CHAPTER TWO

The Use of Concrete Experiences in Early Childhood Mathematics Instruction

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Abstract

Addressed are four key issues regarding concrete instruction: What is concrete? What is a worthwhile concrete experience? How can concrete experiences be used effectively in early childhood mathematics instruction? Is there evidence such experiences work? I argue that concrete experiences are those that build on what is familiar to a child and can involve objects, verbal analogies, or virtual images. The use of manipulatives or computer games, for instance, does not in itself guarantee an educational experience. Such experiences are worthwhile *if* they target and further learning (e.g., help children extend their informal knowledge or use their informal knowledge to understand and learn formal knowledge). A crucial guideline for the effective use of concrete experience is Dewey's principle of interaction—external factors (e.g., instructional activities) need to mesh with internal factors (readiness, interest). Cognitive views of concrete materials, such as the cognitive alignment perspective and dual-representation hypothesis, provide useful guidance about external factors

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but do not adequately take into account internal factors and their interaction with external factors. Research on the effectiveness of concrete experience is inconclusive because it frequently overlooks internal factors.

Piaget's (1936) stages of development helped popularize the use of concrete experiences and manipulatives in early childhood education. If young children were capable of only concrete thinking, then it made sense to focus on "real math" (everyday experiences or objects), not symbolic mathematics that required abstract thinking. Manipulatives—objects purposely structured to more transparently embody a mathematical idea (e.g., relation or procedure) than real objects—were seen as useful tool for introducing mathematics in a developmentally appropriate manner (Mix, 2010). Although young children are now generally considered more capable of abstract thinking and concepts, manipulatives remain a mainstay of early childhood and elementary mathematics education (McNeil, Uttal, Jarvin, & Sternberg, 2009). This chapter addresses four questions: What is concrete? What is a worthwhile concrete experience? How can concrete experiences be used effectively in early childhood mathematics instruction? Is there empirical evidence that such experiences work?

1. WHAT IS CONCRETE?

Theorists have proposed that mathematical thinking progresses from concrete to abstract along basically two different dimensions. One dimension can broadly be characterized as moving from perceptual-motor actions to mental actions. Bruner (1966), for instance, proposed that development involved adding new modes of thinking to previous capabilities. Young children can represent knowledge in terms of motor actions (enactive phase). More advanced children add the capability of representing knowledge as mental pictures involving key characteristics only (iconic phase). In time, children represent ideas using words or written symbols that do not resemble the idea (symbolic phase). Steffe and Cobb (1988) proposed that mathematical thinking moves from a dependence on representations of physical objects or movements triggered only by perceptual input to a reliance on mental objects or actions (concepts evoked independently of perceptions). The educational implication of such perceptual-motor to mental action views is that instruction should start with concrete experiences that involve physically manipulating objects (e.g., create "muscle memory").

An alternative way of conceptualizing the concrete-to-abstract progression is from specific to general thinking. For instance, Resnick (1992) conceptualized concrete-to-abstract development as moving from local (e.g., context- or object-specific) concepts to general concepts (e.g., broad generalizations applied or regardless of context). Put differently, whereas concrete knowledge, which is based on personnel experience, typically can be applied to familiar situations, abstract knowledge, which is based on generalizations, is applicable to unfamiliar problems, as well as familiar situations (Bisanz, Watchorn, Piatt, & Sherman, 2009; Prather & Alibali, 2009). From this perspective, concrete experiences are those that build on young children's existing informal and familiar knowledge gleaned largely from meaningful everyday experiences. Abstract is the formal symbolic knowledge that is general and applicable even beyond personally meaningful contexts.

Each of these perspectives of concrete-to-abstract development seems to apply to particular aspects or levels of development. That is, in some cases, physically acting on objects or physical movement may be needed to promote more advanced thinking (Mix, 2010). In other cases, the particular mode for prompting reflection, insight, and generalization—whether it be physical object, virtual image, verbal analogy, or formal (written) symbols—may be unimportant as long as the context can be related to familiar knowledge. Consider the development of early addition.

- Level 1. Initially, children may need to literally see, for example, one item added to two to make three items to construct a basic understanding of composition or addition. Once such a basic understanding of addition is constructed, children may look for additional examples such as decomposing ••• into •• and • or noticing that two objects and two more make four (Baroody & Rosu, 2006).
- Level 2. About 2.5 years of age, children can solve nonverbal addition (e.g., two items placed on a mat and then hidden and one more item placed next to the mat and then moved under the mat) using a mental representation involving, perhaps, a mental or iconic image (Huttenlocher, Jordan, & Levine, 1994).
- Level 3. Several years later, children solve equivalent word problems—symbolic problems in a verbal format set in a familiar context (Jordan, Hanich, & Uberti, 2003).

- Level 4. Later yet, children solve equivalent formal symbolic problems in a verbal format ("How much is two and one"; Jordan et al., 2003) or a written format (e.g., $2+1$) without a context.[a]

Children at Levels 1 and 2 could be considered as having *sensory-concrete knowledge* and in need of sensory input to make sense of addition or solve for the sum (Clements, 1999; Sarama & Clements, 2009a). Level 3 children, who can solve word problems but not symbolic problems, would have *integrated-concrete knowledge* in they can relate (assimilate) the symbolic language of a word problem to familiar knowledge. Level 4 children are operating at the integrated-concrete knowledge level *if* they must relate the formal expression $2+1$ to familiar knowledge to make sense of the problem and devise or select a strategy for solving it. Level 4 children who do not have to relate the symbolism to a meaningful context to do so are operating at an entirely symbolic level and as having *symbolic knowledge*.

The distinction between sensory-concrete and integrated-concrete knowledge is important because "concrete" is often equated exclusively with physical objects or movements. Consistent with the perceptual-motor to mental actions perspective, physically real materials or experiences may be necessary for novice learners or developmentally less advanced children—especially presymbolic or prelanguage infants and toddlers. However, consistent with specific to general perspective, the same may not true of developmentally more advanced children. For such children, an interpretation of concrete experiences would *not* necessarily exclude using verbal analogies, pictures/diagrams, or virtual-based activities as a basis for instruction, *if* it built on a child's existing knowledge (Baroody, 1989; Clements & McMillen, 1996). There is substantial evidence that children's familiar informal knowledge (including verbal counting-based representations) plays a key role in understanding and learning formal systematic knowledge

[a] Doug Clements (Personal Communication, 1/5/2017) indicated, "We get children to solve verbal (but still abstract) 'two and one more' problems the same time as word problems." According to the specific-to-general hypothesis, the order in which the word problems (WP) and verbal symbolic problems (VSP; e.g., "two and one more") are presented should matter. Specifically, presenting WP should induce better performance on VSP than in VSP-first condition, because the relatively sterile VSP context could be related to familiar and meaningful situations evoked by the WP (i.e., the previously introduced meaningful contexts would give meaning to relatively abstract statements such as "How much is two and one more?"). Moreover, the difference would be particularly pronounced if VSP were even more sterile as those used by Jordan and her colleagues (e.g., "How much is two and one?"). That is, the presence or absence of the term "more" may be important because its presence may better connect a VSP with children's existing informal change add-to understanding of addition (e.g., an initial collection of two is made larger by one *more* item). Clearly, though, these two hypotheses need to be empirically tested.

(Ginsburg, 1977; Ginsburg, Klein, & Starkey, 1998). Verbal analogies—relating unfamiliar symbols to familiar experiences—can be invaluable in understanding formal symbols (e.g., relating the minus sign to "taking away," relating the idea of angle to "the amount of turn," or introducing fractions such as 2/3 in terms of the familiar experience of fair sharing: two pizzas shared fairly among three children; Baroody & Coslick, 1998). Diagrams can be useful in distinguishing types of addition and subtraction and choosing an appropriate strategy (Jitendra et al., 2013). Young children clearly benefit from, for example, the careful use of virtual or computer-based instruction (Clements & Nastasi, 1993; Clements & Sarama, 2003, 2004, 2012). Indeed, Sarama and Clements (2009a) detail seven ways virtual manipulatives afford greater integrated-concrete learning than do physical manipulatives.

However the concrete-to-abstract continuum is conceptualized, the evidence does not support the view that children's thinking is concrete across mathematical domains and adults' thinking is abstract (McNeil et al., 2009). A child's thinking may be relatively abstract in one area such as cardinal numbers (numbers that represent the total of a collection) but relatively concrete in another such as subtraction or the composition of shapes.[b] Indeed, a 3-year old may have a relatively abstract understanding of (the cardinal number) "two" but a relatively limited and inexact understanding of "ten." This perspective underscores the need for educators to understand a child's developmental level within a particular domain when considering whether an instructional activity is sufficiently concrete. This point is underscored in the next two sections of this chapter.

2. WHAT IS A WORTHWHILE CONCRETE ACTIVITY?

In his analysis of why Progressive Education was unsuccessful, Dewey (1963) concluded that instruction should not consist of a conglomeration of activities without clear educational purposes. For example, concrete experiences such as everyday calendar activities are often used without a clear, justifiable goal, or good effect (Clements, Baroody, & Sarama, 2013; National Research Council, 2009). Similarly, using the same M&M activity to teach probability for four consecutive grade levels without modification

[b] The flipside is that even adults can frequently benefit from having unfamiliar aspects of formal mathematics related to their familiar knowledge and can benefit from reflecting on manipulative models (McNeil et al., 2009).

or extension is not likely to result in new learning. Manipulatives used in isolation (e.g., left in a learning center without guidance or feedback) are not likely to achieve worthwhile instructional goals (Uttal, 2003).

Dewey (1963) argued that teachers must avoid *miseducative experiences* (activities for the sake of activity and that may actually impede development) and strive to provide educative experiences (experiences that lead to worthwhile learning or a basis for later learning; see also Sophian, 2004). Concrete experiences and manipulative use need to be guided by a plan for achieving useful instructional goals. According to Dewey, educative experiences result "from an interaction of external factors, such as the nature of the subject matter and teaching practices, and internal factors, such as a child's [developmental readiness] and interests" (Baroody, 1987, p. 37). Teachers, then, must ensure that external and internal factors mesh.

Piaget's principles of meaningful learning illustrate how the external factor of introducing formal symbols needs to fit the internal factor of a child's developmental level. His principle of assimilation specifies that new information is understood and interpreted in terms of existing knowledge. For example, introducing the formal symbolism for subtraction by relating $5-3=?$ to "What is five minus three?" may not make sense to children who do not know the meaning of the formal term "minus." As a result, even though they might be able to solve an analogous word problem, students may be unresponsive, simply guess, or assimilate the formal expression to what do they understand (i.e., assume it represents addition and answer "8"). In contrast, relating $5-3=?$ to their informal knowledge of subtraction (e.g., "Five candies take away three candies leaves what?") can enable them to understand and learn that the minus sign means subtract. Piaget's moderate novelty principle prescribes that information that is neither too familiar nor unfamiliar will pique interest and engagement. For instance, for a child who can already count to thirty-one and read the written numbers to 31, the task of counting up to today's date on a calendar can be readily assimilated but is uninspiring. In contrast, the question "How many tens does 31 show?" asked of the same child without grouping and place-value knowledge cannot be assimilated to prior knowledge and is, thus, incomprehensible and uninteresting. However, a novel compare-type problem that builds on the child's familiar knowledge (e.g., "If today is December 5 and our field trip is December 11, how many more days until our trip?") might make sense and spark interest in solving it. Dewey's principle of interaction is perhaps even more closely akin to Vygotsky's (1962) zone of proximal development: The domain-specific—or even activity-specific—competence a child can exhibit with careful and minimal assistance.

Hypothetical learning trajectories (HTLs) can serve as an invaluable tool for implementing Dewey's (1963) principle of interaction—for ensuring external factors (e.g., instructional goals and activities) effectively interact with the internal factor of a child's developmental level. HLTs—which include theoretically and empirically based developmental levels or steps—(a) define goals for meaningful instruction; (b) provide assessments for identifying a pupil's current developmental level and the next (developmentally appropriate) instructional step; and (c) detail instructional activities to help the pupil achieve this next level (Daro, Mosher, Corcoran, & Barrett, 2011; Sarama & Clements, 2009b, 2009c). HTLs are invaluable in readiness or remedial instruction in that they lay out in sequential order the developmental prerequisites and accompanying instructional activities needed to achieve a target level of instruction. In effect, they can provide guidance in what scaffolding children at a particular developmental level need.

