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Early childhood teacher education: the case of geometry

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Abstract For early childhood, the domain of geometry and spatial reasoning is an important area of mathematics learning. Unfortunately, geometry and spatial thinking are often ignored or minimized in early education. We build a case for the importance of geometry and spatial thinking, review research on professional development for these teachers, and describe a series of research and development projects based on this body of knowledge. We conclude that research-based models hold the potential to make a significant difference in the learning of young children by catalyzing substantive change in the knowledge and beliefs of their teachers.

Keywords Scaling up professional development · Geometry · Spatial reasoning · Learning trajectories · Early childhood

Introduction

Early childhood teacher education: the case of geometry

For early childhood, the domain of geometry and spatial reasoning is an important area of mathematics learning (NCTM 1991, 2006). Viewed broadly, geometric and spatial thinking are not only important in and of themselves, but they also support number and arithmetic concepts and skills (Arcavi 2003). Research even suggests that the ability to represent *magnitude* is dependent on visual-spatial systems in regions of the parietal cortex of the brain (Geary 2007; Pinel et al. 2004; Zorzi et al. 2002). Unfortunately, geometry and spatial thinking are often ignored or minimized in both early education (Sarama and Clements 2009) and in the professional development of early childhood teachers (H. P. Ginsburg et al. 2006). In this article, we build a case for the importance of geometry

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and spatial thinking, review research on professional development for these teachers, and describe a series of research and development projects based on this body of knowledge.

The importance of geometry and spatial thinking for young students

Some mathematicians have claimed that, except for simple calculation, geometric concepts underlie all of the mathematical thought (e.g., Bronowski 1947). Smith (1964) argued that mathematics is a special kind of language through which we communicate ideas that are essentially spatial. From number lines to arrays, even quantitative, numerical, and arithmetical ideas rest on a geometric base. Cross-cultural research substantiates that core geometrical knowledge, like implicit basic number or quantitative knowledge, appears to be a universal capability of the human mind (Dehaene et al. 2006).

Geometry can serve as a core-relating science and mathematics. Two of the most prominent physicists of the last 100 years attributed their advancements to geometry. As a boy, Einstein was fascinated with a compass, leading him to think about geometry and mathematics. He taught himself extensively about geometry by age 12. Later in life, Einstein said that his elements of thought were always initially of a geometric and spatial nature, including "certain…more or less clear images which can be voluntarily reproduced or combined. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the associative play is sufficiently established." Hawking put it this way: "Equations are just the boring part of mathematics. I attempt to see things in terms of geometry" (Larsen 2005, p. 43). These two are not alone, visual thinking played a dominant role in the thinking of Michael Faraday, Sir Fancis Galton, Nikoa Tesla, James D. Watson, René Thom, and Buckminster Fuller, among others (Shepard 1978). This perspective is summarized by a modern geometer as follows:

> Geometry should be a focus at every age, in every grade, every year. Mathematics curricula are often criticized for their insularity—'what does this have to do with the real world?' No mathematical subject is more relevant than geometry. It lies at the heart of physics, chemistry, biology, geology and geography, art and architecture. It also lies at the heart of mathematics, though through much of the twentieth century the centrality of geometry was obscured by fashionable abstraction. This is changing now, thanks to computation and computer graphics which make it possible to reclaim this core without loss of rigor. The elementary school curriculum should give the children the tools they will need tomorrow. (Marjorie Senechal, personal communication, November 18, 2005)

Spatial thinking is an essential human ability that contributes to mathematical ability. It is a process that is distinct from verbal reasoning (Shepard and Cooper 1982) and functions in distinct areas of the brain (Newcombe and Huttenlocher 2000). Further, mathematics achievement is related to spatial abilities (e.g., Ansari et al. 2003). As an example, empirical evidence indicates that spatial imagery reflects not just general intelligence but also a specific ability that is highly related to ability to solve mathematical problems, especially nonroutine problems (e.g., Wheatley et al. 1994). This is particularly important because some individuals are harmed in their progression in mathematics due to lack of attention to spatial skills, benefit from more geometry and spatial skills education (e.g., Casey and Erkut 2005).

