CS146-Problem Set: Bayesian Inference

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1 Bayesian Inference on Call Center Data

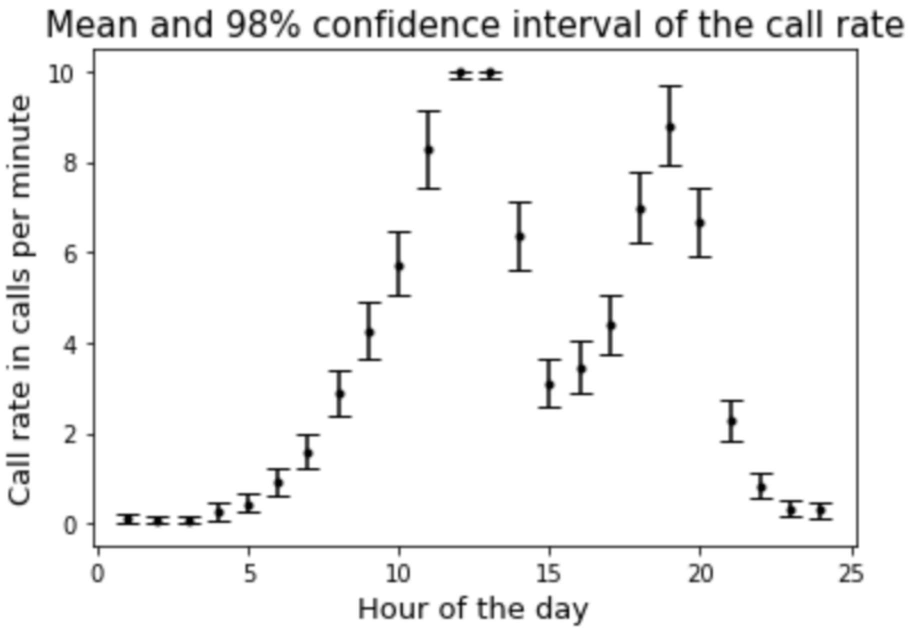


Figure 1: A plot showing the 98% confidence interval and means for the number of calls per minute for each hour.

Looking at the plot in figure 1, We can clearly see that there are some areas of the day where the call rate is fairly stable. Between 9pm and around 5pm my statistical analysis can say with a small level of uncertainty that the call rate will be fairly low, below 2 calls per minute. The same is true about the certainty during the lunch hours from 11 to 1pm, where we consistently get a high volume of calls almost 10 per minute. Between these hours, in the morning and afternoon, I predict more variation in the number of calls coming in. There is a peak in the early evening from around 5pm to 7 pm, but the exact magnitude of this peak will likely vary.[1]

2 Stretch Goal-Explaining the Compute_Posterior Function

```
def compute_posterior(parameter_values, prior, likelihood, data):
    log_prior = np.log(prior(parameter_values))
    log_likelihood = np.array([
        np.sum(np.log(likelihood(param, data)))
        for param in parameter_values])
    unnormalized_log_posterior = log_prior + log_likelihood
    unnormalized_log_posterior -= max(unnormalized_log_posterior)
    unnormalized_posterior = np.exp(unnormalized_log_posterior)
    area = sp.integrate.trapz(unnormalized_posterior, parameter_values)
    posterior = unnormalized_posterior / area
```

[1] **#audience:** My explanatory text keeps in mind the intended audience of a layperson while also showing my understanding of confidence intervals to the professor.

```
11      return posterior
```

1. The purpose of the function is to multiply the prior and likelihood passed as input arguments and to return the posterior as output. Explain how the function achieves this purpose using logarithms.

   Logarithms allow us to turn multiplication problems into addition problems. When two powers are in the same base, multiplying them is like adding their exponents (e.g. $2^4 * 2^5 = 2^9$). This also means that we can take the logarithm in the same base of two very large (or small numbers) and simply add them to multiply the numbers, letting us avoid the numerical errors that come with extremely large or small numbers.

