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Detecting and Correcting Fractions Computation Error Patterns

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ABSTRACT: *This article describes a follow-up analysis of findings from a randomized study that tested the efficacy of a blended version of Enhanced Anchored Instruction (EAI) designed to improve both the computation and problem-solving performances of middle school students with disabilities. The goals of the secondary analysis were to track overall error patterns of students in computing with fractions and to compare the effects of EAI and Business As Usual (BAU) on making these errors. Results showed that students taught with EAI reduced their errors compared to students in BAU classrooms and that reducing the one common error led to improved performance. Error pattern analysis provided clues about how to modify instructional materials for improving computation with fractions.*

The National Assessment of Educational Progress (National Center for Education Statistics, 2011) found that 65% of eighth grade students with disabilities scored below the Basic level compared to 23% of students without disabilities. Contributing to students' low performance was their weak understanding of and computation with rational numbers. This is not surprising because some studies have shown that students with disabilities score only 14.5% and 8.6% on measures assessing decimals and fractions, respectively, and their performance actually worsens in high school (Calhoon, Emerson, Flores, & Houchins, 2007). Concepts such as fraction equivalence are especially difficult for most students, not just for students with disabilities. The 1996 National Assessment of Educational Progress, for example, showed that 35% of students failed to choose 1/3 as the fraction equivalent to 4/12, and 65% of students failed to order five sets of fractions less than 1 from least to greatest (Silver & Kenney, 2000).

At the basic level, difficulties in learning fractions are often related to students' misunderstanding of the relationship between rational and whole numbers (e.g., Baroody & Coslick, 1998; Mack, 1993; National Research Council, 2001), which some math educators have labeled whole-number bias (Ni & Zhou, 2005). As an example, lack of conceptual understanding may lead students to think of 2/3 as two whole numbers, rather than as a relationship between two quantities, or to think of 1/3 as larger than 1/2 because 3 is larger than 2 (Baroody & Coslick, 1998; Fazio & Siegler, 2011). Although students' informal knowledge can and should be built upon, the transition from working exclusively with whole numbers to working with fractions (especially those represented symbolically) is not a trivial one.

Students' confusion with fractions is likely compounded by the complex array of subconstructs (i.e., part-whole/partitioning, ratio, operator, quotient, measure) they represent (Behr, Lesh, Post, & Silver, 1983; Kieren, 1980). A complete description of each subconstruct is beyond the scope of this article, but we provide a few illustrative examples. The *part-whole/partitioning* subconstruct refers to a fraction as the relationship between a unit (partitioned into equal parts) compared to the total number of parts into which the unit is partitioned. A key idea for students to understand is that as the number of parts into which the whole is divided increases, the smaller becomes the size of each part. The *ratio* subconstruct conveys the idea of comparison between two quantities, and students must understand that the relationship does not change when both the numerator and denominator are multiplied by the same number. The *operator* subconstruct requires students to understand that 2/3 can be interpreted as 2 × [1/3] of a unit or 1/3 × [2 units]. The *quotient* subconstruct means that any fraction can be viewed as a division situation. For example, students need to understand that if three pizzas are evenly divided among four people, the pizzas can be divided into four pieces and each person gets three pieces. Last, the *measure* subconstruct requires students to understand that a fraction is a number and can represent a distance from one point to another. Not accepting fractions as numbers leads students to make the common mistake of adding numerators and denominators together (e.g., 3/4 + 1/2 = 4/6). Students should be able to locate a fraction on a number line, which requires mastery of the concept of equivalence and order.

In general education, investigators have shown continued interest in identifying the misconceptions that students have and the errors they make in these areas (e.g., Behr, Harel, Post, & Lesh, 1992; Idris & Narayanan, 2011). Error analysis, as it is formally named, has had a long history in mathematics education. More than 30 years ago, Radatz (1979) speculated that interest in error analysis during his time was growing for several reasons, including limitations in the way test results were used to inform instruction and the implication inferred from group research that the average student exists as a realistic foundation for comparison. Using item-level descriptions of errors, investigators have probed deeper into students' thinking, thereby helping to guide development of more appropriate instructional practices (e.g., Charalambous & Pitta-Pantazi, 2007; Clarke & Roche, 2009; Pitkethly & Hunting, 1996). Much of this work involved small studies that did not test interventions in the context of formalized experiments.

In contrast to this rich history in general education, the research base in special education in teaching and learning fractions is relatively weak. Although promising instructional strategies have been developed in several other important areas such as problem solving (e.g., Jitendra, DiPipi, & Perron-Jones, 2002; Jitendra, Hoff, & Beck, 1999; Montague & Dietz, 2009; Xin, Jitendra, & Deatline-Buchman, 2005) and algebra (e.g., Hutchinson, 1993; Maccini & Hughes, 2000; Scheuermann, Deshler, & Schumaker, 2009; Witzel, Mercer, & Miller, 2003), few intervention studies have been conducted with fractions. Misquitta (2011) located only 10 empirical studies published between 1998 and 2008 that focused specifically on fraction instruction and students with disabilities. The lack of research in this critical area may be one reason why the achievement gap between students with and without disabilities continues to widen (Judge & Watson, 2011). Another explanation for the lack of progress may be due to how the results of studies in special

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education are reported. With notable exceptions (e.g., Empson, 2003), authors typically report their findings as total test scores, but more in-depth analysis at the objective or item level is limited or missing altogether.

In our own research over the past decade, first versions of Enhanced Anchored Instruction (EAI) were designed primarily for teaching problem solving rather than computation. The units typically included 8 to 15 min video-based scenarios in which adolescents are shown attempting to solve an interesting problem. As students searched scenes in the video to identify relevant information for helping the characters solve the subproblems and eventually the overall solution, teachers guided students' understanding by posing probing questions and offering instructional support. Following recommendations in the Institute of Education Sciences *Practice Guide for Organizing Instruction to Improve Student Learning* (Pashler et al., 2007), we added related hands-on applications (e.g., hovercraft building) to help students visualize abstract concepts and to give them additional practice on related problems.