Education in geometry may contribute to a growth in mathematical competence and in other cognitive abilities, including IQ (Clements and Battista 1992; Clements and Sarama 2007b). Geometric knowledge, in particular, is highly related to mathematical reasoning and a host of other mathematics concepts and skills (Tatsuoka et al. 2004), including proportional reasoning, judgmental application of knowledge, concepts and properties, and managing data and processing skills, leading the authors to conjecture that geometry may be a gateway skill to the teaching of higher-order mathematics thinking skills.

Considering these bodies of research and professional judgments, students may benefit from attention to geometric and spatial thinking from the earliest years. Less salient, but perhaps just as important, are the competencies of students' learning of related topics in other subject-matter domains such as computer graphics, navigation, geography, visual arts, and architecture. The U.S. Employment Service estimates that most technical-scientific occupations such as drafter, airplane designer, architect, chemist, engineer, physicist, and mathematician require persons having spatial ability at or above the 90th percentile.

International studies indicate a weakness in students' geometric achievements (Mullis et al. 1997). In the Third International Mathematics and Science Study (TIMSS) research, international performance on geometry and measurement is low, especially for some countries (Beaton et al. 1996; Ginsburg et al. 2005; Lappan 1999). These deficits have been identified in the earliest years of life. Improved teacher education is needed to address limited student learning of geometry (with some empirical support for this causal connection, e.g., van der Sandt 2007).

Professional development

Geometry and present-day preparation of teachers

Although there are exceptions, teachers in many countries, including the U.K. (Jones 2000) and South Africa (van der Sandt 2007), and [throughout the pre-K to grade 12], are not always provided with adequate preparation in geometry and the teaching and learning of geometry. Of all mathematics topics, geometry was the one prospective teachers claimed to have learned the least and believed they were least prepared to teach (Jones et al. 2002).

We shall provide several illustrations of this lack of knowledge, with a caveat: this research base is small and limited in focus. For this reason, the illustrations do not reflect the broad range of concepts, skills, and perspectives that define what we believe to be important in the domains of geometry and spatial thinking. However, they do suggest teachers' unfamiliarity with those domains. For example, many prospective teachers only reach level 1 of the van Hiele (1989) model of geometric thinking—recognizing and categorizing shapes only on the basis of their overall physical similarity to prototypes ("it must be a rectangle, because it looks like a door"). Many are not guided to reach level 2, the descriptive/analytic level, at which people recognize and characterize shapes by their properties (Clements 2003; Swafford et al. 1997). In one study, for example, prospective teachers in the U.K. were at best thinking at level 2 (Fujita and Jones 2006a).

This lack of attention to geometry reveals itself in prospective teachers' responses to tasks in this study (Fujita and Jones 2006a). For example, approximately 13% of prospective teachers in Scotland identified a square as a rectangle and approximately 18% realized that a parallelogram is a trapezium (the authors note that such results contrasts with Kawasaki's (1992) findings that 73% of Japanese prospective teachers define a trapezium correctly, as cited in Fujita and Jones 2006b). Although almost all prospective teachers could draw a square, almost 2/3 could not define it correctly, leaving out any mention of angles (or other constraining properties). Probably, the geometry-deprived

(Fuys et al. 1988) elementary and middle school education that these prospective teachers experienced "fixed" their image and definition of a square, and they see no need to mention angles, even when it is necessary. For all ages, people who know a correct verbal description of a concept but possess a limited visual image (or concept image) associated strongly with the concept may have difficulty learning and applying the descriptions. They are influenced most by the visual image (see the theory of hierarchic interactionalism in Sarama and Clements 2009, and the construct of concept images—a combination of all the mental pictures and properties that have been associated with the concept in a person's memory). In another study, 70% of prospective elementary teachers were below van Hiele level 3, at which people understand relationships between *classes* of figures. Several were at the pre-recognition level (level 0) (Sarama and Clements 2009). Almost two-thirds were at the visual level (1).

Although data on teachers of the youngest children is rare, our research with hundreds of pre-school teachers (discussed in a succeeding section) suggests that most early childhood teachers also have not attained adequate levels of geometric knowledge. This is consistent with the finding that teachers of young children are provided with very limited professional development in mathematics (H. P. Ginsburg et al. 2006).

