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WHY CAN'T JOHNNY REMEMBER THE BASIC FACTS?

Arthur J. Baroody,* Neet Priya Bajwa, and Michael Eiland

Department of Curriculum and Instruction, University of Illinois at Urbana-Champaign, Champaign, Illinois

Memorizing the basic number combinations, such as 9 + 7 = 16 and 16 − 9 = 7, is a punishing and insurmountable task for children with difficulties learning mathematics. Two perspectives on such learning lead to different conclusions about the primary source of this key learning difficulty. According to the conventional wisdom (the Passive Storage View), memorizing a basic fact is a simple form of learning—merely forming and strengthening an association between an expression and its answer. The two primary reasons this simple form of learning does not occur are inadequate practice or, in cases where adequate practice has been provided, a defect in the learner. According to the number sense perspective (Active Construction View), memorizing the basic combinations entails constructing a well-structured or -connected body of knowledge that involves patterns, relations, algebraic rules, and automatic reasoning processes, as well as facts. In effect, fluency with the basic number combinations begins with and grows out of number sense. Aspects of number sense critical to such fluency begin to develop in the preschool years. According to the Active Construction View, the primary cause of problems with the basic combinations, especially among children at risk for or already experiencing learning difficulties, is the lack of opportunity to develop number sense during the preschool and early school years. © 2009 Wiley-Liss, Inc.
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Memorization of the basic number combinations, which include single-digit addition items (e.g., 9 + 3 = 12) and related subtraction items (e.g., 12 − 9 = 3), has been a central goal of elementary instruction since ancient Babylonian times [Baroody, unpublished data]. Despite the recent "math wars," there is general agreement that *all* children need to achieve this goal [National Council of Teachers of Mathematics, 2000; Kilpatrick et al., 2001; National Mathematics Advisory Panel, 2008]. Numerous everyday applications make this basic skill important in its own right. Memorization of the basic number combinations also is important for success with more advanced school mathematics. By freeing cognitive resources so that attention can be focused on more complex matters (e.g., looking for patterns), ready access to number combination knowledge can facilitate problem solving and both mental and written computation with multi-digit and rational numbers [National Mathematics Advisory Panel, 2008].

Despite the importance of the goal and the considerable time and effort teachers devote to achieving it, memorizing the basic combinations is a major stumbling block for many schoolchildren [National Mathematics Advisory Panel, 2008]. For example, using an untimed mental-arithmetic task and self-reports (methods that can seriously overestimate ready knowledge of basic combinations), Henry and Brown [2008] found that the vast majority of 275 first graders, even those from high-performing schools, did not achieve the California state standard (goal) of memorizing the sums to 18 and related differences—despite the instructional emphasis on this goal. The median success rate was a mere 22% of the tested items.

Problems with memorizing the basic combinations are more acute among children at-risk for poor mathematics achievement or already experiencing mathematics difficulties (MD). (At-risk indicators include low family income; a single, poorly educated, or teenage parent; minority status; a physical disability; or emotional difficulties.) Baroody et al. [unpublished data], for instance, found that three-fifths of 133 at-risk first graders had not mastered more than 83% of the $n + 1/1 + n$ combinations by the middle of the school year. Worse, three-eighths had not done so when retested at the end of the school year. Furthermore, a failure to memorize basic number combinations is one of the most pervasive characteristics of children with MD [e.g., Goldman et al., 1988; Geary, 1996; Jordan et al., 2002, 2003; cf. Murphy; De Smedt et al.; O'Hearn, this issue].

Children who do not master the addition combinations, particularly those who have not mastered even the simplest addition combinations by the end of first grade, are handicapped in their efforts to achieve mastery with basic subtraction, multiplication, and division combinations; to perform multi-digit mental or written calculations efficiently; and operate on rational numbers [National Mathematics Advisory Panel, 2008]. Such children, then, are highly susceptible to MD and a spiral of failure and frustration [Ginsburg and Baroody, 1990].

We address the following questions: How do children memorize the basic combinations? How can instruction effec-

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*Correspondence to: Arthur J. Baroody, 310 Education, University of Illinois, Champaign, IL 61820. E-mail: baroody@illinois.edu
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tively facilitate the memorization of basic combinations? Facilitating the memorization of basic combinations should begin when and take how long? Why do many children, particularly those at-risk or already experiencing MD, have difficulty memorizing the basic combinations? What are the theoretical implications and the implications for future research? What are the instructional implications for helping children at-risk or with MD?

HOW DO CHILDREN MEMORIZE THE BASIC COMBINATIONS?

Research indicates that children typically progress through three phases in learning a basic combination or a family of related combinations: *Phase 1: Counting strategies* (using object or verbal counting to determine answers); *Phase 2: Reasoning strategies* (using known facts and relations to deduce the answer of an unknown combination); *Phase 3: Mastery* [efficiently producing answers from a memory network; e.g., Kilpatrick et al., 2001]. The first two phases are characterized by conscious or deliberate and, thus, relatively slow cognitive processes. Phase 3 is characterized by nonconscious or automatic and, thus, relatively fast cognitive processes.

Phase 3 can be achieved by either rote or meaningful memorization [Brownell, 1935]. The first produces *routine expertise*—knowledge that can be applied efficiently and appropriately to familiar tasks but not flexibly to new tasks (*mastery with limited fluency*). Meaningful memorization yields a rich and well-interconnected web of factual, strategic (procedural), and conceptual knowledge. The result is *adaptive expertise*—well-understood knowledge that can be applied efficiently, appropriately, and flexibly to new, as well as familiar, tasks (*mastery with fluency*). The *Passive Storage View* is a description of how basic combinations are memorized by rote; the *Active Construction View*, how they are memorized meaningfully. These views, then, provide different explanations for how Phases 1 and 2 are related to Phase 3 and the nature of Phase 3 itself [Geary, 1996; Baroody and Tiilikainen, 2003].

