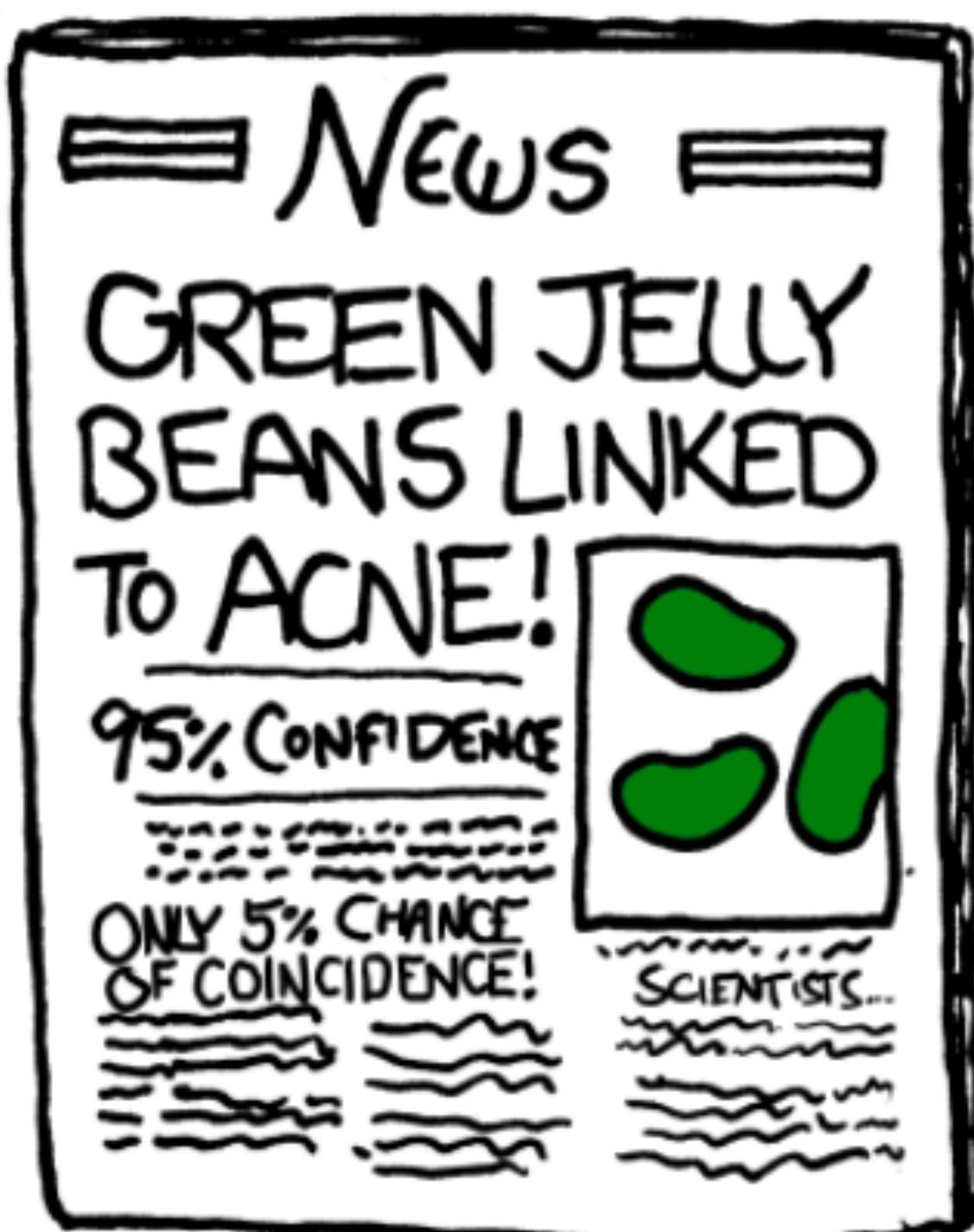


# Lecture 08

10.19.21



# Refresher Quiz

The blood pressure (average of systolic and diastolic measurements) of each of 25 persons were measured. The average was 94.5 mm Hg with a variance of 225. Use a *t*-test to decide if the underlying population could have a mean of 100 mm Hg.

The blood pressure (average of systolic and diastolic measurements) of each of 25 persons were measured. The average was 94.5 mm Hg with a variance of 225. Use a *t*-test to decide if the underlying population could have a mean of 100 mm Hg.

	P							
one-tail	0.1	0.05	0.025	0.01	0.005	0.001	0.0005	
two-tails	0.2	0.1	0.05	0.02	0.01	0.002	0.001	
DF								
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
20	1.325	1.725	2.086	2.528	2.845	3.552	3.85	
21	1.323	1.721	2.08	2.518	2.831	3.527	3.819	
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792	
23	1.319	1.714	2.069	2.5	2.807	3.485	3.768	
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745	
25	1.316	1.708	2.06	2.485	2.787	3.45	3.725	
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707	
27	1.314	1.703	2.052	2.473	2.771	3.421	3.689	
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674	
29	1.311	1.699	2.045	2.462	2.756	3.396	3.66	
30	1.31	1.697	2.042	2.457	2.75	3.385	3.646	
60	1.296	1.671	2	2.39	2.66	3.232	3.46	

The blood pressure (average of systolic and diastolic measurements) of each of 25 persons were measured. The average was 94.5 mm Hg with a variance of 225. Use a *t*-test to decide if the underlying population could have a mean of 100 mm Hg.

**1. Generate a hypothesis and choose a significance level**

$$H_0 : \mu = 100 \quad H_A : \mu \neq 100 \quad \alpha = 0.05$$

**2. Calculate test statistic**

$$t_s = \frac{\bar{y} - \mu}{s/\sqrt{n}} = \frac{94.5 - 100}{\sqrt{225}/\sqrt{25}} = -1.83$$

**3. Calculate the *P*-value**

$H_A : \mu \neq 100$

$\alpha = 0.05$

$t_s = -1.83$

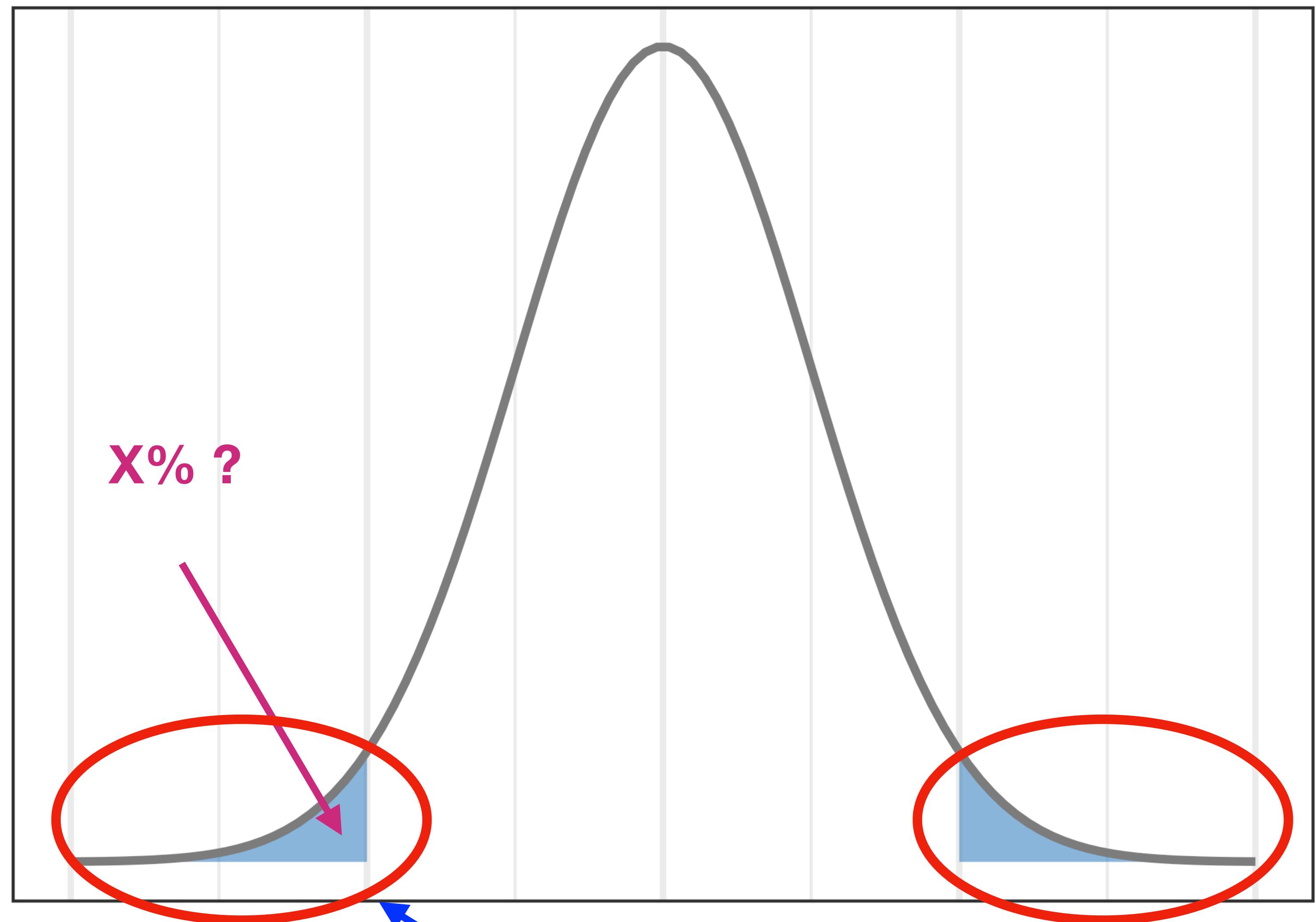
```
> pt(-1.83, 24)*2
```

[1] 0.07969884

$P > \alpha$

**Fail to reject the null!**

### Student's $t$ distribution (df = 37)



$t_s$

$H_A : \mu \neq 100$

$\alpha = 0.05$

$t_S = -1.83$

```
> pt(-1.83, 24)*2
```

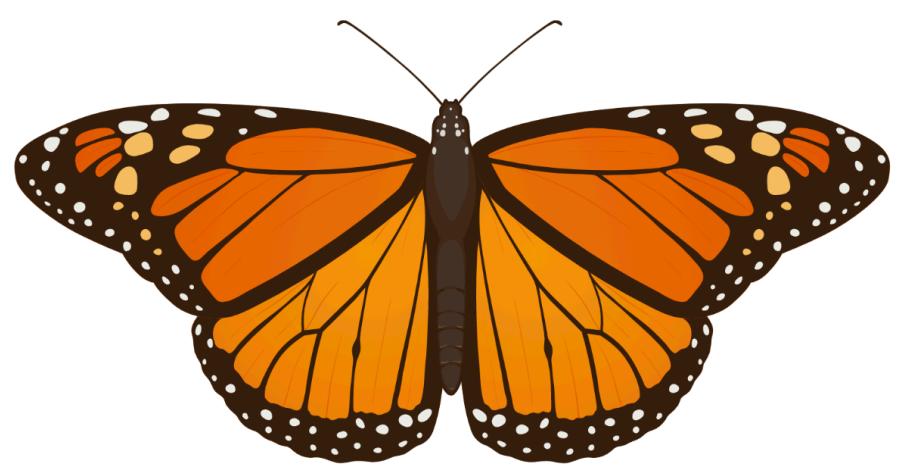
```
[1] 0.07969884
```

$P > \alpha$

**Fail to reject the null!**

	P							
one-tail	0.1	0.05	0.025	0.01	0.005	0.001	0.0005	
two-tails	0.2	0.1	0.05	0.02	0.01	0.002	0.001	
DF								
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
20	1.325	1.725	2.086	2.528	2.845	3.552	3.85	
21	1.323	1.721	2.08	2.518	2.831	3.527	3.819	
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792	
23	1.319	1.714	2.069	2.5	2.807	3.485	3.768	
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745	
25	1.316	1.708	2.06	2.485	2.787	3.45	3.725	
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707	
27	1.314	1.703	2.052	2.473	2.771	3.421	3.689	
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674	
29	1.311	1.699	2.045	2.462	2.756	3.396	3.66	
30	1.31	1.697	2.042	2.457	2.75	3.385	3.646	
60	1.296	1.671	2	2.39	2.66	3.232	3.46	

# The $t$ statistic for hypothesis testing



$$\bar{y} = 32.81 \text{ cm}^2$$

$$s = 2.48 \text{ cm}^2$$

$$\mu = 30 \text{ cm}^2$$

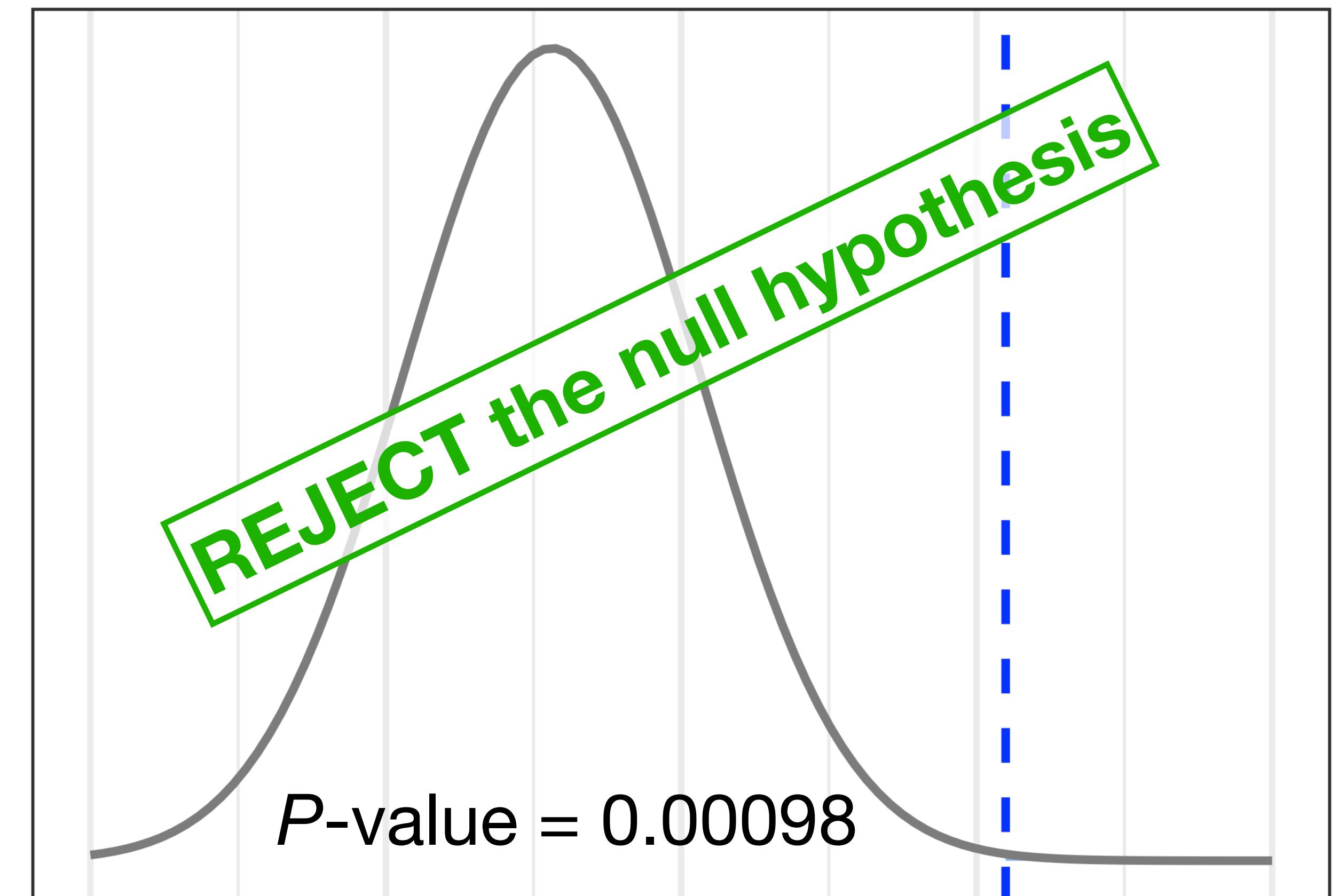
$$\alpha = 0.05$$

$$H_0 : \bar{y} = \mu$$

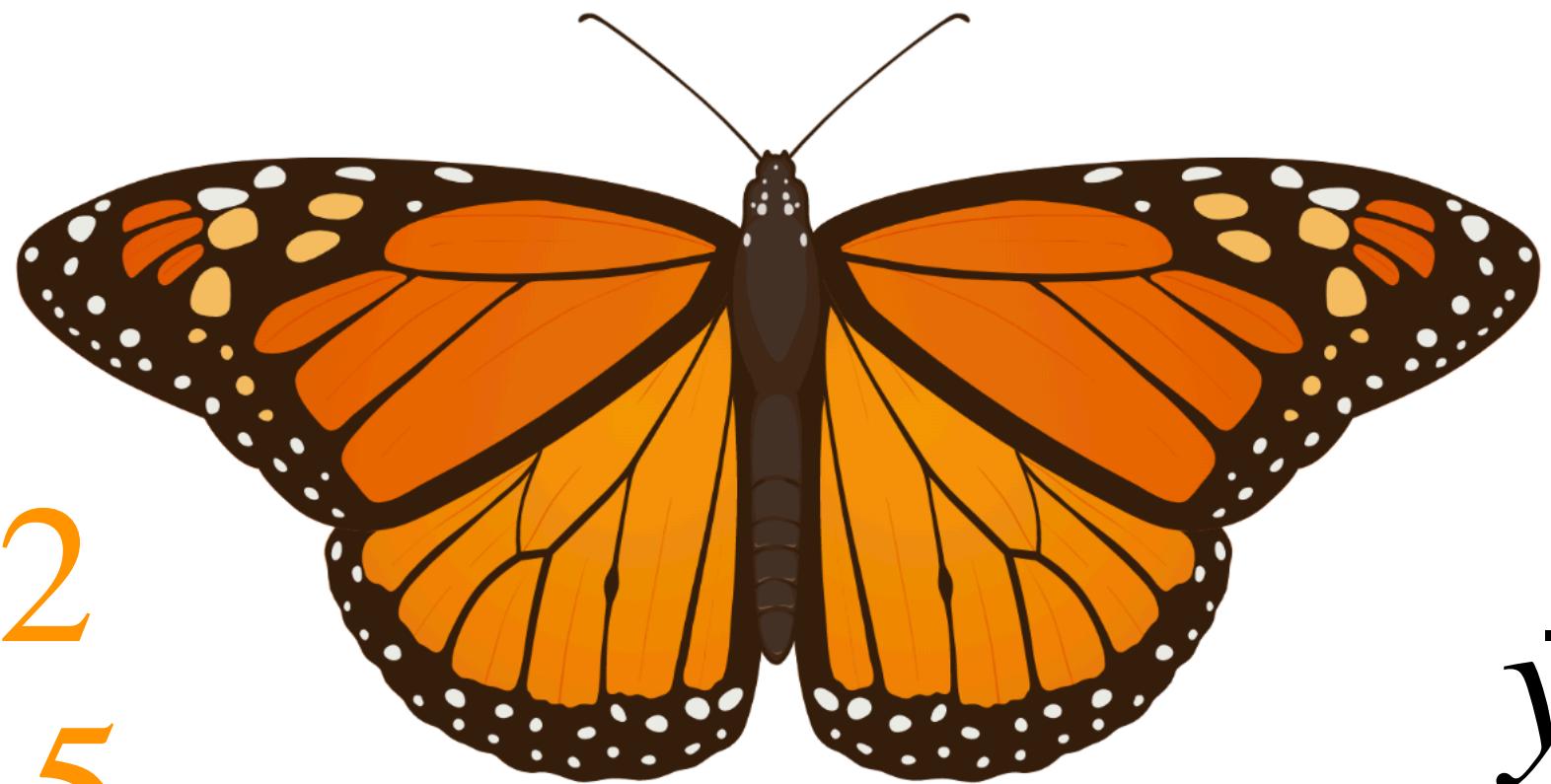
$$H_A : \bar{y} \neq \mu$$

```
> pt(4.23, 13, lower.tail = F)*2
```

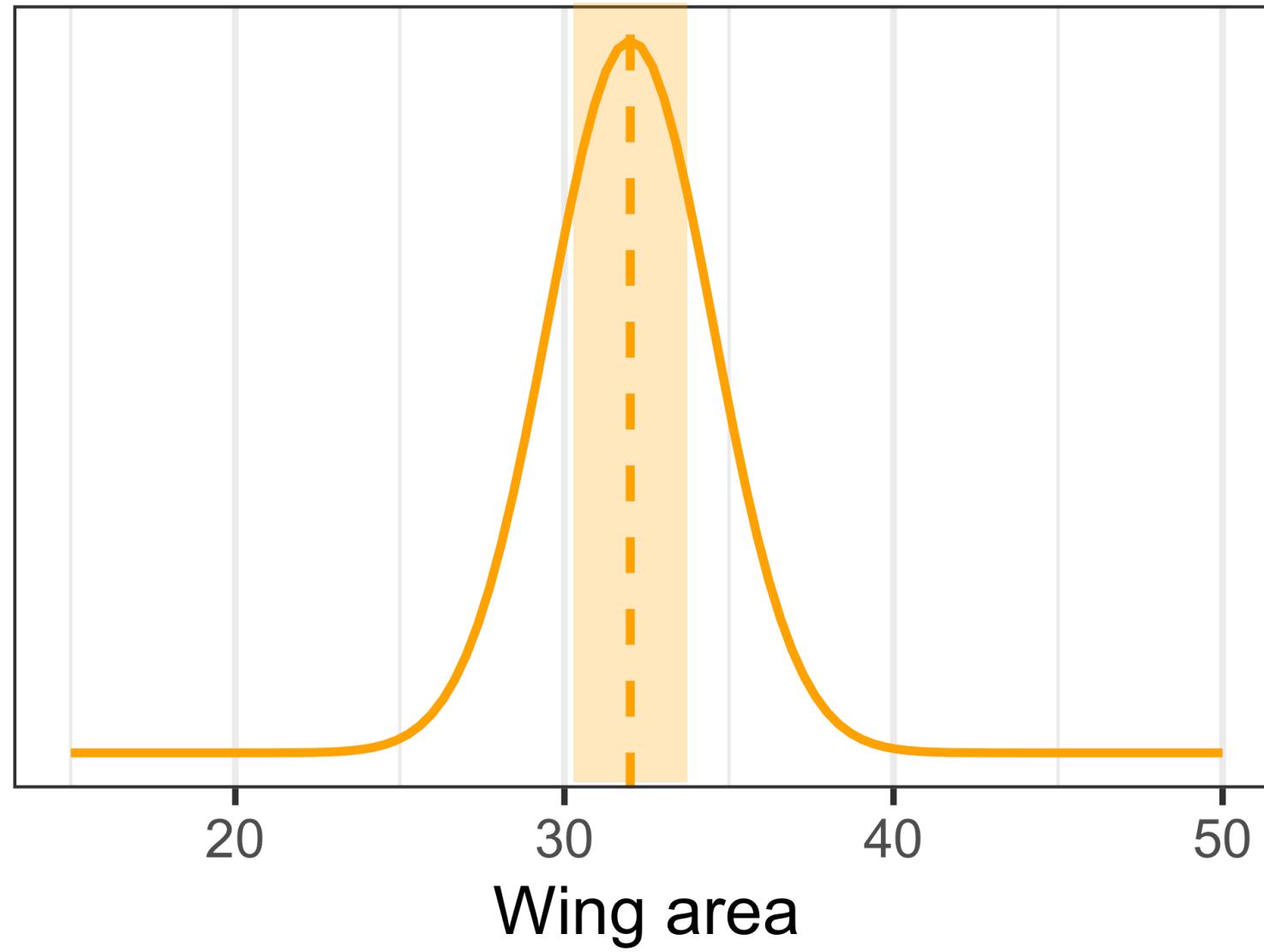
$$t_s = 4.23$$



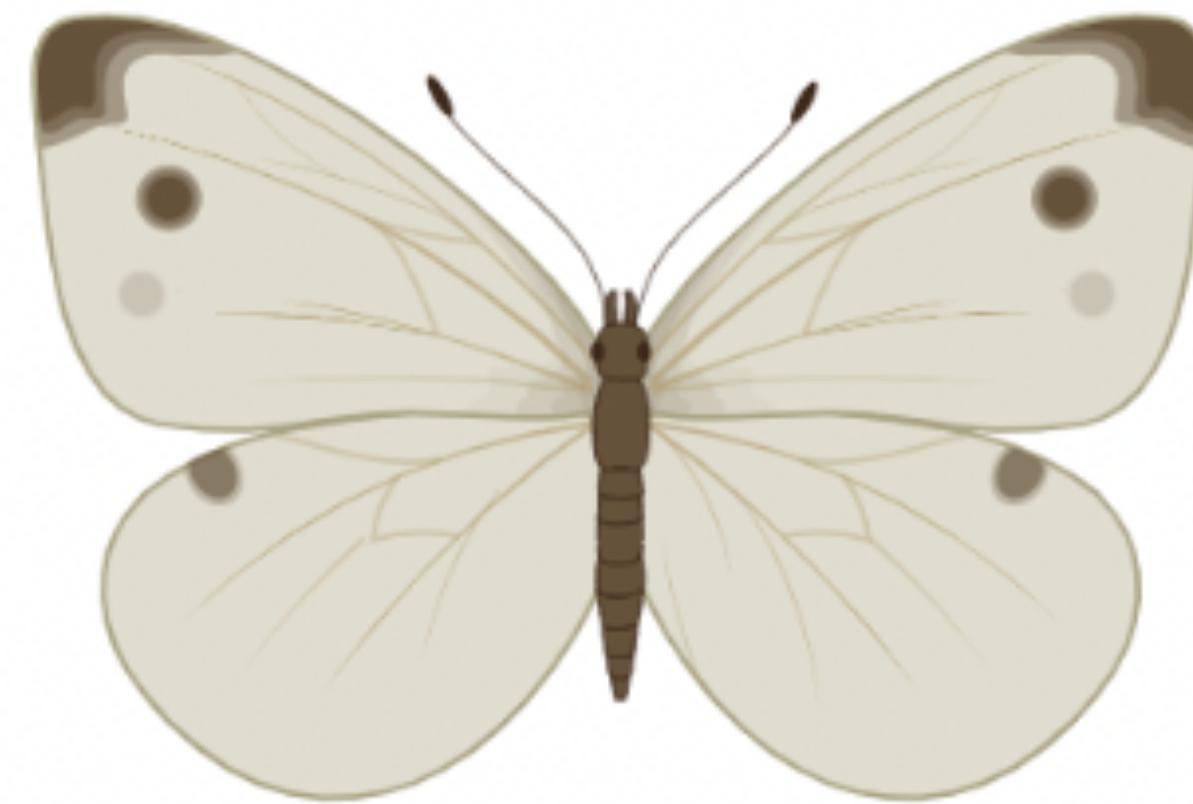
# Comparing populations: difference between means



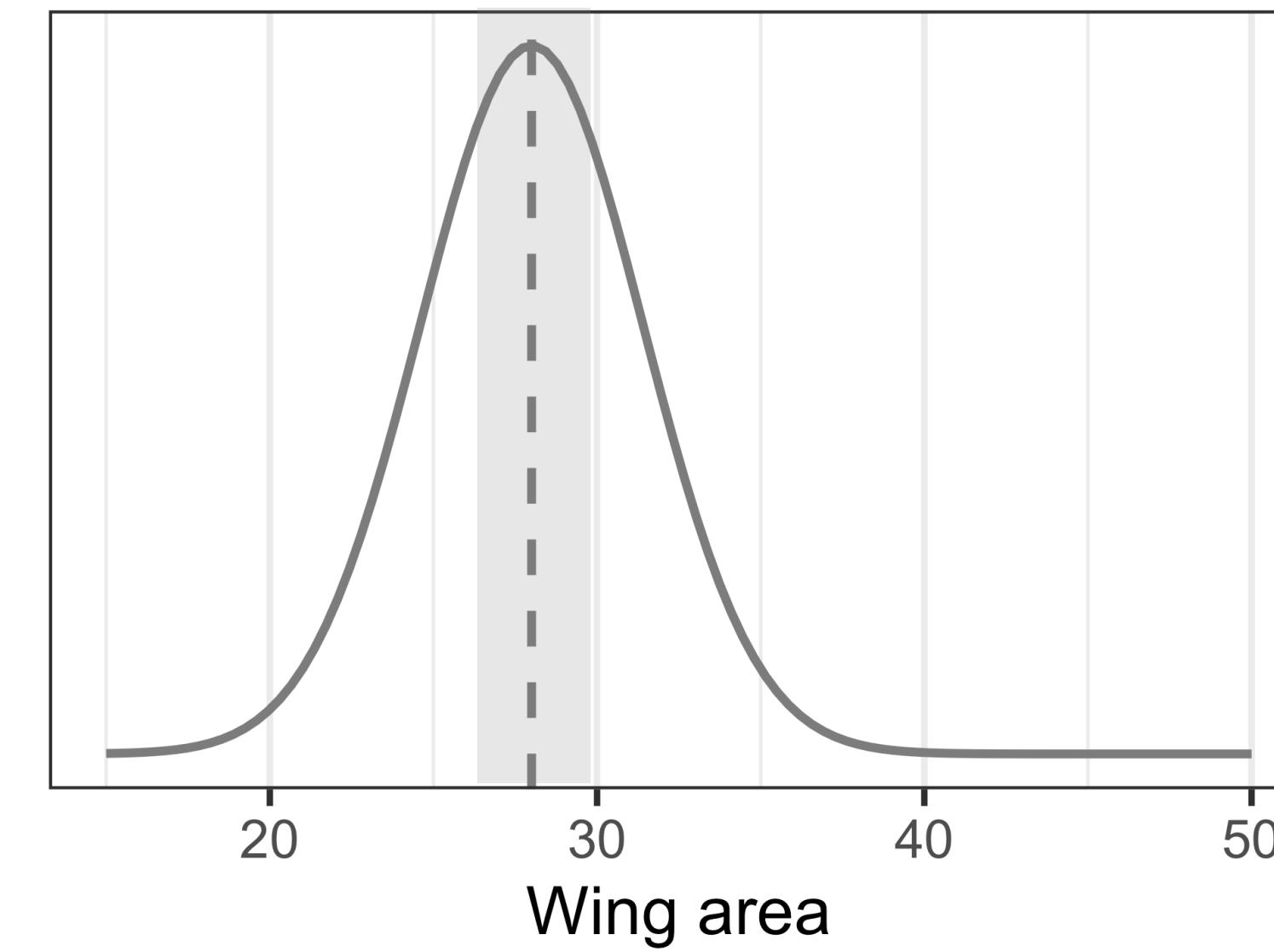
$$\bar{y}_1 = 32$$
$$s_1 = 2.5$$



$$\bar{y}_1 - \bar{y}_2 = 4$$



$$\bar{y}_2 = 28$$
$$s_2 = 3.4$$



# Comparing populations: hypothesis testing

**The null hypothesis:**

$$H_0 : \mu_1 = \mu_2$$

**The alternative hypothesis:**

$$H_A : \mu_1 \neq \mu_2$$

# An introduction to hypothesis testing

The city of Chicago received 48.8 inches of snow last winter (2020-2021) and 34.8 inches of snow the previous winter (2019-2020).

