The Riemann Hypothesis is true.

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Abstract

The Riemann hypothesis states that the real part of all non-trivial zeros of the Riemann zeta function is $\frac{1}{2}$ in the critical strip. In this paper, we provide a way of proving the hypothesis by using the properties of the Riemann zeta functional equation at $\zeta(s) = 0$ and also using the integral representation (Mellin transformation) of the Riemann zeta function in the critical strip. We also take advantage of injectivity of a function of an integral liberated from the Mellin transform.

1. Introduction

The Riemann hypothesis proposed by Bernhard Riemann in his 1859 paper in [2] asserts that the real part of all non-trivial zeros of the Riemann zeta function is $\frac{1}{2}$ on the critical line. It is also known that there are infinitely many non-trivial zeros on the critical line as it was proved by G.H. Hardy in his 1914 paper in [4] and also highlighted in [3]. Proving the hypothesis could have a profound consequence in number theory and also help in understanding the distribution of prime numbers. In this proof, we show that indeed all the non-trivial zeros lie on the critical line in the critical strip of the complex plane.

2. Proof of the Riemann Hypothesis. (Alternative 1)

The functional equation of the Riemann zeta function presented in [2] shows that;

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s)\Gamma(1-s)$$

For $\zeta(s) = 0$, whereby s is a non-trivial zero of the analytically continued Riemann zeta function. Take that s is a complex number $s = a + \boldsymbol{i}b$, for a and b being real numbers with $b \neq 0$ and for a in the region $0 < a < 1$;

$$\zeta(s) = \zeta(1-s) = 0$$

From the Mellin transformation of the Riemann zeta function in the critical strip as shown in [1], we know that;

$$\zeta(s)=\frac{1}{(1-2^{1-s})\,\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}dt$$

When $\zeta(s)=0$,

$$\frac{1}{(1-2^{1-s})\,\Gamma(s)}\neq 0$$

Therefore,

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}dt=0 \qquad \text{(Eq.1)}$$

Let's consider $\zeta(1-\bar{s})=\zeta(\bar{s})=0$ for the conjugate non-trivial zero $\bar{s}=a-\boldsymbol{i}b$.

$$\zeta(1-\bar{s})=\frac{1}{(1-2^{\bar{s}})\,\Gamma(1-\bar{s})}\int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}dt=0$$
$$\Rightarrow \int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}dt=0.(\text{ Eq. }2)$$

But (Eq.1) = (Eq.2) since both equations are equal to zero. Implying that,

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}dt=\int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}dt$$
$$\Rightarrow \int_0^\infty \frac{t^{s-1}-t^{-\bar{s}}}{e^t-1}dt=0 \text{ (Eq.3)}$$

The reason why we choose $\zeta(1-\bar{s})=\zeta(\mathrm{s})$ from the equations we can acquire from $\zeta(\bar{s})=\zeta(1-\bar{s})=\zeta(\mathrm{s})=\zeta(1-\mathrm{s})=0$ is because $\zeta(1-\bar{s})=\zeta(\mathrm{s})$ will help us yield an integral equation that is comfortable to work with as we will see in the next section of this proof.

From (Eq.3), we let $(s-1)=\boldsymbol{m}$ and $(-\bar{s})=\boldsymbol{n}$. Implying that the integral becomes;

$$\int_0^\infty \frac{t^m-t^n}{e^t-1}dt=0 \text{ (Eq. 4)}$$

Observe that for (Eq.4) to hold, $\boldsymbol{m}$ must be equal to $\boldsymbol{n}$ so that the numerator ($t^{\boldsymbol{m}}-t^{\boldsymbol{n}}$) in the integral equates to zero and hence the integral also evaluating to zero. Though (Eq. 4) can be satisfied by some purely real values of $\boldsymbol{m}$ and $\boldsymbol{n}$ such that $\boldsymbol{m}\neq\boldsymbol{n}$, we can't consider them since we know that non-trivial zeros of the zeta function can't be purely real. Another way of verifying the equality of $\boldsymbol{m}$ and $\boldsymbol{n}$ would be by an approach presented below;

For the integral $\int_a^b \frac{t^m-t^n}{e^t-1}dt=0$, where a and b are fixed and let $f_{(m,n)}(t)=\frac{t^m-t^n}{e^t-1}$ for $t>0$, yes $\boldsymbol{m}=\boldsymbol{n}$. A continuous function whose integral vanishes on every interval is necessarily identically zero. Clearly $f_{(m,n)}(t)$ is identically zero only when $\boldsymbol{m}=\boldsymbol{n}$. In case a and b aren't fixed, then $\boldsymbol{m}=\boldsymbol{n}$ is not true. When $\boldsymbol{m}\neq\boldsymbol{n}$, $f_{(m,n)}(t)$ crosses the x axis once at $x=1$ so there's an entire family of intervals about 1 on which the integral of $f_{(m,n)}(t)$ vanishes by the intermediate value theorem.

$$\begin{gathered}\boldsymbol{m} = \boldsymbol{n}\\ s-1 = -\bar{s}\\ a+\boldsymbol{i}b-1 = -(a-\boldsymbol{i}b)\\ a = \frac{1}{2}\end{gathered}$$

*Note that this is a weak way to prove the Riemann Hypothesis. As we continue to the next section titled, "Argument by antisymmetry of a function $I(a)$ and its injectivity", we strongly prove the Riemann Hypothesis beyond reasonable doubt.

3. Argument by antisymmetry of a function $I(a)$ and its injectivity.

From (Eq.3), we had;

$$\int_0^\infty \frac{t^{s-1}-t^{-\bar{s}}}{e^t-1}dt = 0$$

Which can further be evaluated as,

$$\int_0^\infty \frac{t^{a-1+ib}-t^{-a+ib}}{e^t-1}dt = 0 \text{ (Eq. 5)}$$

Let's provide the step-by-step transformation and argument using antisymmetry.
Define:

$$I(a) = \int_0^\infty \frac{t^{a-1+ib}-t^{-a+ib}}{e^t-1}dt$$

Then:

$$I(a) = \int_0^\infty t^{ib}\cdot\left(\frac{t^{a-1}-t^{-a}}{e^t-1}\right)dt = 0$$

This expression isolates the real powers of t and uses the property of the exponents:

$$\begin{gathered}t^{a-1+ib} = t^{ib}\cdot t^{a-1}\\ t^{-a+ib} = t^{ib}\cdot t^{-a}\end{gathered}$$

Thus,

$$I(a) = \int_0^\infty t^{ib}\cdot\left(\frac{t^{a-1}-t^{-a}}{e^t-1}\right)dt = 0$$

Note that the function inside the integrand is antisymmetric in $a \leftrightarrow 1-a$ since: $t^{a-1} - t^{-a}$ becomes $t^{-a} - t^{a-1}$ under $a \leftrightarrow 1-a$.

Computing $I(1-a)$:

$$I(1-a) = \int_0^\infty t^{ib} \cdot \left(\frac{t^{(1-a)-1} - t^{-(1-a)}}{e^t - 1} \right) dt = \int_0^\infty t^{ib} \cdot \left(\frac{t^{-a} - t^{a-1}}{e^t - 1} \right) dt = -I(a) = 0$$

So, it implies that $I(1-a) = -I(a)$ which also implies $I(1-a) = I(a)$ since $I(a) = 0$.

In order to sufficiently prove that for $I(1-a) = I(a)$ implies $1-a = a$, we have to prove that either $I(a)$ or $I(1-a)$ is injective. In other words, prove that the function I is injective.

We will adequately prove that $I(a)$ is injective by the derivative test by showing that the derivative of $I(a)$ with respect to a is greater than zero for all $a \in (0,1)$. We have;

$$I(a) = \int_0^\infty t^{ib} \cdot \left(\frac{t^{a-1} - t^{-a}}{e^t - 1} \right) dt$$

Since $I(a) = 0$, we can also deduce that

$$I(a) = \int_0^\infty \cos(b \ln t) \cdot \left(\frac{t^{a-1} - t^{-a}}{e^t - 1} \right) dt$$

We differentiate both sides of the equation with respect to a (taking note that we can take the derivative under the integral) to acquire,

$$I'(a) = \int_0^\infty \cos(b \ln t) \cdot \left(\frac{\frac{d}{da}\left(t^{a-1} - t^{-a}\right)}{e^t - 1} \right) dt$$

Which eventually gives;

$$I'(a) = \int_0^\infty \cos(b \ln t) \cdot \ln t \cdot \left(\frac{t^{a-1} + t^{-a}}{e^t - 1} \right) dt$$

From sign analysis, we observe that for all real values of $t > 0$ except at $t = 1$ then $I'(a) > 0$ for $a \in (0,1)$. From this, strict monotonicity has been shown which strongly implies that $I(a)$ is injective.

Since we have proven that $I(a)$ is injective, it also implies that $I(1-a)$ is also injective because $I(1-a) = I(a)$ as it was proved already. The definition of injectivity is that a function I is injective when $I(x_1) = I(x_2)$ implies $x_1 = x_2$.

This tells us that from the equation $I(1-a) = I(a)$, we have to assert that $1-a = a$.

