Assignment 3: Predators, Prey, and Foodwebs

1. Community Stability: connections count

With $S = 250, \sigma = 0.3$, how does stability depend on C?

```
library(ggplot2)
```

```
## Warning in register(): Can't find generic `scale_type` in package ggplot2 to
## register S3 method.
```

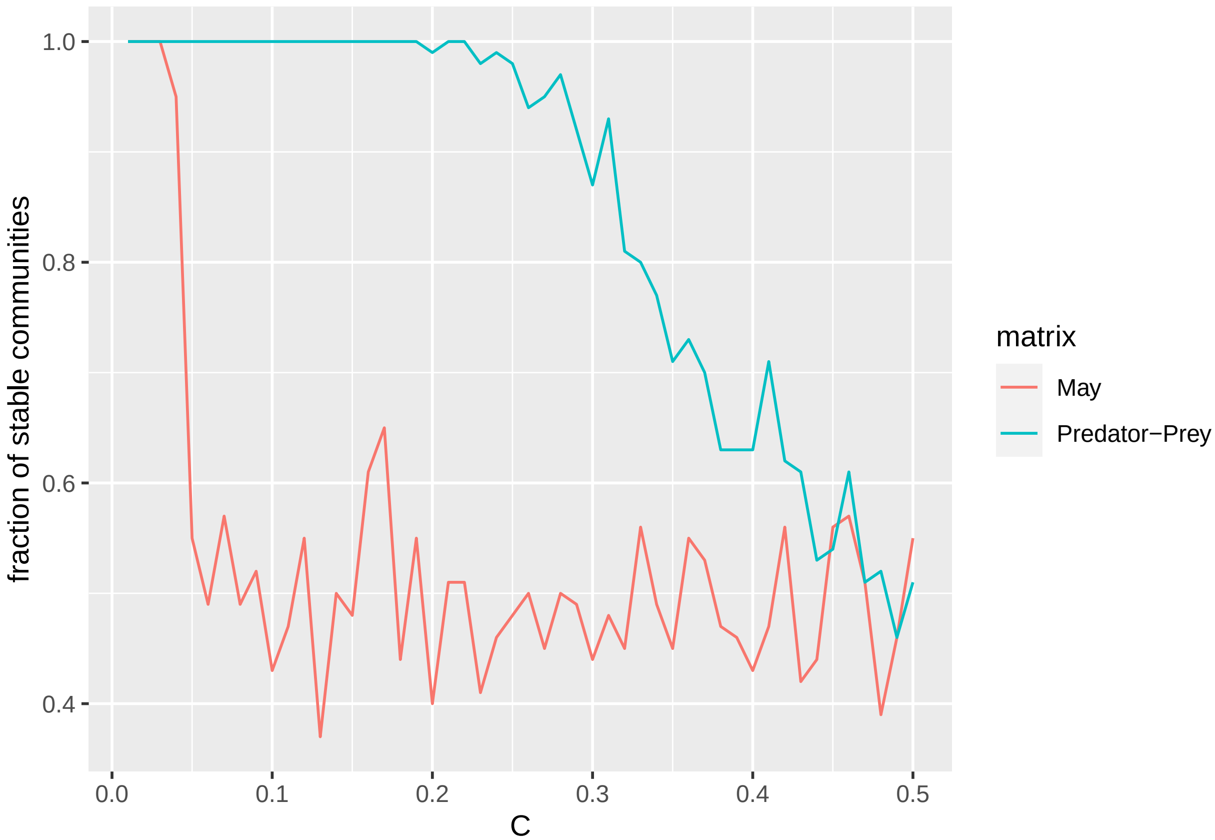
```
library(psych)
```

```
##
## Attaching package: 'psych'
```

```
## The following objects are masked from 'package:ggplot2':
##
##     %+%, alpha
```

```
S <- 250
sigma <- 0.3

MayMatrix<-function(S,C,sigma){
  ## This matrix determines the connections
  A<-matrix(runif(S*S),S,S)
  ## Contains the values for the connections
  B<-matrix(rnorm(S*S,0.0,sigma),S,S)
  A<-(A<= C)*1 # A matrix contains 1 when A[i,j] <= C
  M<-A*B
  diag(M)<- -1
  return(M)
}

PPMatrix<-function(S,C,sigma){
  ## Determine the signs for the connections
  MyS<-sign(rnorm(S*(S-1)/2))
  A<-matrix(0,S,S)
  A[upper.tri(A,diag=F)]<-MyS
  D<-matrix(runif(S*S),S,S)
  D<-(D <= C)*1
  A<-D*A
  A<- A-t(A)
  ## Contains the  values for the connections
  B<-matrix(abs(rnorm(S*S,0.0,sigma)),S,S)
```

```
  M<-A*B
  diag(M)<- -1
  return(M)
}

stableQ<- function(m){
  if(tr(m)<0 & det(m)>0){
    return(1)
  }else{
    return(0)
  }
}


C <-seq(0.01,0.5,by=0.01)

mays <- c()
for (cs in C){
  mays <- append(mays,sum(replicate(100, stableQ(MayMatrix(S,cs,sigma)), simplify = TRUE )))
}

pred <- c()
for (cs in C){
  pred <- append(pred,sum(replicate(100, stableQ(PPMatrix(S,cs,sigma)), simplify = TRUE )))
}

mays <- mays/100
pred <- pred/100


df=data.frame(C = C,
              values=c(mays, pred),
              matrix=c(rep("May",50),rep("Predator-Prey",50))
)


ggplot(df,aes(C,values,col=matrix))+geom_line() +
  ylab("fraction of stable communities")
```