Consider, for instance, the case of 3-year-old Alison (Baroody, 1987). Her father asked, "What number comes after nine when we count?" Stumped, Alison did not reply. Her mother intervened by asking, "What comes after "one, two, three, four, five, six, seven, eight, nine." Alison quickly responded, "Ten." This scenario makes sense in terms of a verbal-counting HLT summarized in Table 1. Alison was at Level 2 in Table 1 (the unbreakable chain level) and had to use a "running start" (count from "one") to indicate the next number in the counting sequence (e.g., What comes after nine?). As children become familiar with the counting sequence, they achieve the breakable chain level (Level 4A in Table 1) and no longer need a "running start" to determine the number that follows another. Instead they can access the counting sequence at the given number and state its successor (e.g., immediately indicate that "ten" comes after "nine"). As her father's question (What comes after nine?) was two levels above Alison's developmental level, she may not have understood the question and certainly did not have the means for answering. Her mother's question (What comes after one, two…nine?) was at her developmental level and thus permitted her to apply what she knew to solving the novel task of answering a number-after question. According to the HLT in Table 1, instruction that might help a child move from Level 2 (the unbreakable chain level) to achieving Level 4 (the breakable chain level) is the transitional Level 3 (determining the number-after with increasingly abbreviated running starts).

Table 1 An Example of a HLT: Verbal-Counting Development

HLT Level	**Conceptual Basis (Fuson, 1988)/Comments**
Level 1: Sing-songer	*String level*: Knows the first few number words as an undifferentiated string of sounds (multisyllabic sound): "onetwothree."
Level 2: Counter from "one"	*Unbreakable chain level*: Differentiates among number words of the sequence but does the have the flexibility to reproduce the sequence other than starting with one (i.e., cannot enter the sequence at any point and count). Thus, to determine the number after five, a child would have to use a running start—that is, count from 1 to 5 (e.g., count: "1–2–3–4–5, 6").
Level 3: Number-after citer with abbreviated running starts	*Transition between unbreakable chain level and breakable chain level*: In order to help children at Level 2, a teacher or parent could pose number-after questions with an increasingly abbreviated running start: "What comes after 2, 3, 4, 5?"; then "What comes after 3, 4, 5?"; and finally "What comes after 4, 5?"
Level 4A: Number-after citer without a running start	*Breakable chain*: Familiarity with the counting sequence permits the child to enter the counting sequence at any point and specify the next number. Thus, a child no longer needs a running start or even an abbreviated running start to determine what comes after any given number (e.g., a child answer the question "What comes after 5?" by saying "5, 6," or "6").
Level 4B: Counter from *n*	Once a child can determine the number after another, they can continue a count from any *n* (e.g., in response to the request "Start with 5 and count," a child counts "5, 6, 7, 8, 9…" or "6, 7, 8, 9…").
Level 5: Counter-on using patterns	*Transition between breakable chain level and numerable chain level*: At this level, a child can count on a specified number of times intuitively—usually only a few times—by means of a pattern (e.g., in response to start with 5 and count two more times, a child counts, "Five, SIX, seven").
Level 6: Counter-on keeping track	*Numerable chain level*: Realizes that counting words themselves can be counted. Requires a keeping-track process to determine when to stop the counting-on process (e.g., for start with five and count four more times: "5, 6 is 1 time, 7 is 2 times, 8 is 3 times, *9* is *4 times*").

The careful use of HLTs can better ensure that concrete experiences at the preschool (age 3–5 years) and primary (grades K to 3) levels are educative. A key cause of mathematical learning difficulties in school, especially among children at risk due to poverty, are gaps in their readiness (informal) knowledge. For preschoolers without rich learning opportunities, informal knowledge can be seriously incomplete or narrow (Baroody, 1987). Identifying where a child is on a HLT can help preschool or primary-level teachers provide the educative experiences that remedy informal deficiencies and extend existing informal knowledge strengths. Another key cause of mathematical learning difficulties in school is a gap between school mathematics instruction and children's existing knowledge (Ginsburg, 1977). Identifying where a child is on an HLT can help preschool to primary-level teachers select the concrete experiences that connect formal mathematics to the child's informal knowledge and enable the child to assimilate symbolic school instruction successfully.

3. HOW CAN CONCRETE EXPERIENCES BE USED EFFECTIVELY IN EARLY CHILDHOOD MATHEMATICS INSTRUCTION?

Since at least the time of Resnick's (1982; Resnick & Omanson, 1987) failed attempt to use base-ten blocks to foster an understanding of base-ten, place-value concepts and skills, it has been clear that manipulatives do not guarantee meaningful learning (Ball, 1992; Baroody, 1989; Clements & McMillen, 1996). That is, children may not spontaneously link concrete models or what they learn from working with concrete objects to other representations, particularly written representations (Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009). How to link a concrete model to symbolic mathematics to effectively illuminate an intended referent remains a critically important challenge for early childhood mathematics instruction. Discussed next are cognitive perspectives, a perspective based on Dewey's interaction principle and learning trajectories, and instructional strategies for ensuring an educative experience.

3.1 Perspectives From Cognitive Psychology

3.1.1 Cognitive Alignment Framework

A key factor in linking concrete models effectively to targeted referent knowledge may be how well the model embodies the relations, steps, or other attributes of the referent knowledge. As Laski and Siegler (2014,

p. 861) noted, "The cognitive alignment framework proposes that the more precise the alignment among the desired mental representation, the physical materials being used to promote learning, and the activities that direct learners' thoughts and actions during the acquisition process, the greater the learning is likely to be." See Clements and Battista (2000) for a discussion on cognitively aligning software.

Consider, for example, children's relative magnitude judgments (see Baroody & Purpura, 2017, for a detailed discussion). Research indicates that initially children do not recognize that number words represent relative magnitude. Theoretically, this would result in assigning all the numbers 1–10 the same level of magnitude or point on a number line (Frame A in Fig. 1). In time, they recognize that smaller, familiar numbers have distinct relative magnitudes but their sense of relative magnitude with larger, unfamiliar numbers is fuzzy. For instance, in finding out that her father was 42, 5.5-year old, Arianne asked in all innocence, "Is that close to a hundred, Daddy?" For younger children, the first few numbers may have distinct relative magnitudes, but even somewhat larger numbers such "six" or "seven" may simply be viewed as "many" or "a lot" and almost indistinguishable from "ten." The result is a response pattern that is logarithmic even for numbers 1–10 (see Frame B in Fig. 1). Children then see each succeeding number in the counting sequence from 1 to 10 as representing a larger amount (a linear representation of magnitude comparisons; Frame C in Fig. 1) and eventually as exactly one larger than its predecessor (i.e., a linear representation with a slope of 1; Frame D in Fig. 1). Research also has indicated that playing a linear numerical board game (*The Great Race*), which directly models a mental left-to-right number list for counting from 1 to 10 (see Fig. 2; Resnick, 1983) significantly improved the linearity of children's magnitude judgments and other aspects of early numeracy (Ramani & Siegler, 2008; Siegler & Ramani, 2008). Indeed, playing *The Great Race* with a linear board significantly produced better results than doing so with a circular board (Ramani & Siegler, 2011; Siegler & Ramani, 2009). Unfortunately, methodological issues cloud what conclusions can be drawn from such research (again see Baroody & Purpura, 2017, for a detailed discussion).

Similarly, playing a semilinear numerical board game using a 0-to-100 hundreds chart (*Race to the Moon*) significantly improved the linearity of children's magnitude judgments to 100 with a counting-on, but not a counting from "one" process (Laski & Siegler, 2014). This was attributed to the fact that, unlike the count-from-1 training, the count-on training directs attention to the numbers in the squares by requiring a child to explicitly name

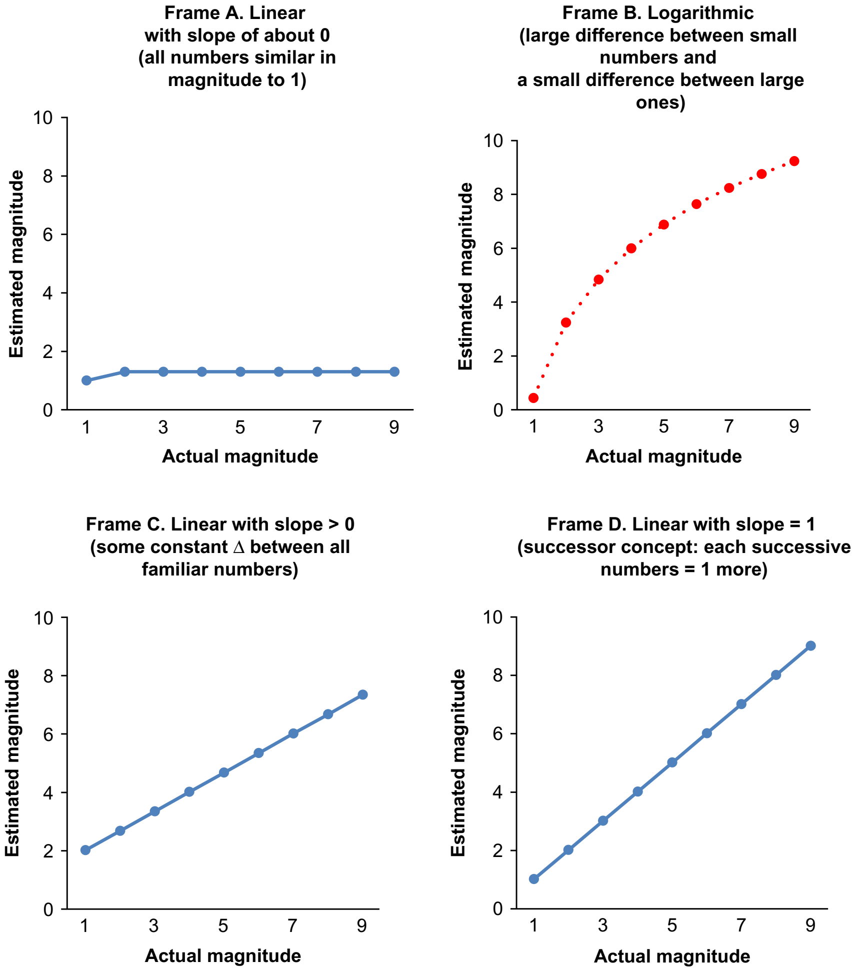
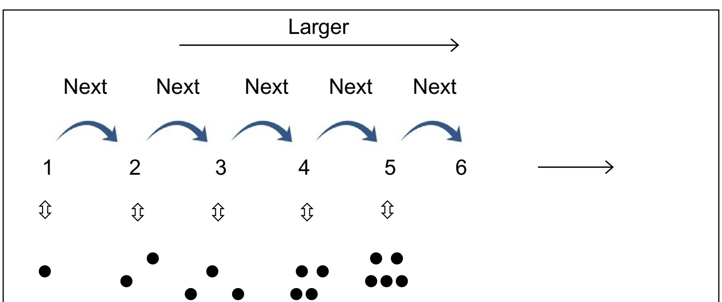


Fig. 1 Different response patterns for magnitude estimation task (where does *n* fit on 1–10 number line?).



Note. As depicted, the figure represents a child who associates specific quantities with the number words “one” to “six” and who understands the increasing magnitude principle—a number further along in the sequence represents a larger quantity. The representation for a child who has discover the successor principle would include the label “next is one more” (instead of “next”).

Fig. 2 A representation of magnitude relations of the counting sequence: a “mental number list” (based on fig. 3.1 in Resnick, 1983).

them. The resulting encoding process "provides the data needed to correlate numbers in different squares with the visuospatial, auditory, kinesthetic, and temporal cues that accompany reaching that square. In contrast, the counting-from-1 procedure does not require, or even encourage, attention to the numbers in the squares" (p. 861). Although counting-on 1 or 2 especially (e.g., moving two spaces from 37 by simply counting "38, 39") may not require an explicit keeping-track process, Baroody and Purpura (2017) hypothesized adding a process (e.g., counting-on "38 is 1 more, 39 is 2 more") might significantly enhance the efficacy of *Race to the Moon*.