Passive Storage View: Phases 1 and 2 can Facilitate, But are Not Necessary for, Memorizing Facts

Memorizing a basic number fact by rote is a simple nonconceptual process of forming a bond or association between an expression such as $7 + 6$ and its answer 13 [Thorndike, 1922; Ashcraft, 1992; Shrager and Siegler, 1998]. Each basic fact is stored discretely in a factual memory network, perhaps as a verbal statement such as, "Five plus three is eight" [Dehaene, 1997]. Phase 3 consists of a single process, fact recall, which entails the automatic retrieval of the associated answer to an expression. Arithmetic knowledge, then, is componential in nature [e.g., Dehaene, 1997]. That is, the fact-retrieval network of the brain is autonomous and operates independently from the conceptual or procedural knowledge [Campbell and Graham, 1985; Crowley et al., 1997; but cf. National Mathematics Advisory Panel, 2008]. Moreover, relatively efficient fact retrieval (Phase 3) replaces relatively inefficient conceptually based counting- or reasoning-based strategies (Phases 1 and 2).

Like other worthwhile knowledge, meaningful knowledge of basic combinations may be a systematic web of understandable facts, principles, and processes.

Early proponents of the Passive Storage View considered counting and reasoning strategies as "immature," "a crutch," and a hindrance—as a bad habit for avoiding the real work of memorizing the basic facts [Smith, 1921; Wheeler, 1939]. This attitude is still prevalent among teachers, principals, and parents today. Modern proponents of the Passive Storage View allow that the first two phases are opportunities to practice basic combinations and to imbue them with meaning (e.g., understand what $7 + 6$ represents) before they are memorized. Even so, Phases 1 and 2 are still viewed as *unnecessary* for achieving Phase 3, because the autonomous mental storehouse of facts (the basis for Phase 3) could theoretically be accomplished directly by extensive practice (e.g., flash card drills) alone [Crowley et al., 1997; Shrager and Siegler, 1998].

Active Construction View: Phases 1 and 2 are Necessary for Meaningful Memorization

Like other worthwhile knowledge, meaningful knowledge of basic combinations may be a systematic web of understandable facts, principles, and processes [Olander, 1931; Brownell, 1935; Baroody, 1985, 1994; National Mathematics Advisory Panel, 2008]. In effect, mastery with fluency grows out of the development of meaningful and well-interconnected knowledge about numbers—*number sense*. For this reason, the Active Construction View is often called the Number Sense View [e.g., Gersten and Chard, 1999]. Phases 1 and 2, then, are critical to creating the rich network of factual, relational, and strategic knowledge that is the basis for mastery with fluency (Phase 3).

Consider two examples. Once children recognize that addition with zero does not change a number, this pattern can be stored as the algebraic rule: "the sum of *any* number and zero is that number" ($n + 0/0 + n = n$ rule). Knowing that a number in the counting sequences is 1 more than the previous number (e.g., "four" is one more than "three") can enable a child to use a known "doubles" combination to logically deduce the sum of a "near double" (e.g., If $3 + 3 = 6$ and 4 is 1 more than 3, then the sum of $3 + 4$ must be 1 more than 6—7). With practice such reasoning strategies become semiautomatic or automatic [Jerman, 1970] and a basis for Phase 3. The memory network of experts may not simply involve discrete individual facts but interconnected concepts, combinations, and automatic rules or reasoning processes. The process of achieving mastery with fluency, then, may involve both passive quantitative changes—the strengthening of associated factual knowledge or increasing efficiency of reasoning processes—and dynamic qualitative changes—a reorganization of memory structures or processes due to an insight [Baroody, 1985, 1994; Butterworth et al., 2003].

Paraphrasing Poincare [1905], mastery with fluency "is built up of facts as a house is of stones, but a collection of facts is no more [such fluency] than a pile of stones is a house" (p 141). As a framework and cement are necessary to transform a pile of stones into a house, patterns and relations can serve to structure and interconnect factual knowledge—to transform an otherwise amorphous bundle of facts into a well-organized body of knowledge.

HOW CAN INSTRUCTION EFFECTIVELY FACILITATE THE MEMORIZATION OF BASIC COMBINATIONS?

Whereas memorization of basic facts by rote can be achieved by drill, meaningful memorization also requires discerning mathematical structure.

Passive Storage View: Phase 3 Instruction Should Entail Well-Designed and Extensive Drill

According to proponents of the Passive Storage View, *the* key factor in memorizing basic facts (by rote)—forming and strengthening specific associations—is practice [Ashcraft, 1992; Crowley et al., 1997; Shrager and Siegler, 1998]. Moreover, Thorndike's (1922) *law of frequency* (the more two stimuli are presented together, the stronger the association) justifies large doses of practice. Siegler and Jenkins [1989] conjectured that seeing or hearing, for instance, "5 + 3" and its sum, "8" thousands of times is necessary to achieve the efficient retrieval of this fact.

In recent years, there has been some concern about a "brute force approach"—memorizing all the basic facts of an operation in relatively short order [e.g., Gersten and Chard, 1999]. Some have recommended focusing on a few facts at a time, ensuring one set of facts be mastered before introducing a new one [e.g., Cooke et al., 1993]. Moreover, proponents of a controlled- or constant-response-time (CRT) approach propose that, beginning in grade 2 (after an opportunity to use informal methods), children should be given only a few seconds to answer and be provided the correct answer if they either respond incorrectly or do not respond within the prescribed time frame [e.g., Goldman and Pellegrino, 1986; Koscinski and Gast, 1993]. These procedures are intended to minimize reinforcing "immature" (counting and reasoning) strategies and associative confusions. The latter entails incorrect responses to 4 + 5, such as stating "6" (the number after five in the counting sequence) or responding "8" (due to miscalculation or recalling the sum of 4 + 4, which shares an addend with 4 + 5). One study indicated that the *FASTT* (Fluency and Automaticity through Systematic Teaching with Technology) program, which builds directly on the CRT philosophy, appears to have been more effective in promoting the fact mastery among children with MD than control training provided peers with our without MD [Hasselbring et al., 1988].

