Wachspress' conjecture

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MATRIX

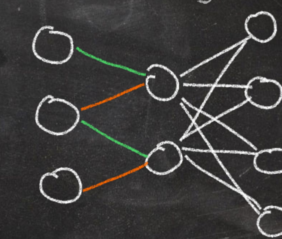
WASP WALLENBERG AI, AUTONOMOUS SYSTEMS AND SOFTWARE PROGRAM

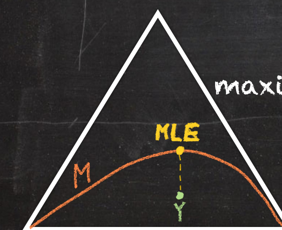
SIMONS FOUNDATION

What I will not talk about...

3D scene reconstruction from 2D images
computer vision

geometry of neural networks
machine learning theory

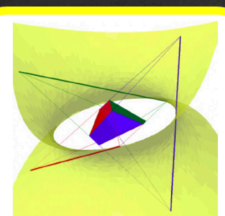




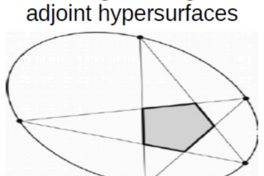
maximum likelihood estimation
algebraic statistics

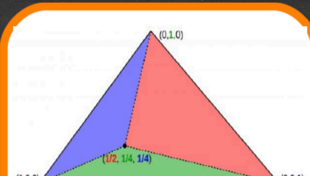
Instead:

Wachspress' adjoint

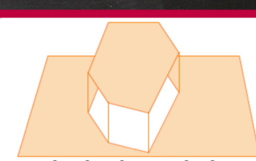


classical algebraic geometry
adjoint hypersurfaces

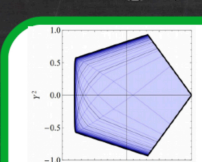




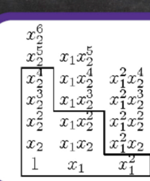
geometric modeling
barycentric coordinates
for arbitrary polytopes



algebraic statistics:
moments of uniform distributions
on polytopes

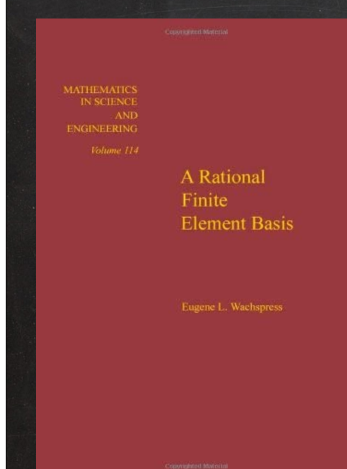


physics
scattering amplitudes



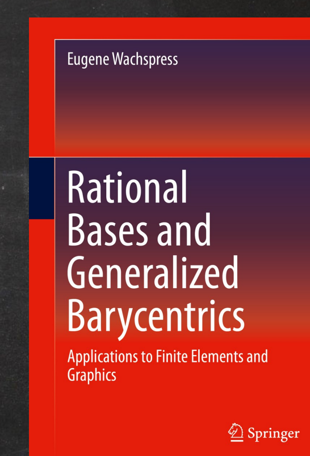
intersection theory:
Segre classes of
monomial schemes

Wachspress' goal



1975

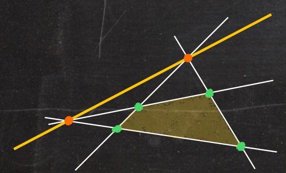
- Generalize barycentric coordinates from simplices to polytopes and further to nonlinear polytopes
- More ambitious: extend finite element method by using (nonlinear) polytopes as basic approximating elements



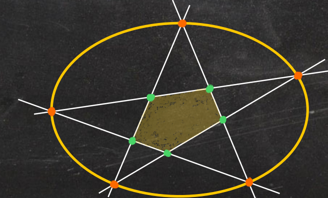
2016

The adjoint curve of a polygon

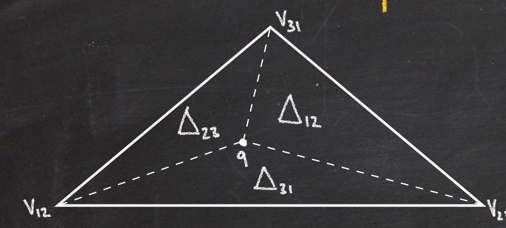
Definition (Wachspress, 1975)
The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