Researchers and educators need to consider carefully what instructional goals they are trying to achieve with particular concrete materials and, as needed, adapt the process by which children interact with the materials. An adaptation of Wynroth's (1986) *Slides and Ladders*, the *Grid Race* (Baroody & Coslick, 1998)—like *Race to the Moon*—utilizes a 0–100 hundreds chart. The goal of *Slides and Ladders* or the *Grid Race*, though, is to foster discovery of mental-addition (and subtraction) rules, such as the decade-after rule (adding 10 to a decade results in the next decade; e.g., $50+10=60$) or the add-10 to a two-digit number rule (the sum is the next decade with the same ones digit; $57+10=67$), and a partial-decomposition mental-addition strategy (e.g., $57+36=57+30+6=87+6=93$). For such goals, the 0-to-5 spinner used in *Race to the Moon* must be replaced by one of several different decks of cards designed to emphasize a particular regularity. For fostering the decade-after rule, the deck would be composed of 0 or 10 cards. For promoting the add-10 to a 2-digit number rule, the deck would be composed of some cards with a number from 1 to 9 and an abundance of cards with 10. For promoting the partial-decomposition strategy, the deck would consist of cards with two-digit numerals (e.g., 14 and 23). For all three versions, the card a player draws would determine the number of spaces a child moves on the hundreds board. For instance, if a child is on space 57 and draws a 10, the child would count by ones 10 spaces to move to space 67. Wynroth's (1986) rationale was that children would discover a shortcut for the laborious process of counting 10 spaces from 57 to 67, namely you can simply move vertically up one row in the table whenever you add 10. Theoretically, this single vertical-move shortcut provides a basis for discovering the decade-change rule for adding 10 to multidigit numbers (e.g., $57+10$ is the next decade, 60, plus the same number of units, seven).

If Laski and Siegler's (2014) hypothesis that count-on procedure requires an encoding process that facilitates attention to number and numerical relations is correct, then, a version of the *Grid Race* that requires a child to specify the problem and outcome (e.g., "57 plus 10 is 67") should be more efficacious in helping children discover mental-addition (and subtraction) shortcuts than a version in which children do not relate their move to a verbal equation. Indeed, encouraging children to use a keeping track process (e.g., "57, 58 is 1 more, 59 is 2 more, 60 is 3 more…67 is 10 more") might be particularly motivating for finding the vertical-jump shortcut or the same ones—next decade rule.

3.1.2 The Dual-Representation Hypothesis

A rationale for using hands-on manipulatives is that the activity of physical exploration can be progressively internalized to form conceptual understanding. Intuitively making manipulatives more perceptually rich should make them more attractive and interesting and help children focus better on a task. Recent research raises questions about these assumptions and even indicates that such physical manipulation and perceptual enrichment of manipulatives may, in many cases, distract from learning (Kaminski, Sloutsky, & Heckler, 2008, 2009, 2013; McNeil & Jarvin, 2007; McNeil et al., 2009).

According to the dual-representation hypothesis, successful use of a symbolic artifact entails mentally representing both its physical and abstract nature. However, such a dual representation is challenging to young children because they are less likely to recognize that a new artifact can have symbolic import. Moreover, they have difficulty simultaneously formulating two active (concrete and abstract) representations of an artifact. "Manipulations that increase the salience of a [model] as a concrete object should decrease children's appreciation and use of it as a [representation], whereas manipulations that decrease children's focus on the object properties of the model should increase their ability to focus on what the [stand-in] represents" (Uttal et al., 2009, p. 157). Consider four examples.

1. DeLoache (2000) hid a small toy dog in a miniature replication of a room and then asked a child to use this model to find a large stuffed dog hidden in the regular size room. In one study, 3-year olds allowed to physically interact with the model during a free play prelude were less successful than those who did not play with the model in using the representation of the hiding place and regular size room to find the hidden larger dog. In another study, 2.5-year olds who viewed the model dog being hid from behind a window were more successful than their peers who had physically interacted with the small toy dog. Marzolf and

DeLoache (1994) found that using a picture as a "map" was superior to using a three-dimensional model. Uttal et al. (2009) concluded that physical interaction interferes with viewing a manipulative as a model.

2. Sarama and Clements (2009a) concluded that computer-based manipulatives may facilitate generalization and transfer more effectively than physical manipulatives. Uttal et al. (2009) concluded, "Using computer-based manipulatives reduces the demands of dual representation, enabling children to focus less on the on-screen objects themselves and more on the connections between the manipulatives and mathematical representations" (p. 138).
3. The results of Uttal et al. (2013) appear to support the view that children have difficulty connecting concrete and abstract representations. In Experiment 1a, these researchers found that manipulative-based instruction on multidigit subtraction with second graders did not transfer to using a written algorithm and instruction on a written algorithm did not transfer to block models.
4. Experiment 1b by Uttal et al. (2013) corroborated the view that highly attractive manipulatives or concrete objects may make it particularly difficult to link a concrete model with an abstract referent. Specifically, second graders who used regular Digi-Blocks designed to teach the base-ten system and how to solve two-digit subtraction seldom used the manipulative inappropriately, whereas most of their peers who used brightly colored and patterned blocks were side tracked.
5. Kaminski et al. (2009) had 11-year olds learn a mathematical concept either concretely with perceptually rich symbols or abstractly with symbolic models. Although the concrete model made learning easier, it resulted in less transfer, whereas the symbolic model made learning harder but resulted in greater transfer. The implication is that concrete representations may limit the generalizations pupils can derive.

Clearly, educators need to use manipulatives thoughtfully, not laden representations with frivolous or unnecessary details, and consider carefully the advantages and disadvantages of manipulatives in various forms. Barmby, Harries, Higgins, and Suggate (2009) reasoned that groups-of or number-line models of multiplication obscure the operation's commutative property. Indeed, it is plausible (based on the authors' informal observations) that children may also better exhibit understanding of multiplicative commutativity in the sterile context of symbolic expressions (e.g., 9×7 and 7×9) than with even rectangular-array models (e.g., 9 rows of 7 items or 7 rows of 9 items) or area models (e.g., rectangles 9- $\times$ 7-linear units and

7- × 9-linear units), because the concrete model of 9 × 7 and 7 × 9 in each case look different.

However, the evidence supporting the dual-representation hypothesis is not clear-cut.

1. A maxim in mathematics education is: If manipulatives are to be used effectively as a teaching tool, provide the manipulatives to children for a period of free exploration. With their curiosity satisfied, they will then be ready to use the manipulatives as intended by the educator—as a tool for prompting reflection and learning. Children in the DeLoache (2000) study were given up to 10 min of free play with the model materials "or any time after 5 min [if] the child seemed to be losing interest" (p. 334). It is unclear whether—consistent with the maxim just described and inconsistent with the dual-representation hypothesis—the children who lost interest performed better than those who did not. It also seems important to distinguish between child-initiated activities and adult-imposed tasks, because children may exhibit a competence on the former not revealed on the latter (Baroody, Lai, & Mix, 2006). Requiring a young child to engage in an adult-imposed task is challenging enough without doing so while the child is still engaged in an interesting child-initiated activity. A way of making adult-imposed testing (or instruction) more palatable and motivating to young children is to make it as similar as possible to child-initiated activities (e.g., present it as a game). As a transition from free play to the experimental task in the DeLoache (2000) study, "the experimenter indicated they would 'do something different now'" followed by an "extensive orientation to the [experimental] task" (p. 334). A perhaps more engaging and potentially more productive transition might be introducing the experimental task as the "Hiding Game." The tester could first show the large room, explain that a large stuffed dog is hidden somewhere in the room, and then offer a clue or hint to the hiding place with toy dog and miniature room. Such a transition would help evaluate whether failure on the standard task when preceded by free play has more to with motivational issues than cognitive limitations. Even if dual representation proves to be a challenge to 2.5-year olds over a variety of conditions (including motivating situations), symbolic representation ability and the ability to discern adult expectations improves with age. Thus, it is not clear that the dual-representation hypothesis is (as) applicable to 7-year olds encouraged to show how base-ten blocks could be used to solve a two-digit subtraction problem involving renaming in the Uttal et al. (2013) study.

2. Direct comparisons of the use of virtual and physical manipulatives are relatively rare, and the advantage of virtual manipulatives may be due to a number of factors other than that they are less distracting than physical manipulatives, which can be touched. See Sarama and Clements (2009a) or Clements and Sarama (2012) for a list of advantages of virtual manipulatives over real manipulatives.
3. A plausible alternative explanation for the results of Uttal et al.'s (2013) Experiment 1a is that the lack of transfer between modes was due to the ineffective nature of the instruction. Both manipulative- and written-based instruction focused on procedures, not the conceptual rationale for the multidigit subtraction algorithm and spanned only two sessions. Brief, meaningless instruction—whether with concrete materials or written symbols—is not likely to promote flexible application of knowledge (Baroody, 1989).
4. Moreover, the key issue may not be as simple as "gaudy vs plain" (as suggested by the result of Uttal et al.'s (2013) Experiment 1b. Petersen and McNeil (2013), for example, found that perceptually rich objects facilitate performance when counting tasks involved unfamiliar objects but hindered performance when the tasks involved familiar objects. Although it would seem that the dual-representation hypothesis would predict that novelty and perceptual richness would combine to render a jazzy, unfamiliar object relatively useless as a learning tool, the authors speculated that the increased attention such objects attracted was channeled into understanding the task, whereas, with familiar objects, it activated existing knowledge of the object and served to draw attention away from the task.

   Perhaps more importantly, there may be value in making a model perceptually rich, interesting, or attractive *if* it helps draw attention to a critical relation or otherwise promotes meaningful learning. For example, the bright colors and patterns of Digi-Blocks in the Uttal et al. (2013) study did not serve any pedagogically useful purpose (e.g., the blocks did not help underscore the grouping or place value ideas underlying the two-digit subtraction procedure). In contrast, using one bright color to represent groups of 10 and a different bright to represent units might help to underscore a grouping concept and the conceptual rationale for regrouping.
5. There are plausible alternative explanations to Kaminski et al.'s (2008, 2009, 2013) evidence. One is the common problem in instructional/curriculum evaluation of using a dependent measure (the test of transfer) that better matched the abstract learning condition. The concrete

training involved showing participants two measuring cups and asking them to determine what would be "left over" after combining the content of one cup of the cups (e.g., 1/3 + 1/3 results in 2/3 leftover; 2/3 + 2/3 results in 1/3 leftover[c]). The abstract training involved generic shapes (e.g., ● + ● results in ◆; ◆ + ◆ results in ●). The transfer task involved "intentionally concrete and contextually rich" displays (e.g.,  +  results in ;  +  results in ), which theoretically should favor the concrete trainees (Kaminski et al., 2009, p. 153). However, whereas the concrete training involved logical reasoning about fraction addition, the abstract training might simply elicit memorization of rules (two circles makes a diamond and vice versa or simply two of a kind make one of a different kind). Whereas applying knowledge of fraction addition is readily apparent in the concrete condition, inducing arbitrary rules might be more challenging. This would account for why the concrete condition resulted in better learning during training. However, on the transfer task, fraction addition is irrelevant but the rule/rules memorized by rote in the abstract condition can be readily applied to two of the four test trials. Indeed, De Bock, Deprez, Van Dooren, Roelens, and Verschaffel (2011) found that the concrete learning context transferred to a similar concrete context and their qualitative analysis raised questions about Kaminski et al.'s (2013) interpretation of what students learned from the abstract training.

It is essential to keep in mind Piaget's caution that it is not the physical activity itself that is important but a child's reflection on the physical activity (Baroody, 1989; Sarama & Clements, 2009a). For research on learning beyond the sensory-concrete level, a focus on whether children touch objects or whether objects are perceptually rich is of secondary, or perhaps little, importance. The key issue is whether a manipulative is pedagogically meaningful—whether it engages children's thinking in ways that prompt reflection on ideas or issues targeted by an educator. That is, a "concrete" model is one that connects with what children already know and prompts them to reconsider and extend this existing knowledge.

[c] The concrete training seems confusing. For example, if the analogy is interpreted as 1/3 in the first cup is poured into the second cup containing 1/3, nothing is left in the first cup and 2/3 is left in the second cup. So, in this case, the answer (what is leftover) seems to reference what is in the *second cup* after combining the two cups. With 2/3 + 2/3, 1/3 would be left in the first cup and the second cup would have 3/3. So in this case, the answer (what is leftover) seems to refer to the *first cup*, not the second cup. An inconsistent analogy stacks the deck against finding a regularity.