2. What is the purpose of np.sum() in line 4?

   The np.sum term adds the log of each of the individual likelihood function for each datum, effectively multiplying them as outlined in 1.

3. Explain why the maximum of the unnormalized log posterior is subtracted in line 7.

   Subtraction in this case is division as we are still dealing with the logarithms, so everything is divided by the largest values, scaling it so the max is now 1.

4. Why do we still have to divided by the area in line 10 even after having subtracted the maximum of the unnormalized log posterior in line 7?

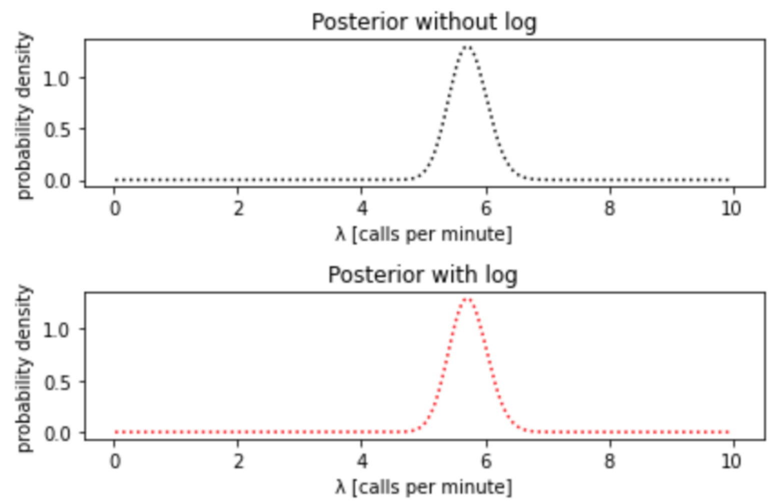
   Because we want a pdf, we need to make sure that the area under the curve is one, not just re-scale the values so the maximum is one.

5. Create an example where not taking logarithms would cause a problem. Create a prior, likelihood, and data set that fails to produce the correct posterior when we don't take logs. Show all your code and visualize your results on one or more plots.

   First, I define an equivalent function to compute_posterior that does not use the logs and make sure it returns the same output for the call center data set.

```
def compute_non_log_posterior(parameter_values, prior, likelihood, data):
    prior_f = prior(parameter_values)
    likelihood_f = np.prod(np.array([np.array(likelihood(param, data))
        for param in parameter_values]),1)
    unnormalized_posterior = prior_f*likelihood_f
    unnormalized_posterior =
        unnormalized_posterior/max(unnormalized_posterior)
    area = sp.integrate.trapz(unnormalized_posterior, parameter_values)
    posterior = unnormalized_posterior/area
    return posterior

# use all the original data and distributions and compare
# the two compute_posterior functions
posterior = compute_posterior(lambdas, prior, likelihood, waiting_times_hour)
posterior_2 = compute_non_log_posterior(lambdas, prior, likelihood,
    waiting_times_hour)

# plot the two posterior distributions.
# They are identical
```

```
fig, (ax1, ax2) = plt.subplots(2)
ax1.plot(lambdas,posterior_2, color='black', linestyle=':')
ax1.set_title('Posterior without log')
ax2.plot(lambdas, posterior, color='red', linestyle=':')
ax2.set_title('Posterior with log')
ax1.set(xlabel='λ [calls per minute]',ylabel= 'probability density')
ax2.set(xlabel='λ [calls per minute]',ylabel= 'probability density')
fig.tight_layout()
plt.show()
```

