An Example of a Hypothetical Learning Progression:

The Successor Principle and Emergence of Informal Mathematical Induction

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Abstract

The development of *mathematical induction*, which entails using a pattern discovered about consecutive integers *and* logical reasoning to create new knowledge, has been little studied. However, some evidence indicates that children as young as 5-years-old can engage in an informal version of such reasoning (Smith, 2002). A hypothetical learning progression is proposed to help account for how informal mathematical induction emerges. By 4 years of age, familiarity with the counting sequence enables children to enter the count sequence at any point and specify the next number instead of always counting from one (Fuson, 1988). This and small number recognition, which (combined with other existing capabilities and knowledge) enables children to see that “two” is one more than “one” and “three” is one more than “two,” may lead to the induction of the *successor principle*: Any number is exactly one more than its counting predecessor. Theoretically, this big idea is the conceptual basis for re-representing the counting sequence as the (positive) *integer sequence* (n, n+1, [n+1]+1) and in a linear manner. These developments, in theory, are a basis for informal mathematical induction, which can be the basis for kindergartners concluding, for example, "There is no largest number” (Baroody, 2005). Educational implications and unresolved issues are discussed.

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Some time ago, I asked Nikki a kindergartner what she thought the largest number was. The girl responded, "A million." I then asked what number she thought came after a million. After a moment's thought, Nikki responded, "A million and one." I next asked what she thought came after a million and one. Again the girl thought for a moment and answered, "A million and two." I pressed Nikki further by asking what she thought came after a million and two. After a few moments of thoughtful reflection, she concluded, "There is no largest number." Nikki's sudden insight essentially entailed the re-discovery of the concept of infinity.

The aim of this paper is to raise questions about the development of a powerful form of reasoning, the basis for which appears to emerge in early childhood. Discussed, in turn, are (a) how the informal version of this type of reasoning contrasts with its formal version; (b) a hypothetical learning progression of how the informal form of this reasoning might develop; (c) educational implications of the learning progression, and (d) issues that need to be addressed to better understand this development, construct an empirically based learning progression, and devise effective instruction for promoting this development.

Formal versus Informal Mathematical Deduction

Mathematicians use various types of reasoning in their work (Clements & Battista, 1992), all of which involve "a process of thought that yields a conclusion from percepts, thoughts, or assertions" (Johnson-Laird, 1999, p. 110, cited by Smith, 2002, on p. 33).

- *Intuitive reasoning* entails playing a hunch—using what is obvious (appearances), what feels right (an assumption), or previous experience (precedent) to draw a conclusion. For example, when I asked Nikki what number came after a million, she probably assumed from her previous experience with questioning adults that her answer (a million is the largest number) was incorrect and that there was an even larger largest number.

- *Empirical induction* entails examining examples (particulars) and discerning a commonality or pattern (discovering a generality). For example, from the succession of questions (examples), Nikki recognized a pattern, namely that even the "largest" numbers have a number after them (induced the generality that every number has a successor).
- *Deductive reasoning* involves reasoning from a premise or premises assumed to be true (a generality or generalities) to logically arrive at a conclusion about a particular case. Asked what comes after a million (or million one), Nikki probably used her knowledge of the recursive nature of the counting sequence to respond. Specifically, she may have reasoned that, after a term starting a new series (e.g., a decade term such as *twenty* or hundred term such as *three hundred*), the rest of the series is generated by combining this first term with the single-digit sequence *one* to *nine* (e.g., after *twenty* comes *twenty-one*, *twenty two*…; after *three hundred* comes *three hundred one, three hundred two*; premise 1) and, if a *million* introduces a new series (premise 2), then the next terms must be a *million one*, a *million two*, and so forth. Unlike the conclusions drawn from intuitive or inductive reasoning, note that Nikki's "deduction" *necessarily* follows from what is given.
- *Mathematical induction* (reasoning by recurrence) combines both empirical induction and deductive reasoning to form a unique type of reasoning[1] (Poincaré, 1905, and Piaget, 1942, cited in Smith, 2008; Smith, 2002). Similar to empirical induction, it entails making a

[1] In contrast, Rips and colleagues (Rips, Asmuth, & Bloomfield, 2006, 2008; Rips, Bloomfield, & Asmuth, 2008) argue from the perspective of logicism that mathematical induction is a form of deductive reasoning (necessitating), which is distinct from empirical induction (universalizing). See Smith (2002) for a critique of logicism.