Overall though, the empirical evidence does not clearly support the proposition that massive practice is the key to combination mastery. The National Mathematics Advisory Panel [2008] did conclude that practice increases efficient recall. However, all but one of the eight studies cited in support of this conclusion either involved adult participants or was based on textbook frequency studies. Consistent with Thorndike's (1922) law of frequency, textbook frequency studies typically indicate that large (relatively difficult) combinations are presented less often than small (relatively easy) combinations. However, frequency itself does not account for many cases of relative difficulty, nor does it account transfer of fluency from practiced items to related but nonpracticed combinations [Baroody, 1994]. Similarly, the effects of (computer-based) drill with children in general and those with MD are inconsistent [compare Kulik and Kulik, 1991, with Fuson and Brinko, 1985, or Vacc, 1992]. For example, Fuchs et al. [2006] found that computer-assisted drill was more effective than business as usual in helping at-risk first graders master addition combinations, but not subtraction combinations. In contrast, Lin et al. [1994] demonstrated the opposite. Although methodologically flawed, some research indicates that fact drill is less effective in fostering combination fluency than instruction that focuses on patterns, relations, and reasoning strategies [see reviews by Baroody, 1985, 1994]. In conclusion, practice undoubtedly plays an important role in fostering combination mastery, but it is not necessarily the key factor or even the most important factor.

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Active Construction View: Instruction for Phases 1, 2, and 3 Should Foster Discovery of the Numerous Patterns and Relations Interconnecting the Basic Combinations

Building the structured (meaningful and well-interconnected) knowledge or number sense that underlies mastery with fluency is a complex and long-term process and, at heart, requires finding patterns and relations. In his 1892 "Talk to Teachers," William James [1958] elegantly summarized the importance of constructing relational knowledge by building on previous knowledge:

"The art of remembering is the art of *thinking*; . . . when we wish to fix a new thing in . . . a pupil's mind, our . . . effort should not be so much to *impress* and *retain* it as to *connect* it with something already there. The connecting *is* the thinking; and, if we attend clearly to the connection, the connected thing will . . . likely [be remembered]" (pp 101–102).

Research indicates that discovering patterns or relations facilitates mastery with fluency. Consider one example of this point and James' point above. Children may first memorize a few $n + 1$ facts by rote. However, once they recognize that such facts are related to their existing counting knowledge—specifically their (already efficient) number-after knowledge—repeated practice of the remaining $n + 1$ combinations is unnecessary for mastery. Discovery of the *number-after rule* ("the sum of $n + 1$ is the number after n in the counting sequence") allows children to efficiently deduce the sum of any $n + 1$ combination for which they know the counting sequence, even those not previously practiced including large combinations such as 1,000,128 + 1 [Baroody, 1992, 1995]. With practice, the number-after rule for $n + 1$ combinations becomes automatic—can be applied quickly, efficiently, and without deliberation.

Focusing on structure, rather than memorizing individual facts by rote, makes the learning, retention, and transfer of any large body of factual knowledge more likely [Katona, 1967]. As with any worthwhile knowledge, meaningful memorization of basic combinations can reduce the amount of time and practice needed to achieve mastery, maintain efficiency (e.g., reduce forgetting and retrieval errors), and facilitate application of extant knowledge to unknown or unpracticed combinations [Baroody, 1985; Carpenter et al., 1989]. For example, recognizing that addition or multiplication is commutative can enable a child to treat an unknown combination as a known combination (e.g., treat 3 + 5 = ? as 5 + 3 = 8) and reduce by nearly half the amount of practice needed to master such combinations [Trivett, 1970; Baroody, 1999c]. This may help to account for why

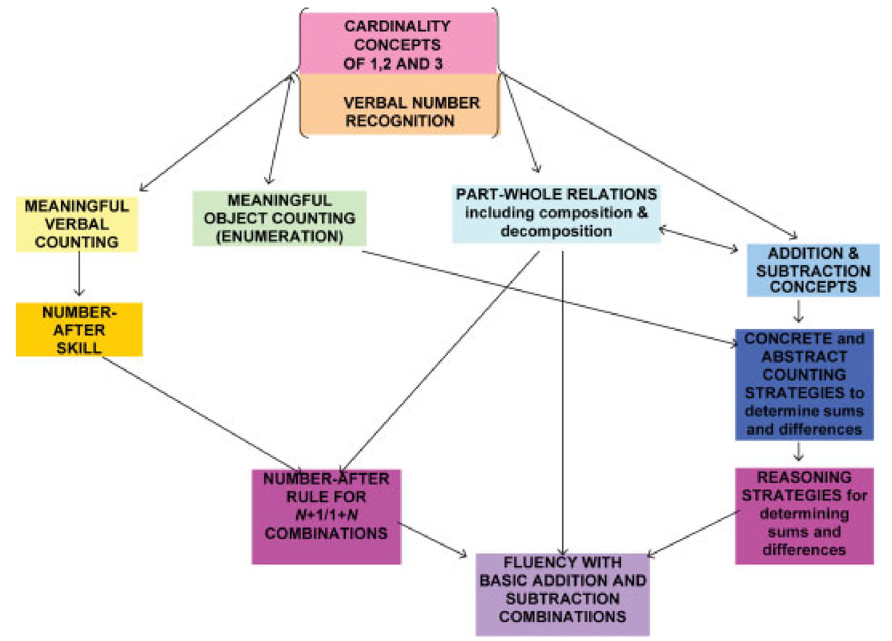

Fig. 1. Learning trajectory of some key number, counting, and arithmetic concepts and skills.

children more easily memorize addition and multiplication combinations than subtraction and division combinations [National Mathematics Advisory Panel, 2008]. Research also indicates that Asian students, who spend more time constructing patterns and relations through guided activities, achieve higher levels of combination mastery than U.S. students do [Stigler and Hiebert, 1999].