We can see in the plot that both methods of generating matrices result in high stability for low values of C. The threshold May proves in his paper $\sqrt{SC} > \frac{1}{\sigma}$, is $C > 2/45 (\approx 0.04444)$ when $S = 250$ and $\sigma = 0.3$. This is consistent with what we simulated. We can see the predator-prey matrix undergoes a similar transition at a much higher value of C, closer to 0.25.

2. Examples of particular dynamics

2.1 damped oscillations of a population to a stable equilibrium point

An example system that will show damped oscillations to a stable equilibrium is the predator-prey model with logistic growth described by the set of equations.

$$\begin{cases} \frac{dX}{dt} = (b-d)X(1-X/K) - \alpha XY \\ \frac{dY}{dt} = \alpha\epsilon XY - mY \end{cases}$$

In general, this system will be stable when the the Jacobian matrix takes the form

$$A = \begin{bmatrix} -a & -b \\ c & 0 \end{bmatrix}$$

First we find a feasible equilibirum for the system:

$$\frac{dY}{dt} = 0 = \alpha\epsilon XY - mY \to X^* = \frac{m}{\epsilon\alpha}$$

and

$$\frac{dX}{dt} = 0 = (b-d)X(1-X/K) - \alpha XY \to Y^* = \frac{b-d}{\alpha}(1-\frac{X}{K}) \to Y^* = \frac{(K\epsilon\alpha - m)(b-d)}{\epsilon\alpha^2 K}$$

Then we find the Jacobian matrix at this feasible equilibrium

$$J_{xx} = \frac{\partial}{\partial X}(b-d)X(1-\frac{X}{K}-\alpha XY) = \frac{-2(b-d)X^*}{K} + (b-d) - \alpha Y^* \rightarrow a_{xx} = J_{xx}|_{X^*,Y^*} = \frac{-(b-d)m}{K\epsilon\alpha}$$

J_{xx} is negative whenever $b > d$ and all constants are positive.

$$J_{xy} = -\alpha X^* \rightarrow a_{xy} = J_{xy}|_{x^*,Y^*} = -\frac{m}{\epsilon}$$

Again, J_{xy} is negative whenever $m > 0$and $\epsilon > 0$.

$$J_{yx} = \alpha\epsilon Y^* \rightarrow a_{yx} = J_{yx}|_{X^*,Y^*} = \frac{(b-d)(K\epsilon\alpha - m)}{\alpha K}$$

This is positive because Y^* has to be positive. Last we we need to show that $a_{yy} = 0$

$$J_{yy} = \alpha\epsilon X^* - m \rightarrow a_{yy} = J_{yy}|_{X^*,Y^*} = \alpha\epsilon\frac{m}{\alpha\epsilon} - m = 0$$

Thus

$$J = \begin{bmatrix} \frac{-(b-d)m}{K\epsilon\alpha} & -\frac{m}{\epsilon} \\ -\frac{(b-d)(K\epsilon\alpha-m)}{\alpha K} & 0 \end{bmatrix}$$

at the the feasible equilibrium whenever all constants, and populations are positive and $b > d$.

