Approximable Structures

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Smoothly Approximable Structures

Definition

A **smoothly approximable** structure is a countable ω-categorical structure $\mathcal{M}$ for which there is an ascending chain of finite homogeneous substructures $\mathcal{N}_i$ such that $M = \bigcup_{i<\omega} N_i$.

- Introduced by Lachlan, theory developed by Cherlin and Hrushovski
- Influential in the development of simple theories

1999 - Nick Granger's Thesis

Example

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- Over a finite field, this would be smoothly approximable
- Approximated using a sequence of finite-dimensional subspaces
- Defined independence relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}$ and showed $\mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}$ in a model $\mathcal{M} \models T_\infty$ is the limit of $\mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}$ in an approximating sequence for $\mathcal{M}$

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Limitation

The definition of $\mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}$ is specific to vector spaces and doesn't make sense outside of this context.

2013 - Gwyneth Harrison-Shermoen's Thesis

- Developed a nice abstract framework for approximation independent of a specific example, defining $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^{\mathrm{lim}}$ as the limit of non-forking independence in a directed system of substructures

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- Showed T_∞ (mostly) fits into this framework, with $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{lim}}_{C} B$ in T_∞ being equivalent to $a \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{\mathrm{acl}(C)} \mathrm{acl}(B)\,\mathrm{acl}(C)$

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Limitation

Theoretically applied much more broadly, but only had one example.

Generalized Approximation Framework

Let $\mathcal{H}$ be a directed system of substructures of a structure $\mathcal{M}$. We say that a property holds "for large enough $\mathcal{N}$" to mean that there is some $\mathcal{N} \in \mathcal{H}$ such that for all $\mathcal{N}' \in \mathcal{H}$ with $\mathcal{N} \subset \mathcal{N}'$, the property holds in $\mathcal{N}'$.
We write $\bar{A} = \mathrm{acl}^{\mathcal{M}}(A)$, where $\mathcal{M}$ is the approximated structure.

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- $A \mathop{\smile\!\!\!\!\!\!|}\,_C^{\lim} B$ if for each finite tuple $a \in A$ there is $\mathcal{N}$ large enough such that $a \mathop{\smile\!\!\!\!\!\!|}\,_{\bar{C} \cap N}^{\mathcal{N}} \bar{B} \cap N$, where $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathcal{N}}$ is non-forking independence in $\mathcal{N}$.

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7. Homogeneity: If $a \equiv^{\mathcal{M}}_{A} b$, then $a \equiv^{\mathcal{N}}_{A \cap N} b$ for all $\mathcal{N}$ containing a, b.

Properties of $\mathop{\smile\!\!\!\!\!\!\,|}^{\text{lim}}$

Theorem (Harrison-Shermoen)

Given a κ-saturated, strongly κ-homogeneous model $\mathcal{M}$ of T approximated by $\mathcal{H}$ where each $\mathcal{N} \in \mathcal{H}$ has a simple theory, $\mathop{\smile\!\!\!\!\!\!\,|}^{\text{lim}}$ has the following properties: invariance, monotonicity, base monotonicity, transitivity, normality, extension, finite character, strong finite character, and symmetry.

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Theorem (Harrison-Shermoen)

If in addition to the assumptions of the previous theorem $\mathcal{H}$ satisfies conditions 6 and 7 and in each $\mathcal{N}$ $\mathop{\smile\!\!\!\!\!\!|}^{f}$ satisfies the Independence Theorem over algebraically closed sets, then $\mathop{\smile\!\!\!\!\!\!|}^{\text{lim}}$ satisfies the Independence Theorem over algebraically closed sets.

Local character of $\mathop{\smile\!\!\!\!\!\!|}^{\text{lim}}$

Lemma

Let $\mathcal{M}$ be a κ-saturated, strongly κ-homogeneous model of T approximated by $\mathcal{H}$ where each $\mathcal{N} \in \mathcal{H}$ has a simple theory. If $\kappa \geq |T|^+$ is a regular cardinal, $\langle A_i : i < \kappa \rangle$ is an increasing continuous sequence of sets of size $< \kappa$, $A_\kappa = \bigcup_{i<\kappa} A_i$, and $|A_\kappa| = \kappa$, then for any finite d there is some $\alpha < \kappa$ such that $d \mathop{\smile\!\!\!\!\!\!|}^{\text{lim}}_{A_\alpha} A_\kappa$.

Examples

It turns out there are actually several examples of NSOP_1 theories approximated by simple structures in the sense of Harrison-Shermoen, suggesting that approximability is a relatively common property in NSOP_1.
The behavior of these examples gives some insight into why it's ultimately difficult to conclude structural properties of NSOP_1 from the approximation by simple.