FACILITATING THE MEMORIZATION OF BASIC COMBINATIONS SHOULD BEGIN WHEN AND TAKE HOW LONG?

Passive Storage View: Mastery can Be Accomplished Quickly in the Primary Grades

According to the Passive Storage View, helping children understand the formal symbolism of addition and subtraction and then getting sufficient practice for memorizing the basic facts should not take long. As mentioned earlier, the California state standards specify that memorizing all 100 sums to 18 and the 100 related differences should be achieved by the end of first grade.

Active Construction View: As Mastery with Fluency Is an Outgrowth of Number Sense [Jordan, 2007], It Should Span All of Early Childhood (Preschool to Grade 3)

Recent research indicates that children begin to construct the foundation (number sense that serves as the developmental basis) for combination fluency as early as 18 months [Baroody et al., 2006a; see Figure 1 for an overview].

Language, in the form of the first few number words, appears critical for constructing the two bases of early number sense shown at the apex of Figure 1: (a) a concept of *cardinal number* (a number can represent the total number of items in a collection) and (b) the skill of *verbal number recognition*—reliably and efficiently recognizing the number of items in small collections and labeling them with the appropriate number word [Baroody et al., 2006a]. The use of "one," "two," and "three" in conjunction with seeing examples and nonexamples of each can help 2 and 3 year olds construct an increasingly reliable and accurate concept of the "intuitive numbers" *one*, *two*, and *three* (an understanding of oneness, twoness, and threeness). By seeing ••, △△, and ∘ ∘ (examples of pairs), for instance, all labeled "two," young children can recognize that the appearance of the items in the collections is not important (shape and color are irrelevant to number) and that a more abstract commonality connects the examples. Initially, children may conclude that the commonality is their *plurality* (each case involves more than one item). In time, their concept of "two" is refined to include only examples of pairs. Seeing •, •••, △, △△△, □□□, and □ □ (nonexamples of pairs) labeled as "not two" or with another number word can be particularly helpful in defining the boundaries of the concept of *two*.

As a basic understanding of cardinal number may not be innate or unfold automatically, young children may need opportunities to construct even the intuitive numbers, as well as the capacity to label them with a number word [cf. Dehaene, 1997; Baroody et al., 2006a]. If so, parents and preschool teachers, then, are important agents in providing the experiences and feedback needed to construct number concepts. They should take advantage of meaningful everyday situations to label (and encourage children) to label small collections (e.g., "How many feet do you have?" "So you need two shoes, not one." "You may take one cookie, not two cookies.") Some children, however, enter kindergarten without being able to recognize and verbally label all the intuitive numbers. Such children are seriously at risk for school failure and need intensive remedial work. Desoete and Gregoire [2006], for instance, found that 33% of 8.5 year olds with MD had a severe verbal number recognition deficit. Kindergarten screening should check for whether children can immediately recognize collections of one to three items and be able to distinguish them from somewhat larger collections of four or five.

As Figure 1 illustrates, the coevolution of cardinal concepts of the intuitive numbers and the skill of verbal number recognition can provide a basis for a wide variety of number, counting, and arithmetic concepts and skills. Specifically, they can provide a basis for *meaningful verbal counting.* Recognition of the intuitive numbers can help children literally see that a collection labeled "two" has more items than a collection labeled "one" and that a collection labeled "three" has more items than a collection labeled "two." This basic ordinal understanding of number, in turn, can help children recognize that the order of number words matters when we count (the *stable order principle*) and that the number word sequence ("one, two, three . . .") represents increasingly larger collections (a basis for using counting to compare collections).

As a child becomes familiar with the counting sequence, they develop the ability to start at any point in the counting sequence and (efficiently) state the next number word in the sequence (*number-after skill*) instead of counting from "one." As noted earlier, this ability is the foundation for discovering the *number-after rule for $n + 1/1 + n$ combinations*, which with practice, becomes automatic and the basis for *fluency with the $n + 1/1 + n$ combinations*.

Verbal number recognition, and the cardinal concept of number it embodies, can also be a basis for functional or *meaningful object counting*—using counting as a tool for determining the cardinal value of a collection, particularly those with more than three items [Benoit et al., 2004]. Children who can immediately recognize "one," "two," and "three" are more likely to benefit from adult efforts to model and teach object counting than those who cannot [Baroody et al., 2006a]. Modeling efforts typically involve saying one number word for each item in a collection and either emphasizing the last number word or repeating it (e.g., "Watch, one, two, t-h-r-e-e kittens!" or "Watch, one, two, three—see there are three kittens"). A child who can immediately recognize collections of up to three or four items is more likely to recognize the purpose of object counting as another way of determining a collection's total (e.g., "Counting is mommy's way of seeing three kittens") and the reason for why others emphasize or repeat the last number word used in the counting process (e.g., "Mommy's voice went up several octaves on 'three' because there are three kittens"). Meaningful object counting is necessary for the invention of *counting strategies* (with objects or number words) to determine sums and differences. The development of an efficient counting strategy can free up attention, which increases the likelihood of discovering patterns and relations. These mathematical regularities, in turn, can serve as the basis for deliberate *reasoning strategies* and automatic reasoning strategies they can serve as components in the *retrieval network*.