For example, Figure 1 shows the isoclines of this system when the birth rate of the prey $b = 2$, the death rate of the prey $d = 0.5$, the prey carrying capacity $K = 15$, , the encounter rate $\alpha = 0.5$, the conversion efficiency of the predator $\epsilon = 0.9$, and the death rate for the predator $m = 0.6$. We can see in this plot that whenever the system starts with $X > 0$ and $Y > 0$, we expect a spiral to the equilibrium. For example, let's start with $X = 10$ and $Y = 10$. The resulting oscillation can be seen in figure 2.

2.2 stable cycles

The most famous example is the simple Lotka-Volterra model we started with described by the following system of equations:

$$\begin{cases} \frac{dN}{dt} = N(a - bP) \\ \frac{dP}{dt} = P(cN - d) \end{cases}$$

Here a is the growth rate of the prey N independent of the predator P (this means exponential growth without the predator). b is the predation rate of the prey by the predator, c is the conversion rate of eaten prey into predator (this is $b * \epsilon = c$, where ϵ is the conversion efficiency of killed prey into predator). Lastly, d is the death rate of the predator without any prey. We can non-dimensionalize using

$$\begin{cases} u(\tau) = \frac{cN(t)}{d} \\ v(\tau) = \frac{bP(t)}{a} \\ \tau = at \\ \alpha = \frac{d}{a} \end{cases}$$

and get

$$\begin{cases} \frac{du}{d\tau} = u(1 - v) \\ \frac{dv}{d\tau} = \alpha v(u - 1) \end{cases}$$

Again, we start by finding the feasible equilibrium where $\frac{du}{d\tau} = 0$ and $\frac{dv}{d\tau} = 0$:

$$\begin{cases} 0 = u(1 - v) \\ 0 = \alpha v(u - 1) \end{cases} \rightarrow \begin{cases} u^* = 1 \\ v^* = 1 \end{cases}$$

