

Model of Neuron and Network: Problem Set 3

Problem 1: The Full Solution

Problem 1.1: The exact solution

$$(u(t)e^{t/\tau})' = (u'(t) + u(t)/\tau)e^{t/\tau}$$

and

$$\frac{u_{stim}(t)}{C} = u'(t) + u(t)/\tau$$

$$\Rightarrow (u(t)e^{t/\tau})' = \frac{u_{stim}(t)}{C} \cdot e^{t/\tau}$$

$$u(t)e^{t/\tau} \Big|_{t_0}^T = \int_{t_0}^T \frac{u_{stim}(t)}{C} \cdot e^{t/\tau} dt$$

$$u(T)e^{T/\tau} - u(t_0)e^{t_0/\tau} = \int_{t_0}^T \frac{u_{stim}(t)}{C} \cdot e^{t/\tau} dt$$

$$u(T)e^{T/\tau} = u(t_0)e^{t_0/\tau} + \int_{t_0}^T \frac{u_{stim}(t)}{C} \cdot e^{t/\tau} dt$$

Divide by $e^{T/\tau}$:

$$u(T) = u(t_0)e^{(t_0-T)/\tau} + \int_{t_0}^T \frac{u_{stim}(t)}{C} e^{(t-T)/\tau} dt$$

Problem 1.2: Response to sinusoidal input

$$u(t) = u(t_0) e^{(t_0 - T)/\tau} + \int_{t_0}^T \frac{u_{\text{sim}}(t)}{c} e^{(t - T)/\tau} dt$$

\Rightarrow for $u_{\text{sim}}(t) = I_0 \sin(2\pi \omega t)$ from $t = 0$ to $t = T$

$$u(t) = u(t_0) e^{(t_0 - T)/\tau} + \int_0^T \frac{I_0 \sin(2\pi \omega t)}{c} e^{(t - T)/\tau} dt$$

Use:

$$\int e^{(t-a)/b} \sin(ct) dt = b e^{(t-a)/b} \frac{-(bc) \cos(ct) + \sin(ct)}{1 + b^2 c^2}$$

$$\begin{aligned} a &= T \\ b &= \tau \\ c &= 2\pi \omega \end{aligned}$$

\Rightarrow

$$u(T) = u(t_0) \cdot e^{(t_0 - T)/\tau}$$

$$+ \frac{I_0}{c} \cdot \tau \cdot e^{(t - T)/\tau} \cdot \left[\frac{-(\tau 2\pi \omega) \cdot \cos(2\pi \omega t) + \sin(2\pi \omega t)}{1 + \tau^2 (2\pi \omega)^2} \right] \bigg|_{t=0}^T$$

Use:

$$A \cdot \sin(x) + B \cdot \cos(x) = \sqrt{A^2 + B^2} \cdot \sin(x + \tan^{-1}(B/A))$$

$$\begin{aligned} A &= 1 \\ B &= -\tau 2\pi \omega \\ x &= 2\pi \omega t \end{aligned}$$

\Rightarrow

$$u(T) = u(t_0) \cdot e^{(t_0 - T)/\tau} + \frac{I_0}{c} \cdot \tau \cdot e^{(t - T)/\tau}$$

$$\cdot \left[\frac{\sqrt{1^2 + (-\tau 2\pi \omega)^2} \cdot \sin(2\pi \omega t + \tan^{-1}(-\tau 2\pi \omega))}{1 + \tau^2 (2\pi \omega)^2} \right] \bigg|_{t=0}^T$$

\Rightarrow calculate...

Problem 2: The solution at Late Times

Problem 2.1: The simplified solution

$$u(T) = u(t_0) e^{(t_0 - T)/\tau} + \int_{t_0}^T \frac{I_{stim}(t)}{c} e^{(t - T)/\tau} dt$$

with $t_0 = -\infty$ and $T \rightarrow \infty$

$$u(T) = u(-\infty) \underbrace{e^{(-\infty - T)/\tau}}_0$$

$$+ \int_{-\infty}^T \frac{I_{stim}(t)}{c} e^{(t - T)/\tau} dt$$

$$= \int_{-\infty}^T \frac{1}{c} e^{-(T - t)/\tau} \cdot I_{stim}(t) dt$$

t will never be $> T \Rightarrow T - t \geq 0$

$$\Rightarrow u(T) = \int_{-\infty}^T G(T - t) I_{stim}(t) dt$$

Problem 2.2: The significance of $G(t)$

$$u(T) = \int_{-\infty}^T G(T-t) \cdot u_{\text{in}}(t) dt$$

$$u_{\text{in}} = \delta(t)$$

$$\Rightarrow u(T) = \int_{-\infty}^T G(T-t) \cdot \delta(t) dt$$

For $t \leq T$

$$u(T) = \int_{-\infty}^T \frac{1}{c} e^{-(T-t)/\tau} \cdot \delta(t) dt$$

Integral with δ -function gives value of function
at $t=0$

$$\Rightarrow u(T) = \frac{1}{c} e^{-T/\tau}$$

Significance: $G(t)$ translates the input into output
at later times

Problem 2.3: Linearity and time-invariance

Linearity:

Show that

$$\begin{aligned} & \int_{-\infty}^T G(T-t) \cdot (\alpha f(t) + \beta g(t)) dt = \\ & \alpha \int_{-\infty}^T G(T-t) \cdot f(t) dt + \beta \int_{-\infty}^T G(T-t) \cdot g(t) dt \\ & = \int_{-\infty}^T G(T-t) \cdot \alpha f(t) + G(T-t) \cdot \beta g(t) dt \\ & = \int_{-\infty}^T G(T-t) \cdot \alpha f(t) + \int_{-\infty}^T G(T-t) \cdot \beta g(t) dt \\ & = \alpha \int_{-\infty}^T G(T-t) \cdot f(t) dt + \beta \int_{-\infty}^T G(T-t) \cdot g(t) dt \end{aligned}$$

Time-invariance:

$$g(t) \equiv f(t + \tau_0)$$

$$u_g(T) = u_g(T + \tau_0)$$

$$1. u_p(T + \tau_0) = \int_{-\infty}^{T + \tau_0} G(T + \tau_0 - t) \cdot f(t) dt$$

$$2. u_g(T) = \int_{-\infty}^T G(T-t) \cdot g(t) dt = \int_{-\infty}^T G(T + \tau_0 - t - \tau_0) \cdot f(t + \tau_0) dt$$

$$\text{Substitution: } u = t + \tau_0 \quad \frac{du}{dt} = 1 \quad \frac{du}{1} = dt \Rightarrow du = dt$$

$$\Rightarrow u_g(T) = \int_{-\infty}^{T + \tau_0} G(T + \tau_0 - u) \cdot f(u) du$$

Problem 2.4: Late-time solution to the sinusoidal input

$$G(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\tau} e^{-t/\tau} & \text{if } t \geq 0 \end{cases}$$

Late-time solution $\rightarrow t \geq 0$

$$G(t) = \frac{1}{\tau} e^{-t/\tau} = \frac{\tau}{\tau} \cdot \frac{1}{\tau} \cdot e^{-t/\tau}$$

$$\hat{G}(\omega) = \frac{\tau}{\tau} \cdot \frac{1}{(1 + 2\pi i \omega \tau)}$$

$$= \frac{\tau}{\tau} \cdot \frac{1}{(1 + 2\pi i \omega \tau)}$$

$$= \frac{1}{C(1/\tau + 2\pi i \omega)}$$