> generalization about a reoccurring pattern (recursive property) for an integer n and its successor ($n + 1$), but unlike empirical induction and like deductive reasoning, the conclusion about n and $n + 1$ (induced property) is necessarily true of all integers. For example, Nikki initially appeared to conclude that a million (n) was the number after the next-largest number, and if a million is the largest number, then there is no number after it. However, when she allowed that there is a number after a million, she concluded that it is not the largest number but that a million one ($n + 1$) is the largest number. But there is also a number after a million and one, namely a million two, and so $n + 1$ is not the largest number. Nikki concluded (deduced) that this process could, in principle, go on forever (i.e., for any number named, there is a successor and thus the counting sequence is infinite).

Like deductive reasoning, then, mathematical induction is creative in that it allows a mathematician to use observations and existing knowledge to create new knowledge—extend understanding beyond the limits of extant knowledge. As the case of Nikki illustrates, a primitive or informal version of mathematical induction can be an important process in young children's meaningful mathematical learning. Indeed, it even enabled her to comprehend what she could not experience directly (infinity).

Differences between informal and formal mathematical deduction. Obvious differences between informal and formal mathematical induction are that the latter requires (a) formal and advanced mathematical training and (b) the formulation of a formal or logical proof. Another possible difference is that, like Nikki, young children may need to consider more than a single case (an integer and its successor) to detect a pattern. In Nikki's case, she considered a million and a million one (first case), then a million one and a million two (second case), and was in the process of considering a third case, a million two and a million three when she had her insight regarding indefinite succession.

Smith (2002, 2008) suggested that—consistent with Poincaré's (1905) and Rips, Bloomfield, and Asmuth's (2008) criteria for mathematical induction—an operational definition of informal mathematical induction should entail (a) establishing a base equality or inequality; (b) assessing universality about number; and (c) gauging the necessity about number. It seems reasonable to assume that Nikki understood that the "largest number" was larger than its predecessor, that any number would have a successor, and that this would be necessarily true of any number.

Existing Research on Informal Mathematical Induction

Despite the potential importance of informal mathematical induction, almost no research has examined the development of this type of reasoning in early childhood. Although empirical induction and deductive reasoning have long been a concern to developmental psychologists and mathematics educators (see, e.g., Ennis, 1969; Evans, 1982; Morris, 2000, 2002), mathematical induction has largely been ignored, perhaps partly because of the assumption that children were not capable of such advanced reasoning (see, e.g., Rips, Bloomfield, & Asmuth, 2008).

Inhelder and Piaget (1963; cited by Smith, 2002) concluded from their study of (informal) mathematical induction that children aged 5 to 7 years old could make such inferences on the basis of iterative actions and that their reasoning was modal (i.e., understood their conclusions were necessarily true). In his replication and adaptation of this research, Smith (2002) came to a similar conclusion. Rips, Bloomfield, and Asmuth (2008) argued that Smith's (2002) results actually bear on universal generalization (empirical induction), not mathematical induction. Smith (2008) countered that the three questions used in his 2002 study (regarding the base equality/inequality through serial additions, universality about number, and necessity about number were operational matches to Poincaré and Rips et al.'s criteria for mathematical induction and that his evidence on the first two questions was significant and evidence regarding the third (modality or necessity) were promising. In brief, Smith (2008) concluded that his evidence indicated that 5- to 7-year olds were capable of a primitive (if fallible) version of mathematical induction—but noted that psychologically rule-learning was a fallible process for everyone.

