Discrete Homework #7

Who ever you are...

Fall 2015

1. Please determine the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = n$, where $x_i \in \mathbb{N}$, 2 divides x_1, $0 \leq 2$, 3 divides x_3, and $x_4 \in 0, 1$.

$$\sum_{n\geq 0}(x_1)_n y^n = g_1(y) = \frac{1}{1-y^2}$$

$$\sum_{n\geq 0}(x_2)_n y^n = g_2(y) = (y^0 + y^1 + y^2) = \frac{1-y^3}{1-y}$$

$$\sum_{n\geq 0}(x_3)_n y^n = g_3(y) = \frac{1}{1-y^3}$$

$$\sum_{n\geq 0}(x_4)_n y^n = g_2(y) = (y^0 + y^1 + y^2) = \frac{1-y^2}{1-y}$$

$$\begin{aligned}
\sum_{n\geq 0} s_n y^n = g(x) &= g_1(x)\cdot g_2(x)\cdot g_3(x)\cdot g_4(x) \\
&= \frac{1}{1-y^2}\cdot\frac{1-y^3}{1-y}\cdot\frac{1}{1-y^3}\cdot\frac{1-y^2}{1-y} \\
&= \frac{1}{(1-x)^2} \\
&= \frac{1}{x}\cdot\frac{x}{(1-x)^2} \\
&= \frac{1}{x}\sum_{n\geq 0} nx^n \\
&= \sum_{n\geq 0} nx^{n-1} \\
&= \sum_{n\geq 0}(n+1)x^n \\
s_n &= n+1
\end{aligned}$$

2. Use generating functions to find a closed formula for the sequence $(a_n)_{n\geq 0}$ that is defined by the recurrence relation $a_n = 2a_{n-1} + n$, for $n > 0$. and $a_0 = 1$.

$$\begin{aligned}
g(x) &= \sum_{n\geq 0} a_n x_n \\
a_n &= 2a_{n-1} + n + [n = 0] \\
\sum_{n\geq 0} a_n x_n &= 2\sum_{n\geq 0} a_{n-1}x^n + \sum_{n\geq 0} nx^n + \sum_{n\geq 0}[n = 0]x^n \\
&= 2x\sum_{n\geq 0} a_{n-1}x^{n-1} + \sum_{n\geq 0} nx^n + \sum_{n\geq 0}[n = 0]x^n \\
g(x) &= 2xg(x) + \frac{1}{(1-x)^2} + x^0 \\
g(x) - 2xg(x) &= +\frac{x}{(1-x)^2} + 1 \\
g(x)(1-2x) &= +\frac{x}{(1-x)^2} + \frac{(1-x)^2}{(1-x)^2} \\
g(x) &= \frac{x + (1-x)^2}{(1-2x)(1-x)^2} \\
&= \frac{3}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} \\
&= 3\sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} x^n - \sum_{n\geq 0}(n+1)x^n \\
a_n &= 3\cdot 2^n - 1 - (n+1) \\
&= 3\cdot 2^n - n - 2
\end{aligned}$$

3. Suppose you have 2 bananas, an even number of apples, and 2 pears. How many ways can you make a collection of fruit with n pieces of fruit?

$$\begin{aligned}
f(x) &= (1 + x + x^2)(1 + x + x^2)(\frac{1}{1-x^2}) \\
&= \frac{9}{2(1-x)} + \frac{1}{2(1+x)} - x^2 - 2x - 4 \\
&= \frac{9}{2(1-x)} + \frac{1}{2(1-(-1)x)} - x^2 - 2x - 4 \\
\sum_{n\geq 0} a_n x^n &= \frac{9}{2}\sum_{n\geq 0} x^n + \frac{1}{2}\sum_{n\geq 0}(-1)^n x^n - \sum_{n\geq 0}[n = 2]x^n - 2\sum_{n\geq 0}[n = 1]x^n - 4\sum_{n\geq 0}[n = 0]x^n \\
a_n &= \frac{9}{2} + \frac{1}{2}(-1)^n - [n = 2] - 2[n = 1] - 4[n = 0], \text{ for } n \geq 0 \\
a_n &= \frac{9}{2} + \frac{1}{2}(-1)^n, \text{ for } n \geq 3
\end{aligned}$$

4. Use an *exponential generating function* to find the closed formula for the sequence in HW#6, problem 2: $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $L_0 = 2$ and $L_1 = 1$.

ToDo:

5. Find a closed formula (giving *exact* values) for the sequence $g_n = g_{n-1} + g_{n-2} + n$ for $n \geq 2$ with $g_0 = 1$ and $g_1 = 2$.

ToDo: