ON THE METAMATHEMATICS OF ALGEBRA

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PREFACE

The formal analysis of the logical foundations and methods of a mathematical discipline may be regarded as interesting in itself, and Abstract Algebra offers ample opportunities for such investigations. However, the present work goes beyond mere analysis and sets out to make a positive contribution to Algebra using the methods and results of Symbolic Logic.

At the present stage of the development of Mathematics, the second aim may appear to be unduly ambitious. Indeed, a book on heuristic methods published recently by a distinguished analyst (G. Polya, 'How to Solve it') makes no mention of Symbolic Logic. And going back one generation we may quote H. Poincaré's comments on Couturat's claim that Symbolic Logic could fertilise the spirit of mathematical invention.

'En ce qui concerne la fécondité il semble que M. Couturat se fasse de naïves illusions. La logistique, d'après lui prête a l'invention 'des échasses et des ailes', et à la page suivante 'Il y a dix ans que M. Peano a publié la première édition de son formulaire'. Comment, voilà dix ans que vous avez des ailes et vous n'avez pas encore volé!'. (H. Poincaré, Science et Méthode, ref. 57). Continuing in this vein, Poincaré then suggests that so far from lending it wings, Symbolic Logic has only provided Mathematics with fetters.

However, it will be shown that the synthesis of more recent results of Symbolic Logic and of the concepts and methods of Algebra leads to developments which justify Couturat's more optimistic point of view. Moreover, it appears that by a metamathematical theory such as is described in the present work we can hope to subordinate the multiplicity of algebraic structures which have been introduced in the last one hundred years (fields, rings, groups, lattices, quasi-fields and semi-groups of various descriptions, multiple groups, etc.), to a general theory in which the features which all these structures have in common appear not only as analogues but as applications of the same principle.

The investigations which are contained in this treatise were undertaken between September 1947 and April 1949. Some of the earlier results, including theorems 3.2.3. and 5.8.1., were communicated in a lecture to the Birkbeck College (London University) Mathematics Seminar in February 1948. Later lectures on the subject were delivered to Mathematical seminars at the Universities of London and Manchester, and the work was also described in a paper which was read by title at the Annual Meeting of the Association for Symbolic Logic in December 1948 (see ref. 61). The present monograph is identical with a thesis which was submitted to London University in May 1949, except for minor verbal modifications and the addition of an example in section 7.3. Also, the proofs of theorems 3.1.1., 5.8.2., 6.4.16., 7.3.7., 7.3.11. and 10.5.2., which are given in the thesis, have been omitted here for the sake of brevity. It appears from papers published since that the first of the two theorems mentioned above (3.2.3). has been proved independently by L. Henkin (ref. 44) while theorem 5.8.1. has been stated by A. Tarski (compare ref. 62).

I am indebted to Professor P. Dienes, Professor M. H. A. Newman and Professor A. Church who have shown their interest in my work in various ways. My thanks are due to Professor L. E. J. Brouwer, Professor A. Heyting, Professor E. W. Beth and to the North-Holland Publishing Company for inviting me to contribute this monograph to the series 'Studies in Logic'.

Finally, I should like to acknowledge on this occasion the debt of gratitude which I owe to my teacher and friend, Prof. A. Fraenkel of the Hebrew University of Jerusalem under whose guidance I made my first original contributions to Mathematics and to Symbolic Logic.

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I

GENERAL INTRODUCTION

1.1. *Purpose of treatise.* The principal object of the present work is the analysis and development of Algebra by the methods of Symbolic Logic. In fact, in view of their transparent logical character, the algebraic theories of fields, rings, and of similar structures appear to be eminently suited to such treatment, more perhaps than any other branch of Mathematics.

Two main lines of attack suggest themselves, both of which make the present argument appear as a natural and inevitable development of, or sequel to, orthodox Mathematics.

Instead of formulating and proving individual theorems as in orthodox Mathematics, we may consider statements about theorems in general. In particular, we may be able to show that any theorem (of a certain class) which is true for one type of mathematical structure is also true for another type of mathematical structure. An instance of such a metamathematical statement is provided by the classical principle of duality in Projective Geometry. However the demonstration of that principle is so simple from a logical point of view that it was actually given prior to the development of Formal Logic.

Again, instead of discussing the properties of algebraic structures defined by specific sets of axioms, we may consider structures given by different sets of axioms, simultaneously. However, in order to obtain positive results we shall have to limit the metamathematical features of such axiomatic systems. To give a concrete example, we might be able to go some way towards developing a theory common to all structures which are given by axiomatic systems characterised solely by the fact that they are formulated within the restricted calculus of predicates (functional calculus of the first level). In actual fact, we shall find that the existence of a relation of equality within the axiomatic systems under consideration appears to be vital for the development of such a theory (see section 1.5. below).

Both lines of attack follow the marked tendency of contemporary mathematical research towards ever increasing generality. However, it is believed that the danger inherent in this tendency — ever increasing triviality of results obtained — has been avoided in the present case.

1.2. *Fundamental logical outlook.* The formal language used in the argument is based on the restricted calculus of predicates with certain extensions which prove natural in the course of the analysis. Although the tendency of the work described is mathematical and not philosophical, it becomes necessary to adopt a definite attitude towards the logical controversies of the present time. Without wishing to commit himself definitely to one school of thought or the other, the present writer wishes to state his belief in as liberal and as unfettered a point of view as possible. He believes that this attitude is quite as consistent as any more restrictive view point such as is expressed, for example, by the various forms of intuitionism.

Thus, in Mathematics, the writer accepts the axiom of choice and its consequences, and the theory of transfinite cardinal and ordinal numbers. In Formal Logic, he accepts such constructs as infinite conjunctions and disjunctions, and believes that it is legitimate to ascribe definite truth values to such constructs in the usual way. Similarly, he accepts the existence of languages containing a non-countable number of propositions, just as in Mathematics he accepts the existence of non-countable sets of numbers. However, it may be worth pointing out that the introduction of these concepts and constructs is not due to an unbridled desire for generalisation, but that they are required for the formulation and proof of theorems which are in themselves quite orthodox. For instance, Archimedes' axiom can be formulated naturally in terms of an infinite disjunction.

Nevertheless, a considerable proportion of the results obtained remains valid from a more restrictive point of view.

1.3. *Metamathematical theorems in Algebra.* Three outstanding examples of theorems which will be proved following the first line of attack are as follows. —

Any theorem formulated in the restricted calculus of predicates (in terms of equality, addition, and multiplication) which is true for all commutative fields of characteristic 0, is true for all commutative fields of characteristic $p > p_0$ where p_0 is a constant depending on the theorem.

Any theorem of the restricted calculus of predicates which is true for all non-Archimedean ordered fields is true for all ordered fields.

Any theorem of the restricted calculus of predicates which is true for the field of all algebraic numbers is true for any other algebraically closed field of characteristic 0.

The proofs of the first and third theorems are based on the fact that the concepts of a 'field of characteristic 0' and of 'an algebraically closed field' can be formulated in terms of countable, but not in terms of finite, sets of propositions in the restricted calculus of predicates, while on the other hand any proof within that calculus can involve only a finite number of such propositions. The proof of the third theorem also requires some results of algebraic field theory. The proof of the second theoıem is slightly more intricate from a logical point of view.

1.4. *Generalised concepts of Algebra.* Following the second line of attack, we shall consider properties of structures of which we know only that they satisfy some specific system of axioms formulated in the restricted calculus of predicates, and that a relation of equality (see section 1.5. below) is defined in the system. We shall investigate various concepts which are parallel to certain standard concepts of Algebra. It is of decisive importance that these general concepts are not merely analogous to their algebraic counterparts but that in the particular cases of the algebraic systems from which they are borrowed, they actually reduce to these counterparts.

Outstanding examples of concepts which will be so generalised are — the concept of an algebraic number and, more generally, of a number which is algebraic with respect to a given commutative field; the concept of a polynomial ring $R(x_1, \ldots, x_n)$ of n variables $x_1, \ldots, x_n$ adjoined to a given commutative ring R; the concept of an ideal. The fact that the above concepts can be formulated in this way is interesting in itself, proving as it does that they can be abstracted from the specific arithmetical operations with which they are normally associated. However, we shall go further than that and shall show that a number of their properties in standard Algebra can be transferred to the more general case considered here.

1.5. *Definition of Equality.* We shall say that a relation of equality is defined in a given system of axioms if it includes a relation $E(x, y)$ which is symmetrical, reflexive and transitive, and such that for every relation $F(x_1, \ldots, x_n)$ included in the system, it can be proved that equal objects can be substituted for one another as arguments,

$$(x_1) \ldots (x_n)(y_1) \ldots (y_n)[[E(x_1, y_1) \wedge E(x_2, y_2) \wedge \ldots \wedge E(x_n, y_n)] \supset [F(x_1, \ldots, x_n) \supset F(y_1, \ldots, y_n)]]$$

where $\wedge$ and $\supset$ denote conjunction and implication respectively.

The above definition of equality, while it does not exhaust the full meaning of identity, will be adequate for our purposes.

1.6. *Example of a generalised concept in Algebra.* The definition of the concept of an algebraic element in a general system may serve as a concrete example. Given an axiomatic system A as indicated in sections 1.4. and 1.5. above, let M be one of its models (realisations). Another model M_1 will be called an extension of M, $M_1 \succ M$ or $M \prec M_1$, if all the elements of M are elements of M_1 such that whenever any relation between such elements holds in M it also holds in M_1 and vice-versa.

The language in which the axiomatic system A is formulated will contain a number of symbols for the relations involved, and in addition it may also contain object symbols, denoting individual elements. To these we add symbols for all the individual elements of M if they are not contained in the language already. We shall consider properties (predicates of one variable) defined in terms of this extended language. A property will be called persistent if, whenever it is satisfied by an element $a \varepsilon M_1$, $M_1 \succ M$ where M_1 also is a model of A, then it is satisfied by a in all models of A which include M_1 and contain a. For instance, let A be an axiomatic system of the concept of a commutative field, while M is the field of rational numbers. Then the property of being the square root of

—1 is persistent, since it is satisfied by the numbers i and $-i$ in all the fields containing these elements. On the other hand, the property of not being the square of another number is not persistent since it is satisfied by 2 in M but it is not satisfied by 2 in the field obtained from M by the adjunction of $\sqrt{2}$.

A property will be called saturable if whenever there is a chain of models of A,

$$M \lessdot M_1 \lessdot M_2 \lessdot \ldots \lessdot M_n \lessdot \ldots$$

and a sequence of elements $a_n \varepsilon M_n$ such that every a_n satisfies the given property in M_n, then there exists a positive integer n_0 such that all the a_n are elements of M_{n_0} or are equal to elements of M_{n_0}.

We then define: an element $a \varepsilon M_1$, $M_1 \gtrdot M$, where M and M_1 are models of an axiomatic system A will be called algebraic with respect to M if it satisfies a saturable persistent property with respect to M.

It will be shown that if M and M_1 are models of an axiomatic system representing the concept of an algebraic field, $M_1 \gtrdot M$, then every element of M_1 which is algebraic with respect to M in the sense defined also is algebraic with respect to M in the mathematical sense, and vice versa.

There are other definitions which reduce to the notion of an element being algebraic with respect to a certain field in the classical case. However it is a familiar phenomenon in abstract Algebra that a standard concept in one system may split up into two or more concepts in a more general system. For instance, the concepts of prime ideals and of maximal ideals are identical in the ring of positive integers although they are distinct in general commutative rings.