Baroody (2005) noted four limitations of the Smith (2002) study. (a) The mathematical induction task ("Recurrence Task") included a question regarding logical necessity ("Does there have to be the same in each or not?"). However, a correct response to such a yes-no question clearly does necessarily indicate a child was convinced the necessity of an argument. This is particularly true where children's consistency apparently was not evaluated. (b) The data appear to have been collapsed over participants for the data analyses. It does not appear that an effort was made to see if individual participants performed in a consistent and logical manner. (c) Only a single example of mathematical induction was studied. Although this is enough to confirm an existence hypothesis (young children are capable of mathematical induction), it limits generalizability. (d) Although Smith (2002) attempted to assess the role of counting in the development of (informal) mathematical induction, the tasks used raised a number of concerns and his results are not persuasive.[2]

Gelman and colleagues (Evans, 1983; Evans & Gelman, 1982; Harnett & Gelman, 1998) did not examine informal mathematical deduction directly, but they found evidence that primary-

[2] As one measure of counting, Smith used the Same Task, which he noted was *compatible* with the ability to count to ten. This simple matching task did involve 10 counters, which were put in two rows of three and a row of four all aligned on the left. Children were asked, "Which two rows had the same?" and, after one of the rows of three was removed, they were further asked whether the two remaining rows had the same or different number of items. Not surprisingly nearly all participants were successful on the task. However, because children can successfully complete such a simple matching task with or without counting skill, this success says little about a child counting of up to 10 items, (c) The Teams Task, which was used to gauge numerical identity in a counterfactual context, was highly verbal in nature, was confusing to this reader, may or may not have been interpreted as counterfactual by participants, and did not clearly distinguish correct answers and true acts of judgment. (d) The data from the Counting Task (gauging the equivalence of two counted rows of item before and after one row is lengthened) seemed counterintuitive. Contrary to previous research and Smith's (2002) data on the number conservation task, children performed better after the perceptually misleading transformation.

level children understood the idea of indefinite succession. Specifically, Harnett and Gelman (1998) found that the majority of second graders understand that every natural number has a successor and that even a quarter of their kindergarten participants appeared to understand this concept and half were classified as “waverers.”

Hypothetical Learning Progression

Although some evidence indicates that 5- to 7-year olds are capable of informal mathematical induction, no clear evidence exists about how children become ready for such reasoning. In this section, we first describe a basic assumption of a hypothetical learning progression for informal mathematical induction and then use existing developmental evidence and theory to describe a plausible progression.

The proposed hypothetical learning progression is based on the assumption that children typically construct knowledge in a bottom-up, not a top-down, manner. Although probably an exaggeration, Colburn (1828) observed that “it is not too bold an assertion to say, that no man ever actually learned mathematics in any other method than by analytic [empirical] induction; that is, by learning the principles by the examples he performs; and not by learning principles first, and then discovering by them how the examples are to be performed.” In contrast Rips and colleagues (Rips, Asmuth, & Bloomfield, 2006, 2008; Rips, Bloomfield, & Asmuth, 2008) offer a critique of this “bootstrap” developmental view (e.g., Carey, 2004; Piantadosi, Tenenbaum, & Goodman, 2012) and suggest that number and arithmetic are built in a more top-down fashion by constructing mathematical schemas that permit mathematical inferences. One particularly relevant argument for rejecting experience with physical quantities as the basis for a concept of the natural numbers is that such experiences are always finite and natural numbers are infinite. Smith (2002) concluded that, for Piaget, mathematical induction entailed both intuition (empirical induction) and logic (deductive reasoning) and the source of the former, as the case of

Nikki illustrates, is reflection on repeated mental actions (see also Sophian, 2008). For others' reasons to question the top-down view and consider the bottom-up alternative, see, for example, Andres, Di Luca, and Presenti (2008); Barner, 2008; Carey, 2008; Cowan, 2008; Smith, 2008; Sophian, 2008).

Our hypothetical learning progression is summarized in Table 1.

