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Flat Foldability of Origami

The problem of flat foldability of origami has confounded both professional origami artists as well as computer science theorists for many years. Two theorists, Barry Hayes and Marshall Bern, have written a paper, "The Complexity of Flat Origami", regarding the complexity of this problem and how it relates to the NP-Complete theory in computer science. The paper states that when given a crease pattern (a square sheet of paper with lines designating folds), determining flat foldability of an origami figure (when the origami is compressed, it will be essentially flat) about a single vertex can be done in linear time, and determining flat foldability of an entire figure is an NP-Complete problem. "The Complexity of Flat Origami" tackles these proofs in three parts: proving flat foldability about each vertex, about an entire figure not given fold orientations nor overlap order, and about an entire figure given fold orientations but not fold overlap order. The two authors of the proof cleverly applied ideas from computer science theory to reduce a tangible puzzle to a complex math problem.

The first of these brilliant computer scientists, Marshall Bern, developed a passion for solving difficult problems through math and computer logic in his years as a student at Yale, University of Texas, and University of California Berkeley where he studied math, applied math, and computer science respectively. Among paper folding, other problems he uses this logic to solve include developing algorithms for protein identification and bioinformatics. It was during his PhD years at Berkeley when he first met and worked with his partner Barry Hayes who at the time was a PhD student of Eric Roberts at Stanford University. Barry Hayes currently works with his math and computer skills to develop wind energy for grid applications and for distribution network analysis. The two first came across the problem when Bern redeveloped an interest in a problem that he had previously solved: that any disk packing can be extended to one with only 3-sided gaps. He had used recursive subdivision to solve this problem, but another person asked this question in the context of modern origami. The website where he saw this question also had a paper about approximate conditions for flat foldability which Bern had on his dining room table one day when Hayes paid a visit. It was at this moment in 1995 when Bern challenged Hayes to prove the NP-Completeness of flat foldability, which engendered “The Complexity of Flat Origami”.

To understand what flat-foldability is, we first define a few key terms. A crease pattern will be defined as a pattern of lines on a square paper that describes where the folds lie. Some crease patterns will describe the orientation of the folds to be either mountain or valley fold. A mountain fold is when the folded vertex is higher than each adjacent side, and a valley fold is when the folded vertex is lower than each adjacent side. In other words, a mountain fold is shaped like ^ when viewed from the side, and a valley fold is a V shape when viewed from the side. In addition, the overlap order describes the sequence in which each flap will be folded. For instance, if the paper is folded into three equal rectangles, determining whether the leftmost rectangle will fold over or under the rightmost rectangle describes overlap order. Finally, flat foldability is the ability of a crease pattern to fold to a flat surface when compressed, given the fold orientations and overlap order. For example, a paper box would not be considered flat foldable, whereas a triangle formed by folding along the diagonal of a square paper would be flat foldable.

Perhaps the most important term to understand, however, is the concept of NP-Completeness. NP-Completeness is a class of decision problems, virtual or real, that cannot be solved by a computer in polynomial time. In the case of flat foldability, the decision problem is deciding whether not a given crease pattern may be folded into an origami structure that can compress into a seemingly flat state upon completion. NP-Complete problems are different from normal NP problems in that if you prove one NP-Complete problem to be solvable in polynomial time, they are all solvable in polynomial time. This is because all NP-Complete problems are a reduction of some similar function, namely 3-SAT. In proving flat foldability of a figure to be NP-Complete, Bern and Marshall show the reduction from “not-all-same” 3-SAT (a slight variation from true 3-SAT in which you cannot have 3 variables in a clause with the same truth value) using what they call wires which represent the variables of 3-SAT clauses and gadgets where wires interact or change values. Some other notorious NP-Complete problems such as the Knapsack Problem or the Traveling Salesman Problem have similar structures in proving relations to 3-SAT. This paper will take a closer look at these wires and gadgets, and how they help prove not-all-same 3-SAT in later sections. However, before getting into this dense proof, one must first understand flat foldability about a single vertex, which is not-so-coincidentally the first part of Bern’s and Hayes’ paper.

We will first break down the problem into the simplest case - the complexity of flat origami involves determining foldability about a single vertex. A vertex is defined as the point where two or more folds meet. Two conditions essentially determine foldability around a vertex: first, that the difference between the number of mountain folds and the number of valley folds should be either two, zero, or negative two, and second, that the sum of alternating angles around the vertex must be a scalar multiple of pi.

The proof for the first conditions is as follows: first, let an angle that is smaller than either of its adjacent angles x, and let the edges that make up x be x-1 and x+1 respectively. Then, the edge x-1 and x+1 must have different fold orientations. We can determine this intuitively; if both edges were the same orientation, one edge would not fold inside the other, therefore making it not flat foldable. For example, let a crease pattern describe three rectangles on a square paper defined by two parallel creases. Given that the middle rectangle is thinner than its two adjacent rectangles, we determine both of its adjacent rectangles to be valley folds and are consequently folding over the center rectangle. Then, the left flap cannot fit under the right flap because it is thicker than the middle rectangle, and vice versa, so two valley folds cannot fold flat. Since these fold orientations must be different all the way around the vertex, this describes the first condition mentioned earlier.

With this condition satisfied, we move on to prove the second condition. First, we subtract the measure of the small angle from one of its adjacent angles and form a new angle, deleting both the original small angle and the sector we previously subtracted out from the adjacent angle. Then, we merge each adjacent angle together. For example, if we have angle measurements of sixty degrees, twenty degrees, and ninety degrees, we will subtract twenty degrees from the ninety degree angle. After forming the new seventy degree angle, we merge the sixty and seventy degree angle together, and repeat the process with other angles around the vertex. We recursively perform this procedure until we are left with two angles. For this resulting figure to be flat foldable, the last two angles must have equal measures, which describes the first condition. This method can be programmed in linear time, and thus, determining flat foldability around a single vertex can be run in polynomial time.

Applying this calculation to a larger perspective, determining the flat foldability of all vertices simultaneously on a crease pattern can also be calculated in polynomial time. In this portion, we will only consider flat foldability around each vertex, and the edges or planes that connect the vertices do not have to be flat yet. Determining whether all vertices fold flat simultaneously requires relatively the same procedure as determining flat foldability around one vertex. The additional problem lies in having to merge each vertex such that they will can all occur on the same paper. Thus, given that all vertices meet the two conditions described in the previous paragraph, we consider two different cases for each edge. Two vertices may share an edge, and the fold orientation of the edge from each vertex may be the same. For example, two vertices may share an edge that is a mountain fold. If all edges are consistent around each vertex, then the figure will be flat foldable because each vertex is flat foldable. However, if the edge contradicts from each vertex, for instance the edge is a mountain fold going from one vertex but is determined to be a valley fold going from the other vertex, then we run into a problem. In order for this figure to be vertex flat foldable, these contradictions must all cancel each other out. For example, another edge may also be a mountain fold from one end and a valley fold from the other end, and this contradiction will cancel out the previous contradiction, forming a figure that is vertex flat foldable. Because this calculation requires a union of the linear time operations in the single-vertex flat foldability calculations, determining flat foldability around all vertices also runs in polynomial time.

With a basic understanding of how folds work and how vertices interact, we can now expand our horizons and look at the flat foldability of an entire figure. The bulk of Bern's and Hayes' work was spent applying this basic knowledge of flat foldability on a large complex scale to prove that deciding flat foldability for a given crease pattern is NP-Complete. They accomplished this feat by showing how the not-all-same 3-SAT problem could be reduced down to origami terms which is how most, if not all, problems are proved to be NP-Complete. To do this, they created a set of axioms that would allow them to more easily prove the similarities to 3-SAT. The two NP-Complete problems that the computer scientists examined are assigned different axioms because they both have slightly different conditions.

The first proof we will look at is flat foldability of a figure given a crease pattern without fold orientations or overlap order. The axioms Bern and Hayes use to prove relation to not-all-same 3-SAT are as follows. First, one should understand the concept of a “wire”. One wire represents one variable in a given clause, and each wire (or variable) is given a true or false value. Bern and Hayes represent each wire with two very close folds that run parallel with each other throughout the entire crease pattern. A wire is said to hold a value of true if the right fold (going into the paper) is a valley fold and the left fold is a mountain fold. A wire is false if the right fold is mountain and the left fold is valley. Each of these wires containing two close parallel folds enters the paper in groups of three. These groups of three represent the clauses of a not-all-same 3-SAT problem. Because these wires can enter from different sides of the paper, they have the potential to intersect. Where these wires intersect we have different geometric shapes called “gadgets”. These gadgets can be thought sort of as the “and” operators that connect the clauses of a not-all-same 3-SAT problem. Bern and Hayes use three different kinds of gadgets, each with a unique set of axioms that must be fulfilled in order for the figure to fold flat. If all of these axioms are met and all gadgets fold flat, the entire figure will fold flat.

The first of these gadgets is an equilateral triangle in which three wires of equal width intersect with certain truth values. Each side of the triangle will be a valley fold if the wire intersecting that side is true and a mountain fold if the wire intersecting that side is false. The triangle gadget is said to fold flat if two of the sides are valley folds (two incoming true wires) or if two of the sides are mountain folds (two incoming false wires). In other words, the triangle cannot have three incoming wires of the same value or it will not fold flat. The second gadget is called a “reflector”, that is, an isosceles triangle in which two wires of equal width intersect at the equal sides of the triangle and a wider wire intersects the larger side of the triangle. Again, the sides are valley or mountain folds based on the values of the incoming wires. The reflector gadget is said to fold flat if the two wires of equal width have the same value and the wider wire has the opposite value. Reflectors should be thought of less as an intersection of three wires and more as one wire changing directions with the same value after hitting the wide side of the triangle where the wider wire is a bi-product of the direction change. These bi-product wires are referred to as “noise” because they are simply the result of a gadget rather than a contributor to the gadget’s flat foldability. They simply run off to an edge of the paper, sometimes intersecting with other wires. The third and final gadget is called a “cross-over”: a parallelogram in which 2 wires intersect at adjacent sides and continue out of the opposite sides with the same value. The figure is said to fold flat if each pair of incoming and outgoing wires have the same value.

While the specificities of these gadgets seem trivial, they all serve an important purpose: they allow the intersection and manipulation of wires which in turn allows origami artists to make more complex crease patterns and still prove them to be flat foldable. The combination of wires and gadgets lead to complex maps with input wires whose values determine the flat foldability of the gadgets and thus of an entire figure. Because the values of these clauses determine the value of the entire figure, the model is quite similar to the model of not-all-same 3-SAT. Deciding what the values of the wires and clauses should be is essentially the NP-Complete problem, as it is in not-all-same 3-SAT..

Determining flat foldability given the fold orientations is similar to the previously discussed case, except that we now concern ourselves with pleats instead of edges. Since we know the fold orientation of each edge, we have to determine whether each pleat will fold flat. We will redefine wires as four creases in a mountain, valley, valley, and mountain pattern respectively. The wire will be true if the left pleat folds over the right pleat, and false if the left pleat folds under the right pleat. Again, these definitions correspond to the clauses of the not-all-same 3-SAT problem. We determine the truth setting of one input wire, and since we are given the fold orientations, the truth setting of all other wires may be determined with regards to the input wire truth setting. However, we encounter a problem when we determine all true or all false settings for each wire, because this produces a spiral shape that cannot be compressed to a flat structure. As a result, we introduce tabs, which are rectangular flaps folded into the paper. These tabs restrict the set of possible overlap orders, consequently avoiding the truth settings that cause the paper to not fold flat. The truth setting of the input wire determines the overlap order of the tabs of the input wire, which determines the overlap order of the remaining tabs, and ultimately forces the correct overlap order of the other wires. Therefore, this determines the overlap order of the given crease pattern.

By first proving vertex flat foldability, flat foldability by determining fold orientation and overlap order, and flat foldability given fold orientation, we conclude that determining flat foldability of origami is NP-Complete. In addition to adding origami to the realm of NP-Complete problems, the property of origami to fold flat has proved itself to have both theoretical and practical contributions. A program called the TreeMaker, by Dr Robert Lang, generates a crease pattern with assigned fold orientations with an input of a stick figure specifying the lengths and connections between flaps. This is revolutionary for origami artists because, although it is still possible to form new origami designs, the TreeMaker makes designing much easier because it generates a relatively accurate base for the figure. In addition, it paves the way for future origami and computer interaction, as it introduces the concept of using computer algorithms to calculate origami folds.

The concept of flat foldability can also take a more practical approach and be applied to products used in our lives. For example, an unexpected use of origami lies in manufacturing airbags. The optimal airbags must generally be able to fold inside a small space, but they must also be able to deflate very quickly. Therefore, the flat foldability of the airbag must be calculated, but each flap must also be minimized, since large flats take longer to deploy. Origami principles consequently aid in determining the crease pattern and overlap order of the airbag. Another example of the practical application of origami flat folding lies in the Eyeglass telescope. The Eyeglass telescope was planned to be roughly the size of a football field, but the problem lay in transporting such a big instrument to space. Origami principles were again used to compress and fold the telescope to an appropriate size for delivery. Therefore, flat foldability can be applied to both theoretical and practical aspects.

Until the 1960s, origami has generally remained stagnant in its practice because it stayed within its own discipline. However, as people began applying computer science theories and mathematical principles to origami, the realm of possibilities has expanded exponentially by proving itself to be a useful tool in manufacturing products among other things. Bern's and Hayes' work has played a significant role in this advancement of origami, simply by proving how hard it is to meet a specific condition (flat foldability) when not given much information (crease patterns and overlap order). By understanding the difficulty of modeling origami to meet certain needs, origami artists, theorists, manufacturers and innovators will all be able to approach origami in more diligent and creative manner. Every advancement in origami also inspires others to push the boundaries on what origami can do, leading to more beautiful art and revolutionary applications to solving real world problems. This has been the goal of applying mathematical and computer science theories to the ancient Japanese art of paper folding and is what Bern and Hayes sought to prove by investing their efforts to prove flat foldability and, if anything, showing that computer science has the ability to expand seemingly unrelated fields.