Such limited education, from their own early years on, leaves teachers under-prepared for teaching geometry. Although level 3 relational thinking in geometry and spatial sense (relations between classes of shapes) is not necessarily a goal of early childhood education, there are several reasons that teachers' low level of geometric thinking is a concern. First, teachers at the pre-recognition (0) and even visual (1) levels cannot adequately assess and teach children at any level, as such thinking is not consistent with mathematical properties (effective teachers must know considerably more than simply the content of the age/grade they teach, National Mathematics Advisory Panel 2008). Second, even if goals for children do not immediately address level 3 thinking, we wish to *do no harm.* Teachers who ask children to go on a "shape hunt" for rectangles, and reject a child's choice because she found a square, make it necessary to carry out later on the difficult task of "unteaching." Third, as stated previously, such limits in levels of thinking reliably reflect a general lack of awareness of the broad domains of geometry and spatial thinking. Thus, there is a need for substantial professional development for teachers of young children.

A lack of knowledge of geometry and geometry education affects new generations. For example, an early study found that kindergarten children had a great deal of knowledge about shapes and matching shapes before instruction began. In one episode, the teacher tended to elicit and verify children's prior knowledge but did not seem to add or develop new knowledge. That is, about two-thirds of the interactions between the teacher and children had children repeat what they already knew in a repetitious format as in the following exchange: Teacher: Could you tell us what type of shape that is? Children: A square. Teacher: Okay. It's a square (Thomas 1982). Along with the researcher's pre-test, this illustrates the common finding that children may already possess the information the teacher is trying to present. When teachers did elaborate, their statements were often filled with mathematical inaccuracies. For instance, teachers claimed that all "diamonds" (rhombi) are squares, that two triangles put together always make a square, and that a square cut in half always yields two triangles.

A mathematics survey was sent to 3,000 teachers and administrators from day care, family care, traditional nursery schools, Head Start, and public and parochial pre-schools in two states in the United States, of which over 400 complete surveys were returned. Asked about their main mathematics activities, 67% of early childhood care providers chose counting; 60% chose sorting; 51%, numeral recognition; 46%, patterning; 34%, number concepts; 32%, spatial relations; 16%, making shapes; and 14%, measuring (Sarama 2002; Sarama and DiBiase 2004). Geometry and measurement concepts were the least popular. This is another reason professional development is so important for teachers for young children.

Strategies for effective professional development

The single most dominant factor affecting students' academic progress is the effectiveness of their teachers (Wright et al. 1997). The presence of *cumulative* effects of teachers on student achievement—with little evidence of compensatory effects of more effective teachers in later grades (Sanders and Rivers 1996)—indicates that teachers of the early years must be effective to serve children well, especially those at the lower end of entering knowledge. Unfortunately, as we have seen, there is little in the literature that would indicate that there are many effective early childhood teachers of geometry. Therefore, professional development for these teachers is critical.

Good professional development in geometry may not be easy to conduct. Some professional development programs do nothing to increase geometric knowledge (Fujita and Jones 2006b; Jones 2000), and teachers' knowledge can even degrade during such programs (van der Sandt and Nieuwoudt 2004). It appears that hearing, reading, and memorizing formal definitions makes little or no impact on these prospective teachers. Even in some intervention projects with an extensive professional development component, teachers improve only in their teaching of number tasks to young children—the children's geometry and geometry reasoning were not affected (Campbell and Rowan 1995). Geometry knowledge may have been inadequately addressed. Still there are indications that some professional development can make a difference to both teachers and their students (Jacobson and Lehrer 2000; Swafford et al. 1997).

The difference may lie in the nature of the professional development. General requirements and measures such as certification alone are not reliable predictors of high-quality teaching (Early et al. 2007; National Mathematics Advisory Panel 2008). More direct measures of what the teachers know about mathematics and the learning and teaching of mathematics do predict the quality of their teaching (National Mathematics Advisory Panel 2008). For example, the mathematics achievement gains of 1st and 3rd graders were significantly related to their teachers' mathematical knowledge for teaching (Hill et al. 2005).