Small number recognition. Piaget (1965) hypothesized that counting was not necessary to construct an understanding of number. This view is consistent with both current competing views of early number development. According to nativists, the privileged-domain hypothesis, or the continuity hypothesis, a concept of number, relatively exact representations of the intuitive numbers one to three, and the principles of counting are innate and this nonverbal knowledge and verbal number skills such as counting directly map this nonverbal knowledge (Gallistel, 2007; Gallistel & Gelman, 1992; Wynn, 1998). Put differently, nonverbal knowledge provides a direct basis for relatively quickly learning verbal number skills. However, the empirical evidence does not clearly support this perspective (e.g., Le Corre & Carey, 2007, 2008; Le Corre, Van de Walle, Brannon, & Carey, 2006; Mix, Huttenlocher, & Levine, 2002; Simon, 1997).

According to empiricists or the discontinuity hypothesis, however, preverbal capacities provide a basis for constructing a number concept (e.g., a basic understanding of singular and plural instances) but symbolic tools such as language permit children to construct exact number concepts and counting skills (Mix, Sandhofer, & Baroody, 2005; Palmer & Baroody, 2011; Spelke, 2003). Number words are a part of children's vocabularies almost from the time they begin to speak at 18 to 24 months (Fuson, 1988). As with other words, children seem to first use "two" and "three" with little specific meaning or to indicate "many" instead of a specific quantity (Mix et al., 2002; Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007; von Glasersfeld, 1982; Wagner & Walters, 1982; Wynn, 1992).

As with many other concepts, children may construct number concepts via an inductive process involving visual examples and non-examples, not the relatively complicated process of counting (see Baroody, Lai, & Mix, 2006, for a review).[3] Similarly, as children see a number word used in conjunction with various visual examples, are corrected for misapplying the term to non-examples, and observe other number words used with non-examples (e.g., "that's three, not two, cookies"), they can construct a well-defined cardinal number concept. For example, seeing different pairs of items labeled "two" can prompt the discovery that appearances (physical characteristics such as shape, color, or arrangement) are not relevant and reveal the commonality that all such examples involve more than one (a plurality). Recognizing that "two" does not apply to collections of three or more items can further prompt them to induce that "two" applies only to couples or pairs. As a result of piecemeal context-specific mappings between concrete instances and a number word (or other symbol such as a numeral), children may simultaneously develop an accurate and general cardinal number concept and "verbal subitizing" or small number recognition (Mix 2009). Exact cardinal concepts of the intuitive numbers and small number recognition anchor the proposed learning progression and provide a basis for meaningful learning key number and counting concepts and skills in the progression.

Meaningful object counting. Small number cardinality concepts and recognition enable children to understand the whys and the hows of enumeration. By observing others enumerate small collections they can numerically recognize, preschoolers can discover that enumeration is another process by which to label the total number of items in a collection with a number word (i.e., determine its cardinal value). Similarly, they can construct an understanding of the how-to-count principles that underlie meaningful one-to-one object counting:

[3] Words can facilitate infants' and toddlers' formulation of kind concepts—conceptual classes based on causal relations or functions. Common labels (e.g., the same noun used to name different examples of a class) appear to support kind categorization by infants (Balaban & Waxman, 1997; Dueker & Needham, 2003; Waxman & Markow, 1995) or toddlers (e.g., Sandhofer & Smith, 1999).

- By observing adults enumerate small collections they can recognize, children may be more likely to abstract the one-to-one principle—to understand why an adult labels each item with a single number word. Doing otherwise results in a different outcome than the cardinal value determined by small number recognition).
- By observing others model counting with collections they visually recognize, children can discover the cardinality principle—that the last number word used represents the total number items or cardinal value of the collection (Baroody et al., 2006). A child who can recognize a collection of "three" better understands why others emphasize or repeat the last number word used to count a collection of three items.
- By observing adults enumerate a heterogeneous collection and labeling it with the cardinal value that can be "seen," small number recognition may also underlie children's discovery of the abstraction principle.
- By observing adults or themselves enumerate small numerically identifiable collections in different directions or arrangements, they may also induce the order-irrelevance principle.