$\deg A_P = |V(P)| - 3$



Barycentric Coordinates



$$\beta_{v_{ij}}(q) = \frac{\text{area}(\Delta_{ij})}{\text{area}(\Delta_{12}) + \text{area}(\Delta_{23}) + \text{area}(\Delta_{23})}$$

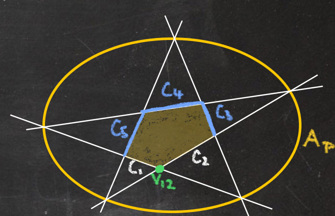
Def: Let $P \subseteq \mathbb{R}^2$ be a convex polygon. A set of function $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called generalized barycentric coordinates for P if, for all $q \in P^\circ$,
a) $\forall u \in V(P): \beta_u(q) > 0$
b) $\sum_{u \in V(P)} \beta_u(q) = 1$, and
c) $\sum_{u \in V(P)} \beta_u(q) u = q$.

Barycentric coordinates for triangles are uniquely determined by a)–c).

This is not true for other polygons!

Barycentric Coordinates

Let $P \subseteq \mathbb{R}^2$ be a convex polygon defined by lines $C_1, \dots, C_k$ and vertices $v_{12}, v_{23}, \dots, v_{k1}$.
- Fix $l_i \in \mathbb{R}[x,y]$ such that $C_i = Z(l_i)$.
- Fix $\alpha_P \in \mathbb{R}[x,y]$ such that $A_P = Z(\alpha_P)$.



Def: The Wachspress coordinates of P are

$$\beta_{v_{ij}}(q) = n_{ij} \frac{\prod_{\substack{m=1 \\ m \neq i,j}}^{k} l_m(q)}{\alpha_P(q)}.$$

$\beta_{v_{12}} = \frac{l_3 l_4 l_5}{\alpha_P}$

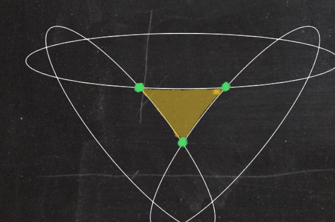
n_{ij}: normalization constant such that $\beta_{v_{ij}}(v_{ij}) = 1$.

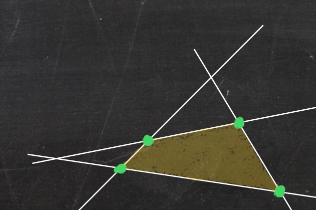
The adjoint is in the denominator and hence must not vanish in the interior of P !!
(Wachspress proved this for polygons)