It will be seen that the above definition involved the introduction of separate symbols for all the elements of M. This implies that if M is non-countable, a non-countable number of object symbols must be available. The necessity for this step will be discussed later in some detail. It appears to be merely the formal consequence of common algebraic practice, but it is also conditioned independently by semantic considerations (see section 2.8. below).

Another interesting feature of the example given above is the simultaneous use of the properties of an axiomatic system and of the properties of its models. It is in fact essential for present developments that an equal degree of reality should be attributed to the axiomatic systems and to the models in which they are realised.

1.7. *Extension of models.* While it is the main object of the present paper to apply Logic to Mathematics, a number of results which are of interest to Mathematical Logic are obtained incidentally. For instance, it will be shown that every model of an axiomatic system formulated in the restricted calculus of predicates can be extended (in the non-trivial sense of the addition of elements which are not equal to any elements within the given system). We conclude that such a model cannot satisfy an axiom of completeness in Hilbert's sense (ref. 1). Since the concept of an ordered field can be formalised within the restricted calculus of predicates while the concept of an Archimedean ordered field does in fact possess a model which is complete in Hilbert's sense (the field of all real numbers), it follows that Archimedes' axiom cannot be formalised within the restricted calculus of predicates. This is the basis of one of two methods, available for the proof of the second theorem mentioned in section 1.3. above.

The above examples will suffice to show that Symbolic Logic can be an effective tool for the discovery and proof of mathematical theorems: while on the other hand the analysis of the procedures used in a highly developed mathematical discipline throws light on points of Symbolic Logic which might be overlooked in purely abstract investigations.

1.8. *Summary.* The following summary is expressed in very general terms and the relevant chapters should be consulted for more precise definitions.

In the second chapter we consider the object language which is used later to describe the mathematical structures in which we are interested. In particular we lay down the rules of formation, of deduction, and of semantic interpretation of the language. In the third chapter, we investigate the mutual relations between deductive and semantic concepts, and we prove that the deductive calculus is complete within the scope required for the sequel. We also consider deductive systems, in Tarski's sense. In the fourth chapter we formulate axiomatic systems for groups, rings, and fields, and in the fifth chapter we prove various metamathematical theorems concerning fields, as exemplified by 1.3. above.

In the sixth chapter, we begin to approach the subject along the second line mentioned in section 1.1. We define the concepts of an 'algebra of axioms' and of an 'algebraic structure' as systems within which relations of equality involving substitutivity are defined. We then consider concepts of isomorphism of mathematical structures, and similarly questions of equivalence of sets of axioms. Finally, we consider the extension of any given structure, and arrive at the result indicated in section 1.7. above. In the seventh chapter, we introduce the metamathematical counterpart of the polynomial extension of an algebraic ring, as well as a more general related concept, and in the eighth chapter we consider generalisations of polynomial equations. In particular, we define 'algebraic predicates', which in the special case of a commutative algebraic field apply to the algebraic numbers of the field only (see 1.6. above). In the ninth chapter we consider properties of convex systems (a system of axioms is called convex, broadly speaking, if the meet of any number of models of the given system is again a model). We also introduce the metamathematical counterparts of separable and perfect fields, and we consider properties of symmetrical predicates in such systems.

The remaining two chapters are dedicated to a metamathematical theory of ideals, including a discussion of generalised algebraic varieties.

1.9. *Conclusion.* Some observations on the methods of proof adopted in the present work may not be out of place. It is natural that a certain proportion of our proofs are transcribed from Mathematics proper, while others are merely applications of the restricted calculus of predicates in Symbolic Logic. But, in addition, the synthesis of Algebra and Metamathematics also gives rise to methods of proof which would appear to represent a new type of deductive technique. (See e.g., the proof of Theorem 11.3.7.). Again, some of the results on convex systems which are given at the beginning of the last chapter require a novel type of argument because the condition of convexity is formulated in terms of the models of the system of axioms under consideration, while the proof makes use of deductive theory within the object language of the given set of axioms.

It will be clear that the present work was not meant to be exhaustive, and that a number of the points dealt with warrant further research. Moreover, in selecting our applications we have confined ourselves to the simplest classical mathematical structures, although the scope of the general theory is far more comprehensive. Some of it, such as the ideal theory of chapters 10 and 11 admits of immediate application to various problems, while other branches of the theory, such as the developments of chapters 7, 8 and 9 may still require considerable work for an effective application. For example, although the theories of algebraic predicates, and of polynomial structures apply to general (non-commutative) fields, it may be quite difficult in practice to obtain a detailed idea of the algebraic significance of these concepts such as is possible in the case of commutative fields.

It would be idle to try and enumerate all the possible fields of application and of extension of the present theory, but we might mention particularly the subject of Topological Algebra. Direct applications to Abstract Geometry may produce interesting axiomatic results, but perhaps not any significant development of the theory. A more fruitful approach may be provided through Algebraic Geometry. In fact, the theory of algebraic varieties in general rings may be considered a promising start in this direction.

However, apart from the potential applications and extensions of the present theory, the development and improvement of our fundamental methods would also be of interest. For example, further research on the connection between concepts such as persistence and convexity, which are formulated in terms of models, and the form and deductive properties of the set of axioms to which they apply, may produce useful results. Again, it may be worthwhile to introduce functors (i.e., descriptions generated by relations such as 'the sum of x and y') as separate entities. This would facilitate a link-up with the theory of general algebras as considered by Birkhoff (ref. 2). (Birkhoff's theory, though dealing with very general types of algebras, does not involve the use of Symbolic Logic).

Another modification, which might help to streamline the argument, would be the elimination of the distinction between objects and relations on one hand and object and relative symbols on the other. In the present work, it was considered appropriate to maintain the differentiation between the former, as elements of mathematical structures, and the latter, as elements of Language.

In conclusion we may point out that in our approach we have translated the partly synthetic, partly arithmetical methods of contemporary Algebra to a different plane. In this respect our procedure differs fundamentally from the methods of decision theory in which the place of a proof is taken by an arithmetical process of compelling validity. Our present aim, on the other hand, is at once more modest and more ambitious. For while we do not try to establish universal rules for deciding whether or not any given statement is valid, we attempt to make a contribution not only to the deductive but also to the creative and heuristic side of mathematical theory.

II

CONSTRUCTION OF A FORMAL LANGUAGE

2.1. *Introduction.* We begin by constructing the object language L on which our investigation will be chiefly based. While this language is largely identical with that of the 'restricted calculus of predicates' in the form in which it was presented by the Hilbert School, there are important differences, both in the fundamental conception and in various details, which make it necessary for us to sketch the development of the subject ab initio.

We conceive a language as an aggregate of statements involving certain primary, or atomic, symbols combined in accordance with a definite set of rules. We do not stipulate that the entire aggregate can actually be written down explicitly (and is therefore finite), nor that it can be written down to any required extent (and is therefore countable). Thus while we retain the word 'symbol' for some of the prime units of our language, inasmuch as they may correspond to — and in that sense, designate — certain counterparts contained in 'actual' mathematical models, we do not identify them with the signs or shapes by which they are denoted on paper. Accordingly, their number may be non-countable, countable, or finite, as the case may be.

In addition to the symbols which are defined in the subsequent sections the following notation will be used for concepts relating to sets in general. —

$a \varepsilon A$ indicates that a is an element of the set A, $A = B$ and $A \neq B$ state that the sets A and B are equal or unequal as the case may be. We write $A \subseteq B$ or $B \supseteq A$ if A is a subset of B and reserve $A \subset B$ and $B \supset A$ for the case that A is a subset of B while at the same time $A \neq B$. The union, meet (intersection), and difference of two sets A and B will be denoted by $A \cup B$, $A \cap B$ and $A - B$ respectively. The sign of equality, $=$, will also be used to define one symbol in terms of another, or in terms of a group of symbols, e.g., $X = [[A] \supset [B]]$.

2.2. *Symbols.* The atomic symbols of our language L are listed below. —

2.2.1. Object symbols, denoted by $a, b, c, \ldots$ (small Roman letters) with or without suffices. These symbols are supposed to constitute a well-defined set of arbitrary transfinite cardinal number.

2.2.2. Dummy symbols, denoted by the letters $u, v, w, x, \ldots$ with or without suffices. These symbols are supposed to form a countable set.

2.2.3. Relative symbols, divided into classes R_n, $n = 0, 1, 2, \ldots$ (relative symbols or order n). Relative symbols of order $n \geqslant 1$ will be denoted by $A(, , \ldots,)$, $B(, , \ldots,)$, $\ldots$ (capital Roman letters followed by n empty spaces, separated by commas, in round brackets). Relative symbols of order 0 will be denoted simply by $A, B, C, \ldots$. The classes R_n will be supposed to constitute well-defined sets whose cardinal numbers are specified but arbitrary.

2.2.4. Copulae. These are the four symbols denoted by $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\supset$ (implication).

2.2.5. Quantifiers. These are, the universal quantifier, denoted by ( ) and the existential quantifier, denoted by $(\exists\ )$.

2.2.6. Separation symbols, denoted by [(left square bracket) and] (right square bracket).

2.3. *Formulae.* 'Prime formulae' are obtained by the juxtaposition of a relative symbol of order n, $n = 1, 2, \ldots,$ and of n object or dummy symbols in any arbitrary but definite order. For instance, if $A(, ,)$ belongs to R_3, then $A(a, b, a)$ and $A(b, a, z)$ are both prime formulae. Relative symbols of order 0 will be considered as prime formulae by themselves, by definition.

'Significant formulae', sometimes called briefly 'formulae', will now be defined recurrently. They will be denoted by the letters $V, W, X, \ldots$ (Roman capitals).

2.3.1. Prime formulae bracketed by square brackets are significant formulae, e.g. $[A(b, a, z)]$ and $[B]$, where $A(, ,)$ and B are of orders 3 and 0 respectively.

2.3.2. If X is a formula, $[\sim X]$ is a formula. If X and Y are formulae, $[X \wedge Y]$, $[X \vee Y]$, and $[X \supset Y]$ are all formulae.

2.3.3. If X is a formula, $[(v)X]$ and $[(\exists v)X]$ are both formulae, provided X contains the dummy symbol v but does not already contain one of the quantifiers (v) or $(\exists v)$.

It will be understood without detailed explanation what is meant by the phrase 'X contains the symbol v', etc. We shall also say that X contains a formula Y if X is constructed from Y and other formulae by one of the rules of 2.3.2. or of 2.3.3.

We shall always assume that the language with which we are dealing contains at least one formula.

2.4. *Statements and predicates.* We now classify all significant formulae as complete formulae, or statements, and incomplete formulae, or predicates, in the following way:

A formula X will be called complete if, whenever a dummy symbol, say v, is contained in it, then v is contained in a formula Y, such that Y occurs in X in one of the forms $[(v)Y]$ or $[(\exists v)Y]$. If v occurs in X more than once (excepting the cases where it occurs within the brackets of a quantifier), then the above condition is supposed to be satisfied whenever v occurs. For example, $[(v)[[A(v)] \wedge [B(v)]]]$ and $[[(v)[A(v)]] \supset [(Ev)B(v)]]]$ are both complete while $[[(v)[A(v)]] \wedge [B(v)]]$ is incomplete.