Results showed that early versions of EAI were effective in helping students develop rather sophisticated problem-solving approaches, but we were disappointed over the weak effects EAI had on students' fractions computation skills (e.g., Bottge, Heinrichs, Mehta, & Hung, 2002; Bottge, Rueda, Serlin, Hung, & Kwon, 2007). We were probably naïve to expect change in computing performance given the complexity of fractions described previously and the serious learning difficulties of the students. To better address this weakness in our instructional materials, we developed a set of computer-based and hands-on computation units for use alongside the problem-solving lessons. By adding multiple representations of fractions and more interactive media (e.g., Mayer & Moreno, 2003; Mousavi, Low, & Sweller, 1995; Tabbers, Martens, & van Merrienboer, 2004), we hoped to guard against overloading students' limited working memory (Chandler & Sweller, 1991; Sweller, 1988). We saw some evidence of this in a small exploratory study (Bottge, Rueda, Grant, Stephens, & LaRoque, 2010).

This article describes a follow-up analysis of findings from a randomized intervention host study that was designed to test the efficacy of our latest version of EAI on the math performance of low-achieving middle school students with disabilities. Our reasons for conducting this secondary analysis were twofold. First, we wanted to track by type and frequency the errors students in the EAI and Business As Usual (BAU) conditions made for each item on a Fractions Computation Test administered prior to and following instruction. Second, we wanted to show what effects, if any, EAI had on the error patterns of students compared to the error patterns of students in the BAU condition. We hoped that tracking students' errors would reveal performance patterns that could eventually lead to more effective instructional methods for deepening conceptual understanding of fractions and procedural accuracy in computing them. Specifically, we designed our study to answer the following research questions:

- What are the most common types of systematic errors students with disabilities make when adding and subtracting fractions?
- What are the differential effects, if any, of EAI and BAU instruction on the specific errors students make when adding and subtracting fractions?
- How does eliminating the most common error result in improved performance on subscales of a Fractions Computation Test?

HOST STUDY OVERVIEW

The host study (Bottge et al., in press) from which these error data were drawn employed a pretest-posttest, cluster-randomized school-based trial to test the efficacy of the two instructional conditions (EAI vs. BAU) on students' ability both to compute with fractions and solve problems. We conducted the random assignment at the school level (not at the class or student level) because administrators and teachers typically schedule students to classes and there was the possibility of treatment contamination within schools due to teachers sharing their instructional materials and strategies.

In addition to the fractions computation instruction, four EAI units were taught consisting of computer-based interactive lessons, video-based anchored problems, and hands-on applied projects. We provided teachers in the EAI group daily lesson plans that included the lesson objectives, materials needed, warm-up exercises, details of the main lesson, and wrap-up questions. Teachers assigned to the BAU condition followed their regular school math curriculum, which had been developed to align with the principles contained in the Kentucky Department of Education Combined Curriculum Document. Instruction in both conditions targeted several standards in the Common Core State Standards Initiative-Mathematics (CCSSI, 2010), especially Ratios and Proportional Relationships, Number System, Statistics and Probability, and Geometry. The overall mean number of instructional days was 94.1 and 93.7 for EAI and BAU classes, respectively.

Researcher-developed tests (e.g., Fractions Computation) and two standardized achievement subtests (Computation, Problem Solving) administered over 3 consecutive days prior to and following the instructional period assessed the effects of EAI on the students' computation and problem-solving skills. Results of the three-level hierarchical linear model used to analyze the test data showed that the gain scores of EAI students were higher than the gain scores of students in the BAU classrooms on the researcher-developed Problem-Solving Test, the Fractions Computation Test, and the Standardized Computation Test. No difference between groups was found on the standardized problem-solving measure.

METHOD

Participants

A total of 308 middle school students with math disability in 15 EAI schools (23 teachers, 33 resource rooms) and 16 BAU schools (26 teachers, 31 resource rooms) completed the study (see Table 1). Parents, students, and teachers signed consent forms approved by the University Institutional Review Board for participating in the study. Students were receiving special education services for mild mental disability, other health impairment, specific learning disability, autism spectrum disorder, or emotional and behavioral disorder. Students were comparable across groups in gender, ethnicity, subsidized lunch, and disability area. Descriptions of disability categories are not included here due to space considerations but are available at the National Dissemination Center for Children With Disabilities (see http://nichcy.org/disability/categories#id).

Students received their math instruction from special education teachers in self-contained special education resource rooms because the Admissions and Release Committee at each school judged their skill levels too low for successful inclusion in general education math classes. The class size of the EAI resource rooms ($M = 4.82$, $SD = 1.36$) and BAU resource rooms ($M = 5.68$, $SD = 2.67$) did not differ, $t(49.41) = 1.87$, $p = .07$.

Two standardized subtests of the Iowa Tests of Basic Skills (ITBS; Form C, Level 12; University of Iowa, 2008) were administered as pretests and helped to confirm the students' low achievement in math: Mathematics Computation ($M = 10.74 <$ ITBS mean raw score of 18.2) and Problem Solving and Data Interpretation ($M = 10.11 <$ ITBS mean raw score of 15.0). According to the ITBS (Form C, Level 12) National Norms (Dunbar et al., 2008, p. 26), students in this study were on average 1.3 *SD*s below the national norm in computation and 1.0 *SD* below the national norm in problem solving for students in Grade 6. Prior to instruction, BAU and EAI groups were comparable on both standardized measures and on the researcher-developed Fractions Computation Test.