Truly,

$$a = \frac{1}{2}$$

4. Asymptotic behavior as a tends to $\frac{1}{2}$.

As we continue from (Eq.5) which is; $\int_0^\infty \frac{t^{a-1+ib}-t^{-a+ib}}{e^t-1}dt = 0$, we let

$$f(t) = t^{a-1+ib} - t^{-a+ib}$$

We note that for $0 \le t \le 1$, we have $\frac{1}{e^t-1} - \frac{1}{t} = \frac{t+1-e^t}{t(e^t-1)}$ is continuous at 0 so it is $O(1)$ and clearly $|f(t)| \le t^{a-1} - t^{-a}$ is integrable on $[0,1]$ too as long as $0 < a < 1$ so with h integrable

$$\int_r^1 \frac{t^{a-1+ib} - t^{-a+ib}}{e^t - 1} dt = \int_r^1 \left(t^{a-2+ib} - t^{-a-1+ib}\right) dt + \int_r^1 h(t) dt$$

But the first integral is

$$-\left(\frac{r^{a-1+ib}}{a-1+ib} + \frac{r^{-a+ib}}{-a+ib}\right) + c$$

And if $a \neq \frac{1}{2}$, it clearly diverges as $r \to 0$. In particular, the integrals do not make sense for any $0 < a < 1$ when $a \neq \frac{1}{2}$ and it is meaningless to talk about them being zero or not. If we use analytic continuation, we get an expression involving the zeta function, namely;

$$F(s) = \left(1 - 2^{1-s}\right) \zeta(s) \Gamma(s) - \left(1 - 2^{\bar{s}}\right) \zeta(1 - \bar{s}) \Gamma(1 - \bar{s})$$

Which is a harmonic function well defined in the strip, but that is not the integral in any meaningful sense, except when $a = \frac{1}{2}$ when it reduces identically to zero and of course the question of its zeros is tangled with Riemann Hypothesis since any $\zeta(s) = 0$, $s = a + ib$ satisfies also $\zeta(1 - \bar{s}) = 0$ so is a zero of F.

Note that if $\frac{1}{2} < a < 1$ we have,

$$|\zeta(1 - a + ib)| \sim \left(\frac{|b|}{2\pi}\right)^{a-\frac{1}{2}} |\zeta(a + ib)|$$

also

$$\left|\frac{\Gamma(a + ib)}{\Gamma(1 - a + ib)}\right| \sim |b|^{2a-1}$$

And so asymptotically we have;

$$|F(a + ib)| \ge |\zeta(a + ib)||\Gamma(a + ib)| \left(1 - |b|^{\frac{1}{2}-a}\right)$$

Hence asymptotically F cannot have other zeros than the ones of the zeta function and probably using more precise asymptotics for both Γ and χ(χ is a Dirichlet character of the Dirichlet eta function) one can give estimates. So, it is quite unlikely that nonexistence of zeros of F outside $a = \frac{1}{2}$ which clearly implies the Riemann hypothesis.

5 . Verification of $a = \frac{1}{2}$ via Fourier transformation.

In this section, we derive $a = \frac{1}{2}$ from a Fourier transform whose function in the integrand is identically zero when (Eq.5) holds. We begin from (Eq.5) that we had acquired earlier on, which was;

$$\int_0^\infty \frac{t^{a-1+ib} - t^{-a+ib}}{e^t - 1} dt = 0$$

Let

$$x = \ln t$$

This implies that,

$$e^x = t$$

Which also implies that

$$\frac{dt}{dx} = e^x$$

And

$$dt = e^x dx$$

We now acquire,

$$\int_0^\infty \frac{t^{a-1+ib} - t^{-a+ib}}{e^t - 1} dt = \int_{-\infty}^\infty e^{x(1+ib)} \cdot \left(\frac{e^{x(a-1)} - e^{-xa}}{e^{e^x} - 1} \right) dx = 0$$
$$\Rightarrow \int_{-\infty}^\infty \left(e^x \cdot \frac{e^{x(a-1)} - e^{-xa}}{e^{e^x} - 1} \right) e^{ibx} dx = 0$$

By analysis, the Fourier transform $\int_{-\infty}^\infty \left(e^x \cdot \frac{e^{x(a-1)} - e^{-xa}}{e^{e^x} - 1} \right) e^{ibx} dx = 0$ only vanishes to zero when the integrand itself is identically zero due to the "uniqueness theorem of the Fourier transform". One can also claim that this argument is correct by checking or proving injectivity of the function inside the integral in terms of a via the derivative test. This implies that;

$$e^x \cdot \left(\frac{e^{x(a-1)} - e^{-xa}}{e^{e^x} - 1} \right) = 0$$

Which further evaluates to

$$e^{x(a-1)} = e^{-xa}$$

Introducing natural logarithms on both sides of the equation simply gives us, $a - 1 = -a$.

Then

$$a = \frac{1}{2}$$

Indeed $a = \frac{1}{2}$ hence the Riemann hypothesis.

We also have to note that a direct substitution $a = \frac{1}{2}$ clearly satisfies (Eq. 5). Indicating and giving us highlights that the Riemann hypothesis might actually be true. Such convergence tells us something crucial about the non-trivial zeros and the result is completely analogous to what Hardy proved in [4].

6. Conclusion

The Riemann hypothesis is correct since we have proven that the real part of all the nontrivial zeros of the Riemann zeta function must be $\frac{1}{2}$ on the critical line in the critical strip. From this proof of the Riemann hypothesis, we have used the Mellin transformation of the Riemann zeta function in the critical strip, utilized the properties of the zeta functional equation at $\zeta(s) = 0$.

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