We define the order of a significant formula as the number of pairs of square brackets contained in it. Thus, the order of $[A(b)]$ is 1, the order of $[(\exists x)[A(x)]]$ is 2 and the order of $[[(x)[B(x)]] \vee [C(a)]]$ is 4. The order of a formula constructed by negation or quantification (see rules 2.3.2. and 2.3.3. above) exceeds by one the order of the formula from which it is constructed. The order of a formula constructed by conjunction, disjunction, or implication, exceeds by one the sum of the orders of the two formulae from which it is constructed.

The language which has been defined is uneconomical in two respects. First, it includes implication as a fundamental concept although, as is well known, it can be expressed in terms of negation and disjunction. This has been done with a view to future developments in which formulae involving conjunction and implication, but not negation and disjunction, will play a special part. Second, a large number of brackets is required even for relatively simple formulae. The present notation was chosen because it indicates quite clearly the structure of a formula and in this way facilitates the definition of order. However, later we shall simplify our notation, largely following accepted procedure, by introducing the following rules.

In the successive construction of a significant formula from a set of prime formulae, the following square brackets may be omitted.

2.4.1. Square brackets enclosing prime formulae.

2.4.2. Square brackets following a negation provided they enclose a negation. For example $[\sim[\sim X]]$ can be replaced by $[\sim\sim X]$.

2.4.3. Square brackets preceding or following the sign of conjunction, provided they enclose a negation.

2.4.4. Square brackets preceding or following the sign of disjunction, provided they enclose a conjunction or a negation. E.g., $[[X \wedge Y] \vee [\sim Z]]$ is replaced by $[X \wedge Y \vee \sim Z]$.

2.4.5. Square brackets preceding or following the sign of implication, provided they enclose a disjunction or a conjunction or a negation.

2.4.6. In any sequence of quantifications such that both the quantifiers on the left and the brackets on the right follow immediately upon one another, all the square brackets may be omitted, with the exception of the innermost and the outermost pairs (which may however be removable by virtue of another rule). For example, $[(x)[(\exists y)[(z)[\sim X]]]]$ becomes $[(x)(\exists y)(z)[\sim X]]$.

2.4.7. Finally, the outermost brackets in a significant formula may also be dropped. For example,

$$[(x)[(y)[(z)[[[A(x, y)] \wedge [A(y, z)]] \supset [A(x, z)]]]]]$$

will be written in a simplified form as

$$(x)(y)(z)[A(x, y) \wedge A(y, z) \supset A(x, z)].$$

The above rules are framed in such a way that any formula which has been simplified by the use of some or all of them can be restored to its fully bracketed form without ambiguity. Rules for the omission of brackets in successive conjunctions or disjunctions are not included, since they involve the validity of the associative law for these operations. However, later on (Chapter 4 et seq.) we shall introduce the further simplification of denoting the conjunction and disjunction of any number of statements taken in any order of association by $[X_1 \wedge X_2 \wedge \ldots \wedge X_n]$ and $[X_1 \vee X_2 \vee \ldots \vee X_n]$ respectively. Any statement or argument involving such expressions will then be meant to apply when the latter are replaced by fully bracketed expressions. Thus $[X_1 \vee X_2 \vee X_3]$ should be replaced by $[X_1 \vee [X_2 \vee X_3]]$ or by $[[X_1 \vee X_2] \vee X_3]$, where the particular choice of the fully bracketed expression is irrelevant. Moreover, different fully bracketed expressions may be selected for the same simplified formula if the latter occurs more than once.

Finally, it may be mentioned that some of the subsequent work is simplified if we restrict the range of significant formulae by the adoption of the following rule. —

Given the formulae X and Y, the expressions $[X \wedge Y]$, $[X \vee Y]$ and $[X \supset Y]$ are significant formulae only if a dummy symbol which is included in a quantifier in one of the two formulae, X or Y, does not occur in the other at all.

This rule does not in any way restrict the scope of the language, but we are following current practice in refraining from its introduction.

2.5. *Calculus of deduction.* From the set of complete formulae, or statements, we now select a set of 'valid statements' by a purely formal procedure.

Given any statements X, Y, Z in a language L, the following are valid statements. —

2.5.1. Statements of implication

$$[X \supset [Y \supset X]]$$
$$[[X \supset [X \supset Y]] \supset [X \supset Y]]$$
$$[[X \supset Y] \supset [[Y \supset Z] \supset [X \supset Z]]]$$

2.5.2. Statements of conjunction

$$[[X \wedge Y] \supset X]$$
$$[[X \wedge Y] \supset Y]$$
$$[[X \supset Y] \supset [[X \supset Z] \supset [X \supset [Y \wedge Z]]]]$$

2.5.3. Statements of disjunction

$$[X \supset [X \vee Y]]$$

$[Y \supset [X \vee Y]]$
$[[X \supset Z] \supset [[Y \supset Z] \supset [[X \vee Y] \supset Z]]]$

2.5.4. Statements of negation
$[[X \supset Y] \supset [[\sim Y] \supset [\sim X]]]$
$[X \supset [\sim [\sim X]]]$
$[[\sim [\sim X]] \supset X]$

In simplified notation, the last statement of 2.5.2., for instance, becomes

$$[X \supset Y] \supset [[X \supset Z] \supset [X \supset Y \wedge Z]]$$

If a significant formula X contains an object symbol a, we may indicate this occasionally by denoting it by $X(a)$. Similarly, if X contains object symbols a and b, we may write it as $X(a, b)$. This notation will be used even if X contains other object symbols apart from those mentioned. We also write $X(v)$ if the dummy variable v is contained in X, provided however that v is not quantified in X (does not appear under a quantifier) while all the other dummy variables are quantified wherever they appear in X. Under similar conditions we write $X(v, w)$ for dummy variables v, w involved in a formula X, etc.

Subject to these conditions, the following are supposed to be valid statements, for arbitrary X, a, and v.

2.5.5. Statements of quantification

$$[[(v)[X(v)]] \supset X(a)]$$

$$[[X(a)] \supset [(\exists v)\, X(v)]]$$

Next we introduce the following rules of inference. —

2.5.6. If X and $[X \supset Y]$ are valid statements, then Y is a valid statement.

2.5.7. If the statement $[X \supset Y(a)]$ is valid, where X does not contain a, so is the statement $[X \supset [(v)Y(v)]]$.

2.5.8. If the statement $[Y(a) \supset X]$ is valid where X does not contain a, so is the statement $[[(\exists v)Y(v)] \supset X]$.

Finally we have the rule of substitution.

2.5.9. From any valid statement X which contains a quantifier with dummy symbol v, another valid statement is obtained by replacing v by any other dummy symbol, both in the bracket of the quantifier and wherever v appears within the square brackets following the quantifier. If the same dummy symbol occurs in more than one quantifier then the operation may be carried out with respect to any one quantifier (and the expression in the brackets following it) alone. However, the rule applies only if the results of the substitution is a significant formula at all.

2.6. *Calculus of deduction*, continued. Rules 2.5.1.—2.5.4. are based on the deductive system for the calculus of propositions given by Hilbert and Bernays (ref. 3, p. 66). The only difference is that the equivalence ($\equiv$) has not been introduced as a fundamental copula. Accordingly, the three propositions relating to equivalence in the system of Hilbert and Bernays are omitted here. The additional rules for the calculus of predicates are essentially due to Bernays and Hilbert–Ackermann (ref. 4), and are an amplification of the rules suggested by Whitehead and Russell (ref. 5). However, the rules of the calculus of propositions as well as the rules of quantification on which the system of Hilbert–Ackermann is based are supposed to refer — in our terminology — only to specific relative symbols of orders 0 and 1. As a result, three further rules of substitution are required which in our system can be deduced from the rules laid down already. One of these rules of substitution, which will be used in the sequel, states, in our terminology: —

2.6.1. In any valid statement, we may replace a statement which is obtained by bracketing a relative symbol of order 0 by an arbitrary statement. The result is a valid statement provided it is a significant formula at all (ref. 4).

We shall also apply some of the standard formulae and results of the calculus of predicates relying on the proofs given for them elsewhere. In particular, we shall require the following results

2.6.2. If

$$[[[\ldots[X_1 \wedge X_2] \wedge X_3] \wedge \ldots \wedge X_n] \supset Y_m],\ m = 1, 2, \ldots k$$

and

$$[[[\ldots[Y_1 \wedge Y_2] \wedge Y_3] \wedge \ldots \wedge Y_k] \supset Z]$$

are all valid statements, then

$$[[[\ldots[X_1 \wedge X_2] \wedge X_3] \wedge \ldots \wedge X_n] \supset Z]$$

is a valid statement.

2.6.3. For every statement X which contains dummy symbols, and therefore quantifiers, there exists a statement X' containing the same relative and object symbols as X such that $[X \supset X']$ and $[X' \supset X]$ are both valid, where X' is of 'normal form', viz.,

$$[q_1[q_2[q_3 \ldots [q_n Y] \ldots]]]$$

In this formula, the q_k denote quantifiers with respect to different dummy symbols, while no quantifiers are contained in Y. A statement which does not contain any quantifiers will be said to be of normal form, by definition (ref. 3, pp. 132—144).

Let K be a set of statements in a given language L. We shall say that a statement Y can be deduced from K (or, is a consequence of K), if there exists a finite sequence of statements in K,

$$X_1, X_2, \ldots X_n,$$

say, such that

$$[[[\ldots [X_1 \wedge X_2] \wedge X_3] \ldots \wedge X_n] \supset Y]$$

is a valid statement in L. The set of statements in L which can be deduced from K will be denoted by $S(K)$. Clearly, $S(K)$ includes $K, S(K) \supseteq K$. The subset of $S(K)$ which consists of statements which can be deduced from K, and which contain only object and relative symbols which appear in at least one statement of K will be denoted by $S'(K)$. Again $S'(K) \supseteq K$. Also, using 2.6.2. we can show that for all sets of statements K, $S(S(K)) = S(K)$ and $S'(S'(K)) = S'(K)$.

A set of statements K will be called contradictory if $S(K)$ includes all the statements of L (e.g., statements of the form $[X \wedge [\sim X]]$), otherwise K will be called consistent.

Two relative symbols A and B of the same order ($n \geqslant 0$) will be said to be co-extensive with respect to a set of statements K, if the statement

2.6.4. $$(u_1) \ldots (u_n) [[A(u_1, \ldots, u_n) \supset B(u_1, \ldots, u_n)] \wedge [B(u_1, \ldots, u_n) \supset A(u_1, \ldots, u_n)]]$$

can be deduced from K.

2.7. *Descriptive interpretation.* We now come to the descriptive, or semantic, interpretation of the statements of a given language, i.e., the branch of Metamathematics that deals with the relation between statements and the mathematical structures which they (may) describe.

A mathematical structure M which can be described by statements in the language defined in the preceding sections, is of the following type.