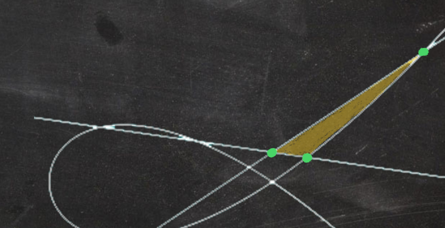
Polypols

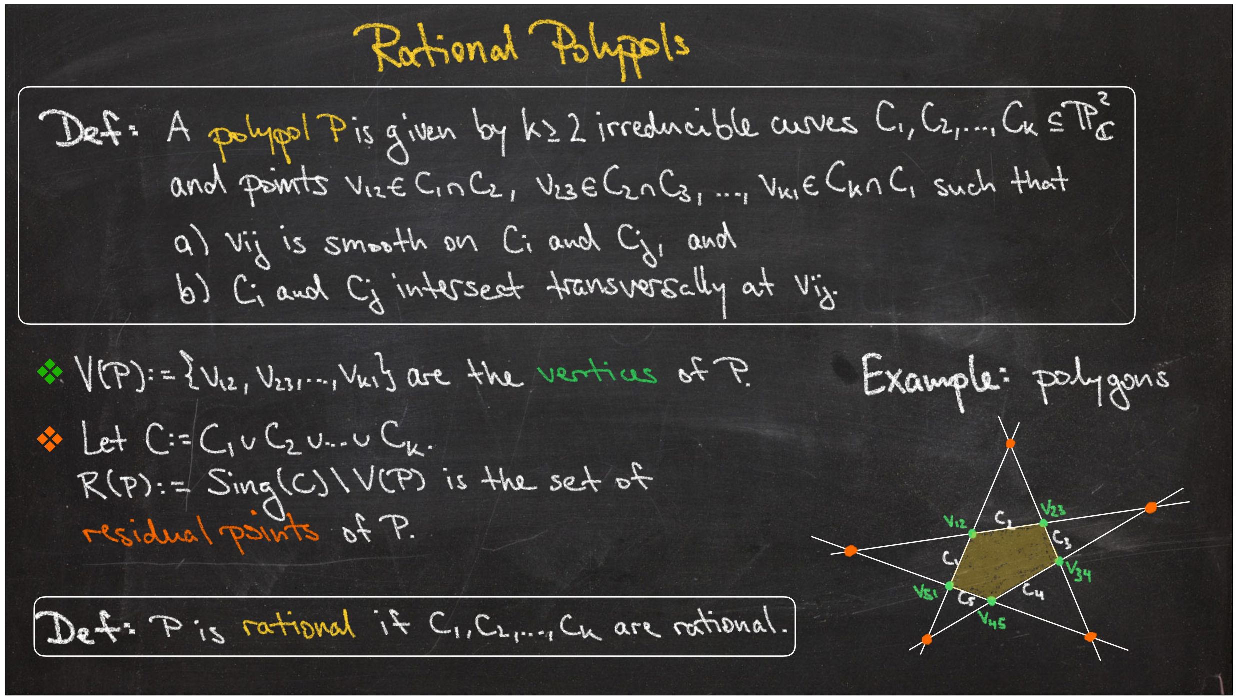
Eugene Wachspress 1975: A polypol is a "closed planar figure bounded by algebraic curves."

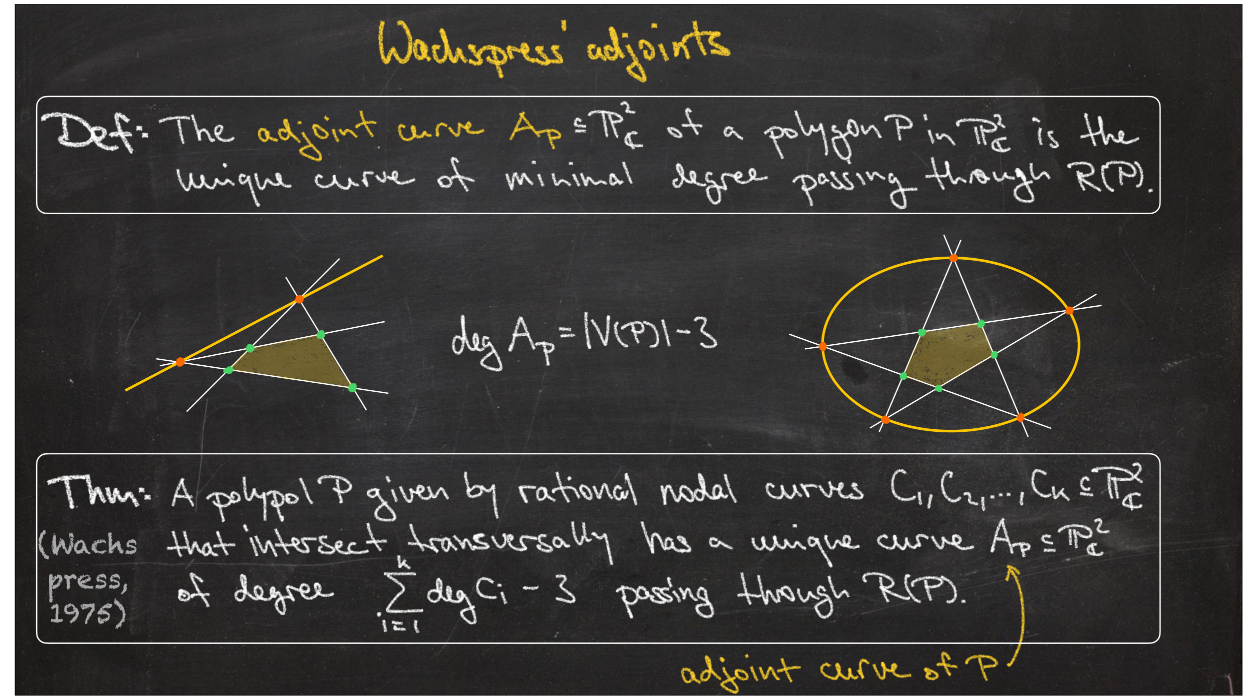
Example: • polygons
• polycons = polypols bounded by lines and conics