Verbal number recognition can enable a child to see one item and another as "two"; one item, another, and yet another as "three," or two items and another as "three" and the reverse (e.g., "three" as one and one and one or as two and one). This can facilitate constructing an understanding of *composition and decomposition* (a whole can be built up from, or broken down into, individual parts, often in different ways). Repeatedly seeing the composition and decomposition of *two* and *three* can lead to *combination fluency with the simplest addition and subtraction combinations* (e.g., "one and one is two," "two and one is three," and "two take away one is one"). Repeatedly decomposing *four* and *five* with feedback (e.g., labeling a collection of four as "two and two," and hearing another person confirm, "Yes, two and two makes four") can lead to *combination fluency with the simplest sums to five* and is another way of discovering the *number-after rule for $n + 1/1 + n$ combinations* [Baroody, unpublished data].

The concepts of composition and decomposition are central to inventing other reasoning strategies. Research with typically developing elementary-level children in Japan indicates that learning and applying a *make-ten strategy* (e.g., $9 + 4 = 9 + [1 + 3] = [9 + 1] + 3 = 10 + 3$) is difficult because the results of decomposing a number into 1 and another part is not obvious to these children [Murata, 2004]. For $9 + 4$ for example, although Japanese children have no problem recognizing that 1 must be added to 9 to make 10, they do not easily recognize the result of 4 take way 1 (that 4 decomposes into 1 and 3). This may be due to a lack of emphasis on composing and decomposing numbers, a deficiency common to early childhood mathematics instruction in the U.S.

The concept of cardinality, verbal number recognition, and the concepts of composition and decomposition can together provide the basis for constructing a basic *concept of addition and subtraction*. For example, by adding an item to a collection of two items, a child can literally see that the original collection has been transformed into a larger collection of *three*. These competencies can also provide a basis for constructing a relatively concrete, and even a relatively abstract, understanding of arithmetic concepts, such as the following [Baroody et al., in press,b]:

- *Concept of subtractive negation:* For example, when children recognize that if you have two blocks and take away two blocks this leaves none, they may induce the pattern that *any number take away itself leaves nothing*.
- *Concept of additive and subtractive identity:* For example, when children recognize that two blocks take away none leaves two blocks, they may induce the regularity that *any number take way none leaves the number unchanged*.

The concepts of subtractive negation and subtractive identity can provide a basis for combination fluency with the $n-n = 0$ and $n-0 = n$ families of subtraction combinations, respectively.

WHY DO CHILDREN HAVE DIFFICULTY LEARNING THE BASIC COMBINATIONS?

Unlike their peers, children with poor mathematics achievement—particularly those with mathematical learning disabilities (MLD)—often do not proceed beyond Phase 1. Put differently, they remain overly reliant on relatively slow counting strategies [Mazzocco et al., 2008]. Although children with MD can be taught reasoning strategies [e.g., Thornton et al., 1983; Torbeyns et al., 2004; Tournaki, 2003], they do not spontaneously invent the strategies—that is, the children fail to independently achieve Phase 2 [Swanson and Cooney, 1985; Swanson and Rhine, 1985]. They typically are unable to retain basic combinations, particularly larger combinations with sums over 10 [Smith, 1921; Kraner, 1980]. However, children with MLD have trouble with even easy combinations, even at 8th grade [Mazzocco et al., 2008]. When recalling facts, children with MD are prone to "associative confusions" and thus exhibit an unusually high error rate. Deficits in the recall of basic facts persist. Indeed, Geary [1990] found that, even after a year of remedial services, children with MD continued to make significantly more retrieval errors than normally achieving peers. Chong and Siegel [2008] found that recall deficits in Grade 2 persisted until at least Grade 5. In brief, children with MD typically fail to attain Phase 3.

Serious delays or difficulties in learning may be due to either cognitive impairment or inadequate educational opportunities [e.g., Baroody, 1996; Robinson et al., 2002; Geary et al., 2004]. Proponents of the *passive* and *active storage views* emphasize one or the other of these causes to explain why children with MD tend to get stuck in Phase 1.

Passive Storage View: Difficulties Mastering Basic Combinations by Children with MD are Typically Caused by Defects Inherent in the Learner

When children do not memorize the basic facts by rote by the simple process of associative learning, a common explanation is that they have not been provided adequate practice. However, many children with MD fail to do so even when given massive practice. In such cases the blame is affixed to the child. For example, Hopkins and Lawson, 2006] attributed the ineffectiveness of practice among children with MD to slow processing (counting) speed. In effect, excessively deliberate processing interferes with strengthening the association between the computed answer and the (forgotten) expression. Chong and Siegel [2008] found that children with a recall deficit exhibited persistent cognitive deficits in working memory and phonological skills, as well as processing speed.

Indeed, the common characteristics of children with MD—an inability to apply (transfer) knowledge to even moderately new problems or tasks; forgetfulness; and confusion—are often attributed to cognitive dysfunction or delay, such as attention or memory deficits or slow neurological development [e.g., Webster, 1980; Geary, 1996; Jordan et al., 2003; Sliva, 2004; Swanson and Jerman, 2006]. For example, Geary et al. [2000] concluded that such children are prone to associative confusion because of a "retrieval-inhibition deficit" (e.g., responding to 4 + 6 with the counting-string associate "7" stems from an irrelevant associations with the counting sequence "6, 7"). For all the reasons discussed in this and the previous paragraph, "over learning" in the form of massive practice is often recommended for children with MD [e.g., Goldman and Pellegrino, 1986; Sliva, 2004].