Examples - T_∞

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- Approximate by finite dimensional subspaces
- Satisfies conditions 1-4, 6-7. Condition 5 is true under the restriction to singletons on the left, finite-dimensional sets on the right, and small sets in the base.

Examples - ω-free PAC fields

Example

The theory of a PAC field whose absolute Galois group is free on countably many generators.

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The theory of a PAC field whose absolute Galois group is free on countably many generators.

- Approximate by e-free PAC fields. This satisfies conditions 1-7.
- The approximated structure will not be saturated, because every element in the approximation has bounded Galois group and the type "I am an element with an unbounded Galois group" can't be expressed as a formula.

Examples - T^*_{feq}

Example

The generic theory of a parameterized equivalence relation. Given an object sort O and a parameter sort P, the relation $E_x(y,z)$ is defined such that for each $p \in P$, $E_p(y,z)$ is an equivalence relation on O.

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- Approximate by substructures K_n given by restriction to at most n classes with respect to each parameter. This satisfies conditions 1-7.

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- Approximate by substructures K_n given by restriction to at most n classes with respect to each parameter. This satisfies conditions 1-7.
- Due to trivial forking in the approximating theories, $A \mathop{\smile\!\!\!\!\!\!\mid}^{\lim}_{C} B \iff A \cap B \subseteq C$.

Stabilization of $\mathrm{acl}^{\mathrm{eq}}$

Definition

We say $\mathcal{H}$ has stabilization of algebraic closure in T^{eq} if whenever $A = \mathrm{acl}^{\mathrm{eq}}(A)$ as computed in $\mathcal{M}$, then for large enough $\mathcal{N}$, $A \cap N = \mathrm{acl}^{\mathrm{eq}}(A \cap N)$ as computed in $\mathcal{N}$.

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Proposition

Suppose $M \models T$ is κ-saturated, strongly κ-homogeneous, and eliminates hyperimaginaries, $\mathcal{H}$ satisfies conditions 1-7 and stabilization of $\mathrm{acl}^{\mathrm{eq}}$, and the elements of $\mathcal{H}$ have theories that are simple. Then $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{lim}}$ satisfies the Independence Theorem. That is, if $B \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{lim}}_A C$, $d \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{lim}}_A B$, $e \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{lim}}_A C$, and $\mathrm{Ltp}^{\mathcal{M}}(d/A) = \mathrm{Ltp}^{\mathcal{M}}(e/A)$, then there exists a such that $a \equiv_{AB} d$, $a \equiv_{AC} e$ and $a \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{lim}}_A BC$.

Simply-approximated theories are simple

Theorem

If $\mathcal{H}$ approximates $\mathcal{M}$ in the sense of Harrison-Shermoen, additionally satisfies stabilization of $\mathrm{acl}^{\mathrm{eq}}$ *and homogeneity, and the approximating structures have simple theories, then* $\mathrm{Th}(\mathcal{M})$ *is simple.*

Failures of Approximation

The initial goal was to show that structures approximated by simple theories were NSOP_1. In fact, they are simple. What is interesting then, is how each of our strictly NSOP_1 examples fail to meet the conditions for approximation (in T^{eq}).

- T_∞: partial failure of strong finite character
- ω-free PAC fields: failure of saturation
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Question

What conclusions can we make without strong finite character of $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{lim}}$?

Stationarity

Theorem

Suppose $\mathcal{M} \models T$, $\mathcal{H}$ satisfies conditions 1 through 3 along with condition 7 (homogeneity), and the elements of $\mathcal{H}$ have stable theories. Then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{lim}}$ is stationary. That is, if $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{lim}}_C b$, $a' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{lim}}_C b$, $a' \equiv^{\mathcal{M}}_C a$, and $C \subseteq a \cap b$, then $ab \equiv^{\mathcal{M}}_C a'b$.

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Proof: By homogeneity and definition of $\mathop{\smile\!\!\!\!\!\!|}^{\lim}$, for large enough $\mathcal{N}$ we get $a \mathop{\smile\!\!\!\!\!\!|}^{\mathcal{N}}_{C \cap N} b \cap N$, $a' \mathop{\smile\!\!\!\!\!\!|}^{\mathcal{N}}_{C \cap N} b \cap N$, and $a' \equiv^{\mathcal{N}}_{C \cap N} a$. Stationarity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathcal{N}}$ implies $a(b \cap N) \equiv^{\mathcal{N}}_{C \cap N} a'(b \cap N)$. Then $ab \equiv^{\mathcal{M}}_C a'b$; otherwise we get a contradiction to convergence of truth value.

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Conjecture

If $\mathcal{M}$ is approximated by a directed system of substructures that all have stable theories, then the theory of $\mathcal{M}$ is NSOP$_4$.

Next Steps

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- What conditions guarantee that a structure **can't** be approximated?
- What can we say about definable groups in an approximated theory?