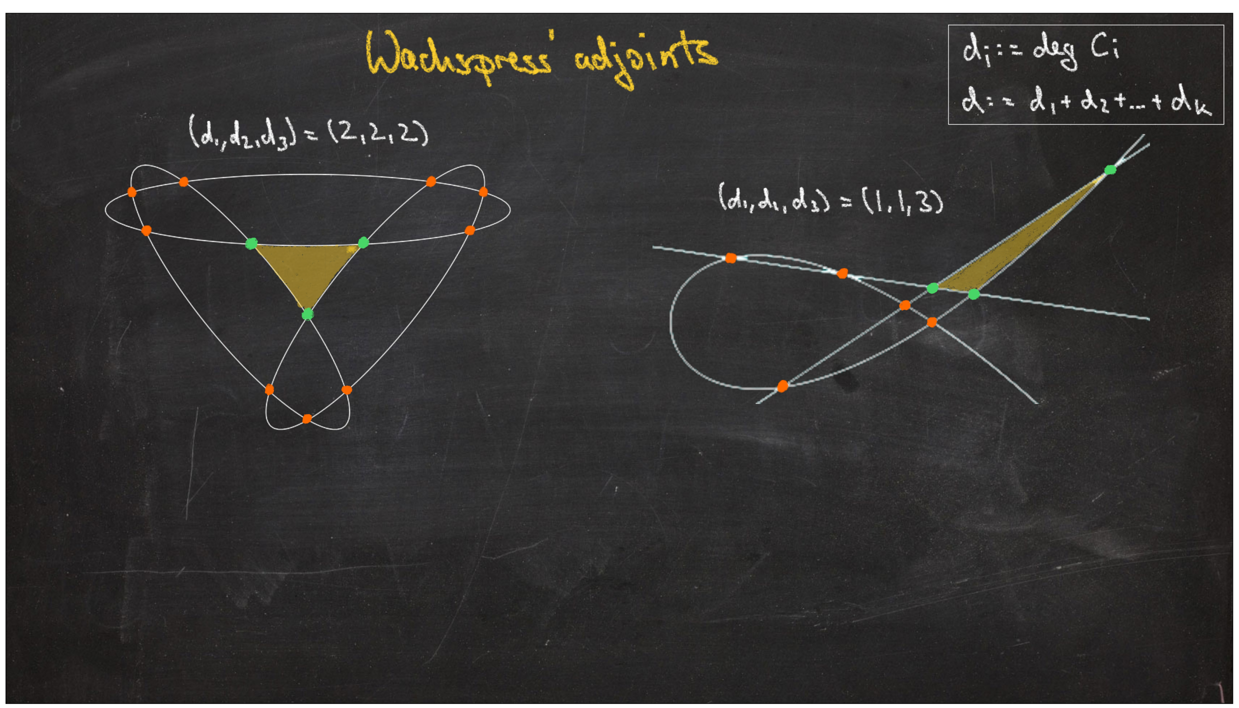


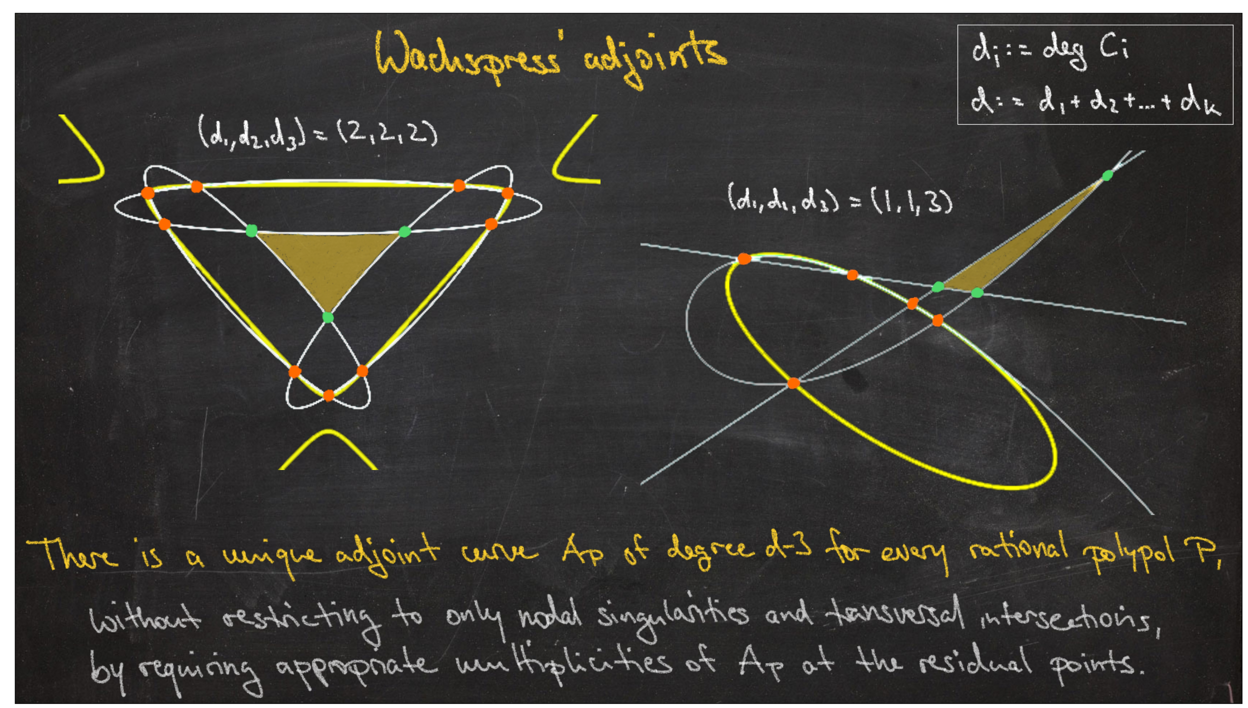





Rational Polypols
Def: A polypol P is given by $k \geq 2$ irreducible curves $C_1, C_2, \dots, C_k \subseteq \mathbb{P}^2_{\mathbb{C}}$ and points $v_{12} \in C_1 \cap C_2$, $v_{23} \in C_2 \cap C_3$, …, $v_{k1} \in C_k \cap C_1$ such that
a) v_{ij} is smooth on C_i and C_j, and
b) C_i and C_j intersect transversally at v_{ij}.
$V(P) := \{v_{12}, v_{23}, \dots, v_{k1}\}$ are the vertices of P.
Let $C := C_1 \cup C_2 \cup \dots \cup C_k$.
$R(P) := \mathrm{Sing}(C) \setminus V(P)$ is the set of residual points of P.
Def: P is rational if $C_1, C_2, \dots, C_k$ are rational.
Example: polygons
v_{12}
C_1
v_{23}
C_3
C_2
v_{34}
v_{51}
C_5
C_4
v_{45}


Wachspress' adjoints
Def: The adjoint curve $A_P \subseteq \mathbb{P}^2_{\mathbb{C}}$ of a polygon P in $\mathbb{P}^2_{\mathbb{C}}$ is the unique curve of minimal degree passing through R(P).
$\deg A_P = |V(P)| - 3$
Thm: A polypol P given by rational nodal curves $C_1, C_2, \dots, C_k \subseteq \mathbb{P}^2_{\mathbb{C}}$
(Wachspress, 1975)
that intersect transversally has a unique curve $A_P \subseteq \mathbb{P}^2_{\mathbb{C}}$ of degree $\sum_{i=1}^{k} \deg C_i - 3$ passing through R(P).
adjoint curve of P


Wachspress' adjoints
$d_i := \deg C_i$
$d := d_1 + d_2 + \dots + d_k$
$(d_1, d_2, d_3) = (2,2,2)$
$(d_1, d_2, d_3) = (1,1,3)$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$(d_1, d_2, d_3) = (2,2,2)$
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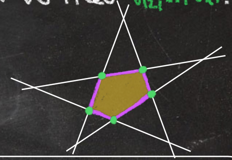
There is a unique adjoint curve A_P of degree d-3 for every rational polypol P,
without restricting to only nodal singularities and transversal intersections,
by requiring appropriate multiplicities of A_P at the residual points.

Wachspress' Conjecture

Conj: The adjoint curve A_P of a regular rational polypol $P \subseteq \mathbb{R}^2$ does not intersect its interior.

Let P be a rational polypol defined by real curves $C_1, \dots, C_k$ and real vertices $v_{12}, \dots, v_{k1}$.

