Mathematical Determination of Righting Moment Vs. Heel Angle

Introduction

When a torque is applied to a boat to make it turn in the water, sometimes the boat will tilt back to its original orientation, and sometimes it will continue to tilt until it capsizes. It turns out that this behavior can be explained and predicted by physics The center of mass, depicted in figure 1 as a black dot, is the average of where the force of gravity acts on the boat. The center of buoyancy, depicted as a blue dot, is the average of where the force of buoyancy acts on the boat. You will notice that the center of buoyancy is at the center of the part of the boat that is underwater. That is because the buoyant force on the boat is equal to the weight of the water that the boat displaces.

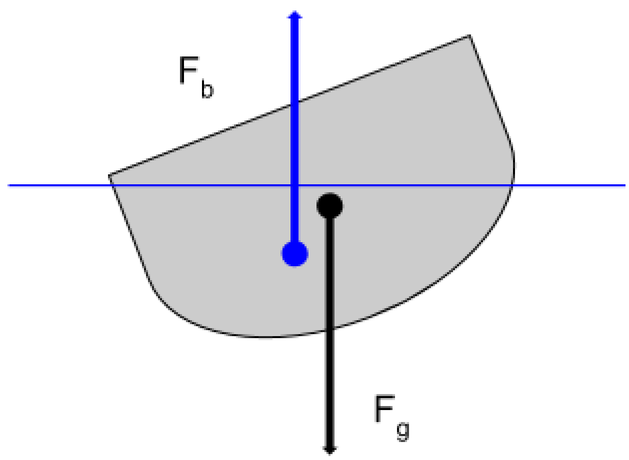

Figure 1: Free Body Diagram of a boat experiencing a righting moment

The diagram shows the boat tilted at an angle such that the center of buoyancy and center of mass are not on the same vertical line. This means that the boat must be experiencing a torque That torque is called the righting moment. The righting moment can be calculated using the center of mass and center of buoyancy of a boat.

Calculate Boat Mass and Center of Mass

The first step in determining the righting moment of the boat is to calculate the mass of the boat based off the density, and use that value, along with the mass and location of ballast, to determine the center of mass of the boat. First, since we know the density, we can use the equation

$$Mass\ of\ Boat\ = z\ distance * \int_{xmin}^{xmax} \int_{f(x)}^{ymax} Boat\ Density\ dydx$$

to find the boat's mass. In this equation, the z distance is the length of the boat in the z direction, the xmax, xmin, and ymin are the bounds of the boat, and the equation f(x) defines the curve of the boat's hull. The value of boat density that we are integrating does not include the ballast or any other mass. Once we have found the mass of the boat, we are able to calculate the center of mass. Since we need to know the center of mass in both the x and y directions, it is subjectively easiest to use two separate equations to calculate the center of mass. One of the equations calculates the center of mass in the x direction:

$$x\ Center\ of\ Mass\ =\ (1 \div Mass\ of\ Boat) * z\ distance * \int_{xmin}^{xmax} \int_{f(x)}^{ymax} x * Boat\ Density\ dydx$$

and the other calculates the center of mass in the y direction:

$$y\ Center\ of\ Mass\ =\ (1 \div Mass\ of\ Boat) * z\ distance * \int_{xmin}^{xmax} \int_{f(x)}^{ymax} x * Boat\ Density\ dydx$$

We will assume that the z center of mass is at zero. Once we have determined the x and y center of mass, we will use these values, along with the coordinates and mass of the ballast to calculate the total center of mass of the boat. We will do so by multiplying each mass by the x or y coordinate of it's center of mass, and summing those values Finally, we will multiply that sum by 1 divided by the total mass.

Calculating Center of Buoyancy

Now that we have the center of mass of the boat and ballast, we just need to calculate the center of buoyancy before we can calculate the righting moment. First, we will define the region of water in terms of draft. This means that if the boat is flat at an angle of zero, we can use the draft as the upward bound of the region in the y direction. However, if the boat is floating at an a angle θ, we will need to use the equation

$$y\ =\ Tan(\theta) * x + draft$$

as the upper bound in the y direction. In this equation, $Tan(\theta)$ acts as the slope of the line, and the draft is the y coordinate of the line when y = 0. We keep the boat constant and tilt the water in this frame of reference because it makes the calculations easier. It would be difficult to redefine the center of mass every time the boat tilted. Next, we will define the submerged area of the boat as an intersection between the water and the boat. This means that the lower bound in the y direction would be $f(x)$, the function defining the hull of the boat.

