Problem Set 6:
Stochastic Theory of Spike Production

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In the last sheet, we looked at how we could use stochastic processes, namely shot noise, to model the random walk of a neuron's membrane potential in response to the integration of input spikes. This week, we will look at how we can use stochastic theory to model and understand a neuron's production of spikes.

Problem 1: Rolling a dice — 5 points

To motivate what comes next, first observe what happens to the distribution of dice roll averages as you roll more and more dice. In other words, observe how the distribution of $\bar{X}_i = (X_1 + \ldots + X_n)/n$ behaves as $n \to \infty$, for X_i the random variable sampling from the discrete uniform distribution over the space $\{1, 2, 3, 4, 5, 6\}$ (a dice). Plot the distribution (histogram of means obtained) of $\bar{X}_i$ for 2 until 10 dice. What do you notice about how the distribution (and the moments of the distribution) changes as you increase the number of dice? Can you explain how you would calculate this evolution of distributions? You don't actually need to do the calculations.

Please don't waste an evening doing this problem with real dice and excel, use Python. You can however construct the distributions by hand by looking at the number of combinations that give the different averages, without doing experiments.

Problem 2: The central limit theorem — 8 points

We can make sense of the last question by appealing to the central limit theorem:

Let $\{X_1, ..., X_n, ...\}$ be independent and identically distributed random variables, each with mean μ and variance σ^2. Due to the law of large numbers, the mean of the sample $\bar{X}_n \equiv (X_1 + ... + X_n)/n$ converges to μ in the $n \to \infty$ limit.
The theorem describes the fluctuations of the sample mean $\bar{X}_n$ about μ during this convergence. It says that in the $n \to \infty$ limit, the distribution of the sample average tends to the normal distribution with mean μ and variance σ^2/n.

Note, this theorem is so useful and significant because it holds irrespective of the distribution of the random variables X_i, as long as it has finite variance. We would like to prove this theorem, given its ubiquity in probability theory. Try to do so by appealing to the characteristic function of the random variables

$$Z_n \equiv \frac{X_1 + ... + X_n - n\mu}{\sqrt{n\sigma^2}} \equiv \sum_{i=1}^{n} \frac{1}{\sqrt{n}} Y_i,$$

where $Y_i = (X_i - \mu)/\sigma$ are newly defined random variables. For reference, the characteristic function of a Gaussian is $\hat{f}(t) = e^{-\mu t} e^{-(\sigma t)^2/2}$ for mean μ and variance σ^2.

Hint: Use the result that having finite k^{th} order moment means that the characteristic function has a k^{th} order derivative, allowing you to expand the expression for the characteristic function using Taylor's theorem. Also, you might find this property of variance useful for relating the new random variable Z_n to the mean $\bar{X}_n$: $\mathbb{V}ar[aX + bY] = a^2\mathbb{V}ar[X] + b^2\mathbb{V}ar[Y] - 2ab\,Cov[X, Y]$, *for constants a, b and random variables X, Y.*

Problem 3: The diffusion approximation 3 points

Back to neuroscience. The diffusion approximation of Stein's model[1] is used to increase the model's tractability. In this approximation, the incoming excitatory and inhibitory jumps are replaced by a Gaussian white noise diffusion term. Summarise why and when (what types of input) this approximation is well founded by making reference the central limit theorem.

[1] A leaky integrate and fire model with excitatory and inhibitory input producing "jumps" in the membrane potential, see lecture 7.