Although children's reflection can and should be guided by a teacher or activity, it probably cannot be forced or imposed. As with instruction in general, imposing how to use manipulatives may be no more effective than imposing symbols and written algorithms on children (Baroody, 1989; Gravemeijer, 2002). As McNeil et al. (2009, p. 139) elegantly noted: Some evidence supports that "students may not actively process the consequences of actions on concrete objects unless they, themselves, are the directors of that activity (Martin, 2009) …" Future research must investigate the best ways for teachers to strike a balance between providing direction and allowing students to regulate and direct their own activity.

3.2 Perspective Based on Dewey's Interaction Principle and Learning Trajectories

Cognitive perspectives (the cognitive alignment framework and dual-representation hypothesis) focus on external factors. To better understand the effective use of concrete materials, it is essential to take into account the interaction of external and internal factors and children's development or readiness as embodied by HLTs. However closely a concrete representation aligns with a mathematical idea, it does not guarantee that it will be an appropriate or effective model for a child *if* the formal idea is developmentally too advanced. For this reason, children may need to begin with a model that is aligned with their existing developmental level of mathematical understanding and thinking and only approximates the formal mathematical idea. In effect, educational researchers and practitioners might better consider a series of concrete models that build on children's informal understanding and thinking and increasingly becomes aligned with the formal mathematical idea. In this subsection, the following five areas of early childhood mathematics education are analyzed in terms of Dewey's interaction principle and HLTs: patterning, a linear representation of the counting numbers and its hypothesized developmental prerequisites, using number lines to formally introduce magnitude comparisons and arithmetic, and base-ten place-value concepts and skills.

3.2.1 Patterning

In *Workjobs* published in 1972, Baratta-Lorton recommended using letters to label, identify, and discuss repeating patterns. For example, she suggested labeling ○□○□○□, up-down-up-down-up-down, blue-red-blue-red-blue-red, and long-short-long-short-long-short all ABABAB or, for short, AB patterns and ○□□○□□○□□, up-down-down-up-down-down-up-down-down, blue-red-red-blue-red-red-blue-red-red, and long-short-short-long-short-short-long-short-short all ABBABBABB or, for short, ABB patterns. Note that the short version serves to identify a repeating pattern's "core"—the basic elements that are repeated or part that repeats.

Doug Clements, who was a kindergarten teacher at the time, recalls being highly skeptical of Baratta-Lorton's (1972) recommendation to label patterns with letters because it struck him as too abstract. Nevertheless, he tried the letter-labeling method with his classes and became convinced of its value. The method is now a popular feature of early childhood mathematics programs, such as the *Building Blocks* program (Clements & Sarama, 2013).

Baroody (1993, p. 2–84) offered the following explanation for why the letter-labeling method may be effective:

> *"Labeling patterns can help children find repeating patterns and discover commonalities among such patterns … using letters can help students analyze and identify patterns. Moreover, coding patterns with letters gives children a convenient way of explicitly describing patterns … such letter codes can help children see that patterns constructed of different materials can* share the same structure *(emphasis added). Understanding that a pattern can be embodied in various, even different-looking, ways represents an important advance in children's mathematical thinking. There comes a dawning recognition that mathematics is a search for underlying structure that transcends appearances."*

Note that the last insight could arguably be classified as a really "big idea."

Recently, Fyfe, McNeil, and Rittle-Johnson (2015) adduced evidence that labeling concrete material abstractly with letters (e.g., "ABABAB") is significantly more powerful than doing so concretely with a physical characteristic such as color names (e.g., "blue-red-blue-red-blue-red") in fostering patterning among preschoolers. They concluded: "Using abstract language to describe patterns facilitates children's pattern abstraction" (p. 5) and was particularly beneficial when children correctly adopted the letter-labeling method.

Fyfe et al.'s (2015) results reinforce the point that manipulatives themselves do not guarantee learning (Ball, 1992; Baroody, 1989; Clements & McMillen, 1996) but that what is critical is how concrete materials are used. Specifically, they concluded that their results are consistent with the cognitive alignment framework (Laski & Siegler, 2014) and the dual-representation hypothesis (Kaminski et al., 2013; Uttal et al., 2009). Fyfe et al.'s (2015) research, though, raises four interesting questions that underscore the importance of Dewey's interaction principle.

1. Is there a critical age or developmental level for successfully introducing the letter-labeling method?

   Fyfe et al. (2015) found a significant age effect—older, but not younger, children tended to benefit. (Participants ranged in age from 3.6 to 4.9 years, average = age 4.4, SD = 0.4.) Perhaps there is a critical age that is necessary to benefit from the letter-labeling training. Research needs to address this potentially critical internal factor. Perhaps more likely, though, children may need to reach a certain developmental level before the letter-labeling method makes sense. Research is needed to ascertain the parameters of this critical internal factor—including any developmental prerequisites for understanding and using the letter-labeling method.
2. Where in a patterning HLT should the letter-labeling method be introduced?

   The learning trajectory detailed by Sarama and Clements (2009b) indicates that the ability to translate a pattern in one medium into another (e.g., translate a pattern comprised of geometric shapes into numbers or letters) develops relatively late (Level 4 in Table 2) and seems to imply teaching the letter-labeling method should not be taught before 6 years of age. Unclear, though, is whether Levels 1–3 are developmental prerequisites for learning the letter-labeling method or whether these levels might benefit from introducing this method first. Note that the Level 1 (Pattern Recognizer) activity of describing the pattern on a strip (e.g., square, circle, square, circle, square, circle, etc.) may not necessarily involve recognizing a pattern. A child could just name each shape in turn. Note that the Level 2 (Pattern Duplicator) activity of fixing or duplicating a pattern such as square, circle, square, circle, square, and circle may simply involve matching a shape on the strip with the shape of a block. So in a real sense, a child could duplicate a pattern by using a nonpattern-based process. The Level 2 activity of extending a pattern would seem to require at least an intuitive understanding of a repeating core. As the next question suggests, it may be particularly useful to introduce the letter-labeling method before introducing more complex patterns (Level 3 in Table 2).[d]

[d] Nicole McNeil (Personal Communication, 3/23/2017) noted that it would be interesting to compare musical sounds, such as da-dum, with the letter-labeling method to see if the former might help young children even more in identifying pattern cores.

Table 2 Selected Levels From the Patterning Learning Trajectory From Sarama and Clements (2009a)

Level Name	Description of Developmental Level
0. Preexplicit Patterner	Implicit pattern detection and use. Labels a visual, rhythmic, or other regularity a "pattern" (e.g., "My shirt has pattern")
1. Pattern Recognizer	Explicitly recognizes and labels a simple sequential repeating pattern (e.g., "I am wearing a blue, red, blue, red blue, red pattern")
2. AB Pattern Fixer, Duplicator, Extender	Can fill in a missing element of a ABABAB pattern, then copy it, and finally, extend it
3. Pattern Fixer, Duplicator, Extender of More Complex Repeating Patterns	Can fill in a missing element of, for example, ABBABBABB pattern; then copy it, and finally, extend it
4. Pattern Unit Recognizer	Identifies the core (smallest unit) of a repeating pattern and can translate patterns into new media (e.g., identify ○□○□○□ as an AB pattern)

3. Is the letter-labeling method effective with all types of repeating patterns?

As Fig. 3 indicates, the impact of the abstract letter-labeling method varied with the type of pattern. Specifically, it had the least impact with AB and ABC patterns, substantial impact with ABB and AAB patterns, and the greatest impact with the AABB pattern. With the concrete method of labeling patterns with physical characteristics, the AB pattern was at least somewhat more salient than the other patterns and, thus, provided less room for improvement. An AB pattern's saliency may be due to its simplicity—a core of two different elements that alternate. In contrast, ABB, AAB, and AABB patterns have a core of two different elements that repeat in a more complex manner, and the ABC pattern have a core of three different elements. The letter-labeling method may have a limited impact on ABC pattern recognition perhaps for the reasons discussed next.

4. Are the results sustainable and generally applicable?

The Fyfe et al. (2015) study involved a single training session of 20 min. So it is unclear whether the results would persist over time and how much instruction is needed to maintain gains. Importantly,

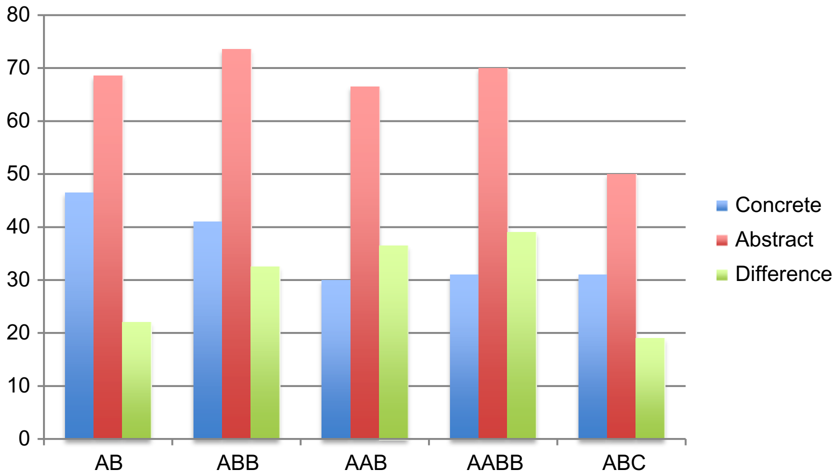


Fig. 3 Summary of Fyfe et al.'s (2015) training experiment on patterns.

unclear is whether the brief training would achieve the hallmark of meaningful instruction, namely spontaneous and successful application of the letter-labeling method to a new pattern, especially less salient regularities such ABC patterns, and how much instruction is needed to promote such transfer. As the small sample ($n=64$) involved middle class children, will the abstract letter-labeling method be as successful with other less-advantaged populations of children (e.g., preschoolers from low-income families, or ELL children)?

3.2.2 A Linear Representation and Its Hypothesized Developmental Prerequisites

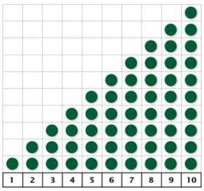
Table 3 illustrates a HLT for children's relative magnitude judgments of the number words 1–10. A series of increasing abstract models that take into account developmental level may promote the levels of this HLT more effectively than relatively abstract linear numerical board games such as *The Great Race*.

3.2.2.1 Increasing Magnitude Principle

A plausible developmental prerequisite for making linear magnitude judgments is the increasing magnitude principle: numbers coming later in the counting sequence represent larger quantities than predecessors (e.g., nine is more than four because it comes after four when we count; Level 2 in Table 3). A cardinality chart—separate numeral cards with interlocking blocks above each numeral to represent relative quantity—might be even more effective than the number-list model alone (e.g., *The Great Race*) in

Table 3 HLT of Mental Magnitude Comparisons

Level 0: Number words without magnitude	Children learn the cardinal meaning of the first few number words—provides a basis for *verbal subitizing* (e.g., "two" represents pairs of objects) but does not realize that "two" is more than "one").
Level 1. Subitizing-based relative magnitude	Verbal subitizing provides a basis for recognizing that "two is more than one" and "three is more than two" (ordinal meaning of the first few number words), because the child can directly see the differences in the relative magnitude to two small collections.
Level 2. Increasing magnitude principle—relative magnitude of distal numbers	The insight that the order of the number words for small numbers (one, two, three) reflect increasing magnitude provides a basis for the generalization that the counting sequence represents increasingly larger quantities—the increasing magnitude principle. The principle and *familiarity with the counting sequence* and *the increasing magnitude concept* provide a basis for comparing two numbers at obviously different positions in the counting sequence (i.e., to make gross comparisons of 2 or 7, 10 or 3, 9 or 5, and 4 or 8).
Level 3. Mental comparisons of neighboring or successive numbers (number after equals more)	Once children become sufficiently familiar with the counting sequence, they can enter the sequence at any point and *specify the next number* instead of always counting from one. This permits children to apply the *increasing magnitude principle* to the task of *mentally comparing even close numbers*, such as the larger of two neighboring numbers (e.g., immediately indicate that eight is more than seven because it follows seven in the counting sequence).
Level 4. Successor principle (number after equals 1 more) and a linear representation of the counting sequence	Verbal subitizing enables children to see that "two" is exactly one more than "one" item and that "three" is exactly one more than "two" items, and this can help them induce the *successor principle* (each successive number in the counting sequence is exactly one more than the previous number). The *successor principle*, in turn, enables children to view the counting sequence as n, $n+1$, $[n+1]+1$, … (the *positive integer sequence*)—a linear representation of number.

Most concrete (most directly models an informal counting-based view of the number sequence)	⟷	Most abstract (least directly models an informal counting-based view of the number sequence)
Discrete-quantity model (e.g., Cardinality chart)	Semidiscrete-quantity model (e.g., Number list)	Continuous-quantity model (e.g., Number line)
		

Note: The three models of number embody different meanings of number:

- A discrete-quantity model such as a cardinality chart represent what she called a *cardinal meaning of number* and corresponds to children's informal counting-based view of the number sequence.
- A continuous-quantity model such as a number line (linear length) represents what Fuson (1988) called a *measurement meaning of number*, which presumbaly requires a formal understanding of the concept of a unit of measurement.
- A semidiscrete-model such as a number list approximates both measurement and cardinal meanings.

Fig. 4 Three models of number.

fostering the principle, because—in terms of young children's existing informal (discrete-quantity-based) view of the counting sequence—it is a concrete and more direct model of relative magnitude (see Fig. 4; Baroody & Purpura, 2017; Frye et al., 2013).

3.2.2.2 Successor Principle

A plausible conceptual basis for a linear representation of the counting sequence and hence linear magnitude judgments is the successor principle (Level 4 in Table 3). This principle entails the insight that each successive number in the counting sequence is exactly one more than its predecessor (e.g., nine which immediately follows eight when we count is one more than eight; Izard, Pica, Spelke, & Dehaene, 2008). Currently, almost no research has been reported on how to promote the successor principle and whether such instruction also fosters a linear representation. Reid, Baroody, and Purpura (2013) compared computer-based training that focused directly on successor principle and that involving *The Great Race*. The successor training consisted of children playing three games involving a hungry monkey swinging through tree branches on which a number line was superimposed to retrieve a banana. The games targeted the number-after knowledge (*what is the number after 3?* Level 4A in Table 1), successor understanding (*how many more branches must the monkey swing to get from 3 to 4? Level 4 in Table* 3), and identifying correct application of the successor principle (*the monkey thinks he needs to swing one more branch to get from 3 to 4, is he correct?*). The direct successor training was significantly more effective than *The Great Race* in promoting successor learning but not linear magnitude comparisons.

3.2.3 Introducing Addition and Subtraction Using a Number-Line Model

Whether a number-line model should be used to formally introduce children to operations on numbers needs to take into account the internal factor of children's understanding of numbers. In recent years, it has become popular to recommend using a number line as a basis for formally introducing young children to addition and subtraction (Bell et al., 2004; National Mathematics Advisory Panel, 2008). For instance, the supplemental *Bridges in Mathematics* (Math Learning Center, 2009) program introduces an addition item such as $5+3=8$ as hopping to the right 5 and then 3 more. The inverse of this—the related subtraction item $8-3=5$—is shown next by hopping from 8 in the opposite direction (to the left) three times.

Although it is crucial that children see addition and subtraction as related operations (e.g., that addition and subtraction of the same number undo each other), the number-line model used by *Bridges in Mathematics* may be too abstract for young children for two reasons (Baroody, 2016a). Consistent with physical to mental embodiments view, one is that it initially connects symbolic equations such as $5+3=8$ and $8-3=5$ to a relatively sterile iconic representation (number line) instead of an enactive representative (e.g., modeling inversion the addition and subtraction with objects) or a "transitional" representation (e.g., modeling inversion the addition and subtraction of the same number with pictures of objects).

Consistent specific to general (familiarity) view, a second reason is that a number line may be inconsistent with young children's informal view of numbers. As Fig. 4 suggests, a number line represents a linear extent (a continuous quantity) and thus embodies what Fuson (1988) identified as a *measurement meaning of number*. Evidence indicates that young children view the number sequence as representing a series of increasingly larger discrete quantities—as embodying what Fuson (1988) identified as a *cardinal meaning of number*. For example, they typically (a) do not spontaneously break up continuous quantities into countable units (Fuson, 1988; Huntley-Fenner, 2001) and (b) misconceive number lines as representing discrete quantities, as evidenced by them counting the numbers or hash marks on the number line, not the linear extents between hash marks (e.g., counting 0 or its hash mark as "one"; Lehrer, 2003). A key source of the difficulties is that children construct a concept of linear unit relatively late. Indeed, Saxe et al. (2010) found that fifth graders struggled with number-line-based problems that required the application of such a concept. For instance, shown a number line with 8 and 10 marked and labeled, many participants incorrectly considered the distance between 8 and 10 as one unit and located 11 at the position for 12. For such reasons, mathematics educators have long recommended against using number lines as basis for introducing formal mathematics instruction (Ernest, 1985; Fuson, 2009), and the National Research Council (2001) recommend against using these models as instructional tool before grade 2.

Two qualifications are in order. *If* instruction helps students construct a solid concept of linear units, the use of a number line for teaching addition and subtraction would be consistent with Dewey's (1963) interaction principle. Using the number line to introduce two-digit mental addition and subtraction as done in Realistic Math may also be consistent with this principle and a potentially useful technique. As Frame A of Fig. 5 illustrates, the number line in this program is initially related to children's informal discrete-quantity view of numbers by showing 10 circles between each hash mark (Blöte, Klein, & Beishuizen, 2000). As Frame B of Fig. 5 illustrates, the empty number line is essentially used as a notation system for aiding students' calculations that can be based on their discrete-quantity view of numbers (Blöte, Van der Burg, & Klein, 2001).

3.2.4 Base-Ten and Place-Value Concepts and Skills

Like the previous section, this subsection illustrates why it is crucial to consider internal (psychological) factors when considering the use of concrete experiences and progressively adapting these experiences as children achieve higher levels on a HLT.

3.2.4.1 Understanding Multidigit Numbers in Terms of Base-Ten/Place-Value Concepts

Children informally view multidigit numbers such as 23 in terms of 23 (countable) units. They must construct base-ten concepts (e.g., 10 ones can be grouped together to form a larger unit called "ten") and place-value concepts (e.g., the value of the digit 2 in 23, by virtue of its position in the tens place, represents 2 tens (2×10) and 3, by virtue of its position in the ones place, represents 3 ones (3×1).

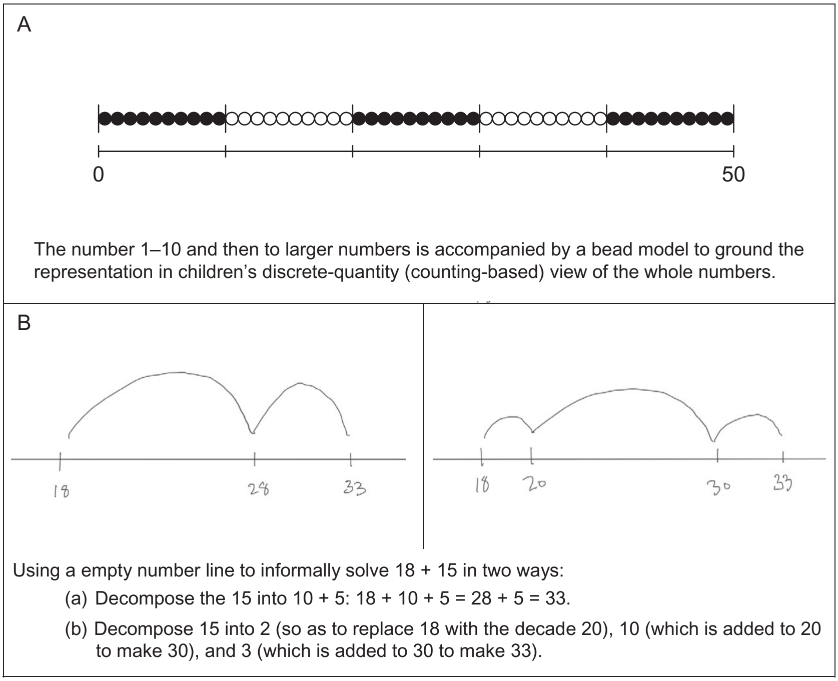
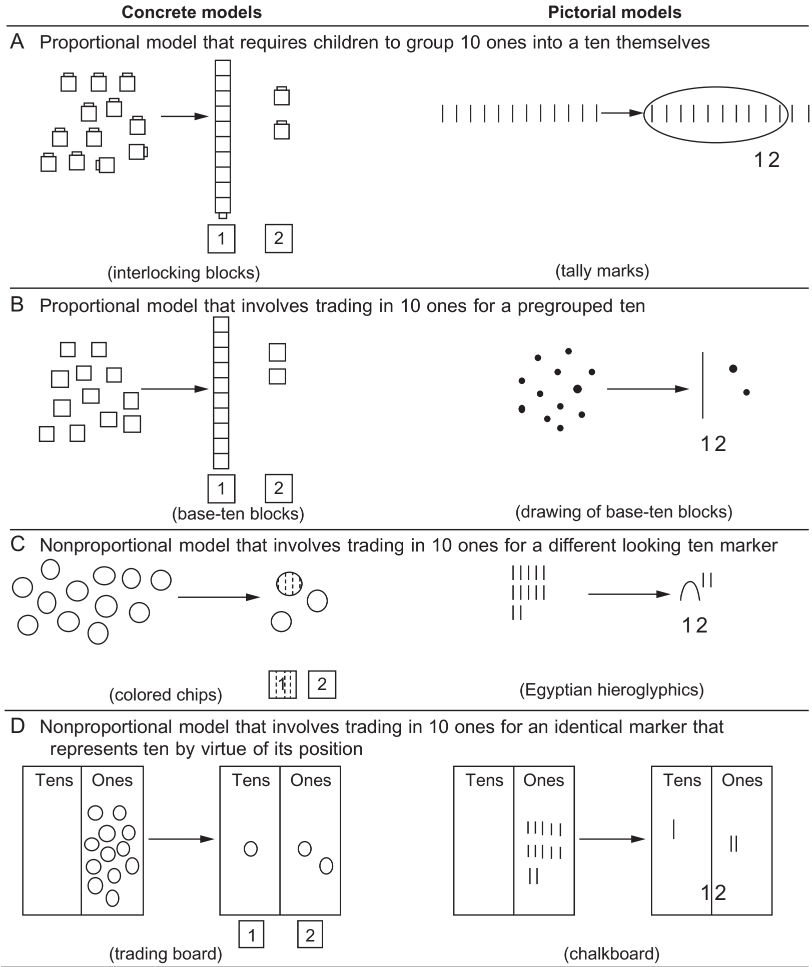


Fig. 5 Use of the number line in Realistic Math to teach two-digit addition and subtraction.

Baroody (1990) adduced a sequence of increasingly abstract manipulative- or picture-based models for fostering the meaningful learning of grouping-by-ten (base-ten) and place-value concepts. Frame A of Fig. 6 illustrates the most concrete multidigit number model (e.g., interlocking blocks) in the sense that it builds directly on children's informal counting-based view of multidigit numbers in two ways: (a) an interlocking-block model is *proportional* (e.g., the representation of a ten is 10 times larger than that for one) and (b) it requires children to *physically put together* 10 ones to create a ten. Although also *proportional*, Frame B models (e.g., base-ten blocks) are somewhat more abstract in that they entail trading 10 embodiments of ones for a *pregrouped embodiment of ten*. That is, unlike using interlocking blocks, which literally entails the experience of constructing a ten from 10 ones, using base-ten blocks entails a child recognizing that a 10-stick or "long" could theoretically be constructed by gluing 10 units together. Conversely, a "ten stick" composed of interlocking blocks can physically be decomposed into 10 unit blocks, whereas a long can only be traded for 10 cubes.



Note. Montessori Beads are between Frame A and B in abstractness. Although 10 discrete beads are strung together to form a ten string (and 10 strings of ten are combined to form block of 100), it may not be practical for children to do this. Yet, the ten string is more clearly made up of 10 discrete units than the long (ten-stick) of a base-ten model.

Fig. 6 Concrete-to-abstract models of a base-ten and place-value meaning of two-digit numbers.

Both Frame A models (e.g., interlocking blocks) and Frame B models (e.g., base-ten blocks) are more concrete than *nonproportional* models shown in Frames C and D of Fig. 6. Frame C models, which involve *trading 10 embodiments of one for a single different-looking embodiment of 10*) are less abstract than Frame D models, which involve *trading 10-for-1 embodiments that are all identical save for the position on say a place-value trading board*. Note that while Frame A, B, C, and D models all model grouping (albeit at increasingly abstract levels), only Frame D models represent an effort to directly embody a place concept (i.e., the position of a digit determines it value; e.g., whether it is a ones or tens).

The key unresolved pedagogical issue is whether it is more efficacious to use a series of increasingly abstract models that start by building directly on children's informal counting-based view of numbers or—as suggested by cognitive alignment perspective—start with a model that directly represents both the grouping and place-value characteristics of multidigit numbers. Other unresolved questions include: Consistent with an interaction perspective and contrary to the dual-representation hypothesis (DeLoache, 2000; Kaminski et al., 2013; Uttal et al., 2009), is using Frame A models in which, for instance, children physically construct a ten from units and then Frame B models (e.g., trading 10 unit blocks a long) more efficacious than starting with Frame B models? Similarly, are using proportional models and then nonproportional models more efficacious than starting with nonproportional models? Are using less abstract models and then the relatively abstract Model D more powerful than starting with Model D? Do the results for all of the previous questions with advantaged children also apply to various at-risk populations, such as children from low-income families or those with mathematical learning difficulties?

3.2.4.2 Magnitude Comparisons and a Linear Representation to 100

Even after children have constructed an understanding of the successor principle (e.g., recognize in principle that a number is one more than its predecessor in the counting sequence), they continue to exhibit a logarithmic response pattern on magnitude comparison tasks involving larger and relatively unfamiliar numbers. Various factors have been adduced to account for such a nonlinear response pattern (Anobile, Cicchini, & Burr, 2012; Barth & Paladino, 2011; Slusser, Santiago, & Barth, 2013). Landy, Charlesworth, and Ottmar (2016, p. 21) concluded:

> *It appears, however, that human representations of natural numbers, at least beyond a paltry few hundred thousand iterations, rely on resources quite distinct from successorship or even a metric "number line," and impose substantial additional structure given by the numeral systems. A fundamental mistake made by classical empiricism was to assume that the inner representations were iconic—that they were like the outer represented. When reasoning about large numbers, we appear to rely on representations that are fundamentally unlike the numbers themselves.*

Perhaps unlike smaller familiar numbers, larger numbers may not be represented in the form of a mental number line as suggested by Resnick (1983).

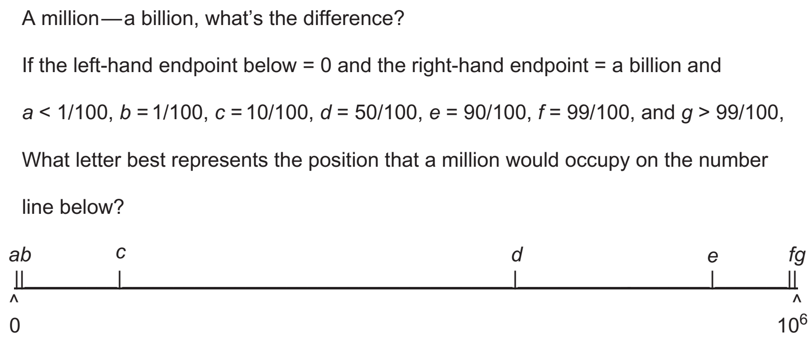


Fig. 7 The million vs a billion problem.

Instead, their representation may be based on an understanding of the structure of the number system. For example, correctly judging the relative magnitude of a million and a billion (see Fig. 7) depends on understanding that a billion is 1000 times larger than a million.

For children who are just becoming familiar with the numbers to 100, the *Grid Race* might be more efficacious than *Race to the Moon* in exploring the structure of the number from 0 to 100 and, thus, have a greater impact on various aspects of numeracy, including a linear representation of numbers. Specifically, whereas counting-on 1 or 2 from a two-digit number is effectively geared toward discovering the (already known) successor principle, it—unlike counting 10 spaces from a decade or other two-digit number—is less likely to reinforce the repetitive structure of the numbers from 0 to 100 (e.g., each decade is 10 more than the preceding decade or repeatedly subtracting 10 changes only the decade not the number of ones). Whereas reinforcing the successor principle is not likely to improve the linearity of magnitude comparisons, a better grasp of how two-digit are interrelated might.

3.3 Instructional Strategies for Ensuring Educative Experiences

3.3.1 Eight Guidelines

Mix (2010) cited five (largely external) dimensions or variables that might impact the effectiveness of manipulatives: variations in structure, the amount of contact, the relation of a model to written symbols, the amount of exposure, and the number of models. These variables are related to Points 1, 2, 5, 6, and 7, respectively. In addition, Points 3, 4, and 8 are important applications of Dewey's (1963) interaction principle. Note that for helping children overcome the long-recognized difficulty of connecting physical models of mathematical ideas or processes to formal written representations (Ginsburg, 1977; Resnick, 1982; Uttal, Liu, & DeLoache, 2006), Points 3, 4, and 5 are particularly important.

1. *Structured use*. A Piagetian approach based on free play, once the mantra of radical constructivists and many early childhood educators, is now generally not considered a viable instructional approach. Although free play is important in its own right, the presence of manipulatives alone in a free play context does not guarantee an educative experience (Baroody, Clements, & Sarama, in press; Fuson, 2009). For example, Clements, Copple, and Hyson (2002) concluded that it is unlikely incidental experiences will foster educative experiences and that young children are ready for organized, sequenced experiences embedded in specific activities (e.g., reading children's literature, music or art activities), play (e.g., math games or physical activities), or projects. Frye et al. (2013) further concluded that, in addition to embedded mathematics activities, early childhood instruction should include daily, targeted mathematics instruction (i.e., dedicated time for structured, mathematics-focused activities). In brief, ensuring educative experience entails using manipulatives with a purpose (e.g., clear goals), direction (e.g., scaffolding), and feedback (including questions that prompt reflection such as "Why do you think that makes sense?" or "Does everyone agree?"). As Brown, McNeil, and Glenberg (2009, p. 161), noted, "Without appropriate structure, learners may fail to discover the target concept."
2. *Active use*. However, not all structured instruction is equally effective. Chi (2009) hypothesized that constructive activities (which elicit responses involving ideas that go beyond the information provided) are more effective than active activities (physically doing something), which in turn are more effective than passive activities (e.g., listening or watching). Consider how this general framework applies to the case of using concrete materials. Using manipulatives to actively involve children in discovery learning—exploring a phenomenon, noticing patterns or relations, and generalizing such regularities—is more effective than either their unguided use or adult-directed use in advancing children's understanding (see Alfieri, Brooks, Aldrich, & Tenenbaum, 2011). Prompting the imitation of a teacher-taught manipulative-based procedure or simply watching a teacher model a procedure with manipulatives may be meaningful to children and promote their successful adoption and application of the procedure when it builds on existing conceptual

understanding but when new conceptual insight is needed (Baroody, Tiilikainen, & Tai, 2006).

The use of concrete experiences involving structured or guided discovery has a number of advantages over direct instruction (see, e.g., Alfieri et al., 2011; Ambrose, Baek, & Carpenter, 2003; DeCaro & Rittle-Johnson, 2012; Fyfe, DeCaro, & Rittle-Johnson, 2014; Rittle-Johnson, Fyfe, Loehr, & Miller, 2015; Schwartz, Chase, Chin, & Oppezzo, 2011; Shafto, 2014). One is that first discovering concepts or inventing self-styled solution strategies is more likely to result in understanding a concrete model and, thus, greater learning or more flexible application of procedures than first learning procedures. For instance, Thompson (1992) found that students who had already learned teacher-taught renaming procedures did not improve in computational accuracy after an intervention with either base-ten blocks or virtual base-ten blocks (Blocksworld). Learning concepts (e.g., via "meaningful" manipulatives) after learning a standard procedure is inconsistent with the moderate novelty principle for many students because of the belief they already know how or what to do.

For a thoughtful discussion of how Froebel and Montessori artfully used carefully designed protocols to guide the manipulatives-based discovery of mathematical ideas, see Balfanz (1999) and Mix (2010). One effective way of involving children in guided discovery learning is to provide them materials that embody examples and nonexamples of a concept so that they can discover for themselves the critical (defining) attributes of the idea (Durkin & Rittle-Johnson, 2012; Rittle-Johnson & Star, 2007, 2009, 2011; Schwartz et al., 2011). For example, a game for helping preschoolers construct a concept of "three" (subitize collections of three) is the game *Is It Three?* Children are shown different examples of three *and* nonexamples of three (e.g., collections of 2 and 4) and asked, "Is this three?" The variety of examples aid in helping children in abstracting the defining property of three (a trio) by discounting other properties such as shape, color, and size, and the nonexamples help define the boundaries of the concept (e.g., •••, but not •• [fewer] or •••• [more], is "three"; Baroody, Lai, et al., 2006; Frye et al., 2013). See Baroody and Coslick (1998) for a variety of example lessons that employ examples and nonexamples. In sum, similar to Martin and Schwartz (2005), Brown et al. (2009, p. 161) observed, "educators need to find an appropriate balance between structure and spontaneity," as learners need some freedom to construct meaningful knowledge for themselves.

3. *Reflective use*. A particularly important aspect of the active use of concrete materials and the construction of meaningful knowledge is thoughtful reflection on the experience—mental, as well as physical, engagement (Baroody, 1989; Sarama & Clements, 2009a). Indeed, consistent with Chi's (2009) distinction between active and constructive activities, Piaget (1964) differentiated between a *physical experience* and a *logical-mathematical experience*. The former "consists of acting upon objects and drawing some knowledge about the objects (e.g., noticing that a steel ball is heavier than a rubber ball of the same size; p. 11). A logical-mathematical experience is knowledge that is drawn from thinking about the actions on the objects. As an example, Piaget cited the experience of a roughly 4-year-old child who counted a row of pebbles first in one direction, then in the opposite direction. Next the child arranged the pebbles in a circle and counted them clockwise and then counter clockwise. After experimenting with yet other arrangements, the child discovered that, regardless of order or arrangement, counting the collection (accurately) always had the same result. In effect, by reflecting on his experience, the child discovered the order-irrelevance and number-constancy principles applicable to any collection. In sum, experiences with concrete materials are more likely to be educative if children spontaneously reflect on their physical actions or are prompted to do so by a teacher or peers asking them to explain or justify their actions. Points 4 and 5 are especially important for prompting reflective use.
4. *Connecting to meaningful knowledge (e.g., capitalizing on familiar analogies)*. To prompt thoughtful reflection, discovery learning or any form of instruction needs to be meaningful, which requires carefully building on what children know (Bruner, 1961; Fyfe, Rittle-Johnson, & DeCaro, 2012). Encouraging children to use manipulatives to solve mathematics problems with understanding needs to take into account Piaget's principle of assimilation and the moderate novelty principle, and a key tool for doing so with learners of all ages is using meaningful analogies. Building on existing and meaningful knowledge (e.g., a familiar analogy) is particularly important for a guided discovery approach, as it enables children to invent their own informal strategy for solving a problem. Building on children's existing knowledge and strategies is the rationale for the *Everyday Math* program (Bell et al., 2004) and can serve as basis for understanding and reinventing formal procedures (see Baroody & Coslick, 1998). Doing so minimizes the need for

students to memorize a teacher-taught problem-solving procedure for a particular class of problems—an advantage that may be particularly important for children with mathematical learning difficulties.

