Coisotropic Hypersurfaces in the Grassmannian

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Abstract

To every projective variety X, we associate a family of hypersurfaces in different Grassmannians, called the coisotropic hypersurfaces of X. These include the Chow form and the Hurwitz form of X. Gel'fand, Kapranov and Zelevinsky characterized coisotropic hypersurfaces by a rank one condition on tangent spaces. We present a new and simplified proof of that result. We show that the coisotropic hypersurfaces of X equal those of its projectively dual variety $X^\vee$, and that their degrees are the polar degrees of X. Coisotropic hypersurfaces of Segre varieties are defined by hyperdeterminants, and all hyperdeterminants arise in that manner. We derive new equations for the Cayley variety which parametrizes all coisotropic hypersurfaces of given degree in a fixed Grassmannian. We provide a `Macaulay2` package for transitioning between X and its coisotropic hypersurfaces.

1 Chow Forms, Hurwitz Forms and Beyond

For any projective variety X, its *Chow form* is a unique polynomial in the coordinate ring of a Grassmannian that represents X. The degree of the Chow form is the degree of X, and the dimension of X determines the Grassmannian. In that sense, Chow forms parametrize algebraic cycles of fixed dimension and degree in projective space. This was first introduced by Cayley [Cay60] to study curves in 3-space. The generalization to aribitrary varieties was given by Chow and van der Waerden [CvdW37]. For a given variety $X \subseteq \mathbb{P}^n$ of dimension k, projective subspaces of dimension $n-k-1$ have typically no intersection with the variety, but those subspaces that do intersect X form a hypersurface in the corresponding Grassmannian; the defining polynomial of this hypersurface is the Chow form.

Consider now the same given variety X, but projective subspaces of dimension $n-k$. These are expected to have a finite number of intersections points with X, given by the degree d of X. The subspaces which do not intersect X in d reduced points form again a hypersurface in the corresponding Grassmannian, as long as $d \geq 2$. The defining polynomial of this hypersurface is called the *Hurwitz form* [Stu]. This hypersurface can also be defined as the closure of the set of all $(n-k)$-subspaces that have a non-transversal intersection at a smooth point of the given variety. In this sense, it plays a crucial role in the study of the condition of intersecting a projective variety with varying linear subspaces [Bü]. Moreover, the Chow forms of curves and the Hurwitz forms of surfaces in 3-space characterize exactly the self-dual hypersurfaces in the Grassmannian of lines in $\mathbb{P}^3$ [Cat14].

A natural generalization of the above described hypersurfaces is studied in Chapters 3.3 and 4.3 of [GKZ94]: The *i-th higher associated hypersurface* of a projective variety X of dimension k is defined as the closure of the set of all $(n-k-1+i)$-dimensional subspaces that intersect X at a smooth point non-transversally. This article is devoted to the study of these hypersurfaces: In particular, it is shown that their degrees are the polar degrees, and that the hypersurfaces associated to the projectively dual variety $X^\vee$ are essentially the hypersurfaces associated to X in reversed order. Moreover, a new proof of the main result in [GKZ94] about these hypersurfaces will be given: The higher associated hypersurfaces are exactly the *coisotropic hypersurfaces*. The proof in [GKZ94] uses machinery from Lagrangian varieties, whereas our proof is direct and self-contained. Due to this result, a more meaningful name for the main objects of study will be used: In the following, the i-th higher associated hypersurface of X is called the *i-th coisotropic hypersurface* of X. Hence, the Chow form and the Hurwitz form are the defining polynomials of the zeroth and the first coisotropic hypersurface.

Section 3 shows that the coisotropic hypersurfaces of a projective variety are essentially projectively dual to the Segre product of the variety with projective spaces. With this, we determine for which indices i the i-th coisotropic hypersurface has indeed codimension one in the Grassmannian. It is proven in Section 4 that the degrees of the coisotropic hypersurfaces coincide with the polar degrees. Section 5 revisits the main theorem of [GKZ94] about coisotropic hypersurfaces: Coisotropy can be defined independently of the underlying projective variety by a certain rank one characterization of matrices formed by first partial derivatives of an equation defining the hypersurface. The coisotropic hypersurfaces of a variety and its projectively dual will be related in Section 6. It is shown in Section 7 that hyperdeterminants are a special case of coisotropic forms. The Cayley variety – formed by all coisotropic hypersurfaces of fixed degree in a fixed Grassmannian – as well as characterizations for coisotropy in Plücker coordinates are studied in Section 8. Finally, we shortly describe in Section 9 a `Macaulay2` package for computations with coisotropic hypersurfaces. As a preparation, naming conventions for different coordinate systems of Grassmannians and some further notation will be fixed in the following section.

2 Preliminaries

This article focusses on projective varieties over $\mathbb{C}$. Let V be an $(n+1)$-dimensional complex vector space, and $\mathbb{P}(V)$ be its projectivization. If $V = \mathbb{C}^{n+1}$, we will simply write $\mathbb{P}^n := \mathbb{P}(\mathbb{C}^{n+1})$ for this complex projective space. By $\mathbb{P}(V)^*$ we denote the projectivization of the dual space V^*, which is formed by hyperplanes in $\mathbb{P}(V)$. The two projective spaces $\mathbb{P}(V)^*$ and $\mathbb{P}(V)$ can be identified by sending every point $y = (y_0 : \ldots : y_n) \in \mathbb{P}(V)$ to the hyperplane

$$\{y = 0\} := \left\{ x \in \mathbb{P}(V) \;\middle|\; \sum_{i=0}^{n} y_i x_i = 0 \right\} \in \mathbb{P}(V)^*.$$

Let $X \subseteq \mathbb{P}(V)$ be an irreducible non-empty projective variety. We denote the *embedded tangent space* of X at a smooth point $x \in X$ by $T_x X$. It can be explicitly computed as

$$T_x X := \left\{ y \in \mathbb{P}(V) \;\middle|\; \forall f \in I(X) : \sum_{i=0}^{n} \frac{\partial f}{\partial X_i}(x) \cdot y_i = 0 \right\},$$

where $I(X) \subseteq \mathbb{C}[X_0, \ldots, X_n]$ denotes the vanishing ideal of X. A hyperplane in $\mathbb{P}(V)$ is called *tangent* to X at a smooth point $x \in X$ if it contains $T_x X$. The closure $X^\vee$ of the set of all hyperplanes that are tangent to X at some smooth $x \in X$ is called the *projectively dual variety* of X; formally

$$X^\vee := \overline{\{H \in \mathbb{P}(V)^* \mid \exists x \in X \text{ smooth } : T_x X \subseteq H\}} \subseteq \mathbb{P}(V)^*.$$

It is known that $X^\vee$ is an irreducible variety if X is irreducible [GKZ94, Ch. 1, Prop. 1.3]. We will heavily use the identification of a projective space and its dual space to view the dual variety $X^\vee$ as a subvariety of $\mathbb{P}(V)$. With this, we will also make frequent use of the following biduality of projective varieties over $\mathbb{C}$, which is also known as reflexivity.

Theorem 1 ([GKZ94, Ch. 1, Thm. 1.1]). *For every projective variety $X \subseteq \mathbb{P}(V)$, we have $(X^\vee)^\vee = X$. More precisely, if $x \in X$ is smooth and $H \in X^\vee$ is smooth, then H is tangent to X at x if and only if x – regarded as a hyperplane in $\mathbb{P}(V)^*$ – is tangent to $X^\vee$ at H.*

2.1 Coordinate Systems

The Grassmannian of all projective subspaces in $\mathbb{P}^n$ of dimension l is denoted by $G(l+1, n+1)$. Different coordinates on Grassmannians are discussed in Section 1 of Chapter 3 in [GKZ94]. We follow the conventions used in [Stu]. There are six different ways of specifying a point $L \in G(l+1, n+1)$.

First, let $A \in \mathbb{C}^{(n-l)\times(n+1)}$ be such that L is the projectivization of the kernel of A. The entries of A are the *primal Stiefel coordinates* of L, and the maximal minors $p_{i_1 \ldots i_{n-l}}$ of A are the *primal Plücker coordinates* of L. Up to scaling, the Plücker coordinates are unique, whereas the Stiefel coordinates are clearly not: Multiplying A with any invertible $(n-l)\times(n-l)$ matrix does not change its kernel. Hence, when denoting by $S(n-l, n+1)$ the *Stiefel variety* of all complex $(n-l)\times(n+1)$ matrices of full rank, the Grassmannian $G(l+1, n+1)$ can be seen as the quotient $S(n-l, n+1)/GL(n-l)$.

Pick now a maximal linearly independent subset of columns of A, indexed by $\{i_1, \ldots, i_{n-l}\}$, and multiply the inverse of this submatrix by A itself. The resulting matrix has the same kernel as A and its columns indexed by $\{i_1, \ldots, i_{n-l}\}$ form the identity matrix. The remaining entries of this new matrix are the *primal affine coordinates* of L. These give a unique representation of L in the *primal affine chart*

$$U_{i_1 \ldots i_{n-l}} := \left\{ A \in \mathbb{C}^{(n-l)\times(n+1)} \;\middle|\; \begin{array}{l} i_j\text{-th column of } A \text{ is standard basis vector } e_j \\ (\text{for } j = 1, \ldots, n-l) \end{array} \right\}$$

of the Grassmannian $G(l+1, n+1)$.

Secondly, let $B \in \mathbb{C}^{(l+1)\times(n+1)}$ such that L is the projectivization of the row space of B. The entries of B are the *dual Stiefel coordinates* of L, and the maximal minors $q_{j_0 \ldots j_l}$ of B are the *dual Plücker coodinates* of L. As above, for every maximal linearly independent subset of columns of B, indexed by $\{j_0, \ldots, j_l\}$, the subspace L has unique *dual affine coordinates* in the *dual affine chart* $U_{j_0 \ldots j_l}$ of the Grassmannian $G(l+1, n+1)$.

We will now argue that the primal coordinates of $G(l+1, n+1)$ are the dual coordinates of $G(n-l, n+1)$. The isomorphism between $G(l+1, n+1)$ and $G(n-l, n+1)$ sends a linear subspace to its orthogonal complement (*not* with respect to the complex scalar product $(x, y) \mapsto \sum x_i \overline{y_i}$, but with respect to the non-degenerate bilinear form $(x, y) \mapsto \sum x_i y_i$). If $L \in G(l+1, n+1)$ is the row space of $B := (I_{l+1}|M)$, where I_k denotes the k-dimensional identity matrix and M is an $(l+1)\times(n-l)$-matrix, then $L^\perp$ is the row space of $A := (-M^T|I_{n-l})$. Moreover, the kernel of A is L and the kernel of B is $L^\perp$, since we have

$$\ker(N) = \operatorname{rs}(N)^\perp$$

for any matrix N with row space $\operatorname{rs}(N)$. Thus the isomorphism between $G(l+1, n+1)$ and $G(n-l, n+1)$ on the same coordinates in both Grassmannians coincides with the map between primal and dual coordinates within the same Grassmannian. For the Plücker coordinates it follows that

$$q_{j_0\ldots j_l} = s(i_1,\ldots,i_{n-l})\cdot p_{i_1\ldots i_{n-l}}, \tag{1}$$

where $i_1,\ldots,i_{n-l}$ form the complement of $\{j_0,\ldots,j_l\}$ in strictly increasing order, and $s(i_1,\ldots,i_{n-l})$ denotes the sign of the permutation $(i_1,\ldots,i_{n-l},j_0,\ldots,j_l)$.

Example 2. Consider the following polynomial in the primal Plücker coordinates of the Grassmannian $G(2,4)$ of lines in $\mathbb{P}^3$, which will appear again in Examples 7 and 19:

$$\begin{aligned}
&(p_{01}^6+p_{02}^6+p_{03}^6+p_{12}^6+p_{13}^6+p_{23}^6)\\
+2&(p_{10}^3p_{02}^3+p_{10}^3p_{03}^3+p_{20}^3p_{03}^3+p_{01}^3p_{12}^3+p_{01}^3p_{13}^3+p_{21}^3p_{13}^3)\\
+2&(p_{02}^3p_{21}^3+p_{02}^3p_{23}^3+p_{12}^3p_{23}^3+p_{03}^3p_{31}^3+p_{03}^3p_{32}^3+p_{13}^3p_{32}^3)\\
+2&\left(p_{01}p_{23}(p_{03}^2p_{12}^2-p_{02}^2p_{13}^2)-p_{02}p_{13}(p_{01}^2p_{23}^2-p_{03}^2p_{12}^2)+p_{03}p_{12}(p_{02}^2p_{13}^2-p_{01}^2p_{23}^2)\right).
\end{aligned} \tag{2}$$

To display the symmetry of the polynomial, the convention $p_{ji}=-p_{ij}$ for $i<j$ is used. The change of coordinates

$$p_{01}\mapsto q_{23}, p_{02}\mapsto -q_{13}, p_{03}\mapsto q_{12}, p_{12}\mapsto q_{03}, p_{13}\mapsto -q_{02}, p_{23}\mapsto q_{01},$$

yields the polynomial in dual Plücker coordinates:

$$\begin{aligned}
&(q_{01}^6+q_{02}^6+q_{03}^6+q_{12}^6+q_{13}^6+q_{23}^6)\\
-2&(q_{10}^3q_{02}^3+q_{10}^3q_{03}^3+q_{20}^3q_{03}^3+q_{01}^3q_{12}^3+q_{01}^3q_{13}^3+q_{21}^3q_{13}^3)\\
-2&(q_{02}^3q_{21}^3+q_{02}^3q_{23}^3+q_{12}^3q_{23}^3+q_{03}^3q_{31}^3+q_{03}^3q_{32}^3+q_{13}^3q_{32}^3)\\
+2&\left(q_{01}q_{23}(q_{03}^2q_{12}^2-q_{02}^2q_{13}^2)-q_{02}q_{13}(q_{01}^2q_{23}^2-q_{03}^2q_{12}^2)+q_{03}q_{12}(q_{02}^2q_{13}^2-q_{01}^2q_{23}^2)\right).
\end{aligned} \tag{3}$$

The polynomial in primal or dual Stiefel coordinates is obtained by substituting the 2×2-minor given by the columns i and j of a general 2×4-matrix $\left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\end{smallmatrix}\right)$ into p_{ij} or q_{ij}, respectively. Similarly, one gets the polynomial in primal or dual affine coordinates by using a matrix of the form $\left(\begin{smallmatrix} 1 & 0 & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\end{smallmatrix}\right)$, i.e., by substituting

$$p_{01}\mapsto 1, p_{02}\mapsto a_{23}, p_{03}\mapsto a_{24}, p_{12}\mapsto -a_{13}, p_{13}\mapsto -a_{14}, p_{23}\mapsto a_{13}a_{24}-a_{23}a_{14}.$$

3 Cayley Trick

Throughout the rest of this article, let $X\subseteq\mathbb{P}^n$ be an irreducible non-empty projective variety over $\mathbb{C}$ with $k:=\dim(X)<n$. For $0\le i\le k$ and a projective subspace $L\subseteq\mathbb{P}^n$ of dimension $n-k+i-1$, the dimension of L intersected with T_xX equals $i-1$ for almost all $x\in X$. Those subspaces that have a larger intersection with some T_xX (for $x\in X$ smooth) form a subvariety of the respective Grassmannian.

Definition 3. For $i\in\{0,\ldots,k\}$, the *i-th coisotropic variety of X* is defined as

$$CH_i(X) := \overline{\{L\in G(n-k+i,n+1)\mid \exists x\in X \text{ smooth}: x\in L, \dim(L\cap T_xX)\ge i\}}.$$

In [GKZ94] these varieties are called *higher associated hypersurfaces* (cf. Chapter 3, Section 2E and Chapter 4, Section 3). For $i=0$ the above definition reduces to

$$CH_0(X) = \{L\in G(n-k,n+1)\mid X\cap L\neq\emptyset\}.$$

As we will see in Corollary 5, the variety $CH_0(X)$ is an irreducible hypersurface for every X. Thus it is defined by an irreducible polynomial in the Plücker coordinates of $G(n-k,n+1)$, which is known as the *Chow form* of X. The case $i=1$ is studied in [Stu]. The variety $CH_1(X)$ is an irreducible hypersurface if $\deg(X)\geq 2$, and its corresponding polynomial in the Plücker coordinates of $G(n-k+1,n+1)$ is called the *Hurwitz form* of X. For general i, the condition $\dim(L\cap T_xX)\geq i$ is equivalent to $\dim(L+T_xX)\leq n-1$, meaning that L intersects X at x non-transversally.

The Chow form can be constructed as the dual of a Segre product, which is known as the *Cayley trick* [GKZ94, Ch. 3, Thm. 2.7]. This works for all coisotropic varieties: Consider the Segre embedding

$$\mathbb{P}^{k-i}\times X\hookrightarrow \mathbb{P}^{k-i}\times\mathbb{P}^n\hookrightarrow \mathbb{P}^{(k-i+1)(n+1)-1}.$$

With this, we will show that the projectively dual variety $(\mathbb{P}^{k-i}\times X)^\vee$ equals $CH_i(X)$, when the latter is interpreted in the primal matrix space $\mathbb{P}(\mathbb{C}^{(k-i+1)\times(n+1)})$. Formally, consider the following projection from the Stiefel variety $S(k-i+1,n+1)$ of all complex $(k-i+1)\times(n+1)$-matrices of full rank onto the ambient Grassmannian of $CH_i(X)$:

$$\begin{aligned} p: S(k-i+1,n+1) &\longrightarrow G(n-k+i,n+1),\\ A &\longmapsto \operatorname{proj}(\ker A). \end{aligned}$$

Then take the closure $\overline{p^{-1}(CH_i(X))}$ in the primal matrix space, which is identified with the dual space $(\mathbb{P}^{(k-i+1)(n+1)-1})^*$.

Proposition 4. *The varieties $\overline{p^{-1}(CH_i(X))}$ and $(\mathbb{P}^{k-i}\times X)^\vee$ in the matrix space $\mathbb{P}(\mathbb{C}^{(k-i+1)\times(n+1)})$ are equal.*

Proof. This proof follows the lines of the proof of Theorem 2.7 in Chapter 3 of [GKZ94].

Using the above identification, the variety $(\mathbb{P}^{k-i}\times X)^\vee$ is the closure of the set of all $A\in\mathbb{P}(\mathbb{C}^{(k-i+1)\times(n+1)})$ such that the hyperplane

$$\{A=0\}:=\left\{M\in\mathbb{P}(\mathbb{C}^{(k-i+1)\times(n+1)})\;\middle|\;\sum_{i,j}a_{ij}m_{ij}=0\right\}$$

is tangent to $\mathbb{P}^{k-i}\times X$ at some point (y,x) with $x\in X$ smooth. Denote by $\boldsymbol{x}\in\mathbb{C}^{n+1}$, $\boldsymbol{y}\in\mathbb{C}^{k-i+1}$ and $\boldsymbol{A}\in\mathbb{C}^{(k-i+1)\times(n+1)}$ affine representatives of x, y and A, respectively. Moreover, let $\operatorname{cone}(X)$ be the affine cone over X. Then we have

$$\begin{aligned} T_{(\boldsymbol{y},\boldsymbol{x})}\operatorname{cone}(\mathbb{P}^{k-i}\times X) &= \{(\boldsymbol{y}\otimes v)+(w\otimes\boldsymbol{x})\mid v\in T_{\boldsymbol{x}}\operatorname{cone}(X), w\in\mathbb{C}^{k-i+1}\}\\ &= (\boldsymbol{y}\otimes T_{\boldsymbol{x}}\operatorname{cone}(X))+(\mathbb{C}^{k-i+1}\otimes\boldsymbol{x})\subseteq\mathbb{C}^{(k-i+1)\times(n+1)}. \end{aligned}$$

Now consider $\overline{p^{-1}(CH_i(X))}$, which is the closure of the set of all $A\in\mathbb{P}(\mathbb{C}^{(k-i+1)\times(n+1)})$ with full rank such that $\boldsymbol{x}\in\ker\boldsymbol{A}$ and $\dim(\ker(\boldsymbol{A})\cap T_{\boldsymbol{x}}\operatorname{cone}(X))\geq i+1$ for some smooth point $x\in X$. Hence, it is enough the show the following two equivalences:

$$\begin{gathered} \{\boldsymbol{A}=0\}\supseteq\mathbb{C}^{k-i+1}\otimes\boldsymbol{x}\Leftrightarrow\boldsymbol{x}\in\ker\boldsymbol{A},\\ \exists\boldsymbol{y}\in\mathbb{C}^{k-i+1}\setminus\{0\}:\{\boldsymbol{A}=0\}\supseteq\boldsymbol{y}\otimes T_{\boldsymbol{x}}\operatorname{cone}(X)\Leftrightarrow\dim(\ker(\boldsymbol{A})\cap T_{\boldsymbol{x}}\operatorname{cone}(X))\geq i+1. \end{gathered}$$

Due to $\ker\boldsymbol{A}=\{u\in\mathbb{C}^{n+1}\mid\forall w\in\mathbb{C}^{k-i+1}:\sum_{i,j}a_{ij}w_iu_j=0\}$, the first equivalence is obvious. Consider the linear map $\psi: T_{\boldsymbol{x}}\operatorname{cone}(X)\to\mathbb{C}^{k-i+1}, v\mapsto\boldsymbol{A}v$ for the second equivalence. Since $\dim(T_{\boldsymbol{x}}\operatorname{cone}(X))=k+1$, the dimension of $\ker\psi=\ker(\boldsymbol{A})\cap T_{\boldsymbol{x}}\operatorname{cone}(X)$ is at least $i+1$ if and only if ψ is not surjective, which is equivalent to the existence of some non-zero $\boldsymbol{y}\in\mathbb{C}^{k-i+1}$ orthogonal to $\operatorname{im}\psi$, i.e., $\sum_{i,j}a_{ij}y_iv_j=0$ for all $v\in T_{\boldsymbol{x}}\operatorname{cone}(X)$. $\square$

The construction in Proposition 4 will be the main ingredient for many of the following proofs. Furthermore, it shows that the defining polynomials of the coisotropic varieties (in case they are all hypersurfaces) interpolate from the Chow form via the Hurwitz form to the X-*discriminant* which is the defining polynomial of $X^\vee$.

This raises immediately the next question when the i-th coisotropic variety indeed is a hypersurface in the Grassmannian.

Corollary 5. *$CH_i(X)$ is an irreducible hypersurface if and only if $i \leq k - \operatorname{codim} X^\vee + 1$.*

Proof. A fundamental theorem on the product of varieties is proven in [GKZ94, Ch. 1, Thm 5.5,]: For any irreducible variety $X \subseteq \mathbb{P}^n$, define $\mu(X) := \dim X + \operatorname{codim} X^\vee - 1$. Then for the product $X \times Y \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$of two such varieties $X_1 \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ we have

$$\mu(X \times Y) = \max\{\dim X + \dim Y, \mu(X), \mu(Y)\}. \tag{4}$$

Hence, the dual $(X \times Y)^\vee$ is a hypersurface in $(\mathbb{P}^{(n+1)(m+1)-1})^*$ if and only if the above maximum equals $\dim X + \dim Y$. This is in particular true for the variety $\mathbb{P}^{k-i}$ embedded into itself such that $(\mathbb{P}^{k-i})^\vee = \emptyset$ [GKZ94, Ch. 1, Cor. 5.9]. By convention, $\dim(\mathbb{P}^{k-i})^\vee = -1$, so $\mu(\mathbb{P}^{k-i}) = 2(k-i)$. Then $(\mathbb{P}^{k-i} \times X)^\vee$ is a hypersurface if and only if $2k - i \geq k + \operatorname{codim} X^\vee - 1$. $\square$

This motivates the following definition.

Definition 6. For $i \in \{0, \ldots, k - \operatorname{codim} X^\vee + 1\}$, the hypersurface $CH_i(X)$ is called the *i-th coisotropic hypersurface of X*. Its defining polynomial in the Plücker coordinates of $G(n-k+i, n+1)$ is called the *i-th coisotropic form of X*.

Example 7. Let $X \subseteq \mathbb{P}^3$ be the surface defined by the Fermat cubic $x_0^3+x_1^3+x_2^3+x_3^3$. The projectively dual of X is also a surface. Therefore, the surface X has three coisotropic hypersurfaces. The Chow form of X in dual Plücker coordinates of $G(1,4) \simeq \mathbb{P}^3$ is just the Fermat cubic itself. The Hurwitz form of X in primal and dual Plücker coordinates of $G(2,4)$ is given by the polynomials in Example 2. This was computed with `Macaulay2` [GS]. Finally, the second coisotropic form of X in primal Plücker coordinates p_i of $G(3,4)$ is the following polynomial of degree 12, which is also the defining equation of $X^\vee$:

$$\begin{aligned} &6(z_0^4+z_1^4+z_2^4+z_3^4) - 8(z_0^3+z_1^3+z_2^3+z_3^3)(z_0+z_1+z_2+z_3) \\ &+(z_0^2+z_1^2+z_2^2+z_3^2)^2 + 2(z_0^2+z_1^2+z_2^2+z_3^2)(z_0+z_1+z_2+z_3)^2 - 40z_0z_1z_2z_3, \end{aligned}$$

where $z_i := p_i^3$ for $0 \leq i \leq 3$.

4 Polar Degrees

After studying the dimension of the coisotropic hypersurfaces, the next focus will lie on their degrees. In fact, these degrees agree with the well-studied polar degrees [Pie78, Hol88]. As before, let $X \subseteq \mathbb{P}^n$ be a variety of dimension k. Moreover, let $0 \leq i \leq k$ and $V \subseteq \mathbb{P}^n$ be a projective subspace of dimension $n-k+i-2$. For almost all $x \in X$, the dimension of V intersected with T_xX equals $i-2$. Define the *i-th polar variety of X* as

$$P_i(X,V) := \overline{\{x \in X \text{ smooth} \mid \dim(V \cap T_xX) \geq i-1\}}.$$

Given a general X, the i-th polar variety has codimension i in X for almost all choices of V. Furthermore, for any X there exists an integer $\delta_i(X)$ that is equal to the degree of $P_i(X,V)$ for almost all V. Hence, we can define the *i-th polar degree of* X independently of the choice of a subspace.

Definition 8. For $i \in \{0,\dots,k\}$, the *i-th polar degree of* X is denoted by

$$\delta_i(X) := \deg P_i(X,V),$$

where $V \subseteq \mathbb{P}^n$ is a generic projective subspace of dimension $\dim V = n-k+i-2$.

These degrees satisfy a lot of interesting properties:

1. $\delta_i(X) > 0$ if and only if $i \leq k - \operatorname{codim} X^\vee + 1$.
   (Note that this coincides with the range of indices where the coisotropic varieties of X are hypersurfaces.)
2. $\delta_0(X) = \deg X$.
3. $\delta_{k-\operatorname{codim} X^\vee+1}(X) = \deg X^\vee$.
4. $\delta_i(X) = \delta_{k-\operatorname{codim} X^\vee+1-i}(X^\vee)$.
5. $\delta_i(X\cap H) = \delta_i(X)$ for any $0 \leq i \leq k-1$ and any generic hyperplane $H \subseteq \mathbb{P}^n$.
6. $\delta_i(\pi(X)) = \delta_i(X)$ if $\operatorname{codim} X \geq 2$, where the irreducible subvariety $\pi(X) \subseteq \mathbb{P}^{n-1}$ is the image of a general linear projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$.

One can also define the polar degrees via the *conormal variety*

$$\mathcal{N}_X := \overline{\{(x,y) \in \mathbb{P}^n \times \mathbb{P}^n \mid x \in X \text{ smooth}, T_xX \subseteq \{y=0\}\}}.$$

The *multidegree* (defined in Section 8.5 of [MS04]) of the $\mathbb{Z}^2$-graded ring $\mathbb{C}[x,y]/I(\mathcal{N}_X)$ is a homogeneous polynomial of degree $n+1$ in two new variables whose non-zero coefficients are the polar degrees. Using the command `multidegree` in `Macaulay2`, this gives a practical way to compute the polar degrees of a given variety X.

Now another property will be added to this list, namely that the degree of the i-th coisotropic hypersurface of X is the i-th polar degree of X. On first sight, this is remarkable since the coisotropic hypersurfaces are subvarieties of a Grassmannian, whereas the polar varieties are subvarieties of the projective variety $X \subseteq \mathbb{P}^n$. The degree of a hypersurface $\Sigma \subseteq G(l+1,n+1)$ is defined as

$$\deg \Sigma := |\{L \in \Sigma \mid N \subseteq L \subseteq M\}|,$$

where $N \subseteq M \subseteq \mathbb{P}^n$ is a generic flag of $(l-1)$-dimensional and $(l+1)$-dimensional projective subspaces. Proposition 2.1 in Chapter 3 of [GKZ94] shows that every irreducible hypersurface of degree d in a Grassmannian is given by a polynomial of degree d in the Plücker coordinates, and this polynomial is unique up to the Plücker relations and constant factors.

Theorem 9. *For $0 \leq i \leq k-\operatorname{codim} X^\vee+1$, the degree of the i-th coisotropic hypersurface of X equals the i-th polar degree of X, i.e.,*

$$\deg CH_i(X) = \delta_i(X).$$

Proof. Let $0 \leq d \leq k$. For $0 \leq i \leq k-d$ and a generic subspace $M \subseteq \mathbb{P}^n$ of codimension d we have $\delta_i(X \cap M) = \delta_i(X)$ by applying the fifth property above several times. Fix now $0 \leq i \leq k - \operatorname{codim} X^\vee + 1$ and set $d := k-i$. Choose a generic M of codimension d as well as a generic subspace $N \subseteq M$ of dimension $\dim N = \dim M - \dim(X \cap M) + i - 2 = n-k+i-2$. Then the i-th polar degree of X equals $\deg P_i(X \cap M, N)$. Since $X \cap M$ is i-dimensional, it follows from the first and the third property above that the dual $(X \cap M)^\vee$ is a hypersurface in M^* with degree $\delta_i(X \cap M)$. To sum up,

$$\delta_i(X) = \deg P_i(X \cap M, N) = \delta_i(X \cap M) = \deg(X \cap M)^\vee.$$

The degree of the hypersurface $(X \cap M)^\vee$ is also the number of hyperplanes in M that are tangent to some smooth point of $X \cap M$ and that contain N, but these hyperplanes are exactly the subspaces in $CH_i(X)$ with $N \subseteq L \subseteq M$. $\square$

Remark 10. For general projective varieties X, we can give another geometric argument to show Theorem 9. As above, let V be a generic projective subspace of dimension $n-k+i-2$. Consider the variety $S_i(X, V) \subseteq \mathbb{P}^n$ formed by the union of all lines through V and the i-th polar variety $P_i(X, V)$. For general X, the i-th polar variety has codimension i in X and $S_i(X, V)$ is a hypersurface of degree $\delta_i(X)$. The i-th coisotropic form of X in dual Stiefel coordinates is a polynomial in the entries of a general $(n-k+i) \times (n+1)$-matrix B. Substituting the last rows of that matrix by a basis of V yields a homogeneous polynomial $F \in \mathbb{C}[b_{0j} \mid 0 \leq j \leq n]$, whose degree is the degree of the i-coisotropic hypersurface of X. This polynomial defines an irreducible hypersurface in $\mathbb{P}^n$, which is in fact $S_i(X, V)$. This shows Theorem 9 for general X.

To see that $S_i(X, V)$ and the zero locus of F are the same, it is enough to show that F vanishes at every point in $S_i(X, V)$. This is clear for all points in $V \subseteq S_i(X, V)$. For a point $y \notin V$ on the line between $x \in P_i(X, V)$ and some point in V we have that F vanishes at y if and only if it vanishes at x. If x is a smooth point of X such that the dimension of $V \cap T_x X$ is at least $i-1$, then the projective span of V and x is a point in $CH_i(X)$ and F vanishes at x. Since the set of all those x is open and dense in $P_i(X, V)$, all points in $P_i(X, V)$ are in the zero locus of F.

5 Rank One Characterizations

In the following, several equivalent characterizations for coisotropic hypersurfaces will be given. A very fundamental equivalence statement is proven in Theorem 4.13 in Chapter 4 of [GKZ94]. This proof contains many geometric ideas by taking a detour over conormal varieties and Lagrangian varieties in general. Here a new and direct proof will be presented, using just the Cayley trick from Proposition 4. For this, another notion of coisotropy will be defined first, which is for now not associated to any projective variety X:

Definition 11. Let U and V be finite dimensional vector spaces. Define $W := \operatorname{Hom}(U, V)$, and identify W^* with $\operatorname{Hom}(V, U)$ via $\operatorname{Hom}(V, U) \ni \phi \mapsto \operatorname{tr}(\cdot \circ \phi)$. A hyperplane $H \subseteq W$ is called *coisotropic* if its defining equation has rank one in $\operatorname{Hom}(V, U)$.

More precisely, choose bases for U and V, and consider the matrix E_{ij} with exactly one 1-entry at position (i, j) (0-entries otherwise) as an element in W. Let $\psi \in W^*$ be the equation for the hyperplane H, and define N_ψ as the matrix whose (i, j)-th entry equals $\psi(E_{ji})$. The coisotropy condition for H means exactly that the rank of N_ψ equals one. This definition can be extended to hypersurfaces in a Grassmannian, since the tangent space of $G(l+1, n+1)$ at some L is naturally isomorphic to $\operatorname{Hom}(L, \mathbb{C}^{n+1}/L)$.

Definition 12. An irreducible hypersurface Σ in the Grassmannian $G(l+1, n+1)$ is called *coisotropic* if, for every smooth $L \in \Sigma$, the tangent hyperplane $T_L\Sigma$ is coisotropic in $T_L G(l+1, n+1)$.

To be more concrete, let $L \in G(l+1, n+1)$ be given in primal affine coordinates by an $(n-l) \times (l+1)$-matrix M_L. This means that for some fixed $i_1, \ldots, i_{n-l}$, the subspace L is the kernel of an $(n-l) \times (n+1)$-matrix A whose columns indexed by $i_1, \ldots, i_{n-l}$ are standard basis vectors and the remaining columns form M_L. Let $\rho := \rho_{i_1,\ldots,i_{n-l}}$ be the map that sends a given matrix M_L to its corresponding primal Plücker coordinates, namely the maximal minors of A. Thus, if Q denotes the polynomial in primal Plücker coordinates that defines Σ, then $Q \circ \rho$ is the equation for Σ in the primal affine chart $U_{i_1,\ldots,i_{n-l}}$. Identifying the tangent hyperplane of Σ at L with

$$T_L\Sigma = \left\{ \dot{M} \in \mathbb{C}^{(n-l)\times(l+1)} \;\middle|\; \sum_{i,j} \frac{\partial(Q \circ \rho)}{\partial a_{ij}}(M_L) \cdot \dot{M}_{ij} = 0 \right\}$$

yields that $T_L\Sigma$ is coisotropic if and only if the rank of the matrix with entries $\frac{\partial(Q\circ\rho)}{\partial a_{ij}}(M_L)$ is one. To ease notation, we denote this matrix by $J_{Q\circ\rho}(M_L)$; in general:

Definition 13. For a polynomial $F \in \mathbb{C}[a_{ij} \,|\, 1 \le i \le l, 1 \le j \le m]$, let J_F be the $(l \times m)$-matrix of all first-order partial derivatives of F, i.e., the (i,j)-th entry of J_F is $\frac{\partial F}{\partial a_{ij}}$.

Theorem 3.14 in Chapter 4 of [GKZ94] states that the two notions of coisotropy given in Definitions 3 and 12 are in fact the same:

Theorem 14. *1. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety with $k := \dim X$, and let $i \in \{0, \ldots, k - \operatorname{codim} X^\vee + 1\}$. Then $CH_i(X)$ is coisotropic.*

2. Let $\Sigma \subseteq G(l+1, n+1)$ be an irreducible coisotropic hypersurface. Then there exists an irreducible projective variety $X \subseteq \mathbb{P}^n$ such that $\Sigma = CH_{\dim X + l + 1 - n}(X)$.

This is remarkable since Definition 12 does not depend on the underlying projective variety X. Before proving the theorem, some helpful and easy characterizations of coisotropy will be established. The first one says that it is enough to check coisotropy for a hypersurface in the Grassmannian on *one fixed affine chart* of the Grassmannian to deduce coisotropy for the whole hypersurface. This statement can also be found as Proposition 3.12 in Chapter 4 of [GKZ94].

Proposition 15. *Let $\Sigma \subseteq G(l+1, n+1)$ be an irreducible hypersurface, given by a homogeneous polynomial Q in primal Plücker coordinates. Moreover, fix a primal affine chart $U_{i_1,\ldots,i_{n-l}}$ of $G(l+1, n+1)$ together with the map $\rho := \rho_{i_1,\ldots,i_{n-l}}$ sending $(n-l)\times(l+1)$-matrices to their corresponding primal Plücker coordinates. Then Σ is coisotropic (in the sense of Definition 12) if and only if the 2×2-minors of $J_{Q\circ\rho}(M)$ are zero for all $M \in V(Q \circ \rho)$.*

Proof. Assume first that Σ is coisotropic. For every smooth point $L \in \Sigma$ representable in the affine chart $U_{i_1,\ldots,i_{n-l}}$ by the $(n-l) \times (l+1)$-matrix M_L, we have that the rank of $J_{Q\circ\rho}(M_L)$ is one. For all singular points this rank is zero. This proves one direction.

For the other direction, assume that $J_{Q\circ\rho}(M)$ has rank at most one for all $(n-l)\times(l+1)$-matrices $M \in V(Q \circ \rho)$, i.e., that $M \in V(Q \circ \rho)$ has rank exactly one for all smooth M.

If $Q = p_{i_1 \dots i_{n-l}}$, then one can directly calculate that the coisotropy condition is also satisfied on all other affine charts; hence Σ is coisotropic. Otherwise, let $L \in \Sigma$ smooth be representable in the affine chart $U_{i'_1,\dots,i'_{n-l}}$ by the $(n-l) \times (l+1)$-matrix M_L, and let $\rho' := \rho_{i'_1,\dots,i'_{n-l}}$ be the corresponding Plücker map, where $(i'_1, \dots, i'_{n-l})$ is different from $(i_1, \dots, i_{n-l})$. It is left to show that the rank of $J_{Q \circ \rho'}(M_L)$ is one. Since the smooth points in Z form an open subset of Z, there is an open neighborhood $\mathcal{U}$ of M_L such that all $M \in \mathcal{U} \cap V(Q \circ \rho')$ correspond to smooth points in Z. Due to the assumption that $Q \neq p_{i_1 \dots i_{n-l}}$, every open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of M_L contains a point $M \in \mathcal{V} \cap V(Q \circ \rho')$ whose corresponding smooth point in Z can be represented in the affine chart $U_{i_1,\dots,i_{n-l}}$. By assumption, $J_{Q \circ \rho'}(M)$ has rank one for all those M. It follows from the continuity of the 2×2-minors of $J_{Q \circ \rho'}$ that the rank of $J_{Q \circ \rho'}(M_L)$ is also one. $\square$

It is sometimes more practical to work in Stiefel coordinates than in an affine chart. For example, the defining polynomial of a hypersurface in a Grassmannian is still homogeneous when written in Stiefel coordinates, but generally not in affine coordinates. Furthermore, the Cayley trick from Proposition 4 uses Stiefel coordinates. To give the coisotropic characterization in Stiefel coordinates (Proposition 17), a technical lemma is needed.

Lemma 16. *Denote by* pl *the map that sends a given matrix to its maximal minors. Let Q be a homogeneous polynomial in Plücker coordinates in $G(l+1, n+1)$, and let $N \in \mathbb{C}^{(n-l)\times(n+1)}$ as well as $U \in \mathbb{C}^{(n-l)\times(n-l)}$ with $\det(U) \neq 0$. Then we have*

$$J_{Q \circ \mathrm{pl}}(UN) = \det(U)^{\deg(Q)-1} \mathrm{adj}(U)^T J_{Q\circ \mathrm{pl}}(N). \tag{5}$$

Proof. For $m \in \mathbb{N}$, let $[m] := \{1, \dots, m\}$. Note first that

$$\frac{\partial \det}{\partial a_{ij}}(A) = (-1)^{i+j} \det(A_{[n-l]\setminus\{i\},[n-l]\setminus\{j\}}) = \mathrm{adj}(A)_{ji},$$

holds for all matrices $A \in \mathbb{C}^{(n-l)\times(n-l)}$. This identity implies

$$\begin{aligned} \frac{\partial \det}{\partial a_{ij}}(UA) &= \mathrm{adj}(UA)_{ji} = (\mathrm{adj}(A)\mathrm{adj}(U))_{ji} = \sum_{m=1}^{n-l} \mathrm{adj}(A)_{jm}\mathrm{adj}(U)_{mi} \\ &= (-1)^{j+i} \sum_{m=1}^{n-l} \det(A_{[n-l]\setminus\{m\},[n-l]\setminus\{j\}}) \det(U_{[n-l]\setminus\{i\},[n-l]\setminus\{m\}}). \end{aligned}$$

For $I \subseteq [n+1]$ with $|I| = n-l$ and $j \in I$, let $\mathrm{pos}(j, I) \in [n-l]$ denote the position of $j \in I$ when I is ordered strictly increasing (i.e., j is the $\mathrm{pos}(j, I)$-th element of I). Then:

$$\begin{aligned} \frac{\partial (Q \circ \mathrm{pl})}{\partial a_{ij}}(N) &= \sum_I \frac{\partial Q}{\partial p_I}(\mathrm{pl}(N)) \frac{\partial p_I}{\partial a_{ij}}(N) \\ &= \sum_{I : j \in I} \frac{\partial Q}{\partial p_I}(\mathrm{pl}(N))(-1)^{i+\mathrm{pos}(j,I)} \det(N_{[n-l]\setminus\{i\},I\setminus\{j\}}). \end{aligned}$$

Finally, set $d := \deg(Q) - 1$ and show that the entries of the two matrices in (5) are equal:

$$\frac{\partial (Q \circ \mathrm{pl})}{\partial a_{ij}}(UN)$$

$$
\begin{aligned}
&= \sum_{I:j\in I} \frac{\partial Q}{\partial p_I}(\det(U)\mathrm{pl}(N))(-1)^{i+\mathrm{pos}(j,I)} \sum_{m=1}^{n-l} \det(U_{[n-l]\setminus\{i\},[n-l]\setminus\{m\}}) \det(N_{[n-l]\setminus\{m\},I\setminus\{j\}}) \\
&= \det(U)^d \sum_{m=1}^{n-l} \det(U_{[n-l]\setminus\{i\},[n-l]\setminus\{m\}}) \\
&\quad \cdot (-1)^{i+m} \sum_{I:j\in I} \frac{\partial Q}{\partial p_I}(\mathrm{pl}(N))(-1)^{m+\mathrm{pos}(j,I)} \det(N_{[n-l]\setminus\{m\},I\setminus\{j\}}) \\
&= \det(U)^d \sum_{m=1}^{n-l} \det(U_{[n-l]\setminus\{i\},[n-l]\setminus\{m\}})(-1)^{i+m} \frac{\partial(Q\circ \mathrm{pl})}{\partial a_{mj}}(N) \\
&= \det(U)^d \sum_{m=1}^{n-l} \mathrm{adj}(U)_{mi} \frac{\partial(Q\circ \mathrm{pl})}{\partial a_{mj}}(N).
\end{aligned}
$$
□

Proposition 17. *Let $l \geq 1$, and let $\Sigma \subseteq G(l+1, n+1)$ be an irreducible hypersurface, given by a homogeneous polynomial Q in primal Plücker coordinates. Then Σ is coisotropic (in the sense of Definition 12) if and only if the 2×2-minors of $J_{Q\circ\mathrm{pl}}(N)$ are zero for all $N \in V(Q\circ \mathrm{pl})$.*

Proof. First assume that the rank of $J_{Q\circ\mathrm{pl}}(N)$ is at most one for all $N \in V(Q\circ\mathrm{pl})$. By Proposition 15, it is enough to consider the affine chart $U_{1,\dots,n-l}$ together with the Plücker map $\rho := \rho_{1,\dots,n-l}$. Hence, it is sufficient to show that the rank of $J_{Q\circ\rho}(M)$ is at most one when $M \in V(Q\circ\rho)$. Indeed, $(I_{n-l}|M) \in V(Q\circ\mathrm{pl})$ and the rank of $J_{Q\circ\mathrm{pl}}(I_{n-l}|M)$ is at most one by assumption. Since the last $l+1$ columns of $J_{Q\circ\mathrm{pl}}(I_{n-l}|M)$ coincide with $J_{Q\circ\rho}(M)$, it follows that $J_{Q\circ\rho}(M)$ has rank at most one.

Secondly, let Σ be coisotropic, and let $N \in V(Q\circ\mathrm{pl})$. One can assume that all $(n-l)\times(n-l)$-minors of N are non-zero, since this always holds up to a change of coordinates. We now fix a 2×2-minor of $J_{Q\circ\mathrm{pl}}$ and show that it vanishes on N. Without loss of generality, this minor does not contain entries from the first $n-l$ columns of $J_{Q\circ\mathrm{pl}}$ (by the assumption $l \geq 1$). Hence, it is enough to show that the last $l+1$ columns of $J_{Q\circ\mathrm{pl}}(N)$ form a matrix of rank at most one. Since the first $n-l$ columns of N form an invertible matrix U, one can write $N = U\cdot(I_{n-l}|M)$ where M is an $(n-l)\times(l+1)$-matrix. Moreover, $\rho := \rho_{1,\dots,n-l} = \mathrm{pl}(I_{n-l}|\cdot)$ implies

$$
\frac{\partial(Q\circ\rho)}{\partial m_{ij}}(M) = \frac{\partial(Q\circ\mathrm{pl}(I_{n-l}|\cdot))}{\partial m_{ij}}(M) = \frac{\partial(Q\circ\mathrm{pl})}{\partial m_{ij}}(I_{n-l}|M).
$$

Together with Lemma 16 and $M \in V(Q\circ\rho)$, this leads to

$$
\begin{aligned}
\mathrm{rank}\left(J_{Q\circ\mathrm{pl}}(N)_{\{n-l+1,\dots,n+1\}}\right) &= \mathrm{rank}\left(\det(U)^d \mathrm{adj}(U)^T J_{Q\circ\mathrm{pl}}(I_{n-l}|M)_{\{n-l+1,\dots,n+1\}}\right) \\
&\leq \mathrm{rank}\left(J_{Q\circ\mathrm{pl}}(I_{n-l}|M)_{\{n-l+1,\dots,n+1\}}\right) \\
&= \mathrm{rank}\left(J_{Q\circ\rho}(M)\right) \leq 1,
\end{aligned}
$$

where $A_{\{n-l+1,\dots,n+1\}}$ denotes the last $l+1$ columns of an $(n-l)\times(n+1)$-matrix A. □

The above proposition uses the assumption $l \geq 1$ to place a 2×2-minor outside of a maximal square submatrix. Note that for $l = 0$, every hypersurface in $G(1, n+1) \simeq \mathbb{P}^n$ is coisotropic. With this, it can be proven that the two given definitions of coisotropy coincide.

Proof of Theorem 14. For the first part, let Q be the defining polynomial of $CH_i(X)$ in the primal Plücker coordinates of $G(n-k+i, n+1)$. By Proposition 4, $Q \circ \mathrm{pl}$ is the defining equation of $(\mathbb{P}^{k-i} \times X)^\vee$. If $n-k+i \leq 1$ or $n-k+i \geq n$, then $CH_i(X)$ is trivially coisotropic. Thus assume $1 < n-k+i < n$. By Proposition 17, it is enough to show that the rank of $J_{Q\circ \mathrm{pl}}(N)$ equals one for all $N \in (\mathbb{P}^{k-i} \times X)^\vee$ smooth. For such an N, denote by H_N the tangent hyperplane to $(\mathbb{P}^{k-i} \times X)^\vee$ at N. Then H_N is a point in $\left((\mathbb{P}^{k-i} \times X)^\vee\right)^\vee \simeq \mathbb{P}^{k-i} \times X$, by biduality. Since the defining equation for the hyperplane H_N is given by $J_{Q\circ \mathrm{pl}}(N)$, i.e.,

$$H_N = \left\{ M \in \mathbb{P}\left(\mathbb{C}^{(k-i+1)\times(n+1)}\right) \;\middle|\; \sum_{i,j} \frac{\partial(Q\circ \mathrm{pl})}{\partial a_{ij}}(N) \cdot M_{ij} = 0 \right\},$$

the point in $\mathbb{P}^{k-i} \times X$ corresponding to H_N is $J_{Q\circ \mathrm{pl}}(N)$. Hence, $J_{Q\circ \mathrm{pl}}(N) \in \mathbb{P}^{k-i} \times X$, which implies that the rank of $J_{Q\circ \mathrm{pl}}(N)$ is one.

Consider now the second part of the theorem. Let again Q be the defining polynomial of $\Sigma \subseteq G(l+1, n+1)$ in primal Plücker coordinates, such that $Q\circ\mathrm{pl}$ is the defining equation in primal Stiefel coordinates. If $l = 0$, then $\Sigma \subseteq \mathbb{P}^n$ and $CH_0(\Sigma) = \Sigma$. Therefore, assume that $l \geq 1$. Since $V(Q\circ \mathrm{pl})$ is a hypersurface, its dual consists of all tangent hyperplanes at its smooth points. This means that this dual can be identified – as above – with

$$V(Q\circ \mathrm{pl})^\vee = \overline{\{J_{Q\circ\mathrm{pl}}(N) \mid N \in V(Q\circ \mathrm{pl}), \mathrm{rank}\,(J_{Q\circ\mathrm{pl}}(N)) \neq 0\}}.$$

By Proposition 17, the rank of $J_{Q\circ\mathrm{pl}}(N)$ is at most one for all $N \in V(Q\circ\mathrm{pl})$, which implies that $V(Q\circ \mathrm{pl})^\vee \subseteq \mathbb{P}^{n-l-1} \times \mathbb{P}^n$. For all smooth $N \in V(Q\circ\mathrm{pl})$, denote by $x_N \in \mathbb{P}^n$ the projective point spanning the 1-dimensional affine row space of $J_{Q\circ\mathrm{pl}}(N) \in \mathbb{C}^{(n-l)\times(n+1)}$. This defines an irreducible projective variety

$$X := \overline{\{x_N \mid N \in V(Q\circ \mathrm{pl}), \mathrm{rank}\,(J_{Q\circ\mathrm{pl}}(N)) \neq 0\}} \subseteq \mathbb{P}^n$$

such that $V(Q\circ\mathrm{pl})^\vee \subseteq \mathbb{P}^{n-l-1} \times X$. In fact, equality holds: $V(Q\circ\mathrm{pl})^\vee = \mathbb{P}^{n-l-1} \times X$. To see this, consider $N \in V(Q\circ\mathrm{pl})$ smooth and $y = (y_1 : \ldots : y_{n-l}) \in \mathbb{P}^{n-l-1}$. Without loss of generality, $y_1 = 1$ and the m-th column of $J_{Q\circ\mathrm{pl}}(N)$ – denoted by $v \in \mathbb{C}^{n-l}$ – is not 0. Then we have for all $0 \leq j \leq n$ that the j-th column of $J_{Q\circ\mathrm{pl}}(N)$ is a scalar multiple $\mu_j v$ such that $(\mu_0 : \ldots : \mu_n) = x_N$. Pick now a basis $(w^{(2)}, \ldots, w^{(n-l)})$ of $\{w \in \mathbb{C}^{n-l} \mid \sum_{i=1}^{n-l} \overline{w_i} v_i = 0\}$, and let $a_1 := \overline{v}$ and $a_i := y_i\overline{v} + \overline{w^{(i)}}$ (for $2 \leq i \leq n-l$) be the rows of a matrix $A \in \mathbb{C}^{(n-l)\times(n-l)}$. Then A is invertible and $A \cdot J_{Q\circ\mathrm{pl}}(N)$ equals $y \otimes x_N$ (up to scaling). Its adjugate matrix $U := \mathrm{adj}(A^T)$ is also invertible, $UN \in V(Q\circ\mathrm{pl})$ smooth, and by Lemma 16,

$$J_{Q\circ\mathrm{pl}}(UN) = \det(U)^{\deg(Q)-1} \det(A)^{n-l-2} A \cdot J_{Q\circ\mathrm{pl}}(N)$$

is equal to $y \otimes x_N$ (up to scaling). It follows that $V(Q\circ\mathrm{pl}) = (\mathbb{P}^{n-l-1} \times X)^\vee$, which concludes the proof using the Cayley trick in Proposition 4. $\square$

Theorem 14 and Propositions 15 and 17 provide practical tools to check for coisotropy of a given hypersurface in a Grassmannian. Moreover, the the Cayley trick in Proposition 4 and the proof of Theorem 14 give ways to recover the underlying projective variety.

Example 18. All coisotropic forms in the Grassmannian $G(2, 4)$ of lines in $\mathbb{P}^3$ are either Chow forms of curves or Hurwitz forms of surfaces.

Example 19. As in Example 7, let $X \subseteq \mathbb{P}^3$ be the surface defined by the Fermat cubic $x_0^3 + x_1^3 + x_2^3 + x_3^3$. The polynomials (2) and (3) given in Example 2 are the Hurwitz form of X in primal and dual Plücker coordinates. Thus both polynomials define the same hypersurface $CH_1(X) \subseteq G(2,4)$. Now consider the hypersurface Σ in the same Grassmannian $G(2,4)$ whose defining equation in primal Plücker coordinates is the second polynomial (3). Then its defining equation in dual Plücker coordinates is the first polynomial (2). Geometrically this hypersurface Σ is obtained from $CH_1(X)$ by sending every line $L \in CH_1(X)$ to its orthogonal complement. Using the characterization of coisotropy in Proposition 15, it follows that Σ is also coisotropic. By Theorem 14, the hypersurface Σ has an underlying projective variety: It turns out that Σ is the first coisotropic hypersurface of the surface $X^\vee$ dual to X, which has degree 12 and is given by the polynomial in Example 7. Hence, the two similar polynomials of degree 6 from Example 2 are the Hurwitz forms of a surface of degree 3 and a surface of degree 12.

6 Duality

So far only characterizations of coisotropy using primal coordinates were treated. The case of dual coordinates will be considered now by proceeding as in Example 19. For a hypersurface $\Sigma \subseteq G(l+1, n+1)$, denote by $\Sigma^\perp \subseteq G(n-l, n+1)$ the corresponding hypersurface given by sending every $L \in \Sigma$ to its orthogonal complement $L^\perp$. If Σ is given by a homogeneous polynomial Q in primal Plücker coordinates, the equation for $\Sigma^\perp$ is derived by applying the change of coordinates in (1) to Q (which is then also the equation for Σ in dual Plücker coordinates). It follows immediately from Proposition 15 that Σ is coisotropic if and only if $\Sigma^\perp$ is coisotropic. Furthermore, since changing between the two above Grassmannians is the same as changing between primal and dual coordinates, it follows that Propositions 15 and 17 also hold for dual (instead of primal) coordinates. This raises the question of how the underlying projective varieties of Σ and $\Sigma^\perp$ are related.

Theorem 20. *Let $\Sigma \subseteq G(l+1, n+1)$ be an irreducible hypersurface. Then Σ is coisotropic if and only if $\Sigma^\perp \subseteq G(n-l, n+1)$ is coisotropic. In that case, their underlying projective varieties are projectively dual to each other. More precisely,*

$$CH_i(X)^\perp = CH_{\dim X - \operatorname{codim} X^\vee + 1 - i}(X^\vee),$$

where $X \subseteq \mathbb{P}^n$ is the irreducible variety such that $\Sigma = CH_i(X)$ for $i := \dim X + l + 1 - n$.

Proof. This proof uses the ideas and notations in the proof of the Cayley trick (Proposition 4). Consider again the projection $p : S(l+1, n+1) \to G(n-l, n+1), A \mapsto \operatorname{proj}(\ker A)$. By the Cayley trick, it is sufficient to show that

$$\overline{p^{-1}(\Sigma^\perp)} = (\mathbb{P}^l \times X^\vee)^\vee.$$

The variety $\overline{p^{-1}(\Sigma^\perp)}$ is the closure of the set of all $A \in \mathbb{P}(\mathbb{C}^{(l+1)\times(n+1)})$ with full rank such that the row space $\operatorname{rs}(\boldsymbol{A})$ of an affine representative $\boldsymbol{A}$ satisfies for some smooth $x \in X$:

$$\boldsymbol{x} \in \operatorname{rs}(\boldsymbol{A}) \text{ and } \dim(\operatorname{rs}(\boldsymbol{A}) \cap T_{\boldsymbol{x}}\operatorname{cone}(X)) \geq i+1. \tag{6}$$

The variety $(\mathbb{P}^l \times X^\vee)^\vee$ is the closure of the set of all A such that

$$\{\boldsymbol{A} = 0\} \supseteq \left(\mathbb{C}^{l+1} \otimes \boldsymbol{u}\right) + (\boldsymbol{v} \otimes T_{\boldsymbol{u}}\operatorname{cone}(X^\vee)). \tag{7}$$

for some $(v, u) \in \mathbb{P}^l \times X^\vee$ smooth. Hence, it is left to show that the projectivizations of all full rank matrices satisfying (6) for some smooth $x \in X$ lie in $(\mathbb{P}^l \times X^\vee)^\vee$, and that all projectivizations of matrices satisfying (7) for some smooth $(v, u) \in \mathbb{P}^l \times X^\vee$ are contained in $\overline{p^{-1}(\Sigma^\perp)}$.

Fix first a full rank matrix $\boldsymbol{A}$ and some $x \in X$ smooth with (6). The dimension of $\mathrm{rs}(\boldsymbol{A}) \cap T_{\boldsymbol{x}}\mathrm{cone}(X)$ is at least $i + 1$ if and only if the dimension of $\mathrm{rs}(\boldsymbol{A}) + T_{\boldsymbol{x}}\mathrm{cone}(X)$ is at most n. This is equivalent to the existence of a non-zero vector $\boldsymbol{u}$ orthogonal to $\mathrm{rs}(\boldsymbol{A}) + T_{\boldsymbol{x}}\mathrm{cone}(X)$. This means exactly that u is tangent to X at x and $\boldsymbol{u} \in \ker \boldsymbol{A}$, i.e., $\{\boldsymbol{A} = 0\} \supseteq \mathbb{C}^{l+1} \otimes \boldsymbol{u}$. Without loss of generality, u is smooth in $X^\vee$. Due to biduality, x is tangent to $X^\vee$ at u, and hence $\boldsymbol{x}$ orthogonal to $T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$. Because of $\boldsymbol{x} \in \mathrm{rs}(\boldsymbol{A})$, it further holds that $\boldsymbol{x}$ is orthogonal to $\ker(\boldsymbol{A})$. Therefore, the dimension of $\ker(\boldsymbol{A}) + T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$ is at most n, and the dimension of $\ker(\boldsymbol{A}) \cap T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$ is at least $\dim(X^\vee) - l$. This is equivalent to the existence of some $v \in \mathbb{C}^{l+1} \setminus \{0\}$ with $\{\boldsymbol{A} = 0\} \supseteq \boldsymbol{v} \otimes T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$. Thus, the matrix $\boldsymbol{A}$ satisfies (7) and its projectivization lies in $(\mathbb{P}^l \times X^\vee)^\vee$.

For the other direction, the complete argument can be reversed: Fix a full rank matrix $\boldsymbol{A}$ and some $(v, u) \in \mathbb{P}^l \times X^\vee$ smooth with (7). Due to $\{\boldsymbol{A} = 0\} \supseteq \boldsymbol{v} \otimes T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$, the dimension of $\ker(\boldsymbol{A}) + T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$ is at most n, which implies the existence of a non-zero vector $\boldsymbol{x}$ orthogonal to $\ker(\boldsymbol{A}) + T_{\boldsymbol{u}}\mathrm{cone}(X^\vee)$. Hence, $\boldsymbol{x} \in \mathrm{rs}(\boldsymbol{A})$ and x is tangent to $X^\vee$ at u. Without loss of generality, x is smooth in X. Then u is tangent to X at x, and $\boldsymbol{u}$ is orthogonal to $T_{\boldsymbol{x}}\mathrm{cone}(X)$. Because of $\{\boldsymbol{A} = 0\} \supseteq \mathbb{C}^{l+1} \otimes \boldsymbol{u}$, it also holds that $\boldsymbol{u}$ is orthogonal to $\mathrm{rs}(\boldsymbol{A})$. Hence, the dimension of $\mathrm{rs}(\boldsymbol{A}) \cap T_{\boldsymbol{x}}\mathrm{cone}(X)$ is at least $i+1$. This shows that the matrix $\boldsymbol{A}$ satisfies (6), and its projectivization is contained in $\overline{p^{-1}(\Sigma^\perp)}$. $\square$

Note that this gives an alternate proof that the polar degrees of a projective variety and its dual are the same but in reversed order, as stated in the forth property of polar degrees in Section 4.

Example 21. The Chow form of a non-degenerate curve in $\mathbb{P}^3$ is the Hurwitz form of its dual surface, after the change of coordinates in (1). Thus the variety of lines intersecting the curve is essentially the same as the the variety of lines tangent to the dual surface. For example, the dual of the twisted cubic is the surface cut out by the discriminant of a cubic univariate polynomial. The Chow form of the twisted cubic in primal Plücker coordinates is the determinant of the Bezout matrix B:

$$B := \begin{pmatrix} p_{01} & p_{02} & p_{03} \\ p_{02} & p_{03}+p_{12} & p_{13} \\ p_{03} & p_{13} & p_{23} \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} q_{23} & -q_{13} & q_{12} \\ -q_{13} & q_{03}+q_{12} & -q_{02} \\ q_{12} & -q_{02} & q_{01} \end{pmatrix},$$

and the Hurwitz form of the discriminant in primal Plücker coordinates surface is the determinant of the matrix on right.

Example 22. The Hurwitz form of a general surface in $\mathbb{P}^3$ is the Hurwitz form of its dual surface, up to the coordinate change (1). Consider for example the self-dual Segre-variety $\mathbb{P}^1 \times \mathbb{P}^1$. Its Hurwitz form is the determinant of the matrix $\left(\begin{smallmatrix} 2p_{02} & p_{12}+p_{03} \\ p_{12}+p_{03} & 2p_{13} \end{smallmatrix}\right)$, which stays invariant with respect to the change of coordinates (1).

This phenomenon was also observed in Example 19: Theorem 20 explains why the Hurwitz form of the Fermat cubic surface $X \subseteq \mathbb{P}^3$ and the Hurwitz form of its dual surface $X^\vee$ of degree 12 agree up to coordinate change, and why the second coisotropic hypersurface of X is exactly $X^\vee$. Moreover, it follows that the second coisotropic hypersurface of $X^\vee$ is the Fermat cubic surface X.

Example 23. In $G(3,5)$, there are three cases for coisotropic hypersurfaces: Chow forms of curves, Hurwitz forms of surfaces, and second coisotropic forms of threefolds. On the other hand, there are just two cases for $G(2,5)$, namely Chow forms of surfaces and Hurwitz forms of threefolds. The following table summarizes which forms coincide, depending on the dimension of the variety X and its dual $X^\vee$.

$X \mid X^\vee$	curve	surface	threefold
curve		$CH_0(X)^\perp = CH_0(X^\vee)$	$CH_0(X)^\perp = CH_1(X^\vee)$
surface	$CH_0(X)^\perp = CH_0(X^\vee)$	$CH_0(X)^\perp = CH_1(X^\vee)$	$CH_0(X)^\perp = CH_2(X^\vee)$
		$CH_1(X)^\perp = CH_0(X^\vee)$	$CH_1(X)^\perp = CH_1(X^\vee)$
threefold	$CH_1(X)^\perp = CH_0(X^\vee)$	$CH_1(X)^\perp = CH_1(X^\vee)$	$CH_1(X)^\perp = CH_2(X^\vee)$
		$CH_2(X)^\perp = CH_0(X^\vee)$	$CH_2(X)^\perp = CH_1(X^\vee)$

7 Hyperdeterminants Revisited

The purpose of this section is to derive and discuss the following result.

Proposition 24. *The i-th coisotropic form of the Segre variety $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_d}$ in $\mathbb{P}^{(n_1+1)\cdots(n_d+1)-1}$, in primal Stiefel coordinates, coincides with the hyperdeterminant of format $(n_1 + \ldots + n_d - i + 1) \times (n_1 + 1) \times \ldots \times (n_d + 1)$. All hyperdeterminants arise in that manner.*

Chapter 14 of [GKZ94] is devoted to the study of hyperdeterminants. They are defined as follows: For $n_1, \ldots, n_d \geq 1$, the variety $X := \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_d}$ characterizes all tensors of format $(n_1 + 1) \times \ldots \times (n_d + 1)$ having rank at most one. Whenever the dual variety $X^\vee$ is a hypersurface, its defining polynomial is called the *hyperdeterminant of format* $(n_1 + 1) \times \ldots \times (n_d + 1)$. Analogously to Corollary 5, one can derive the condition for $\operatorname{codim} X^\vee = 1$: Recall that $\mu(Y) := \dim Y + \operatorname{codim} Y^\vee - 1$ for every irreducible projective variety $Y \subseteq \mathbb{P}^n$. The equality (4) proven in [GKZ94, Ch. 1, Thm. 5.5] generalizes by induction to

$$\mu(X_1 \times \ldots \times X_d) = \max\{\dim X_1 + \ldots + \dim X_d, \mu(X_1), \ldots, \mu(X_d)\}.$$

Hence $X^\vee$ is a hypersurface if and only if $2n_i \leq n_1 + \ldots + n_d$ for all $i = 1, \ldots, d$, and

$$\operatorname{codim} X^\vee = \max\{1, 2\max\{n_1, \ldots, n_d\} - (n_1 + \ldots + n_d) + 1\}.$$

Example 25. In the special case $d = 2$ of matrices, the projectively dual variety $X^\vee$ is given by all matrices that do not have full rank. This variety is a hypersurface if and only if the matrices have square format ($n_1 = n_2$). In the case of square matrices the defining polynomial of $X^\vee$ is the usual determinant. Otherwise the codimension of $X^\vee$ equals $|n_2 - n_1| + 1$.

Proof of Proposition 24. Let $0 \leq i \leq n_1 + \ldots + n_s - \operatorname{codim} X^\vee + 1$. By the Cayley trick in Proposition 4, the i-th coisotropic form of X written in primal Stiefel coordinates is exactly the hyperdeterminant of format $(n_1 + \ldots + n_d - i + 1) \times (n_1 + 1) \times \ldots \times (n_d + 1)$. It is clear that all hyperdeterminants arise in that way as coisotropic forms of the varieties of tensors with rank at most one. □

Remark 26. Note that even the usual determinant of square matrices is given by the Chow form of $\mathbb{P}^n$. Using the duality explained in Theorem 20, the hyperdeterminants can also be characterized as the coisotropic forms (written in dual Stiefel coordinates) of the varieties of *degenerate tensors*.

If all inequalities $2n_i \leq n_1 + \ldots + n_d$ are satisfied (which means that $\operatorname{codim} X^\vee = 1$) such that at least one of them holds with equality, the hyperdeterminant is said to be of *boundary format*. An example for this is the determinant of square matrices. The hyperdeterminants of boundary format can also be characterized in terms of coisotropic forms. This is also studied in Section 3C of Chapter 14 in [GKZ94], but here this naturally and immediately follows from the duality studied in Theorem 20.

Corollary 27. *The Chow form of the Segre variety $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_d}$ in primal Stiefel coordinates is a hyperdeterminant of boundary format, and – up to permuting the tensor format – all hyperdeterminants of boundary format arise in that manner.*

If $\operatorname{codim} X^\vee \geq 2$*, then the Chow form of $X^\vee$ in dual Stiefel coordinates is a hyperdeterminant of boundary format, and – up to permuting the tensor format – all hyperdeterminants of boundary format arise in that manner.*

Proof. The first part of this proposition is clear. Note that the second part uses the convention that the Chow form of the empty variety $(\mathbb{P}^n)^\vee$ in dual Stiefel coordinates is the usual $(n+1) \times (n+1)$-determinant. If $X^\vee$ is not a hypersurface and $d \geq 2$, exactly two coisotropic forms of X yield hyperdeterminants of boundary format: The Chow form and the $(2 \cdot (n_1 + \ldots + n_d - \max\{n_1, \ldots, n_d\}))$-th coisotropic form, where – by Theorem 20 – the latter is the Chow form of $X^\vee$, up to the coordinate change (1). Hence, although $X^\vee$ is not defining a hyperdeterminant, its Chow form in dual Stiefel coordinates coincides with the hyperdeterminant of boundary format $(2\max\{n_1, \ldots, n_d\} - (n_1 + \ldots + n_d) + 1) \times (n_1 + 1) \times \ldots \times (n_d + 1)$. $\square$

Remark 28. Theorem 3.3 in Chapter 14 of [GKZ94] shows that all hyperdeterminants of boundary format can be written as the usual determinant of a square matrix whose entries are linear forms in the tensor entries. Hence, if $X^\vee$ is not a hypersurface, the Chow forms of X and $X^\vee$ have determinantal representations in their Stiefel coordinates.

Analogously, if $X^\vee$ is a hypersurface and the corresponding hyperdeterminant is of boundary format, the Chow form of X and the $(n_1 + \ldots + n_d)$-th coisotropic hypersurface of X (which is just $X^\vee$) give hyperdeterminants of boundary format. These are the only two coisotropic hypersurfaces of X with that property, and their defining polynomials in Stiefel coordinates have determinantal representations.. Finally, if $X^\vee$ is a hypersurface and its hyperdeterminant is not of boundary format, the Chow form of X is the only coisotropic form which yields a hyperdeterminant of boundary format. In all cases the Chow form of X has a determinantal representation in primal Stiefel coordinates.

Example 29. Consider the case $X = \mathbb{P}^1 \times \mathbb{P}^n$. The variety X of $2 \times (n+1)$-matrices of rank at most one is self-dual and it has three coisotropic hypersurfaces. Hence, after the change of coordinates in (1), the Chow form of X is the same as the second coisotropic form of X.

For $n = 1$, the variety X itself is a hypersurface, given by the 2×2-determinant. Therefore, its Chow form and its second coisotropic form are also the 2×2-determinant in their respective Plücker coordinates. As mentioned in Example 22, the Hurwitz form of X is the determinant of $\left(\begin{smallmatrix} 2p_{02} & p_{12}+p_{03} \\ p_{12}+p_{03} & 2p_{13} \end{smallmatrix}\right)$, which leads after substitution by the 2×2-minors of a general 2×4-matrix to the hyperdeterminant of format $2 \times 2 \times 2$.

Analogously, the Chow form of X in primal Stiefel coordinates is the hyperdeterminant of boundary format $3 \times 2 \times 2$. Hence, this hyperdeterminant can be written as the determinant of $\left(\begin{smallmatrix} p_{012} & p_{013} \\ p_{023} & p_{123} \end{smallmatrix}\right)$, where the p_{ijk} are the 3×3-minors of a general 3×4-matrix. On the other hand, the $3 \times 2 \times 2$-hyperdeterminant has a determinantal representation:

Let $A, B \in \mathbb{C}^{3\times 2}$ be the two slices of a general $3\times 2\times 2$-tensor in the last direction. The determinant of the 6×6-matrix $\left(\begin{smallmatrix} A & B & 0 \\ 0 & A & B \end{smallmatrix}\right)$ is by Laplace expansion in the first three rows equal to $p_{012}p_{123} - p_{013}p_{023}$, where the p_{ijk} are the minors of the 3×4-matrix $(A\ B)$. Moreover, the $2\times 2\times 3$-hyperdeterminant is also given by the second coisotropic form of $\mathbb{P}^1\times\mathbb{P}^2$ in primal Stiefel coordinates, or equivalently by the Chow form of $\mathbb{P}^1\times\mathbb{P}^2$ in dual Stiefel coordinates. Thus, the $2\times 2\times 3$-hyperdeterminant can be obtained by substituting the 2×2-minors of the general 6×2-matrix $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$ into the Chow form

$$\begin{aligned} &q_{13}q_{14}q_{24} - q_{03}q_{14}q_{25} - q_{12}q_{14}q_{25} + q_{02}q_{15}q_{25} - q_{03}q_{14}q_{34} + q_{01}q_{34}^2 + q_{03}q_{04}q_{35} \\ &-q_{02}q_{05}q_{35} + q_{02}q_{14}q_{35} - q_{01}q_{24}q_{35} + q_{12}^2q_{45} - q_{02}q_{13}q_{45} - q_{01}q_{23}q_{45}. \end{aligned}$$

8 The Cayley Variety

Chow forms of space curves and Hurwitz forms of surfaces in $\mathbb{P}^3$ – which are all cases of coisotropic hypersurfaces in $G(2,4)$ – were already studied by Cayley [Cay60]. Therefore the variety $\mathcal{C}(l+1,d,n+1)$ of all coisotropic forms of degree d in the coordinate ring of $G(l+1,n+1)$ is named *Cayley variety* in the following. Its subvariety of all Chow forms was introduced by Chow and van der Waerden [CvdW37] and is called *Chow variety*. The problem of recognizing the Chow forms among all coisotropic forms is addressed in Section 3 of Chapter 4 of [GKZ94]. This goes already back to Green and Morrison [GM86], who gave explicit equations for the Chow variety.

In [BKLS16], the vanishing ideals of the Cayley variety and the Chow variety of hypersurfaces of degree two in $G(2,4)$ were computed: It was found that $\mathcal{C}(2,2,4)$ has degree 92 and codimension 10 in a 19-dimensional projective space. Moreover, the prime decomposition of its radical ideal into Chow forms of curves and Hurwitz forms of surfaces is explicitly computed. In particular, this decomposition led to the interesting observation that every Chow form of a plane conic in $\mathbb{P}^3$ is already contained in the closure of all Hurwitz forms of surfaces in $\mathbb{P}^3$.

In contrast to the previous sections of this article, where hypersurfaces in Grassmannians were mainly studied in affine or Stiefel coordinates, the computation in [BKLS16] used a differential characterization of coisotropy in Plücker coordinates, which goes also back to Cayley: He had already realized that a hypersurface $\Sigma \subseteq G(2,4)$ with defining polynomial Q in Plücker coordinates is coisotropic if the following polynomial vanishes everywhere on Σ:

$$\forall L \in \Sigma : \left(\frac{\partial Q}{\partial p_{01}} \cdot \frac{\partial Q}{\partial p_{23}} - \frac{\partial Q}{\partial p_{02}} \cdot \frac{\partial Q}{\partial p_{13}} + \frac{\partial Q}{\partial p_{03}} \cdot \frac{\partial Q}{\partial p_{12}} \right)(L) = 0.$$

This follows immediately from the affine or Stiefel characterization of coisotropy given in Propositions 15 and 17. In this section, we provide a generalization of Cayley's result to $G(2,n+1)$ for $n \geq 3$, and describe how this can used to compute the vanishing ideal of the Cayley variety $\mathcal{C}(2,d,n+1)$.

For a homogeneous irreducible polynomial Q in dual Plücker coordinates of $G(2,n+1)$ and for $0 \leq j < k < i < m \leq n$, define

$$R^Q_{jkim} := \frac{\partial Q}{\partial q_{jk}} \cdot \frac{\partial Q}{\partial q_{im}} - \frac{\partial Q}{\partial q_{ji}} \cdot \frac{\partial Q}{\partial q_{km}} + \frac{\partial Q}{\partial q_{jm}} \cdot \frac{\partial Q}{\partial q_{ki}}.$$

Note that this polynomial of degree $2(\deg Q - 1)$ is a differential version of the usual Plücker relations. To allow permutations of the indices, define $R^Q_{\pi(jkim)} := \operatorname{sgn}(\pi) R^Q_{jkim}$ for $\pi \in S_4$.

Theorem 30. *Let $n \geq 3$, and let $\Sigma \subseteq G(2, n+1)$ be an irreducible hypersurface, given by a homogeneous polynomial Q in dual Plücker coordinates. Then Σ is coisotropic if and only if for all $0 \leq i < m \leq n$, the following polynomial in dual Plücker coordinates vanishes everywhere on Σ:*

$$\forall L \in \Sigma : \left(\sum_{0 \leq j < k \leq n, j,k \notin \{i,m\}} q_{jk} R^Q_{jkim} \right)(L) = 0.$$

Proof. This proof relies on Proposition 17. Let again pl be the map that sends a matrix to its maximal minors, such that $\mathrm{pl}(A)_{ij} = a_{0i}a_{1j} - a_{0j}a_{1i}$ denotes the minor given by the i-th and j-th column of a $2 \times (n+1)$-matrix A. For shorter notation, write $\beta_{ij} := \frac{\partial Q}{\partial q_{ij}}(\mathrm{pl}(\cdot))$, as well as $q_{ij} = \mathrm{pl}(\cdot)_{ij}$. This will be used to compute the (2×2)-minors of $J_{Q \circ \mathrm{pl}}$. For this, pick two columns with indices $0 \leq i < m \leq n$. The set of remaining column indices is denoted by $S := \{0, \ldots, n\} \setminus \{i, m\}$. The chain rule for partial derivatives gives

$$\frac{\partial (Q \circ \mathrm{pl})}{\partial a_{0i}} = -\sum_{j \in S} \beta_{ji} a_{1j} - \beta_{mi} a_{1m}, \qquad \frac{\partial (Q \circ \mathrm{pl})}{\partial a_{1m}} = \sum_{j \in S} \beta_{jm} a_{0j} + \beta_{im} a_{0i}.$$

Hence, the (2×2)-minor of $J_{Q \circ \mathrm{pl}}$ given by the columns i and m equals

$$\begin{aligned}
&\frac{\partial (Q \circ \Psi)}{\partial a_{0i}} \cdot \frac{\partial (Q \circ \Psi)}{\partial a_{1m}} - \frac{\partial (Q \circ \Psi)}{\partial a_{1i}} \cdot \frac{\partial (Q \circ \Psi)}{\partial a_{0m}} \\
=& a_{0i}a_{1m}\beta_{im}^2 + \sum_{j \in S} a_{0j}a_{1m}\beta_{im}\beta_{jm} - \sum_{j \in S} a_{0i}a_{1j}\beta_{im}\beta_{ji} - \sum_{j,k \in S} a_{0k}a_{1j}\beta_{ji}\beta_{km} \\
&- a_{0m}a_{1i}\beta_{im}^2 + \sum_{j \in S} a_{0j}a_{1i}\beta_{im}\beta_{ji} - \sum_{j \in S} a_{0m}a_{1j}\beta_{im}\beta_{jm} + \sum_{j,k \in S} a_{0j}a_{1k}\beta_{ji}\beta_{km} \\
=& q_{im}\beta_{im}^2 + \sum_{j \in S} q_{jm}\beta_{im}\beta_{jm} + \sum_{j \in S} q_{ji}\beta_{im}\beta_{ji} + \sum_{j,k \in S, j \neq k} q_{jk}\beta_{ji}\beta_{km} \\
=& \beta_{im} \left(\sum_{0 \leq j < k \leq n} q_{jk}\beta_{jk} - \sum_{j,k \in S, j<k} q_{jk}\beta_{jk} \right) + \sum_{j,k \in S, j<k} q_{jk}\beta_{ji}\beta_{km} - \sum_{j,k \in S, j<k} q_{jk}\beta_{ki}\beta_{jm} \\
=& \beta_{im} \sum_{0 \leq j < k \leq n} q_{jk}\beta_{jk} - \sum_{j,k \in S, j<k} q_{jk} \left(\beta_{jk}\beta_{im} - \beta_{ji}\beta_{km} + \beta_{jm}\beta_{ki} \right) \\
=& \beta_{im} \sum_{0 \leq j < k \leq n} q_{jk}\beta_{jk} - \sum_{j,k \in S, j<k} q_{jk} R^Q_{jkim}.
\end{aligned}$$

Since Q is homogeneous, we have for all $N \in V(Q \circ \mathrm{pl})$ that $\sum_{0 \leq j < k \leq n} (q_{jk}\beta_{jk})(N) = 0$. Now the corollary follows from Proposition 17. $\square$

Note that for $n = 3$, the above corollary yields exactly Cayley's differential characterization: $R^Q_{0123}(L) = 0$ for all $L \in \Sigma$. For $n = 4$, one gets the following 10 polynomials in dual Plücker coordinates:

$$\begin{aligned}
q_{01}R^Q_{0134} + q_{02}R^Q_{0234} + q_{12}R^Q_{1234}, &\qquad q_{01}R^Q_{0124} - q_{03}R^Q_{0234} - q_{13}R^Q_{1234}, \\
q_{01}R^Q_{0123} + q_{04}R^Q_{0234} + q_{14}R^Q_{1234}, &\qquad -q_{02}R^Q_{0124} - q_{03}R^Q_{0134} + q_{23}R^Q_{1234}, \\
-q_{02}R^Q_{0123} + q_{04}R^Q_{0134} - q_{24}R^Q_{1234}, &\qquad q_{03}R^Q_{0123} + q_{04}R^Q_{0124} + q_{34}R^Q_{1234}, \\
q_{12}R^Q_{0124} + q_{13}R^Q_{0134} + q_{23}R^Q_{0234}, &\qquad q_{12}R^Q_{0123} - q_{14}R^Q_{0134} - q_{24}R^Q_{0234}, \\
-q_{13}R^Q_{0123} - q_{14}R^Q_{0124} + q_{34}R^Q_{0234}, &\qquad q_{23}R^Q_{0123} + q_{24}R^Q_{0124} + q_{34}R^Q_{0134}.
\end{aligned} \tag{8}$$

Theorem 30 gives a method to compute the vanishing ideal of the Cayley variety $\mathcal{C}(2, d, n+1)$: For positive integer N and D, denote by $(\binom{N}{D}) := \binom{N+D-1}{D}$ the multiset coefficient i.e., the number of mononomials of degree degre D in N variables.

Corollary 31. *Consider the Cayley variety $\mathcal{C}(2, d, n+1) \subseteq \mathbb{P}(\mathbb{C}[G(2, n+1)]_d)$, and let $\boldsymbol{c}$ be a vector of homogeneous coordinates on $\mathbb{P}(\mathbb{C}[G(2, n+1)]_d)$. There are $\binom{n+1}{2}$ matrices of size*

$$\left(\!\!\left(\begin{array}{c}\binom{n+1}{2}\\ 2d-1\end{array}\right)\!\!\right) \times \left[1 + \left(\!\!\left(\begin{array}{c}\binom{n+1}{2}\\ d-1\end{array}\right)\!\!\right) + \binom{n+1}{4} \cdot \left(\!\!\left(\begin{array}{c}\binom{n+1}{2}\\ 2d-3\end{array}\right)\!\!\right)\right] \tag{9}$$

whose entries are polynomials in $\boldsymbol{c}$, such that the ideal generated by the maximal minors of these matrices defines – up to saturation – the Cayley variety $\mathcal{C}(2, d, n+1)$. Moreover, these minors have degree

$$2 + \left(\!\!\left(\begin{array}{c}\binom{n+1}{2}\\ d-1\end{array}\right)\!\!\right) \tag{10}$$

in the $\dim(\mathbb{C}[G(2, n+1)]_d)$ *many unknowns* $\boldsymbol{c}$.

Proof. Let Q be a general homogeneous polynomial of degree d in the dual Plücker coordinates q_{ij} of $G(2, n+1)$. Denote the coefficient vector of the polynomial Q by $\boldsymbol{c}$. The entries of $\boldsymbol{c}$ serve as homogeneous coordinates on $\mathbb{P}(\mathbb{C}[G(2, n+1)]_d)$, although – due to the Plücker relations – they are not independent unknowns. The characterization in Theorem 30 states that the equation

$$C_{i,m} := \sum_{0 \le j < k \le n,\, j,k \notin \{i,m\}} q_{jk} R^{Q}_{jkim}$$

vanishes everywhere on the hypersurface defined by Q, for all $0 \le i < m \le n$. Equivalently, the polynomial $C_{i,m}$ is contained in the radical of the ideal generated by Q and the Plücker relations. Under the assumption that this ideal is already radical, we get the condition

$$C_{i,m} - F^{(d-1)} \cdot Q - \sum_{0 \le \alpha < \beta < \gamma < \delta \le n} G^{(2d-3)}_{\alpha\beta\gamma\delta} \cdot \mathcal{R}_{\alpha\beta\gamma\delta} = 0, \tag{11}$$

where $\mathcal{R}_{\alpha\beta\gamma\delta}$ denotes the quadratic Plücker relation $q_{\alpha\beta}q_{\gamma\delta} - q_{\alpha\gamma}q_{\beta\delta} + q_{\alpha\delta}q_{\beta\gamma}$, and $G^{(2d-3)}_{\alpha\beta\gamma\delta}$ and $F^{(d-1)}$ are homogeneous polynomials of degree $2d-3$ and $d-1$, respectively. Let $\boldsymbol{a}$ denote the coefficient vector of $F^{(d-1)}$, and let $\boldsymbol{b}$ denote the vector of all coefficients of all $G^{(2d-3)}_{\alpha\beta\gamma\delta}$. The coefficient of each monomial of (11), where the variables are the Plücker coordinates q_{ij}, has quadratic terms in $\boldsymbol{c}$ (coming from $C_{i,m}$), multilinear terms in $\boldsymbol{a}$ and $\boldsymbol{c}$ (coming from $F^{(d-1)} \cdot Q$), and linear terms in $\boldsymbol{b}$ (coming from $\sum G^{(2d-3)}_{\alpha\beta\gamma\delta} \cdot \mathcal{R}_{\alpha\beta\gamma\delta}$). Hence, we can represent such a coefficient as a vector: The quadratic terms in $\boldsymbol{c}$ are the first entry. For each coefficient in $\boldsymbol{a}$ we add an entry, namely the corresponding linear form in $\boldsymbol{c}$. Finally, we add the constant factor of each coefficient in $\boldsymbol{b}$.

In this way, we get a vector for each monomial in (11). Let $M_{i,m}$ be the matrix whose rows are given by these vectors. To sum up, the rows of $M_{i,m}$ are indexed by the monomials of (11), its columns are indexed by the entries in the vector $(1, \boldsymbol{a}, \boldsymbol{b})$, and the entries of $M_{i,m}$

are (at most quadratic) polynomials in $\boldsymbol{c}$: The first column contains quadrics, the columns corresponding to $\boldsymbol{a}$ consist of linear forms, and the remaining columns (corresponding to $\boldsymbol{b}$) contain constants. In particular, the matrix $M_{i,m}$ has size (9), and its maximal minors have degree (10) in $\boldsymbol{c}$. The condition (11) is equivalent to that the vector $(1, \boldsymbol{a}, \boldsymbol{b})$ is contained in the kernel of the matrix $M_{i,m}$. Hence, for all $0 \leq i < m \leq n$, the maximal minors of $M_{i,m}$ give basic equations for the vanishing ideal of the Cayley variety $\mathcal{C}(2, d, n+1)$, but one still has to do some careful computational work to compute the actual vanishing ideal. There are three reasons for this: First, the maximal minors of $M_{i,m}$ also capture vectors in the kernel of $M_{i,m}$ that are of the form $(0, \boldsymbol{a}, \boldsymbol{b})$. Thus one still has to saturate by minors of the matrix that is obtained by deleting the first row from $M_{i,m}$. Secondly, we assumed the ideal generated by Q and the Plücker relations to be radical. Therefore, this method might not characterize all coisotropic forms. Finally, the maximal minors of the matrices $M_{i,m}$ already lead to extraneous factors that arise since Theorem 30 requires Q to be irreducible. In particular, all squares trivially satisfy the condition (11). $\square$

Example 32. The above method was explicitly computed in [BKLS16] for the Cayley variety $\mathcal{C}(2, 2, 4)$ of quadratic coisotropic forms in $G(2, 4)$. In this case, condition (11) reduces to $R^Q_{0123} - s \cdot Q - t \cdot \mathcal{R}_{0123}$, for constants s and t. We get only one matrix M with 3 columns and 21 rows. This matrix is given explicitly in Figure 1 of [BKLS16] (but with columns in reversed order as described here). In this case, we do not need to compute saturations, since it cannot happen that the kernel of M contains vectors of the form $(0, \boldsymbol{a}, \boldsymbol{b})$. By Proposition 1 of [BKLS16] the 3×3-minors of M form the vanishing ideal of the Cayley variety $\mathcal{C}(2, 2, 4)$, up to the extraneous factor of all quadrics that are squares modulo the Plücker relation. These are $\binom{21}{3}$ equations of degree 3 in the 21 unknowns $\boldsymbol{c}$, which are in fact just $20 = \dim \mathbb{C}[G(2, 4)]_2$ unknowns due to the Plücker relation.

Example 33. Consider the Cayley variety $\mathcal{C}(2, 3, 5)$ of cubic coisotropic forms in $G(2, 5)$. We have the ten equations in (8) of degree 5 in the 10 variables q_{ij}. For each such equation, the condition 11 contains 2002 monomials. The quadric $F^{(2)}$ has 55 monomials and the cubics $G^{(3)}_{\alpha\beta\gamma\delta}$ have 220 monomials each. This leads to ten matrices with 2002 rows and $1156 = 1 + 55 + 5 \cdot 220$ columns. The first column of each matrix consists of quadratic forms in $\boldsymbol{c}$, the next 55 columns contain linear form in $\boldsymbol{c}$, and the remaining columns have only constants. The maximal minors of these matrices are thus $10 \cdot \binom{2002}{1156}$ equations of degree 57 in the 220 unknowns $\boldsymbol{c}$, which are in fact just $175 = \dim \mathbb{C}[G(2, 5)]_3$ unknowns due to the Plücker relations. Hence, the computation of the vanishing ideal of the Cayley variety $\mathcal{C}(2, 3, 5)$ is a hard computational task.

9 Computations

A `Macaulay2` package for calculating coisotropic hypersurfaces and recovering their underlying varieties can be obtained at

`page.math.tu-berlin.de/~kohn/packages/Coisotropy.m2`

To use the package, the user can simply start `Macaulay2` from the same directory where the file was saved and then use the command `loadPackage "Coisotropy"`. After that, the following commands are available:

`dualVariety I`: Computes the ideal of the projectively dual variety of the projective variety given by the ideal `I`.

`polarDegrees I`: Computes a list whose i-th entry is the degree of the i-th coisotropic hypersurface of the projective variety given by the ideal `I`. This is done by computing the multidegree of the conormal variety, as described in Section 4.

`coisotropicForm (I,i)`: Returns the i-th coisotropic form in primal Plücker coordinates of the projective variety given by the ideal `I`. The computation of this form follows essentially Definition 3.

`isCoisotropic (Q,k,n)`: Checks if a hypersurface in $G(\mathtt{k}+1,\mathtt{n}+1)$ is coisotropic. The hypersurface is given by a polynomial `Q` in primal Plücker coordinates. This is implemented by using the characterization of coisotropy in Proposition 15.

`recoverVar (Q,k,n)`: Computes the ideal of the underlying projective variety of a coisotropic hypersurface in $G(\mathtt{k}+1,\mathtt{n}+1)$, which is given by a polynomial `Q` in primal Plücker coordinates. This computation uses the Cayley trick in Proposition 4.

`dualToPrimal (Q,k,n)`: Transforms the polynomial `Q` in dual Plücker coordinates of $G(\mathtt{k}+1,\mathtt{n}+1)$ to a polynomial in primal Plücker coordinates. This can be used to perform the change of coordinates (1) before calling one of the above commands that require primal Plücker coordinates.

`primalToDual (Q,k,n)`: Reverse transformation to `dualToPrimal`.

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