Problem Set 5:
Stochastic Theory of Spike Integration

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Using models such as the leaky integrate–and–fire neuron in Problem Sheet 4, we saw that postsynaptic potentials produced by afferent spike trains (or any arbitrary input for that matter) can be computed by summing the individual responses of the membrane potential to individual spike inputs. This simply amounts to a convolution of the spike train $P(t)$ with the the impulse response function $h(t)$ of the membrane (otherwise known as response kernel). In this sheet, we will look more closely at the statistics of shot noise, the random walk (of the membrane potential for example) obtained as the output of a system with response kernel $h(t)$ and a Poisson point process as input.

Problem 1: The characteristic functional 10 points

Let us first derive the characteristic functional[1] $\hat{f}_S(x) = \mathbb{E}[e^{ixs}]$ of the shot noise, from which we will be able to obtain the moments of the process $S(t)$ amplitude distribution later. In this sheet, we will take the intensity of the afferent Poisson point process λ to be a constant. To clarify, the density function of the shot noise process is the amplitude distribution of $S(t)$, which is the density function of the random values $\{s\}$ that this process $S(t)$ takes over time. In other words:

$$f_S(s)\,\mathrm{d}s = \mathbb{P}[s < S(t) \leq s + \mathrm{d}s].$$

The characteristic functional is then simply the Fourier transform of this.

Problem 1.1: Step one: the conditional characteristic functional 6 points

A key trick here is first noting that the density function and thus the characteristic functional of the shot noise during some interval $(0, T)$ are *conditional* on how many Poisson points $N(T) = n_T$ arrive as input to the system during this interval. In other words, to derive the unconditional characteristic functional $\hat{f}(x)$ we must first find the conditional one $\hat{f}_{S|N}(x|n_T)$. Derive this function using the conditional expectation of $e^{ixS(t)}$.

Hints: Note that the signal $S(t)$ conditional on $N(t) = n_T$ is $S(t) = \sum_{k=1}^{n_T} h(t - \tau_i)$. Also, how are the points τ_i distributed on $(0, T)$ and how do they depend on each other? You may also find the law of the unconscious statistician helpful: for a random variable X, the expected value of some function of this random variable $g(X)$ is $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x$.

Solution.

[1]Note the terminology "functional" is used as the random quantity $S(t)$ in question is not a random variable (with possible outcomes being scalars, for example), but rather a stochastic process where possible outcomes are functions in time. A functional is a mapping from the space of functions to some number set.

The conditional characteristic functional is

$$\begin{aligned}\hat{f}_{S|N}(x|n_T) &= \mathbb{E}[e^{i\,x\,S(t)}|N(t)=n_T] \\ &= \mathbb{E}[e^{i\,x\,\sum_{k=1}^{n_T} h(t-\tau_i)}] \\ &= \prod_{k=1}^{n_T} \mathbb{E}[e^{i\,x\,h(t-\tau_i)}]\end{aligned}$$

where on the last line we used the exponentiation identity $e^{x+y} = e^x e^y$. We now need to know what the expectation of $h(t-\tau_i)$ is, which is where the distribution of the points τ_i becomes important. For a Poisson process, we know that these points are all independent of each other and identically uniformly distributed along $(0,T)$: $f_\tau(x) = 1/T$. Using the law of the unconscious statistician:

$$\begin{aligned}\hat{f}_{S|N}(x|n_T) &= \prod_{k=1}^{n_T} \int_{-\infty}^{\infty} g(t) f_\tau(t)\,\mathrm{d}t \\ &= \prod_{k=1}^{n_T} \int_0^T \left(e^{i\,x\,h(t-\tau_i)}\right)\cdot\left(\frac{1}{T}\right)\mathrm{d}t \\ &= \left(\frac{1}{T}\int_0^T e^{i\,x\,h(u)}\mathrm{d}u\right)^{n_T},\end{aligned}$$

where in the last line we used that all points are identically distributed (hence product simply amounting to raising to the power of n_T), and changed integration variable to u for simplicity.

Problem 1.2: Step two: The unconditional characteristic functional 4 points

Now that we have the conditional characteristic functional, we can simply obtain the unconditional one. Do so.

Hints: The conditional and unconditional forms are related as usual. You will probably find the power series expansion of the exponential function useful: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. *Also, you will need the distribution of the variable* n_T.

Solution.

As for all unconditional expressions, we can simply obtain the unconditional characteristic functional as the probability weighted sum of the conditional counterparts. This, along with the knowledge that n_T is a Poisson random variable with parameter λT for the time interval of length T, allows us to find the characteristic functional:

$$\begin{aligned}\hat{f}_S(x) &= \sum_{n_T=0}^{\infty} \hat{f}_{S|N}(x|n_T)\,\mathbb{P}[N(t)=n_T] \\ &= \sum_{n_T=0}^{\infty} \left(\frac{1}{T}\int_0^T e^{i\,x\,h(u)}\mathrm{d}u\right)^{n_T}\cdot\left(\frac{(\lambda T)^{n_T}\,e^{-\lambda T}}{n_T!}\right)\end{aligned}$$

$$
\begin{aligned}
&= e^{-\lambda T} \sum_{n_T=0}^{\infty} \frac{1}{n_T!} \left(\lambda \int_0^T e^{i\,x\,h(u)} \mathrm{d}u \right)^{n_T} \\
&= e^{-\lambda T} \left(e^{\lambda \int_0^T e^{i\,x\,h(u)} \mathrm{d}u} \right) \\
&= e^{\lambda \int_0^T e^{i\,x\,h(u)} - 1\,\mathrm{d}u},
\end{aligned}
$$

where the power series expansion of the exponential function was used in the second last line, and the $\int_0^T 1\,\mathrm{d}u = T$ was substituted into the last line.