**The null hypothesis:**

$$H_0 : \mu_1 = \mu_2$$

**Chicago received the same amount of snow in 2019 and 2020**

**The alternative hypothesis:**

$$H_A : \mu_1 \neq \mu_2$$

**Chicago received a different amount of snow in 2019 and 2020**

# Comparing populations: hypothesis testing

In a certain clinical trial, 30 patients received the drug treatment and 30 patients received the placebo. After two months of treatment, disease progress was measured.

**The null hypothesis:**

$$H_0 : \mu_1 = \mu_2$$

**The patients in the control group have the same disease progression as those in the treatment group.**

**The alternative hypothesis:**

$$H_A : \mu_1 \neq \mu_2$$

**The patients in the control group have a different disease progression as those in the treatment group.**

# Quick note: association vs. causation

In a certain clinical trial, 30 patients received the drug treatment and 30 patients received the placebo. After two months of treatment, disease progress was measured.

- **Association is not causation**
- Harder to determine cause and effect relationship from observational study
  - *Could be confounding facts*

**The alternative hypothesis:**

$$H_A : \mu_1 \neq \mu_2$$

**The patients in the control group have a different disease progression as those in the treatment group.**

# Comparing populations: hypothesis testing

**The null hypothesis:**

$$H_0 : \mu_1 = \mu_2 \longrightarrow H_0 : \underline{\mu_1 - \mu_2} = 0$$

**The alternative hypothesis:**

$$H_A : \mu_1 \neq \mu_2 \longrightarrow H_A : \underline{\mu_1 - \mu_2} \neq 0$$

***How do we choose between these two hypotheses?***

# Comparing populations: the $t$ statistic

The  $t$  test is a standard method of choosing between these hypotheses

$$H_0 : \mu_1 = \mu_2$$

$$H_A : \mu_1 \neq \mu_2$$

*t is in units of SE*

**Test statistic:**

Variation in differences of  
means from random samples

$$t_s = \frac{\bar{y}_1 - \bar{y}_2}{SE_{\bar{Y}_1 - \bar{Y}_2}}$$

How far the difference  
between the two means  
are from 0 (null hypothesis)

# Adding and subtracting random variables



$$\mu_1 = 300; \sigma_1 = 22$$

$$\mu_2 = 368; \sigma_2 = 26$$

What is the overall difference in means between the populations (black and white)?

What is the variance in the difference of means between these two populations?

$$\begin{aligned}\mu_{1-2} &= \mu_1 - \mu_2 \\ &= -68\end{aligned}$$

$$\sigma_{1-2}^2 = \sigma_1^2 + \sigma_2^2$$

$$\begin{aligned}\sigma_{1-2}^2 &= 22^2 + 26^2 \\ &= 1160\end{aligned}$$

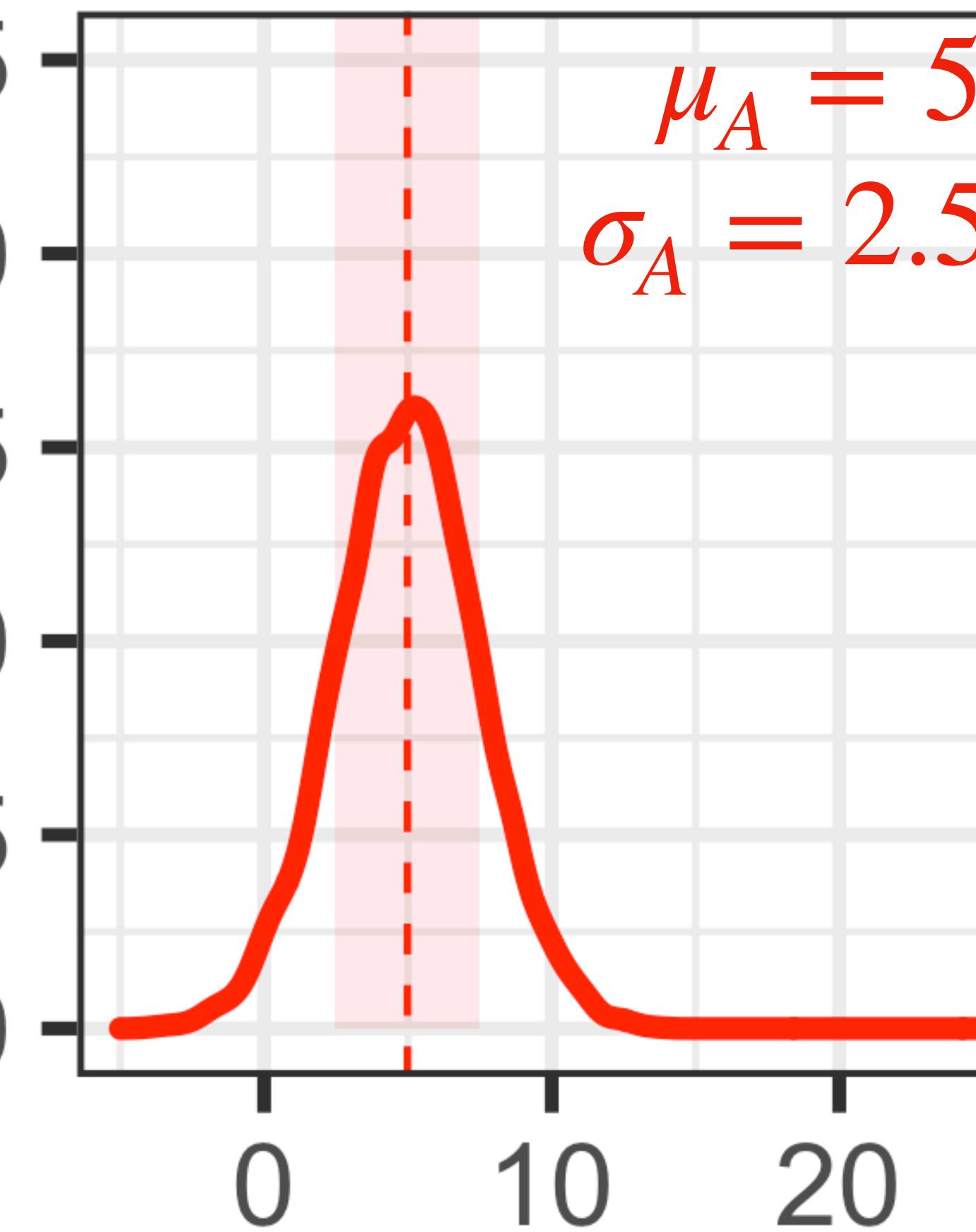
$$\sigma_{1-2} = 34.06$$

If you want to see the math: link [here](#)

Relative frequency

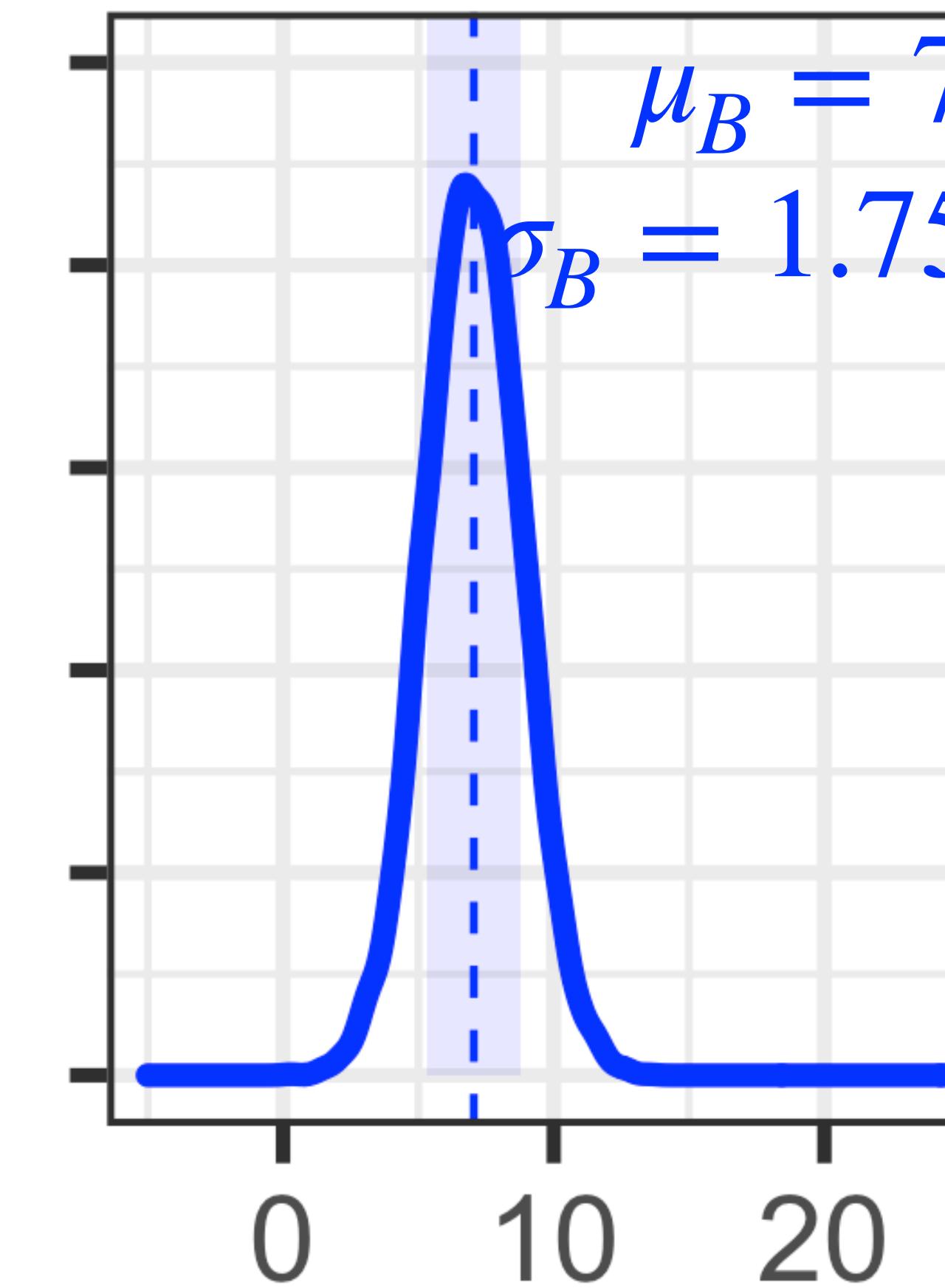
**Var A = {11, 5, 4, 3, 5}**

$$\begin{aligned}\mu_A &= 5 \\ \sigma_A &= 2.5\end{aligned}$$



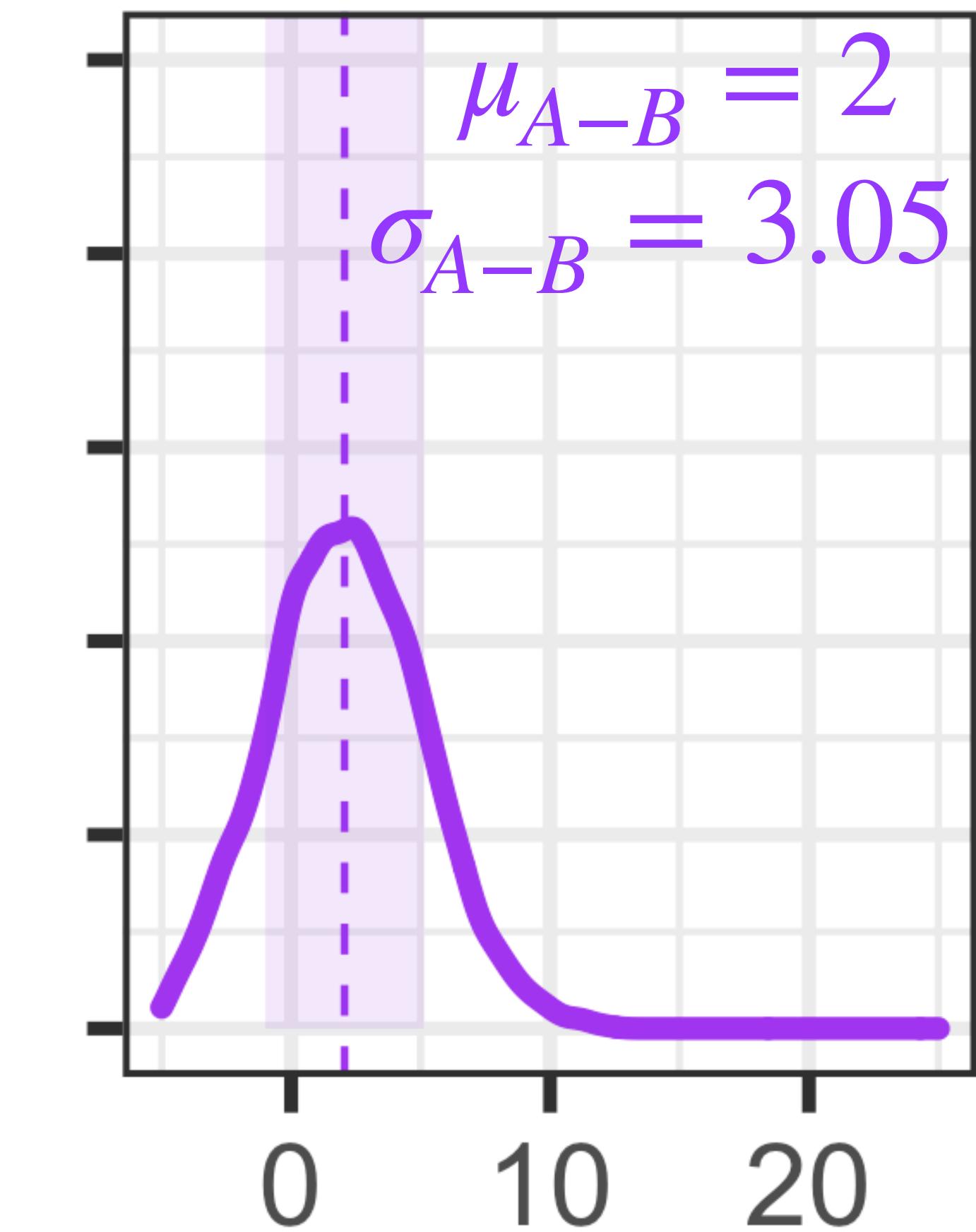
**Var B = {6.5, 6.6, 6.8, 6.5, 6.1}**

$$\begin{aligned}\mu_B &= 7 \\ \sigma_B &= 1.75\end{aligned}$$



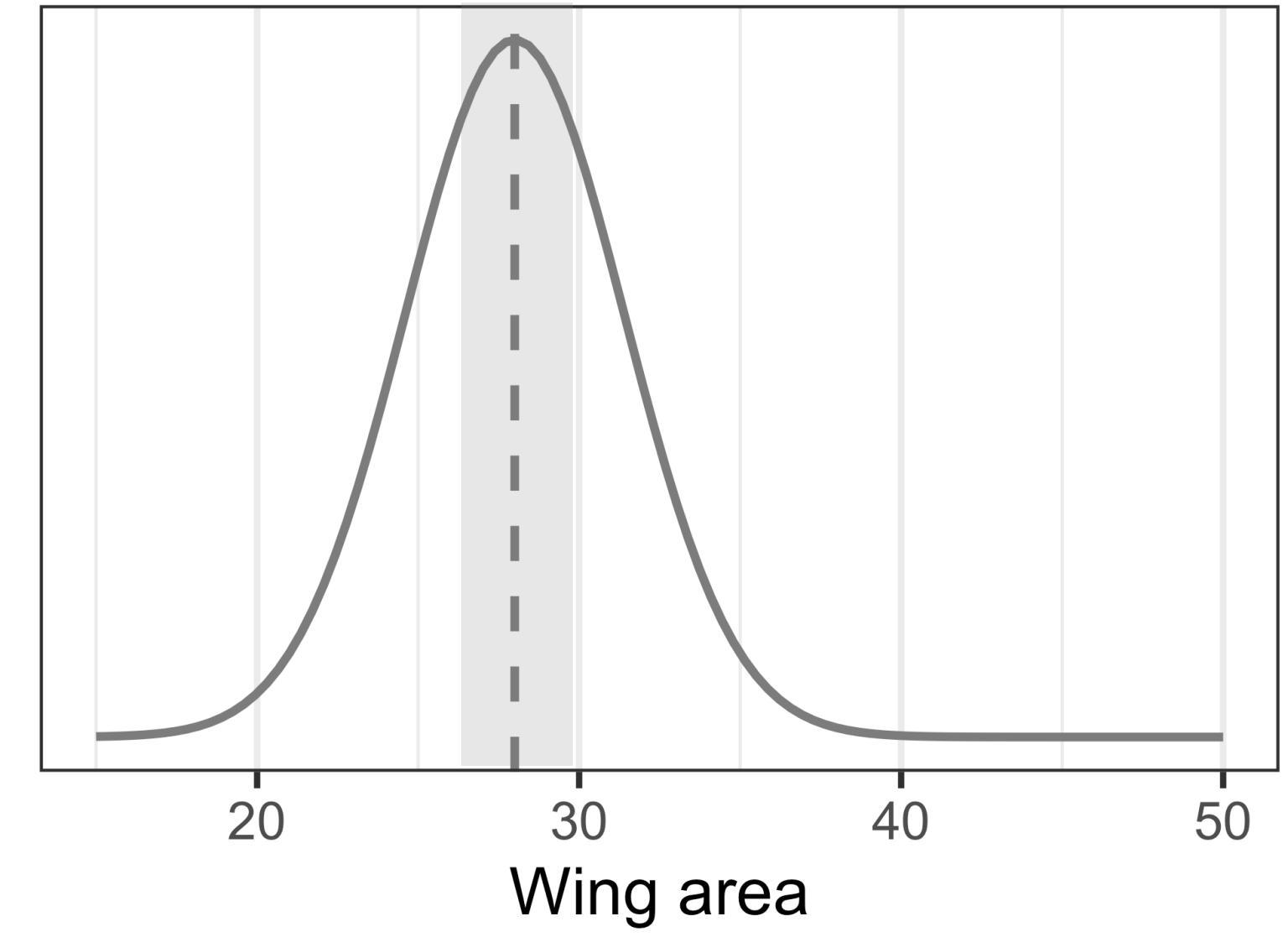
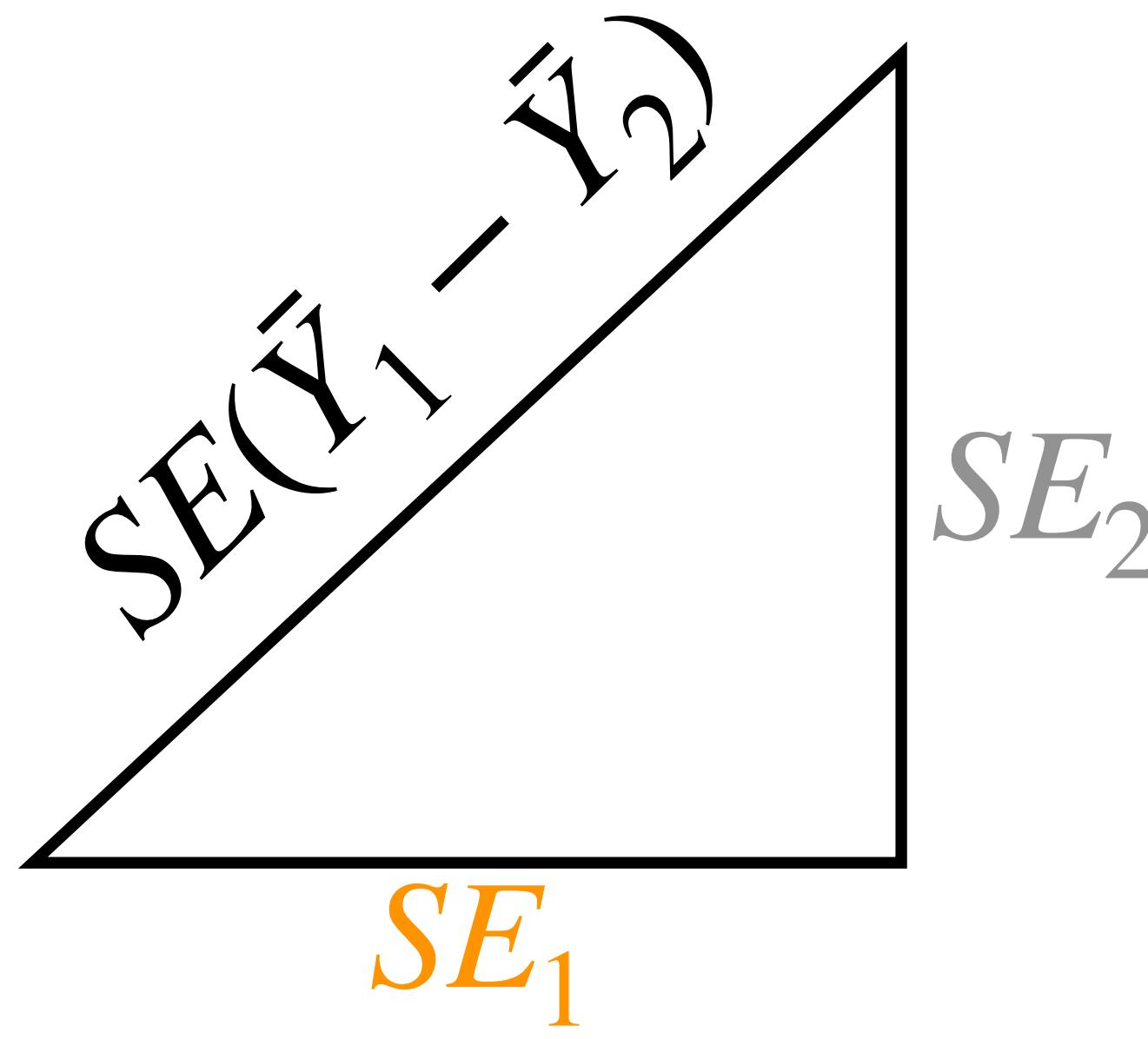
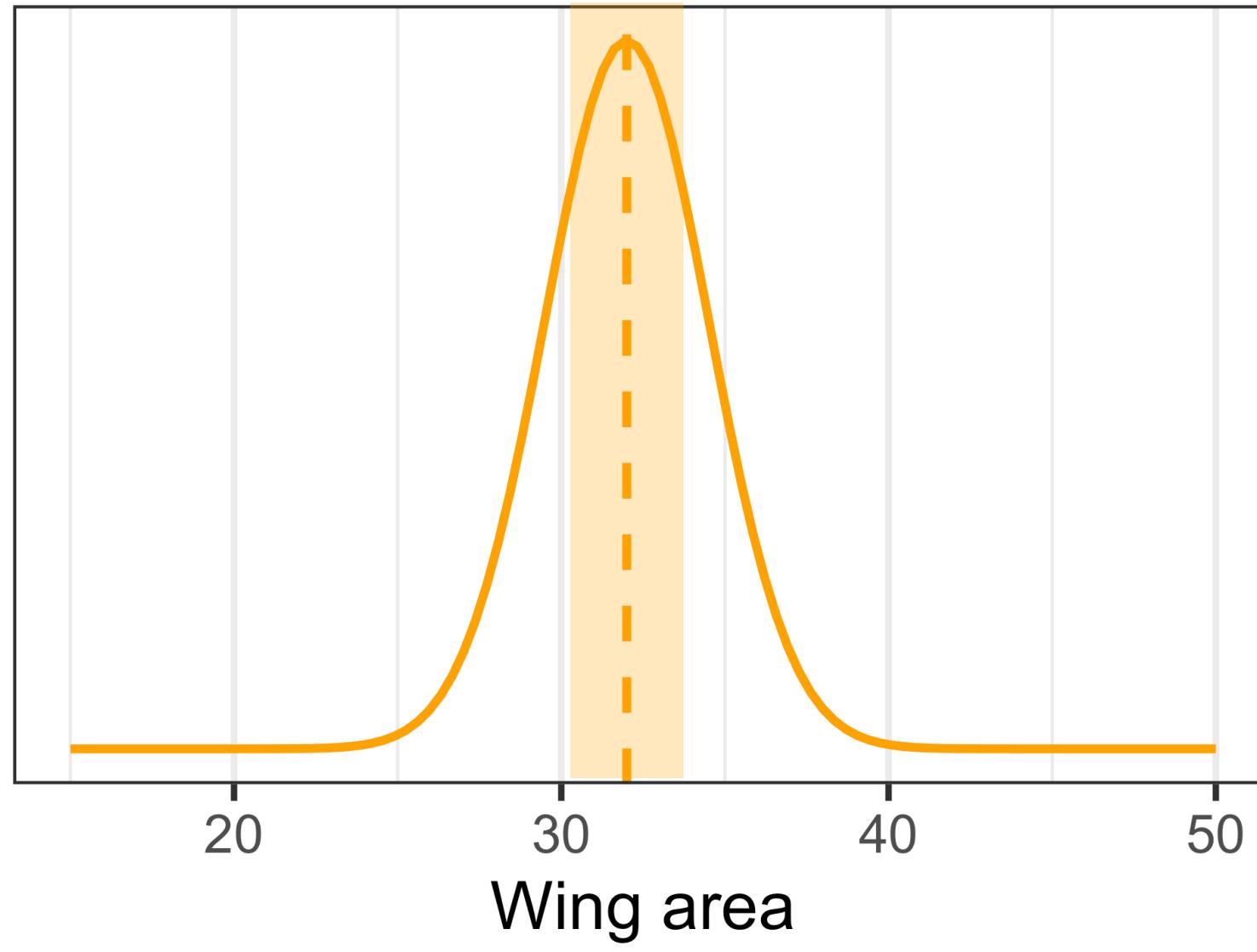
**Var (A+B) = {6.5-11, 6.6-5, 6.8-4, 6.5-3, 6.1-5}**