- The i-th side of P is the real segment of C_i from $v_{i-1,i}$ to $v_{i,i+1}$.
- The union of the sides bounds a simply connected region $P_{\geq 0}$.

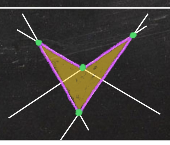


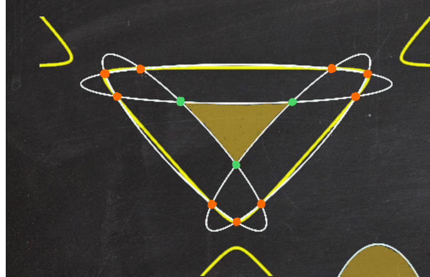
Def: P is regular if

a) all points on its sides, except its vertices, are smooth on $C = C_1 \cup \dots \cup C_k$, and

b) C does not pass through the interior of $P_{\geq 0}$.

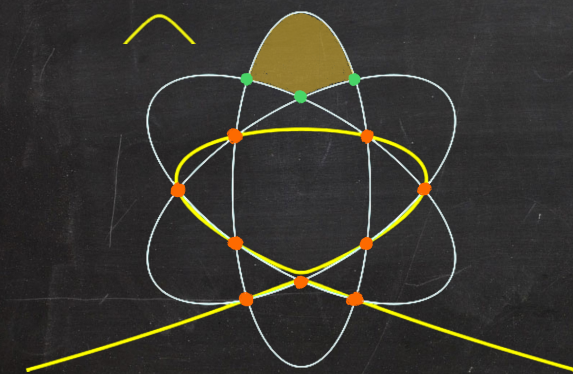
Example: A polygon is regular iff it is convex.

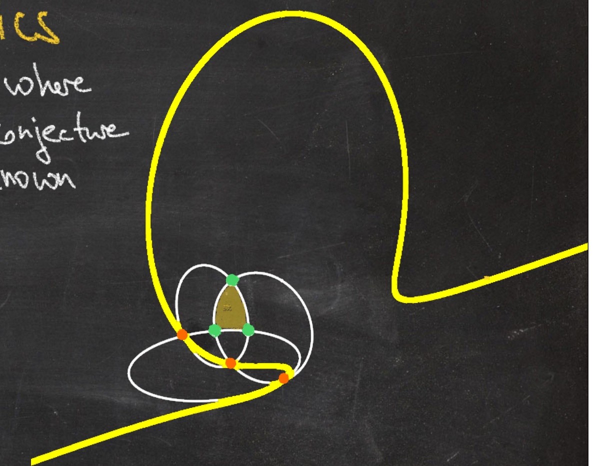




3 conics

First case where Wachspress' conjecture is not known

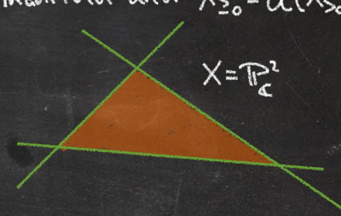




Positive Geometries (Arkani-Hamed, Bai, Lam 2017)

Let:
- X be a projective, complex, irreducible, n-dimensional variety,
- $X_{\geq 0} \subseteq X(\mathbb{R})$ be a closed semi-algebraic subset such that
- $X_{>0} = \mathrm{Int}(X_{\geq 0})$ is an open oriented n-dimensional manifold and $X_{\geq 0} = cl(X_{>0})$.
- $\partial X_{\geq 0} = X_{\geq 0} \setminus X_{>0}$
- ∂X = Zariski closure of $\partial X_{\geq 0}$ in X
  $= C_1 \cup C_2 \cup \dots \cup C_k$ irreducible components
- $C_{i,\geq 0} = cl(\mathrm{Int}(C_i \cap X_{\geq 0}))$

$X = \mathbb{P}^2_{\mathbb{C}}$



Def: $(X, X_{\geq 0})$ is a positive geometry if there is a unique non-zero rational n-form $\Omega(X, X_{\geq 0})$, called its canonical form, satisfying:
- If $n = 0$, $X = X_{\geq 0}$ = point and $\Omega(X, X_{\geq 0}) = \pm 1$.
- If $n > 0$, $(C_i, C_{i,\geq 0})$ is a positive geometry s.t. $\mathrm{Res}_{C_i} \Omega(X, X_{\geq 0}) = \Omega(C_i, C_{i,\geq 0})$ $\forall i$, and $\Omega(X, X_{\geq 0})$ is holomorphic on $X \setminus (C_1 \cup \dots \cup C_k)$.