Professional development ought to focus on geometry and spatial reasoning and be sufficiently intense and extensive. As an example, one 4-week (3 h/day, 4 days per week) intervention program designed to enhance teachers' knowledge of geometry and their knowledge of research on student cognition in geometry resulted in significant positive changes in content knowledge and van Hiele level for middle school teachers, as well as in what they taught and how they taught (Swafford et al. 1997). Initially, participants functioned at low van Hiele levels, with 79% being at the first three levels, 75% were at the top two levels after professional development (formal deduction and rigor). Progression to a higher van Hiele level may be rapid if facilitated by high-quality instruction for adult learners.

Developing and scaling up effective professional development for early childhood geometry

Our research for the last decade has addressed how to design and implement high-quality professional development that improves the competencies of the participating teachers and, through them, of their students. We focused equally on the domains of number and geometry and spatial reasoning. In this paper, we mostly describe elements concerned with geometry and spatial reasoning.

The Building Blocks: learning trajectories

Our first development task was to design, in cooperation with pre-school teachers, an early childhood mathematics curriculum, *Building Blocks* (Clements and Sarama 2007a), around a core of *learning trajectories*. Learning trajectories have three parts: (1) a goal (that is, an aspect of a mathematical domain children should learn), (2) a developmental progression, or learning path through which children move through levels of thinking, and (3) instruction that helps them move along that path (for a complete description, see Sarama and Clements 2009). Formally, hypothetical learning trajectories are descriptions of children's thinking as they learn to achieve specific goals in a mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking (Clements and Sarama 2004b). These learning trajectories are hypothetical because they must be realized in classrooms through the interaction of teachers, students, and materials. That is, teachers assess children's level of thinking and guide children, as they think appropriate, through the learning trajectory, Thus, each manifestation will differ.

As an example of a hypothetical learning trajectory, we constructed a three-step developmental progression for shape composition (Clements et al. 2004; Sarama et al. 1996), in which children advance through levels of trial and error, partial use of geometric attributes, and mental strategies to synthesize shapes into composite shapes. We then designed a sequence of instructional tasks requiring children to solve shape puzzles off and on the computer, the structures of which correspond to the levels of this developmental progression (Clements and Sarama 2007c).

Learning trajectories are useful pedagogical, as well as theoretical, constructs (Clements and Sarama 2004a; Simon 1995; Smith et al. 2006). Knowledge of developmental progressions—levels of understanding and skill, each more sophisticated than the last—is essential for high-quality teaching based on understanding both mathematics and children's thinking and learning. Early childhood teachers' knowledge of young children's mathematical development is related to their students' achievement (Carpenter et al. 1988; Peterson et al. 1989). Further, researchers suggest that professional development focused on developmental progressions increases not only teachers' professional knowledge but also their students' motivation and achievement (e.g., Clarke 2004; Fennema et al. 1996). Thus, learning trajectories can facilitate developmentally appropriate teaching and learning for children (c.f. Brown et al. 1995).

The first evaluation of the *Building Blocks* curriculum supported the efficacy of this approach with pre-schoolers (3.5–4.5 years of age; the pre-kindergarten year). Two teachers worked with the *Building Blocks* curriculum (after working with the developers for a half day examining the materials) and two others used the regular school curriculum. One of the reasons for the name *Building Blocks* was that the materials emphasize the development of basic *mathematical building blocks* organized into two areas: spatial and geometric competencies and concepts and numeric and quantitative concepts. Although effect sizes in the first study were large and significant for all mathematical topics, the effect size for geometry (1.47) was larger than that for number (.85) (Clements and Sarama 2007c).

In geometry, the relative effect on the turn item was small (and higher for the comparison group) and the effect on congruence was positive, but small. Effects on construction of shapes and spatial orientation were large. The largest relative gains in geometry were achieved on shape identification and composition of shapes—two subtopics with the most well-developed learning trajectories. For example, the Building Blocks children made four times the gains as the comparison group in geometric composition (on a subtest with items for grades pre-K to grade 2 with a maximum of 11 points, the control increased to 2.01 from a pre-test score of 1.23, whereas the Building Blocks group increased to 4.47 from a pre-test score of 1.07). The curriculum engages children in several activities to develop this competence, including creating free-form pictures with a variety of shape sets, such as pattern blocks and tangrams, and solving outline puzzles with those same shape sets. Informal work with three-piece foam puzzles and clay cutouts is conducted for several weeks during mid-Fall. In April, teachers introduced the outline puzzles, which provided the most guidance along the learning trajectories. Most children worked about 2 days on the puzzles designed for children at the first level of the our theoretical learning trajectory for geometric composition (Clements et al. 2004) called the *Pre-Composer* level, 3 days at the *Piece Assembler* level, and two at the *Picture Maker* level. Only about a third of the children completed all those puzzles and thus could be confidently classified as operating at the *Shape Composer* level or above. Four children appeared to operate at best at the *Piece Assembler*. Teachers guided individual children to work on puzzles at the level just above that they appeared to have mastered (these puzzles are provided by the curriculum, classified by level).

The computer similarly monitored their progress on the "Piece Puzzler" software to assign levels of difficulty based on children's success; teachers checked and adjusted this level after conducting their assessments. The combination of off- and on-computer activities at an appropriate, progressive developmental level appeared to facilitate children's development of the mental actions on objects that engendered thinking at each subsequent level. This is shown in the strategies they employed. Almost 90% of the children placed shapes together without leaving gaps, an indication of thinking at the Picture Maker level or above. About 67% of the children turned shapes into the correct orientation *before* physically placing them within the puzzle outline. The same percentage appeared to search for "just the right shape" that they "knew would fit." These behaviors are criteria for the Shape Composer level. Only about 56%, however, showed immediate, confident, systematic completion of puzzles. Children's strategies therefore suggest that on the assessment roughly 10% were at the Piece Assembler level, 23% were operating at the Picture Maker level, 11% were in transition to the Shape Composer level, and 56% were at the Shape Composer level (or above; subsequent levels were not assessed). In contrast, the comparison group had roughly 77% at the Pre-Composer or Piece Assembler levels, 15% at the Picture Maker level, and 8% in transition to the Shape Composer level, consistent with developmental averages for this age group (Clements et al. 2004).

Scaling up

A strength of this phase of research (Larsen 2005), but a limitation for its generalizability, (Clements et al. 2004) is that we worked closely with only two teachers. Thus, fidelity to the curriculum and the learning trajectories core was relatively easy to maintain. How can such a successful approach be scaled up across a large number of diverse populations and contexts, avoiding the dilution and pollution that usually plagues such efforts? Difficulties are compounded in the area of geometry, where teachers may have a weak background and dislike teaching it.

To address these difficulties, we used research-based guidelines to create and test our TRIAD (*T*echnology-enhanced, *R*esearch-based, *I*nstruction, *A*ssessment, and professional *D*evelopment) intervention. TRIAD's theoretical framework (Sarama et al. 2008) is an elaboration of the *Network of Influences* framework (Sarama et al. 1998). As this framework is based on extensive reviews of theory and research as well as our own work, it is consistent with, but extends in levels of detail, such theories as diffusion theory and the overlapping spheres of influence (Cooper 1998; Denton and Fletcher 2003; Epstein 1987; Rogers 2003; Showers et al. 1987). Thus, scale-up can be seen as the effort to maintain the integrity of the vision and practices of an innovation through increasingly numerous and complex socially mediated filters, through phases of introduction, initial adoption, implementation, and institutionalization. Of the ten research-based guidelines in the TRIAD model (Sarama et al. 2008 has a complete description, as does UBTRIAD.org), the two most relevant to this article are the following:

- Focus on instructional change that promotes depth and quality of children's thinking, placing research-based learning trajectories at the core of the teacher/child/curriculum triad to ensure that curriculum, materials, instructional strategies, and assessments are aligned (Ball and Cohen 1999; Clements and Sarama 2004b; National Mathematics Advisory Panel 2008; Raudenbush 2008; Sowder 2007).
- Provide professional development that is multifaceted, extensive, ongoing, reflective, focused on common actions and problems of practice and especially children's thinking, grounded in particular curriculum materials, and, as much as possible, situated in the classroom. Encourage sharing, risk taking, and learning from and with peers (Elmore 1996; Hall and Hord 2001; Showers et al. 1987; Sowder 2007).

To elaborate the first bulleted guideline, we determined that we would weave specific content knowledge into our professional development on the geometry learning trajectories. For example, as part of the TRIAD professional development sequence, we discuss geometric and pedagogical terms with teachers. Both the literature (Sarama and Clements 2009) and our previous experience has revealed many lacunae and errors in teachers' knowledge of geometry content; therefore, we begin by guiding teachers to build such "common content knowledge" (Ball et al. 2008, p. 399). We help teachers define *attributes* to mean any characteristic of a shape. Some are *defining attributes.* To be a square, a shape must have *straight* sides. Others are *nondefining attributes*, such as a shape having a certain color or constructed out of a specific material (cf. Vinner and Hershkowitz 1980). Some defining attributes describe the parts of a shape. Others are special attributes we call *properties,* which describe a *relationship between parts*. Teachers sort manipulative shapes into Venn diagram "loops" labeled with different properties.

We also strive to develop teachers' specialized content knowledge (Ball et al. 2008, p. 399). For example, terms such as "straight" and "angle" are defined. Knowing these definitions may be considered common content knowledge. We then explicitly compare the definitions with common usage of such terms that may confound or conflict with mathematical usage. These comparisons add to the teachers' specialized content knowledge. One activity is to have teachers describe a shape, such as a triangle, and, whenever possible, draw a nontriangle that fits that description (e.g., "it has three lines" might be drawn as |||). They then discuss whether the drawing accurately fit the description, why it was not a triangle, and how the description could be improved to constrain the next drawing to be a triangle. Further discussions include the different ways different cultures and subcultures define and label shapes. For example, in the U.S., a disjoint classification of rectangles and squares is typical in early childhood classes. On the other hand, in East Asia, children are initially introduced to "four-side-shapes" and only later on are introduced to subcategories of four-sided shapes, with similarly accurate, descriptive names such as "parallel-four-sided-shape" (Han and Ginsburg 2001).

This knowledge is then connected to knowledge of content and students (Ball et al. 2008), through the *developmental progression* component of our learning trajectories. For example, we discuss that at certain levels of shape identification, children often classify shapes by nondefining attributes such as orientation, saying that the shape is not a triangle because it is not "right-side up." Another common confusion is between "side" and "angle." Children may also misunderstand concepts, such as "right angle," confusing it with "facing to the right" or with horizontality (Sarama and Clements 2009).

These insights into children's geometric thinking are structured into broad developmental levels of geometric thinking—the van Hiele model (van Hiele 1986). For example, we contrasted the visual (or syncretic, see Sarama and Clements 2009) level of geometric thinking with the descriptive/analytic level, in which people recognize and characterize shapes *by their defining attributes*. For instance, a child might think of a square as a figure that has four equal sides and four right angles. We illustrate that in later grades, students see relationships between *classes of figures* (abstract/relational, this has been called horizon knowledge, Ball et al. 2008). For example, due to cultural as well as developmental factors, most U.S. children (and adults) incorrectly believe that a figure is not a rectangle because it is a square (incorrectly believing these are mutually exclusive categories).

The last component of the learning trajectories construct involves instruction (knowledge of content and teaching, Ball et al. 2008). We discuss that geometric attributes and properties might be learned by observing, measuring, drawing, and making models. We note and provide illustrations of the research finding that if the examples and nonexamples students experience are rigid, not representing the range of the shape category, so will be their concepts. We discuss with teachers that this is not a developmental limitation; in a study, one of the youngest 3-year-olds scored higher than every 6-year-old on shape recognition tasks (Clements et al. 1999). Concepts of two-dimensional shapes begin forming in the pre-K years and stabilize as early as age 6 (Gagatsis and Patronis 1990; Hannibal and Clements 2010), so early experiences are important. This learning will be more effective if it includes a range of examples (including nonprototypical examples) and distractors (including those visually similar to examples) to build valid and strong concept images, including dynamic and flexible imagery (Owens 1999).

As we develop teachers' concepts and vocabulary, we also discuss specific pedagogical implications, such as developing children's language. For example, many children describe triangles as having "three points and three sides," but up to half, were not sure what a "side" was. (Clements et al. 1999). As with the number word sequence, the English language presents challenges. In the East Asian languages, for example, every "quadrilateral" is called simply a "four-side-shape." For most terms, judges evaluated the East Asian versions to have more mathematical clarity; for example, "sharp angle" vs. English's "acute angle" or "parallel-four-sided-shape" vs. English's "parallelogram" (Han and Ginsburg 2001).

