

Models of Neurons and Networks: Problem Set 3

Problem 1: The Full Solution

Problem 1.1: The exact solution

$$(u(t)e^{t/\tau})' = (u'(t) + u(t)/\tau) e^{t/\tau}$$

and

$$\frac{u_{\text{stim}}(t)}{C} = u'(t) + u(t)/\tau$$

$$\Rightarrow (u(t)e^{t/\tau})' = \frac{u_{\text{stim}}(t)}{C} \cdot e^{t/\tau}$$

$$u(t)e^{t/\tau} \Big|_{t_0}^T = \int_{t_0}^T \frac{u_{\text{stim}}(t)}{C} \cdot e^{t/\tau} dt$$

$$u(T)e^{T/\tau} - u(t_0)e^{t_0/\tau} = \int_{t_0}^T \frac{u_{\text{stim}}(t)}{C} \cdot e^{t/\tau} dt$$

$$u(T)e^{T/\tau} = u(t_0)e^{t_0/\tau} + \int_{t_0}^T \frac{u_{\text{stim}}(t)}{C} \cdot e^{t/\tau} dt$$

Divide by $e^{T/\tau}$:

$$u(T) = u(t_0)e^{(t_0-T)/\tau} + \int_{t_0}^T \frac{u_{\text{stim}}(t)}{C} e^{(t-T)/\tau} dt$$

Problem 1.2: Response to sinusoidal input

$$u(t) = u(t_0) e^{(t_0-t)/\tau} + \int_{t_0}^t \frac{u_{\text{lim}}(e)}{\tau} e^{(t-e)/\tau} dt$$

\Rightarrow for $u_{\text{lim}}(t) = b \sin(2\pi\omega t)$ from $t=0$ to $t=T$

$$u(t) = u(t_0) e^{(t_0-t)/\tau} + \int_0^T \frac{b \sin(2\pi\omega t)}{\tau} e^{(t-T)/\tau} dt$$

Use:

$$\int e^{(t-a)/b} \sin(ct) dt = b e^{(t-a)/b} \frac{-(b/c) \cos(ct) + \sin(ct)}{1 + b^2 c^2}$$

$$a = T$$

$$b = \tau$$

$$c = 2\pi\omega$$

\Rightarrow

$$u(t) = u(t_0) e^{(t_0-t)/\tau} + \frac{t_0}{\tau} \cdot \tau \cdot e^{(t-T)/\tau} \cdot \left[\frac{-(\tau 2\pi\omega) \cdot \cos(2\pi\omega t) + \sin(2\pi\omega t)}{1 + \tau^2 (2\pi\omega)^2} \right] \Big|_{t=0}^T$$

\Rightarrow Use:

$$A \cdot \sin(x) + B \cdot \cos(x) = \sqrt{A^2 + B^2} \cdot \sin(x + \tan^{-1}(B/A))$$

$$A = 1$$

$$B = -\tau 2\pi\omega$$

$$x = 2\pi\omega t$$

\Rightarrow

$$u(t) = u(t_0) e^{(t_0-t)/\tau} + \frac{t_0}{\tau} \cdot \tau \cdot e^{(t-T)/\tau} \cdot \left[\frac{\sqrt{1^2 + (-\tau 2\pi\omega)^2} \cdot \sin(2\pi\omega t + \tan^{-1}(-\tau 2\pi\omega))}{1 + \tau^2 (2\pi\omega)^2} \right] \Big|_{t=0}^T$$

\Rightarrow calculate ...