Ordinal number concept of small numbers. Small number recognition enables children to see that "two is *more* than one" item and that "three is more than "two" items and what adults mean by the term "more." This enables children to recognize that numbers have an ordinal, as well as cardinal, meaning.

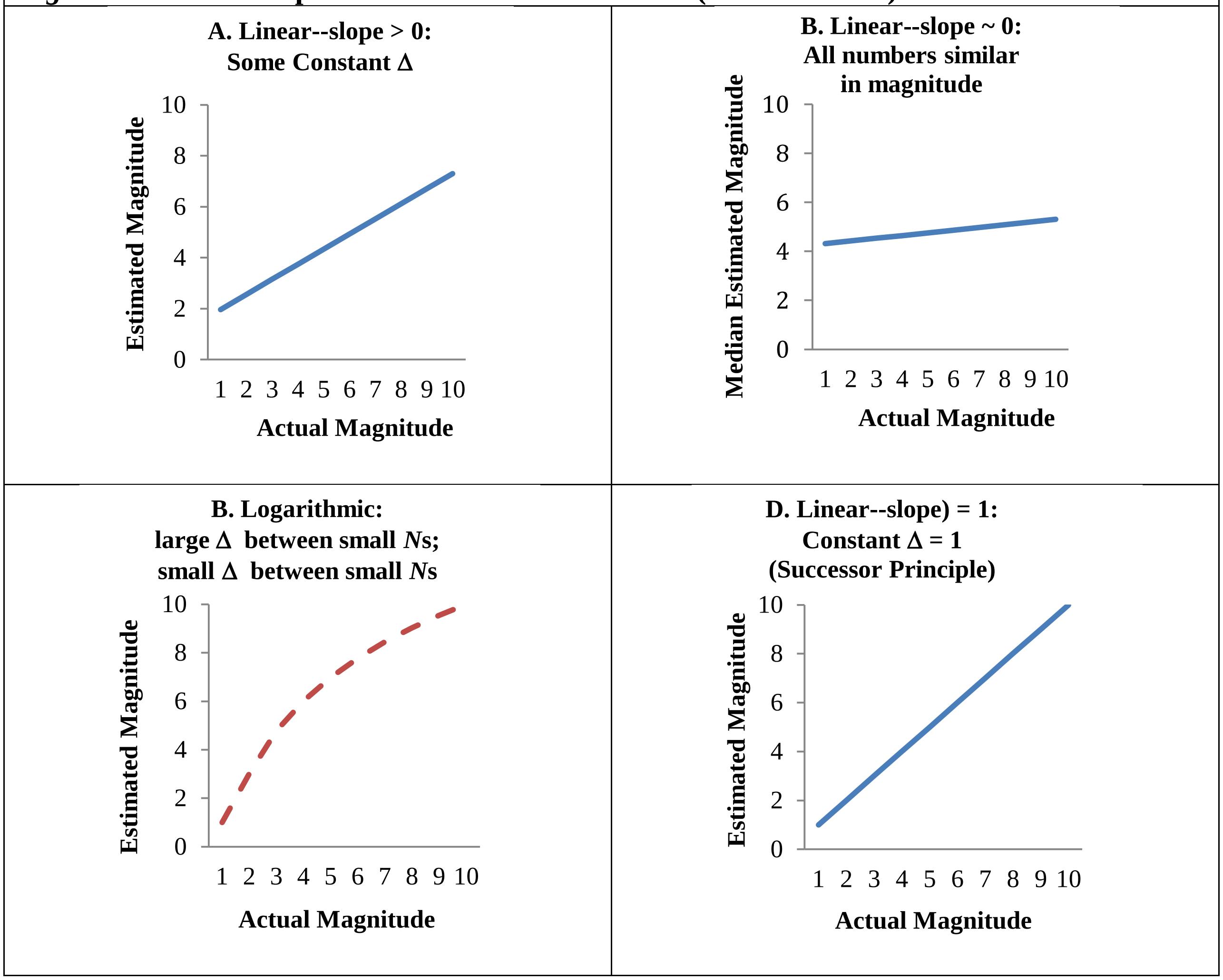
Increasing magnitude principle and counting-based number comparisons. Between 3.5 to 6 years of age, children's thinking about the counting numbers from one to ten undergoes critically important changes. By 4 years of age, children apply their ordinal concept of small numbers to their knowledge of the counting sequence. Specifically, they appreciate that the number words in the counting sequence, not only represent different, specific quantities but, quantities of increasingly larger size—the "increasing magnitude" principle (Sarnecka & Carey, 2008). Put differently, they learn that a number word further along in the counting sequence than another represents the larger collection (e.g., Schaeffer, Eggleston, & Scott, 1974). This enables children to determine the larger of two collections by counting them—the collection that requires the longer count is "more."