In a similar vein, early childhood curricula traditionally introduce shapes in four basic level categories: circle, square, triangle, and rectangle. The idea that a square is not a rectangle takes root by age five (Clements et al. 1999; Hannibal and Clements 2010). We discuss the need to re-think our presentation of squares as an isolated set. If we try to teach young children that "squares are rectangles," especially through direct telling, confusion is likely. If, on the other hand, we continue to teach "squares" and "rectangles" as two separate groups, we may block children's transition to more flexible categorical thinking (cf. Gagatsis 2003). We present the *Building Blocks* approach, which is to engage children with many examples of squares and rectangles, varying orientation, size, and so forth, including squares as examples of rectangles (with double naming—"it's a square-rectangle"). Older children can discuss "general" categories, such as quadrilaterals and triangles, counting the sides of various figures to choose their category. Also, teachers might encourage them to describe why a figure belongs or does not belong to a shape category.

The second bulleted guideline emphasizes children's thinking and learning trajectories, but also posits that effective professional development ought to be extensive, ongoing, and reflective. In our present TRIAD model, we work with teachers for 7–8 full days during their first year of implementation, and five full days the second year (of which about a third deal mainly with geometry). In between these sessions, teachers work with mentors (e.g., TRIAD or school staff trained in the TRIAD model).

Further, consistent with research on adult learners, the TRIAD professional development "discussions" previously described are neither lectures nor limited to talk. Tasks and small-group work dominate the professional development sessions. For example, teachers explore the *Building Blocks* shape set, a collection of widely varying geometric figures. They sort the shapes by attributes, sometimes using Venn diagrams, and participate in other activities such as "Back-To-Back," in which one teacher chooses a shape from one shape set at random and then must sufficiently describe it without using its name so that another teacher can choose it from a matched shape set. They construct shapes from "parts" (e.g., sticks and angle connectors) and by composing and decomposing shapes. The sessions continue to provide hands-on experiences in rooms set up to mirror the structure of early childhood classrooms, with an emphasis on interactions with peers around common issues. In the classroom, coaches and mentors work with teachers throughout the year, visiting teachers in their classrooms no less than once per month, discussing children's learning and teaching based on the learning trajectories.

To develop teachers' knowledge of the learning trajectories, a main activity uses the technological tool *Building Blocks Learning Trajectory (BBLT)* web application. *BBLT* provides scalable access to the learning trajectories via descriptions, videos, and commentaries. Each aspect of the learning trajectories—*developmental progressions* of children's thinking and connected *instruction*—are linked to the other. That is, teachers might choose the "instruction" view, then click an activity and not only see an explanation and video of the activity "in action," but also immediately see the level of thinking that activity is designed to develop, in context of the entire learning trajectory.

The level of thinking is illustrated with assessment tasks and video of classroom activities in which children illustrate thinking at each level (the icons above the video allow the selection of alternative video), an approach that has received empirical support (Klingner et al. 2003). Finally, this developmental level is connected to all correlated activities. Thus, a user in this view could jump to the "Make Buildings" activity or any of the activities whose goal is develop that level of thinking.

Teachers also can test themselves on determining levels of the developmental progressions by attempting to classify the level of thinking displayed by a child in a video clip; assistive feedback is offered on these attempts.

In this way, teachers may view a piece of video as an example of a curriculum activity and, later, when studying developmental sequences, see it again as an example of a particular child's level of thinking within. In each case, supporting text directs attention to each perspective and the connections between them. Such rich learning experiences could promote flexible, integrated knowledge, as teachers learn to see and to integrate, the teaching and learning aspects of education. This may help teachers to develop necessary multiple representations of, and perspectives on, complex phenomena. In turn, this may promote the successful application of theories to concrete cases (Feltovich et al. 1997). The resulting cognitive flexibility positively impacts the variety of teaching strategies that people develop and ease with which they acquire new repertoires (Showers et al. 1987).

BBLT provides professional development by bringing participating teachers into intimate contact with "best practice" classrooms, including instruction and assessment. BBLT is used in four related ways. First, it aids presentations of the trajectories and activities to teachers. Second, teachers observe, react to, test themselves on, and discuss (often online) specific trajectory levels, activities, or the relationship between the two. Third, coaches and mentors use the site in talking to teachers, often in their classrooms, about the trajectories, activities, or the relationship between the two. This is especially valuable in situations in which a teacher says, or demonstrates, that she or he did not fully understand a given activity's goals or structure. Fourth, teachers may voluntarily consult BBLT when they wish to refresh their memories on a particular activity they are to teach or delve more deeply into understanding their children's thinking.