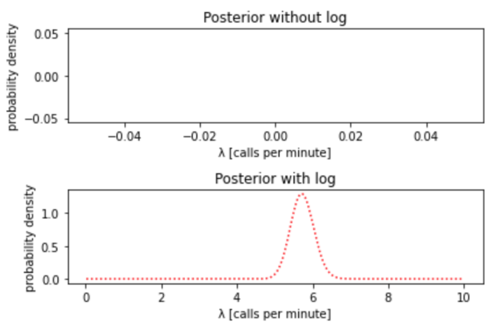


Next, I simply change one of the functions to be larger than python can handle. In this case, I multiply the likelihood function by a large integer since it does not have to be a valid probability distribution. Now, python throws an error and is no longer capable of producing the output and simply returns an error message. Even when using this same ridiculous likelihood function with the log function, it will return the correct result (because of re-scaling, the result is even equivalent to the previously computed function

```
# define a new likelihood function that multiplies the original by a large inte
def broken_likelihood(lambda_, datum):
    return 9999999999999*sts.expon.pdf(datum, scale=1/lambda_)

# compute the posterior with the two methods
# and the broken likelihood function
log_posterior = compute_posterior(lambdas, prior, broken_likelihood,
    waiting_times_hour)
wrong_posterior = compute_non_log_posterior(lambdas, prior,
    broken_likelihood, waiting_times_hour)

# plot the two results (one of them was not able to be computed)
fig, (ax1, ax2) = plt.subplots(2)
ax1.plot(lambdas,wrong_posterior, color='black', linestyle=':')
ax1.set_title('Posterior without log')
```

```
ax2.plot(lambdas, log_posterior, color='red', linestyle=':')
ax2.set_title('Posterior with log')
ax1.set(xlabel='λ [calls per minute]',ylabel= 'probability density')
ax2.set(xlabel='λ [calls per minute]',ylabel= 'probability density')
fig.tight_layout()
plt.show()
```

```
/usr/local/lib/python3.7/site-packages/numpy/core/fromnumeric.py:87: RuntimeWarning: overflow encountered in reduce
  return ufunc.reduce(obj, axis, dtype, out, **passkwargs)
/usr/local/lib/python3.7/site-packages/ipykernel_launcher.py:6: RuntimeWarning: invalid value encountered in true_div
ide
```



3 Code Appendix

```
import numpy as np
import scipy as sp
import scipy.stats as sts
import matplotlib.pyplot as plt
%matplotlib inline

# Load the data set containing durations between calls arriving at the call
# center during 1 day. All values are in minutes.
waiting_times_day = np.loadtxt('call_center.csv')

# Display some basic information about the data set.
print('Size of data set:', len(waiting_times_day))
print('First 3 values in data set:', waiting_times_day[:3])
print('Sum of data set:', sum(waiting_times_day))
```

Ouput:

```
Size of data set: 5856
First 3 values in data set: [30.   3.4  3.2]
Sum of data set: 1441.6838153800093
```

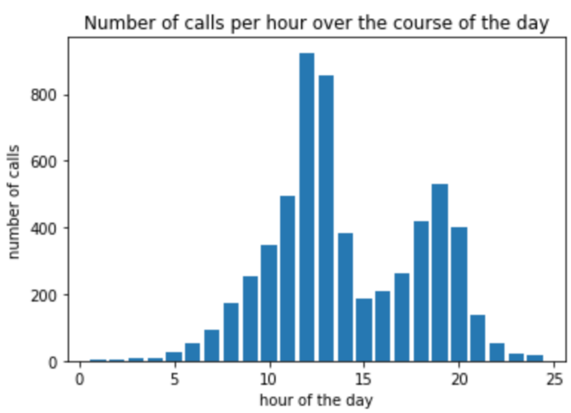
```
# Make 24 empty lists, one per hour.
waiting_times_per_hour = [[] for _ in range(24)]

# Split the data into 24 separate series, one for each hour of the day.
current_time = 0
for t in waiting_times_day:
    current_hour = int(current_time // 60)
    current_time += t
    waiting_times_per_hour[current_hour].append(t)

for hour, calls_in_hour in enumerate(waiting_times_per_hour):
    print(f'{hour:02}:00-{hour + 1:02}:00 - {len(calls_in_hour)} calls')
```

Ouput:

```
00:00-01:00 - 5 calls
01:00-02:00 - 4 calls
02:00-03:00 - 6 calls
03:00-04:00 - 8 calls
04:00-05:00 - 26 calls
05:00-06:00 - 53 calls
06:00-07:00 - 93 calls
07:00-08:00 - 173 calls
08:00-09:00 - 254 calls
09:00-10:00 - 345 calls
10:00-11:00 - 496 calls
11:00-12:00 - 924 calls
12:00-13:00 - 858 calls
13:00-14:00 - 382 calls
14:00-15:00 - 185 calls
15:00-16:00 - 207 calls
16:00-17:00 - 263 calls
17:00-18:00 - 419 calls
18:00-19:00 - 531 calls
19:00-20:00 - 400 calls
20:00-21:00 - 137 calls
21:00-22:00 - 51 calls
22:00-23:00 - 20 calls
23:00-24:00 - 16 calls
```

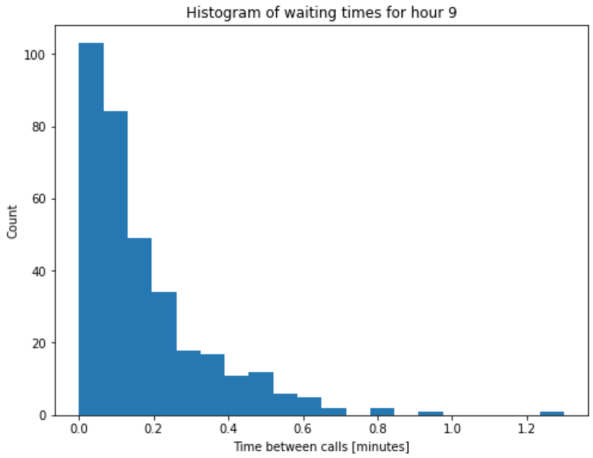
```python
# Plot the number of calls per hour as a bar plot
calls_per_hour = [len(hour) for hour in waiting_times_per_hour]
plt.bar(range(1,25), calls_per_hour)
plt.xlabel('hour of the day')
plt.ylabel('number of calls')
plt.title('Number of calls per hour over the course of the day')
plt.show()
```

Ouput:



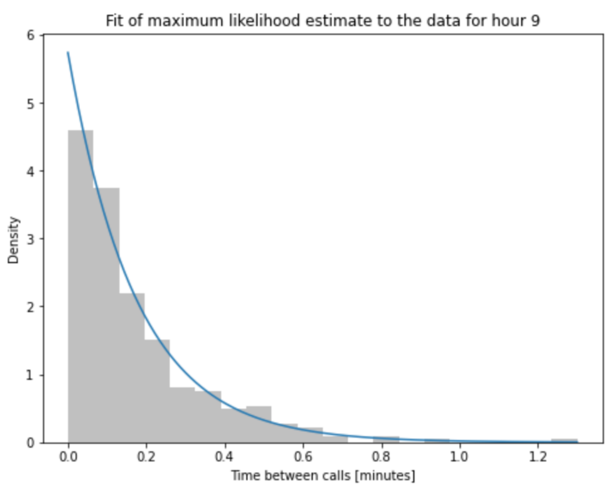

```python
# Plot histogram of waiting times for one hour
hour_index = 9
waiting_times_hour = waiting_times_per_hour[hour_index]

plt.figure(figsize=(8, 6))
plt.hist(waiting_times_hour, bins=20)
plt.xlabel('Time between calls [minutes]')
plt.ylabel('Count')
plt.title(f'Histogram of waiting times for hour {hour_index}')
plt.show()
```