Active Construction View: Difficulties Mastering Basic Combinations by Children with MD are Typically due to Inadequate Opportunities to Develop Number Sense

Although some children with MD certainly have cognitive impairments potentially related to a MLD, many, perhaps most, have difficulty with mathematics because they do not have the chance to construct a rich number sense. Put differently, the primary cause of poor mathematics achievement is inadequate preschool (informal) experiences, ineffective school (formal) instruction, or both [Baroody, 1996]. The symptoms commonly associated with MD (discussed in the previous paragraph), then, are often or even usually the by-product of weak number sense and routine expertise.

A weak number sense and consequent poor mathematics achievement all too often begin with inadequate informal knowledge. Preschoolers' informal knowledge can differ significantly [e.g., Dowker, 2005; Baroody et al., 2006a; National Mathematics Advisory Panel, 2008]. Gaps in everyday knowledge can interfere with understanding formal instruction or engaging in mathematical thinking (e.g., devising problem-solving strategies, reasoning logically about problems) and, thus, greatly delay or hamper the learning of school mathematics [Baroody, 1987, 1996, 1999b; Jordan et al., 2003; Canobi, 2004].

Indeed, early number sense predicts mathematics achievement in school. Duncan et al. [2008] found that preschool and kindergarten mathematical knowledge strongly predicts mathematical performance as late as eighth grade. Early number sense predicts mathematics achievement in school over and above measures of cognitive competence, such as verbal, spatial, and memory skills [Mazzocco and Thompson, 2005; Jordan and Locuniak, 2008; but cf. Passolunghi et al., 2008] or background predicators, such as family income or reading ability [Jordan et al., 2007]. For example, Aunola et al. [2004] found that initial performance in school mathematics was higher and progress in acquiring formal knowledge was faster among those who entered kindergarten with a high level of informal mathematical knowledge and was best predicted by counting ability. Moreover, Mazzocco and Thompson [2005] found that, although no single criterion or combination of criteria in kindergarten consistently predicted which children would be categorized as having persistent poor mathematics achievement in Grades 2 or 3, a score of less than the tenth percentile on the *TEMA-2* [Ginsburg and Baroody, 1990] was the most consistent criteria. They attributed this result to the fact that the *TEMA-2* assessed a wide range of informal, as well as formal, concepts and skills. Mazzocco and Thompson [2005] further found that a composite of three informal items (cardinality, comparisons of one-digit numbers, and mentally adding one-digit numbers) and one formal item (reading numerals) from the *TEMA-2* were predictive of which children would be designated mathematically "learning disabled" (LD) during both Grades 2 and 3. The lack of opportunity to develop informal knowledge is especially likely among at-risk children, particularly those from low-income families or with poorly educated parents [Jordan and Levine, this issue].

> ***Difficulties mastering basic combinations by children with MD are typically due to inadequate opportunities to develop number sense. . .*** **inadequate preschool (informal) experiences, ineffective school (formal) instruction, or both.**

More specifically, as illustrated in Figure 1, weak informal knowledge or number sense can prevent or seriously interfere with memorizing the basic combinations. At-risk children are particularly likely to lack the informal knowledge that serves as the developmental prerequisites for mental addition and subtraction and for readily memorizing even the simplest addition combinations. Jordan et al. [2006] found that "basic weaknesses in number sense, such as the inability to grasp counting principles or manipulate quantities mentally," result in difficulties learning the basic number combinations (p 175). They found, for example, that kindergartners who struggled with solving word problems and verbally stated addition problems (e.g., "four and three is how much altogether") had significantly more difficulty learning basic combinations in first grade. Children who cannot meaningfully count collections, mentally represent and operate on collections, or understand and represent word problems or verbally stated addition expressions do not have a basis for even Phase 1 (counting-based addition) strategies.

Even if at-risk children achieve Phase 1, they often lack the informal knowledge or number sense to advance within or beyond this phase. Baroody et al. [2006b] distinguished between two major categories of counting-based addition strategies. Concrete strategies entail representing one or both addends with objects before beginning the sum count. For 3 + 5 for example, the common concrete counting-all strategy entails counting out three blocks (or putting up three fingers), counting out five blocks (or putting up five fingers), and then counting all the items. In contrast, abstract strategies entail representing the addends during the sum count. For 3 + 5 for instance, the relatively advanced counting-on strategy might involve stating the cardinal value of the larger addend ("Five") and then counting three times, perhaps using fingers to keep track of three: "Six (one finger raised), seven (two fingers raised), eight (three fingers raised). In a real sense, abstract strategies represent the first stage in performing mental addition (arithmetic) at relatively abstract level (with numbers instead of object or even mental images of objects).

The transition from using concrete counting-based addition strategies to abstract strategies is a major challenge, particularly for those at risk or already experiencing MD [Baroody et al., in press,a]. Although concrete strategies are relatively easy to learn (either through self-invention or via direct instruction such as modeling), even among children with MD, more efficient abstract strategies are not, particularly among children with learning difficulties [e.g., Baroody, 1996, 1999b]. The key reason for this disjuncture is that concrete strategies require relatively basic informal knowledge to construct or learn, whereas abstract strategies necessitate relatively advanced conceptual knowledge of counting and numbers [Baroody et al., 2006b]. Not surprisingly, children from low-income families, who are more likely to have weak informal knowledge, tend to rely on less advanced or efficient counting strategies for determining sums [Baroody et al., 1982; Jordan et al., 2008].

A problem with not advancing beyond concrete strategies (not devising more efficient abstract strategies) is that it makes it less likely that children will have the cognitive resources for inventing reasoning strategies. The use of relatively inefficient concrete strategies reduces the chances that working memory will be freed to discover patterns and relations—the bases for more relatively efficient reasoning strategies. Moreover, because concrete strategies entail determining the sum *after* independently representing the addends, a child may not connect the sum with expression. In contrast, abstract strategies require a child to determine the sum while simultaneously representing the addends—a process that may enhance a child associating a sum and an expression.

Indeed, weaknesses in informal knowledge or reliance on concrete strategies can prevent at risk children from discovering even those patterns and relations most salient or obvious to typically developing children [Baroody, 1992, 1996] and, thus, interfere with learning even the most basic combinations (those involving zero or one). In a yearlong training study [Baroody et al., in press], only 11% and 1% of 4 and 5 year olds who received either structured or unstructured discovery achieved fluency (including transfer) with $n + 0/0 + n$ combinations and the $n + 1/1 + n$ combinations, respectively. Key reasons for disappointing results were that most participants (a) were still struggling to master more concrete addition concepts/skills and (b) had not mastered the number-after-n relations (a key developmental prerequisite for discovering the number-after rule for $n + 1$ combinations). A subsequent training experiment with developmentally more advanced children (kindergartners) indicated that structured discovery learning of the $n + 0/0 + n = n$ pattern, the number-after relation for $n + 1/1 + n$ items helped kindergartners achieve significant improvement in combination fluency with these families [Baroody and Eiland, unpublished data].

Desoete and Gregoire [2006] found that numerical competence in kindergarten was associated with arithmetic difficulties in Grade 1 and that kindergartners who struggled with basic numerical skills were significantly more likely to have a difficulty in learning and remembering basic combinations in Grade 1. In particular, they found that prefirst graders who had limited knowledge of the counting word sequence and who could not count from a given number to another number (e.g., start with six and count up to nine) exhibited delays in Grade 1. Note that the number-after skill is a necessary first step for the count-from-a-given-number-to-another-number task and for the number-after rule for $n + 1 / 1 + n$ combinations. These facts may help to explain why was the former predictive of difficulties learning the basic combinations.

A second key cause of weak number sense and consequent MD, particularly among at-risk children, is ineffective formal instruction [e.g., Lubienski and Shelley, unpublished data]. Teachers in schools with large populations of at-risk children are among the most poorly trained, particularly in regards to mathematics and methods for fostering fluency [e.g., Lubienski, Unpublished data]. Such teachers are especially prone to teach as they were taught—in a traditional manner [Struchens and Silver, 2000]. With its focus on memorizing facts by rote, traditional instruction discourages looking for patterns and relations—key bases for a rich number sense and combination fluency. Because memorizing numerous isolated facts by rote is extremely difficult, many children give up on mastering combinations or forget or confuse much of what they do learn. Instruction that promotes routine expertise, then, makes mastering basic facts unduly difficult and often creates the symptoms of learning difficulties frequently attributed to children with MD. Because of mile-wide curricula and incomplete professional training, gaps in informal knowledge and deficiencies in basic formal knowledge often go undiagnosed and unremedied [Ginsburg and Baroody, 1990]. As a result, many at-risk children fall further behind, and their learning deficiencies persist or get worse throughout the school years.

WHAT ARE THE THEORETICAL IMPLICATIONS AND THE IMPLICATIONS FOR FUTURE RESEARCH?

As the vast majority of research has not used path analysis [Passolunghi et al., 2008], training experiments, or other techniques for gauging causal links, it is premature to draw conclusions about the primary cause of MLDs in general and memorizing the basic combinations in particular. Even brain imagining techniques merely indicate what is, not what could be. That is, they indicate current structure and functioning, which may be the result of learning difficulties, not their cause [Baroody, 2003; Geary, 2003].

In recent years, efforts have focused on distinguishing between classes of MLD, specifically those with deficient mathematics achievement and those with simply low (below average) but not deficient achievement [e.g., Mazzocco et al., 2008]. This is a useful distinction, because these two groups may differ in significant ways. Even so, future research should focus on differentiating children whose alleged "MLD" is due to weak or inadequate number sense (the lack of opportunity) from those with a MLD associated with one or more cognitive deficits. Although it is reasonable to assume that children with deficient achievement are more likely than children with low achievement to have cognitive deficits, some children in either group may simply have lack the opportunity to develop number sense. One approach to making this distinction is the syndrome research exemplified throughout this volume. Another will require research that includes examining children's learning potential with effective intervention (e.g., developmentally appropriate and engaging instruction). A third would involve a combination of the two.

WHAT ARE THE INSTRUCTIONAL IMPLICATIONS FOR HELPING CHILDREN AT-RISK FOR OR WITH MD?

Why Is Early Screening and Intervention Important?

Research indicates that children with early MD are not doomed to a spiral of failure and frustration—that some do catch up with their peers [Geary et al., 2000; Gersten et al., 2005]. Early screening and remedial efforts are essential for ensuring all children, including those at-risk schoolchildren, have the opportunity to develop number sense—a basis for mastery with fluency. For example, in regard to screening, Jordan and Locuniak [2008] found that "number sense screening in kindergarten, using "'at risk' versus 'not at risk' criteria, successfully ruled out 84% of the children who did not go on to have calculation fluency difficulties and positively identified 52% of the children who later showed fluency difficulties" (p 451). What is unclear is whether all or only a portion of those who did not catch up had a MLD.

Early intervention that promotes fluency with basic addition and subtraction combinations is of particular interest because of the current focus in special education on preventing, rather than remedying, learning difficulties [e.g., Sliva, 2004]; the growing concern about equity [e.g., National Council of Teachers of Mathematics, 2000; Kilpatrick et al., 2001]; and worries about whether reform-based instruction is appropriate for children with special learning needs [e.g., Vaughn, et al., 2000]. A disproportionate number of at-risk children are from (urban or rural) low-income families or a minority group (see Jordan and Levine, this volume). A lack of mathematical proficiency means most are denied an equal opportunity to obtain professional and technology-related jobs. If a way could be found to help at-risk children achieve fluency with basic addition and subtraction combinations, many such children might be able to achieve normal or near-normal success in school mathematics and avoid debilitating life long academic failure/poor achievement [Baroody, 1996; Dev et al., 2002]. This might significantly reduce the number of children inappropriately labeled MLD or even LD. Moreover, effective early intervention with the basic combinations may help children with an organically based (genuine cognitive) disability maximize their potential. It may also help at-risk, low achieving, and LD children, who otherwise do not thrive with reform or standards-based curricula [Fuson et al., 2000], develop a basis for the adaptive expertise required by these more demanding curricula [cf. Vaughn et al., 2000]. Such expertise can promote other aspects of computational fluency with which at-risk children typically have difficulty (e.g., solving problems, estimation, or multi-digit mental arithmetic).

