Problem Set 8:
Correlated Spike Trains

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In all preceding problem sets, when we have considered multiple random variables, we have understood them to be independent. In this sheet, we will look at how one can define ideas such as statistical dependence and correlation by focusing on correlated spike trains. Indeed, a neuron in a network will not receive completely uncorrelated spike trains as input, since these trains often come from the same recurrent network of neurons.

Let us consider the single interaction process (SIP) model (see Lecture 9). In this model, n individual spike trains are modelled as $x_i(t) = w_u(t) + w_i(t)$ for $i = 1, \ldots, n$, where $w_u(t) = \sum_k \delta(t - t_k)$ is a single realisation of a stationary Poisson point process with rate α common to all spike trains $x_i(t)$, and $w_i(t)$ is a new independent realisation of a Poisson point process of rate β unique to that neuron. Remember that the sum of Poisson processes is also a Poisson process of rate $r = \alpha + \beta$.

Obviously, these spike trains $x_i(t)$ and measures of the trains, such as spike counts, are not independent. Let us quantify this dependence with the use of some statistical measures.

Problem 1: Covariance and correlation — 4 points

We will first look at statistical measures of random variables, rather than stochastic processes which are functions of time. As an example, let N_i be the event counts of the respective spike train $x_i(t)$ during an observation of duration of length T. To look at the dependence of these random variables, let us first define **covariance** as

$$\mathbb{C}\text{ov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \tag{1}$$

For independent random variables X and Y, $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$, such that $\mathbb{C}\text{ov}[X, Y] = 0$. Note, however, that one cannot determine that two random variables are independent if their covariance vanishes.

Since the covariance is bounded by $\mathbb{C}\text{ov}[X, Y]^2 \leq \mathbb{V}\text{ar}[X]\,\mathbb{V}\text{ar}[Y]$, the **correlation** coefficient $\mathbb{C}\text{or}[X, Y]$ between two variables provides a normalised notion of covariance. It is defined as

$$\rho_{XY} \equiv \mathbb{C}\text{or}[X, Y] = \frac{\mathbb{C}\text{ov}[X, Y]}{\sqrt{\mathbb{V}\text{ar}[X]\,\mathbb{V}\text{ar}[Y]}}, \tag{2}$$

which is always falls in the interval $[-1, 1]$.

Problem 1.1: The covariance of counts N_i — 4 points

Calculate the covariance between the counts N_i and N_j of two processes $x_i(t)$ and $x_j(t)$ $(i \neq j)$ from the SIP model over some time interval of length T. Use this to calculate the correlation coefficient between the counts. What are the bounds of the correlation?

Problem 2: Cross- and autocorrelation 8 points

Let us now turn to the random processes themselves. The **crosscorrelation** of two *real* stochastic processes $R(s)$ and $S(t)$ is defined as

$$\rho_{RS}(s,t) = \mathbb{E}[R(s)S(t)], \tag{3}$$

providing a measure of similarity of two processes as a function of their evolutions in time t and s.

If both processes are *jointly stationary*, such that their joint distribution remains unchanged under time shifts (that is, it doesn't suffice that the individual processes are stationary, but their dependence must also be unchanging in time), the crosscorrelation is then just a function of the displacement $\Delta = s - t$ of one process relative to the other:

$$\rho_{RS}(\Delta) = \mathbb{E}[R(t+\Delta)S(t)]. \tag{4}$$

Comparing a process to itself, one obtains the **autocorrelation**

$$\rho_{SS}(s,t) = \mathbb{E}[S(s)S(t)], \qquad \text{or} \qquad \rho_{SS}(\Delta) = \mathbb{E}[S(t+\Delta)S(t)], \tag{5}$$

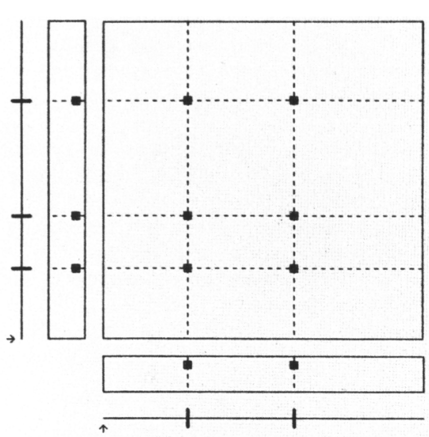
where again the second equality holds if the autocorrelation is simply a function of the displacement Δ rather than absolute time t, s.

