The Neural Code

Summer term 2022, Christian Leibold, Yuk-Hoi Yiu

Problem Set 8 — **Mutual Information** — **05.07.2022**

Due date: 12.07.2022

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1. (Gaussian Channel) We consider a system where the output Y is a sum of input X and noise N,

$$Y = X + N \ ,$$

and both, X and N ar normally distributed with mean 0 and variance σ_X^2 and σ_N^2, respectively.

(a) Explain why $p_Y(y) = \int \mathrm{d}\, p_X(x)\, p_N(y-x)$!

(b) The Fourier transform of a normal distribution with 0 mean equals $\hat{p}(\omega) = \exp[-\frac{1}{2}\,(\omega\sigma)^2]$. Use the Fourier-convolution Theorem to show that $p_Y(y)$ is a 0-mean normal distribution with $\sigma_Y^2 = \sigma_X^2 + \sigma_N^2$!

(c) Compute the entropy $H = -\int \mathrm{d}t\, p(t)\, \ln p(t)$ of a normal distribution!

(d) Show that the mutual information equals

$$I(X,Y) = H(Y) - H(Y|X) = \ln\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right)/2 \ .$$

(e) Why is the Gaussian channel information used as an upper bound for all systems $Y = X + N$ with arbitrary distribution of X?

2. (Density-Based Method)

The file `singleunit.py` contains a function that simulates spike count output $\vec{k}$ of two neurons for a stimulus with a single parameter $\theta_n = (n\,/12)\,\pi \ , \ n = 0, ..., 11$.

(a) Sample $\vec{k}$ from 100 repetitions of each of the 12 stimuli (uniform distribution of θ).

(b) Find $\vec{\lambda}(\theta)$ using a maximum (log)likelihood approach for the model distribution

$$p(\vec{k}|\theta) = \frac{[\lambda_1(\theta)]^{k_1}\,[\lambda_2(\theta)]^{k_2}}{k_1!\,k_2!}\ \exp[-\lambda_1(\theta) - \lambda_2(\theta)] \ .$$

Use `scipy.optimize.fmin` to fit the parameters.

(c) Compute the entropy $H(K_i)$ of the marginal distribution of $p(k_i) = \sum_j p(k_i|\theta_j)p(\theta_j)$ by numerical summation over all N samples $k_i^{(n)}$.

$$H(K_i) = -N^{-1}\sum_{n=1}^{N} \ln\left[\sum_j p(k_i^{(n)}|\theta_j)p(\theta_j)\right]$$

(d) Compute the mutual information $I(\vec{K},\theta) = H(\vec{K}) - H(\vec{K}|\theta) = \sum_i [H(K_i) - H(K_i|\theta)]$ with the noise entropy

$$H(K_i|\theta) = N^{-1}\sum_n [-k_i^{(n)}\ \ln(\lambda_i(\theta^{(n)})) + \lambda_i(\theta^{(n)}) + \ln(k_i^{(n)}!)]$$

(e) Compute the upper bound

$$I(\vec{K},\theta) = \ln\left[1 + \frac{\mathrm{var}(\theta)}{\mathrm{var}(k_1) + \mathrm{var}(k_2)}\right]/2$$

(f) Repeat your computations for a prior distribution $p(\theta)$ in which θ_6 occurs 10 times as often as all the other θs.