Ouput:




```
# Exponential distribution with maximum likelihood fit to the data
lambda_ = 1 / np.mean(waiting_times_hour)
distribution = sts.expon(scale=1/lambda_)

plt.figure(figsize=(8, 6))
plt.hist(waiting_times_hour, bins=20, density=True, color='#c0c0c0')
x = np.linspace(0, max(waiting_times_hour), 200)
y = distribution.pdf(x)
plt.plot(x, y)
plt.xlabel('Time between calls [minutes]')
plt.ylabel('Density')
plt.title(f'Fit of maximum likelihood estimate to the data for hour {hour_index}')
plt.show()
```

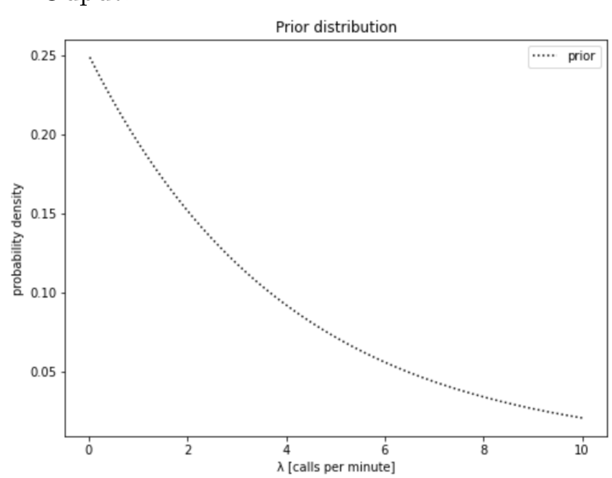
Ouput:




```
# DEFINE THE PRIOR DISTRIBUTION

# This function takes 1 input, namely the parameter value (λ) at which to
# compute the prior probability density. You need to evaluate the distribution
```

```
# Gamma(λ | α, β).

def prior(lambd):
    # Gamma distribution
    distribution = sts.gamma(a=1, scale=1/0.25)
    return distribution.pdf(lambd)

# PLOT THE PRIOR

lambdas = np.linspace(0, 10, 500)[1:]

plt.figure(figsize=(8, 6))
plt.plot(lambdas, prior(lambdas), color='black', linestyle=':', label='prior')
plt.xlabel('λ [calls per minute]')
plt.ylabel('probability density')
plt.title(f'Prior distribution')
plt.legend()
plt.show()
```

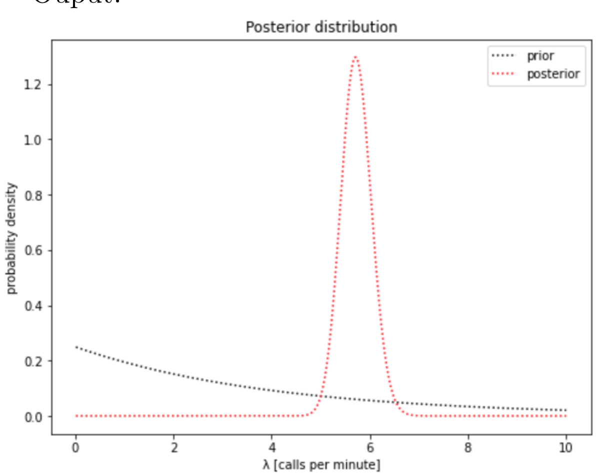
Ouput:




```
# DEFINE THE LIKELIHOOD FUNCTION
#
# This function takes 2 inputs, namely the parameter (λ) value at which to
# compute the likelihood and a value from the data set. You need to evaluate
# the exponential distribution of the datum given the parameter value.

def likelihood(lambda_, datum):
    return sts.expon.pdf(datum, scale=1/lambda_)
    # return sts.expon(scale=1/lambda_).pdf(datum)

# THE POSTERIOR DISTRIBUTION
# (COMPUTED AUTOMATICALLY USING THE PRIOR AND LIKELIHOOD)
#
# The function below is provided to help make computing the posterior easier. It
```

```
# follows the same pattern as in the previous class, where we multiple the prior
# and the likelihood evaluated at various values of the parameter (λ).
#
# You will see the function uses the logarithms of various distributions. This
# is for numerical reasons. When you multiply lots of likelihood values, the
# result can become very large or very small, causing numerical overflow or
# underflow in Python. Taking logarithms avoids this problem.

def compute_posterior(parameter_values, prior, likelihood, data):
    log_prior = np.log(prior(parameter_values))
    log_likelihood = np.array([
        np.sum(np.log(likelihood(param, data)))
        for param in parameter_values])
    unnormalized_log_posterior = log_prior + log_likelihood
    unnormalized_log_posterior -= max(unnormalized_log_posterior)
    unnormalized_posterior = np.exp(unnormalized_log_posterior)
    area = sp.integrate.trapz(unnormalized_posterior, parameter_values)
    posterior = unnormalized_posterior / area
    return posterior

posterior = compute_posterior(lambdas, prior, likelihood, waiting_times_hour)

# YOU NEED TO PLOT THE PRIOR AND POSTERIOR ON THE SAME AXES.
plt.figure(figsize=(8, 6))
plt.plot(lambdas, prior(lambdas), color='black', linestyle=':', label='prior')
plt.plot(lambdas, posterior, color='red', linestyle=':', label='posterior')
plt.xlabel('λ [calls per minute]')
plt.ylabel('probability density')
plt.title(f'Posterior distribution')
plt.legend()
plt.show()
```