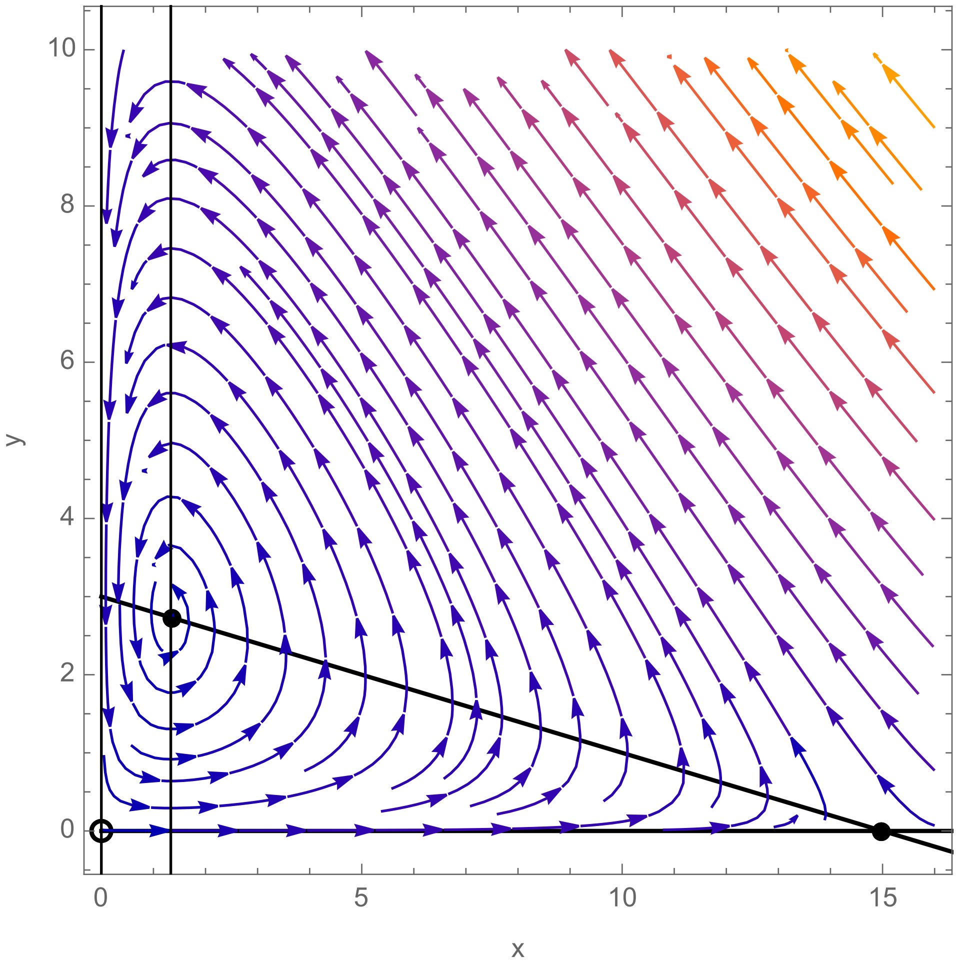


Figure 1: The dynamics of the predator prey model with logistic growth

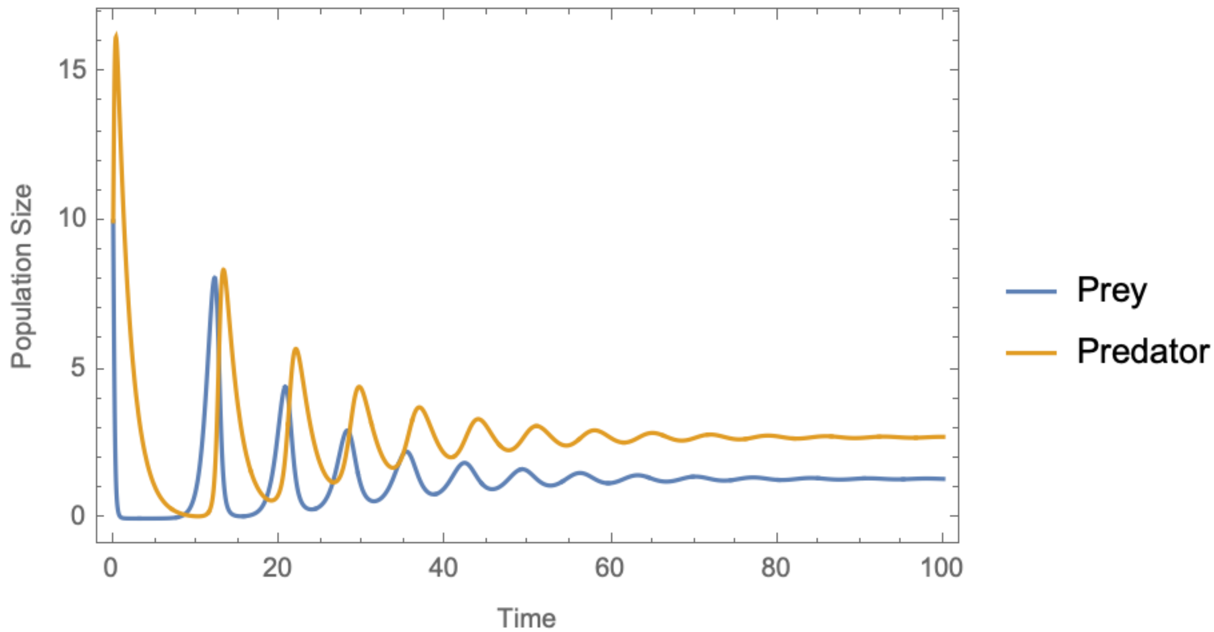


Figure 2: The dynamics of the predator prey model with logistic growth

The Jacobian matrix at this feasible equilibrium is

$$J = \begin{bmatrix} 1-v & -u \\ \alpha v & \alpha u - \alpha \end{bmatrix} A = J|_{u=u^*, v=v^*} = \begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix}$$

Here, $Tr(A) = 0, Det(A) > 0$ so we know there is a fixed point center and thus oscillations are stable with the magnitude determined by the constants.

Figure 3 shows the isoclines and figure 4 the dynamics of this system when $a = 2, b = 1, c = 0.9, d = 0.5, N[0] = 5, P[0] = 2$

2.3 unstable dynamics away from a nontrivial equilibrium

An example of a model where we get an unstable nontrivial equilibrium is two consumers competing for the same resource where the intraspecific competition is larger than the interspecific competition. This can be described by the system:

$$\begin{cases} \frac{dN_1}{dt} = N_1(r_1 - \alpha_{11}N_1 - \alpha_{12}N_2) \\ \frac{dN_2}{dt} = N_2(r_2 - \alpha_{22}N_2 - \alpha_{21}N_1) \end{cases}$$

N_1 and N_2 are the competitors, r_1 and r_2 their growth rates, α_{11} and α_{22} the strength of their intraspecific competition and α_{12} and α_{21} the strength of interspecific competition of species 2 on species 1 and species 1 on species 2 respectively.