It consists of a set of objects (which like the object symbols will be denoted by small Roman letters $a, b, \ldots$) and of sets of relations of order n ($A(\ ), \ldots, B(,\ , \ldots), \ldots$) such that for every relation A of order n, defined in M, and for every ordered n-ple $a_1, \ldots, a_n$ of different or identical objects of M, the term $A(a_1, \ldots, a_n)$ either holds or does not hold. We do not ascribe any intensional meaning to the fact that A holds, or does not hold between the objects $a_1, \ldots, a_n$. We may imagine that this is simply specified by a function $T(A(a_1, \ldots, a_n))$ which takes one of the two values 1 ('holds') or 0 ('does not hold') as the argument varies over all the possible combinations of relations of order n and of n-ples in M. We do not identify the relations as such with ordered n-ples of objects in M, so that we may in fact have

$$T(A(a_1, \ldots, a_n)) = T(B(a_1, \ldots, a_n))$$

for all n-ples $a_1, \ldots, a_n$ in M, for two different relations A and B. In that case the two relations will be called co-extensive in M. We also accept the possibility of relations of order 0 being included in M, such that $T(A) = 1$ or $T(A) = 0$ (i.e., A holds or does not hold) in M without reference to the objects of M. Such relations do not in fact appear in the familiar mathematical structures which will be considered. They do appear as the elements of structures defined in connection with the calculus of propositions in which, on the other hand, ordinary relations ($n \geqslant 1$) and objects are not normally present.

Let C be a one-to-one correspondence under which there corresponds to every object of a structure M just one object symbol of a language L, while at the same time the relations in M correspond to relative symbols in L, such that every relation is paired with a relative symbol of the same order. Such a correspondence will be called 'complete'. Let $K(C)$ be the set of significant formulae in L

which contain only object symbols and relative symbols in C, although they may contain any number of dummy symbols. To every prime formula appearing in $K(C)$ which does not contain any dummy symbols there corresponds the expression of a relation between certain objects of M, which either holds or does not hold in M.

We shall say that a statement of $K(C)$ either holds or does not hold in M (under the correspondence C), according to the following rules.

2.7.1. A statement X of order 1 holds if and only if the expression in M which corresponds to the prime formula in L by whose bracketing X is obtained, holds in M.

Given two statements of $K(C)$, X and Y,

2.7.2. $[X \wedge Y]$ holds in M if and only if both X and Y hold in M.

2.7.3. $[X \vee Y]$ holds in M if and only if at least one of the two statements X or Y holds in M.

2.7.4. $[X \supset Y]$ holds in M if Y holds in it, and also if X does not hold in it.

2.7.5. $[\sim X]$ holds in M if and only if X does not hold in it.

Given any predicate $X(v)$ of $K(C)$,

2.7.6. $[(v)X(v)]$ holds in M if and only if $X(a)$ holds in M for all the object symbols a in L which correspond to objects in M.

2.7.7. $[(\exists v)X(v)]$ holds in M if and only if $X(a)$ holds in M for at least one object symbol a in L which corresponds to an object in M.

The above set of rules is consistent, i.e., its successive application cannot lead to the conclusion that some statement of $K(C)$ both holds and does not hold in M. This follows from the fact that each one of the rules 2.7.2.—2.7.7. bases the decision as to whether a statement does, or does not, hold on conditions regarding statements of lower order, while the decision as to whether a statement of order 1 holds or not depends directly on M (rule 2.7.1.). In this way the decision whether the statements of $K(C)$ hold can be made successively and unambiguously, although in the case of rules 2.7.6. and 2.7.7. the decision may depend on an infinite and even on a non-countable set of statements of lower order.

2.8. *Descriptive interpretation*, continued. Now let C be a one-to-one correspondence established between some of the relative symbols of a language L and the relations of a mathematical structure M, and between some of the object symbols of L and *some* (but not all) the objects of M. Such a correspondence will be said to be incomplete. We again denote by $K(C)$ the set of significant formulae which contain object symbols and relative symbols involved in C.

Let S be the set of objects in M which do not correspond to any object symbols under C, and conversely, let T be the set of object symbols in L which do not correspond to any objects under C. If the cardinal number of the set S is greater than the cardinal number of T, then we consider a language L' which, in addition to all the object symbols and relative symbols of L contains a set of object symbols T' such that the cardinal number of $T \cup T'$ is at least equal to the cardinal number of S. (In accordance with what was said in the introduction we do not question the existence of such a language for any given structure). On the other hand if the cardinal number of S' is not greater than the cardinal number of T, we simply put $L' = L$.

To define whether a statement of $K(C)$ holds or does not hold in M, under present conditions, we reduce this case to that considered earlier, in the following way.

We supplement C by a one-to-one correspondence between the elements of S and some of the elements of $T \cup T'$ (or of T in the case $L' = L$) and thus obtain a one-to-one correspondence C' between all the objects of M and some of the object symbols of L' and between the relations of M and some of the relative symbols of L'. This is a correspondence of the type considered in section 2.7, and by the rules of that section, we can decide whether any given statement of $K(C')$ holds or does not hold in M. But $K(C)$ is easily seen to be a subset of $K(C')$, and we now define that a statement of $K(C)$ will be said to hold, or not to hold in M under the correspondence C according as it holds or does not hold in M under the correspondence C'. Given C, this definition is independent of the particular choice of the language L' and of the extended correspondence C', subject to the conditions laid down above.

The set of statements of $K(C)$ that hold in M will be denoted by $K^*(C)$. Also, given a set of statements K in a language L, a mathematical structure M will be said to be a model of K if there is a correspondence C as above such that K belongs to $K^*(C)$.

The semantic interpretation of the restricted calculus of predicates is considered in detail by Carnap (refs. 6, 7) from a somewhat different point of view. Carnap restricts the discussion to languages in which the number of object symbols (in our terminology) is countable. However, it appears from the present argument that the decision whether a specific statement that begins with a quantifier holds in a non-countable structure M, must involve — though possibly only in passing — a language which contains an object symbol for every object of M, i.e., whose set of object symbols is itself non-countable.

In fact, consider one of the two rules 2.7.6. or 2.7.7., e.g., the former. The 'ordinary' interpretation of the statement $[(v)X(v)]$ is that X holds for all argument values. But the argument values of X must be elements of the language L, since $X(a)$ is defined only for object symbols a and not for objects a. Hence, in order to express the idea that X is universally true in M we must have at our disposal one object symbol in L for every object in M, as asserted. However, it is in keeping with the definition of a language given here that the objects and relations of M should themselves serve as object and relative symbols of a language L at the same time.

In particular, suppose that the objects and relations of a structure M are at the same time object and relative symbols in a language which includes the set of statements K. Then M may be called an internal model of K if it is a model of K under a correspondence under which the object and relative symbols of K coincide with the corresponding objects and relations in M.

Finally we note that if two relative symbols are co-extensive with respect to a set of statements K, then the two corresponding relations in any model M of K are co-extensive in that model.

2.9. *Infinite conjunctions and disjunctions.* For certain purposes, it becomes necessary to consider a language which is based on the same types of atomic symbols as the language(s) considered hitherto, but whose set of significant formulae is more extensive by virtue of the following rules which are additional to those of section 2.3. above.

Let $X_1, X_2, X_3 \ldots$ be an infinite (countable) sequence of (significant) formulae. Then

2.9.1. The expression denoted by $[\prod_{n=1} X_n]$ or alternatively by $[X_1 \wedge X_2 \wedge X_3 \wedge \ldots]$ is a formula.

2.9.2. The expression denoted by $[\sum_{n=1} X_n]$ or alternatively by $[X_1 \vee X_2 \vee X_3 \vee \ldots]$ is a formula.

The division of significant formulae into complete formulae (statements) and incomplete formulae (predicates) is retained. We shall not require a theory of deduction for the extended language in connection with our present purposes. The semantic interpretation of the statements of the extended language is based on a correspondence C with a mathematical structure M as described in section 2.7. above. To the rules of that section we now add

2.9.3. A statement of the form $[\prod_{n=1} X_n]$ holds in M if all the X_n hold in M, $n = 1, 2, \ldots$.

2.9.4. A statement of the form $[\sum_{n=1} X_n]$ holds in M if at least one X_n holds in M, $n = 1, 2, \ldots$.

Languages involving infinite conjunctions and disjunctions are considered by Carnap in the book quoted above (ref. 7).

When referring to 'a language L' in future we shall think of a language as considered in the preceding sections (2.1.—2.8.). The language of this last section will be referred to as 'an extended language'.

III

RELATIONS BETWEEN DEDUCTIVE AND DESCRIPTIVE CONCEPTS

3.1. *Compatibility of deductive and descriptive concepts.* In this chapter we shall be concerned with the relation between the deductive concepts developed in sections 2.5. and 2.6. and the descriptive, or semantic, concepts of sections 2.7. and 2.8.

We shall omit the proofs of the following theorem (see preface).

3.1.1. Let $K(C)$ be the set of statements defined in a language L with respect to a correspondence C with a model M. Then every statement of $K(C)$ which is valid in L holds in M.

3.1.2. *Theorem.* Given L, M, and C, as in the first theorem, it will now be proved that any statement of $K(C)$ that can be deduced from statements which hold in M, must itself hold in M.

In fact, let Y be a statement of $K(C)$ that can be deduced from a set of statements $X_1, X_2, \ldots, X_n$ which hold in M. The statement

$$[[[\ldots [X_1 \wedge X_2] \wedge X_3] \ldots \wedge X_n] \supset Y]$$

is then valid in L and therefore holds in M. Referring to 2.7.2. we also see that the statements

$$[X_1 \wedge X_2], [[X_1 \wedge X_2] \wedge X_3], \ldots, [[\ldots [X_1 \wedge X_2] \wedge X_3] \ldots \wedge X_n],$$

all hold in M. Hence, by 2.7.4., Y holds in M, as asserted

3.1.3. *Theorem.* The statements of a contradictory set K in a language L cannot all hold simultaneously in any structure M with respect to any correspondence C.

In fact, by assumption, we consider only languages which contain at least one statement e.g., X. Since K is contradictory, $[X \wedge [\sim X]]$ can be deduced from K, so that if all the statements of K hold in M, $[X \wedge [\sim X]]$ also holds in M, by theorem 3.1.2., above. But by 2.7.2., this can be the case only if X and $[\sim X]$ both hold in M. And this is impossible by 2.7.5.

3.2. *Completeness of the calculus of deduction.* The following theorems are complementary to the theorems of section 3.1.

3.2.1. *Theorem.* If a statement X of a language L holds in every structure for which it is defined — i.e., if $X \varepsilon K(C)$ entails $X \varepsilon K^*(C)$ for all M and C — then X is valid in L.

This theorem states that the theory of deduction given in section 2.3. is 'complete' according to one of the various meanings attributed to this word, i.e., it shows that a statement which is 'universally true' is valid in L.

3.2.2. *Theorem.* If a set of statements K and a statement Y in a language L are such that Y is defined and holds in all structures for which all the statements of K are defined and hold, then Y can be deduced from K.

3.2.3. *Theorem.* For every consistent set of statements K in a language L there exists a structure M such that all the statements of K hold in M, for some correspondence C.

Theorem 3.2.1. can be reduced to 3.2.3. First of all, assume that the statement X mentioned in theorem 3.2.1. is not valid in L. In that case we are going to show that $[\sim X]$ cannot be contradictory. In fact, if $[\sim X]$ — more correctly, the unit set containing $[\sim X]$ — were contradictory, then $[[\sim X] \supset Y]$ would be valid for all statements Y in L, hence $[[\sim X] \supset X]$ would be valid, hence $[[\sim [\sim X]] \vee X]$, $[X \vee X]$, and finally X would all be valid, according to the rules of the calculus of propositions, contrary to assumption. Again, if $[\sim X]$ is not contradictory, then by theorem 3.2.3. there is a structure in which $[\sim X]$ holds, and in which, therefore, X does not hold. This is contrary to the assumption of theorem 3.2.1., that X holds in all structures in which it is defined. It follows therefore that X must be valid in L, as required.

Theorem 3.2.2. can also be reduced to 3.2.3. Let $X_1, X_2, \ldots, X_n$ be any finite number of statements of K. If Y cannot be deduced from these statements, then

$$[[[\ldots [X_1 \wedge X_2] \wedge X_3] \ldots \wedge X_n] \supset Y]$$

cannot be valid in L. Hence we can show, similarly as before, that the conjunction

$$[[\ldots [X_1 \wedge X_2] \wedge X_3] \ldots \wedge [\sim Y]]$$

is not contradictory, and thence that the set $\{X_1, X_2, \ldots, X_n, [\sim Y]\}$ is not contradictory. But this means that if Y cannot be deduced from the statements of K, then the set K augmented by $[\sim Y]$ is consistent. By theorem 3.2.3., there therefore exists a structure in which all the statements of K and also $[\sim Y]$ hold, i.e., Y does not hold. This is contrary to the hypothesis of theorem 3.2.2. and it follows that Y can in fact be deduced from K, as asserted.

3.3. *Proof of the completeness theorem for special cases.* Theorem 3.2.3. of section 3.2. will first be proved for the particular case that K involves only relative symbols of order 0, besides copulae and square brackets. We than have to find a set of relations of order 0 in one-to-one correspondence with the relative symbols of K, and which either hold or do not hold in a structure M in such a way that the statements of K, interpreted according to the rules of section 2.7., all hold in M. To simplify matters, we may identify the relations in question with the relative symbols A of order 0 of K. We then assign values 'holds' ($T(A) = 1$) or 'does not hold' ($T(A) = 0$) to these relative symbols in such a way that the statements of K all hold. (The 'structure M' then consists simply of the set of relative symbols in association with the specified function $T(A)$). The fact that this can be done when K contains a single (and by assumption non-contradictory) statement again can be accepted as a well known result of the calculus of propositions (ref. 4). The case where K contains a finite number of statements can be reduced to that just mentioned by replacing the statements of K by their conjunction (taken in any arbitrary order). For the case that K is countable, the construction of the appropriate $K(A)$ was indicated by Gödel. The proof for general infinite K which will now be given includes the possibility that K is countable as a special case.

We require some of the definitions and results of the theory of abstract sets (refs. 8, 9). In particular, we recall that a section of a well-ordered set P is a well-ordered set constituted from all the elements preceding any specific element of P and ordered according to the same law as in P. We also recall that an initial ordinal number is the first ordinal number corresponding to any specific transfinite cardinal number. A well-ordered set of transfinite cardinal number

p has an initial ordinal number if and only if all its sections have cardinal numbers less than p. From this fact, we easily deduce the following lemma.

3.3.1. Let P be a well-ordered set of disjoint finite ordered sets Q_ν some of which may be empty. (There is a slight philosophical difficulty about the assumption that a set may contain more than one empty set: this however is easily removed, and there will be no need to go into details). Let Q be the union of the sets Q_ν ordered according to the following law. An element r of Q shall precede an element s of Q according as the Q_ν containing r precedes the Q_ν containing s in P. If r and s are both contained in the same Q_ν then they shall precede each other according to their order in that Q_ν. Q is readily seen to be well-ordered according to this law. Then the lemma states that the ordinal of Q is smaller than or equal to the ordinal of P provided the latter is an initial number.

If the cardinal q of Q is smaller than the cardinal p of P, then the lemma is obvious. Assume then that $q \geqslant p$. Now let Q' be any section of Q. Then it will be seen that the elements of Q' are all contained in the elements Q_ν of a section P' of P. But the cardinal number p' of P' is smaller than p. Also all the Q_ν are finite so that the cardinal number q' of Q' is smaller than or equal to $p'\aleph_0 = p'$. Hence all the sections of Q have a smaller cardinal than $p \leqslant q$, showing that Q is well-ordered according to the initial number of q, which therefore equals p.

3.4. *Proof of the completeness theorem for special cases,* continued. If theorem 3.2.3. is not true for all K involving only relative symbols of order 0, then there must be a K of smallest transfinite cardinal p for which it does not hold, though holding for all sets of smaller cardinal number. Select such a set K and assume that it has been ordered according to the initial ordinal of p, ψ, say. For every statement X_ν in K, $0 \leqslant \nu < \psi$, we define the finite ordered set Q_ν as the set of relative symbols which occur in X_ν, but not in any preceding element of K, the elements of Q_ν being taken in the order in which they occur for the first time in X_ν when that statement is read from left to right. Let P be the set of the Q_ν in the order of increasing ν, so that the ordinal number of P is ψ. The union of the sets Q_ν then constitutes the set Q of all the relative symbols involved in K. Let us order Q in accordance with the rule given in the lemma 3.3.1. (i.e. according to the order of the corresponding Q_ν or according to the order within a given Q_ν, as the case may be). According to 3.3.1., the ordinal number of Q, χ, is then smaller than or equal to the ordinal number of P, ψ. Let R be the ordered set of sections P'_ν of P, $0 \leqslant \nu < \psi$. The ordinal of R is again ψ so that its cardinal is p.

Denote by A_ν the elements of Q, i.e., the relative symbols involved in K. We are going to show that we can define the function $T(A)$ with functional values 1 (A_ν holds) and 0 (A_ν does not hold) for all the elements of Q in such a way that the statements of K all hold. This is contrary to the assumption that the theorem does not hold for the set K in question and therefore proves that the theorem holds for all sets which involve only relative symbols of order 0.

For any section K'_ν of K, we denote by Q'_ν the set of relative symbols which are involved in K'_ν. Clearly, $Q'_\nu \subseteq Q'_\mu$ for $\nu < \mu$ and $Q'_\nu \subseteq Q$ for all ν ($\nu < \psi$). The definition of a function $T(A)$ with functional values 1 or 0 for the A of a subset of Q will be called a valuation of that subset. A valuation of a set Q'_ν will be called admissible if all the statements of K'_ν hold with respect to A. Since all the K'_ν are consistent and of cardinal smaller than p, it follows that there exists admissible valuations for all the Q'_ν, $0 < \nu < \psi$. Given an admissible valuation for a set Q', the valuation yielded by it for the elements of any Q'_μ, $\mu < \nu$, clearly is admissible as well.

We are going to show —

3.4.1. Either A_0 holds in at least one admissible valuation of every Q'_ν, $0 < \nu < \psi$, or A_0 does not hold in at least one admissible valuation of every Q', or both.

Assume that A_0 does not hold in any admissible valuation of some specific Q'_μ. Then it cannot hold in any admissible valuation of any Q'_ν, $\nu > \mu$, since this would yield an admissible valuation for Q'_μ in which A_0 holds. Hence A_0 does not hold in any admissible valuation of Q'_ν, $\nu \geqslant \mu$. But every admissible valuation of Q'_μ yields admissible valuations for all Q'_ν, $\nu < \mu$, in which A_0 does not hold. It follows that A_0 does not hold in at least one admissible valuation of every Q'_ν, $C < \nu < \psi$, proving the assertion.

We now define a valuation of the set Q by (finite or transfinite) induction. We define $T(A_0) = 1$ if A_0 holds in at least one admissible valuation of every Q'_ν, $1 \leqslant \nu < \psi$, otherwise $T(A_0) = 0$. In that case, as shown above, A_0 does not hold in at least one admissible valuation of every Q'_ν.

We denote by S_ν the sections of Q, so that S_0 is empty, S_1 contains A_0 only, etc. Assume that we have already defined the function T for all the A_ν of a section S_μ, $1 \leqslant \mu < \psi$, is such a way that this valuation of S_μ agrees with at least one admissible valuation of every Q'_σ, $\sigma < \psi$ which includes S_μ, with regard to the elements of that section (i.e. $T(A_\nu)$ is the same for the two valuations — of S_μ and of Q'_σ — for all $A_\nu \varepsilon S_\mu$). This condition is satisfied by the valuation of S_1, as defined above. We may then show similarly as in 3.4.1. above, that either the definition $T(A_\mu) = 1$, together with the given valuation of S_μ, agrees with at least one admissible valuation of every Q'_σ, $\sigma < \psi$, or the same applies to $T(A_\mu) = 0$, (or both). In the first case we define $T(A_\mu) = 1$, in the alternative case $T(A_\mu) = 0$. However, in order to prove that this procedure of induction can be carried on indefinitely, we still have to show that for every μ, the valuation of S_μ found in this way does in fact agree with at least one admissible valuation of every Q'_σ as explained above. As mentioned, the condition is satisfied for S_1. Assume that there exist sections of Q for which it is not satisfied, and let S_μ be the first section of that description. The ordinal μ cannot be of the first kind: for it follows directly from our construction that if the condition is satisfied for any given section, it is also satisfied for the subsequent section. Assume then that μ is a number of the second kind (limit number).

Let S' be the well-ordered sequence of statements obtained from S_μ by replacing every element A_ν of S_μ by $[A_\nu]$ if $T(A_\nu) = 1$ in the given valuation, and by $[\sim [A_\nu]]$ if $T(A_\nu) = 0$ in that valuation. We now consider the set of well-ordered sets K''_ν, $0 \leqslant \nu < \psi$, where $K''_\nu = S' + K'_\nu$. The cardinal number of every K'_ν is smaller than p and the cardinal number of S' (which is the cardinal number of S_μ, which is a section of Q) is also smaller than p, hence the cardinal numbers of all the K''_ν are smaller than p. Next, we are going to prove that the sets K''_ν are all consistent. For this purpose, we only have to show that every finite subset, e.g. S'', of any particular $K''_\nu = S' + K'_\nu$ is consistent. Since the number of statements contained in S'' is finite, while on the other hand μ is a limit number, it follows that there is a section S_σ, $\sigma < \mu$ which contains all the relative symbols involved in the statements of S''. Now the valuation of S_σ, arrived at in the process of our definition by induction, is, by assumption, part of an admissible valuation for K'_ν (all the statements of K'_ν hold with respect to it). It also forms part of the given valuation of S_μ, with respect to which the statements of S' clearly hold, by construction. It follows that all the statements of S'' hold for the given valuation of S_σ and hence, by theorem 3.1.3., that S'' is consistent. Hence also, K''_ν is consistent and since its cardinal number is smaller than p, there exists a valuation V' of the relative symbols involved in K''_ν with respect to which all the statements of K''_ν hold. But since S' forms part of K'', clearly V' must agree with the original valuation so far as the elements of S_μ are concerned. Since this is true for arbitrary ν, the given valuation of S_μ forms part of at least one admissible valuation for every K'_ν i.e., of every Q'_ν. But this is precisely the condition which according to our original assertion is true for all S_μ. The assertion is therefore proved, and the definition of a valuation of Q can be carried on indefinitely, finally leading to a value 0 or 1 of $T(A_\nu)$ for every A_ν of Q.

Every statement X of K holds with respect to this valuation since it must be contained in a section K'_ν of K, and the valuation of Q is defined in such a way that all the statements of every section of K, e.g. of K'_ν, hold with respect to it.