Ouput:




```
def compute_percentile(parameter_values, distribution_values, percentile):
    '''
```

```python
    Compute the parameter value at a particular percentile of the given
    probability distribution values. This function uses the cumulative trapezoid
    integrator in SciPy.

    Inputs:

        parameter_values (array of float) This is the list of parameter values
          at which the probability distribution has been evaluated.

        distribution_values (array of float) This is the list of values of the
          probability density function evaluated at the parameter values above.

        percentile (float) This is the value between 0 and 1 of the percentile
          to compute.

    Returns: (float) The parameter value at the given percentile.
    '''
    cumulative_distribution = sp.integrate.cumtrapz(
        distribution_values, parameter_values)
    percentile_index = np.searchsorted(cumulative_distribution, percentile)
    return lambdas[percentile_index]

# YOU HAVE TO USE THE FUNCTION ABOVE TO COMPUTE THE CONFIDENCE INTERVAL
print("95% confidence interval from",compute_percentile(lambdas, posterior, 0.025),
      "to",compute_percentile(lambdas, posterior, 0.975))
```

Ouput:

```
95% confidence interval from 5.130260521042084 to 6.332665330661323
```

```python
# Define empty lists to fill with the 24 values for lower and upper bounds
# of the 98% confidence interval as well as the means
confidence_intervals = []
means = []
for hour in waiting_times_per_hour:
    # compute the new posterior with the same prior and likelihood
    # depending on the new data
    posterior = compute_posterior(lambdas, prior, likelihood, hour)
    # lower bound is the 0.01 percentile, upper bound the 0.99 percentile
    confidence_intervals.append(
        [compute_percentile(lambdas, posterior, 0.01),
        compute_percentile(lambdas, posterior, 0.99)])
    # the mean is the dot product of lambdas and the posterior
    # divided by the sum of the posterior
    means.append(np.dot(posterior,lambdas)/np.sum(posterior))

# get just the lower and upper bounds of the 98% confidence intervals
lower = np.array(confidence_intervals)[:,0]
upper = np.array(confidence_intervals)[:,1]
```

```
# Plot the means and their confidence intervals over time
plt.errorbar(range(1,25),means,yerr=np.array([means-lower,upper-means]),fmt='.k',ca
plt.title("Mean and 98% confidence interval of the call rate",size=15)
plt.ylabel("Call rate in calls per minute",size=13)
plt.xlabel("Hour of the day",size=13)
plt.show()
```

Ouput: