Problem Set 4: Stochastic Point Processes

Problem 1: The Poisson Process

Problem 1.1: Counting vs interval statistics

Distribution of spike counts: Poisson distribution

$$P[N=k] = \frac{\Lambda^k}{k!} e^{-\Lambda} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

1. Probability of no spike in interval of length t is same as obtaining an ISI $> t$

$$P[N(\Lambda)=0] = P[t_1 > t]$$

2. $P[N=0] = \frac{\Lambda^0}{0!} e^{-\lambda t} = e^{-\lambda t}$

3. $P[t_1 > t]$ is the probability of obtaining any ISI larger than t which is the "inverse" of an interval in the range of $(0, t]$

$$P[t_1 > t] = 1 - \int_0^t f(s)\, ds$$

4. $$e^{-\lambda t} = 1 - \int_0^t f(s)\, ds$$

$$\int_0^t f(s)\, ds = 1 - e^{-\lambda t}$$

$$f(t) = \lambda e^{-\lambda t}$$ because:

$$f(t) = \frac{1}{dt} \int_0^t f(s)\, ds$$

Problem 1.2: Doubly stochastic process

$$E[N(0 \to t) \mid \lambda(t)] = \int_0^t \lambda(s)\, ds \equiv \Lambda$$

Expected value:

$$E[X] = E[E[X \mid Y]]$$

$$\Rightarrow E[E[N(0 \to t) \mid \lambda(t)]] = E[N(0 \to t)]$$

$$\Leftrightarrow E[\Lambda] = E[N(0 \to t)]$$

Variance:

$$Var[X] = E[Var[X \mid Y]] + Var[E[X \mid Y]]$$

$$\Rightarrow Var[N(0 \to t)] = E[Var[N(0 \to t) \mid \lambda(t)]] + Var[E[N(0 \to t) \mid \lambda(t)]]$$

$$= E[\Lambda] + Var[\Lambda]$$

$\uparrow$

variance
of Poisson
distribution
is Λ

Problem 2: Renewal Processes

Problem 2.1: Interval density vs. hazard function

1. Find a

$$1 = \int_0^{\infty} \frac{\lambda}{a} e^{-\lambda t} \, dt \qquad \text{should be normalized}$$

$$1 = \left[-\frac{1}{a} e^{-\lambda t} \right]_D^{\infty}$$

$$1 = 0 + \frac{1}{a} e^{-\lambda D}$$

$$a = e^{-\lambda D}$$

$$\Rightarrow f(t) = \begin{cases} 0 & \text{if } t \leq D \\ \frac{1}{e^{-\lambda D}} & \text{if } t > D \end{cases}$$

2. Find survivor function $F(t)$

$$F(t) = 1 - \int_0^t f(s) \, ds$$

If $t \leq D$:

$$F(t) = 1 - 0 = 1$$

If $t > D$:

$$F(t) = 1 - \int_0^t \frac{1}{e^{-\lambda D}} e^{-\lambda s} \, ds$$

$$= 1 - \left[-\frac{\lambda}{e^{-\lambda D}} e^{-\lambda s} \right]_0^t$$

$$= 1 - \left[-\frac{\lambda}{e^{-\lambda D}} e^{-\lambda t} + \frac{\lambda}{e^{-\lambda D}} e^{-\lambda D} \right]$$

$$= 1 + \frac{\lambda}{e^{-\lambda D}} e^{-\lambda t} - \frac{\lambda}{e^{-\lambda D}} e^{-\lambda D}$$

3. Find hazard function $h(t)$

$$h(t) = f(t) \,/\, F(t)$$

If $t \leq D$:

$$h(t) = \frac{0}{1} = 0$$

If $t > D$:

$$h(t) = \frac{\frac{1}{e^{-\lambda D}} e^{-\lambda t}}{F(t)}$$

Problem 2.2: The waiting time paradox

1. Waiting time if bus comes every 15 min:
   on average 7.5 min

2. Waiting time for Poisson process with
   $\lambda = 1/15 \text{ min}^{-1}$
   $\Rightarrow$ find expected value of interval distribution

$$E[f(t)] = \int_0^\infty \lambda e^{-\lambda t} \cdot t \, dt$$

$$= \left[-e^{-\lambda t} \cdot t\right]_0^\infty - \int_0^\infty -e^{-\lambda t} \, dt$$

Integration by parts

$$= 0 - 0 - \left[\frac{1}{\lambda} e^{-\lambda t}\right]_0^\infty$$

$$= -\Big(\underbrace{\frac{1}{\lambda} e^{-\lambda \infty}}_{0} - \frac{1}{\lambda} e^{-\lambda 0}\Big)$$

$$= \frac{1}{\lambda}$$

$$= \lambda^{-1}$$

$\Rightarrow \lambda^{-1} = 15 \text{ min}$

$\Rightarrow$ The mean waiting time will not be reduced, it is even longer!