Number-after knowledge. As children become familiar with the counting sequence, they no longer have to start with "one" and use a running to start to determine the number after another (e.g., "After four is "one, two, three, four—*five*."). Instead, they access the counting sequence at the given number and state its successor. This skill is critical for the (efficient) application of the next two steps in the learning progression.

Mental comparisons of close/neighboring numbers. The application of the increasing magnitude principle and number-after knowledge enables children to efficiently and mentally compare even close numbers, such as the larger of two neighboring numbers, without counting (e.g., "Which is more seven or eight?—eight").

Successor principle. The increasing magnitude principle, number-after knowledge, and the mental ability to compare even neighboring numbers, however, does not necessarily entail a <u>*linear representation of numbers of numerical magnitude*</u>*—an understanding that each successive number increases by a constant amount* (e.g., that the difference between 8 and 9 is the same as that between 2 and 3). Such an understanding is embodied by a straight-line graph with a slope that is appreciably more than 0 (see, e.g., Frame A of Figure 1). (A straight line with a slope that is basically 0, as illustrated in Frame B of Figure 1, indicates viewing all numbers as having essentially the same magnitude.) Indeed, some research suggests that children begin with a *logarithmic representation of number* (e.g., Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010). That is, their representation of the counting numbers seems to involve large magnitude differences between small numbers and increasingly smaller ones between progressively larger

***Figure 1*. Different representations of numbers 1-10 (Δ = difference).**



numbers. For example, the estimated difference between 8 and 9 is smaller than that between 2 and 3 (see Frame C of Figure 1). Young children may see “really large numbers” such as 8, 9, and 10 as almost indistinguishable in magnitude.

In time, preschoolers use small number recognition to see that “two” is exactly one more than “one” items and that “three” is exactly one more than “two” items. By generalizing this induced pattern to the counting sequence as a whole, children construct the <u>*successor principle*</u>: Any number is exactly 1 more than its predecessor in the counting sequence.

Reconceptualization of the counting sequence as the (positive) integer sequence.

Theoretically, the successor principle is the conceptual basis and impetus for re-representing the counting sequence as the (positive) integer sequence: n, $n+1$, $[n+1]+1$, … (Sarnecka & Carey, 2008) and in a linear manner. As Frame D in Figure 1 illustrates, knowledge of the successor principle should result in magnitude estimates that are not only *linear* in shape but have *a slope of 1*.

Informal mathematical induction. Theoretically, the successor principle is also the developmental prerequisite for informal mathematical induction because such reasoning is not possible without a representation of the integer sequence.

Infinite succession principle (concept of infinity). As the case of Nikki illustrates, informal mathematical induction can lead a child to conclude that the counting or natural numbers are unending or infinite (Baroody, 2005; cf. Rips, Bloomfield, & Asmuth, 2008)—a concept that appears to be within the grasp of many, if not most, primary-level children (e.g., Harnett & Gelman, 1998).

Educational Implications

Three implications follow from our review.

1. *Informal mathematical induction needs to be included as a goal for primary-level instruction.* Although mathematics educators and others interested in reforming the mathematics curriculum argue that empirical induction and deductive reasoning are important instructional goals for elementary-level children (if mathematical reasoning is discussed at all), mathematical induction is seldom recommended as such (but cf. Smith, 2002, 2003). Although the National Council of Teachers of Mathematics (NCTM, 2000) Standards includes discussions of looking for patterns (empirical induction) and logical (deductive) reasoning, no explicit mention is made of mathematical induction. By extension, the NCTM (2006) *Curriculum Focal Points* does not either. In *Adding It Up: Helping Children Learn Mathematics*, the National Research Council

(2001) includes adaptive reasoning (defined as logical reasoning) as a key component of mathematical proficiency, but gives short shrift to empirical induction and does not mention mathematical induction at all. With the exception of problem solving, the report of the National Mathematics Advisory Panel (2008) focused on mathematical content, not processes. Similarly, the Common Core State Standards (2011) focus almost exclusively on mathematical content.

2. *Instruction on informal mathematical induction needs to be based on an empirically and theoretically based learning progression*. Clearly, young children will not be ready to develop such a relatively sophisticated form of reasoning without adequate preparation. Research is needed to evaluated the proposed learning progression in this paper or otherwise identify the developmental prerequisites for informal mathematical induction and their developmental relations. This knowledge then needs to be translated into curricula and teacher guidelines so that such reasoning can be encouraged in a developmental appropriate manner.

3. *Teachers need to actively encourage informal mathematical induction*. Smith (2002) argued that Piaget's views on education have often been misinterpreted or misrepresented. For example, although some strict constructivists eschew the term *discover* and criticize the *transmission* of knowledge (e.g., Kamii, 1985), Piaget did not. As the case of Nikki illustrates, Piaget believed teachers should be active agents in children's education, not passive or laissez faire observers, as his views of teachers are often characterized.

Issues

A number of issues need to be resolved or researched.

1. *Does the definition of informal mathematical induction presented in the present paper on pp. 4 to 6 of the "Formal versus Informal Mathematical Deduction" section make sense?* Should it be modified and, if so, how? For example, is there anything that should be added or subtracted from the definition presented? Might informal mathematical induction be better characterized as

developing in phases? For instance, in light of Smith's (2002) tentative results regarding modality, might an understanding that a conclusion is necessarily true in all applicable cases represent a relatively advanced level of proficiency with this type of reasoning?

2. *Are current tasks reasonable measures of informal mathematical induction and, if not, how might these tasks be improved?* The recurrence task Smith (2002) adapted from Inhelder and Piaget (1963) involved an initial state consisting of two transparent plastic containers ("pots"), which were either empty (equivalent start) *or* had one item (a "green cat") out in one (non-equivalent start). A child was asked Question 1 (Q1): "*Is there the same in each, or is there more in one than the other?*" The second step involved actual and observed additions to the pots. After the child put three to eight green cats in one pot one at a time and simultaneously did the same with orange cats and the other pot, s/he was asked Q1 followed by Question 2 (Q2): "*Does there have to be the same in each, or not?*" The third step entailed actual but non-observed additions. This step was identical to the second, except that the child could not view the plastic containers and their contents. The fourth step involved hypothetical additions. The three key hypotheticals were (a) "*pretend to add six to each pot at the same time*"; (b) "*pretend to add a great number to each pot at the same time*"; and "*pretend to add any number at all to one pot and the same number to the other.*" The first hypothetical was followed by Q1 and Q2 served as a warm-up. The second and third hypothetical served as the penultimate and ultimate directives and were followed by Question 3 (Q3): "*If you add [a great number/any number] here [pointing to one pot] and the same number to that [pointing to the other pot], would there be the same in each, or would there be more in one than the other?*" Might Rips, Bloomfield, and Asmuth (2008) be correct that this task actually bears on universal generalization (empirical induction), not mathematical induction? Put differently, is it mathematical induction if a child correctly answers Q3 in response to the penultimate and ultimate directive if the child induced the pattern on

previous trials or steps and is simply applying it to answer Q3 (i.e., what new knowledge has the child generated as a result of using a discovered pattern AND reasoning logically)?

Might the “largest number task” (the series of simple questions asked of Nikki) be adequate to gauge informal mathematical induction or does the task need to be expanded (cf. Harnett & Gelman’s (1998) extensive protocol of questions)? Might the “counting-pattern task” described next be a good candidate for an informal mathematical induction measure for young children? Define an *even number* informally in terms of a collection that can be shared fairly between two people with nothing left over and an *odd number* as a collection that leaves one leftover item after fairly sharing as many items as possible. Have the child determine whether each of two successive numbers such as 4 and 5 is even and odd. Have the child use a fair sharing (divvying-up) strategy to answer and record the results on a number list (e.g., green highlight for the 4 square to indicate even; red highlight for 5 square to indicate odd). Then ask if 8 or 9 are each even or odd. If needed, repeat this sharing process and feedback and proceed with the next example (e.g., 13 and 14). The dependent measure is whether a child will realize that for any pair of successive numbers one must be even and the other must odd.

3. *Are there additional tasks that could or should be used with primary-level children to gauge whether they can learn or use informal mathematical induction?* In addition to the few examples of informal mathematical induction cited in the present paper, are there other examples of such reasoning that 5 to 8 year olds might reasonable be expected to be successful?

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Table 1: Hypothetical Learning Progression	
Cardinal concept of small numbers + small number recognition **(1 and 2 first, then 3, and—in time—4 to 6)**	By seeing different examples of a number labeled with a unique number word (e.g., "*two* eyes," "*two* hands," "*two* socks," "*two* shoes," "*two* cars") and non-examples labeled with other number words ("take one cookie, not two"), children construct of precise cardinal concepts one, two, and three. These concepts permit ***small number recognition*****:** The ability to immediately recognize and label small collections with an appropriate number word.
Meaningful object counting **(including the count-cardinality principle)**	***Small number recognition*** enables children to understand the principles underlying ***meaningful counting:*** *stable order, one-to-one*, and *cardinality principles*. For example, by watching an adult count a small collection a child can recognize as "three," s/he can understand why the last number word in the count is emphasized or repeated—it represents the total or how many (the cardinal value of the collection).
Ordinal concept of small numbers **(comparisons of collections composed of 1 to about 3 items)**	***Small number recognition*** enables children to see that "two is *more* than one" item and that "three is more than "two" items (understand the term "more" and that numbers have an ***ordinal meaning***, as well as cardinal meaning.
Increasing magnitude principle **+** ***counting-based number comparisons*** **(especially collections larger than 3)**	***Small number recognition*** and ***ordinal number concept*** permit discovery of the **increasing magnitude principle:** the counting sequence represent increasingly larger quantities. This enables them to use the ***meaningful object counting*** to determine the larger of two collections (e.g., 7 items is more than 6 items because you have to count further to get to *seven* than you do for *six*).
Number-after knowledge **of the counting sequence**	Familiarity with the counting sequence enables a child to enter the sequence at any point and ***specify the next number*** instead of always counting from one.
Mental comparisons of close/neighboring number **(number after = more)**	The use of the ***increasing magnitude principle*** and ***number-after knowledge*** enables children to determine efficiently and ***mentally compare even close numbers*** such as the larger of two neighboring numbers (e.g., "Which is more seven or eight?—eight")
Successor Principle **(Number after = 1 more)**	**Small number recognition** enables children to see that "two" is exactly one more than "one" items and that "three" is exactly one more than "two" items, and this can help them understand the ***successor principle*** (each successive number in the counting sequence is exactly one more than the previous number).
Reconceptualization of the counting sequence as the (positive) integer sequence	The ***successor principle*** enables children to view the counting sequence as n, n+1, [n+1]+1, … (***positive integer sequence***)—a linear representation of number.
Informal mathematical induction	An understanding of the ***successor principle*** and a representation of the ***(positive) integer sequence*** provide a basis for informal mathematical induction.
Infinite succession principle (concept of infinity)	***Informal mathematical induction enables children to use their number-after knowledge of the counting sequence and the successor principle to realize that the positive integer sequence can, in principle, go on forever.***