We can solve for the equilibria and get four equilibria of which only the fourth is non-trivial.

$$\begin{cases} N_1^* = 0 \\ N_2^* = 0 \end{cases} \begin{cases} N_1^* = 0 \\ N_2^* = -\frac{r_2}{a_{22}} \end{cases} \begin{cases} N_1^* = \frac{r_1}{a_{11}} \\ N_2^* = 0 \end{cases} \begin{cases} N_1^* = -\frac{a_{22}r_1 - a_{12}r_2}{a_{12}a21 - a_{11}a_{22}} \\ N_2^* = -\frac{-a_{21}r_1 + a_{11}r_2}{a_{12}a_{21} - a_{11}a_{22}} \end{cases}$$

We can then look at the Jacobian to determine the conditions for the feasible equilibrium to be unstable. To simplify, we will take the Jacobian of the per capita growth rates of both species.

$$J = \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix}$$

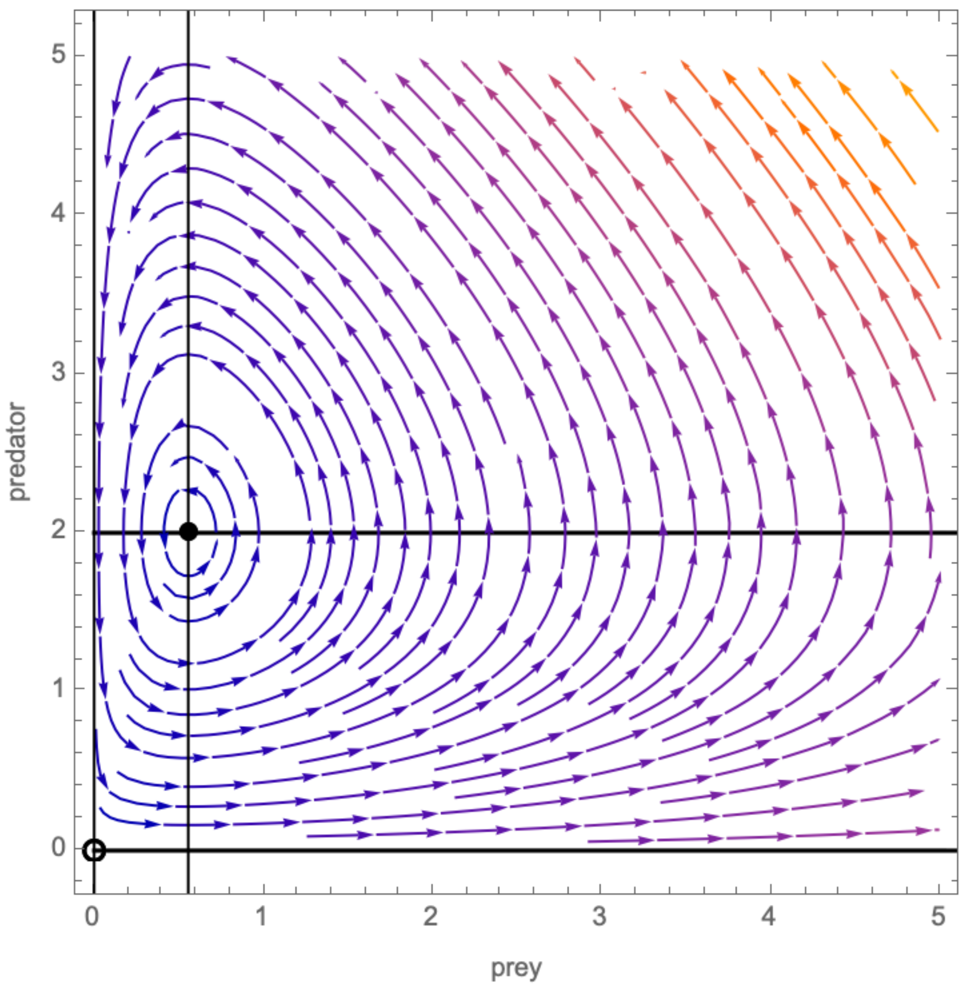


Figure 3: The dynamics and isoclines of the basic, Lotka-Volterra predator-prey model

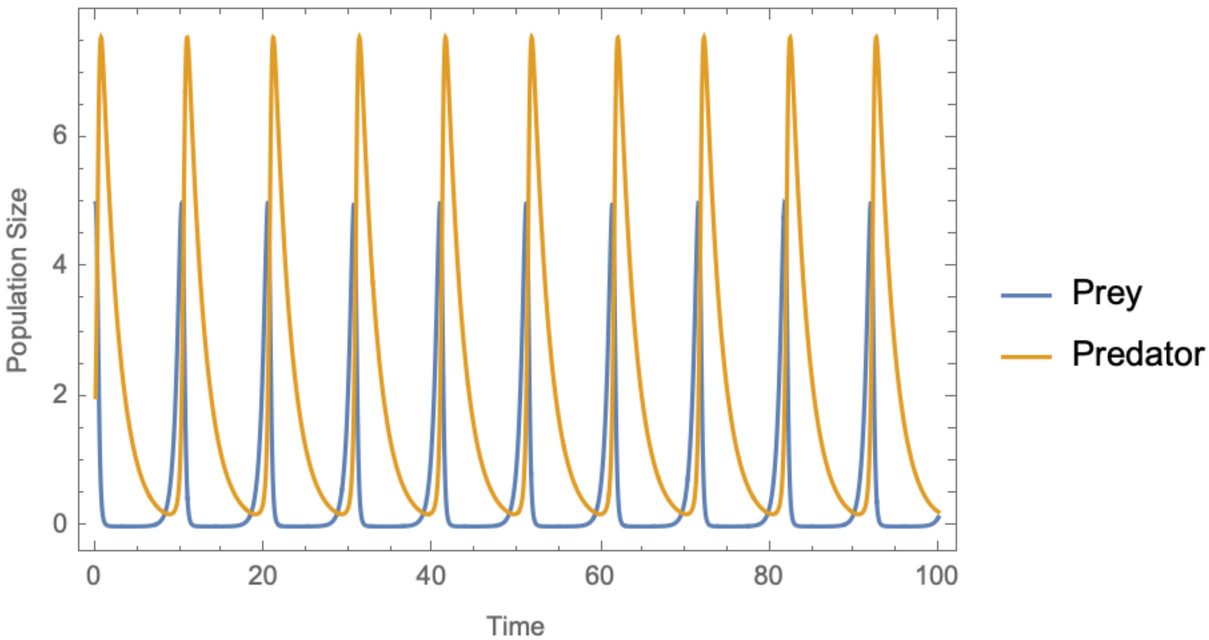


Figure 4: The dynamics of the basic, Lotka-Volterra predator-prey model

For this system to be stable, the real parts of the eigenvalues needs to be negative. The eigenvalues are

$$\lambda_1 = \frac{-\alpha_{11} - \alpha_{22} + \sqrt{\alpha_{11}^2 - 2\alpha_{11}\alpha_{22} + \alpha_{22}^2 + 4\alpha_{12}\alpha_{21}}}{2} \lambda_2 = \frac{-\alpha_{11} - \alpha_{22} - \sqrt{\alpha_{11}^2 - 2\alpha_{11}\alpha_{22} + \alpha_{22}^2 + 4\alpha_{12}\alpha_{21}}}{2}$$

Thus, the system is unstable whenever $\alpha_{11}\alpha_{22} < \alpha_{12}\alpha_{21}$.

For example, let $r_1 = r_2 = 2$, $\alpha_{11} = 0.5$, $alpha_{12} = 0.7$, $\alpha_{22} = 0.6$, and $\alpha_{21} = 0.8$. The resulting stream plot shows an unstable equilibrium and no matter how close to this equilibirum the population starts, it will converge to one of the single-species equilibria.

2.4 chaotic dynamics

The probably most famous example of a simple equation leading to chaotic behavior is Robert May's logistic map model of population dynamics in a single species x.

$$x_{n+1} = rx_n(1 - x_n)$$

Once again, r represents a growth rate for the population but this time x is not the absolute population size but rather a fraction representing the proportion of the maximum (carrying capacity) population size possible.