EAI Instruction

The first unit included a series of computer-based modules and concretized manipulatives designed to develop students' understanding and procedural competence with rational numbers. We developed the instructional package called Fractions at Work (FAW), which contains seven chapters divided into self-contained units. The first chapter describes properties of fractions, including their purpose and function. The goal of the second chapter is to help students understand the concept of equivalence. Using fractions indicated on an interactive tape measure, the computer screens show how the value of fractions depends on the number of parts into which an inch is divided (i.e., denominator) and the number of these parts

TABLE 1

Teacher and Student Characteristics

	Teachers BAU (n = 26)	Teachers EAI (n = 23)	χ^2	t	p
Gender			0.13[a]		.76
Male	8	6			
Female	18	17			
Ethnicity			0.18[a]		.67
Caucasian	25	20			
African American	1	2			
Asian	0	1			
Years teaching special education				1.62	.11
M	12.21	9.07			
Median	11.50	8.00			
SD	6.86	6.68			
Range	2–27	1–28			
Highest degree earned			0.06[a]		.99
BA, BS	6	6			
MA, MS	20	17			

	Students BAU (n = 163)	Students EAI (n = 145)	χ^2	t	p
Gender			2.07[a]		.18
Boys	102	102			
Girls	61	43			
Grade			2.33[a]		.14
6	2	6			
7	83	57			
8	78	82			
Ethnicity			2.66		.27
Caucasian	126	116			
African American	26	25			
Latino	6	3			
Native American	3	0			
Biracial & Other	2	1			
Disability/service area			1.93		.95
MMD	62	56			
OHI	56	44			
SLD	24	24			
ASD	12	12			
EBD	9	9			
Subsidized lunch[b]	122	109	0.17[a]		.79
Math achievement (pretest *M*)					
Fractions Computation Test	3.06	2.36		1.51	.13
ITBS Computation	10.40	11.17		1.60	.11
ITBS Problems Solving and Data Interpretation	10.31	9.88		1.03	.31

Note. BAU = business as usual; EAI = enhanced anchored instruction; MMD = mild mental disability; OHI = other health impairment; SLD = specific learning disability; ASD = autism spectrum disorder; EBD = emotional and behavioral disorders. ITBS = Iowa Tests of Basic Skills. Less populated categories combined for Chi-square test: Teacher ethnicity reduced to Caucasian and Other; Grade reduced to 6, 7, and 8; Student ethnicity reduced to Caucasian, African American, and Other.
[a]Fisher's Exact Test. [b]Five students did not have a subsidized lunch status.

available (i.e., numerator). In the third chapter, students are shown how to add simple fractions that have the same denominator. Multiple examples are used to emphasize why it is not appropriate to add denominators. Chapters 4 and 5 describe how to add simple fractions with unlike denominators. Students are first taught to check whether they can multiply the smaller denominator by some number to make it equal to the larger denominator. In Chapter 6, students are led through a series of steps on how to add and subtract mixed numbers, including procedures for renaming and simplifying fractions. The last chapter summarizes the content in the previous chapters and includes an assessment to check for understanding.

As students worked on the lessons, teachers used concrete materials such as fraction strips to help their students understand difficult concepts such as equivalence. For example, teachers gave narrow strips of tag board to students and asked them to imagine that one of the strips was a long candy bar that needed to be shared by two people. Students represented how to share the candy bar by folding the strip in half. The teachers then asked them to label the fold 1/2. The students repeated the process with the other three strips (candy bars that needed to be shared among four, eight, and 16 people) using repeated folding to show each segment, which they then labeled. When students had labeled all four of the fraction strips, teachers posed questions about equivalence, relative size, and the function of and relationship between numerator and denominator: For example, "What are other names for 1?" and "What is another name for 3/8?" In the days that followed, students used their fraction strips to solve problems such as: "If you have 3/4 of a candy bar and your friend has 1/2 of a candy bar, who has more?" and "How much more does she have?" Students used their strips to show and discuss how to compute fractions.

All fraction subconstructs were addressed in some way with FAW, but much of the instruction was devoted to three main concepts. In particular, the part-whole/partitioning subconstruct was emphasized because we found in previous studies with low-performing students that the symbolic representation of fractions puzzled them and they had little or no understanding of the purpose and function of fractions. In one example, the media made use of animated representations of size as the denominator number increased and decreased. The ratio subconstruct was also emphasized because students often have a poor understanding of the relationship between numerator and denominator, especially when finding equivalent fractions. Some of this instruction was related to markings on a ruler so students could relate equivalence to an applied use. Fractions At Work also devoted much of the instructional time to the measure subconstruct because two of the problem-solving units required measurement skills with fractions. Students took an average of 20 instructional days to complete FAW.

Based on previous studies with EAI, we knew teachers needed quality professional development in the appropriate use of the multimedia-based and hands-on units. The main activity was a 2-day workshop conducted by a middle school math teacher who had taught with EAI for several years. He modeled teaching strategies and gave teachers opportunities to play the part of students in sample lessons. Teachers had ample opportunity to discuss issues related to specifics of the lessons (e.g., creating hands-on materials) and technology use. The workshop proceedings were video recorded and placed on a server available for EAI teachers to review during the study.

BAU Instruction

The learner objectives in BAU classrooms paralleled those of the EAI units. According to the data retrieved from teachers' self-reports, lesson plans, and confirmed by classroom observers, BAU instruction focused on analyzing proportional relationships, solving real-life problems with whole numbers and fractions, graphing ordered pairs, working on concepts related to geometry, and understanding pre-algebraic expressions. Most teachers taught fractions with textbooks used in the general education math classes. BAU teachers also used computer-based units they displayed on interactive whiteboards and manipulatives for teaching.

Because the BAU lessons varied across classes, we provide one sample of a typical teaching lesson collected from a direct classroom observation. The lesson objective consisted of comparing and ordering fractions and solving word problems. The teacher projected a web-based video on the screen and students were directed to display their fraction pieces on their desks. Students placed five strips (1 whole, 1/2, 1/4, 1/8, 1/16) on their desks and then the teacher played a video (i.e., animated cartoon) that described procedures for rounding numbers. After the video ended, the teacher projected word problems on the screen that the students had been taught the previous day. The teacher asked volunteers to read the problems out loud and to share their problem-solving strategies. The first problem read: "A full tank of gas holds 14 gallons. The fuel gauge reads 1/4 full. About how many gallons are left in the tank? How many more miles can you drive?" A student described the calculator procedures needed to multiply a fraction by a whole number (1/4 of 14). The teacher then told students that at that rate they could travel 200 miles on a full 14-gallon tank of gas. She then asked a student to find 1/4 of 200 miles in order to solve the second part of the problem. To solve the second part of the problem using a calculator, she instructed a student to type "1 b/c 4 times 200" (b/c is a fraction button on advanced calculators). Other lessons focused on making predictions using data and graphs. Calendar Math (e.g., Gillespie & Kanter, 2000) was also a key part of the daily math instruction in several BAU classrooms.

Classroom Observations

Trained observers (principal investigator, graduate assistants, project manager) collected whole-class observations and recorded field notes directly into customized templates using FileMaker Pro 10 (FileMaker Pro 10, 2009). The template included fixed spaces for recording demographic information (e.g., date, school, condition, unit of instruction), boxes for indicating level of treatment fidelity (e.g., surface features, quality of implementation), and open spaces for describing classroom activities. The content of the templates for recording treatment fidelity of EAI was derived from the daily lesson plans that teachers were provided during their training. Primary observers conducted 173 whole-class observations in the EAI classrooms. Primary observers indicated that teachers taught activities in the warm-up 70% of the time and the main lesson 95% of the time. A secondary observer was present during 34 of the observed class periods (19.6%). Interobserver agreement was 71% for the warm-up and 94% for the main part of the lesson. Observers reported that on several occasions teachers were unable to finish the entire lesson plan in one day for various reasons (e.g., shorter class time, behavioral problems, other school activities) so the lesson was continued on the following day. To a large extent, these variations in schedule explained inconsistencies between what the lesson plans prescribed and how they were carried out.

We also collected a total of 117 observations from BAU classes, which included nine observations from secondary observers. The FileMaker templates included large open sections where observers wrote detailed descriptions of the lessons, which we used to help confirm that BAU and EAI teachers were addressing similar math content.

Instrumentation

The Fractions Computation Test was administered prior to and following the instructional period, which also included the problem-solving units in the host study. The Fractions Computation Test consisted of 20 items (14 addition, 6 subtraction) to measure students' ability to add and subtract fractions. Items asked students to add and subtract fractions with like denominators and unlike denominators where the larger denominator could serve as the common denominator (e.g., 7/8 – 1/4). All but one of the items included fractions that could be found on a ruler. The test included simple fractions and mixed numbers as well as addition of three fractions. Students were told to reduce their answers to simplest form and to show all their work. On 18 of the items, students earned 1 point for correct work and 1 point for the correct answer. On two items with mixed numbers that required renaming prior to subtracting, students were awarded an additional point. Calculator use was not allowed. Internal consistency estimates (α) of previous versions of this test were .91 (Bottge et al., 2002) and .97 (Bottge et al., 2007). Internal consistency estimates for this sample were .81 at pretest and .96 at posttest. Trained graduate assistants scored the tests. Interrater agreement based on 20% of the pretests and posttests was 97%.

In addition to computing total scores, doctoral students who worked as research assistants identified and coded the primary error students made on each incorrect item. Prior to coding, the project faculty provided research assistants with sufficient background as to the types of errors they might encounter. The error codes generated in this study were defined by a systematic procedure that first had primary and secondary scorers independently identify error patterns on a sample of pretests. Next, they discussed the errors they had detected, resolved any discrepancies, developed descriptor language for each type of error, and then assigned each error a code label. Last, the research assistants scored a second sample of tests and computed interrater reliability. Successive samples of tests were scored until an acceptable level of agreement was achieved (i.e., 90%). This process resulted in a final total of 11 codes, including an Other category where student work did not suggest an error pattern and a No Response category where the student made no attempt to compute the answer. Based on the pretest sample, prior math research, and our previous findings, we anticipated students would make the most mistakes in Combining (C: Student combines numerators together and denominators together), Add All (AA: Student adds together all the numbers of the fractions and finds a total sum), Select Denominator (SD: Student uses one of the denominators in the answer without finding the equivalent fraction), and Equivalent Fractions (EQ: Student makes an error when attempting to find an equivalent fraction). Two research assistants independently identified and coded item errors on 20% of the Fractions Computation Test pretests and posttests across both conditions. Interrater agreement was 96% (EAI = 98%, BAU = 93%). Note that the slightly higher agreement for EAI was likely due to EAI students making fewer errors on the posttest.

ANALYSIS

Our host research design inherited data hierarchy with students (level 1) nested within classrooms (level 2) nested within schools (level 3). Hierarchical linear modeling was thus employed to analyze data gathered in the host study from the experimental design. Specifically, we adopted a three-level hierarchical linear model for the error analysis. This analysis was based on whether a student became worse or better in making errors from pretest to posttest and enabled us to assess the effects of worsening or improving over the same error in a subscale. We also examined whether there were differences between BAU and EAI students in the effects of worsening or improving errors on performance.

RESULTS

The logic behind our error analysis at the item level is straightforward. Our goals were to identify the most popular errors that students made in fraction computation, examine the treatment effects between EAI and BAU students making those errors, and highlight mathematical characteristics of items in which EAI students noticeably reduced those errors from pretest to posttest compared to BAU students. This process enabled us to pinpoint the types of mathematical problems on which students in the EAI classes demonstrated the largest effects.

Identifying the Most Common Errors: Research Question 1

Table 2 shows frequencies (counts) of item-level errors of EAI and BAU students on the Fractions Computation Test at pretest and posttest. Based on the descriptive numbers and error trends, we found that the C error was the most common mistake that EAI and BAU students made across all items at both testing times. As a reminder, students who committed this error added (or subtracted) the numerators and the denominators. A similar error was AA where students added all numerators and denominators to find a total sum, but because this was far less common we chose to target the more populated C error in our analysis.

At first glance of the frequencies in Table 2, it may appear that EAI students made more errors than BAU students. For example, on the pretest 51 BAU students (31%) made the C error on the first item compared to 63 EAI students (43%). On the posttest, however, 46 BAU students

TABLE 2

Item-Level Frequencies of Fraction Computation Errors by Time of Test (Pre = T1, Post = T2) and Group (BAU = 163, EAI = 145)

				Error																			
		No error		C		AA		SD		AC		EQ		CE		WO		O		L/S		RN		NR	
Item	Group	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
1	BAU	64	86	51	46	21	8	2	0	5	2	1	0	2	1	0	0	17	20	0	0	0	0	0	0
	EAI	41	118	63	14	19	4	0	0	4	1	0	0	0	1	0	0	16	7	0	0	0	0	2	0
2	BAU	61	85	52	46	20	10	2	0	6	6	1	0	2	1	0	0	19	15	0	0	0	0	0	0
	EAI	44	116	62	14	17	4	0	2	4	1	0	1	0	0	0	0	15	7	0	0	0	0	3	0
3	BAU	5	11	89	94	19	10	14	10	6	7	2	3	0	0	0	0	23	25	0	0	0	0	5	3
	EAI	1	69	89	21	18	4	11	20	3	1	1	10	0	3	0	0	16	11	0	0	0	0	6	6
4	BAU	4	11	83	92	19	10	17	11	9	5	2	4	0	0	0	0	25	26	0	0	0	0	4	4
	EAI	1	66	84	24	18	4	12	19	2	1	1	9	0	5	0	0	20	12	0	0	0	0	7	5
5	BAU	1	6	85	95	16	10	13	13	5	6	2	4	3	0	0	0	33	25	0	0	0	0	5	4
	EAI	0	54	85	21	17	4	11	19	4	2	1	13	0	7	0	0	20	17	0	0	0	0	7	8
6	BAU	3	5	86	92	16	10	12	16	6	7	2	5	1	0	0	0	34	25	0	0	0	0	3	3
	EAI	0	54	82	24	19	4	12	25	6	1	1	17	0	4	0	0	17	9	0	0	0	0	8	7
7	BAU	3	4	77	92	15	6	15	14	5	7	3	2	0	1	0	0	38	31	0	0	0	0	7	6
	EAI	1	52	71	23	15	3	17	15	3	0	1	22	0	5	0	0	23	15	0	0	0	0	14	10
8	BAU	1	4	80	89	11	7	13	16	6	7	2	3	1	2	0	0	38	28	0	0	0	0	11	7
	EAI	0	49	71	22	16	3	8	18	5	1	1	17	1	6	0	0	29	16	0	0	0	0	14	13
9	BAU	3	3	73	84	11	5	13	14	4	6	1	5	0	2	0	0	49	38	0	0	0	0	9	6
	EAI	0	48	67	21	15	3	15	17	3	1	3	20	0	5	0	0	25	15	0	0	0	0	17	15
10	BAU	0	6	78	89	14	5	13	12	5	7	1	3	3	0	0	0	37	35	0	0	0	0	12	6
	EAI	0	47	70	20	17	3	9	15	3	2	3	15	0	10	0	0	26	18	0	0	0	0	17	15
11	BAU	2	7	84	89	18	10	15	17	4	3	3	1	0	0	0	0	31	31	0	0	0	0	6	5
	EAI	1	54	72	23	18	5	10	11	3	0	1	16	1	4	0	0	26	18	0	0	0	0	13	14

continues

TABLE 2. *Continued*

		No error		Error																			
				C		*AA*		*SD*		*AC*		*EQ*		*CE*		*WO*		*O*		*L/S*		*RN*		*NR*	
Item	*Group*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*	*T1*	*T2*
12	BAU	2	5	69	87	17	8	17	12	5	3	1	2	0	1	0	0	40	39	0	0	0	0	12	6
	EAI	1	50	73	22	16	4	14	17	2	0	1	19	0	2	0	0	24	16	0	0	0	0	14	15
13	BAU	2	2	72	78	11	7	15	14	3	3	2	3	0	2	0	0	44	45	0	0	0	0	14	9
	EAI	1	41	65	18	14	3	13	20	2	0	1	17	0	8	0	0	30	19	0	0	0	0	19	19
14	BAU	1	3	74	85	12	7	18	19	3	4	2	1	1	2	0	0	39	35	0	0	0	0	13	7
	EAI	0	40	68	20	12	4	14	19	2	0	2	16	0	9	0	0	25	20	0	0	0	0	22	17
15	BAU	53	69	52	49	7	2	2	2	3	1	0	1	2	6	1	4	32	27	0	0	0	0	11	2
	EAI	40	104	44	7	4	1	3	1	1	1	0	1	1	0	0	2	33	13	0	0	0	0	19	15
16	BAU	2	6	84	88	8	1	15	15	2	2	2	0	0	2	0	2	35	41	0	0	0	0	15	6
	EAI	1	52	70	21	3	0	13	22	1	9	3	0	0	4	0	2	30	13	0	0	0	0	24	22
17	BAU	24	39	55	55	7	1	2	4	2	0	0	1	8	8	6	6	45	41	0	0	0	0	14	8
	EAI	29	73	41	8	2	1	3	0	1	0	0	1	4	17	1	4	37	21	0	0	0	0	27	20
18	BAU	1	4	2	4	6	1	0	1	0	0	0	1	0	2	5	3	65	64	61	68	2	5	21	10
	EAI	1	22	4	1	1	1	1	0	1	0	0	1	0	0	1	2	48	17	60	74	0	4	28	23
19	BAU	3	3	63	74	7	2	17	10	2	0	1	2	0	1	0	3	53	59	0	0	0	0	17	9
	EAI	1	41	60	16	1	1	9	19	1	1	2	19	0	4	0	3	46	18	0	0	0	0	25	23
20	BAU	1	3	68	63	8	1	11	9	4	0	0	1	0	1	0	3	50	61	4	15	0	0	17	6
	EAI	0	11	54	13	2	1	8	9	1	1	1	9	0	1	0	1	52	28	2	41	0	5	25	25

Note. BAU = business as usual; EAI = enhanced anchored instruction; C = combining; AA = add all; SD = select denominator; AC = add components; EQ = equivalent fractions; CE = computation; WO = wrong operation; O = other; L/S = large over small; RN = renaming; NR = no response.

(28%) made the same error compared to only 14 EAI students (10%), which was an improvement of 33% from pretest to posttest for EAI students compared with only 3% for BAU students. Most of the other item performances show a similar pattern of improvement in favor of the EAI students across test times.

Assessing Instructional Impact on Errors: Research Question 2

In contrast to BAU students, who showed a consistent pattern and number of mistakes from pretest to posttest, the posttest patterns of EAI students looked much different. Most important, far fewer EAI students made the C error on the posttest than on the pretest compared to BAU students. Close inspection of the posttests also showed that when EAI students continued to make errors, many of the errors had to do with finding a common denominator. This is shown by a consistent pattern of more numerous errors in the posttest error columns SD and EQ, which is not the case for BAU students who continued to make the C and AA errors. The error pattern of EAI reveals students' emerging conceptual knowledge of fractions, but it also shows how an incomplete conceptual understanding of equivalence limits students' ability to compute accurately.

Hierarchical linear modeling analyses tested the differential effects of making C errors on gains from pretest to posttest between students who were in the EAI and BAU instructional groups. Table 3 displays the results for the addition, subtraction, denominator, and rewriting subscales. On the addition subscale, irrespective of instructional method, decreasing C errors on the posttest was associated with larger overall gains in the addition subscale scores for the total group of students, although effect sizes (ESs) were relatively small (i.e., .20, .26, .23). Much larger ESs were found for students in the EAI group, both for overall addition and for addition with mixed numbers. That is, fewer C errors in these areas by students in the EAI group contributed to significant improvements on test items related to adding overall (ES = .83) and to adding mixed numbers (ES = .95). This was also the case for the subscales of unlike denominator, no rewriting, and rewriting, with all showing large ESs. The exception was in the no rewriting subscale, which showed a significant but moderate ES.

Assessing Effects of Reducing the Most Common Error: Research Question 3

We also found that making more C errors from pretest to posttest had statistically significant, negative effects on performance on addition items. A worsening performance of C was associated with much smaller gains in overall addition, and the impact on the total group of students resulted in a relatively large ES (.51). A statistically significant difference was also found between BAU and EAI students meaning that a worsening performance of BAU students on C had a much greater negative impact on their test scores (ES = .57). In sum, by correcting their C errors, EAI students showed much greater gains in addition than BAU students. Even when the performance of some EAI students worsened (i.e., made more C errors), the negative effects on their overall test scores were smaller than those of the BAU students.

Although the majority of treatment effects showed that eliminating C errors helped EAI students achieve higher test scores, we did find a couple instances that showed overcoming C errors helped BAU students. Specifically, overcoming C errors was associated with a greater gain in the like denominator subscale for BAU students (ES = .41). Making more C errors was associated with less gain in the simple fraction subtraction and like denominator subscales for EAI students than for BAU students, with large ESs in both cases, 1.04 and .60, respectively.

In summary, positive effects of improving performance on C errors were greater for EAI students on five subscales (overall addition, mixed numbers addition, unlike denominator, no rewriting, and rewriting) and for BAU students on one subscale (like denominator). Negative effects of making more C errors had less impact on total test scores for EAI students on three subscales (overall addition, mixed numbers addition, and unlike denominator) and for BAU students on two subscales (simple fraction subtraction and like denominator). Treatment effects of eliminating the C error clearly favored the EAI student group.

TABLE 3

HLM Results Estimating Effects of All Students and EAI Students Making Fewer or More Posttest Combining Errors

	Fewer C Errors – Total			*Fewer C Errors – EAI*			*More C Errors – Total*			*More C Errors – EAI*		
Errors	b	SE	*ES*	b	SE	*ES*	b	SE	*ES*	b	SE	*ES*
Addition												
Overall	1.84*	.72	.20	7.66***	1.32	.83	–4.73**	1.28	.51	5.25**	1.67	.57
Simple Fraction	1.38***	.41	.26	1.80	.86	.33	–2.82***	.49	.52	–1.47	1.00	.27
Mixed Number	.95*	.38	.23	3.90***	.64	.95	–1.52*	.58	.37	1.61*	.73	.39
Subtraction												
Overall	.32	.48	.08	1.45	.89	.38	–2.11***	.48	.56	–.54	.80	.14
Simple Fraction	.60**	.19	.40	.10	.37	.06	–.87***	.19	.58	–.90*	.40	.60
Mixed Number	.07	.24	.03	.77	.42	.30	–1.23**	.33	.48	–.06	.54	.02
Denominator												
Like	2.06***	.29	.65	–1.30*	.54	.41	–2.94***	.45	.93	–3.29**	.93	1.04
Unlike	1.26	.99	.12	8.54***	1.79	.80	–4.57**	1.45	.43	4.46*	1.99	.42
Rewriting												
No rewriting	.79	.58	.12	2.87*	1.19	.45	–2.37***	.55	.37	–.61	1.12	.10
Rewriting	.99	.64	.16	4.23**	1.14	.67	–3.12**	.86	.49	2.60	1.32	.41

Note. HLM = hierarchical linear modeling; EAI = enhanced anchored instruction; C = combining. Effect size (ES) computed as Hedges' g = $\frac{b}{\sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{(n_1+n_2-2)}}}$

*$p < .05$. ** $p < .01$. *** $p < .001$.

Table 4 identifies the mathematical characteristics of each of the computation items (numbers of C errors in this table are derived from Table 2). EAI students made improvement on all items, and their gains followed a consistent pattern on four items in particular. For example, 89 EAI students made a C error on Item 3 of the pretest, but only 21 students made the same error on the posttest, representing an improvement of 68 counts. By comparison, 89 BAU students made a C error on the pretest, and 94 students made the same error on the posttest, which amounts to a worse performance of five counts. Items 4, 5, and 6 followed a similar pattern. Overall, EAI students as a group got much better at avoiding the C error, whereas the performances of BAU students stayed much the same or deteriorated.

Treatment effects of eliminating the C error clearly favored the EAI student group.

All four items are addition problems consisting of two simple fractions in a vertical format with unlike denominators where one denominator is a multiple of the other. Two items required rewriting the answer and two items did not require rewriting. The cognitive processes needed for solving these problems are quite complex. First, students needed to recognize that both fractions were simple fractions and that the denominators needed to be changed if they were not alike. Next, they had to recognize that one denominator was a multiple of the other and then create like denominators by multiplying both the numerator and denominator of one fraction by the same multiple. To compute the answer, students had to add the numerators together over the same like denominator, which was especially challenging because the fractions were aligned vertically. Last, students had to judge whether the fraction in the answer needed rewriting to produce a mixed number or no rewriting to remain as a simple fraction.

DISCUSSION

In this article we extended our analysis of results from a larger host study to investigate the error patterns in adding and subtracting fractions of students who were in EAI and BAU instructional conditions. Close inspection of item response patterns revealed that students in both instructional groups made more C errors than other types of errors, which is consistent with findings from recent studies (e.g., Idris & Narayanan, 2011). This pattern of errors remained for BAU students on the pretest and posttest, whereas EAI students made far fewer C errors on the posttest than they made on the pretest. The reduction of C errors was associated with improvement in the performance of EAI students on five subscales, including addition items, items with like and unlike denominators, and answers that could be rewritten.

Despite the extent of their disabilities and low skill levels, many EAI students became remarkably proficient in navigating the series of procedural steps necessary for finding common denominators and computing fractions correctly. For example, the number of EAI students making the C error on Item 3 from pretest to posttest was reduced from 89 students to 21 students. Thus, almost half the EAI students corrected this error (68 out of 145 students) compared to BAU students who continued to make C errors from pretest (89) to posttest (94). Similar patterns were observed for other items. An interesting artifact in the overall improvement of the EAI students was an increasing error trend by a few students on the posttest who presumably knew they needed to find a new denominator but made mistakes attempting to find it. We note this as a positive finding because students at least recognized the need for finding common denominators prior to adding or subtracting.

We attribute our results to two main factors. First, we paid close attention to the development of the instructional materials based on previous studies with EAI. The FAW software was designed for use with manipulatives (e.g., paper fraction strips), which afforded opportunities for students to go back and forth between electronic and hands-on formats, as needed, for deeper conceptual understanding. A narrator in FAW described concepts in language that was meaningful

TABLE 4

Mathematical Characteristics of Items Showing Improvement of EAI From Pretest to Posttest

Item	*Improvement* *EAI*	*BAU*	*Mathematical Characteristics*
1	49	5	2 simple fractions, vertical addition, like denominators, no rewriting
2	48	6	2 simple fractions, vertical addition, like denominators, rewriting
3	**68**	**–5**	**2 simple fractions; vertical addition; even, unlike denominators; 1 multiple of the other; no rewriting**
4	**60**	**–9**	**2 simple fractions; vertical addition; even, unlike denominators; 1 multiple of the other; no rewriting**
5	**64**	**–10**	**2 simple fractions; vertical addition; even, unlike denominators; 1 multiple of the other; rewriting**
6	**58**	**–6**	**2 simple fractions; vertical addition; even, unlike denominators; 1 multiple of the other; rewriting**
7	48	–15	2 mixed numbers; vertical addition; even, unlike denominators; 1 multiple of the other; no rewriting
8	49	–9	2 mixed numbers; vertical addition; even, unlike denominators; 1 multiple of the other; rewriting
9	46	–11	2 mixed numbers; vertical addition; even, unlike denominators; 1 multiple of the other; no rewriting
10	50	–11	2 mixed numbers; vertical addition; even, unlike denominators; 1 multiple of the other; rewriting
11	49	–5	3 simple fractions; vertical addition; even, unlike denominators; 2 multiples of the other; rewriting
12	51	–18	3 simple fractions; vertical addition; even, unlike denominators; 2 multiples of the other; no rewriting
13	47	–6	3 mixed numbers; vertical addition; even, unlike denominators; 2 multiples of the other; no rewriting
14	48	–11	3 mixed numbers; vertical addition; even, unlike denominators; 2 multiples of the other; rewriting
15	37	3	2 simple fractions, vertical subtraction, like denominators, rewriting
16	49	–4	2 simple fractions; vertical subtraction; even, unlike denominators; 1 multiple of the other; no rewriting
17	33	0	2 mixed numbers, vertical subtraction, like denominators, no borrowing, rewriting
18	**3**	**–2**	**2 mixed numbers, vertical subtraction, rename from whole number, no rewriting**
19	44	–11	2 mixed numbers; vertical subtraction; even, unlike denominators; one multiple of the other; no renaming; no rewriting
20	41	5	2 mixed numbers; vertical subtraction; even, unlike denominators; 1 multiple of the other; renaming from whole number; no rewriting

Note. EAI = enhanced anchored instruction; BAU = business as usual; Values under Improvement are frequencies representing reductions in number of students who made combining errors from pretest to posttest. Bold rows identify items that demonstrate the largest (3, 4, 5, 6) and smallest (18) improvements.

to students with special care given to using wording that articulated valid and precise explanations of math concepts (see Seethaler, Fuchs, Star, & Bryant, 2011). That is, principles guiding its development emphasized the meaning of fractions, the proportional characteristics of rational numbers, natural ways students may view problems and solutions, and alternate forms of visual representations (Moss & Case, 1999). The instructional units were focused on helping students understand the meaning of fraction notation and equivalent fractions so that fraction addition and subtraction would make more intuitive sense (Hiebert & Carpenter, 1992; Star, 2005). This was important because it could effectively reduce the number of procedures students would have to remember as they added and subtracted fractions.

We note this as a positive finding because students at least recognized the need for finding common denominators prior to adding or subtracting.

Second, we provided quality training to teachers in the use of the materials. The main training activity was a 2-day summer workshop (16 hr) conducted by a middle school math teacher who had taught with EAI for several years. He had been our primary advisor in creating the lesson plans after using prototypes of the lessons in his classroom. During the training sessions, he modeled teaching strategies and gave teachers opportunities to play the part of students during sample lessons. They also had ample opportunity to discuss issues related to lesson study, assessment techniques, and technology use. The sessions were video recorded and placed on a server available for EAI teachers to review during the school year. Classroom observations showed that the training had been effective as most teachers implemented the EAI lessons with a high degree of fidelity. BAU teachers followed the curriculum specified by their local school administrators. For the most part, BAU teachers had received ongoing training for several years to implement the lessons. Thus, compared to EAI teachers who received only 2 days of training, BAU teachers would be expected to have the advantage in teaching their lessons as intended.

Of course a substantial number of students continued to make errors, which we had anticipated. Fractions have a long history of posing serious obstacles to the mathematical development of children with and without disabilities (Behr, Wachsmuth, & Post, 1985; Woodward, Baxter, & Robinson, 1999). Even among some of the EAI-trained teachers, observers reported that the high level of conceptual and procedural math complexity coupled with the serious nature of the students' learning problems contributed to difficult teaching situations. This was the case especially for teachers who admitted they did not have a deep knowledge of fractions or only had minimal experience in teaching them. A steady stream of research suggests that these instructional problems with fractions are not unique to teachers in this study (see Clarke & Roche, 2009; Post, Harel, Behr, & Lesh, 1991).

LIMITATIONS

Despite the valuable information yielded by the error analysis and the overall positive effects of EAI in reducing errors, we acknowledge limitations of the study. First, we relied on indirect methods for identifying errors. That is, test scorers looked at student work to judge the type of errors students made in contrast to some researchers who use student talk-alouds or interviews to identify procedural mistakes. Despite missing the qualitative richness that more direct methods could convey, we judged our process to be valid and also appropriate for the scope and design of the study. We identified and coded the types of errors students made based on the work of other math researchers, our own teaching experiences with similar students, and our previous research.

Although the host study spanned 18 weeks, the part of the study involving explicit fractions instruction with FAW spanned only 4 weeks. This is a relatively brief time period for students with math disability to gain a full understanding of fractions concepts and computation procedures. During the problem-solving units, it is probable that some students were able to use their partial understanding of fractions to help them solve the problems in these units. However, judging from the errors some students continued to make, they would have profited by more work on the FAW units.

Last, some students moved out of the resource room prior to taking the posttest for a variety of reasons (e.g., family circumstances, changed class placements, health issues). We anticipated fluid numbers of participants, especially considering the length of the study. Because our method of analysis required a complete data set (pretests and posttests), we could use scores from 308 out of the 407 students who were originally enrolled in the study. A comparison of completers and noncompleters (BAU = 45, EAI = 54) showed no systematic patterns of performance in pretest scores or student profiles (i.e., primary/service area) between groups and thus we are confident that the study sample is representative of the larger group.

CONCLUSIONS

The central instructional issue in the host study was how to deliver engaging instruction that valued procedural fluency (e.g., computing with fractions) while also giving attention to conceptual understanding (e.g., the meaning behind computational rules). The EAI materials were specially designed to deepen the understanding of students' conceptual knowledge of fractions. According to math educators such as Ma (1999), deep understanding of fundamental math concepts has been lacking among many American students, and this can be attributed in part to the surface-level knowledge of their math teachers. Detailed error analysis conducted within the context of a large intervention study provides important clues about how to make additional revisions to our teacher training and instructional materials so teachers receive the background knowledge they need and more students benefit.

We are encouraged by these findings because students need to become proficient with fractions if they are to pursue additional mathematics (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Wilson, 2009). Students who cannot master fractions will likely have difficulty learning algebra (Crawford, 2002) because the procedures used in adding and subtracting rational expressions are similar to those for adding and subtracting fractions.

Based on what we have learned in this study from teachers and their students, we will incorporate into the new version of EAI even more hands-on problems and richer problem representations as investigators suggest (e.g., Fazio & Siegler, 2011). The new version of EAI will also include more innovative ways for students to demonstrate their understanding in more sensitive assessment tools, which are key principles in Universal Design for Learning (Center for Applied Special Technology, 2011). Our challenge is to develop powerful contexts for learning and assessing fractions because as Ceci and Roazzi (1994) point out, "we cannot conclude that children lack certain cognitive abilities just because they do not exhibit them in a given context" (p. 93). Our challenge is to design new contexts to help those students who learn in diverse ways.

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