Consider the case of division. Primary-grade children often readily devise a divvy-up strategy for fair-sharing problems (Amount ÷ number of shares = size of shares; see Table 4; Hiebert & Tonnessen, 1978; Kouba & Franklin, 1993). In contrast, Squire and Bryant (2002) noted they often have difficulty solving more generic partitive problems (Amount ÷ number of groups = size of each group) and especially measurement problems (Amount ÷ size of each group = number of groups; again see Table 4). Capitalizing on children's everyday experience of fair sharing can help them understand, compare, and solve more challenging (i.e., generic partitive and measurement) division word problems. The following guidelines outline how instruction can successfully foster the use of manipulatives to informally solve division word problems:

Table 4 Types of Division Word Problems, Examples, and Children's Informal Modeling Strategy

Type of Problem[a]	Example Word Problem[b]	Informal Strategy for Directly Modeling the Meaning of the Problem
Fair sharing	How much candy did each of three friends get if they shared 12 candies fairly?	*Divvy-up*: • Count out 12 chips, • Then distribute a chip to each of three piles until all the chips are gone, and • Finally, count the chips in one of the piles to determine the answer.
Partitive	Mandy has 12 candies. She puts 4 candies in each bag. How many bags can she fill?	
Modified fair sharing	How many friends can share 12 candies fairly if each share is 3 candies?	*Measure-out*: • Count out 12 chips, • Then create piles of three until all the chips are gone, and • Finally, count the number of piles to determine the answer.
Measurement	Mandy has 3 bags of candies and 12 candies altogether. If there is the same number of candies in each bag, how many candies are in each bag?	

[a]Fair-sharing problems are familiar examples of the more generic partitive class of division problems. Modified fair-sharing problems are familiar examples of the more generic measurement class of division problems.

[b]The examples of the partitive and measurement problems are modeled after those illustrated at http://www.math.niu.edu/courses/math402/packet/packet-4.pdf (examples that had been adapted from Cognitively Guided Instruction, University of Wisconsin-Madison, 1992).

- Introduce division with highly familiar fair-sharing problems. By devising a divvying-up strategy with manipulatives themselves, children will understand, own, and apply effectively apply the strategy.
- Relate less familiar generic partitive problems to fair sharing. For instance, encourage a child who does not understand such problems (e.g., does not spontaneously apply a divvy-up strategy) to translate them into fair-sharing terms: The amount is analogous to knowing the total number of items, the number of groups is comparable to the number of people (the number of shares), and the unknown (the size of each group) corresponds to the size of each share.
- Introduce the moderately novel modified fair-sharing problem (Amount ÷ size of each share = number of shares) to foster the invention of an informal measure-out strategy, see Table 4). Solving such problems before introducing generic measurement problems will provide a basis for understanding and solving the latter, more unfamiliar and challenging problems.
- Relate less familiar generic measurement problems to modified fair sharing: Measurement division is analogous to knowing the total amount and the size of each share and not knowing the number of people with whom the total amount can be shared (number of shares). This will help children recognize that their informal measure-out strategy is applicable to such problems.

As illustrated in Fig. 8, the familiar and meaningful analogy of (modified) fair sharing can also be invaluable in helping students

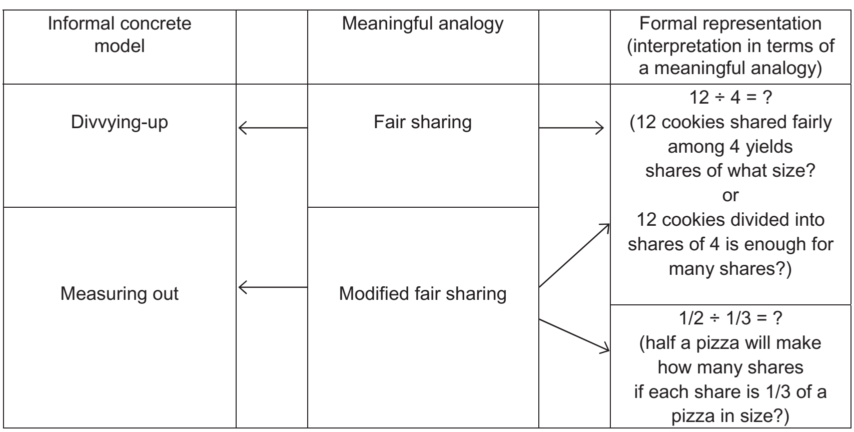


Fig. 8 Meaningful analogies as the bridge between informal concrete models of division and formal symbolic representations of division.

making sense of formal representations of division and bridging the gap between written division expressions (e.g., $12 \div 3$ or $1/2 \div 1/3$) or equations (e.g., $12 \div 3 = ?$ or $1/2 \div 1/3 = ?$) and their informal models or solution strategies. That is, the analogy provides the missing connection between formal representations of division and children's knowledge of how to use manipulatives to informally solve division problems. With older elementary or even college-level students, the modified fair-sharing analogy for measurement division can provide the crucial basis for making sense of fraction division. For instance, for many students, it is a mystery why $1/2 \div 1/3$ results in the larger number 3/2 or 1–1/6 (not a smaller number as with the division of whole numbers). The result makes sense if you consider that 1/2 of pizza allows can be used to create one full share and a part (1/6) of another share.

5. *Explicit connections*. Many manipulative-based activities do not include corresponding written representations or procedures. Moreover, children often do not spontaneously relate concrete models to abstract formalisms or vice versa (Resnick, 1982; Resnick & Omanson, 1987; Uttal et al., 2013). Frequently, then, concrete materials need to be explicitly linked to the abstract idea such material are intended to represent to ensure thoughtful reflection (Brown et al., 2009; Fyfe et al., 2015). As a first step, self-explanations may be particularly valuable in facilitating learning both concepts and strategies because they prompt a pupil to explicitly summarize new information and thus consciously reflect on it (Berthold & Renkl, 2009; Matthews & Rittle-Johnson, 2009; Rittle-Johnson, 2006; Rittle-Johnson et al., 2015; Star & Rittle-Johnson, 2009). Consistent with the Common Core State Standards Math Practices Standard 3 (http://www.corestandards.org/Math/Practice/) encouraging children to share, justify, and compare their informal strategies can also be valuable in fostering explicit reflection and understanding, including gaps or inconsistencies in their thinking (Baroody & Coslick, 1998; Battista, 2016; Piaget, 1928). So as to foster connections with symbolic procedures, teachers can encourage children to translate informal manipulative-based strategies into a written procedure or vice versa (e.g., by asking them to evaluate a formal procedure in terms of informal concepts and procedures). Finally, teachers can explicitly summarize pupil's discoveries and explanations and, as needed, fill in gaps to ensure coherency. See Brown et al. (2009) for a discussion of how artfully used gestures can assist in guiding children's attention to

the connection between concrete representations or action and abstract ideas.

6. *Extended use*. Uttal (2003, p. 111) noted: "Several lines of research have shown that for manipulatives to be effective, they must be used repeatedly for the same concept." Children may need time using manipulatives to discover regularities or otherwise make sense of their intended purpose. Unfortunately, formal instruction often limits manipulative exposure (Mix, 2010).
7. *Multiple representations*. Using multiple representations can serve to promote abstraction or generalization of a discovered regularity or deepen conceptual understanding of a symbolic representation (Barmby et al., 2009; Kaminski et al., 2009; McNeil et al., 2009; Moseley, 2005; Steinbring, 1997). Moreover, translating formal symbolism into multiple concrete embodiments (as well as the reverse) can further ensure multiple connections and a deeper understanding of an idea (Baroody & Coslick, 1998). However, research has shown that simply accessing multiple representations does not necessarily lead to improved understanding (Ainsworth, 2006). Underscoring the importance of helping students explicitly connects various representations of an idea, Seufert (2003, p. 228) noted: "Learners must interconnect the external representations and actively construct a coherent mental representation in order to benefit from the complementing and constraining functions of multiple representations."

   A concern with multiple models is that it reduces the exposure to any one model and may confuse students (Mix, 2010; Uttal, 2003). For this reason, Asian schools focus on a single concrete model (Stevenson & Stigler, 1992). Concerns with the single-model approach, though, are an overly local concept or inflexible strategy use. This may contribute to an education that undermines pupils' disposition, creativity, autonomy, and initiative (Zhao, 2014).
8. *Purposeful use*. Manipulatives may be particularly engaging when children have a purpose of their own for using them or an adult creates a reason. Math games may be a particularly useful way to create a real need to use and reflect on the use of concrete experiences (for examples, see Baroody, 2016b; Baroody et al., in press; Baroody & Coslick, 1998; Clements & Sarama, 2013). Dramatic play (e.g., a store scenario) also provides a structured yet purposeful way of creating a situation where children have the opportunity to use concrete materials to explore, reflect on, and extend mathematical concepts or apply mathematical skills.

3.3.2 A Case in Point

Consider the research and recommendations by Kaminski and Sloutsky (2013) regarding graph instruction for primary-grade pupils. Participants were randomly assigned to an extraneous information condition, which involved a combination of pictogram and bar graph (see Frame A in Fig. 9), or a no extraneous information condition, which involved a solid bar graph (see Frame B in Fig. 9). All participants were taught how to read a bar graph by relating the *x*-axis (time in weeks) to the *y*-axis (the number of shoes). Children in the extraneous condition were posttested with a combination of a misleading pictogram but accurate bar graph (see Frame C of Fig. 9, and those in no extraneous condition were tested with a solid bar graph (see Frame D of Fig. 9).[e] Whereas, the vast majority of kindergartners and first graders in the extraneous condition counted the items in the misleading pictogram and thus responded incorrectly, the vast majority of those in the no extraneous condition used the taught procedure and responded correctly. On a posttest task involving novel (pattern-filled) bar graphs (see Frames E and F of Fig. 9), kindergarten and first graders in the extraneous information condition used far less counting, and the majority responded correctly. However, those in the no extraneous information condition performed significantly better.

Kaminski and Sloutsky (2013) drew two conclusions:

1. If pictograms are used in primary instruction, then teachers should explicitly direct children's attention to the relation between the *x*-axis and *y*-axis so that they do not have to rely on a counting strategy.
2. Although pictograms are often recommended as a scaffold for helping young child make sense of bar graphs (e.g., Choate & Okey, 1981), such scaffolding is not necessary for successfully learning how to read bar graphs.

If a child repeatedly failed to see the connection between counting the items in a pictogram and the corresponding value on the *y*-axis, then Kaminski and Sloutsky's (2013) first recommendation is consistent Point 5—help make connections explicit. However, the second recommendation should be treated with caution or even skepticism for several reasons. Consistent with Point 4 (connect new instruction to prior knowledge), real graphs, picture graphs, and pictograms (especially with grid) are recommended as scaffolds

[e] The condition-specific task for the Extraneous Information Condition of the Kaminski and Sloutsky (2013) study (see Frame C of Fig. 9) is not ecologically valid, as teachers and textbook publishers (typically) do not use pictograms that depict an incorrect number of items.

