Problem Set 2

Katherine Cheng, Richard Davis, Marty Keil

April 16, 2015

Problem 3: Tiling with Triominoes

Theorem. *It is always possible to tile a grid of size $2^n \times 2^n$ that's missing exactly one square with right triominoes.*

Proof. By induction. Let $P(n)$ be "it is always possible to tile a grid of size $2^n \times 2^n$ that's missing exactly one square (it doesn't matter which square) with right triominoes." We will prove that $P(n)$ is true for all natural numbers n.

As our base case, we prove that $P(1)$ is true; that is, that it is possible to tile a grid of size $2^1 \times 2^1$ with any square missing with right triominoes. There are four possible squares in a 2×2 grid that can be missing: (a) top-left, (b) top-right, (c) bottom-right, (d) bottom-right. In each of these cases, we can place a single right triomino so that all the non-missing squares are covered (Figure 1).

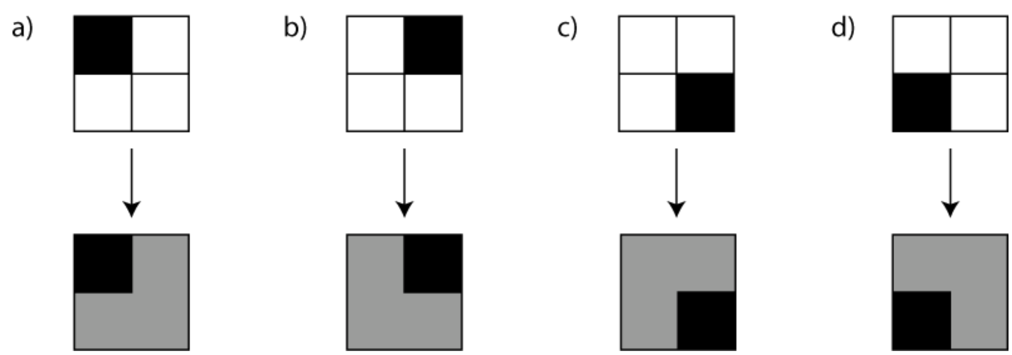

Figure 1: Base Case, black square is missing

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ is true. This means that it is possible to tile a $2^k \times 2^k$ grid with any square missing with right triominoes. We will prove that $P(k+1)$ is true, that it is possible to tile a $2^{k+1} \times 2^{k+1}$ grid with any square missing with right triominoes.

Start with a $2^{k+1} \times 2^{k+1}$ grid with no squares missing, then remove a single square at any point in the grid. We will show that it is always possible to recreate this board by combinging four $2^k \times 2^k$ boards with a single square missing in each.

The larger board is made up of four quadrants, each of size $2^k \times 2^k$. The missing square must reside in one of these quadrants. From our inductive hypothesis we know that, taken alone, it is possible to tile this quadrant with right triominoes.

What about the other three quadrants? Also from our inductive hypothesis, we know that it is possible to tile each of these with right triominoes as long as we remove a single square from each. It does not matter which square we remove. So, we remove a single square from the corner of each of these three quadrants.

We now have four separate $2^k \times 2^k$ grids, each with a single square missing and each tiled with right triominoes. We can now recombine these grids to recreate the $2^{k+1} \times 2^{k+1}$ grid. We leave the first grid (the one containing the original missing square) where it is and rotate the three grids (the ones with corner pieces missing) until the missing squares all come together in the center. These missing squares will always form a missing space that can be tiled with a right triomino (Figure 2).

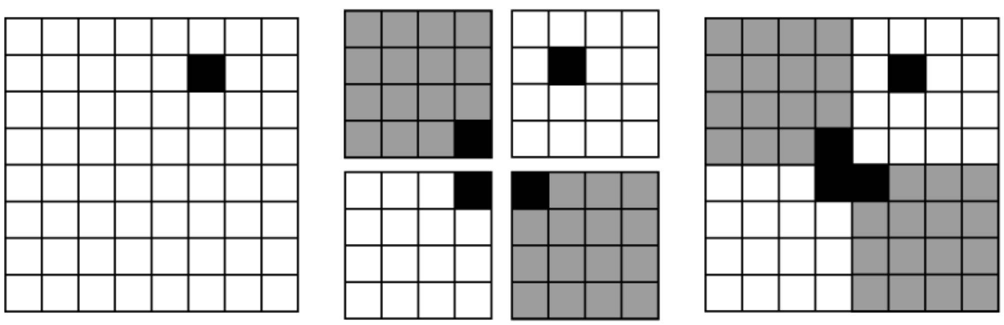

Figure 2: Creating a $2^3 \times 2^3$ board from four $2^2 \times 2^2$ boards.

When that triomino is placed on the board, we will have successfully tiled the original $2^{k+1} \times 2^{k+1}$ board with a single square missing with right triominoes.

$\square$

Problem 4: The Circle Game

Suppose that you have a circle with $2n$ arbitrary-chosen points on its circumfrence. n of these points are labeled $+1$ and n are labeled -1. Define this circle to be C_n. You can play a game where you choose one of these points as your starting point and then start moving clockwise around C_n. As you go, you'll pass through some number of $+1$ points and some number of -1 points. You lose the game if at any point in your journey you pass through more -1 points than $+1$ points. You win the game if you get all the way around to your starting point without losing.