Why Is Structured Discovery Learning of Patterns and Relations Essential for Fostering Number Sense and Mastery with Fluency?

It follows from the Active Construction View that the basis for helping students build both number sense in general and combination fluency in particular is creating opportunities for them to discover patterns and relations, particularly relations to existing knowledge. For example, learning subtraction combinations can be facilitated if children see the connection between subtraction and addition [e.g., if $5 + 3 = 8$, then $8 - 5$ is 3; Putnam et al., 1990; Baroody, 1999a], and $2\text{x}n/n\text{x}2$ combinations can be meaningfully memorized by recognizing the relation between multiplication and (repeated) addition of a like term [e.g., $2 \times 7 = 7 + 7 = 14$; Baroody, 1993]. Ideally, learning opportunities should be purposeful (personally relevant and engaging), meaningful (build on what children know), and inquiry-based (directed at promoting autonomous mathematical problem solving and reasoning). Such instruction may be particularly important for and can be effective with children with MD [Baroody, 1987, 1999b; Bottge et al., 2007].

Is Structured Discovery Learning of Patterns and Relations Appropriate for Children Classified as MLD, LD, or Globally Developmental Delayed (GDD)?

Put differently, should children with developmental disabilities be taught in a different manner than typically developing peers—one tailored to their particular difficulty? Clearly, children whose mathematical difficulties are compounded by other problems, such as language or reading difficulties, need help in both areas and may require accommodations until or if these other problems are resolved. That said, until proven otherwise, it seems prudent and fair to assume that children with developmental disabilities learn according to the same learning principles as their peers and should be taught in a manner similar to them. Indeed, there is empirical evidence even children with serious developmental disabilities can induce or discover important arithmetic regularities such as additive commutativity, the $n + 0/0 + n = n$ rule, the number-after rule for adding one, and subtraction as addition—and apply such knowledge to novel or unpracticed items [see Baroody, 1999b, for a review of the GDD literature].

Might Direct Instruction Also Be Effective in Promoting Number Sense and Mastery with Fluency?

The short answer is yes possibly but typically no. Put differently, it is difficult to impose number sense on children. Baroody et al. [2006b] noted that direct instruction can be effective when a child is developmentally ready and receptive and when the instruction clearly and directly builds on what the child knows. For example, modeling a concrete counting all strategy can be highly effective with kindergartners and children with serious MLD, because the strategy corresponds clearly with their informal concept of addition. However, modeling an abstract counting-on strategy typically fails because comprehending and learning such an advanced strategy seems to require relatively advanced or sophisticated concepts of number and addition, concepts that children may need to discover for themselves [cf. Crowley et al., 1997].

Similarly, efforts to use direct instruction to foster learning of reasoning strategies and mastery of combinations are often unsuccessful. Efforts to model the doubles plus or minus 1 reasoning strategy can backfire, because children do not understand the rationale of the imposed procedure. Specifically, students sometimes use the imposed strategy accurately (e.g., $7 + 8$ is $7 + 7 + 1$ or $8 + 8 - 1$ is 15) but other times inaccurately [e.g., for $7 + 8$: reason incorrectly that $7 + 7 - 1$ is 13 or that $8 + 8 + 1$ is 17; Torbeyns et al., 2005]. In the same vein, in three brief training sessions over several weeks, Booth and Siegler [2008] presented first graders with information about the magnitudes of the addends in single- and multi-digit addition combinations and their sums in form of a length on a number line. Participants in the direct-instruction condition were significantly better in recalling the four trained facts and estimating the sums than those other training or control conditions on the immediate posttest. However, the effect sizes were small. More importantly, no significant improvement was evident for any of the training conditions 2 weeks after the training. In brief, although the training was an effort to improve mental-arithmetic performance by building on children's number sense (mental number line), the short-term efforts to impose knowledge did not produce lasting results.

Is Practice Unimportant?

As general rules of thumb, practice is useful if its primary purpose is to provide students an opportunity to discover patterns or relations or to ensure reasoning strategies become automatic (once a child has discovered patterns and relations).

A Case in Point

Many of the points above are illustrated by the case of Westin, an at-risk first grader [Baroody, unpublished data]. In the context of a computer-based math game, the boy was presented 6 + 6. He determined the sum by counting. Shortly afterward, he was presented 7 + 7. He smiled and answered quickly, "Thirteen." When the computer feedback indicated the sum was 14, he seemed puzzled. A couple of items later, he was presented 8 + 8 and noted, "I was going to say 15, because 7 + 7 was 14. But before 6 + 6 was 12, I thought for sure that 7 + 7 would be 13 but it was 14. So I'm going to say 8 + 8 is 16." As the case of Westin illustrates, giving all children the opportunity to explore numbers and their relations can be beneficial to their mathematical thinking and learning. With the aid of guided discovery learning, Westin realized for himself a valuable lesson—that the sums of consecutive doubles increase by two (and are all even numbers). An effective teacher might follow-up with a probing or thought-provoking question (e.g., "Why do you think that is?") and additional purposeful practice (e.g., in form of a math game or solving engaging problems). ■

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