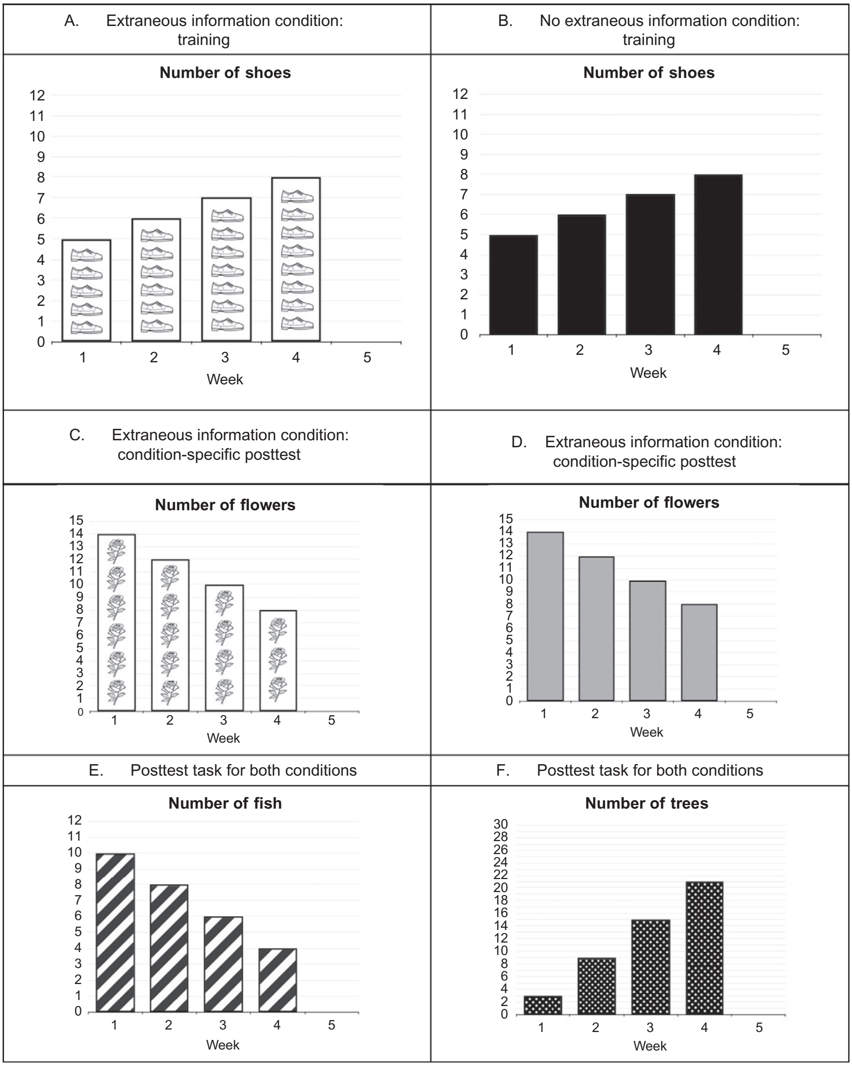


Fig. 9 Graphs similar to those used by Kaminski and Sloutsky (2013) for training and testing (based on their Figs. 1 and 2).

for meaningful introducing bar graphs, in part, because young children have an informal, counting-based (discrete-quantity) view of number and bar graphs entail a measurement (continuous-quantity) model of number. Some primary-grade children, then, may need such scaffolding to *understand* a bar graph and how to read it. Importantly, Kaminski and Sloutsky (2013) apparently did not administer a delayed posttest to assess retention. There are several reasons to believe their intervention involving no extraneous information would not have a lasting impact. Inconsistent with Points 2 and 3, the instruction entailed a passive activity (e.g., listening or watching the trainer read a graph) and an active activity (imitating the trainer's procedure) but not constructive activities (producing responses that entail ideas that go beyond provided information) requiring reflection. Although the instruction refers to common objects and unit of time, it is not clear whether the two variables were related in a meaningful way (e.g., by a familiar story line; Point 4). As a trainer demonstrated a graph-reading procedure four times and a participant practiced it only four times (in same session), the intervention did not involve extended use (Point 6). Finally, the training did not seem purposeful from a child's perspective or particularly engaging (Point 8). All in all, the instruction seems readily forgettable.

4. IS THERE EVIDENCE THAT CONCRETE EXPERIENCES WORK?

Some empirical evidence indicates that, for example, concrete experiences are useful in extending existing informal knowledge by providing young children an opportunity to discover and apply a mathematical regularity or devise and practice an informal strategy—(Boggan, Harper, & Whitmire, 2010; Clements & Sarama, 2012) and that games (Bright, Harvey, & Wheeler, 1985), including computer games (Baroody, Purpura, Eiland, & Reid, 2015; Obersteiner, Reiss, & Ufer, 2013; Shin, Sutherland, Norris, & Soloway, 2012) can be valuable in promoting mathematical learning. However, in light of the preceding discussions on instructional strategies for ensuring effective use of concrete experiences and Dewey's (1963) interaction principle, it should not be surprising that research on the effectiveness of concrete experiences is mixed (see, e.g., reviews by Mix, 2010; Uttal, 2003). In explaining the mixed results of Thompson's (1992) use of the Blocks Microworld program and research on the effectiveness of manipulatives in general, Mix (2010) concluded that whether a model might or might not work depends on such factors as how manipulatives are used, the outcome measure, and the characteristics of the learner.

For instance, Walker, Mickes, Bajic, Nailon, and Rickard (2013) evaluated the relative efficacy of using a conceptual approach (fact triangles) and a drill approach (answer-production [AP] training) to promote fluency with subtraction combinations with grade 1–6 students. Fact triangles

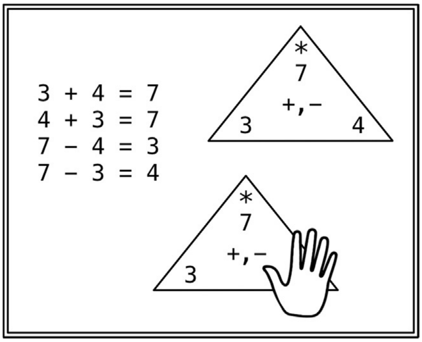


Fig. 10 Example of a fact triangles using 3–4–7.

(e.g., see Fig. 10) are widely used to help children see that subtraction is related to addition and that known sums can be used to reason out unknown differences (e.g., if $3+4=7$, then $7-3=4$). Walker et al. (2013) found that AP training was significantly more efficacious in promoting subtraction fluency with practiced combinations than the fact-triangle intervention but that neither approach promoted transfer of fluency to unpracticed subtraction combinations. These researchers concluded that fact triangles "are not an effective vehicle for fluency training or for establishing flexibly applicable arithmetic skill" and curricula should "deemphasize fact-triangle exercises in favor of more AP training" (p. 30).

However, the fact-triangle intervention in the Walker et al. (2013) study may have failed because of the ineffective manner in which the training was implemented. The model used was (a) perhaps only semiactive (Section 3.3, Point 2), (b) not designed to prompt reflection (Point 3), (c) not particularly meaningful (Point 4), (d) without explicit connections between procedures and concepts (Point 5), (e) relatively short in duration (Point 6), (f) dependent on a single (not multiple) representations (Point 7), and (g) not purposeful and engaging (Point 8). In regard to Point 4, their fact-triangle training involved only two of seven steps in a HLT for fostering the meaningful memorization of subtraction combinations (Baroody, 2016a). In contrast, virtual concrete experiences designed to be consistent with Points 2–4 and 6–8 were significantly efficacious in promoting fluency with unpracticed subtraction combination—transfer, which is a primary goal of education (Baroody, Purpura, Eiland, & Reid, 2014; Baroody, Purpura, Eiland, Reid, & Paliwal, 2016). Moreover, although Walker et al. did attempt to gauge transfer, they did not measure which method was more effective in fostering the conceptual understanding that addition and subtraction are related operations. Finally, aside from a fluency pretest, these researchers did assess internal factors such as developmental readiness to benefit from fact-triangle instruction.

Mix (2010) concluded that manipulatives "play different roles in different situations" (p. 41) and the key question is not "do such educational tools work" but "do these materials used in this particular way activate this particular mechanism in this particular learner?" Moreover, citing Ginsburg and Golbeck (2004), she noted, "almost no research has addressed how or why these materials might help" (p. 41). To address such a question effectively, researchers need to spell out the role manipulatives are presumed to play—a factor that dictates the outcome measures. Importantly, both the theoretical model and intervention effort need to take into account Dewey's (1963) principle of interaction—how external factors are intended to mesh with internal factors. Evaluations of manipulatives involving interventions that violate this principle (e.g., attempt to impose a manipulative procedure via direct instruction quickly) confound instructional approaches with the potential value of the manipulative in a learning environment that honors the principle (e.g., encourages a child to use their existing knowledge to reflect on how to use the manipulative). Furthermore, a fair evaluation of manipulatives should include assessing the developmental readiness of participants along a HLT. To date, such internal factors have been largely overlooked when researchers construct hypotheses regarding why and how manipulatives work and evaluate the impact of manipulatives.

5. CONCLUSIONS

Both educators and researchers need to take into account Dewey's (1963) principle of interaction when considering the use of concrete experiences. The main implication of the interaction perspective is that the choice of concrete experience depends on internal or psychological factors (e.g., a child's developmental level and interests) as well as external factors (e.g., the goal of instruction, the nature of the instructional activity). Indeed, ensuring an educative experience requires that the latter mesh with the former, including a child's developmental level.

What is "concrete" may depend on whether the goal is sensory-concrete, integrated-concrete, or abstract knowledge. "Students may require physically concrete materials to build meaning [sensory-concrete knowledge] initially" (Sarama & Clements, 2009a, p. 146). However, a priori, there is no reason to believe that virtual representations could not provide the sensory input needed to construct mathematical ideas initially. Whether virtual experiences can be designed to promote sensory-concrete knowledge as effectively as (or even more so than) physical materials, is an open and important question. Until such evidence is available, results showing that virtual manipulatives are more effective than physical manipulatives (e.g., Thompson, 1992) are not necessarily inconsistent with perceptual-motor to mental actions perspective (cf. Mix, 2010), *if* the goal is integrated or abstract knowledge. For these goals, concrete is what is meaningful—what can be connected to other familiar ideas or situations—not sensory input from physical objects or movements (Sarama & Clements, 2009a).

A worthwhile concrete experience or activity is educative (e.g., promotes the next level of development in a HLT or lays the groundwork for more advanced levels). "Good [concrete experiences or] manipulatives are those that aid students in building, strengthening, and connecting various representations of mathematical ideas" (Sarama & Clements, 2009a, p. 146). Their good use could benefit from a systematically planned approach to assessment and instruction based on HLTs. Such an approach can help ensure that external mesh with internal factors by identifying a child's level of development and the goal, suggested activities for achieving this goal, and the assessment for gauging whether the goal has been achieved. Although largely overlooked by researchers, an educative experience may also include activities that foster a positive disposition for further learning or problem-solving competence.

The effective use of concrete experiences and manipulatives—whether they work or not—depends on what is arguably the most important and too often overlooked factor in learning, namely internal factors (e.g., a child's developmental level or interests; Fyfe et al., 2012; McNeil & Jarvin, 2007). Key questions both educators and researchers need to ask include: Is the concrete experience or manipulative consistent with the principle of assimilation (e.g., does it fit a child's existing knowledge and is thus meaningful). In particular, does instruction tap a conceptual schema (e.g., a familiar and meaningful analogy) by which children can connect a formal representation or process to their existing informal concrete model or strategy? Is it moderately novel and purposeful and, thus, engaging to children? Is the experience designed to prompt reflection, discovery, or other mental actions, or does it merely entail passively watching and modeling an adult-imposed strategy? Indeed, whether a manipulative is meaningful, engaging, and thought provoking may be appreciably more important factors than its physical appearance (e.g., whether gaudy or sterile). Considering such questions will reduce the chances of concluding that a concrete activity or material is ineffective or attributing its failed use to a child's or children's cognitive limitations when such activities and materials are not meaningful, engaging, or thought provoking.

Taking into account internal factors and how they mesh with external ones may mean using an approach Bruner (1960) called a "spiral curriculum." It may be developmentally inappropriate to start with a model that most closely aligns with an ultimate goal (a formal representation). Instead, increasing abstract and aligned manipulatives may need to be gradually introduced in a stepwise fashion. More specifically, instruction may need to start with a concrete model that is not only *partially* aligned with the ultimate goal of instruction or desired mental representation but is aligned with a child's developmental level. Then a series of increasingly less concrete models can be used to revisit the idea as the child achieves more advanced levels of development. Finally, when the child is developmentally ready, a relatively abstract model that precisely aligns with the ultimate goal of instruction or desired mental representation can be introduced.

For example, an abacus embodies both base-ten and place-value concepts and is a relatively precise concrete model of multidigit numbers, addition, and subtraction. As such, the model meets the criterion for a useful model according to cognitive alignment perspective, especially if related to written representations. However, an abacus corresponds to the most abstract model (Frame D) in Fig. 6. As such, it may not a good tool for introducing a base-ten/place-value meaning of multidigit numbers, whereas bundling 10 items to create a group of ten (a proportional model that entails the physical action of grouping but not place value) may be more concrete (align more closely to children's informal counting-based view of number). Later, though, the abacus (a nonproportional grouping and place-value model)—which aligns more closely with place-value, base-ten meaning of multidigit written numbers—may well be useful.

Clearly, though, the spiral curriculum hypothesis of concrete experience/manipulative use—like so many aspects of the educational use of these tools—needs rigorous empirical verification. Much more research is needed to develop or refine HLTs that guide the effective use of such educational tools. The promise is that concrete experiences can be a valuable means of discovering and generalizing mathematical regularities and extending informal or formal knowledge (for learners of all ages). Such experiences may be a particularly invaluable means of making sense of formal mathematics instruction—of relating the unfamiliar, context-free ideas of school mathematics to familiar, personally meaningful contexts.

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