$$\begin{aligned}\mu_{A-B} &= 2 \\ \sigma_{A-B} &= 3.05\end{aligned}$$



# Standard error of $\bar{Y}_1 - \bar{Y}_2$

$$SE_{\bar{Y}} = \frac{s}{\sqrt{n}}$$

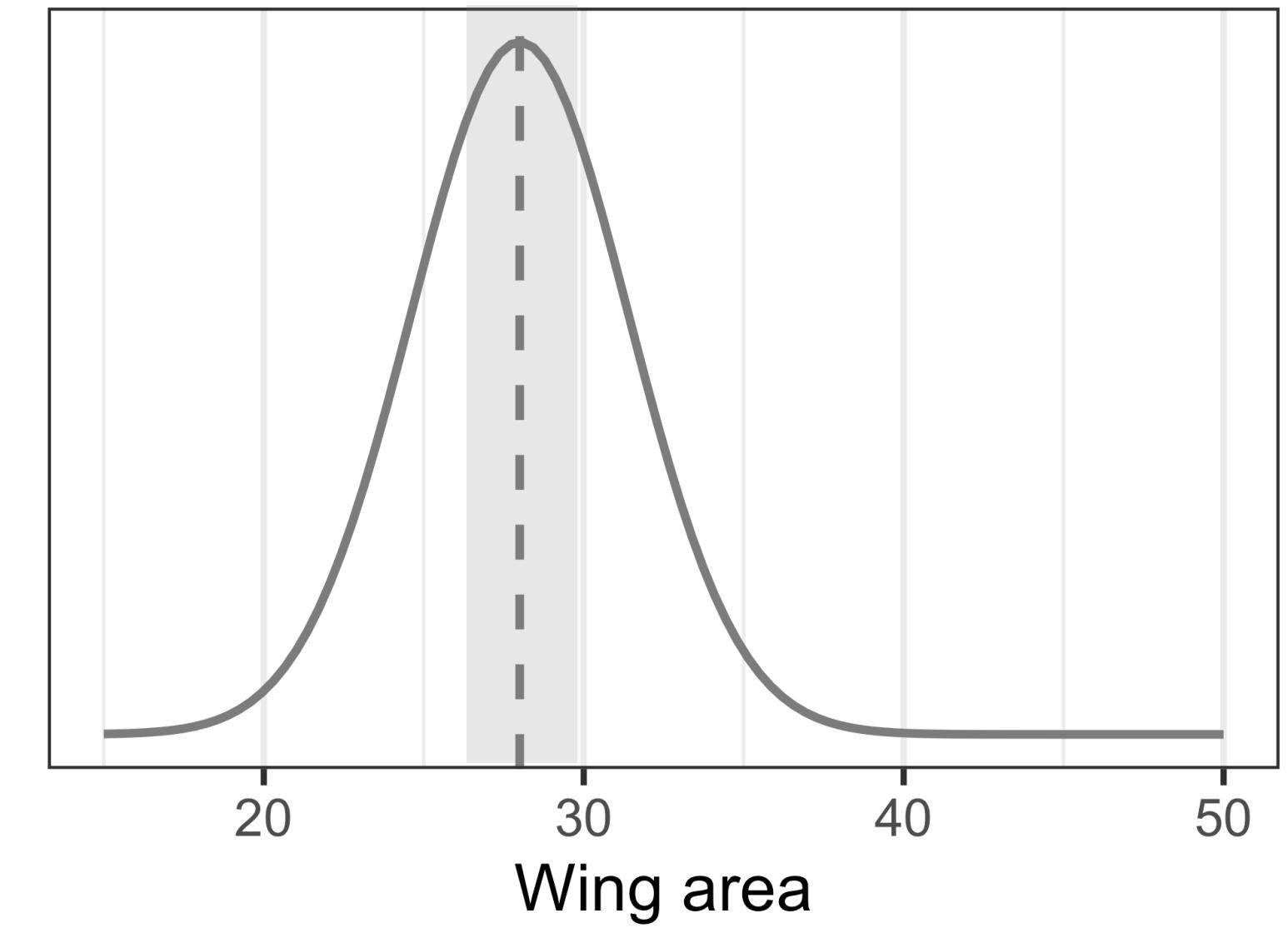
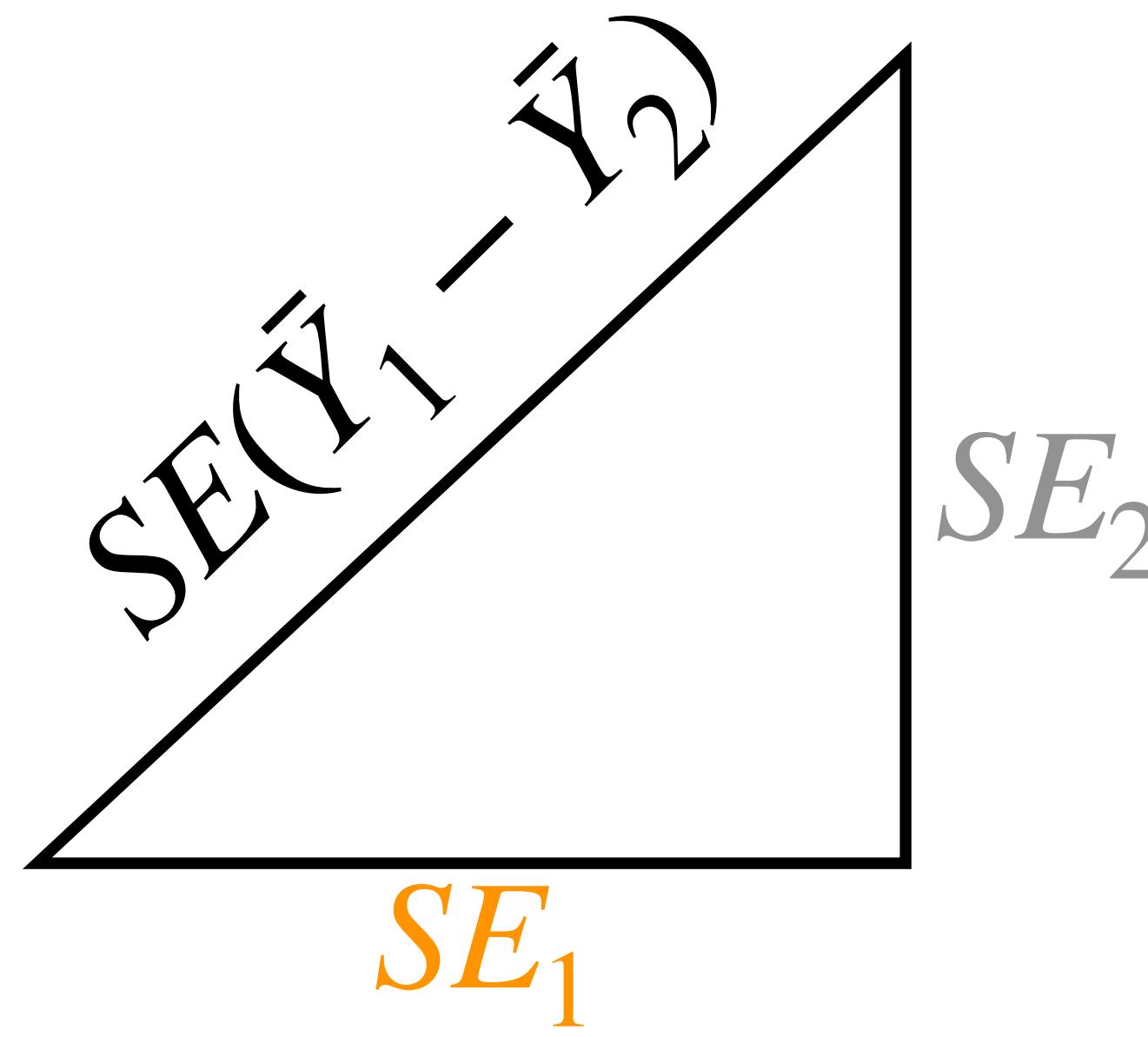
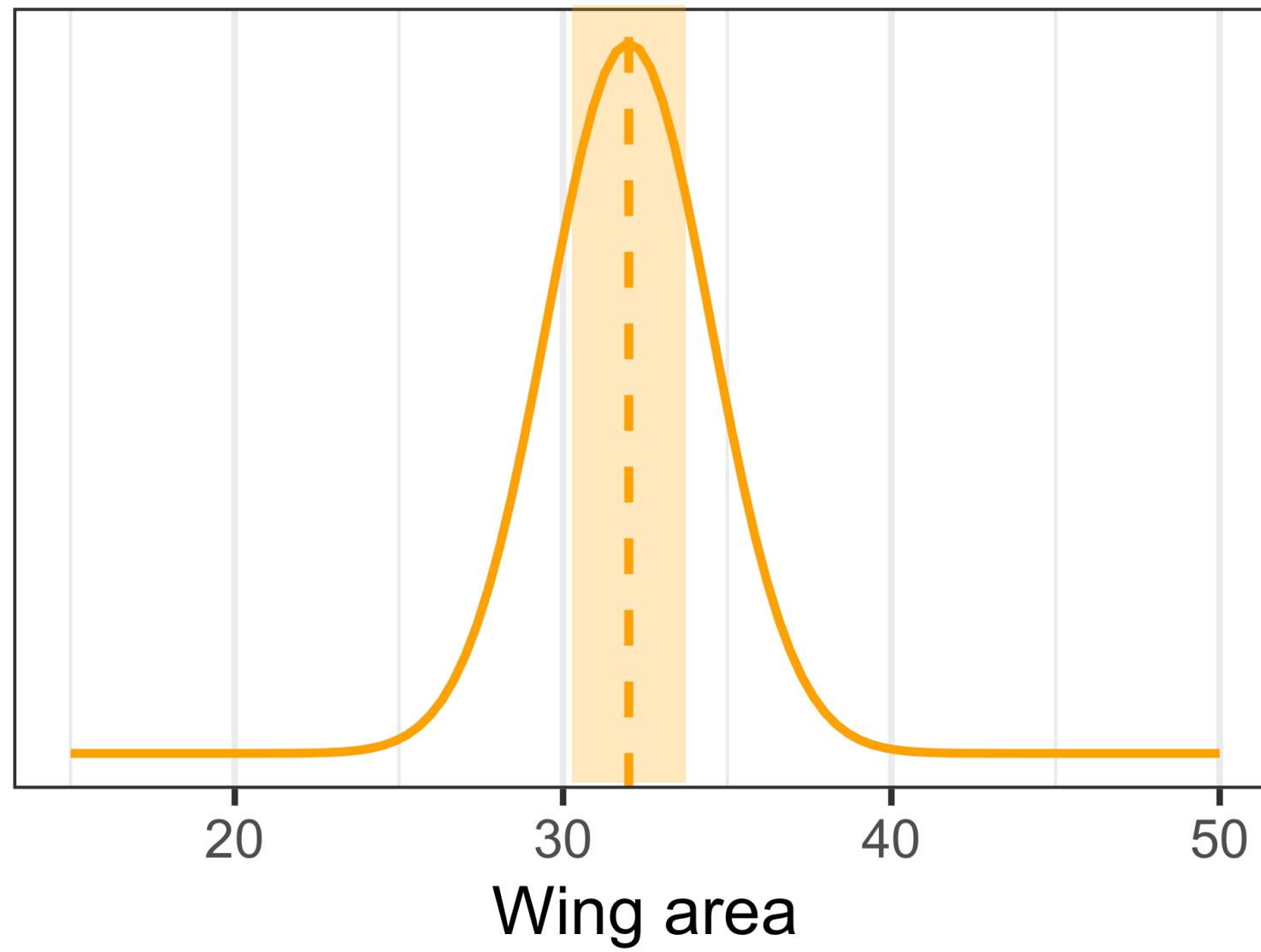


# Standard error of $\bar{Y}_1 - \bar{Y}_2$

$$SE_{\bar{Y}} = \frac{s}{\sqrt{n}}$$

$$SE_{\bar{Y}}^2 = \frac{s^2}{n}$$

(Variance of the mean)

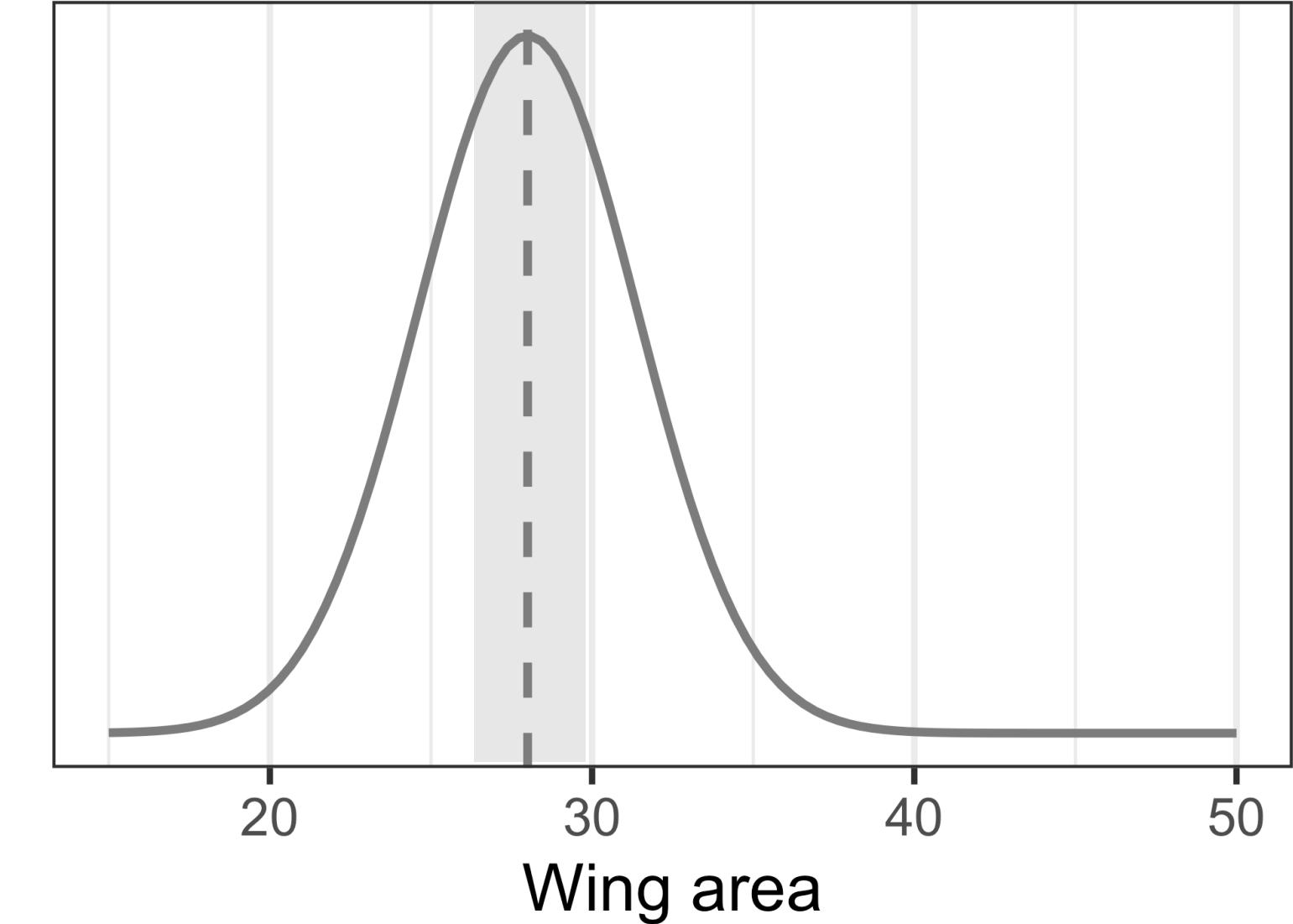
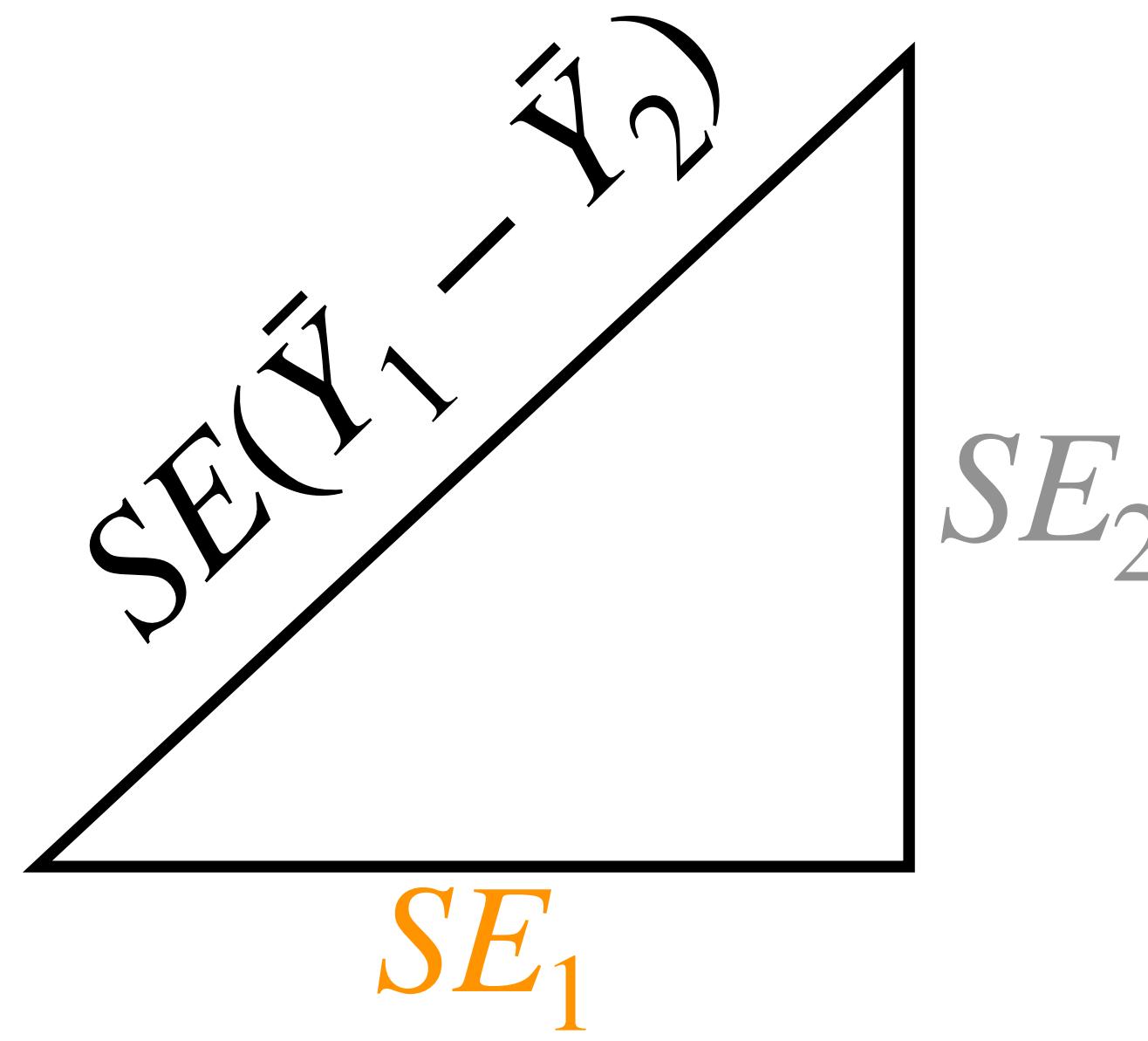
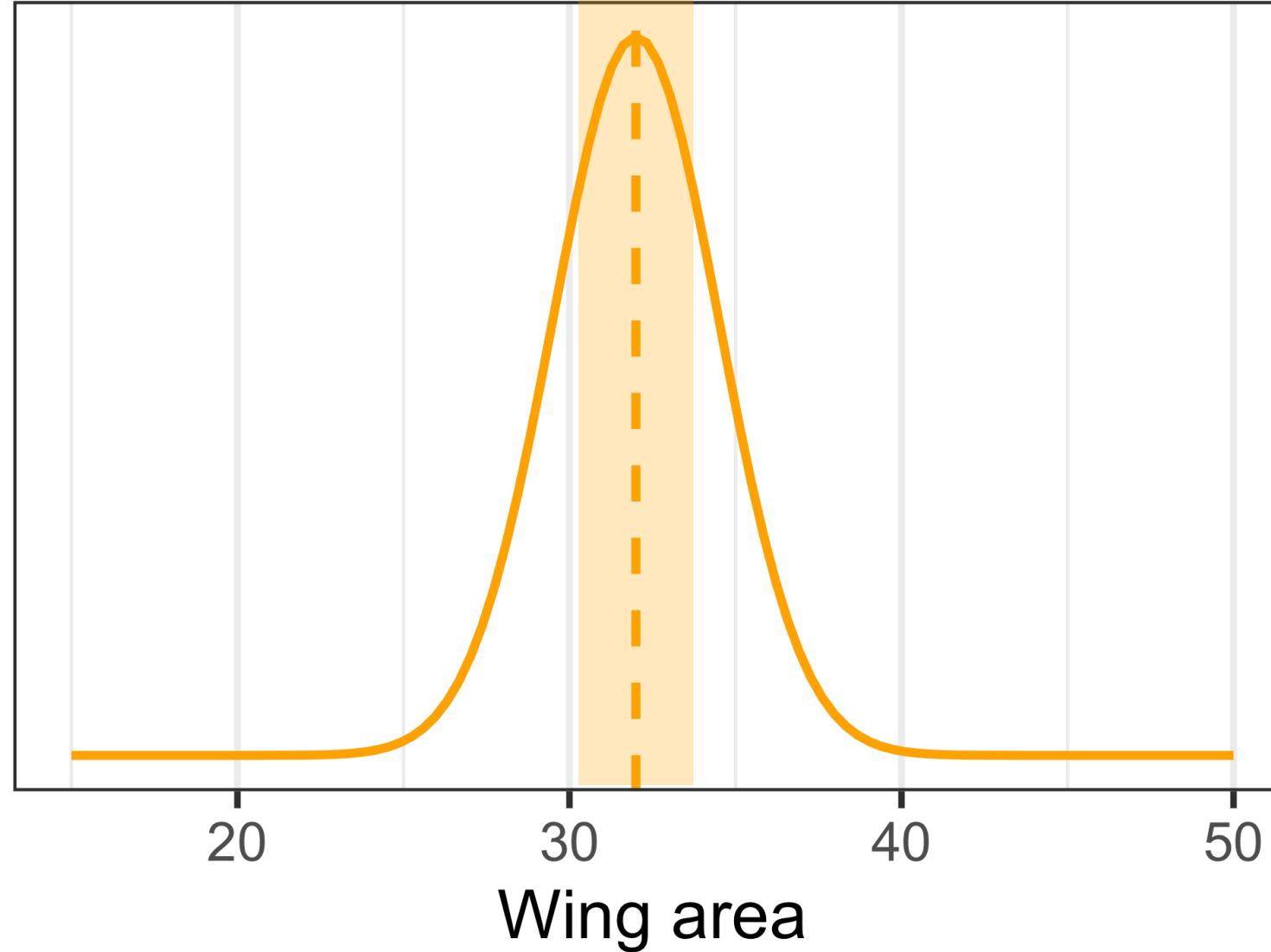


# Standard error of $\bar{Y}_1 - \bar{Y}_2$

$$SE_{\bar{Y}} = \frac{s}{\sqrt{n}}$$

$$SE_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{SE_1^2 + SE_2^2}$$

**Variability in each estimate ADDS to the total variability in the difference**

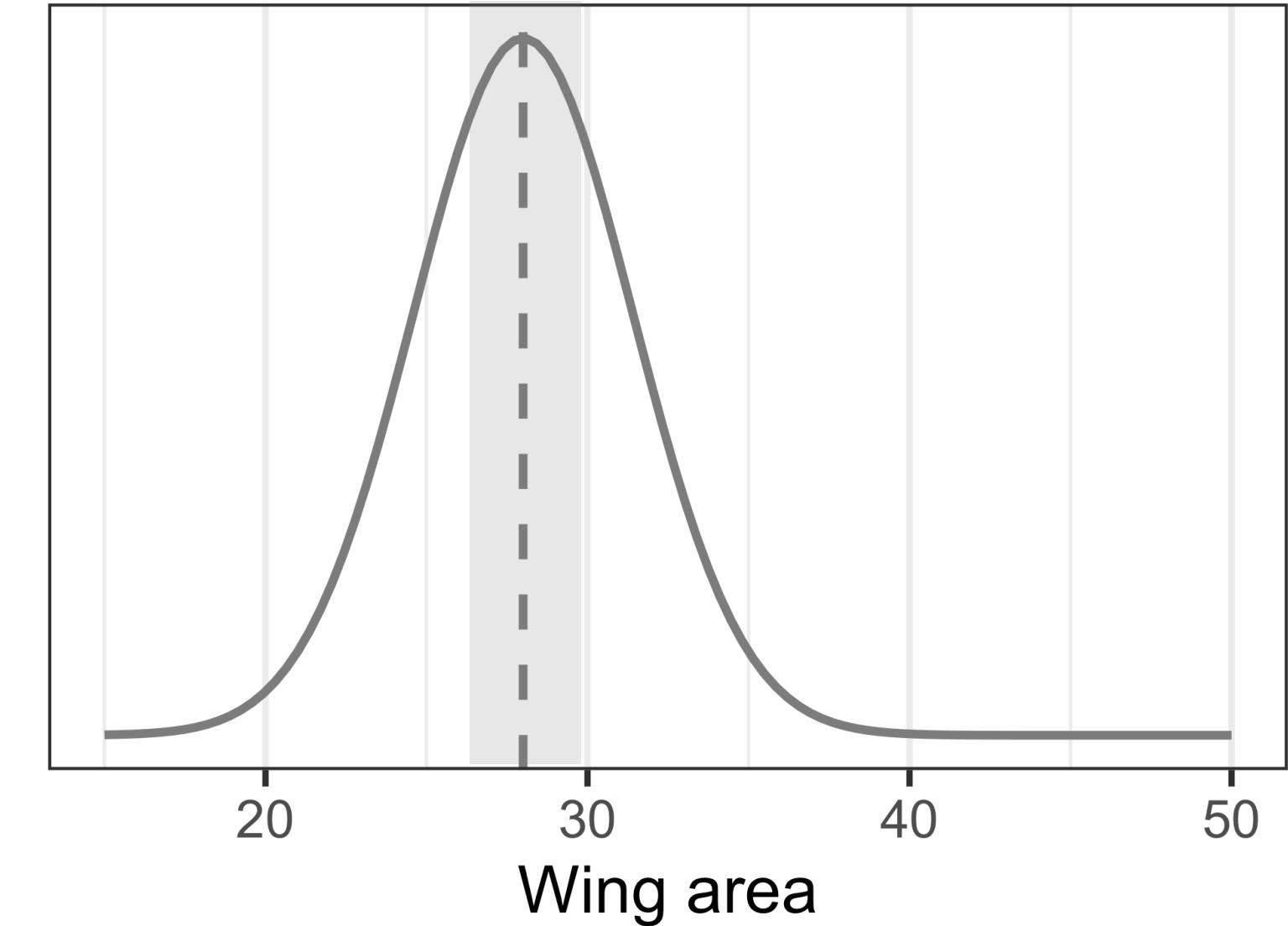
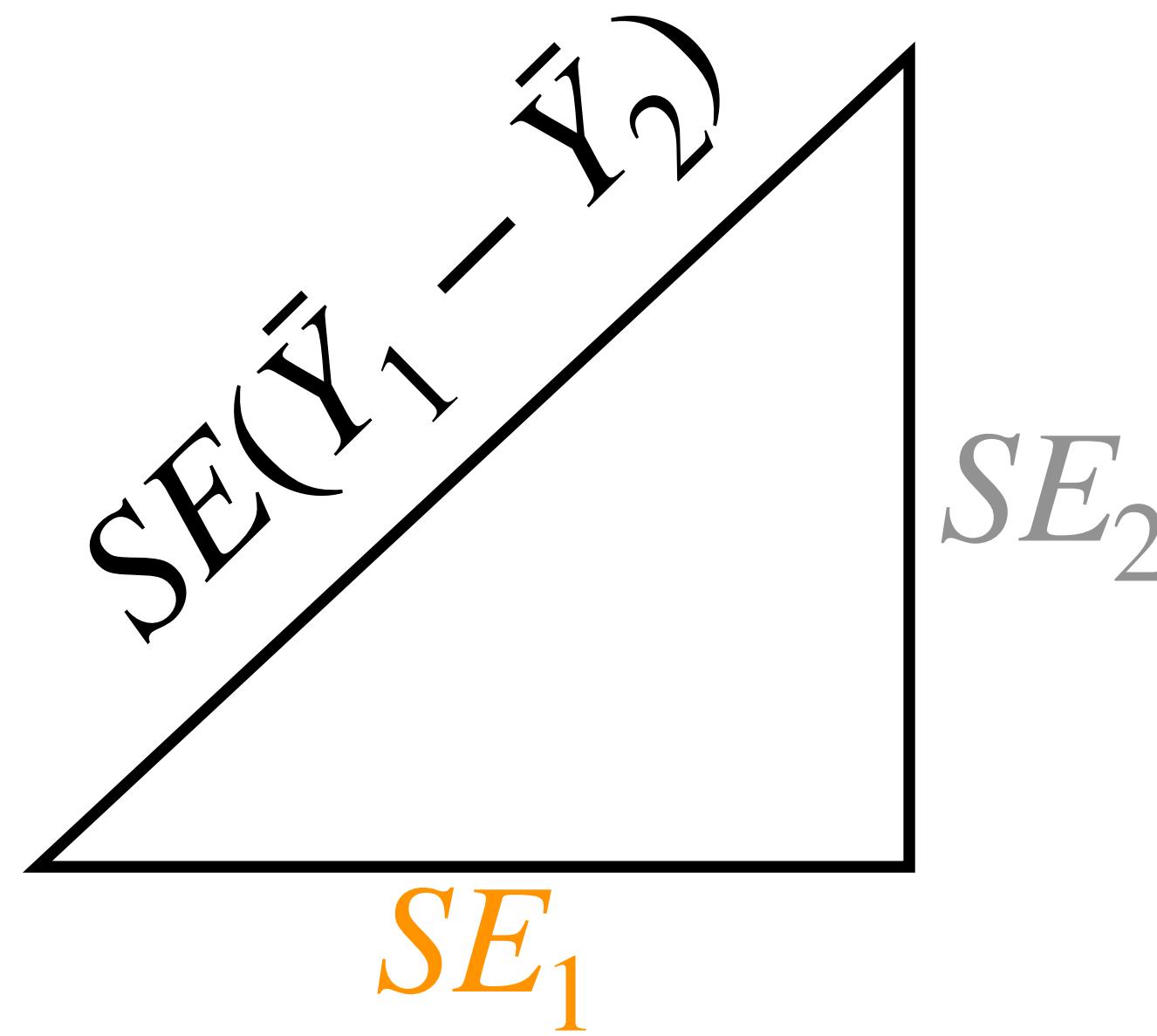
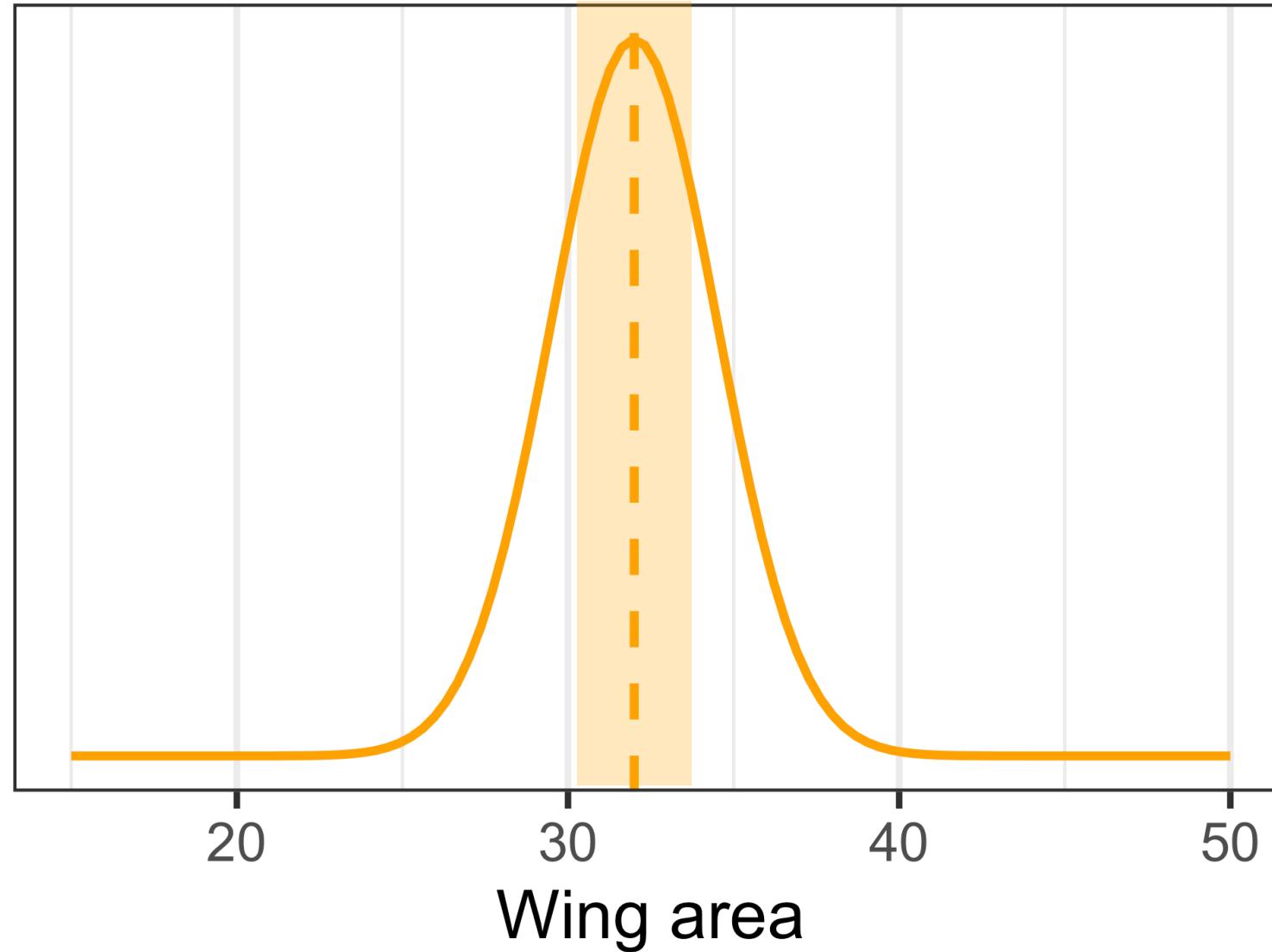


# Standard error of $\bar{Y}_1 - \bar{Y}_2$

$$SE_{\bar{Y}} = \frac{s}{\sqrt{n}}$$

$$SE_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{n}}$$

**Variability in each estimate ADDS to the total variability in the difference**



# Standard error of $\bar{Y}_1 - \bar{Y}_2$

$$SE_{\bar{Y}} = \frac{s}{\sqrt{n}}$$

$$SE_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{n}}$$

***Variability in each estimate ADDS to the total variability in the difference***

- Whether we add  $\bar{Y}_2$  to  $\bar{Y}_1$  or subtract, we **add their variances**
- The “noise” associated with  $\bar{Y}_2$  (i.e.  $SE_2$ ) adds to the overall uncertainty
- The greater the variability in  $\bar{Y}_2$ , the greater the variability in  $\bar{Y}_1 - \bar{Y}_2$

Standard error of  $\bar{Y}_1 - \bar{Y}_2$

$$SE_{\bar{Y}} = \frac{s}{\sqrt{n}}$$

$$SE_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{\frac{2.5^2}{14} + \frac{3.4^2}{12}} = 1.18$$

***Variability in each estimate ADDS to the total variability in the difference***

- Whether we add  $\bar{Y}_2$  to  $\bar{Y}_1$  or subtract, we **add their variances**
- The “noise” associated with  $\bar{Y}_2$  (i.e.  $SE_2$ ) adds to the overall uncertainty
- The greater the variability in  $\bar{Y}_2$ , the greater the variability in  $\bar{Y}_1 - \bar{Y}_2$

# Standard error of $\bar{Y}_1 - \bar{Y}_2$

**Calculate the standard error of the difference between Sample 1 and Sample 2**

$$SE_1 = \frac{s_1}{\sqrt{n_1}}$$

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_2 = \frac{s_2}{\sqrt{n_2}}$$

# Standard error of $\bar{Y}_1 - \bar{Y}_2$

**Calculate the standard error of the difference between Sample 1 and Sample 2**

$$SE_1 = \frac{4.3}{\sqrt{6}}$$

$$SE_1 = 1.75$$

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_2 = \frac{5.7}{\sqrt{12}}$$

$$SE_2 = 1.65$$

$$SE_{1-2} = \sqrt{SE_1^2 + SE_2^2}$$

# Standard error of $\bar{Y}_1 - \bar{Y}_2$

**Calculate the standard error of the difference between Sample 1 and Sample 2**

$$SE_1 = \frac{4.3}{\sqrt{6}}$$

$$SE_1 = 1.75$$

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_2 = \frac{5.7}{\sqrt{12}}$$

$$SE_2 = 1.65$$

$$SE_{1-2} = \sqrt{1.75^2 + 1.65^2}$$

$$SE_{1-2} = 2.41$$

# Standard error of $\bar{Y}_1 - \bar{Y}_2$

**Calculate the standard error of the difference between Sample 1 and Sample 2**

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = \sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{n}}$$

# Standard error of $\bar{Y}_1 - \bar{Y}_2$

**Calculate the standard error of the difference between Sample 1 and Sample 2**

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = \sqrt{\frac{4.3^2}{6} + \frac{5.7^2}{12}}$$

$$SE_{1-2} = 2.41 \checkmark$$

# Comparing populations: the $t$ statistic

The  $t$  test is a standard method of choosing between these hypotheses

$$H_0 : \mu_1 = \mu_2$$

$$H_A : \mu_1 \neq \mu_2$$

*t is in units of SE*

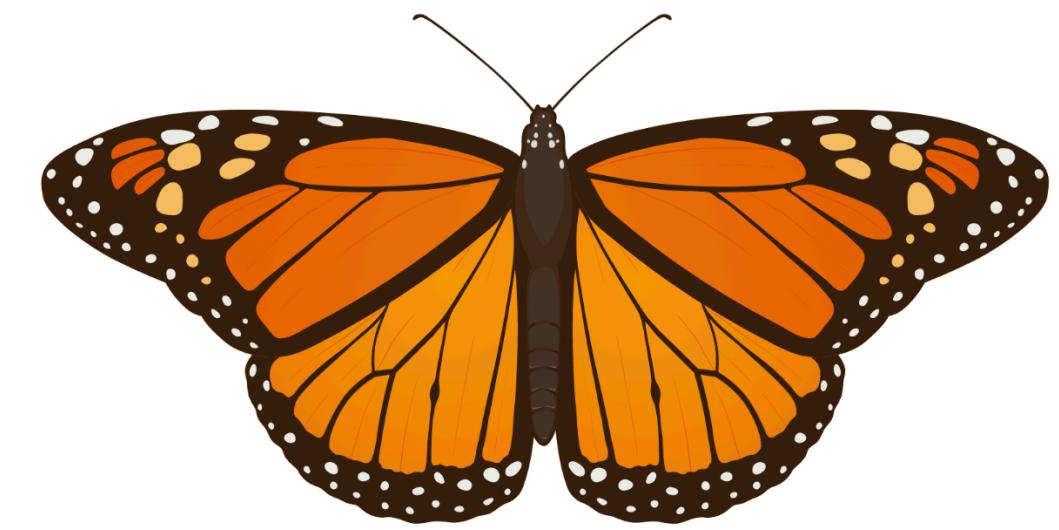
**Test statistic:**

$$t_s = \frac{\bar{y}_1 - \bar{y}_2}{SE_{\bar{Y}_1 - \bar{Y}_2}}$$

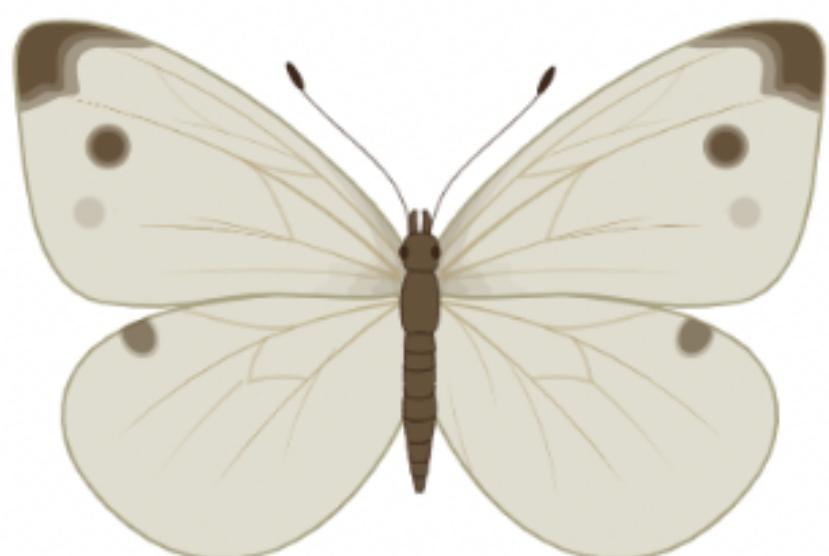
Variation in differences of means from random samples

How far the difference between the two means are from 0 (null hypothesis)

# Comparing populations: the $t$ statistic



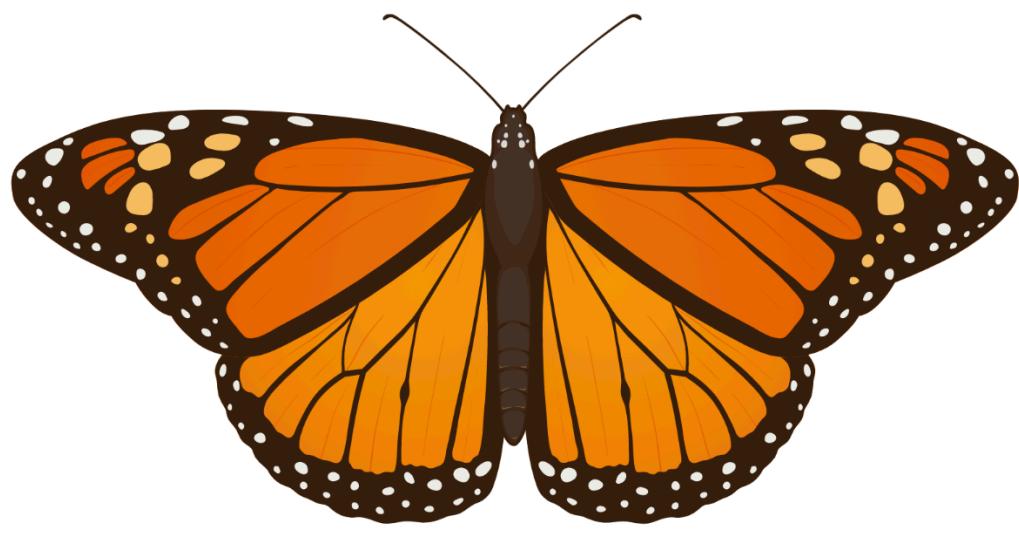
$$\bar{y}_1 = 32$$
$$s_1 = 2.5$$



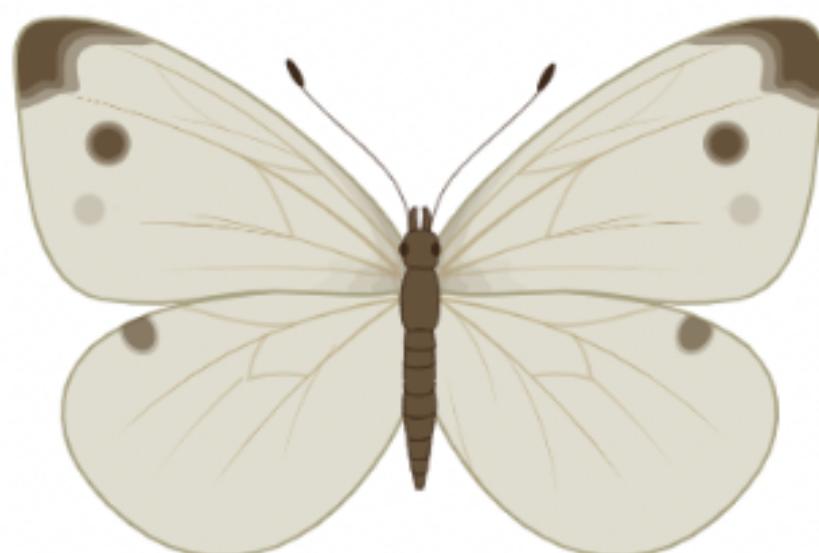
$$\bar{y}_2 = 28$$
$$s_2 = 3.4$$

$$t_s = \frac{\bar{y}_1 - \bar{y}_2}{SE_{\bar{y}_1 - \bar{y}_2}} \quad (SE_{\bar{Y}_1 - \bar{Y}_2} = 1.18)$$
$$t_s = \frac{32 - 28}{1.18} = 3.39$$

# Comparing populations: the $t$ statistic



$$\bar{y}_1 = 32$$
$$s_1 = 2.5$$



$$\bar{y}_2 = 28$$
$$s_2 = 3.4$$

## 1. Generate a hypothesis and choose a significance level

$$H_0 : \bar{y}_1 = \bar{y}_2 \quad H_A : \bar{y}_1 \neq \bar{y}_2 \quad \alpha = 0.05$$

## 2. Calculate test statistic

$$t_s = \frac{32 - 28}{1.18} = 3.39$$

## 3. Calculate the $P$ -value

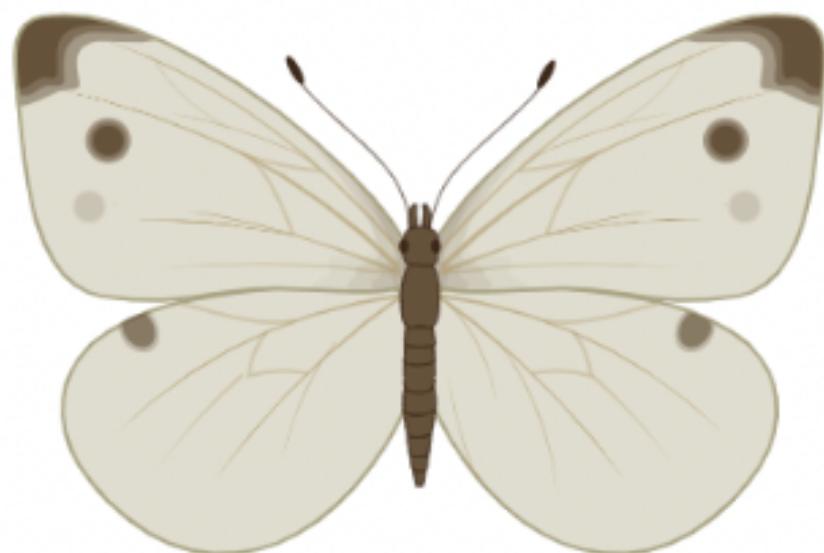
Use smaller of  $n_1-1$  and  $n_2-1$  for degrees of freedom\*

Conservative estimate

# Comparing populations: the $t$ statistic



$$\bar{y}_1 = 32$$
$$s_1 = 2.5$$



$$\bar{y}_2 = 28$$
$$s_2 = 3.4$$

**1. Generate a hypothesis and choose a significance level**

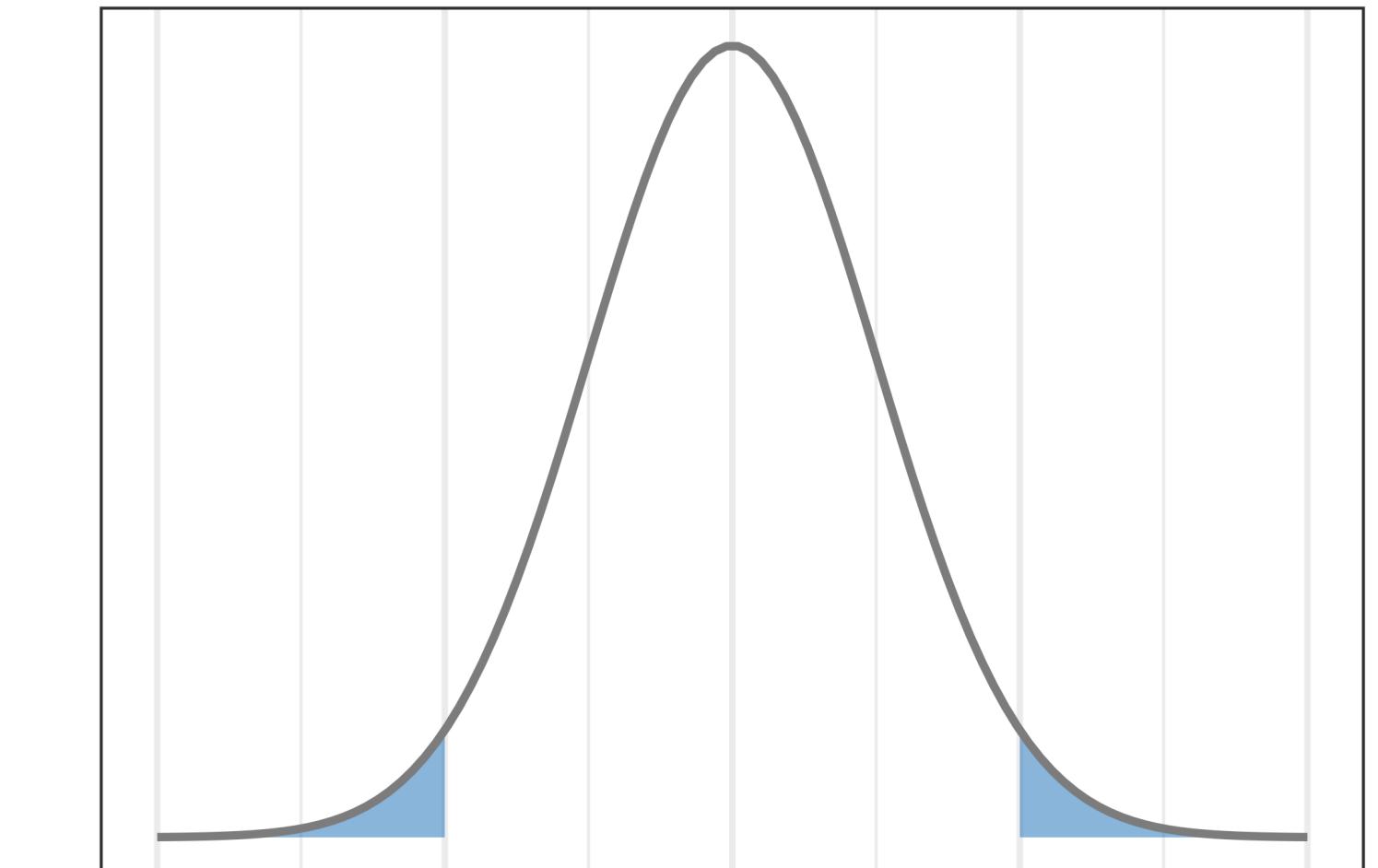
$$H_0 : \bar{y}_1 = \bar{y}_2 \quad H_A : \bar{y}_1 \neq \bar{y}_2 \quad \alpha = 0.05$$

**2. Calculate test statistic**

$$t_s = \frac{32 - 28}{1.18} = 3.39$$

**3. Calculate the  $P$ -value**

$$df_1 = n_1 - 1 = 13$$



$$df_2 = n_2 - 1 = 11$$

$$t_S = 3.39$$

$$df = 11$$

$$\alpha = 0.05$$

$$0.002 < P < 0.01$$

```
> pt(3.39, 11, lower.tail = F) * 2
```

[1] 0.00603



	P	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
one-tail								
two-tails		0.2	0.1	0.05	0.02	0.01	0.002	0.001
DF								
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
6	1.44	1.943	2.447	3.143	3.707	5.208	5.959	
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408	
8	1.397	1.86	2.306	2.896	3.355	4.501	5.041	
9	1.383	1.833	2.262	2.821	3.25	4.297	4.781	
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587	
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437	

	1.782	2.179	2.681	3.055	3.93	4.318	
12	1.771	2.16	2.65	3.012	3.852	4.221	
14	1.345	1.761	2.145	2.624	2.977	3.787	4.14
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073

120	1.289	1.658	1.98	2.358	2.617	3.16	3.373
1000	1.282	1.646	1.962	2.33	2.581	3.098	3.3
Inf	1.282	1.645	1.96	2.326	2.576	3.091	3.291

REJECT the null hypothesis

# Another example

Use a *t*-test to compare the difference of means between Sample 1 and Sample 2

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = \sqrt{\frac{4.3^2}{6} + \frac{5.7^2}{12}}$$

$$SE_{1-2} = 2.41 \checkmark$$

## Use a *t*-test to compare the difference of means between Sample 1 and Sample 2

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = 2.41$$

	P	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
one-tail								
two-tails								
DF								
1		3.078	6.314	12.706	31.821	63.656	318.289	636.578
2		1.886	2.92	4.303	6.965	9.925	22.328	31.6
3		1.638	2.353	3.182	4.541	5.841	10.214	12.924
4		1.533	2.132	2.776	3.747	4.604	7.173	8.61
5		1.476	2.015	2.571	3.365	4.032	5.894	6.869
6		1.44	1.943	2.447	3.143	3.707	5.208	5.959
7		1.415	1.895	2.365	2.998	3.499	4.785	5.408
8		1.397	1.86	2.306	2.896	3.355	4.501	5.041
9		1.383	1.833	2.262	2.821	3.25	4.297	4.781
10		1.372	1.812	2.228	2.764	3.169	4.144	4.587
11		1.363	1.796	2.201	2.718	3.106	4.025	4.437
12		1.356	1.782	2.179	2.681	3.055	3.93	4.318
13		1.35	1.771	2.16	2.65	3.012	3.852	4.221
14		1.345	1.761	2.145	2.624	2.977	3.787	4.14
15		1.341	1.753	2.131	2.602	2.947	3.733	4.073

120	1.289	1.658	1.98	2.358	2.617	3.16	3.373
-----	-------	-------	------	-------	-------	------	-------

## Use a *t*-test to compare the difference of means between Sample 1 and Sample 2

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = 2.41$$

1. Generate a hypothesis and choose a significance level

$$H_0 : \bar{y}_1 = \bar{y}_2 \quad \alpha = 0.05$$

$$H_A : \bar{y}_1 \neq \bar{y}_2$$

2. Calculate test statistic

$$t_s = \frac{y_1 - y_2}{SE_{1-2}}$$

3. Calculate the *P*-value

## Use a *t*-test to compare the difference of means between Sample 1 and Sample 2

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = 2.41$$

$$df_1 = 5$$

$$df_2 = 11$$

1. Generate a hypothesis and choose a significance level

$$H_0 : \bar{y}_1 = \bar{y}_2 \quad \alpha = 0.05$$

$$H_A : \bar{y}_1 \neq \bar{y}_2$$

2. Calculate test statistic

$$t_s = \frac{40 - 50}{2.41} = -4.14$$

3. Calculate the *P*-value

## Use a *t*-test to compare the difference of means between Sample 1 and Sample 2

	Sample 1	Sample 2
n	6	12
y	40	50
s	4.3	5.7

$$SE_{1-2} = 2.41$$

$$t_S = 4.14$$

$$df = 5$$

$$\alpha = 0.05$$

$$0.002 < P < 0.01$$

	P	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
one-tail								
two-tails								
DF								
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
6	1.44	1.943	2.447	3.143	3.707	5.208	5.959	
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408	
8	1.397	1.86	2.306	2.896	3.355	4.501	5.041	
9	1.383	1.833	2.262	2.821	3.25	4.297	4.781	

```
> pt(4.14, 5, lower.tail = F) * 2
```

[1] 0.0089



REJECT the  
null hypothesis

# Confidence interval for $\bar{Y}_1 - \bar{Y}_2$

**95% confidence interval for one sample**

$$\bar{y} \pm t_{0.025} SE_{\bar{Y}}$$

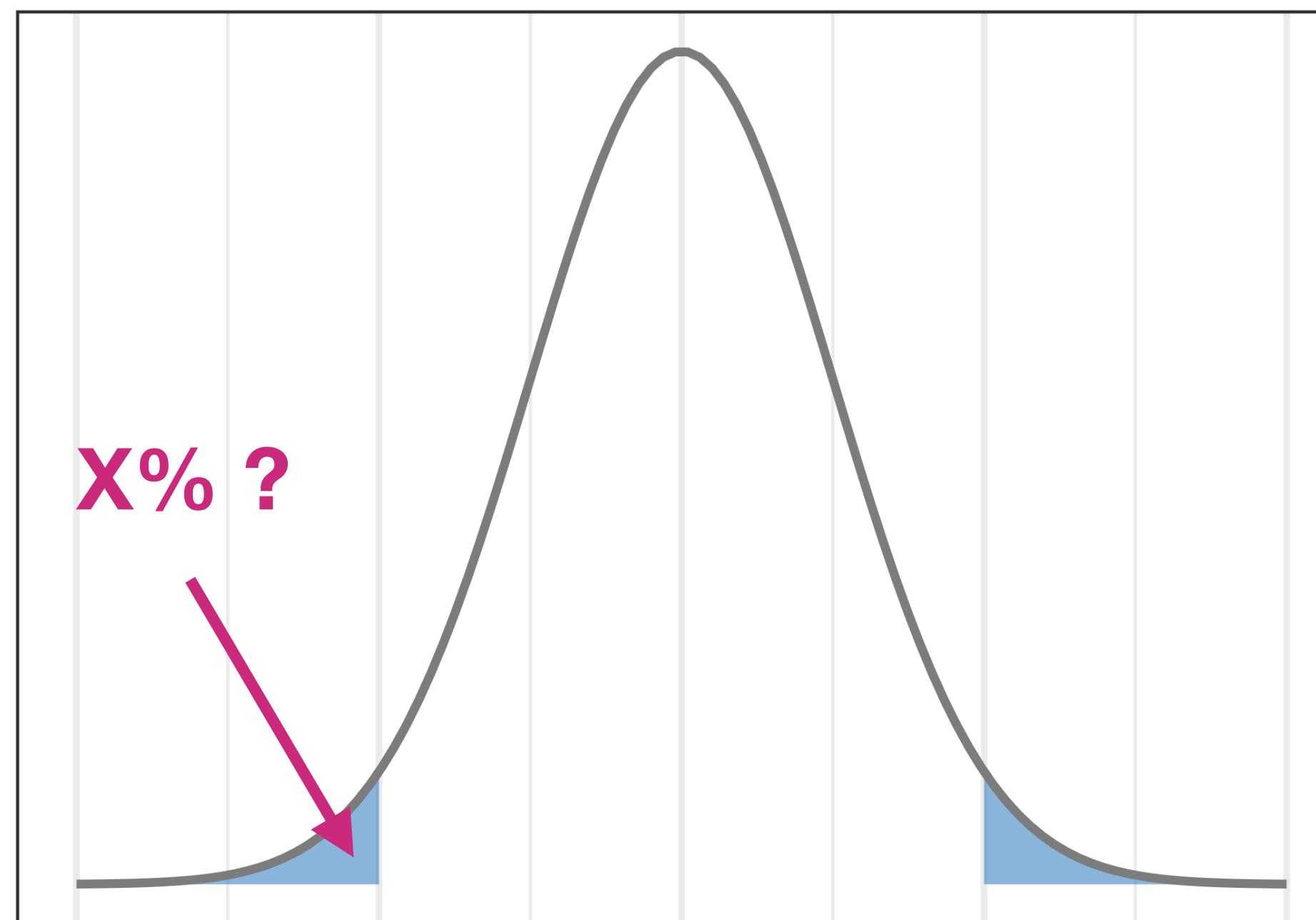
**95% confidence interval for difference of means (two samples)**

$$(\bar{y}_1 - \bar{y}_2) \pm t_{0.025} SE_{(\bar{Y}_1 - \bar{Y}_2)}$$

Use smaller of  $n_1-1$  and  $n_2-1$  for degrees of freedom\*

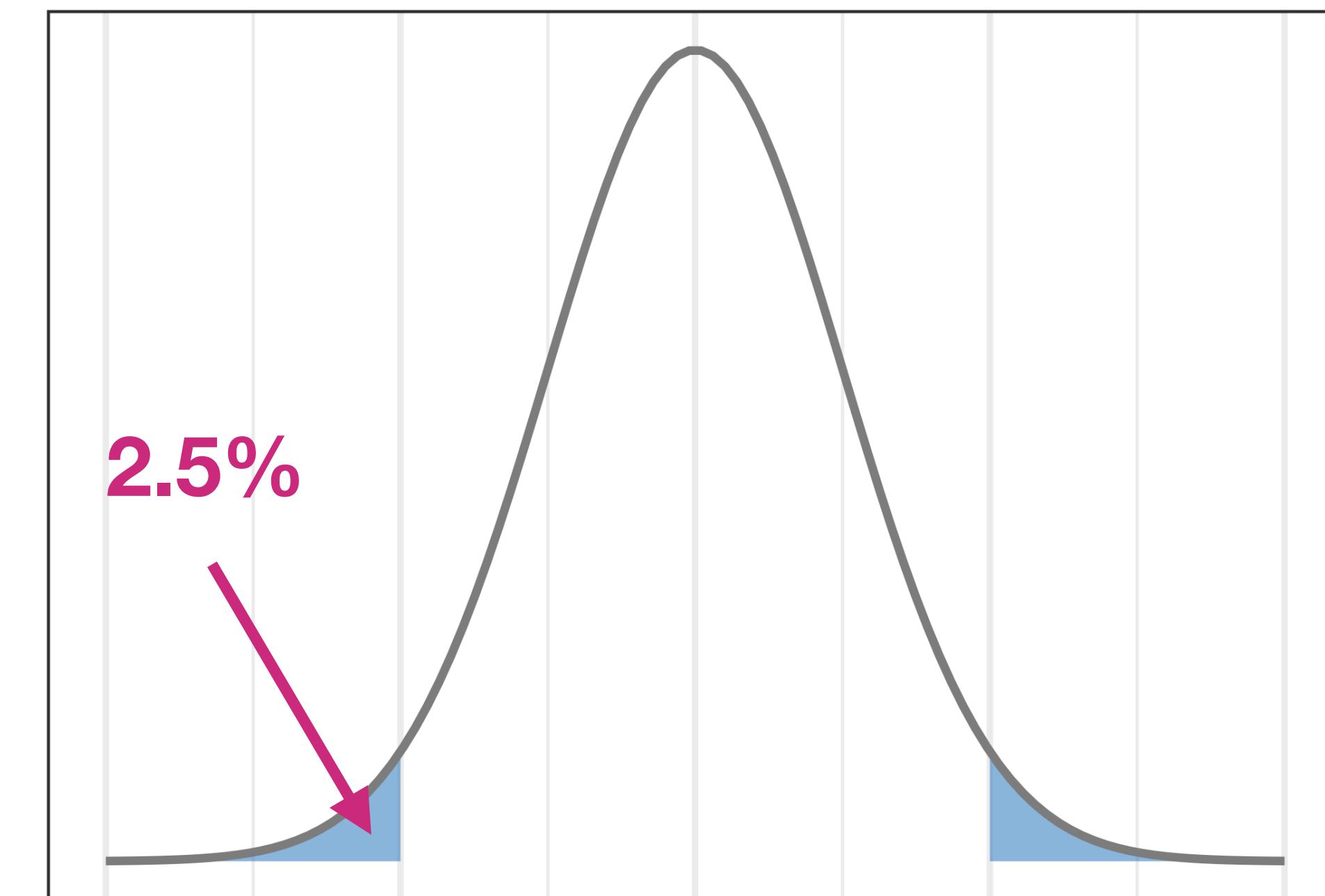
# Comparing $t$ -test and confidence intervals

**$t$ -test**



```
> pt(t, df)*2
```

**Confidence interval**



```
> qt(p, df)
```

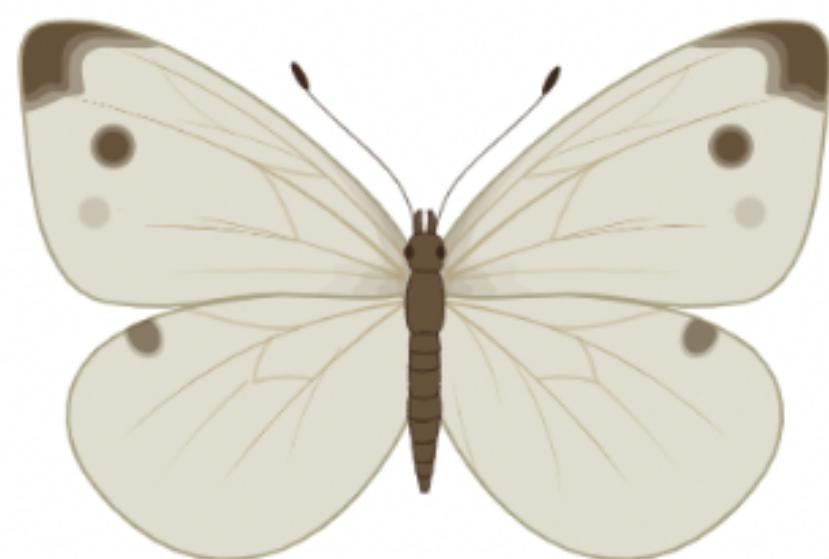
# Confidence interval for $\bar{Y}_1 - \bar{Y}_2$



95% confidence interval for difference of means (two samples)

$$(\bar{y}_1 - \bar{y}_2) \pm t_{0.025} SE_{(\bar{Y}_1 - \bar{Y}_2)}$$

Use smaller of  $n_1-1$  and  $n_2-1$  for degrees of freedom\*



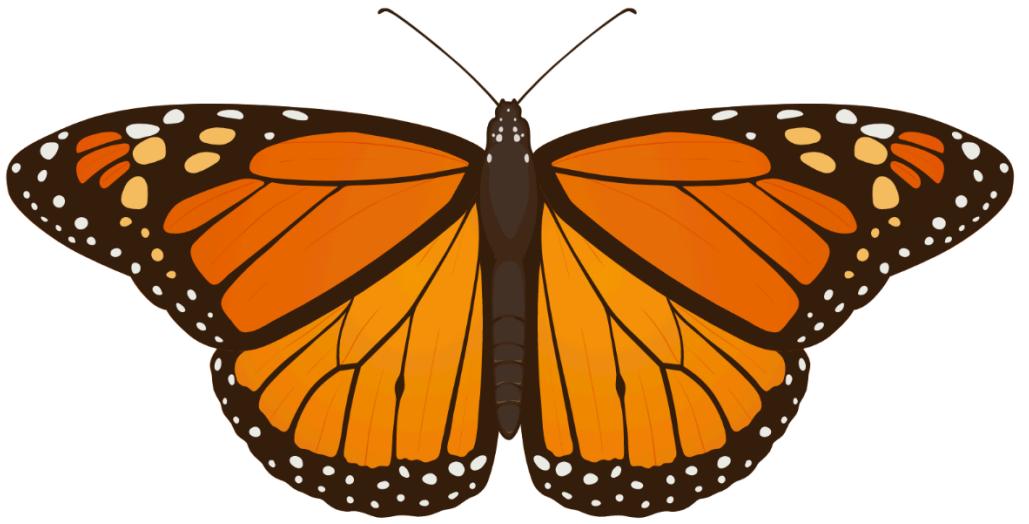
$$(32 - 28) \pm t_{0.025}(1.18)$$

$$df = 11$$

$$\begin{aligned}\bar{y}_1 &= 32 \\ s_1 &= 2.5\end{aligned}$$

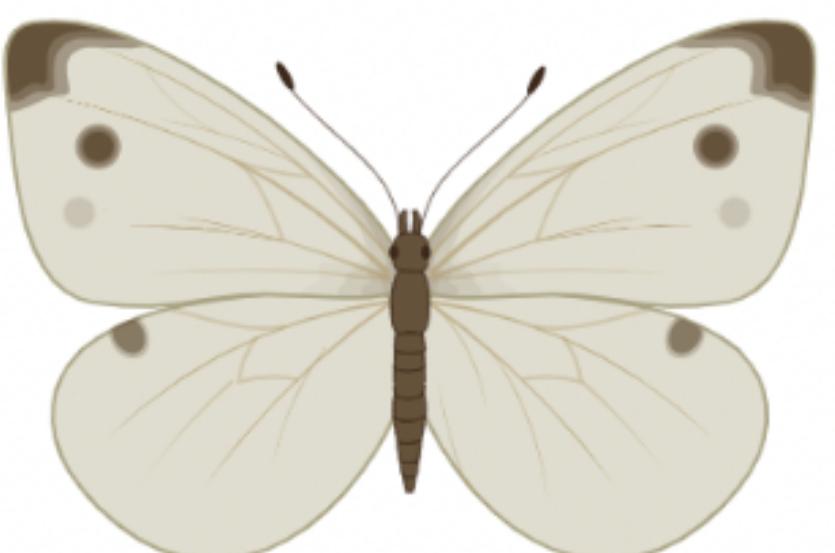
$$\begin{aligned}\bar{y}_2 &= 28 \\ s_2 &= 3.4\end{aligned}$$

# Confidence interval for $\bar{Y}_1 - \bar{Y}_2$



$$\bar{y}_1 = 32$$

$$s_1 = 2.5$$

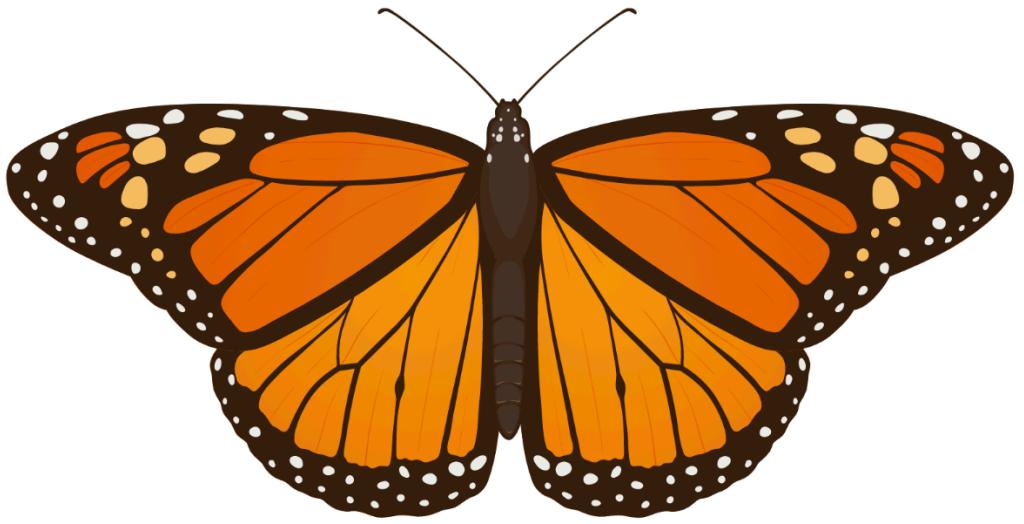


$$\bar{y}_2 = 28$$

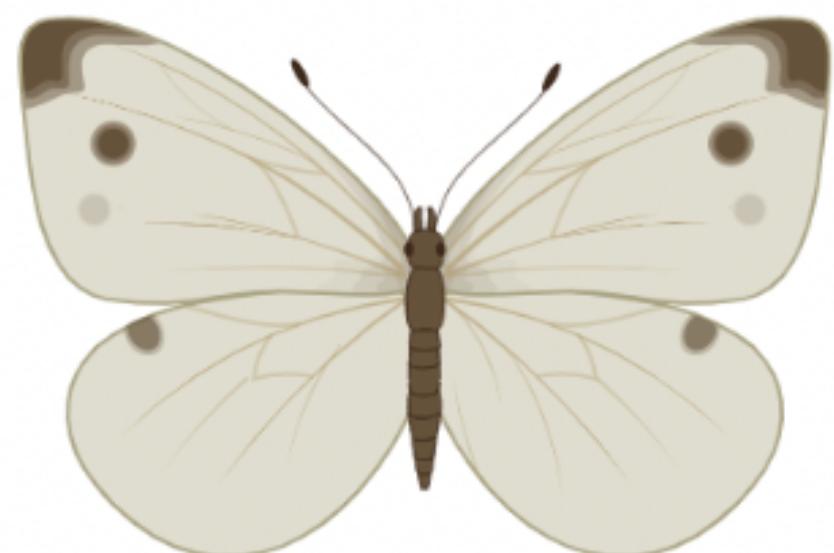
$$s_2 = 3.4$$

	P							
one-tail	0.1	0.05	0.025	0.01	0.005	0.001	0.0005	
two-tails	0.2	0.1	0.05	0.02	0.01	0.002	0.001	
DF								
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
6	1.44	1.943	2.447	3.143	3.707	5.208	5.959	
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408	
8	1.397	1.86	2.306	2.896	3.355	4.501	5.041	
9	1.383	1.833	2.262	2.821	3.25	4.297	4.781	
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587	
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437	
12	1.356	1.782	2.179	2.681	3.055	3.93	4.318	
13	1.35	1.771	2.16	2.65	3.012	3.852	4.221	
14	1.345	1.761	2.145	2.624	2.977	3.787	4.114	

# Confidence interval for $\bar{Y}_1 - \bar{Y}_2$



$$\bar{y}_1 = 32$$
$$s_1 = 2.5$$



$$\bar{y}_2 = 28$$
$$s_2 = 3.4$$

95% confidence interval for difference of means (two samples)

$$(\bar{y}_1 - \bar{y}_2) \pm t_{0.025} SE_{(\bar{Y}_1 - \bar{Y}_2)}$$

Use smaller of  $n_1-1$  and  $n_2-1$  for degrees of freedom\*

$$(32 - 28) \pm t_{0.025}(1.18)$$

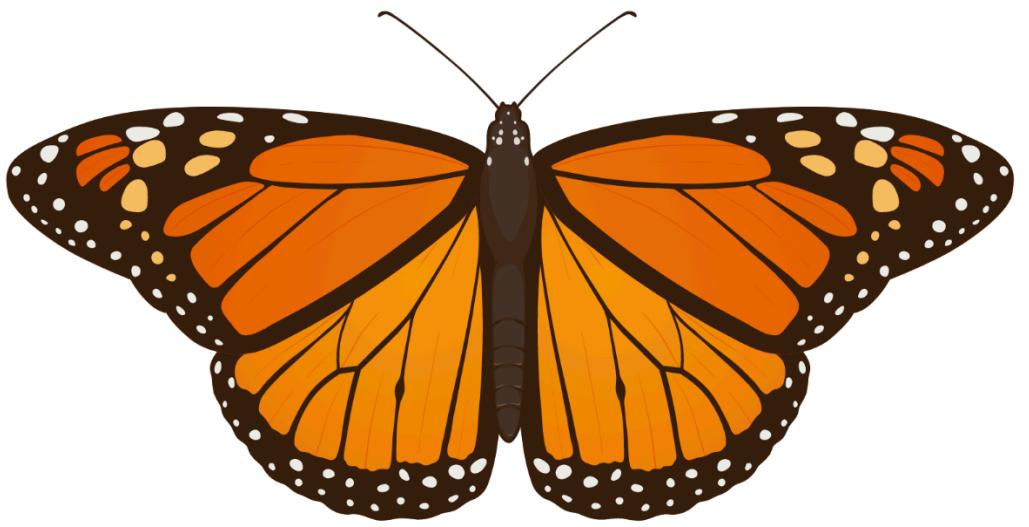
$$df = 11 \quad > qt(0.975, 11) \longrightarrow t_s = 2.2$$

$$4 \pm 2.596 \longrightarrow (1.404, 6.596)$$

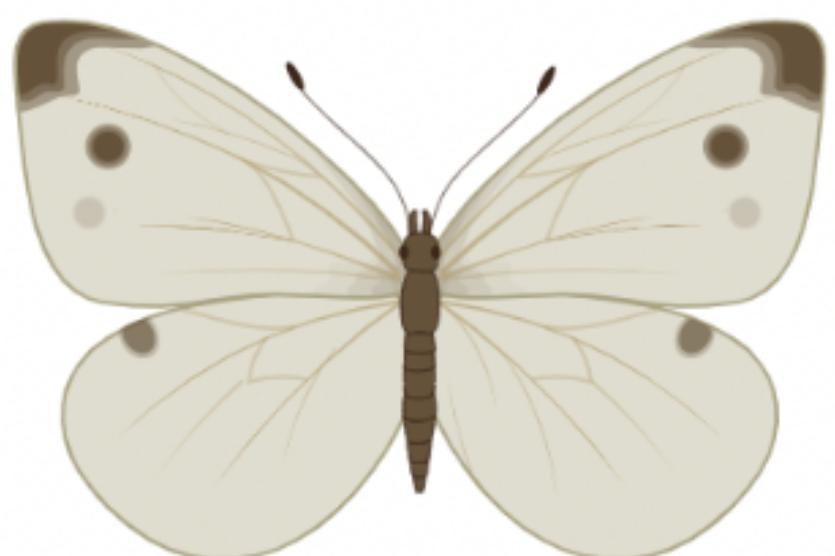
Does NOT include 0!!



# Comparing populations: the $t$ statistic

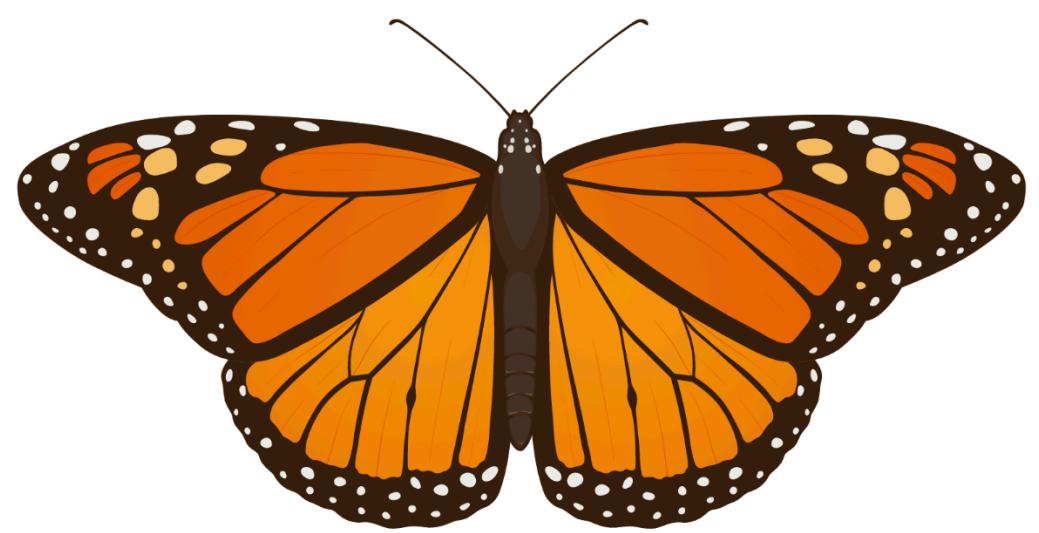


$$\bar{y}_1 = 32$$
$$s_1 = 2.5$$



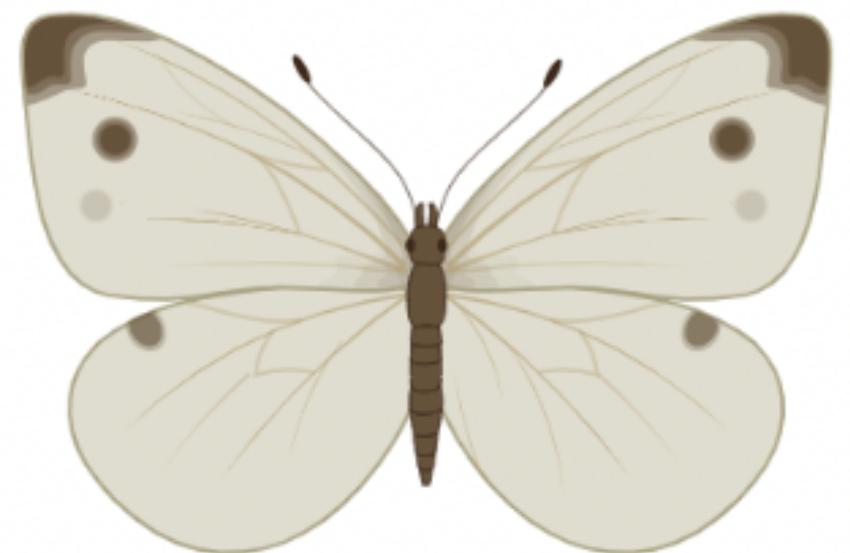
$$\bar{y}_2 = 28$$
$$s_2 = 3.4$$

```
# set random seed  
set.seed(76)  
  
# make populations for butterfly  
y1 <- rnorm(14, 32, 2.5)  
y2 <- rnorm(12, 28, 3.4)  
  
# calculate t test with t.test  
t.test(y1, y2)
```



$$\bar{y}_1 = 32$$

$$s_1 = 2.5$$



$$\bar{y}_2 = 28$$

$$s_2 = 3.4$$

$$t_s = 3.39$$

$$df = 11$$

```
# set random seed  
set.seed(76)  
  
# make populations for butterfly  
y1 <- rnorm(14, 32, 2.5)  
y2 <- rnorm(12, 28, 3.4)  
  
# calculate t test with t.test  
t.test(y1, y2)
```

Welch Two Sample t-test

```
data: y1 and y2  
t = 4.2051, df = 23.351, p-value = 0.0003288  
alternative hypothesis: true difference in means is not equal to 0  
95 percent confidence interval:  
 1.646552 4.829972  
sample estimates:  
mean of x mean of y  
31.37680 28.13854
```

**P = 0.00603**

**(1.404, 6.596)**

# Degrees of freedom: two samples

**Just for example, no need to memorize this equation!!!**

$$df = \frac{(SE_1^2 + SE_2^2)^2}{SE_1^4/(n_1 - 1) + SE_2^4(n_2 - 1)}$$

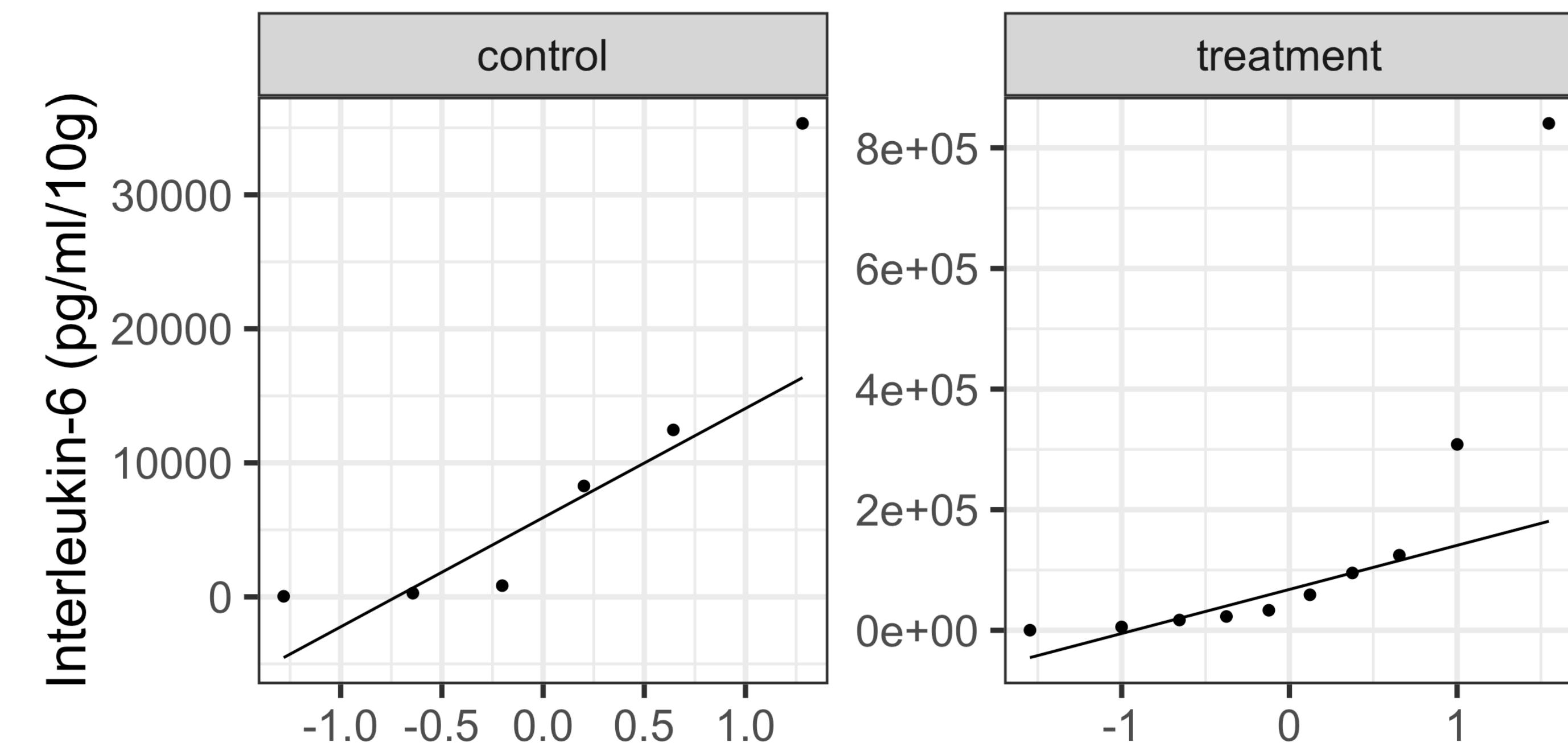
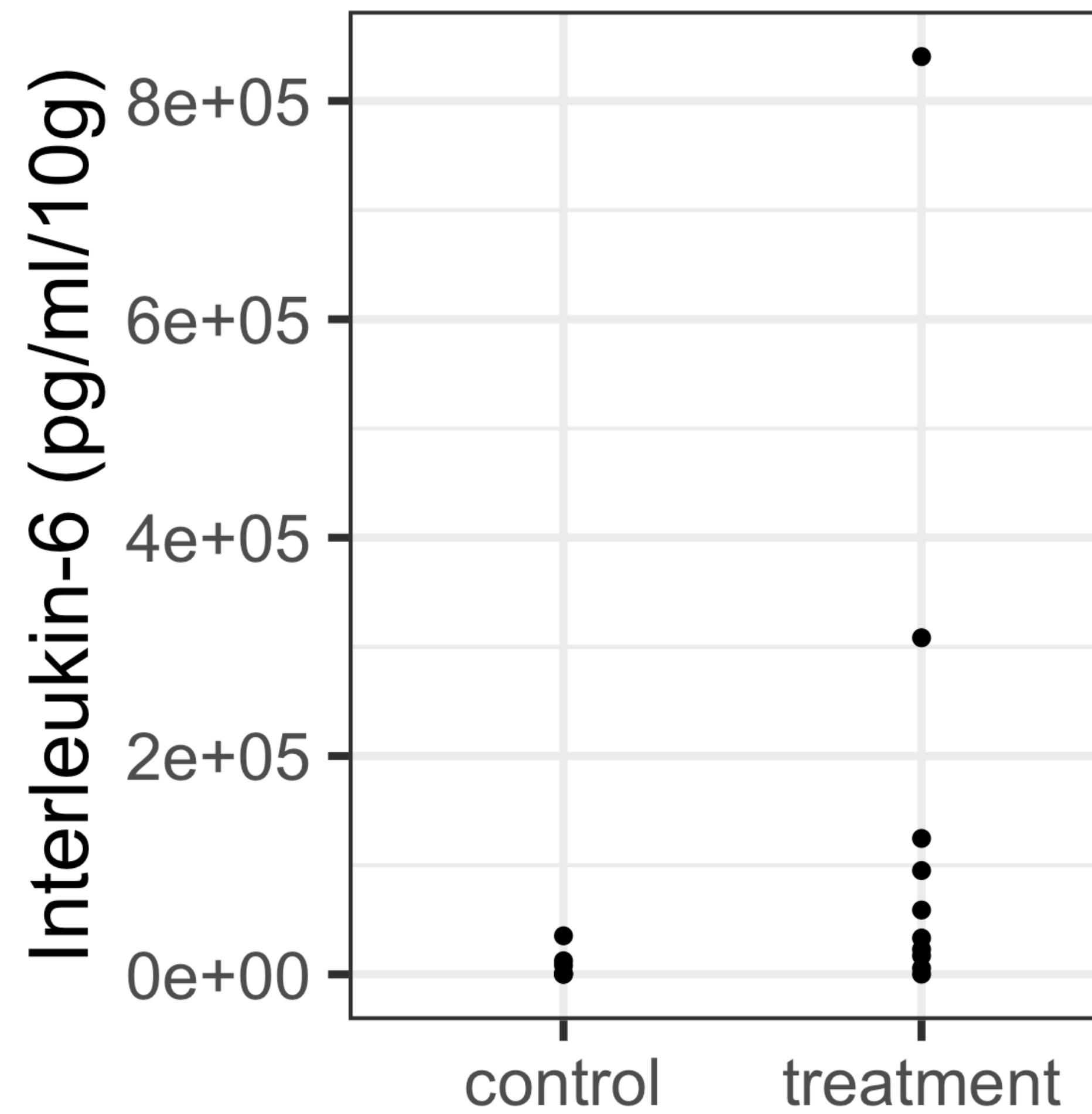
$$df = \frac{(0.668^2 + 0.982^2)^2}{0.668^4/(14 - 1) + 0.982^4(12 - 1)}$$

$$df = 19.2 \text{ (Compared to 23)}$$

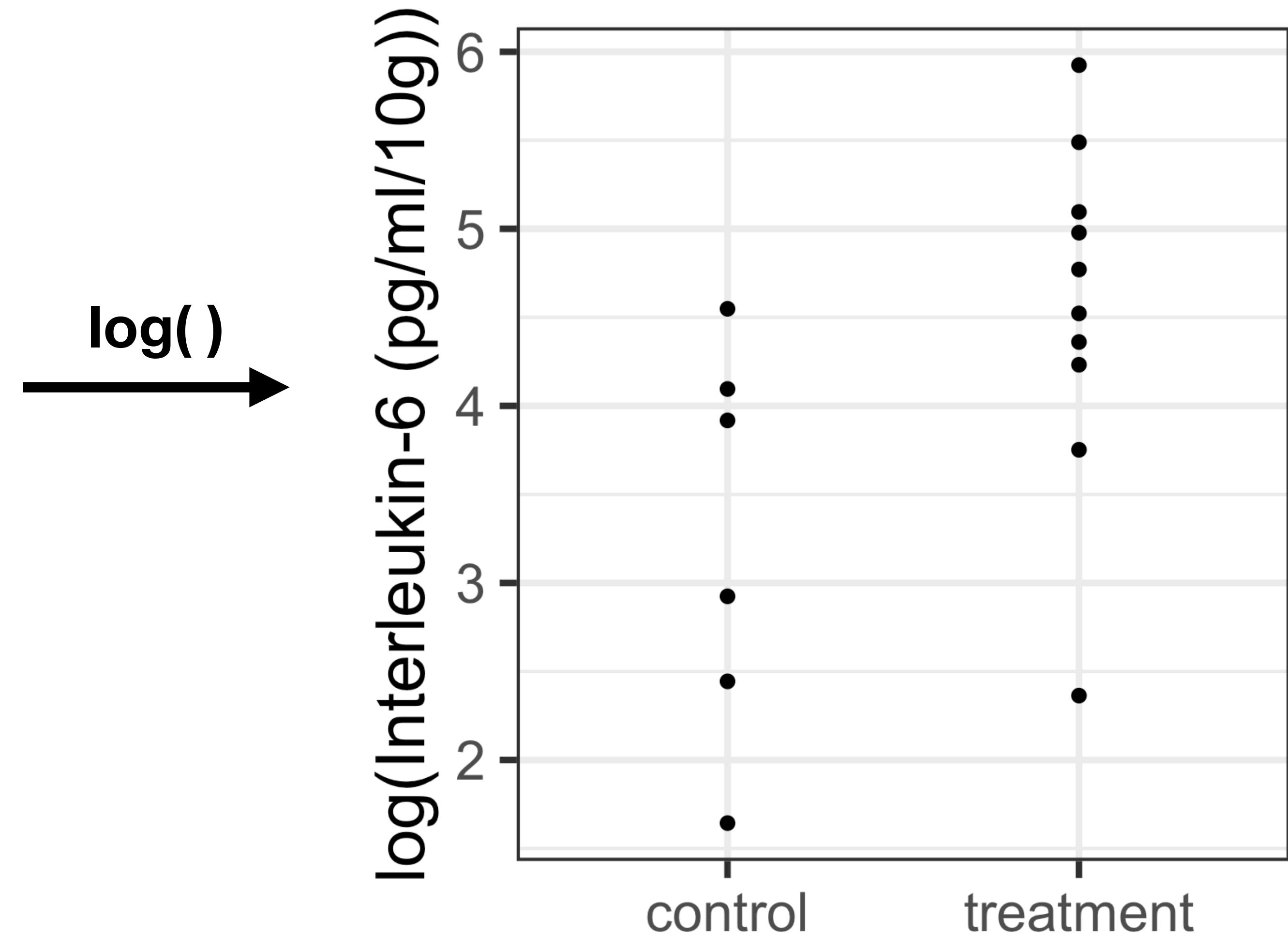
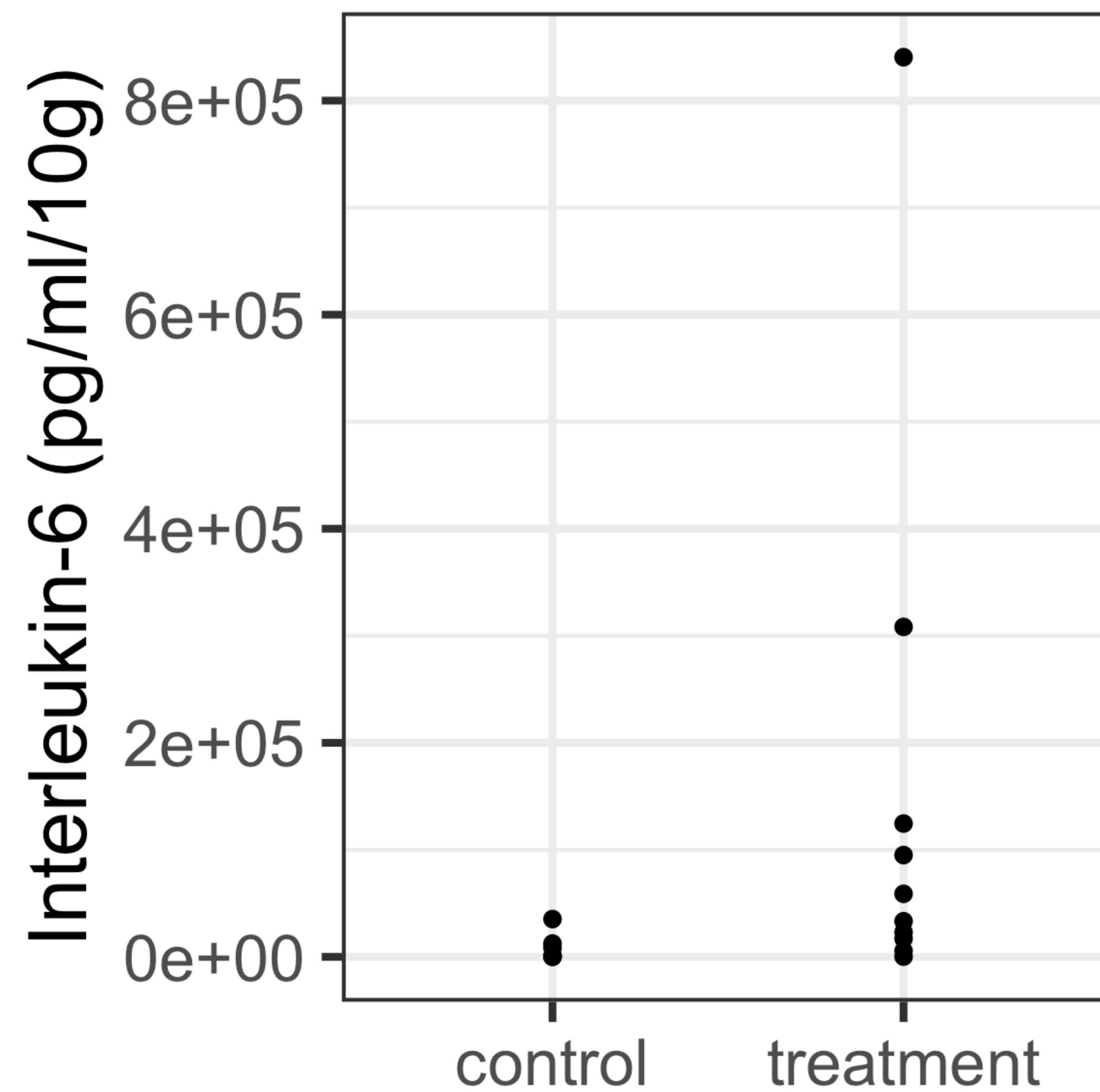
# Assumptions for comparing populations

- **Conditions on the design of the study:**
  - (1) Data is a random sample from respective large populations
  - (2) The two samples must be independent of each other
- **Conditions on the form of the population distribution**
  - (3) If  $n$  is small, the population distribution must be  $\sim$ normal
  - (4) If  $n$  is large, the population distribution doesn't have to be normal

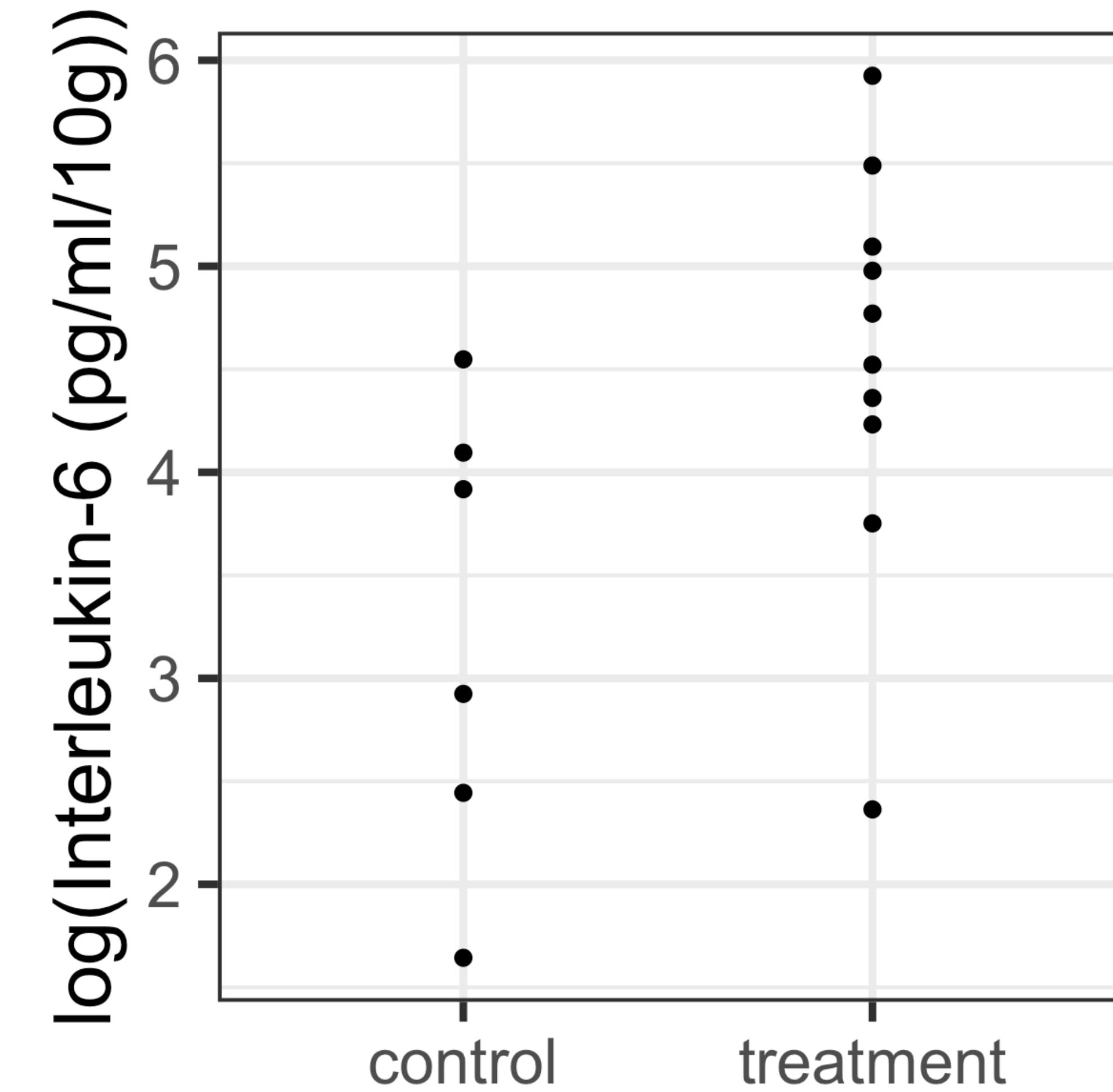
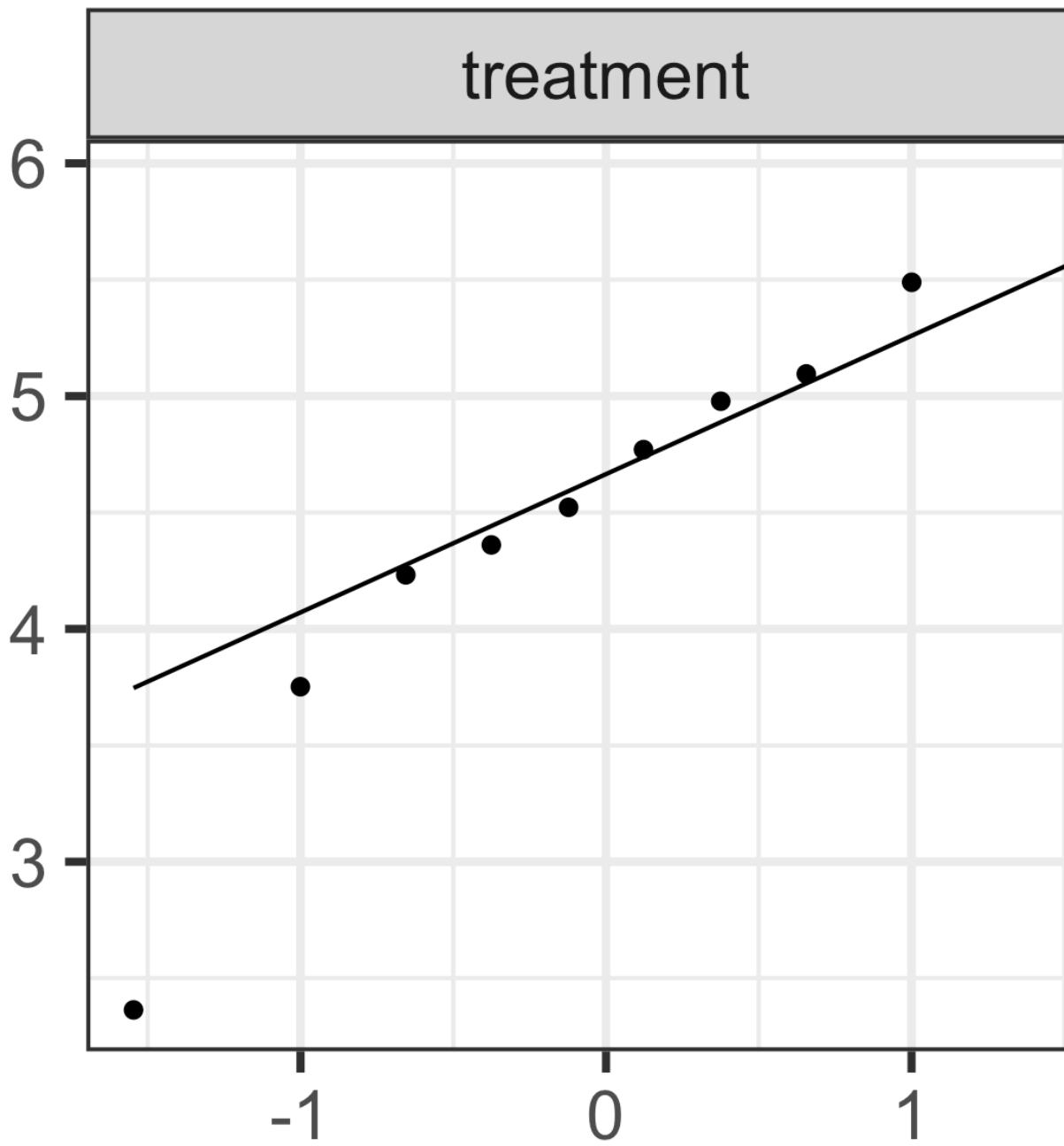
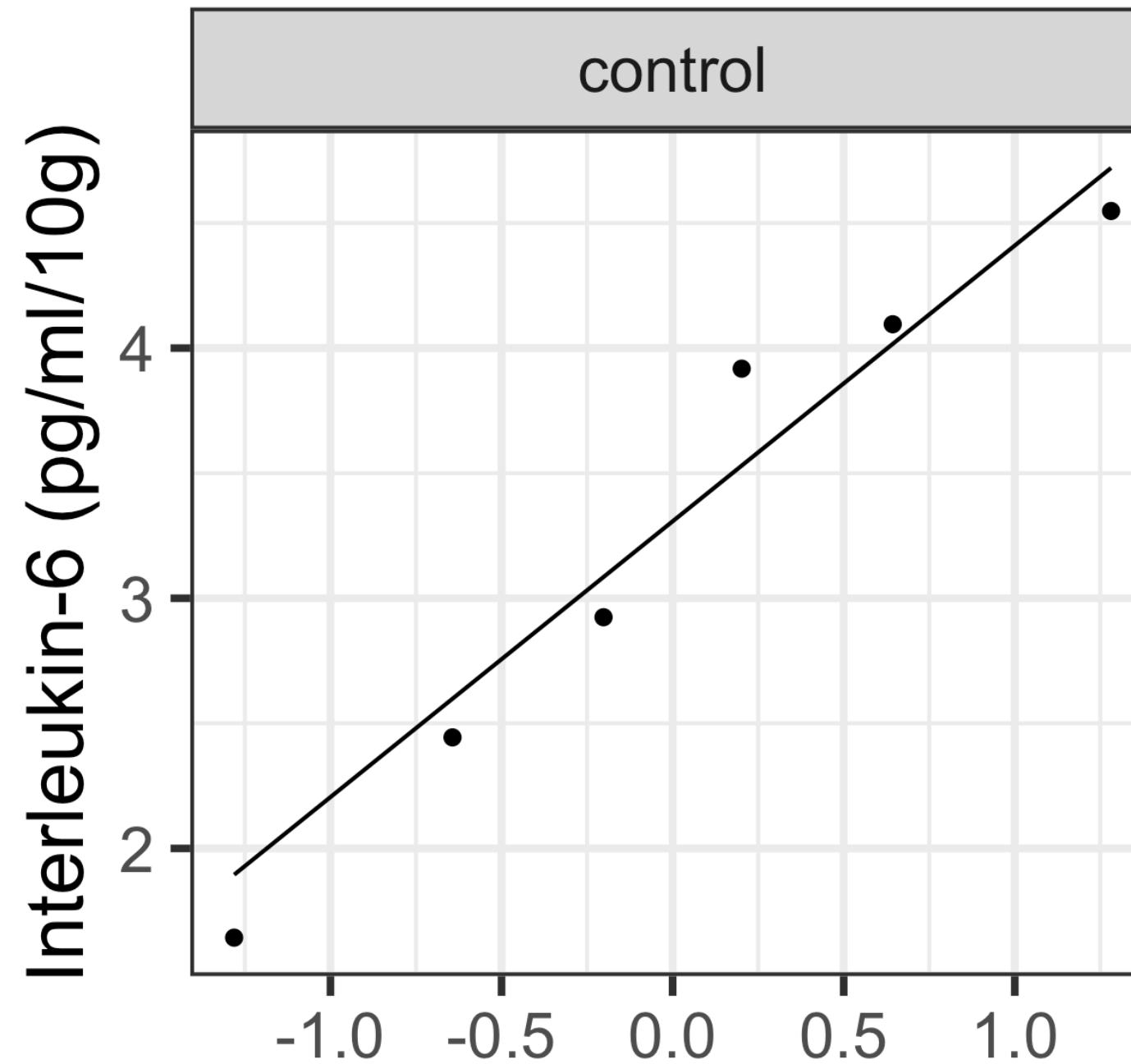
# Assumptions for comparing populations



# Assumptions for comparing populations



# Assumptions for comparing populations



# Assumptions for comparing populations

## Non-transformed

Welch Two Sample t-test

```
data: il$control and il$treatment
t = -1.7179, df = 9.0826, p-value = 0.1196
alternative hypothesis: true difference in means
is not equal to 0
95 percent confidence interval:
-326709.82 44453.02
sample estimates:
mean of x mean of y
9536.5 150664.9
```

## Log-transformed

Welch Two Sample t-test

```
data: i1$log.control and i1$log.treatment
t = -2.3319, df = 9.6879, p-value = 0.04269
alternative hypothesis: true difference in means
is not equal to 0
95 percent confidence interval:
-2.52145436 -0.05188761
sample estimates:
mean of x mean of y
3.262180 4.548851
```

# The paired-sample design

You are performing a study to see how heart rate changes during exercise. You measure 20 individual's heart rates before exercise and directly after five minutes of vigorous cardiovascular exercise.

Independent samples?

Observations = 40

Sample size,  $n$  = 20

Individual	Pre	Post
1	69	143
2	72	150
...	...	...
20	57	170

# The paired-sample design

You are performing a study to see how heart rate changes during exercise. You measure 20 individual's heart rates before exercise and directly after five minutes of vigorous cardiovascular exercise.

Independent samples?

Observations = 40

Sample size,  $n$  = 20

Individual	Pre	Post	Difference (D)
1	69	143	143-69
2	72	150	150-72
...	...	...	...
20	57	170	170-57

# The paired-sample design

You are performing a study to see how heart rate changes during exercise. You measure 20 individual's heart rates before exercise and directly after five minutes of vigorous cardiovascular exercise.

Independent samples?

Use difference ( $D$ ) as a single sample now

Observations = 40

Sample size,  $n = 20$

Individual	Pre	Post	Difference (D)
1	69	143	74
2	72	150	78
...	...	...	...
20	57	170	113

# Examples of paired-sample designs

- **Pre-test & post-test samples:** factor is measured before and after
- **Cross-over trials:** patients switch treatments half-way through a trial
- **Matched samples:** individuals are matched based on characteristics
- **Duplicate measurements:** technical replicates
- **Pairing by time:** observations made at the same time, month, etc.

*Paired-sample designs aim to **reduce bias** and **increase precision***

# Paired-sample $t$ test and confidence

**Hypothesis testing for paired samples is done exactly the same as hypothesis testing for one sample as long as you use the difference sample distribution**

$$\bar{d} = 88.3$$

$$s_D = 5.4$$

$$SE_{\bar{D}} = 1.20$$

$$SE_{\bar{D}} = \frac{s_D}{\sqrt{n_D}}$$
$$= \frac{5.4}{\sqrt{20}}$$

# Paired-sample $t$ test and confidence

**Hypothesis testing for paired samples is done exactly the same as hypothesis testing for one sample as long as you use the difference sample distribution**

$$\bar{d} = 88.3$$

$$s_D = 5.4$$

$$SE_{\bar{D}} = 1.20$$

$$\bar{d} \pm t_{0.025} SE_{\bar{D}}$$

$$(df = n_D - 1)$$

$$\text{qt}(0.975, \text{ df } = 19)$$

$$88.3 \pm 2.09(1.20)$$

# Paired-sample $t$ test and confidence

Hypothesis testing for paired samples is done exactly the same as hypothesis testing for one sample as long as you use the difference sample distribution

$$\bar{d} = 88.3 \pm 2.5$$

$$s_D = 5.4$$

$$SE_{\bar{D}} = 1.20$$

$$\bar{d} \pm t_{0.025} SE_{\bar{D}}$$

$$(df = n_D - 1)$$

$$\text{qt}(0.975, \text{ df } = 19)$$

$$88.3 \pm 2.09(1.20)$$

# Paired-sample $t$ test and confidence

**Hypothesis testing for paired samples is done exactly the same as hypothesis testing for one sample as long as you use the difference sample distribution**

$$\bar{d} = 88.3 \pm 2.5$$

$$s_D = 5.4$$

$$SE_{\bar{D}} = 1.20$$

$$t_s = 73.5$$

$$H_0 : \mu_{\bar{D}} = 0 \quad H_A : \mu_{\bar{D}} \neq 0$$

$$t_s = \frac{\bar{d} - 0}{SE_{\bar{D}}}$$

$$t_s = \frac{88.3}{1.20}$$

# Paired-sample $t$ test and confidence

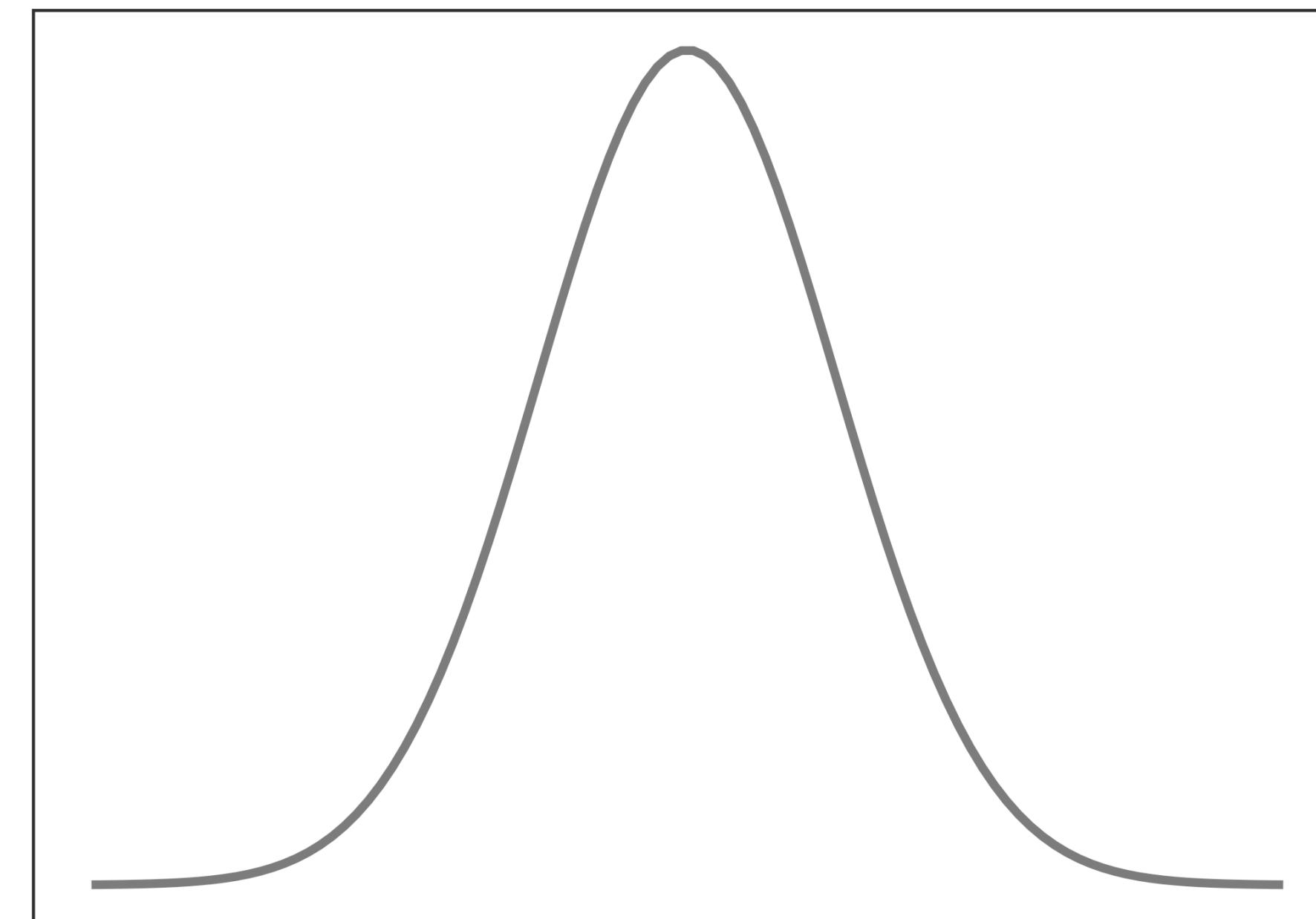
**Hypothesis testing for paired samples is done exactly the same as hypothesis testing for one sample as long as you use the difference sample distribution**

$$\bar{d} = 88.3 \pm 2.5$$

$$s_D = 5.4$$

$$SE_{\bar{D}} = 1.20$$

$$t_s = 73.5$$



$$t_s = 73.5$$

```
pt(73.5, df = 19) = 0
```



cAMP is a substance that can mediate cellular response to hormones. In a certain study, oocytes from four *Xenopus* females were divided into two batches: one batch was exposed to progesterone and the other was not. After 2 minutes, each batch was assayed for its cAMP content. Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

Frog	Control	Progesterone
1	6.01	5.23
2	2.28	1.21
3	1.51	1.40
4	2.12	1.38

cAMP is a substance that can mediate cellular response to hormones. In a certain study, oocytes from four *Xenopus* females were divided into two batches: one batch was exposed to progesterone and the other was not. After 2 minutes, each batch was assayed for its cAMP content. Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

Frog	Control	Progesterone		P	one-tail	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
				two-tails	0.2	0.1	0.05	0.02	0.01	0.002	0.001	
				DF								
1	6.01	5.23		1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	2.28	1.21		2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.51	1.40		3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	2.12	1.38		4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
				5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
				6	1.44	1.943	2.447	3.143	3.707	5.208	5.959	
				7	1.415	1.895	2.365	2.998	3.499	4.785	5.408	
				8	1.397	1.86	2.306	2.896	3.355	4.501	5.041	
				9	1.383	1.833	2.262	2.821	3.25	4.297	4.781	
				10	1.372	1.812	2.228	2.764	3.169	4.144	4.587	

cAMP is a substance that can mediate cellular response to hormones. In a certain study, oocytes from four *Xenopus* females were divided into two batches: one batch was exposed to progesterone and the other was not. After 2 minutes, each batch was assayed for its cAMP content. Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

Frog	Control	Prog.	Diff.
1	6.01	5.23	0.78
2	2.28	1.21	1.07
3	1.51	1.40	0.11
4	2.12	1.38	0.74
Mean	2.98	2.31	0.68
SD	2.05	1.95	0.40

Independent samples?

PAIRED  $t$  test!

1. Generate a hypothesis and choose a significance level

2. Calculate the differences

3. Calculate test statistic

4. Calculate the  $P$ -value

Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

$$\bar{d} = 0.68$$

$$s_d = 0.40$$

$$n_d = 4$$

**1. Generate a hypothesis and choose a significance level**

$$H_0 : \bar{d} = 0 \quad H_A : \bar{d} \neq 0 \quad \alpha = 0.10$$

**2. Calculate the differences**

**3. Calculate test statistic**

$$t_s = \frac{\bar{d}}{SE_{\bar{D}}} = \frac{0.68}{0.40/\sqrt{4}} = 3.4$$

**4. Calculate the  $P$ -value**

$$df = n_d - 1 = 3$$

Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

$$\bar{d} = 0.68$$

$$s_d = 0.40$$

$$n_d = 4$$

$$t_s = 3.4$$

$$df = 3$$

$$0.02 < P < 0.05$$

	P	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
one-tail		0.1	0.05	0.025	0.01	0.005	0.001	0.0005
two-tails		0.2	0.1	0.05	0.02	0.01	0.002	0.001
DF								
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578	
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6	
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924	
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61	
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869	
6	1.44	1.943	2.447	3.143	3.707	5.208	5.959	
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408	
8	1.397	1.86	2.306	2.896	3.355	4.501	5.041	

```
> pt(3.4, 3, lower.tail = F) * 2
```

[1] 0.0424



REJECT the  
null hypothesis

Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

```
# create data table
df <- data.frame(control = c(6.01, 2.28, 1.51, 2.12),
                  progesterone = c(5.23, 1.21, 1.40, 1.38)) %>%
  dplyr::mutate(diff = control - progesterone)
```

```
# calculate t test with t.test
t.test(df$control, df$progesterone, paired = T)
```

Paired t-test

data: df\$control and df\$progesterone  
t = 3.3387, df = 3, p-value = 0.04443



alternative hypothesis: true difference in means is not equal to 0

95 percent confidence interval:

0.0315868 1.3184132

sample estimates:

mean of the differences

0.675

**t = 3.4**

**P = 0.0424**

**Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$**

```
# create data table
df <- data.frame(control = c(6.01, 2.28, 1.51, 2.12),
                  progesterone = c(5.23, 1.21, 1.40, 1.38)) %>%
  dplyr::mutate(diff = control - progesterone)
```

```
# calculate t test with t.test
t.test(df$diff)
```

One Sample t-test

```
data: df$diff
t = 3.3387, df = 3, p-value = 0.04443
```



alternative hypothesis: true mean is not equal to 0

95 percent confidence interval:

0.0315868 1.3184132

sample estimates:

mean of x

0.675

**t = 3.4**

**P = 0.0424**

Use a  $t$  test to investigate the effect of progesterone on cAMP. Let  $H_A$  be nondirectional and let  $\alpha = 0.10$

```
# create data table
df <- data.frame(control = c(6.01, 2.28, 1.51, 2.12),
                  progesterone = c(5.23, 1.21, 1.40, 1.38)) %>%
  dplyr::mutate(diff = control - progesterone)
```

```
# calculate t test with t.test
t.test(df$control, df$progesterone, mu = 1, paired = T)
```

Paired t-test

$$t_s = \frac{\bar{d} - 1}{SE_{\bar{D}}} \\ = -1.6$$

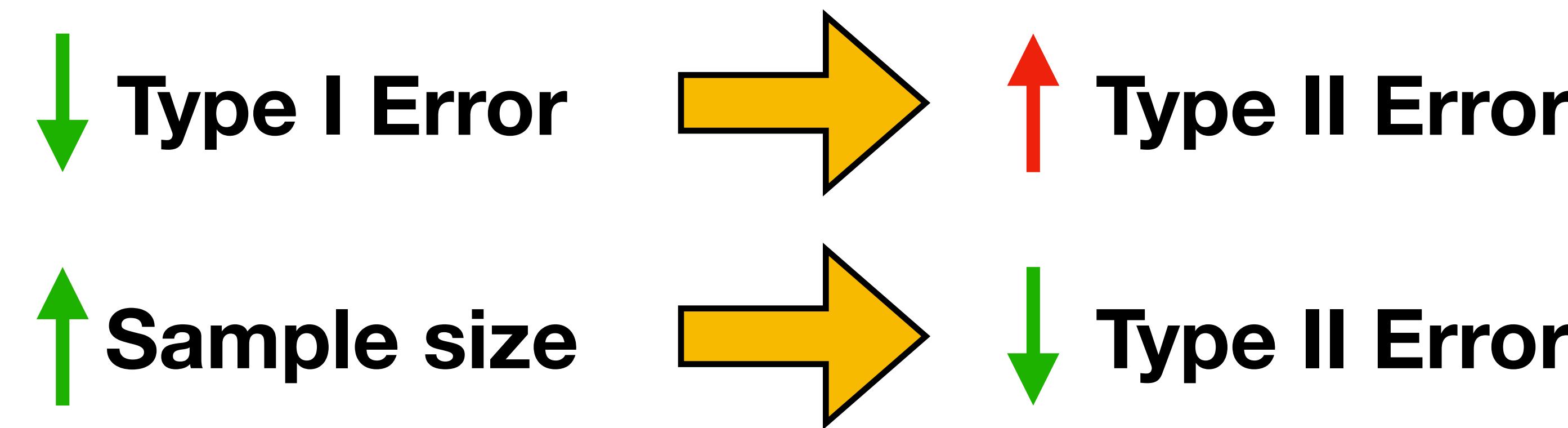
```
Paired t-test

data: df$control and df$progesterone
t = -1.6075, df = 3, p-value = 0.2063
alternative hypothesis: true difference in means is not equal to 1
95 percent confidence interval:
 0.0315868 1.3184132
sample estimates:
mean of the differences
 0.675
```

# Assumptions for paired-sample analysis

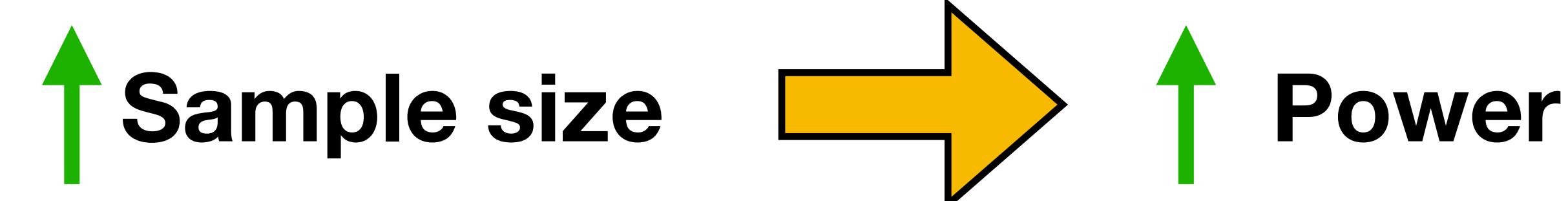
- **Conditions on the design of the study:**
  - The *differences* (D's) must be regarded as a random sample from some large population
- **Conditions on the form of the population distribution**
  - The population distribution of the D's must be normal (or sample size must be large ~ approx. normal)

# Returning to a discussion on power



The **power** of a statistical test is the probability of the null hypothesis is **false** and you **reject** the null hypothesis

$$\text{Power} = 1 - \text{Type II Error}$$

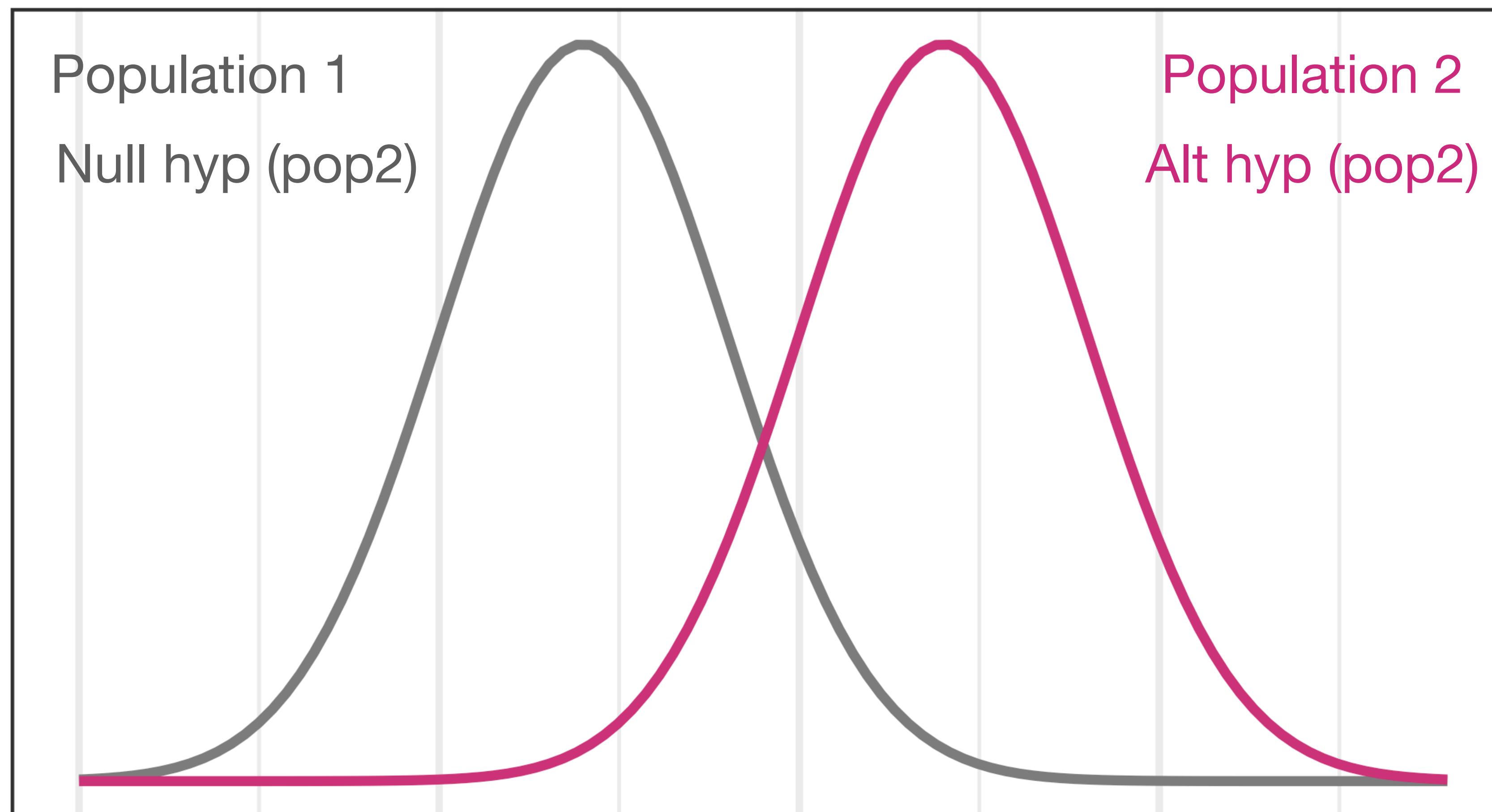


Measure of **sensitivity** of a test: ability to detect a difference between means when the difference truly exists

**Power** = P(rejecting  $H_0$  when  $H_A$  is TRUE)

$$H_0 : Pop1 = Pop2$$

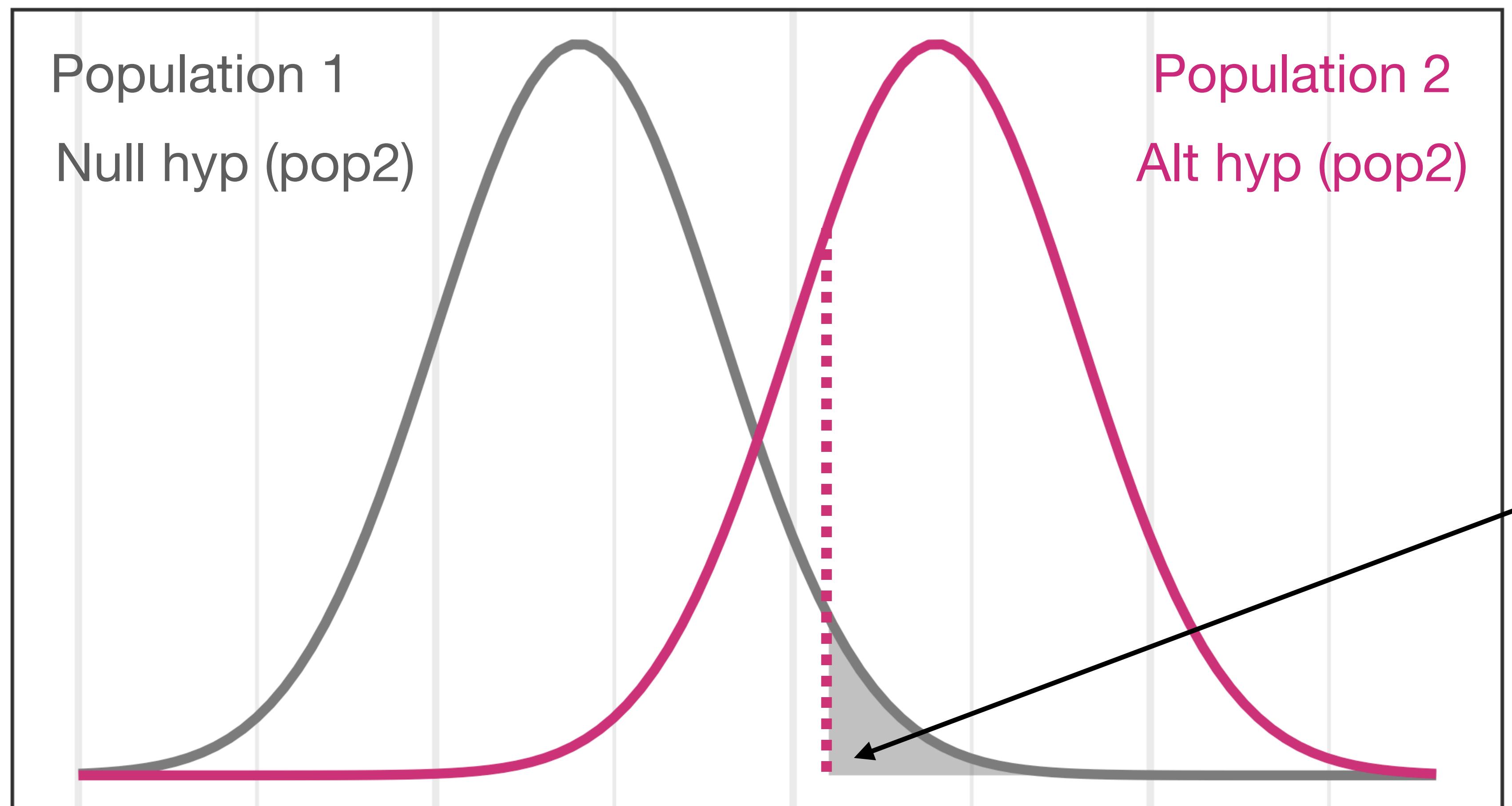
$$H_A : Pop2 > Pop1$$



**Power** = P(rejecting  $H_0$  when  $H_A$  is TRUE)

$$H_0 : Pop1 = Pop2$$

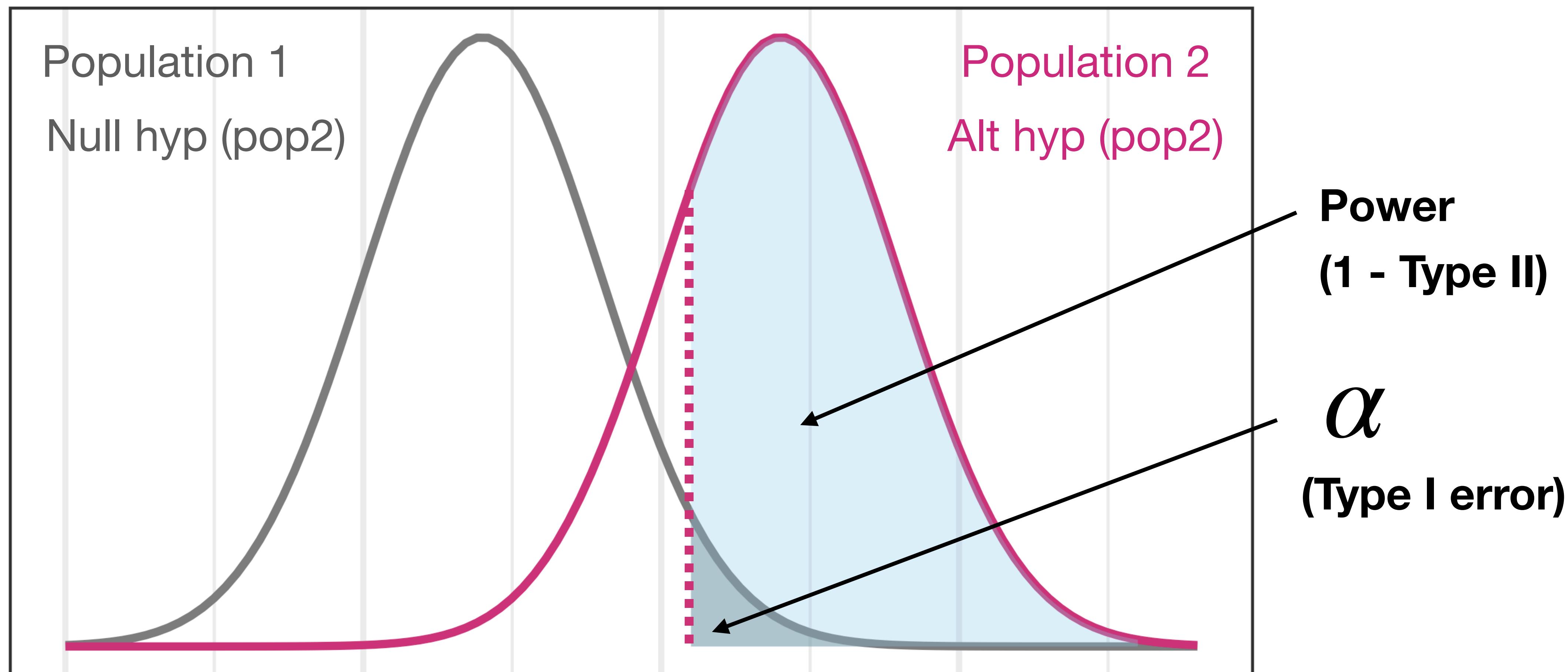
$$H_A : Pop2 > Pop1$$



**Power** = P(rejecting  $H_0$  when  $H_A$  is TRUE)

$$H_0 : Pop1 = Pop2$$

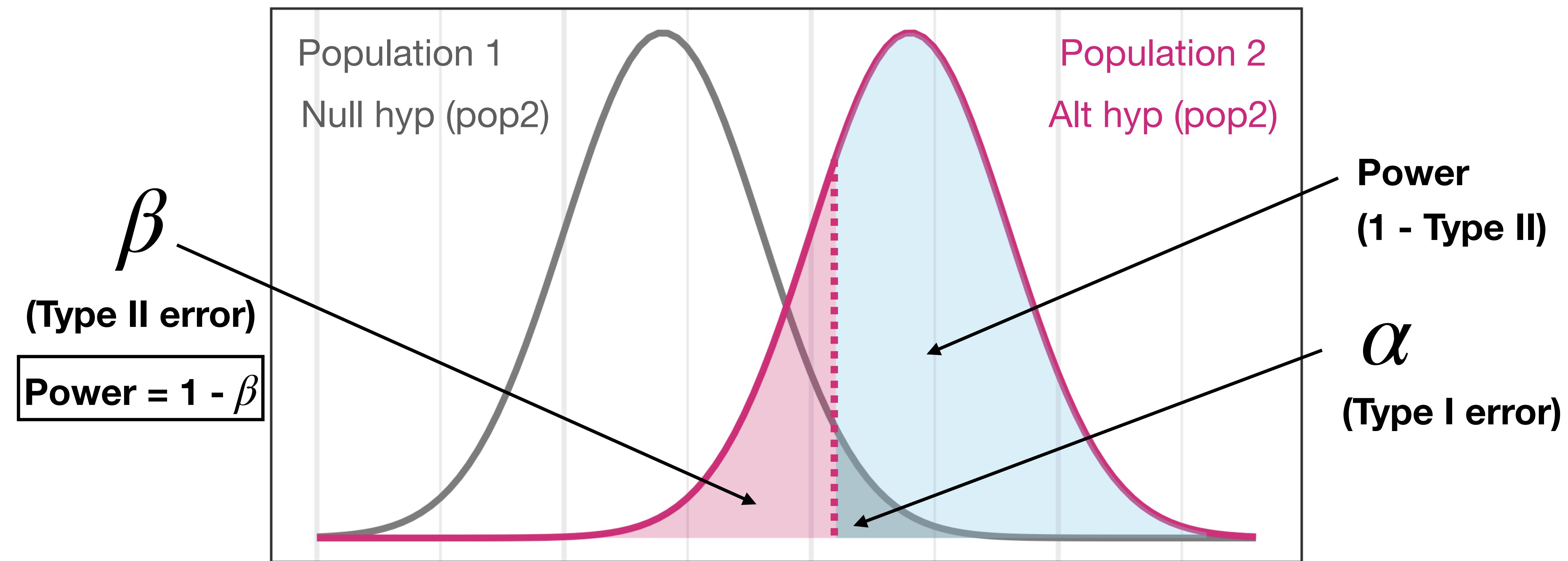
$$H_A : Pop2 > Pop1$$



**Power** = P(rejecting  $H_0$  when  $H_A$  is TRUE)

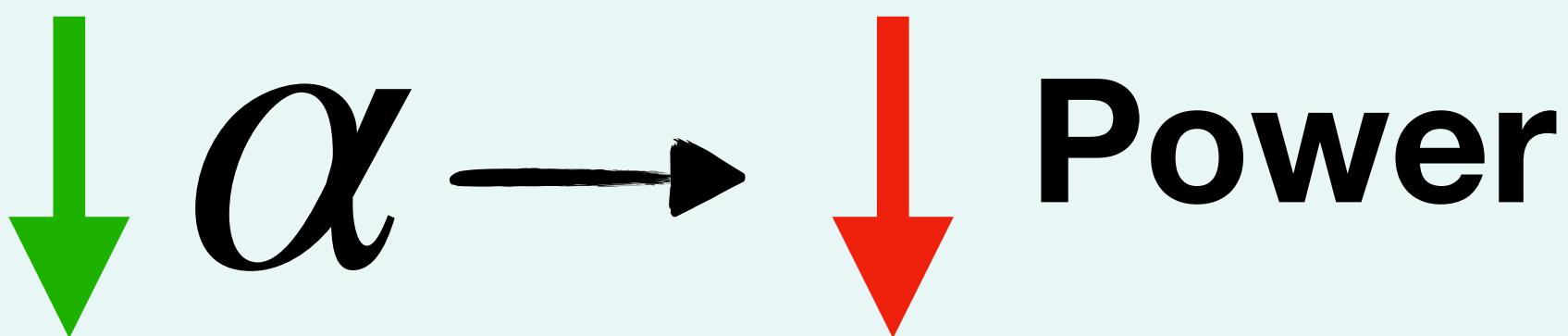
$$H_0 : Pop1 = Pop2$$

$$H_A : Pop2 > Pop1$$



# Planning for adequate power

**Significant level,  $\alpha$**



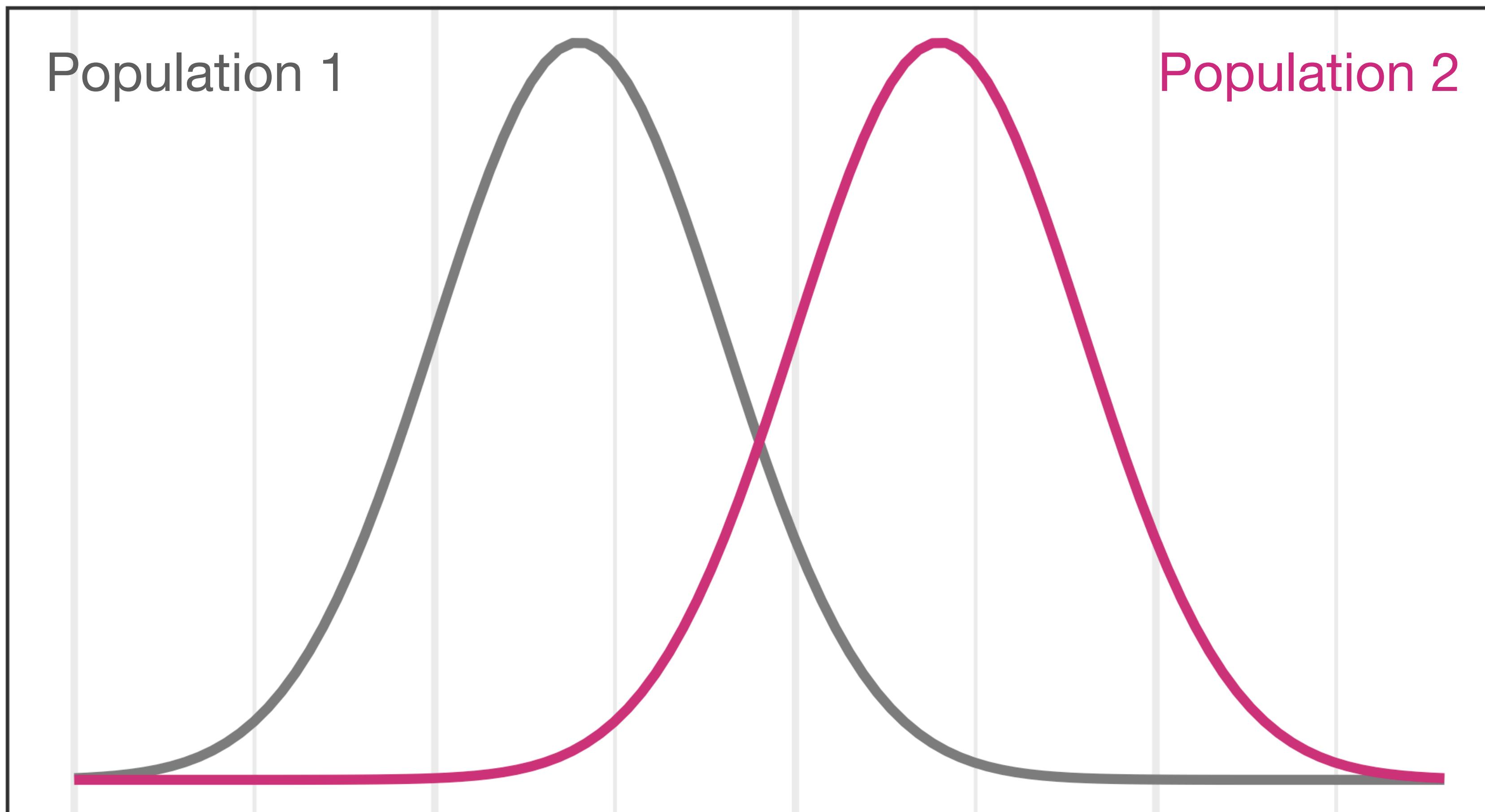
**Standard deviation,  $\sigma$**

**Sample size,  $n$**

**Difference in means,  $\bar{y}_1 - \bar{y}_2$**

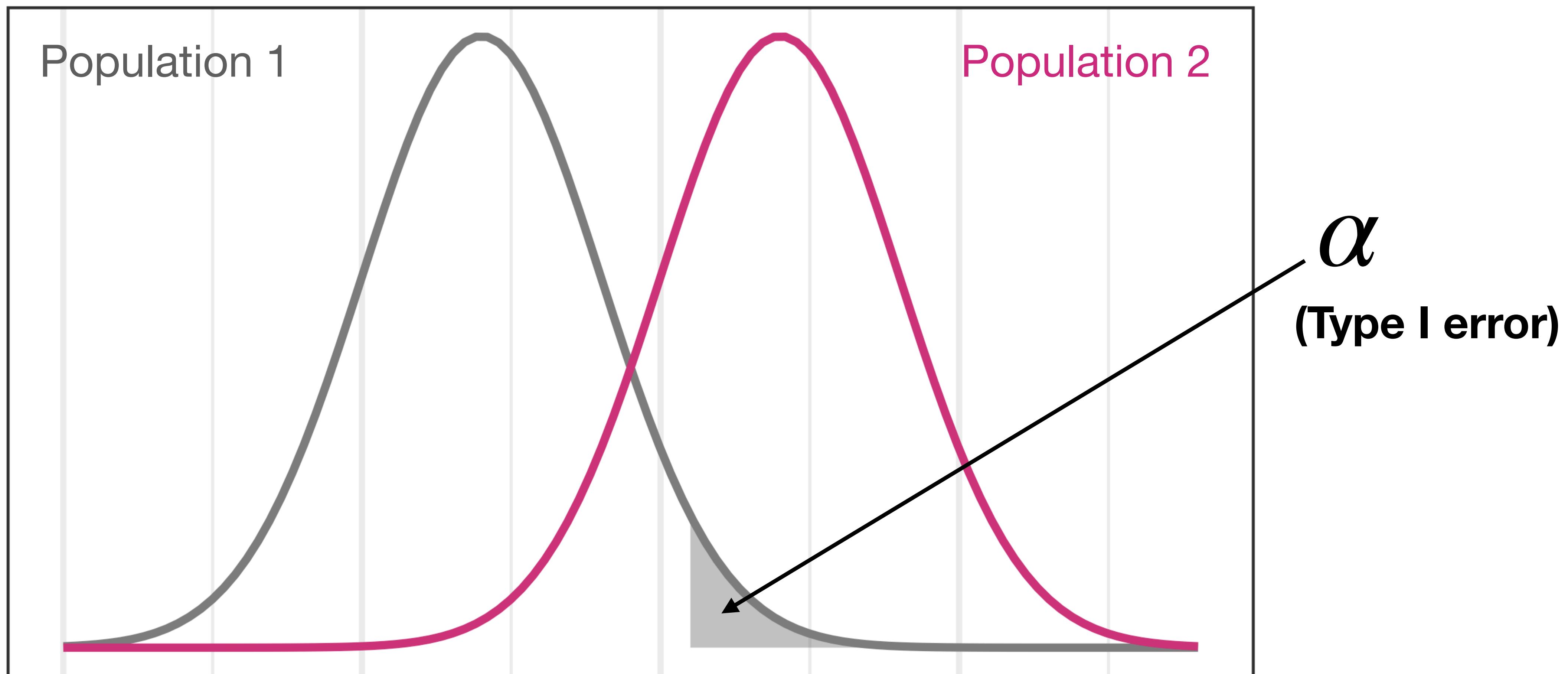
**Power** = 1 - prob(Type II Error)

↓  $\alpha$  → ↓ **Power**



**Power** = 1 - prob(Type II Error)

↓  $\alpha$  → ↓ **Power**



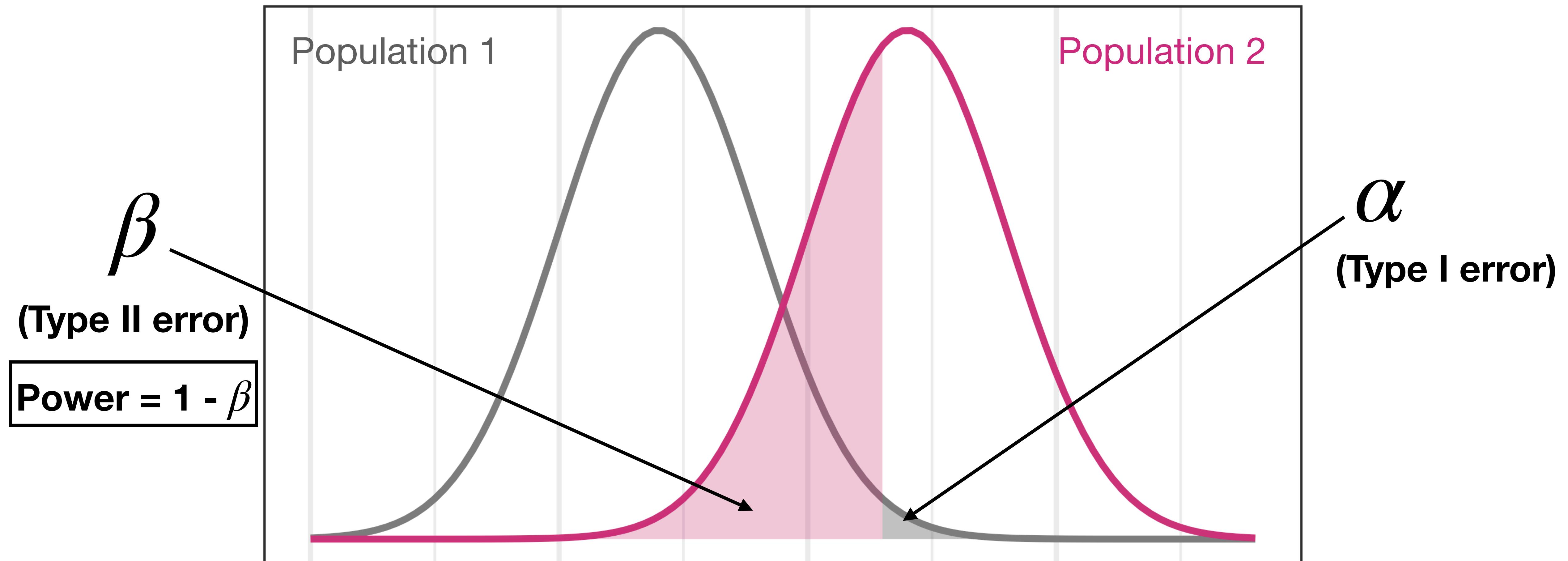
**Power** =  $1 - \text{prob}(\text{Type II Error})$

↓  $\alpha$  → ↓ **Power**



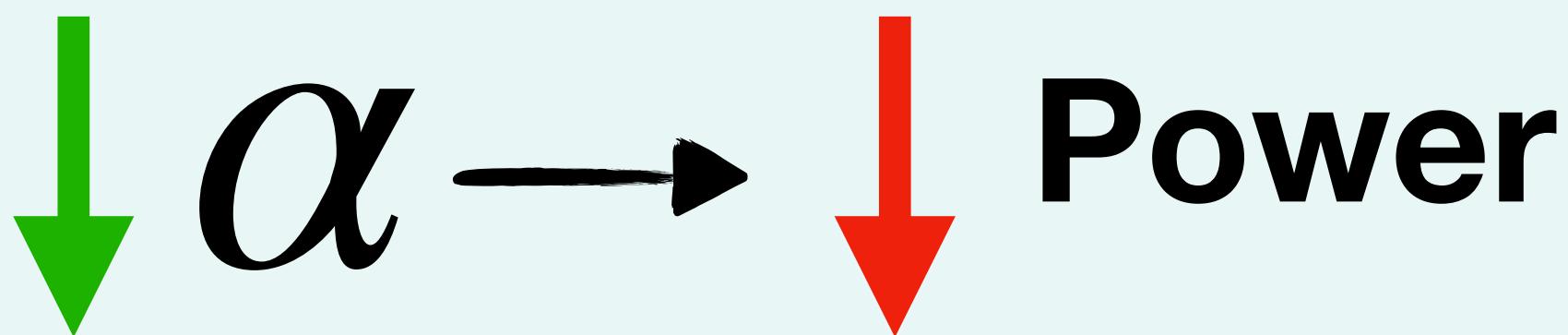
**Power** = 1 - prob(Type II Error)

↓  $\alpha$  → ↓ **Power**

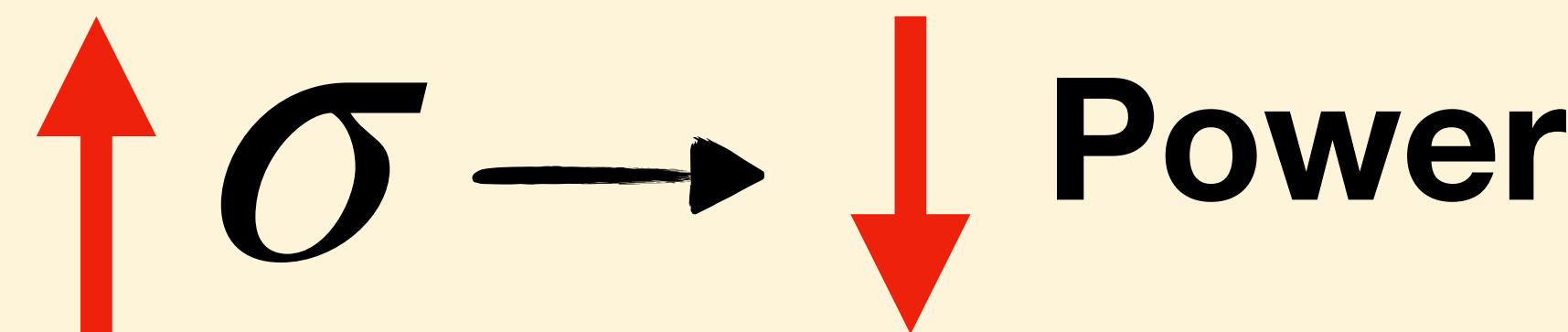


# Planning for adequate power

**Significant level,  $\alpha$**



**Standard deviation,  $\sigma$**

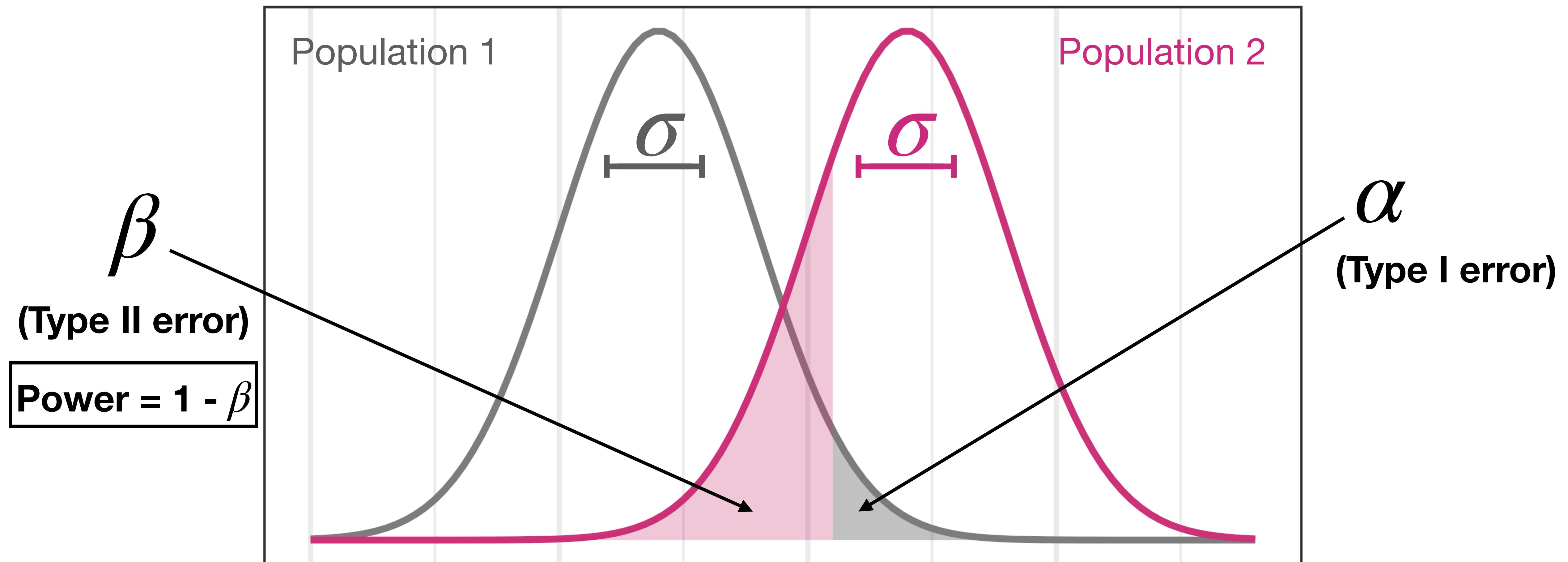


**Sample size,  $n$**

**Difference in means,  $\bar{y}_1 - \bar{y}_2$**

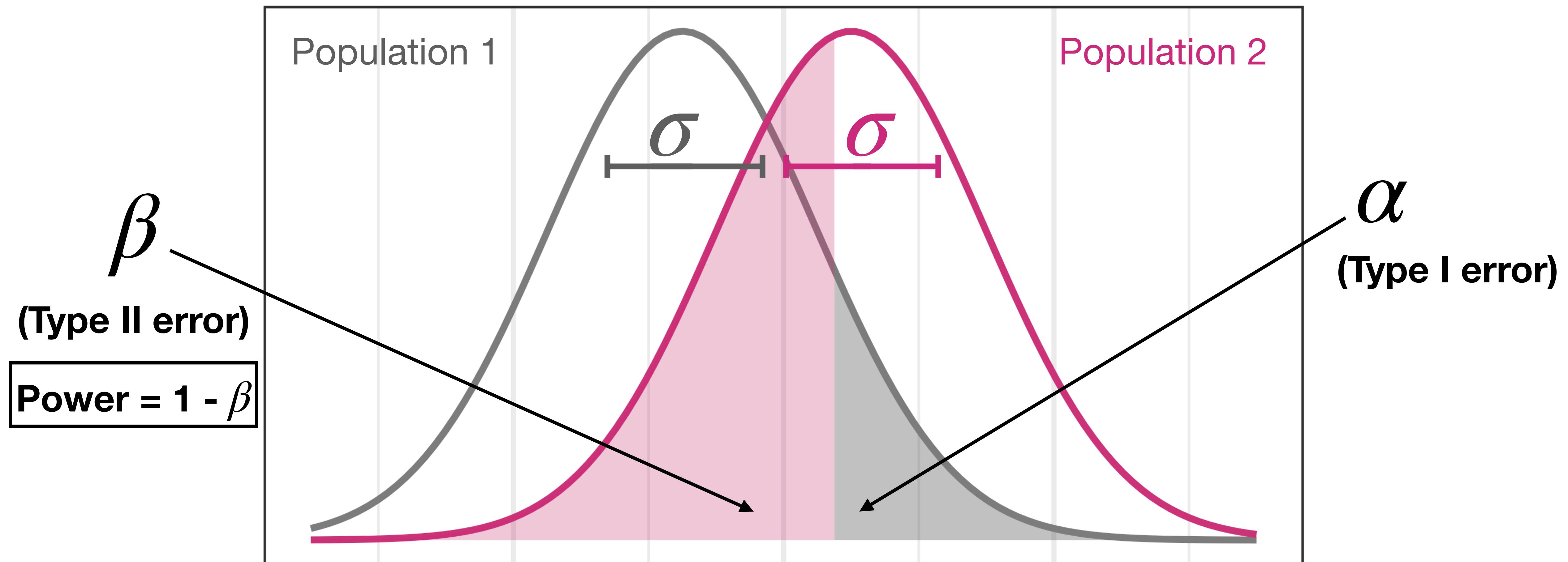
$$\text{Power} = 1 - \text{prob}(\text{Type II Error})$$

$\uparrow \sigma \rightarrow \downarrow \text{Power}$



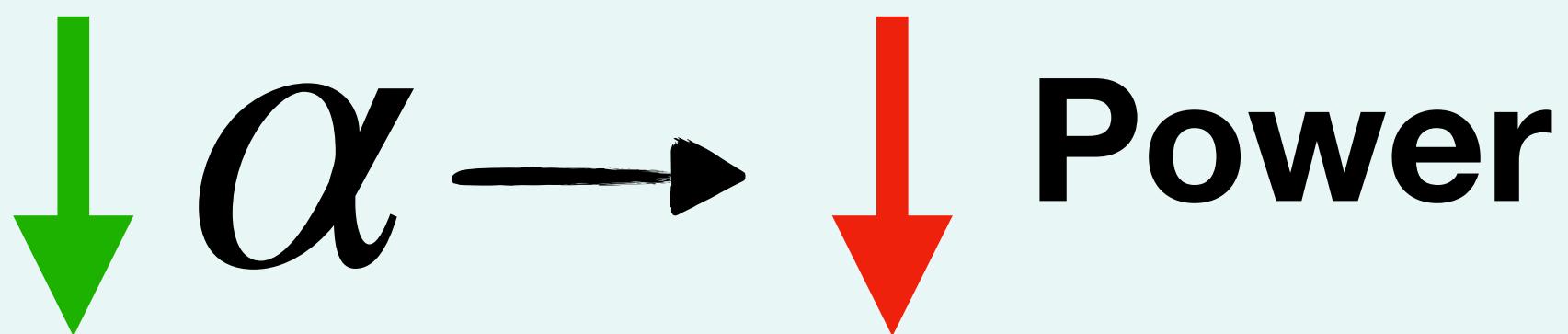
$$\text{Power} = 1 - \text{prob}(\text{Type II Error})$$

$\uparrow \sigma \rightarrow \downarrow \text{Power}$

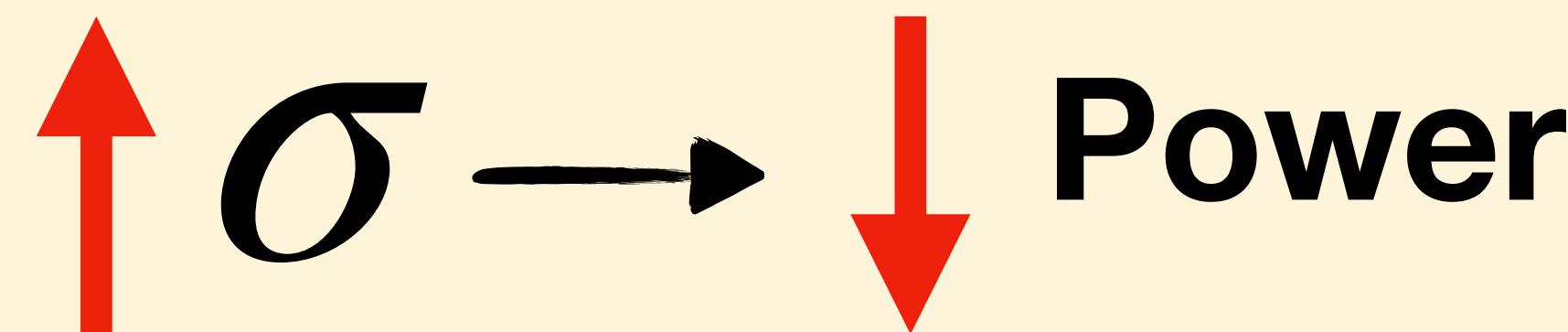


# Planning for adequate power

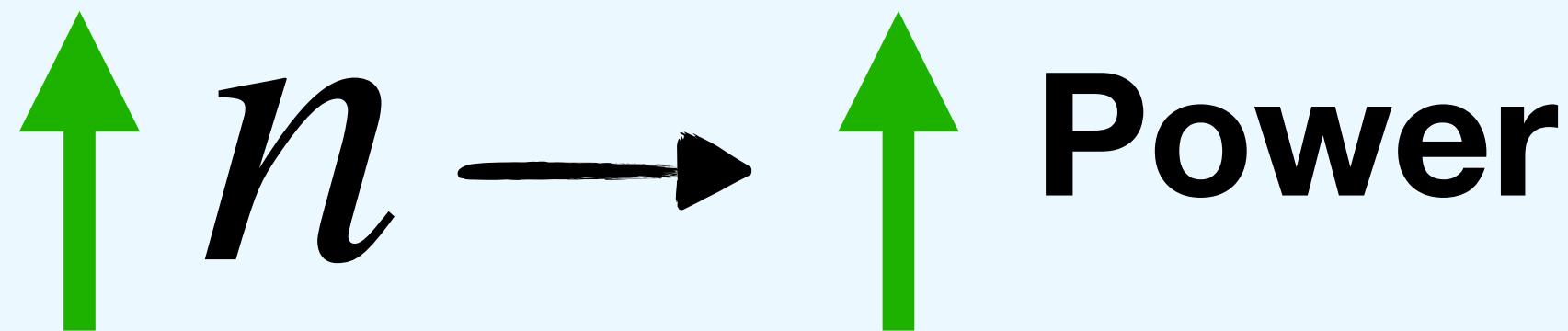
**Significant level,  $\alpha$**



**Standard deviation,  $\sigma$**

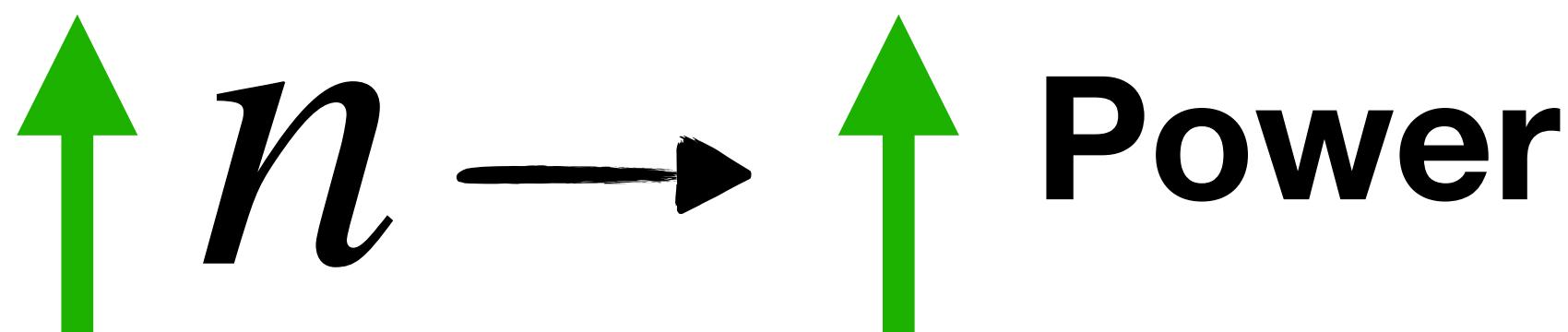


**Sample size,  $n$**



**Difference in means,  $\bar{y}_1 - \bar{y}_2$**

# Planning for adequate power

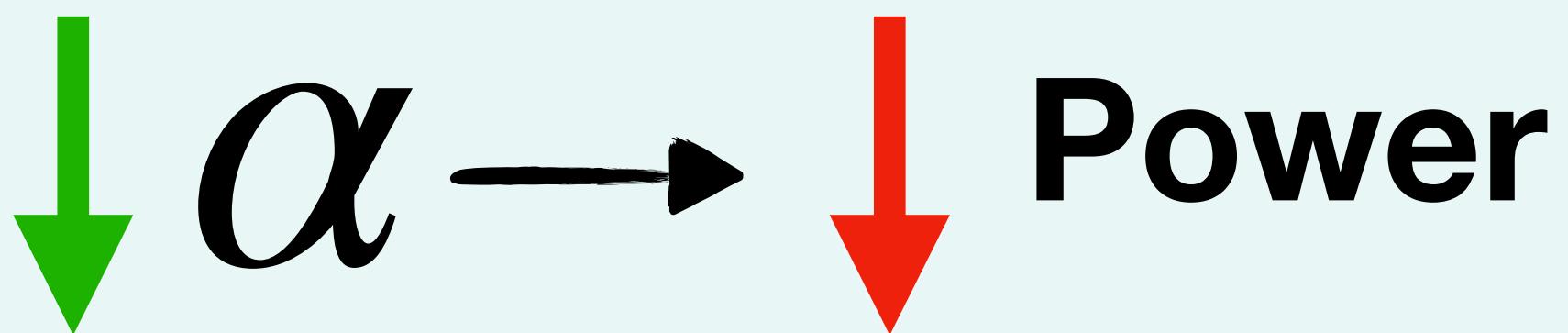


**Explain n**

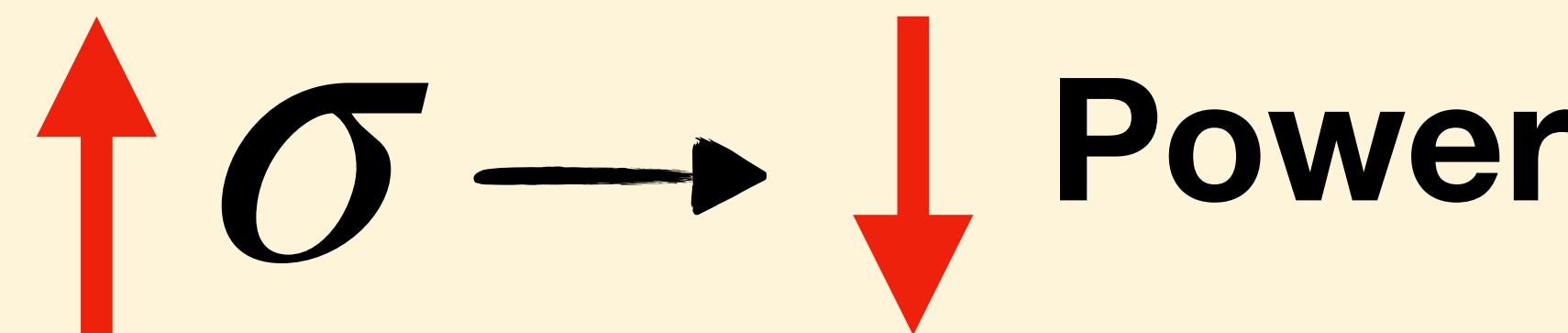
**Also higher power if both samples are same size**

# Planning for adequate power

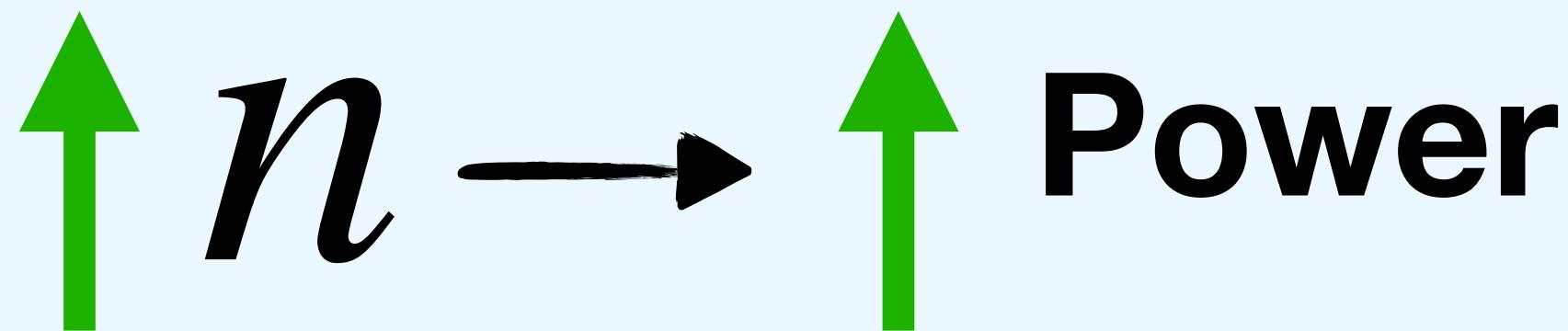
**Significant level,  $\alpha$**



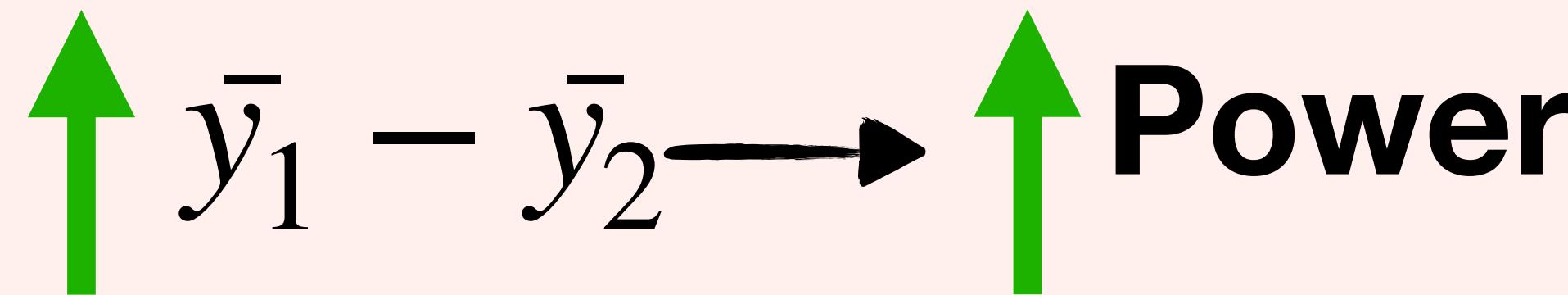
**Standard deviation,  $\sigma$**



**Sample size,  $n$**

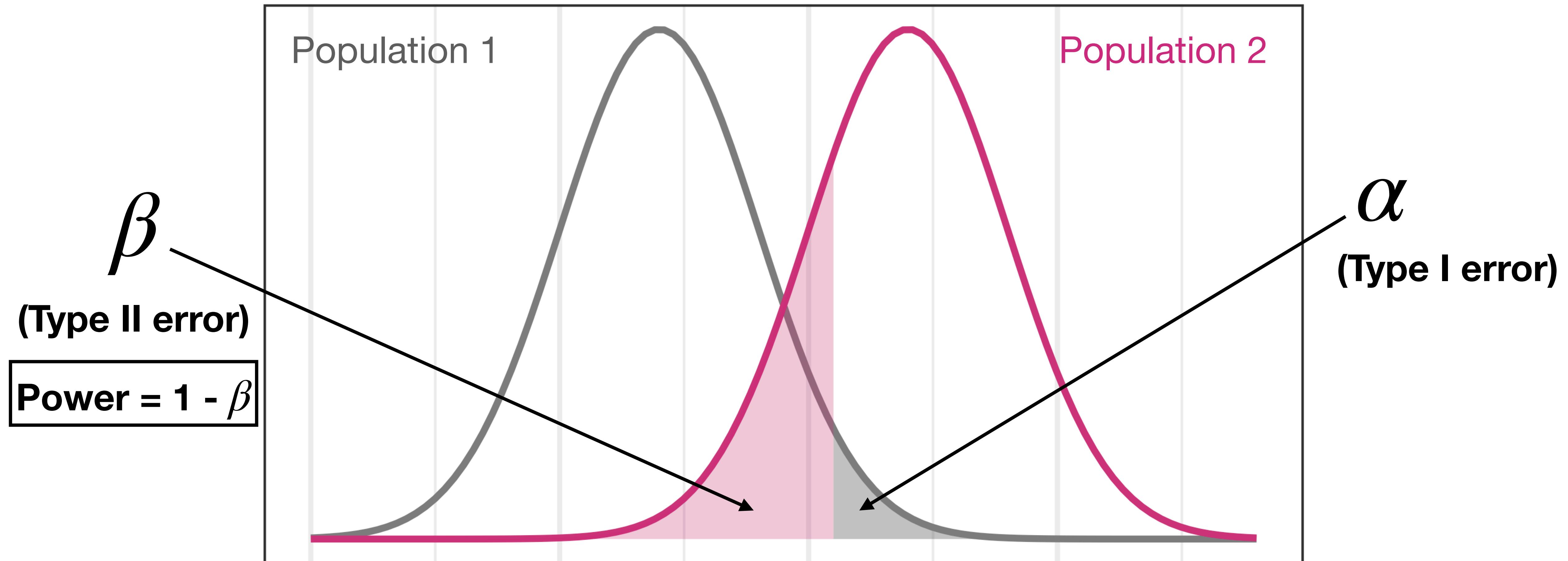


**Difference in means,  $\bar{y}_1 - \bar{y}_2$**



**Power** = 1 - prob(Type II Error)

$$\uparrow \bar{y}_1 - \bar{y}_2 \longrightarrow \uparrow \text{Power}$$



$$\text{Power} = 1 - \text{prob}(\text{Type II Error})$$

$$\uparrow \bar{y}_1 - \bar{y}_2 \rightarrow \uparrow \text{Power}$$



# Planning for adequate power

```
power.t.test(n = 20, delta = 2, sd = 1, sig.level = 0.05)
```

Two-sample t test power calculation

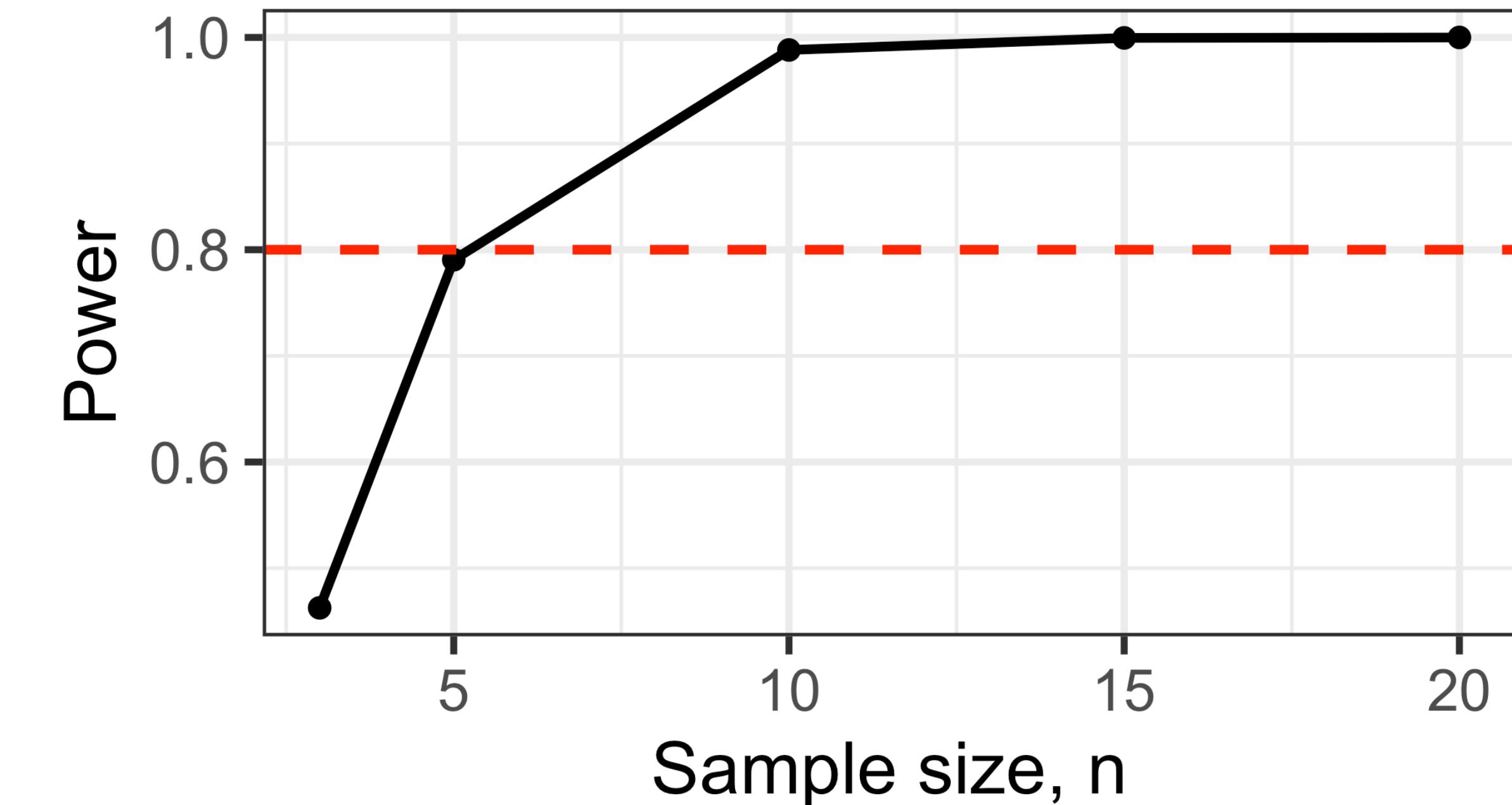
```
    n = 20
    delta = 2
    sd = 1
    sig.level = 0.05
    power = 0.9999866
    alternative = two.sided
```

NOTE: n is number in \*each\* group

# Planning for adequate power

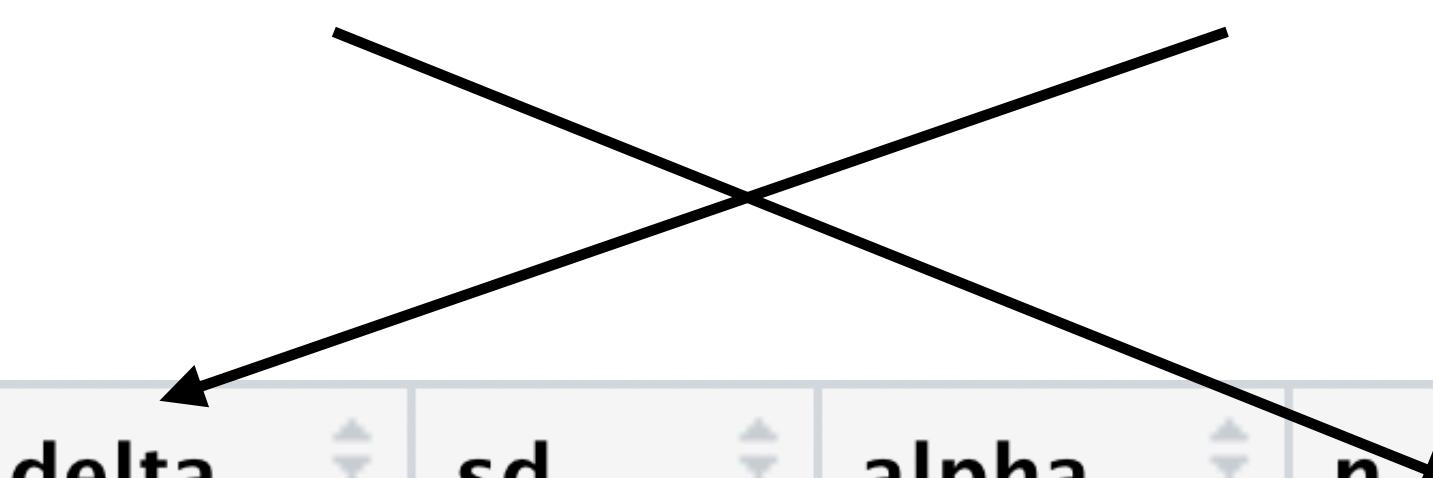
`power.t.test(n = {X}, delta = 2, sd = 1, sig.level = 0.05)`

delta	sd	alpha	n	power
2	1	0.05	3	0.4626081
2	1	0.05	5	0.7905416
2	1	0.05	10	0.9881790
2	1	0.05	15	0.9995525
2	1	0.05	20	0.9999866



# Planning for adequate power

```
power.t.test(n = {X}, delta = {X}, sd = 1, sig.level = 0.05)
```



delta	sd	alpha	n	power
1.50	1	0.05	3	0.29293454
2.00	1	0.05	3	0.46260813
0.25	1	0.05	5	0.05377560
0.50	1	0.05	5	0.10383995
1.00	1	0.05	5	0.28592760
1.50	1	0.05	5	0.54936418

# Planning for adequate power

```
power.t.test(n = {X}, delta = {X}, sd = 1, sig.level = 0.05)
```