Problem 2: Calculating the moments 6 points

Problem 2.1: The raw moments 4 points

Equipped with the characteristic functional, use it to calculate the first two (raw) moments of the shot noise process:

$$
\begin{aligned}
\mathbb{E}[S] &= \frac{1}{i} \hat{f}'(0), \quad \text{and} \\
\mathbb{E}[S^2] &= \frac{1}{i^2} \hat{f}''(0).
\end{aligned}
$$

Solution.

To calculate both moments, we need the calculate the first and second derivative of $\hat{f}(x)$. For the first, we define $\hat{f}(x) \equiv w(z(x))$ where $w(v)) = e^{\lambda v}$ and $z(x) = \int_0^T e^{i\,x\,h(u)} - 1\,\mathrm{d}u$, such that we can use the product rule:

$$
\frac{\mathrm{d}w(z(x))}{\mathrm{d}x} = \frac{\mathrm{d}w(z(x))}{\mathrm{d}(z(x))} \cdot \frac{\mathrm{d}z(x)}{\mathrm{d}x}.
$$

Thus:

$$
\begin{aligned}
\frac{\mathrm{d}\hat{f}(x)}{\mathrm{d}x} &\equiv \frac{\mathrm{d}w(z(x))}{\mathrm{d}x} \\
&= \frac{\mathrm{d}w(z(x))}{\mathrm{d}(z(x))} \cdot \frac{\mathrm{d}z(x)}{\mathrm{d}x} \\
&= \left(\lambda e^{\lambda z(x)} \right) \cdot \left(i \int_0^T h(u) e^{ih(u)x} \mathrm{d}u \right).
\end{aligned}
$$

We can now get the second derivative. We need to use the chain rule $\frac{\mathrm{d}(g(x)h(x))}{\mathrm{d}x} = \frac{\mathrm{d}g(x)}{\mathrm{d}x} h(x) + g(x) \frac{\mathrm{d}h(x)}{\mathrm{d}x}$:

$$
\frac{\mathrm{d}^2 \hat{f}(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\hat{f}'(x) \right)
$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda e^{\lambda z(x)}\right) \cdot \left(i \int_0^T h(u) e^{ih(u)x} \mathrm{d}u\right) + \left(\lambda e^{\lambda z(x)}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(i \int_0^T h(u) e^{ih(u)x} \mathrm{d}u\right) \\
&= \left(\lambda^2 e^{\lambda z(x)} \cdot i \int_0^T h(u) e^{ih(u)x} \mathrm{d}u\right) \cdot \left(i \int_0^T h(u) e^{ih(u)x} \mathrm{d}u\right) \\
&\qquad\qquad\qquad\qquad + \left(\lambda e^{\lambda z(x)}\right) \cdot \left(i^2 \int_0^T h^2(u) e^{ih(u)x} \mathrm{d}u\right) \\
&= -\lambda^2 e^{\lambda z(x)} \left(\int_0^T h(u) e^{ih(u)x} \mathrm{d}u\right)^2 - \lambda e^{\lambda z(x)} \left(\int_0^T h^2(u) e^{ih(u)x} \mathrm{d}u\right).
\end{aligned}
$$

Using the first derivative, the first moment is:

$$
\begin{aligned}
\mathbb{E}[S] &= \frac{1}{i} \hat{f}'(0) \\
&= \lambda e^{\lambda z(0)} \int_0^T h(u) e^{ih(u)\cdot 0} \mathrm{d}u \\
&= \lambda \int_0^T h(u) \mathrm{d}u,
\end{aligned}
$$

where we used $z(0) = \int_0^T e^{i\cdot 0 \cdot h(u)} - 1 \,\mathrm{d}u = 0$.

The second moment is:

$$
\begin{aligned}
\mathbb{E}[S^2] &= \frac{1}{i^2} \hat{f}''(0) \\
&= -\hat{f}''(0) \\
&= \lambda^2 e^{\lambda z(0)} \left(\int_0^T h(u) e^{ih(u)\cdot 0} \mathrm{d}u\right)^2 + \lambda e^{\lambda z(0)} \left(\int_0^T h^2(u) e^{ih(u)\cdot 0} \mathrm{d}u\right) \\
&= \lambda^2 \left(\int_0^T h(u) \mathrm{d}u\right)^2 + \lambda \left(\int_0^T h^2(u) \mathrm{d}u\right) \\
&= \mathbb{E}[S]^2 + \lambda \int_0^T h^2(u) \mathrm{d}u.
\end{aligned}
$$

Note, both of these moments are functions of the interval length T, as long as $T \leq t_f$ where t_f is the maximum time at which the response $h(u)$ is not vanishing (i.e. $h(u) = 0 \;\forall\; u : (t_f < u < 0) \cap t_f > 0$ in that the function has compact support and the response starts at $u = 0$). Both are also obviously functions of the point process intensity λ.

Problem 2.2: The mean and variance **2 points**

Using the raw moments you just calculated, calculate the mean and variance of shot noise.

Solution.

As for the mean and the variance, the mean is simply the first (raw) moment $\mathbb{E}[S]$, while the variance is the second *central* moment:

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}[X] &= \mathbb{E}[S^2] - \mathbb{E}[S]^2 \\
&= \left(\mathbb{E}[S]^2 + \lambda \int_0^T h^2(u) \mathrm{d}u \right) - \mathbb{E}[S]^2 \\
&= \lambda \int_0^T h^2(u) \mathrm{d}u.
\end{aligned}
$$