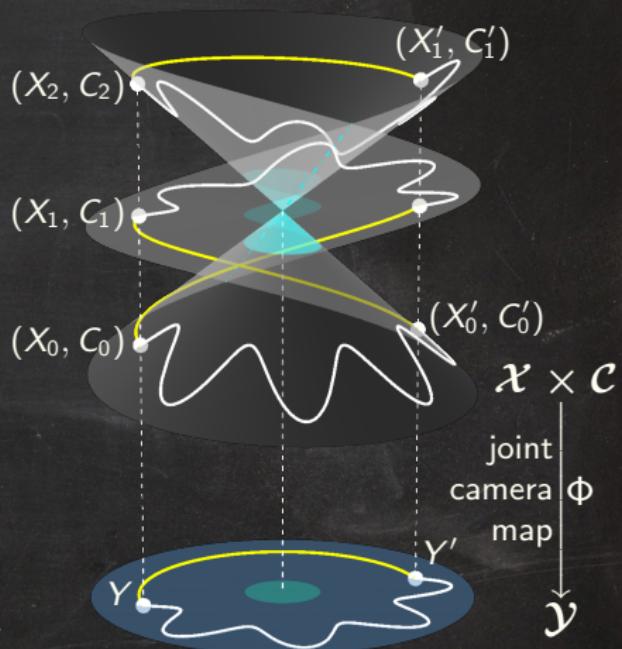
Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$
- ◆ Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X'_0, C'_0) for Y' via **homotopy continuation**
- ◆ Along a random path from Y' to Y track the solution (X'_0, C'_0) for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- ◆ Keep on circulating between Y and Y' until no more solutions for Y are found



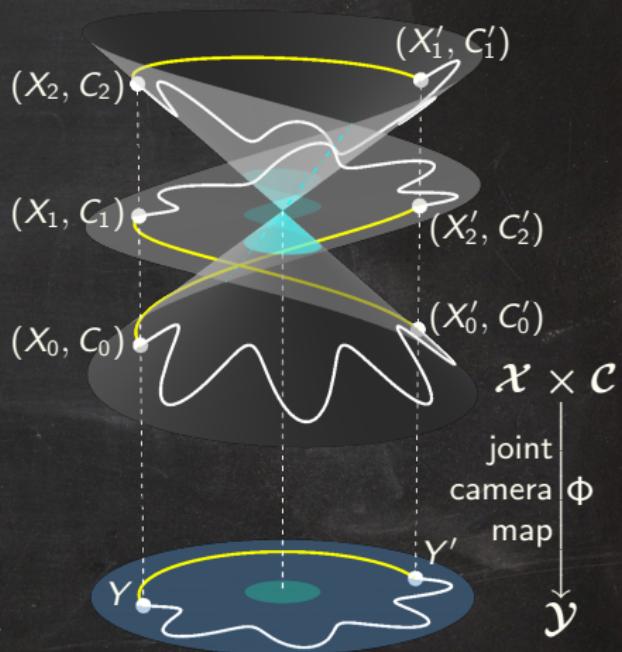
Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$
- ◆ Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X'_0, C'_0) for Y' via **homotopy continuation**
- ◆ Along a random path from Y' to Y track the solution (X'_0, C'_0) for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- ◆ Keep on circulating between Y and Y' until no more solutions for Y are found



Monodromy

- ◆ Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- ◆ Set $Y = \Phi(X_0, C_0)$
- ◆ Pick $Y' \in \mathcal{Y}$
- ◆ Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X'_0, C'_0) for Y' via **homotopy continuation**
- ◆ Along a random path from Y' to Y track the solution (X'_0, C'_0) for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- ◆ Keep on circulating between Y and Y' until no more solutions for Y are found



m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^d l^f l_\alpha^a$	1021 ₁	1013 ₃	1005 ₅	2011 ₁	2003 ₂	2003 ₃	1030 ₀	1022 ₂	1014 ₄	1006 ₆	3001 ₁	2110 ₀	2102 ₁
(p, l, \mathcal{I})													
Minimal Degree	Y $> 450k^*$	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
				11306*	26240*	11008*	3040*	4524*			1728*	32*	544*
m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^f p^d l^f l_\alpha^a$	2102 ₂	1040 ₀	1032 ₂	1024 ₄	1016 ₆	1008 ₈	2021 ₁	2013 ₂	2013 ₃	2005 ₃	2005 ₄	2005 ₅	3010 ₀
(p, l, \mathcal{I})													
Minimal Degree	Y 544*	Y 360	Y 552	Y 480	N	N	Y 264	Y 432	Y 328	Y 480	Y 240	Y 64	Y 216
m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^f p^d l^f l_\alpha^a$	3002 ₁	3002 ₂	2111 ₁	2103 ₁	2103 ₂	2103 ₃	3100 ₀	2201 ₁	5000 ₂	4100 ₃	3200 ₃	3200 ₄	2300 ₅
(p, l, \mathcal{I})													
Minimal Degree	Y 312	Y 224	Y 40	Y 144	Y 144	Y 144	Y 64	N	Y 20	Y 16	Y 12	N	N

Thanks for your attention!