Classification Theory and the Map of the Universe

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Classifying Mathematical Objects

A fundamental mathematical question is when and how can we classify various mathematical objects.

Which is more complex:

- A graph or a group?
- A field or a topological space?

Model theory allows us to make sense of these types of questions.

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Example

The field of real numbers: $(\mathbb{R}, +, -, \cdot, 0, 1)$ is a structure that models the theory of fields. If we extend the language to include the $<$ symbol, then we get an ordered field.

Classifiability

A relatively straightforward way to classify theories is by looking at the number of non-isomorphic models of a given cardinality.

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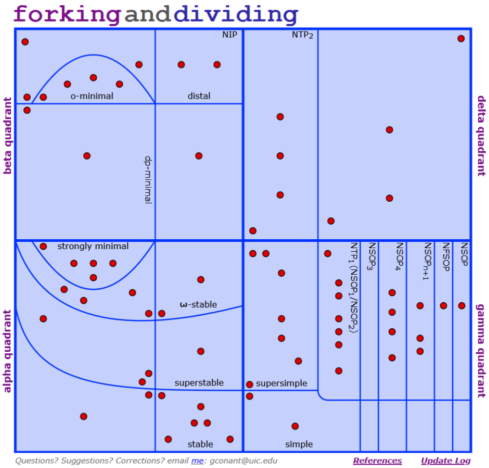
We can also use model theory to determine when theories are **not** classifiable.

The Model-Theoretic Map of the Universe



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Stability

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Definition

A formula $\varphi(x, y)$ has the **order property** if there exist sequences $(a_i)_{i<\omega}, (b_i)_{i<\omega}$ such that $\varphi(a_i, b_j)$ is true if and only if $i \leq j$.

Alternate definition of stability: no formula has the order property.

Example: Dense Linear Orders

A dense linear order $\mathcal{M}$ is a structure in the language $\langle < \rangle$ with the following theory:

- For all $x, y \in M$, exactly one of $x < y$, $x = y$, and $y < x$ holds.
- For all $x, y \in M$, $x < y$ implies there is some $z \in M$ with $x < z < y$.

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What about structures that don't have a built-in order?

Example: The Random Graph

Recall that a graph (V, E) has a vertex set and an edge set, where the edges represent pairs of vertices. The construction of the random graph is as follows: every time we add a vertex to the graph, flip a coin to decide whether to add an edge to each existing vertex.

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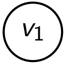

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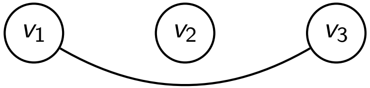

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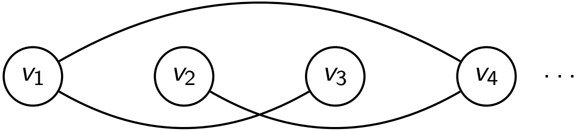

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(v_1, v_4)	Yes
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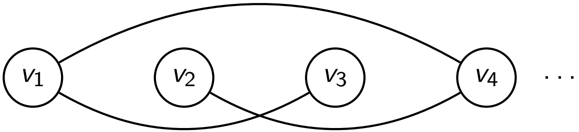

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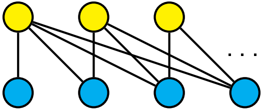
The random graph is characterized by the following axioms:

- $G = (V, E)$ is an infinite graph.
- For all disjoint finite sets of vertices $V_1, V_2 \subset V$, there is a vertex $v \in V$ such that v is connected to every element of V_1 and no elements of V_2.

Ordering in the Random Graph

Definition

A **half-graph** is a bipartite graph on $2n$ vertices $u_1, \dots u_n, v_1, \dots, v_n$, where u_i and v_j are connected by an edge if and only if $i \leq j$.



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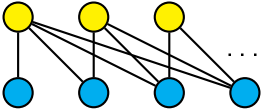

Fact

Every countable graph is contained as a subgraph of the random graph.

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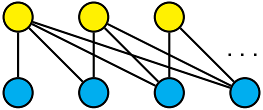

Fact

Every countable graph is contained as a subgraph of the random graph.

In particular, we can find an infinite half-graph inside a model of the random graph. Then taking our sequences to be the two halves of this subgraph, the formula xRy (where R is the edge relation) has the order property.

Outside of Stability: IP and SOP

Definition

A formula $\varphi(x, y)$ has the **strict order property** (SOP) if there is a sequence $(a_i)_{i<\omega}$ such that there exists an x for which $\varphi(x, a_i)$ is false and $\varphi(x, a_j)$ is true if and only if $i < j$.

In DLO: take φ to be $x < y$ and any increasing sequence.

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Definition

A formula $\varphi(x, y)$ has the **independence property** (IP) if for any finite n there exists a sequence $(b_i)_{i<n}$ such that for every $X \subseteq n$ there is some x for which $\varphi(x, b_i)$ is true if and only if $i \in X$.

In RG: take φ to be xRy and it follows immediately from the axioms

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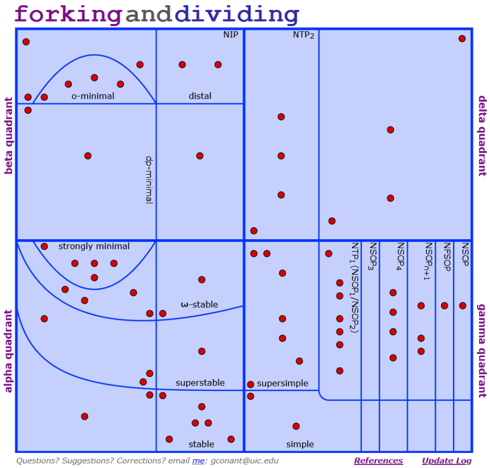

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Question

Suppose $\mathcal{M}$ can be represented as the union of a directed system of substructures $\mathcal{N} \subset \mathcal{M}$, satisfying a particular set of axioms. What properties can we determine about the theory of $\mathcal{M}$ based solely on specific assumptions about the theories of the substructures?

Conclusions Thus Far

Theorem

If $\mathcal{M}$ is approximated by a directed system of substructures that all have simple theories, then the theory of $\mathcal{M}$ is also simple.

Conjecture

If $\mathcal{M}$ is approximated by a directed system of substructures that all have stable theories, then the theory of $\mathcal{M}$ has the property NSOP_4.