In summary, the TRIAD/*Building Blocks* professional development follows research-based guidelines and provides a combination of experiences. Multiple studies indicate it has had strong positive effects on both teachers' practice and their children's achievement; we describe these studies in the following section.

Research on scaling up TRIAD

The TRIAD model has been evaluated in several research projects. In one (Clements and Sarama 2008), 36 pre-school classrooms were randomly assigned to one of three conditions: experimental (TRIAD/*Building Blocks*), comparison (a different pre-school mathematics curriculum), or control. Children were pre- and post-tested with an individual assessment, between which children participated in *Building Blocks* instruction. Two observational measures indicated that the curricula were implemented with fidelity and that the experimental condition had significant positive effects on classrooms' mathematics environment and teaching. The experimental group's overall score (all topics, including number, operations, geometry, measurement, and patterning) increased significantly more than the comparison group score (effect size, .47) and the control group score (effect size, 1.07). In geometry, both intervention groups scored higher than the control group, with little difference between them, on identifying shapes and representing shapes. The TRIAD/*Building Blocks* group also increased in the frequency of completely correct constructions more than the other two groups. On the shape identification items, children gained on most of the individual shapes, with the greatest gains on prototypes and rotated variants of the class for squares and triangles, and for these and particular distractors for rectangles (e.g., avoiding choosing a parallelogram) and rhombuses.

The TRIAD/*Building Blocks* group scored higher than both the comparison and control groups on comparing shape. Children made greater gains increasing their matches of congruent shapes (requiring slides, flips, or turns) than on decreasing erroneous pairing of noncongruent shapes.

The TRIAD/*Building Blocks* group scored higher than the comparison group, which scored higher than the control group, on shape composition. The TRIAD/*Building Blocks* group increased more than the other two groups in completely correct solutions. Similarly, the TRIAD/*Building Blocks* group increased substantially more than the other groups in using more sophisticated strategies, such as rotating shapes into the correct orientation before placing them on the puzzle, searching for specific shapes with intentionality, and, in general, solving the puzzle systematically, immediately, and confidently These results are similar to those of the research on the *Building Blocks* curriculum described previously. Given that both intervention curricula cover these topics in similar proportions, the learning trajectories structure of the *Building Blocks* curriculum may be one reason for the greater gains in these topics. The largest scale-up research project is now underway, involving over 100 teachers in three cities (see http://UBTRIAD.org). First year results are similarly positive (Clements et al. 2011). Other researchers have used ideas similar to learning trajectories with good success. As just one example, the work of the Early Numeracy Research Project similarly shows that children learn geometry considerably earlier and better in educational environments based explicitly on children's development of geometric thinking (Clarke 2004). Clarke's "growth points" are consist with the developmental levels discussed here and are based on the same notion that the van Hiele levels need to be further delineated to serve educational needs, especially in the early childhood. The instructional approach is that there is little value in extensive formal naming of shapes until children are beginning to classify by properties, which helps children organize their knowledge of shapes in a categorization structure. Second graders using this approach were at levels of thinking beyond those in a comparison second grade group. Indeed, children as young as kindergarten, provided these high-quality learning experiences, were achieving what the comparison second graders achieved.

Conclusions

Students' lack of competence in geometry and spatial reasoning is a problem not only for geometric topics, but for other mathematical topics as well as other subject-matter domains. Professional development in early childhood geometry ought to be an international concern. Professional development is not always ongoing, continuous, reflective, and motivating. Research-based suggestions and models (such as the TRIAD model of professional development) hold the potential to make a significant difference in the learning of young children by catalyzing substantive change in the knowledge and beliefs of their teachers. We believe that the TRIAD model's success is due at least in part to the comprehensive inclusion of common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al. 2008), and, even more important, the synthesis of these four categories of knowledge into geometric learning trajectories. However, this is but one model. More research and creative effort are needed so that we may improve the professional development of early childhood teachers in the domain of geometry.

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