Positive Geometries

Example:
- $n = 1 \Rightarrow X$ rational curve
  $\Rightarrow X_{\geq 0}$ = union of closed intervals
  $\Omega(\mathbb{P}^1_{\mathbb{C}}, [a,b]) = \frac{b-a}{(b-x)(x-a)} dx$

- $(Gr(k,n), Gr(k,n)_{\geq 0})$ totally nonnegative Grassmannian

Conjecture: Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^{k+m}$ be a linear map, $m \geq 0$, $k+m \leq n$.
$\rightsquigarrow \tilde{\varphi} : Gr(k,n) \dashrightarrow Gr(k,k+m)$.
The Grassmann polytope (Lam 2015) $(Gr(k,k+m), \tilde{\varphi}(Gr(k,n)_{\geq 0}))$ is a positive geometry.

Special case: If the matrix of φ has positive maximal minors, the Grassmann polytope is called the (tree) amplituhedron $\mathcal{A}_{n,k,m}(\varphi)$.

For $m = 4$, it encodes the scattering amplitude of n interacting particles, $k+2$ have helicity $-$, the others helicity $+$. (Arkani-Hamed, Trnka 2013)

Rational Polypols are Positive Geometries

General problem for positive geometries: find formulae for $\Omega(X, X_{\geq 0})$

Now let $(X, X_{\geq 0})$ be a positive geometry where $X = \mathbb{P}^2_{\mathbb{C}}$.

$\Rightarrow C_1, \dots, C_k$ are rational curves

$\Rightarrow X_{\geq 0}$ is a generalized rational polypol.

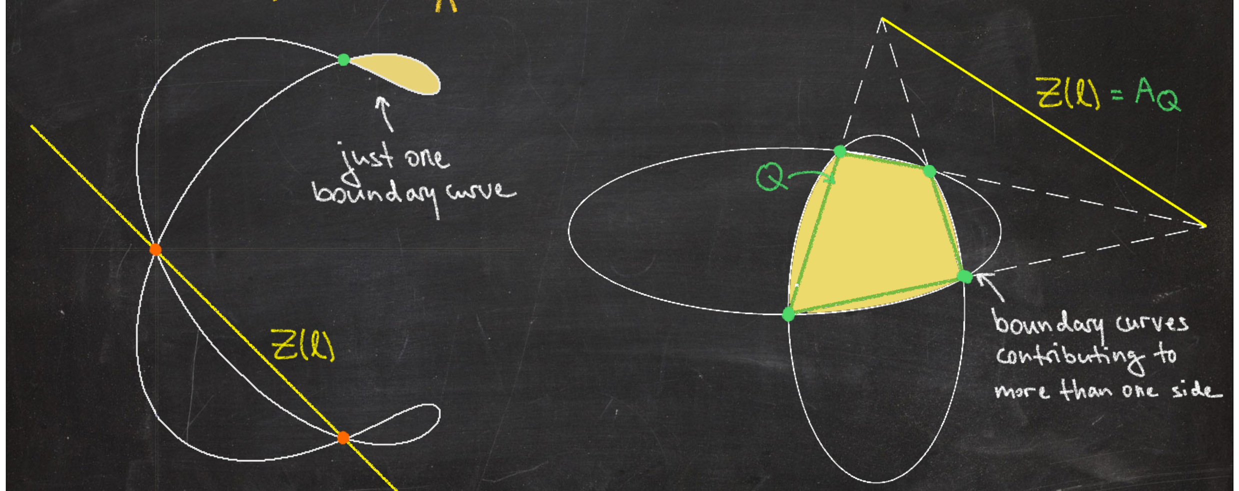
Thm: Let $\mathcal{P}_{\geq 0}$ be a real rational polypol with boundary curves $C_1, \dots, C_k$. Then $(\mathbb{P}^2_{\mathbb{C}}, \mathcal{P}_{\geq 0})$ is a positive geometry with canonical form

$$\Omega(\mathbb{P}^2_{\mathbb{C}}, \mathcal{P}_{\geq 0}) = \eta \frac{\alpha_P}{f_1 f_2 \cdots f_k} dx \wedge dy$$

where $\alpha_P, f_1, \dots, f_k \in \mathbb{R}[x,y]$ such that $Z(\alpha_P) = A_P$, $Z(f_i) = C_i$, and η is a normalizing constant.

