A Lighting-Invariant Point Processor for Shading: Supplementary Material

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Lemma 1 (§4.1). *The transformation $R_{\mathbf{I}}$ always maps I_{yy} to a nonpositive value.*

Proof. Under this transformation,

$$R_{\mathbf{I}} : \mathbf{I} = \begin{pmatrix} I \\ I_x \\ I_y \\ I_{xx} \\ I_{xy} \\ I_{yy} \end{pmatrix} \mapsto \tilde{\mathbf{I}} = \begin{pmatrix} I \\ +\sqrt{I_x^2 + I_y^2} \\ 0 \\ \frac{I_x^2 I_{xx} + 2 I_x I_{xy} I_y + I_y^2 I_{yy}}{I_x^2 + I_y^2} \\ \frac{I_x^2 I_{xy} + I_x I_y (I_{yy} - I_{xx}) - I_{xy} I_y^2}{I_x^2 + I_y^2} \\ \frac{I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2}{I_x^2 + I_y^2} \end{pmatrix} =: \begin{pmatrix} \tilde{I} \\ \tilde{I}_x \\ \tilde{I}_y \\ \tilde{I}_{xx} \\ \tilde{I}_{xy} \\ \tilde{I}_{yy} \end{pmatrix}$$

Over $\mathbb{R}$, $\tilde{I}_{yy}$ is the same sign as its numerator $W := I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2$, so we'll study $\text{sgn}(W)$ as a proxy. Recall that the point processor considers the point $(x, y) = (0, 0)$. By the quadratic patch assumption the true surface $f^*(x, y) = ax + by + \frac{1}{2}\left(c^2 x + 2dxy + ey^2\right)$, so at this point we have $\mathbf{f}^* = (a, b, c, d, e)^T$. Recall also that $I(0, 0) = \rho \mathbf{L} \cdot \mathbf{N}^*(x, y)/||\mathbf{N}^*(x, y)||$, where $\mathbf{N}^* = (-(\partial f/\partial x), -(\partial f/\partial y), 1)^T = (-a, -b, 1)^T$ and WLOG $\rho = 1$. Writing $\mathbf{I}(0, 0)$ as a function of $(\mathbf{f}^*, \mathbf{L}^*)$, we have

$$W = I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2 = \underbrace{\frac{(d^2 - ce)^2}{|(1 + a^2 + b^2)^{9/2} (L_1^2 + L_2^2 + L_3^2)^{3/2}|}}_{\text{strictly positive}} \underbrace{(aL_1 + bL_2 - L_3)}_{\text{strictly negative by assumption}} V,$$

$$\begin{aligned} V &:= L_1^2 + b^2 L_1^2 - 2ab L_1 L_2 + L_2^2 + a^2 L_2^2 + 2a L_1 L_3 + 2b L_2 L_3 + a^2 L_3^2 + b^2 L_3^2 \\ &= (bL_1 - aL_2)^2 + (L_1 + aL_3)^2 + (L_2 + bL_3)^2. \end{aligned}$$

Since V can be written as the sum of squares, the entirety of W is nonpositive; therefore, $\tilde{I}_{yy}$ is always nonpositive. Furthermore, V can only be zero-valued when $(a, b) = (-L_1/L_3, -L_2/L_3)$, which generically will not occur. $\square$