Next, we will find the mass of the water displaced by the boat. According to Archimedes' principle, the mass of the water displaced by the boat will be equal to the buoyant force, which is the value we really need. Then, we will set the buoyant force equal to to the integral

$$Buoyant\ Force = \int_{-g(draft)}^{g(draft)} \int_{f(x)}^{draft} dA$$

which defines buoyant force in terms of draft. In this equation, g(x) is equal to $f^{-1}(x)$, since we solve f(x) for draft. Now that we know buoyant force from determining the mass of the displaced water, we can plug in the buoyant force and solve for draft.

Now that we know the draft, we now know the equation for the top bound of the underwater region of the boat, giving us all that we need to solve for center of buoyancy. The top bound for the underwater region of the boat is $y = Tan(\theta) * x + draft$ and the bottom bound is f(x) which defines the hull of the boat. Now, we can integrate over the submerged area in order to find the center of buoyancy.

Calculating Righting Moment

Now that we know both the center of mass and the center of buoyancy, it is quite easy to find the righting moment. We just need to find the difference between the two centers, and then multiply that difference by the force of buoyancy:

$$Righting\ Moment = (Center\ of\ Mass - Center\ of\ Buoyancy) * (Force\ of\ Buoyancy)$$

Results

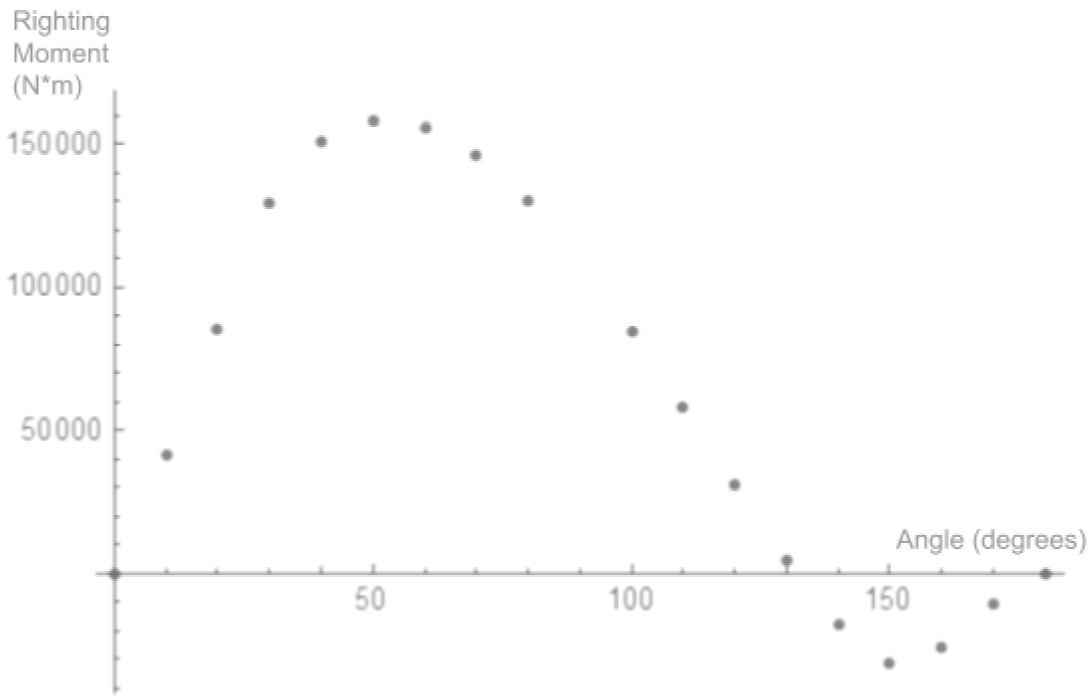

Figure 2: Graph of Angle Vs. Righting Moment

This figure shows an example of a graph of the angle of the boat vs. the righting moment. In this graph, the righting moment crosses the x axis and switches from positive to negative at around 130 degrees. This means that the angle of vanishing stability is 130 degrees, which means that if one were to tip the boat by 129 degrees, the boat would flip back upright, but if the boat was tipped 131 degrees, it would capsize.