Again we find candidates for stability, where $x_{n+1} = x_n = x^*$. This is the case when $x_1^* = 0$ or $x_2^* = 1 - \frac{1}{r}$ (only feasible equilibirum).

We take the derivative to test for stability at the feasible equilibirum.

$$f'(x) = \frac{d}{dx} rx(1 - x) = r(1 - x) f'(x)|_{x_2^*} = r(1 - 2(1 - \frac{1}{r})) = 2 - r$$

If $|2 - r| < 1$, the point is stable, so for $1 < r < 3$. Next, we look at the period-2 stable points, where $x_{n+2} = f(x_{n+1}) = f(f(x_n)) = f^{(2)}(x_n)$.

$$x = f(f(x)) = rf(x)(1 - f(x)) = r^2 x(1 - x)(1 - rx(1 - x)) = -r^2 x(x - 1)(rx^2 - rx + 1)$$

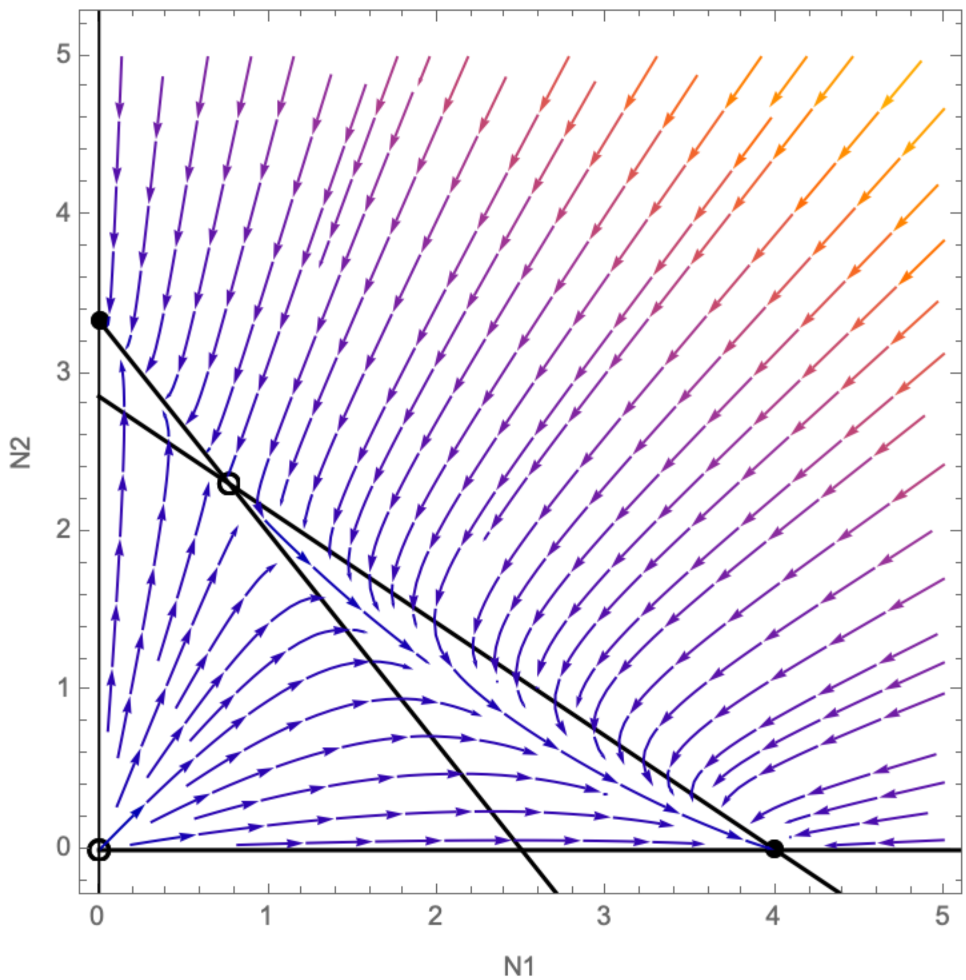


Figure 5: The dynamics of the competition model

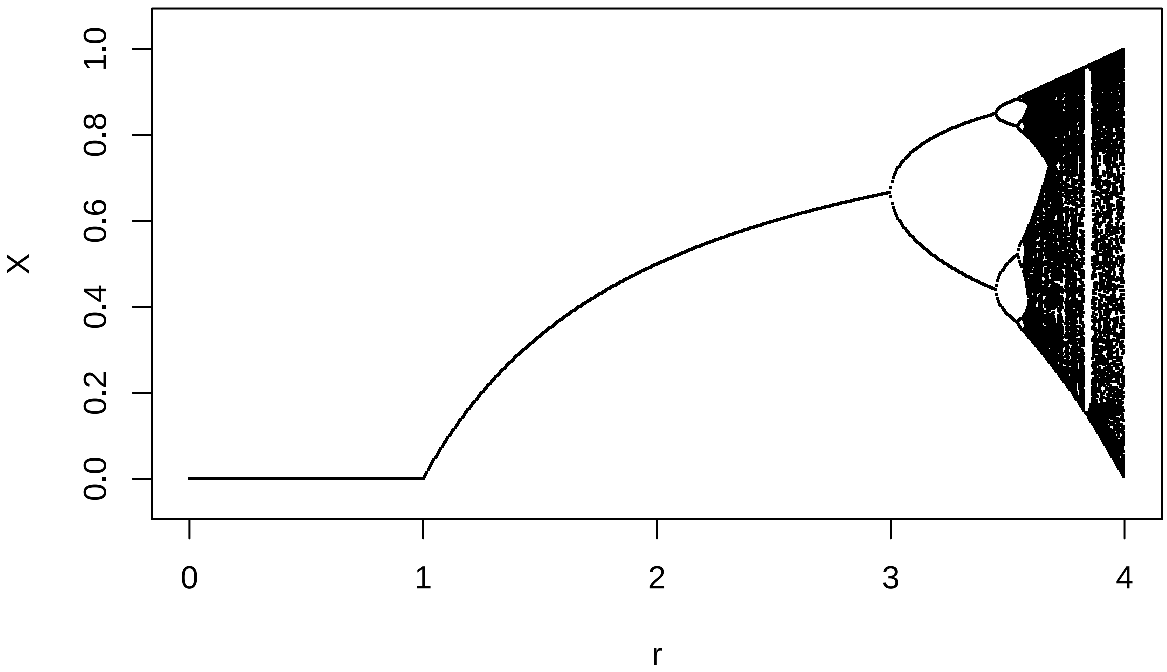
There are three solutions for which $x > 0$ $x = 1 - \frac{1}{r}$ This is the same we found for the stable equilibirum case and we know it is unstable for $r > 3$. The other two solutions are more difficult to predict the behavior for

$$x = \frac{r + 1 \pm \sqrt{(}r^2 - 2r - 3)}{2r} = y_{1,2}$$

We need to look at $|\frac{\partial}{\partial x} f^{(2)}(x)|y_1 < 1$.This eventually will give us another value range similar to $1 < r < 3$ for which there is a stable 2-period solution. Then we repeat the same for period-4, period-8, etc. All of them will occupy smaller and smaller ranges of r approaching 3.57, after which the dynamics are chaotic.

