Quiz 2

PSTAT 120B

Spring 2023

Instructions:

WRITE CLEARLY and neatly on a sheet of paper. Make sure to write your full name, as listed on GauchoSpace, and your perm number at the top of the page in order to receive credit. You should **try not to** use your notes to complete this quiz; we encourage you to attempt it without assistance in order to check your understanding of the material.

Problem:

1. Let Y be a random variable with a density function given by

$$f(y) = \begin{cases} \frac{3}{2}y^2, & -1 \leq y \leq 1, \\ 0, & elsewhere. \end{cases}$$

These three densities will be easiest to find/derive using the method of direct integration (or the method of distribution functions).

(a) Find the density function of $U_1 = 3Y$.

First, we identify the region of support for U_1, which is from -3 (for $y = -1$) to 3 (for $y = 1$). Then we write the CDF of U_1 in terms of Y:

$$\begin{aligned} F(u) &= P(U \leq u) \\ &= P(3Y \leq u) \\ &= P(Y \leq \frac{u}{3}) \end{aligned}$$

Then we use integration to find this; first we have to find the CDF of Y, or $F_Y(y)$:

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{\infty} f(y)dy \\ &= \int_{-1}^{y} \frac{3}{2}y^2 dy \\ &= \frac{3}{2}\left[\frac{y^3}{3}\right]_{-1}^{y} \\ F(y) &= \frac{1}{2}(y^3 - 1) \end{aligned}$$

Then:

$$\begin{aligned} F_{U_1}(u) &= F_Y\left(\frac{u}{3}\right) \\ &= \frac{1}{2}\left(\left(\frac{u}{3}\right)^3 - 1\right) \\ &= \frac{1}{2}\left(\frac{u^3}{18} - 1\right) \end{aligned}$$

and the density is as follows for $-3 \le U_1 \le 3$ and 0 otherwise.

$$\begin{aligned} f_{U_1}(u) &= F_{U_1}^{-1}(u) \\ &= \frac{d}{du_1}\frac{1}{2}\left(\frac{u^3}{18} - 1\right) \\ &= \frac{1}{12}u^2 \end{aligned}$$

(b) Find the density function of $U_2 = 3 - Y$.

First, since Y can range from -1 to 1, U can range from/is supported from 2 to 4. Then:

$$\begin{aligned} F_{U_2}(u) &= P(U_2 \le u) \\ &= P(3 - Y \le u) \\ &= P(-Y \le u - 3) \\ &= P(Y \ge 3 - u) \\ F_{U_2}(u) &= 1 - F_Y(3 - u) \end{aligned}$$

and $F_Y(3-u) = \frac{1}{2}\left[(3-u)^3 - 1\right]$, so $1 - F_Y(3-u) = 1 - \left[\frac{(3-u)^3}{2} - \frac{1}{2}\right]$.

Then $f_{U_2}(u) = F_{U_2}^{-1}(u) = \frac{3}{2}(3-u)^2$ for the region $2 \le U \le 4$.

(c) Find the density function of $U_3 = Y^2$.

First, U_3 can range from 0 to 1. Then:

$$\begin{aligned} F_{U_3}(u) &= P(U_3 \le u) \\ &= P(Y^2 \le u) \\ &= P(-\sqrt{u} \le Y \le \sqrt{u}) \\ &= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) \\ F_{U_3}(u) &= u^{\frac{3}{2}} \end{aligned}$$

Then $f_{U_3}(u) = F_{U_3}^{-1}(u) = \frac{3}{2}\sqrt{u}$ for $0 \le u \le 1$.

(d) Find $E[U_1]$, $E[U_2]$, and $E[U_3]$.

$$
\begin{aligned}
E[U_1] &= \int_{-3}^{3} u_1 f(u_1) du_1 \\
&= \int_{-3}^{3} u_1^3 \frac{1}{12} du_1 \\
&= \frac{1}{12} \left[\frac{{u_1}^4}{4} \right]_{-3}^{3} \\
&= \frac{1}{12}(20.25 - 20.25) \\
E[U_1] &= 0
\end{aligned}
$$

$$
\begin{aligned}
E[U_2] &= \int_{2}^{4} u_2 f(u_2) du_2 \\
&= \int_{2}^{4} u_2 \frac{3}{2}(3 - u_2)^2 \\
&= \left[\frac{3(u_2 - 3)^3(u^2 + 1)}{8} \right]_{2}^{4} \\
E[U_2] &= 3
\end{aligned}
$$

and

$$
\begin{aligned}
E[U_3] &= \int_{0}^{1} u_3 \frac{3}{2} \sqrt{u_3} du_3 \\
&= \frac{3}{2} \int_{0}^{1} u_3^{\frac{3}{2}} \\
E[U_3] &= 0.6
\end{aligned}
$$