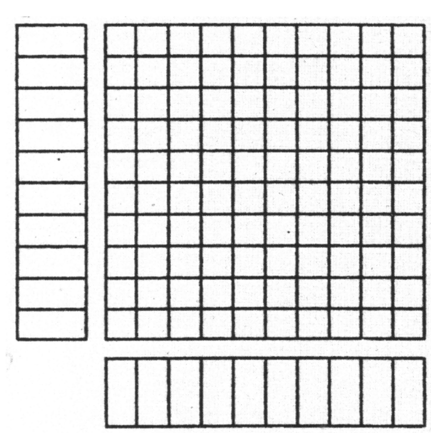
Problem 2.1: Product spike trains 6 points

We can now obtain the auto- and crosscorrelation functions for the Poisson point process as shown in slides 15-17 of Lecture 9. To do so, we define the 2-dimensional (2D) signal given by the product of spikes trains $R(s)$ and $S(t)$ as

$$R(s)S(t) = \Big[\sum_j \delta(s-s_j)\Big]\Big[\sum_k \delta(t-t_k)\Big] = \sum_{jk} \delta(s-s_j)\delta(t-t_k), \tag{6}$$

where the signal only has spikes are points $(s,t) = (s_j, t_k)$ (see Figure 1 (left)). Make a note of how many spikes are in the 2D signal in the area of h^2, given the number of spikes in both 1D signals in their intervals of length of h. We can thus make reference to the counts of the 2D signal $N_{RS}\big((s,s+h] \times (t,t+h]\big)$, which gives the counts of $R(s)S(t)$ in the "area" (as opposed to interval for the 1D signal) defined by the limits $s : (s, s+h]$ and $t : (t, t+h]$. This is thus the 2D equivalent of $N(t)$.





Aertsen, Gerstein, Habib, Palm, 1989

Figure 1: *(Left) The 2D signal arising from the product of two 1D signals. (Right) Bins of width h, thus area h^2, covering the 2D signal space.*

The **crosscorrelation** of two point processes can then be defined as a function of $N_{RS}(s \times t)$ as:

$$\rho_{RS}(s,t) = \mathbb{E}[R(s)S(t)] = \lim_{h\to 0} \frac{1}{h^2}\, \mathbb{E}\left[N_{RS}\big((s, s+h] \times (t, t+h]\big)\right], \tag{7}$$

where h is the width of bins that one uses to count the spikes in the 2D signal (see Figure 1 (right)). Taking the limit ($\lim_{h\to 0}$) of the expected count $\mathbb{E}[N_{RS}]$ divided by the bin area h^2 amounts to finding the expected density of spikes in the 2D signal at time pair (s,t), rather than the count itself $N_{RS}(s,t)$.

Use this definition to obtain the crosscorrelation of two independent spike trains $\rho_{w_i w_j}(s,t)$ and generalise the reasoning to obtain the autocorrelation $\rho_{w_i w_i}(t)$.

Problem 2.2: Putting everything together — 2 points

Use your results for the previous problems to calculate the autocorrelation $\rho_{x_i x_i}(\Delta)$ of one of the inputs $x_i(t)$ and the crosscorrelation $\rho_{x_i x_j}(\Delta)$ between two inputs $x_i(t)$ and $x_j(t)$.

Problem 3: Auto- and crosscovariance — 4 points

One can make an equivalent notion to covariance (equation 1) for processes or signals (functions of time) by calculating the crosscorrelation of the two centred processes $\bar{R}(s) = R(s) - \mathbb{E}[R(s)]$ and $\bar{S}(t) = S(t) - \mathbb{E}[S(t)]$, where $\mathbb{E}[R(s)]$ is the mean of the process R *at time* s, for example. In this way, the **crosscovariance** is defined as

$$\gamma_{RS}(s,t) = \mathbb{E}\left[\big(R(s) - \mathbb{E}[R(s)]\big)\big(S(t) - \mathbb{E}[S(t)]\big)\right] = \mathbb{E}[R(s)S(t)] - \mathbb{E}[R(s)]\,\mathbb{E}[S(t)]. \tag{8}$$

By comparing to equation 1, one sees that the crosscovariance gives the covariance of the two processes at each pair of time points (s,t).

One can define the **autocovariance** by comparing the process to itself

$$\gamma_{SS}(s,t) = \mathbb{E}[S(s)S(t)] - \mathbb{E}[S(s)]\,\mathbb{E}[S(t)]. \tag{9}$$

As for the auto- and crosscorrelation, both of these may merely dependent on the displacement Δ. Note that for $s = t$ (or $\Delta = 0$), the autocovariance gives the variance $\gamma_{SS}(t,t) = \mathbb{V}\mathrm{ar}[S(t)]$ (or $\gamma_{SS}(0) = \mathbb{V}\mathrm{ar}[S]$).

Problem 3.1: First for $w_i(t)$ — 2 points

Use the results from the last problem to calculate the autocovariance $\gamma_{w_i w_i}(\Delta)$ of a Poisson point process $w_i(t)$ and the crosscovariance $\gamma_{w_i w_j}(\Delta)$ between two such independent processes $w_i(t)$ and $w_j(t)$.

Hint: Remember $\mathbb{E}[w_i(t)] = \beta$ for a stationary Poisson proccess with rate β.

Problem 3.2: Now for the inputs $x_i(t)$ **2 points**

Use the results from the last question to calculate the autocovariance $\gamma_{x_i x_i}(\Delta)$ of one of the inputs $x_i(t)$ and the crosscovariance $\gamma_{x_i x_j}(\Delta)$ between two inputs $x_i(t)$ and $x_j(t)$.