To summarize for low values of r, the dynamics are stable, for intermediate values we get oscillations of different periods, but for values of r close to 4, the dynamics become chaotic. We created the bifurcation diagram for these dynamics in Assignment 1 as seen below.

```r
## This function returns the values of the min and max
peaks <- function(x) {
  if (min(x)==max(x)) return(min(x)) ## Does not oscillate
  l <- length(x)
  xm1 <- c(x[-1], x[l])
  xp1 <- c(x[1], x[-l])
  z<-x[x > xm1 & x > xp1 | x < xm1 & x < xp1]
  if (length(z)==0) return(min(x)) ## It has not converged yet
  return (z)
}

## This function creates a simulation of the logistic map
LogisticMap<-function(N0,r,TimeSteps){
  Results<-rep(0,TimeSteps)
  Results[1]<-N0
  for (j in 2:TimeSteps){
    Results[j]<-r*Results[j-1]*(1-Results[j-1])
  }
  return(Results)
}

## Plot the Diagram for Logistic Map
plot(0,0, xlim=c(0,4), ylim=c(-0.05,1.05),type="n", xlab="r", ylab="X")
for (r in seq(0.001,4,0.005)) { # These are the initial and final values for r
  out <- LogisticMap(0.5,r,2500) # Initial conditions
  l <- length(out) %/% 10 # use only the last 250 steps
  out <- out[(9*l):(10*l)]
  p <- peaks(out)
  l <- length(out)
  points(rep(r, length(p)), p, pch=".")
}
```



X
r
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4