[K, Piene, Ranestad, Rydell, Shapiro, Sinn, Sorea, Telen]

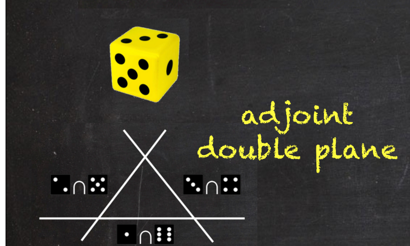
Non-Polypol Positive Geometries

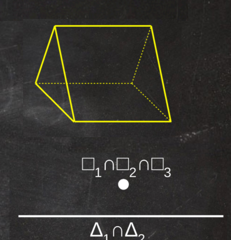


$$\Omega(\mathbb{P}^2_{\mathbb{C}}, \mathcal{P}_{\geq 0}) = \frac{\ell}{f_1} dx \wedge dy \qquad \Omega(\mathbb{P}^2_{\mathbb{C}}, \mathcal{P}_{\geq 0}) = \frac{\ell}{f_1 f_2} dx \wedge dy$$

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

independent of chosen triangulation

Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

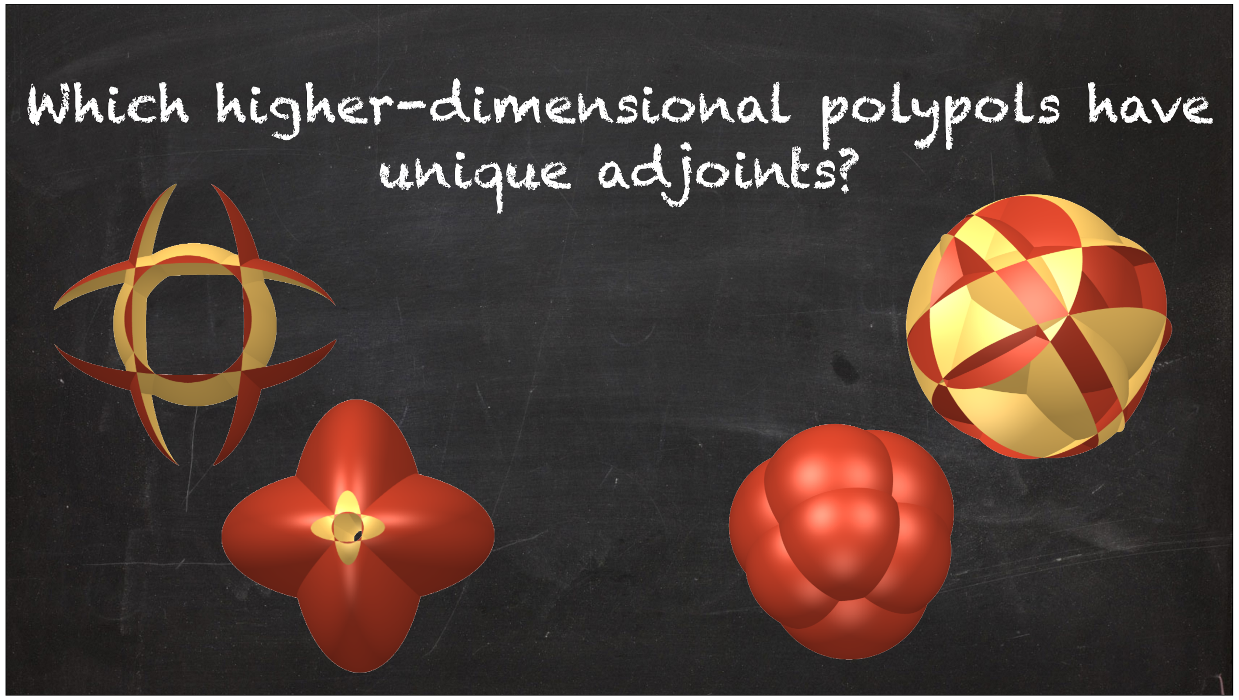
$Z(\mathrm{adj}_P) = A_{P^*}$

where $t = (t_1, \dots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \dots - v_n t_n$.

Warren defines barycentric coordinates with the adjoint in the denominator!

Why does A_P not pass through the interior of P?

(Wachspress' conjecture)


Which higher-dimensional polypols have
unique adjoints?