Theorem 1. *For any $n \in N$ where $n \geq 1$, no matter which of the n points are labeled $+1$ and which are labeled -1, there is always one point you can start at to win the game.*

Proof. By induction. To prove this we will need two additional lemmas.

Lemma 1. *For any $n \geq 1$, there is always at least one $(+1, -1)$ clockwise point pair on C_n.*

Proof. By induction. Let $Q(n)$ be "you will always pass through a $(+1, -1)$ point pair while moving clockwise along C_n." We will prove that this holds for $Q(n + 1)$.

As our base case, we prove that $Q(1)$ is true. C_1 is a circle with a single $+1$ point and a single -1 point. We see that there is, indeed, a $(+1, -1)$ clockwise point pair on this circle.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $Q(k)$ is true. This means that there is at least one $(+1, -1)$ point pair on C_k. We will use this to prove that $Q(k+1)$ is true, that there is at least one $(+1, -1)$ clockwise point pair on C_{k+1}.

We can create C_{k+1} by adding a $+1$ point and a -1 point anywhere on C_k. It is impossible to place either of these points in a way that breaks up the $(+1, -1)$ clockwise point pair that already exists on C_k. If we place the $+1$ between the existing $(+1, -1)$ pair, we create a new sequence $(+1, +1, -1)$ that still contains a $(+1, -1)$ clockwise point pair. If we place the -1 point between the existing $(+1, -1)$ pair, we create a new sequence $(+1, -1, -1)$ that still contains a $(+1, -1)$ clockwise point pair. □

Lemma 2. *If a $(+1, -1)$ point pair is added into a winning path, the path remains a winning path.*

Proof. A winning path is defined as a path that never takes you through more -1 points than $+1$ points at any point in the journey. In other words, at any point the count of $+1$ points passed through p must be $\geq$ the count of -1 points passed through n, or $p \geq n$. If we insert a $(+1, -1)$ clockwise point pair after any arbitrary point in a winning path and this inequality still holds at all points on the path, we will have shown that we can always insert a $(+1, -1)$ clockwise point pair into a winning path and it will remain a winning path.

We insert a $(+1, -1)$ clockwise point pair into a winning path. No matter where we insert this, the inequality $p \geq n$ must hold at the point directly preceding the inserted pair. Passing through the first point in the $(+1, -1)$ clockwise point pair yields $(p+1) \geq n$. Passing throught the second point in the $(+1, -1)$ clockwise point pair yields $(p + 1) \geq (n + 1)$. Subtracting one from both sides gives $p \geq n$. □

Now we can proceed with the proof of theorem 1. Let $P(n)$ be "For any $n \in N$ where $n \geq 1$, no matter which of the n points are labeled $+1$ and which are labeled -1, there is always one point you can start at to win the game." We will prove that $P(n)$ is true for all $n \geq 1$.

As our base case, we prove that $P(1)$ is true. Because C_1 only has a single $+1$ point and a single -1 point, we immediately see that by starting at the $+1$ point we can win the game.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ is true, meaning there is a point you can start at on circle C_k to win the game. Let's start with C_{k+1}, a circle with $k + 1$ positive points and $k + 1$ negtive points. By lemma 1, we know that there is at least one $(+1, -1)$ clockwise point pair on C_{k+1}. If we remove this pair we are left with a new circle C_k. From our inductive hypothesis, we know that there is at least one winning path on this circle. By lemma 2, we know that we can insert a $(+1, -1)$ clockwise point pair anywhere into this winning path and still have a winning path. This means we can insert the previously-removed $(+1, -1)$ clockwise point pair back into its original spot and the winning path will remain a winning path. This proves that there is at least one winning path in C_{k+1}. $\square$

1 Problem 5: Binomial Trees

Binomial trees are a specific family of directed trees defined as follows: a binomial tree of order n is a single node with n children, which are binomial trees of order $0, 1, 2, \ldots, n1$.
Theorem. *A binomial tree of order n has exactly 2^n nodes.*

Proof. By strong induction. Let $P(n)$ be "a binomial tree of order n has exactly 2^n nodes." We will prove that $P(n)$ holds for all natural numbers n.

As our base case, we will prove that $P(0)$ is true. A binomial tree of order 0 is a single node with no child nodes. Because $1 = 2^0$ we show that our base case is true.

For the inductive step, assume that for some $n \in \mathbb{N}$, that for any $k \leq n$, that $P(k)$ holds and that a binomial tree of order k has exactly 2^k nodes.

By definition, we know that a binomial tree of order $k+1$ has $k+1$ children, which are themselves binomial trees of order $0, 1, 2, \ldots, k$. To find the total number of nodes in a binomial tree of order $k+1$ we need to add all of the nodes of the children plus one for the root node. By the inductive hypothesis we know that each of the children trees have $2^0, 2^1, 2^2, \ldots, 2^k$ nodes. Thus, the total number of nodes in the binomial tree of order $k+1$ is given by

$$1 + \sum_{i=0}^{k} 2^i. \tag{1}$$

In page 102 of the Mathematical Foundations of Computing course reader it is proved that

$$\sum_{i=0}^{j-1} 2^i = 2^j - 1.$$

Letting $j = k+1$ gives us

$$\sum_{i=0}^{k+1-1} 2^i = 2^{k+1} - 1$$

Plugging this into 1 gives us

$$1 + 2^{k+1} - 1 = 2^{k+1}$$

$\square$