Homework 0

PSTAT 120B

This assignment is intended to help you recall or review some of the prerequisite material from PSTAT 120A that we'll be using in the course, mainly properties of expectation and variance.

1. (Linearity of expectation I) Let X be a random variable, and a, b be constants. Use properties of integration/summation to show that:

$$\mathbb{E}(aX + b) = a\mathbb{E}X + b$$

Consider both the discrete and continuous cases.

For discrete: $E[aX+b] = E[\sum_{x=1}^{n} aX+b]$

$= a E[\sum_{i=1}^{n} X] + b$

$= a E[X] + b$

For cont: $E(ax+b) = E\int_{-\infty}^{\infty} ax + b$

$= E[\int_{-\infty}^{\infty} ax] + b$

$= a E[\int_{-\infty}^{\infty} x \, dx] + b$

$= a E[x] + b$

2. (Linearity of expectation II) Let X, Y be random variables and a, b, c be constants. Use properties of integration/summation to show that

$$\mathbb{E}(aX + bY + c) = a\mathbb{E}X + b\mathbb{E}Y + c$$

Consider both the discrete and continuous cases.

For discrete: $E(aX+bY+c) = E(\sum_{n=}^{\infty} aX+bY+c)$

$= aE(\sum_{n=1}^{\infty} X) + bE(\sum_{n=1}^{\infty} Y) + c$

$= aEX + bEY + c$

For cont: $E(aX+bY+c)$

$= E[\int_{-\infty}^{\infty} aX + bY + c]$

$= aE\int_{-\infty}^{\infty} X + bE\int_{-\infty}^{\infty} Y + c$

$= aE[X] + bE[Y] + c$

3. (Variance identity) Let X be a random variable. Show that:

$$\text{var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

(Hint: use the covariance identity from the review lecture.)

$Var(X) = cov(X,X)$

$cov(X,X) = E[XX] - E[X]E[X]$

$= E[X^2] - E[X]^2$

4. (Variance of a linear transformation) Let X be a random variable and a, b be constants. Show that:

$$\text{var}(aX + b) = a^2\text{var}X$$

(Hint: use the variance identity and linearity of expectation.)

$= Var(aX) + \cancel{Var(b)}$ ⇒ since its just a digit turns to 0

$= a^2 Var(X)$

5. (Variance of a linear combination) Let X, Y be random variables and a, b, c be constants. Show that:

$$\text{var}(aX + bY + c) = a^2\text{var}X + b^2\text{var}Y + 2ab\text{cov}(X, Y)$$

(Hint: write the variance as a covariance and use bilinearity twice.)

$Var(aX+bY+c) = Var(aX) + Var(bY) + \underbrace{\cancel{Var(c)}}_{0} + 2Cov(aX,Y) + 2Cov(X,bY)$

$= a^2Var(X) + b^2Var(Y) + \quad + 2aCov(X,Y) + 2bCov(X,Y)$

$= a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$

6. (Zero covariance of independent RV's) Let X, Y be independent random variables. Show that:

$$\text{cov}(X, Y) = 0$$

(Hint: show $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$ and use the covariance identity.)

$V(X+Y) = V(X) + V(Y) + 2\,Cov(X,Y)$

$E(XY) - E(X) \cdot E(Y) = Cov(X,Y)$

$\underbrace{E(XY) = E(X)E(Y)}_{\text{the same}}$

$\frac{2\,Cov(X,Y)}{2} = \frac{0}{2}$

7. (Normal distribution) Let $X \sim N(\mu_x, \sigma_x^2)$ $Y \sim N(\mu_y, \sigma_y^2)$ be independent, and a, b, c be constants.

(a) Write the joint density $f(x, y)$.

$f(x,y) = (2\pi\sigma_{x+y}^2)^{-1/2} e^{-\frac{1}{2\sigma_{x+y}^2}(x-\mu_{x+y})^2}$

(b) Find $\mathbb{E}(aX + bY + c)$.

$E(aX+bY+c) = \int_x \int_y aX+bY+c \cdot \left((2\pi\sigma_{x+y}^2)^{-1/2} e^{-\frac{1}{2\sigma_{x+y}^2}(x-\mu_{x+y})^2}\right) dy\, dx$

(c) Find $\mathbb{E}X^2$. (Hint: use the variance identity.)

$E(x^2) = \int_x x^2 \cdot (2\pi\sigma_x^2)^{-1/2} e^{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2}$

(d) Find $\text{var}(aX + bY + c)$.

$\text{Var}(aX+bY+c) = a^2 V(X) + b^2 V(Y)$

$\downarrow$

$a^2(E(X^2) - E(X)^2) + b^2(E(Y^2) - E(Y)^2)$

apply the results of part b, c

8. (Standard normal quantiles) Let $Z \sim N(0,1)$. The (upper) standard normal quantiles are $z_\alpha = \Phi^{-1}(1-\alpha)$. Show that

$$z_\alpha = -z_{1-\alpha}$$

(Hint: use the fact that φ is an even function to argue that $\Phi(x) = 1 - \Phi(-x)$ for any x; then substitute z_α for x and simplify.)

$Z \sim N(0,1)$

$z_\alpha = \Phi^{-1}(1-\alpha)$

$= \Phi^{-1}(1 - \Phi^{-1}(x))$

$= -z_{1-\alpha}$

$\Phi^{-1}(x) = \Phi^{-1}(1 - \Phi(-x))$

$\Phi^{-1}(1 - \Phi^{-1}(x))$