Exam 1

PSTAT 120B

Instructions: the exam is open book and open note and has no strict time limit. You can use any course materials, and you are allowed to use results in the book, lecture notes, and past homeworks and quizzes without repeating proofs or derivations. Please do not consult with other students until after the submission deadline has passed. By submitting your work you are acknowledging that your work is entirely your own.

Background

The Hubble constant is a fundamental cosmological parameter used to describe the size and age of the universe. It relates the distance and velocity of objects relative to earth via the relationship $v = \eta d$. Astrophysicists estimate the Hubble constant η from observed ratios $H = \frac{v}{d}$ for a sample of galaxies. However, observations of H for any given galaxy can vary considerably due to measurement error.

Though seemingly simple, measurements of velocity and distance are challenging to obtain with accuracy, and continue to be refined (see a recent article). Velocity is calculated from spectroscopic measurements, which are less complicated to obtain compared with distance measurements. For the latter, "[r]eliable 'distance indicators', such as variable stars and supernovae, must be found in ... a sample of galaxies that are far enough away that motions due to local gravitational influences are negligibly small" (quote from Space Science Short).

The primary objective of the Hubble Space Telescope (HST), launched in 1990, is to collect data that will lead to more precise estimates of the Hubble constant. In 2001, Freedman *et al.* published a set of measurements based on novel distance indicators and HST data; distance-velocity ratios from measurements for 36 type 1a supernovae using their method of calculation are shown in Figure 1.

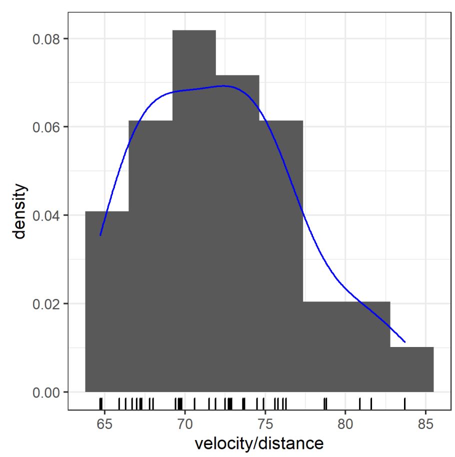

Figure 1: Histogram of H values from 36 type 1a supernovae, with a smooth overlay to aid in visualizing the shape of the distribution. The exact values in the sample are shown as ticks below the histogram. The sample mean and variance are $\bar{H} = 72.1861$ and $S_H^2 = 23.7858$.

Objective

Your goal in this assignment is to estimate the age of the universe based on an estimate of the Hubble constant η and to quantify the quality of the estimate. The age of the universe can be expressed in years as

$$\theta = \frac{c}{\eta}$$

where c is the conversion factor $\frac{(Mpc/km)}{(s/year)} = 978,440,076,094$.

Assumptions

Assume throughout that $H_1, \ldots, H_{36} \overset{iid}{\sim} \text{gamma}(\alpha, \beta)$, so that the common density is:

$$f(h; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} h^{\alpha-1} e^{-\frac{h}{\beta}} \qquad h > 0$$

Note that this exam works with the gamma density as parameterized in the textbook – that is, with shape parameter α and scale parameter β.

Assume further that the common mean is in fact the Hubble constant; denote this by $\eta = \mathbb{E}[H_i]$. Consider the estimators $\hat{\eta} = \bar{H} = \frac{\sum H_i}{n}$ and $\hat{\theta} = \frac{c}{\hat{\eta}}$.

Problems

1. The following parts guide you through finding the distribution of $\hat{\theta}$.

   (a) Show that $c^{-1}\bar{H}$ has a gamma $\left(n\alpha, \frac{\beta}{nc}\right)$ distribution. Write its density.

   (b) Show that $\hat{\theta} = c\bar{H}^{-1}$ has an inverse gamma distribution with density

   $$g(u; \alpha, \beta) = \frac{(cn)^{n\alpha}}{\beta^{n\alpha}\Gamma(n\alpha)} u^{-n\alpha-1} e^{-\frac{cn}{u\beta}}$$

   For short, $U \sim IG\left(n\alpha, \frac{\beta}{cn}\right)$. (Hint: for $x > 0$, the function $\frac{1}{x}$ is monotone decreasing.)

2. The following parts guide you through assessing the bias of $\hat{\theta}$.

   (a) Write the relationship between the parameters α, β and the parameter of interest θ.

   (b) Review the Wikipedia page on the inverse gamma. Pay close attention to how (their) β appears in the density function (we have used a 'shape' parameter and they have used a 'rate' parameter; one can convert directly between the two by substituting the reciprocal of one for the other in the density, CDF, moments, etc.). Find an expression for the expected value of $\hat{\theta}$ in terms of c, α, β, and n.

   (c) Find an expression for bias $\left(\hat{\theta}\right)$ in terms of n, c, α, and β.

3. The following parts have you apply and interpret your results.

   (a) Compute an estimate for the age of the universe from the data provided.

   (b) Do you expect this is an underestimate, an overestimate, or neither? Explain why in 1-2 sentences.

Supernova data from Freedman *et al.*

Supernova ID	Velocity (km/sec)	Distance (Mpc)	H_i ($km/sec/Mpc$)	$\hat{\sigma}_i$
SN1990O	9065.00	134.70	67.30	2.30
SN1990T	12012.00	158.90	75.60	3.10
SN1990af	15055.00	198.60	75.80	2.80
SN1991S	16687.00	238.90	69.80	2.80
SN1991U	9801.00	117.10	83.70	3.40
SN1991ag	4124.00	56.00	73.70	2.90
SN1992J	13707.00	183.90	74.50	3.10
SN1992P	7880.00	121.50	64.80	2.20
SN1992ae	22426.00	274.60	81.60	3.40
SN1992ag	7765.00	102.10	76.10	2.70
SN1992al	4227.00	58.00	72.80	2.40
SN1992aq	30253.00	467.00	64.70	2.40
SN1992au	18212.00	262.20	69.40	2.90
SN1992bc	5935.00	88.60	67.00	2.10
SN1992bg	10696.00	151.40	70.60	2.40
SN1992bh	13518.00	202.50	66.70	2.30
SN1992bk	17371.00	235.90	73.60	2.60
SN1992bl	12871.00	176.80	72.70	2.60
SN1992bo	5434.00	77.90	69.70	2.40
SN1992bp	23646.00	309.50	76.30	2.60
SN1992br	26318.00	391.50	67.20	3.10
SN1992bs	18997.00	280.10	67.80	2.80
SN1993B	21190.00	303.40	69.80	2.40
SN1993O	15567.00	236.10	65.90	2.10
SN1993ag	15002.00	215.40	69.60	2.40
SN1993ah	8604.00	119.70	71.90	2.90
SN1993ac	14764.00	202.30	72.90	2.70
SN1993ae	5424.00	71.80	75.60	3.10
SN1994M	7241.00	96.70	74.90	2.60
SN1994Q	8691.00	127.80	68.00	2.70
SN1994S	4847.00	66.80	72.50	2.50
SN1994T	10715.00	149.90	71.50	2.60
SN1995ac	14634.00	185.60	78.80	2.70
SN1995ak	6673.00	82.40	80.90	2.80
SN1996C	9024.00	136.00	66.30	2.50
SN1996bl	10446.00	132.70	78.70	2.70

Table 1: Velocity and distance measurements for 36 type 1a supernovae. Data from Freedman, Wendy L., et al. "Final results from the Hubble Space Telescope key project to measure the Hubble constant." The Astrophysical Journal 553.1 (2001): 47.