Problem 2: The Solution at Late Times

Problem 2.1: The simplified solution

$$u(T) = u(t_0) e^{(t_0 - T)/\tau} + \int_{t_0}^T \frac{I_{stim}(t)}{\tau} e^{(t-T)/\tau} dt$$

with $t_0 = -\infty$ and $T \rightarrow \infty$

$$\begin{aligned} u(T) &= u(-\infty) e^{\cancel{(t_0 - T)/\tau}} \\ &\quad + \int_{-\infty}^T \frac{I_{stim}(t)}{\tau} e^{(t-T)/\tau} dt \\ &= \int_{-\infty}^T \frac{1}{\tau} e^{-(T-t)/\tau} \cdot I_{stim}(t) dt \end{aligned}$$

t will never be $> T \Rightarrow T-t \geq 0$

$$\Rightarrow u(T) = \int_{-\infty}^T G(T-t) I_{stim}(t) dt$$

Problem 2.2. The significance of $G(t)$

$$u(T) = \int_{-\infty}^T G(T-t) \cdot u_{\text{in}}(t) dt$$

$$u_{\text{in}} = \delta(t)$$

$$\Rightarrow u(T) = \int_{-\infty}^T G(T-t) \cdot \delta(t) dt$$

For $t \leq T$:

$$u(T) = \int_{-\infty}^T \frac{1}{C} e^{-(CT-t)/C} \cdot \delta(t) dt$$

Integral with δ -function gives value of function

at $t=0$

$$\Rightarrow u(T) = \frac{1}{C} e^{-T/C}$$

Significance: $G(t)$ translates the input into output

at late times

Problem 2.3: Linearity and time-invariance

Linearity:

Show that

$$\begin{aligned}
 & \int_{-\infty}^T G(T-\epsilon) \cdot (\alpha f(\epsilon) + \beta g(\epsilon)) d\epsilon = \\
 & \quad \alpha \int_{-\infty}^T G(T-\epsilon) \cdot f(\epsilon) d\epsilon + \beta \int_{-\infty}^T G(T-\epsilon) \cdot g(\epsilon) d\epsilon \\
 & = \int_{-\infty}^T G(T-\epsilon) \cdot \alpha f(\epsilon) + G(T-\epsilon) \cdot \beta g(\epsilon) d\epsilon \\
 & = \int_{-\infty}^T G(T-\epsilon) \cdot \alpha f(\epsilon) + \int_{-\infty}^T G(T-\epsilon) \cdot \beta g(\epsilon) d\epsilon \\
 & = \alpha \int_{-\infty}^T G(T-\epsilon) \cdot f(\epsilon) d\epsilon + \beta \int_{-\infty}^T G(T-\epsilon) \cdot g(\epsilon) d\epsilon
 \end{aligned}$$

Time-invariance:

$$g(t) = f(t + \tau_0)$$

$$U_g(T) = U_g(T + \tau_0)$$

$$1. U_g(T + \tau_0) = \int_{-\infty}^{T+\tau_0} G(T + \tau_0 - \epsilon) \cdot f(\epsilon) d\epsilon$$

$$2. U_g(T) = \int_{-\infty}^T G(T - \epsilon) \cdot g(\epsilon) d\epsilon = \int_{-\infty}^T G(T + \tau_0 - \epsilon - \tau_0) \cdot f(\epsilon + \tau_0) d\epsilon$$

$$\text{Substitution: } u = \epsilon + \tau_0 \quad \frac{du}{d\epsilon} = 1 \quad \frac{du}{1} = d\epsilon \Rightarrow du = d\epsilon$$

$$\Rightarrow U_g(T) = \int_{-\infty}^{T+\tau_0} G(T + \tau_0 - u) \cdot f(u) du$$

Problem 2.3: Late-time solution to the sinusoidal input

$$G(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{C} e^{-t/\tau} & \text{if } t \geq 0 \end{cases}$$

Late-time solution $\rightarrow t \gg 0$

$$G(t) = \frac{1}{C} e^{-t/\tau} = \frac{\tau}{C} \cdot \frac{1}{\tau} e^{-t/\tau}$$

$$\hat{G}(\omega) = \frac{\tau}{C} \cdot \frac{1}{(1 + 2\pi i \omega \tau)}$$

$$= \frac{\tau}{C \tau (1/\tau + 2\pi i \omega)}$$

$$= \frac{1}{C(1/\tau + 2\pi i \omega)}$$