3.5. *Proof of the completeness theorem for special cases*, continued. Next, we shall prove theorem 3.2.3. for sets K the statements of which involve relative symbols of arbitrary order, provided only that they do not contain any dummy symbols. Similarly as before, we shall use the relative symbols and object symbols of the statements of K as the relations and objects of a structure in which K holds. Let Q be the set of prime formulae involved in K. We may assume that in addition to any relative symbols of order 0 which are contained in Q, the language L of K contains a set of relative symbols of order 0 whose cardinal number is at least equal to Q. (Otherwise we extend L as in section 2.8. above). We may therefore find a set Q' of relative symbols of order 0 which are in one-to-one correspondence with the prime formulae of Q, such that Q and Q' have no elements in common. (Two prime formulae of Q differ if they contain different relative symbols or if, for the same relative symbol, they contain different object symbols at corresponding places). Let K' be the set of statements obtained by replacing all the prime formulae of a statement of K by the corresponding relative symbols of Q'. K' cannot be contradictory if K is consistent. For let Y' be any statement of the form $[[A'] \wedge [\sim[A']]]$ where A' is a specific element of Q'. Then if K' is contradictory, it contains a finite set of statements $X'_1, \ldots, X'_n$ such that the statement

$$[[[\ldots[X'_1 \wedge X'_2] \wedge X'_3] \ldots \wedge X'_n] \supset Y']$$

is valid. Let $X_1, X_2, \ldots, X_n$ be the corresponding statements of K while Y is the statement $[[A] \wedge [\sim[A]]]$ where A is the prime formulae of Q which corresponds to A'. Then by the rule of substitution 2.6.1., we obtain a valid statement by replacing $X'_1, X'_2, \ldots, X'_n$, and Y' by $X_1, X_2, \ldots, X_n$, and Y respectively.

$$[[[\ldots [X_1 \wedge X_2] \wedge X_3] \ldots \wedge X_n] \supset Y]$$

It follows that Y can be deduced from K. But Y is of the form $[[A] \wedge [\sim[A]]]$ and $[[X \wedge [\sim X]] \supset Z]$ is known to be a valid formula for arbitrary X and Z. This signifies that K is contradictory, contrary to assumption.

Since K' is consistent, there exists, by the result of action 3.4. a valuation of Q' according to which all the statements of K' hold. We now take the relative symbols and object symbols of Q as the relations and objects of a model M, and we define that a relation of order n, A, shall hold between objects $a_1, \ldots, a_n$ of M, if and only if $A(a_1, \ldots, a_n)$ appears in Q, such that the corresponding relative symbol A' in Q' has the valuation $T(A') = 1$.

The statements of K hold in M under the correspondence C which makes each element correspond to itself. In fact, the decision whether a statement X of K holds in M does not depend on the character of the prime formulae of X, but only on the question whether the corresponding relations hold in M. And these relations hold whenever the corresponding relative symbols of X' hold, where X' is the statement of K' corresponding to X. But X' is known to hold for the given valuation, and it follows therefore that X also holds, as asserted.

3.6. *Proof of the completeness theorem for the general case.* To prove theorem 3.2.3. for a general consistent set of statements K, which involves dummy symbols as well as relative symbols and object symbols, we shall again take the relative symbols of K as the relations of a structure M in which the statements of K hold, while the object symbols of K, if any, will form a subset of the set of objects of M.

We first reduce K to a set K' by replacing each one of its statements X by a corresponding statement of normal form X' (see 2.6.3.), wherever X is not already of that form. Since $[X \supset X']$ is valid for all X, it follows from 2.6.2. that K', like K, is consistent. Also, since $[X' \supset X]$ is valid, it follows (3.1.1.) that it holds in every structure and hence that whenever X' holds in any given structure, so does X (and by the same correspondence). Thus, if there exists a structure in which all the statements of K' hold then all the statements of K hold in the same structure.

It follows that in order to prove our theorem we may confine ourselves from the outset to a set K in which all the statements are of normal form. The 'order of quantifiers' in a given statement of normal form will be taken to be that in which the quantifiers appear in the statement when read from left to right. We shall define a procedure which will be divided into a countable number of stages, where each stage, except the first, consists of a countable number of steps. To carry out this procedure, we shall assume provisionally that we have an indefinite number of object symbols at our disposal within the language L of K. It will be shown later than a set of object symbols of a certain well-defined cardinal number (depending on K) will be adequate for our purpose.

The first stage consists of the definition of a set of objects, P_1, and a set of statements K_1. P_1 will be taken as the set of object symbols involved in K, if any: if K does not involve any object symbols, then we select an arbitrary object symbol of L as the only element of P_1. K_1 is simply defined by $K_1 = K$.

Having gone through the first $(n-1)$ stages of the procedure, leading to the definition of sets of objects (which are also object symbols in L), $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_{n-1}$ and of sets of statements $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_{n-1}$, $n \geqslant 2$, we proceed to carry out the steps of the nth stage.

The first step consists of the following operation applied simultaneously to all the statements X of K_{n-1}. If X does not contain any quantifiers, we leave it unchanged. If X contains quantifiers of which the first is an existential quantifier, then we replace X by another statement X' which is obtained from X by omitting the quantifier in question, and by replacing the corresponding dummy symbol wherever it occurs elsewhere in X by an object symbol which is not contained in P_{n-1}. We also ensure that the object symbols introduced in this way differ from one another for different statements X of K_{n-1}.

On the other hand, if the first quantifier of a statement X of K_{n-1} is a universal quantifier, then we replace X by the statements X' determined by omitting the quantifier and replacing the corresponding dummy symbol wherever it occurs elsewhere in X_1 in turn by all the elements of P_{n-1}. In this way, we obtain from K_{n-1} a set of statements $K_1^{(n)}$, such that to every statement X in K which involves quantifiers, there correspond one or more statements in $K_1^{(n)}$ which involve one quantifier less. The statements of $K_1^{(n)}$ are of normal form, and the order of the quantifiers in any given statement of $K_1^{(n)}$ is the same as that in the corresponding statement of K_1 except for the first quantifier of the original statement which is now missing.

We define $P_1^{(n)}$ as the set obtained from P_{n-1} by the inclusion of the object symbols which were introduced during the first step.

The second step is exactly similar to the first step, except that K_{n-1} is replaced by $K_1^{(n)}$ and P_{n-1} by $P_1^{(n)}$: we then obtain another set of statements $K_2^{(n)}$ and a set of object symbols (or objects), $P_2^{(n)}$, such that $P_2^{(n)} \supseteq P_1^{(n)}$.

In this way, we may continue indefinitely. However, if the number of quantifiers in the statements of K_{n-1} is uniformly bounded by a positive integer m, then $K_m^{(n)} = K_{m+1}^{(n)}$, and $P_m^{(n)} = P_{m+1}^{(n)}$, so that in that case the procedure is effectively finite.

Having obtained the sequences of sets of object symbols $P_1^{(n)} \subseteq P_2^{(n)} \subseteq P_3^{(n)} \subseteq \ldots$ and of sets of statements $K_1^{(n)}, K_2^{(n)}, K_3^{(n)} \ldots$ we then define P_n as the union of the sets $P_m^{(n)}$ and K_n as the union of the sets $K_m^{(n)}$. This completes the nth stage.

3.7. *Proof of the completeness theorem for the general case*, contd. At every stage we therefore obtain a set of object symbols P_n and a set of statements K_n. Let P' be the union of the sets P_n, $n = 1, 2, \ldots$ and K' the union of the sets K_n, $n = 1, 2, \ldots$. We propose to show that K' is consistent.

Let Y be a statement of the form $[W \wedge [\sim W]]$, where W does not involve any object symbols contained in K'. If K' is contradictory then it includes a finite set of different statements, $X'_1, X'_2, \ldots X'_l$ such that

3.7.1. $$[[[\ldots [X'_1 \wedge X'_2] \wedge X'_3] \ldots \wedge X'_l] \supset Y]$$

is valid.

Taking any particular X'_m of this set, we can find a set $K_n = K_{n(m)}$ in which it is contained. Hence also, we can find a set $K_p^{(n)} = K_{p(m)}^{(n(m))}$ in which X'_m is contained, unless X'_m is already an element of $K_1 = K$. We assume that we have chosen $n = n(m)$ as small as possible for every given m and, having determined n, that we have chosen $p = p(m)$ as small as possible, if $n > 1$. For the purposes of the present section only, we define the degree of the finite set $\{X'_1, X'_2, \ldots X'_l\}$ by $d = \sum_{m=1}^{l} n(m)$. We have $d \geqslant l$, and we shall show that if $d > l$ then we can deduce Y from another set of statements of K', whose degree is smaller than d. If $d = l$, then all the X'_m belong to $K_1 = K$, so that K would be contradictory contrary to assumption. Assume then $d > l$. Let X'_k be a statement of the given set, such that $n = n(k)$ is not exceeded by any other $n = n(m)$ of the set. Amongst the statements which satisfy this condition, we select one for which $p = p(k)$ is a maximum. (There may be more than one such statement, and subject to the above conditions, the choice is arbitrary). By the rules of the calculus of propositions, 3.7.1. is valid if and only if

3.7.2. $$[X'_k \supset [[\ldots [X'_1 \wedge X'_2] \ldots \wedge X'_l] \supset Y]]$$

where X'_k has been omitted from the conjunction.

X'_k has been obtained at the $p(k)$th step of the $n(k)$th stage from another statement X^*_k by one of the operations described in section 3.5., above. Assume first that the first quantifier of X^*_k is a universal quantifier, then X'_k is obtained from X^*_k by omitting the quantifier and replacing the corresponding dummy variable by an object symbol of $P^{(n)}_{p-1}$, if $p > 1$, or of P_{n-1} if $p = 1$. It then follows from the first statement of quantification (2.5.6.) that $[X^*_k \supset X'_k]$ is a valid statement, and hence that the replacement of X'_k by X^*_k in 3.7.2. produces a valid statement

3.7.3. $$[X^*_k \supset [[\ldots[X'_1 \wedge X'_2] \ldots \wedge X'_l] \supset Y]]$$

that is to say, Y can be deduced from the set of statements $X'_1, X'_2 \ldots, X^*_k, \ldots X'_l$.

On the other hand, if the first quantifier of X^*_k is an existential quantifier, then X'_k is obtained from it by omitting the quantifier and replacing the corresponding dummy symbol by an object symbol which is not contained in either $X'_1, X'_2, \ldots X'_{k-1}, X'_{k+1} \ldots X'_l$ or Y. Hence, by the third rule of inference (2.5.8.), the validity of 3.7.3. can again be deduced from the validity of 3.7.2. Now X_k may be identical with one of the remaining X'_m, in that case Y can be deduced from the set $\{X'_1, X'_2, \ldots X'_{k-1}, X'_{k+1} \ldots X'_l\}$ above, and the degree of that set is certainly smaller than d. If this is not the case, then we distinguish the following possibilities. Either $p(k) = 1$, and in that case X^*_k belongs to K_{n-1} and therefore to some $K^{(n-1)}_q$, so that the degree of $\{X'_1, X'_2, \ldots, X^*_k, \ldots X'_l\}$ is again smaller than d, or $p(k) > 1$, so that X^*_k belongs to $K^{(n)}_{p-1}$. In that case we repeat the operation of selecting a statement corresponding to maximum n and p— this time from the set $[X'_1, X'_2, \ldots, X^*_k, \ldots X'_l\}$. It will be seen that after a finite number of operations the degree of the resulting set of statements will be necessarily reduced by 1. (The case that X^*_k is identical with X'_k cannot arise, because in that case X'_k would have belonged to lower p or n than specified).

After a finite number of such reductions of degree (i.e., not more than d—1) we finally obtain a set of statements whose degree equals their number, so that all the statements belong to $K_1 = K$. This again implies that K is contradictory, contrary to assumption. We have therefore established the fact that K' is consistent, as asserted.

Now let K^* be the set of statements of K' which are characterised by the property that they do not contain any dummy symbols. K^* is consistent because it is a subset of the consistent K'. By the result of section 3.5., there therefore exists a structure M, consisting of the object symbols of K' as objects, and of the relative symbols of K' as relations, in which all the statements of K^* hold. We shall show that, moreover, all the statements of K' hold in M.

The proof is by induction according to the number of quantifiers contained in a statement. If a statement of K' does not contain any quantifiers then it is included in K^* and therefore holds in M, by assumption. Assume then that we have already proved that all the statements of K^* which contain no more than n quantifiers hold in M. Let X be a statement of K' which contains $(n + 1)$ quantifiers (if there are such statements in K'). Assume that X is of the form $[(\exists v)Y(v)]$. Then, by the construction of section 3.6., K' also contains the statement $Y(a)$ for some object symbol a (corresponding to an object in M, i.e. to itself). $Y(a)$ contains only n quantifiers and therefore holds in M. Hence, by rule 2.7.7., X also holds in M. Again, assume that X is of the form $[(v)Y(v)]$. To prove that this statement holds in M, we have to show, by rule 2.7.6. that $Y(a)$ holds in M for all the object symbols of P'. But since $Y(a)$ contains only n quantifiers it is sufficient for this purpose to show that $Y(a)$ is contained in K' for such general a. Since X belongs to K', it also belongs to some specific K_n; and since a belongs to P', it also belongs to some specific P_m. It then follows from the construction of section 3.6, that $Y(a)$ belongs to K_p where $p = \max(m, n) + 1$, and therefore belongs to K'. Thus, in this case also, X holds in M.

We have shown that all the statements of K' hold in M (by the correspondence by which every object and relative symbol of K' corresponds to itself). But K is a subset of K', so that all the statements of K hold in M. This completes the proof of theorem 3.2.3.

To estimate the number of object symbols required in L, in addition to the object symbols which are contained in K, let k be a transfinite cardinal number such that the joint number of object and relative symbols in K is not greater than k. Then the cardinal number of P_1 is not greater than k, and the same applies to the cardinal number of $K_1 = K$, as can be shown without difficulty. Now at every step of the procedure of section 3.6., if we start from sets $P_m^{(n)}$ and $K_m^{(n)}$ (or P_{n-1} and K_{n-1} for the first step of a stage) whose cardinals are not greater than k, then the cardinals of $P_{m+1}^{(n)}$ and of $K_{m+1}^{(n)}$ (or of $P_1^{(n)}$ and $K_1^{(n)}$) also will not be greater than k. In fact, at any given step we add not more than one element to $P_m^{(n)}$ for each statement of $K_m^{(n)}$ to obtain $P_{m+1}^{(n)}$, showing that the cardinal of $P^{(n)}_{m+1}$ is not greater than $k + k = k$. Also $K^{(n)}_{m+1}$ is obtained by replacing every statement of $K^{(n)}_m$ either by one other statement, or by as many statements as there are object symbols in $P^{(n)}_m$. It follows that there are not more than $\aleph_0 k = k$ statements in $K^{(n)}_{m+1}$. Applying this argument first to the steps of the second stage, beginning with P_1 and K_1 we find that the sets $P^{(2)}_m$ and $K^{(2)}_m$, $m = 1, 2, \ldots$ all have cardinals not greater than k. It follows that the sets P_2 and K_2 also have cardinals not greater than $k + k + k + \ldots$ ($\aleph_0$ times) $= k$. We then conclude in a similar way that the sets P_3 and K_3 have cardinals not greater than k, etc. Finally, we conclude that the sets P' and K' have cardinal numbers not greater than k, which shows that it suffices to assume that L contains k object symbols, in addition to the object symbols of K. We observe however that the selection of object symbols may have to be done in an appropriate manner to avoid exhausting the set of object symbols of L after a finite number of steps or stages.

Since the set of objects in M is a subset of the set of object symbols in L, we have shown that if K involves a finite number of object and relative symbols then it holds in a structure containing a finite or countable number of objects. This is the theorem of Löwenheim and Skolem (compare ref. 10); while, if K involves a transfinite number of object and relative symbols, then it holds in a structure which contains not more objects than there are object and relative symbols (taken jointly) in K. The cardinal number of objects in M may in fact necessarily exceed the cardinal number of object symbols in K, even if the latter is transfinite, if it is exceeded by the number of relative symbols in K.

Theorem 3.2.3. was first proved by Gödel (ref. 11) for finite K. He also stated the theorem for countable K but it appears that his proof applies only when the number of object symbols ('free individual variables', in Gödel's terminology) is finite. The two intermediate stages of our proof were suggested by Gödel's very remarkable paper, although the actual procedures employed for the successive reductions are different.

3.8. *Systems of statements.* A set of statements K within a language L is called a 'system' if it includes all the statements which can be deduced from it, in symbols, if $S(K) = K$. The set of valid statements in L is a system. A system is called maximal, or complete, if it is consistent, and if the only system in which it is included is the set of all statements of L.

An abstract theory of systems has been developed by Tarski (refs. 12, 13) to whom the concept is due.

It is interesting to note that there is, under conditions of wide generality, a certain dual relation between systems of statements and sets of mathematical structures in which they are defined and hold. For the purposes of this section we shall refer to mathematical structures simply as 'models'.

Let K be the set of all statements of a language L on one hand, and let M be the set of all models definable in a class N consisting of a definite set of objects and of definite sets of relations of order n, $n = 0, 1, 2, \ldots$. We shall assume that there are just as many relations of order n in N as there are relative symbols of the same order in L, while there are at least as many objects in N as there are object symbols in L; we shall also assume that a certain definite one-to-one correspondence has been established between the respective classes of relations of N and of relative symbols of L, and between some of the objects of N and all the object symbols of L. Moreover, we shall suppose that the cardinal number of the objects of N which do not correspond to any object symbols under C is transfinite, and that it is not smaller than the total number of object and relative symbols in L. It then follows from the preceding sections that a model can be found within N for every consistent set of statements in L.

A statement X can be deduced from a set of statements K' if and only if it holds in all the models in which K' holds. In fact, if X can be deduced from K' then it holds in all the models in which K' holds, by 3.1.2. On the contrary, if X cannot be deduced from K', then the set K' augmented by $\sim X$ is consistent and therefore holds in a model $P \varepsilon M$. This means that K' holds in P but not X, and shows that the condition is sufficient. It follows that for any set of statements K', $S(K')$ is the set of statements which hold in all the models in which K' holds.

A model $P \varepsilon M$ will be said to be associated with a set of models $M' \subseteq M$, if every statement $X \varepsilon K$ which holds in all the models of M' also holds in P. Given any set of models $M' \subseteq M$, we denote by $S(M')$ the set of models associated with it. We have $S(M') \supseteq M'$. Also if $M' \subseteq M''$ then $S(M') \subseteq S(M'')$ just as for sets of statements $K' \subseteq K''$ implies $S(K') \subseteq S(K'')$. A 'system of models' is a set $M' \subseteq M$ which satisfies $S(M') = M'$.

The set of statements which hold in all the models of a set $M' \subseteq M$ is a system of statements, by 3.1.2. Similarly, the set M' of all models in which all the statements of a set $K' \subseteq K$ hold is a system of models. In fact let P be a model which is associated with M'. All the statements which hold in all the models of M' also hold in P, so that in particular all the statements of K' hold in P. Hence P belongs to M' showing that $S(M') = M'$, so that M' is a system of statements, as required. The fact that a system of statements K' is the set of all statements which hold in a set of models M' will be indicated by $M' \rightarrow K'$. On the other hand, the fact that a system of models M' is the set of all models in which all the statements of a set K' hold, will be indicated by $M' \leftarrow K'$. It will be seen that if $M' \subseteq M''$ and $M' \rightarrow K'$, while $M'' \rightarrow K''$, then $K' \supseteq K''$. Similarly, if $K' \subseteq K''$ and $M' \leftarrow K'$ while $M'' \leftarrow K''$, then $M' \supseteq M''$.

If M' is a system of models, then $M' \rightarrow K'$ entails $M' \leftarrow K'$. To prove this, let $M'' \subseteq M$ be the set of all models in which all the statements of K' hold. We have to show $M' = M''$. Let P'' be an arbitrary element of M'', then all the statements of K' hold in P''. That is to say, all the statements that hold in all the elements of M' hold in P''. Hence $P'' \varepsilon S(M')$ and so, since M' is a system of models, $P'' \varepsilon M' = S(M')$. This shows that M'' is a subset of M'. Again, let P' be an arbitrary element of M'. Then all the statements of K' hold in P' showing that P' also belongs to M''. Thus $M' \subseteq M''$, which in conjunction with $M'' \subseteq M'$ yields $M' = M''$ as required.

There therefore exists a one-to-one correspondence between the systems of models in N and the systems of statements in L, to be denoted by $M' \leftrightarrow K'$, such that K' consists of all the statements which hold in all the models of M, $M' \rightarrow K'$, and M' consists of all the models in which all the statements of K' hold, $M' \leftarrow K'$. The system of models corresponding to the system of valid statements in L, is M, the system of all models. The system of models corresponding to the system of all statements in L is the empty set. Finally, if $M' \leftrightarrow K'$, $M'' \leftrightarrow K''$, and $M' \subseteq M''$, then $K' \supseteq K''$.

At this point, we abandon the development of the dual relationship between systems of statements and systems of models, which, however, can be carried a good deal further still.

3.9. *Lindenbaum's theorem.* A theorem due to Lindenbaum (ref. 13) states that every consistent set of statements K in a language L is included in a complete set, K^*. To prove this theorem, let K' be the set of all statements in L not included in K, and assume that K' has been well ordered according to an ordinal number μ, so that the sequence begins with $X_1, X_2, X_3, \ldots$, etc.

Having put $K_0 = K$, we define K_1 as $K_0 + \{X_1\}$ if X_1 is consistent with K_0, and as K_0 if X_1 is not consistent with K_0. In general, we define K_ν as the union of $\{X_\nu\}$ and of all K_λ, $\lambda < \nu$, if X_ν is consistent with all K_λ, $\lambda < \nu$, and as the union of all K_λ, $\lambda < \nu$, if there is a K_λ, $\lambda < \nu$ with which X_ν is not consistent. In this way we may define a K_ν for every X_ν (i.e., $\nu < \mu$ or $\nu \leqslant \mu$, as the case may be). Finally we define K^* as the union of all K_ν. We are going to show that K^* is consistent and complete. For if K^* were contradictory, then it would contain a finite number of statements $X_{\nu_1}, \ldots, X_{\nu_k}$, $k \geqslant 1$, taken in the given order within K', which are inconsistent with $K = K_0$. We take the set $\{X_{\nu_1}, \ldots, X_{\nu_k}\}$ as a minimal set, so that if we remove X_{ν_k}, the set becomes consistent with K. But X_{ν_k} was included in K_{ν_k} on the assumption that it was consistent with all K_λ, $\lambda < \nu_k$ e.g. with $K_{\nu_{k-1}}$ which includes K together with $\{X_{\nu_1}, \ldots, X_{\nu_{k-1}}\}$ (or with $K = K_0$, if $k = 1$). This shows that K^* is consistent.

K^* is complete. For assume on the contrary that there exists a statement X_ν which is consistent with K^* but is not contained in it. But in that case X_ν is consistent with all K_λ, $\lambda < \nu$, and so X_ν is included in K_ν, by construction. This shows that K^* is actually complete.

IV

SPECIFICATION OF AXIOMATIC SYSTEMS

4.1. *Axioms for groups, rings, and fields.* We now turn to the application of metamathematical theory to concrete algebraic systems. It will be convenient to use accepted terms such as 'axiom' and 'theorem', but it is understood that (except where we refer to metamathematical theorems) these terms are synonymous with the word 'statement' which was defined earlier.

Comparatively little use will be made of the rules of deductive theory as developed in sections 2.5. and 2.6. above. In fact, for our present purpose the chief importance of deductive theory within the restricted calculus of predicates is that it provides a method by which any given 'theorem' which is entailed by a set of 'axioms' in the sense that it holds whenever all the axioms of the set hold, can be deduced formally from a finite number of these axioms. The detailed procedure is less important: instead, most of our argument will be 'intensional' (semantic), i.e., based on the interpretation of any given set of statements given in sections 2.7. and 2.8.

We begin with the specification of system sof axioms for some of the most important concepts of Algebra, viz., groups, rings, and fields. (The word 'system' is used here in its wider sense as being roughly synonymous with 'set' or 'aggregate', not in the specific sense attributed to it by Tarski, see section 3.8. above.) We formulate these concepts within a language L which contains at least two relative symbols, $S(\ ,\ ,)$ and $E(,)$, (of orders three and two respectively) for groups, at least three relative symbols, $S(\ ,\ ,)$, $P(\ ,\ ,)$ and $E(,)$ for rings and fields, and at least four relative symbols, $S(\ ,\ ,)$, $P(\ ,\ ,)$, $E(,)$ and $Q(,)$ for ordered fields.

The axioms for the various algebraic concepts will be selected from the following set. We use the simplified notation described in section 2.4.

Axioms of equality

4.1.1. $(x)[E(x, x)]$
4.1.2. $(x)(y)[E(x, y) \supset E(y, x)]$
4.1.3. $(x)(y)(z)[E(x, y) \wedge E(y, z) \supset E(x, z)]$
4.1.4. $(x)(y)(z)(u)(v)(w)[S(x, y, z) \wedge [E(x, u) \wedge [E(y, v) \wedge E(z, w)]] \supset S(u, v, w)]$
4.1.5. $(x)(y)(z)(u)(v)(w) [P(x,y,z) \wedge [E(x, u) \wedge [E(y, v) \wedge E(z, w)]] \supset P(u,v,w)]$
4.1.6. $(x)(y)(z)(u)[Q(x, y) \wedge [E(x, z) \wedge E(y, u)] \supset Q(z, u)]$

The first three axioms express the properties of reflexivity, of symmetry, and of transitivity of E; the remaining three axioms ensure, briefly, the substitutivity of objects which satisfy (the relation corresponding to) E, in S, P and Q respectively.

Axioms of existence.

4.1.7. $(\exists x)[E(x, x)]$
4.1.8. $(x)(y)(\exists z)[S(x, y, z)]$
4.1.9. $(x)(y)(\exists z)[P(x, y, z)]$
4.1.10. $(x)(y)(\exists z)[S(x, z, y)]$
4.1.11. $(x)(y)(\exists z)[S(x, x, x) \vee P(x, z, y)]$

Axiom 4.1.7. ensures the existence of at least one object, while axioms 4.1.8. — 4.1.11. postulate the existence of specific objects which are in stated relations with certain given objects. 4.1.7. can be deduced from 4.1.1., by 2.5.5. and the rules of the calculus of propositions.

Axioms of arithmetic (or, of identification)

4.1.12. $(x)(y)(z)(u)[S(x, y, z) \wedge S(x, y, u) \supset E(z, u)]$
4.1.13. $(x)(y)(z)(u)[P(x, y, z) \wedge P(x, y, u) \supset E(z, u)]$
4.1.14. $(x)(y)(z)(u)[S(x, y, z) \wedge S(y, x, u) \supset E(z, u)]$
4.1.15. $(x)(y)(z)(u)[P(x, y, z) \wedge P(y, x, u) \supset E(z, u)]$
4.1.16. $(x)(y)(z)(t)(u)(v)(w)[[S(x, y, z) \wedge S(z, t, u)] \wedge [S(y, t, v) \wedge S(x, v, w)]] \supset E(u,w)]$
4.1.17. $(x)(y)(z)(t)(u)(v)(w)[[P(x, y, z) \wedge P(z, t, u)] \wedge [P(y, t, v) \wedge P(x, v, w)]] \supset E(u, w)]$
4.1.18. $(x)(y)(z)(s)(t)(u)(v)(w)[[[P(x, y, z) \wedge [P(x, s, t) \wedge [S(z, t, u) \wedge [S(y, s, v) \wedge P(x, v, w)]]]] \supset E(u, w)] \wedge [[P(y, x, z) \wedge [P(s, x, t) \wedge [S(z, t, u) \wedge [S(y,s,v) \wedge P(v,x,w)]]]] \supset E(u, w)]]$

Axioms 4.1.12. and 4.1.13. state that the objects whose existence is ensured by 4.1.8. and 4.1.9. are unique. The remaining axioms are; the commutative laws for S — (4.1.14) — P — (4.1.15) — the associative laws for S — (4.1.16) — and P — (4.1.17) —, and the distributive law — (4.1.18).

Axioms of order

4.1.19. $(x)(y)[E(x, y) \vee [Q(x, y) \vee Q(y, x)]]$

4.1.20. $(x)(y)[Q(x, y) \supset \sim Q(y, x) \wedge \sim E(x, y)]$

4.1.21. $(x)(y)(z)[Q(x, y) \wedge Q(y, z) \supset Q(x, z)]$

4.1.22. $(x)(y)(z)(u)(v)(w)[[Q(x, y) \wedge Q(z, u)] \wedge [S(x, z, v) \wedge S(y, u, w)] \supset Q(v, w)]$

4.1.23. $(x)(y)(z)(v)(w)[[[Q(x, y) \wedge Q(z, x)] \wedge S(z, z, z)] \wedge [P(x, z, v) \wedge P(y, z, w)] \supset Q(v, w)]$

Axioms 4.1.19. and 4.1.20. state that just one of the relations $E(x, y)$ or $Q(x, y)$ or $Q(y, x)$ holds between any two objects x and y, while axiom 4.1.21. asserts the transitivity of Q. Axioms 4.1.22. and 4.1.23. establish the connection between Q and S and P.

Interpreting $E(x, y)$ here, as everywhere in the sequel, as the relation of equality, while $S(x, y, z)$ stands for 'z is the product of x and y', we see that axioms 4.1.1. — 4.1.4., 4.1.7., 4.1.8., 4.1.10., 4.1.12., 4.1.16., together constitute the axiomatic system of a group. We shall denote this set by A_G. Adding 4.1.14. to A_G, we obtain the axiomatic system A_C of a commutative group. Again axioms 4.1.1. — 4.1.10, 4.1.12. — 4.1.14., 4.1.16. — 4.1.18. constitute an axiomatic system A_R for general rings, where $S(x,y,z)$ now denotes addition, 'z is the sum of x and y', and $P(x, y, z)$ denotes multiplication, 'z is the product of x and y'. An axiomatic system A_{CR} for commutative rings is obtained by adding 4.1.15. to A_R, while axiomatic systems A_S and A_F for skew and commutative fields are obtained by adding 4.1.11. to A_R and A_{CR} respectively. Finally, all the axioms 4.1.1. — 4.1.23. together constitute a system A_Q for the concept of a commutative ordered field, where $Q(x, y)$ is interpreted as 'x is smaller than y'.

4.2. *Characteristic of a field, and Archimedes' axiom.* To formulate the condition that a field or ring have a certain specific characteristic, we define a sequence of predicates $S_0(x, y)$, $S_1(x, y)$, $S_2(x, y)$, ... recursively. Thus, let

4.2.1. $S_0(x, y) = [S(x, y, x)]$,

and

$$S_{n+1}(x, y) = [(\exists z)[S_n(x, z) \wedge S(z, x, y)]] \quad n = 0, 1, 2, 3, \ldots$$

The verbal interpretation of $S_n(x, y)$, $n \geqslant 1$, is 'y is the result of the continued addition of x(n times), $y = x + x + \ldots + x$. It will be observed that while we use the concept of the natural number in our metalanguage, this concept is not as such included in the language L.

For later applications, we define similarly,

4.2.2. $P_0(x, y) = [P(x, y, x) \wedge [(z)[P(z, y, z)]]]$

and

$$P_{n+1}(x, y) = [(\exists z)[P_n(x, z) \wedge P(z, x, y)]] \; n = 0, 1, 2, 3, \ldots$$

'y is the nth power of x'.

We now formulate a sequence of axioms H_n, $n = 2, 3, \ldots$ as follows,

4.2.3. $H_n = [(x)(y)(z)[S_n(x, y) \supset S(y, z, z)]]$.

The axiomatic system A_p of commutative fields of characteristic p, where p is any prime number, is then obtained by adding H_p to the axioms of A_F. On the other hand, the axiomatic system A_0 of fields of characteristic 0 is obtained by adding to A_F the infinite number of axioms $[\sim H_p]$ where p varies over the whole range of prime numbers.

Next, with a view to its application to commutative algebraic fields, we formulate the condition that every polynomial of order n has a root. This condition may be taken as

4.2.4. $E_n = [(x_0)(x_1) \ldots (x_n)(\exists y)(\exists y_1)(\exists y_2) \ldots (\exists y_n)(\exists z_1)(\exists z_2)$
$\ldots (\exists z_n)(\exists u_1)(\exists u_2) \ldots (\exists u_{n-1})[[P_1(y, y_1) \wedge [P_2(y, y_2) \wedge$
$[\ldots \wedge P_n(y, y_n)] \ldots]] \wedge$
$[[P(x_1, y_1, z_1) \wedge [P(x_2, y_2, z_2) \wedge [\ldots \wedge P(x_n, y_n, z_n)] \ldots]]] \wedge$
$[S(z_1, z_2, u_1) \wedge [S(u_1, z_3, u_2) \wedge [\ldots \wedge [S(u_{n-2}, z_n, u_{n-1}) \wedge$
$E(_{n-1}, x_0)] \ldots]]] \vee$
$[S(x_1, x_1, x_1) \wedge [S(x_2, x_2, x_2) \wedge [\ldots \wedge S(x_n, x_n, x_n)] \ldots]]]]$

Assuming that all the axioms of A_F are satisfied, this may be interpreted as follows. 'Given $x_0, x_1, \ldots, x_n$, there exists a y such that $x_1 y + x_2 y^2 + \ldots + x_n y^n = x_0$, unless the coefficients $x_1, x_2, \ldots, x_n$ all vanish'.

An axiomatic system for the concept of an algebraically closed field of specified characteristics is then obtained by adding to A_p (or A_0) all the axioms E_n, $n = 2, 3, \ldots$. We denote the sets of axioms obtained in this way by C_p (or C_0) respectively.

Finally, we wish to formalise Archimedes' axiom. This, however, will be done only within an extended language L^* which includes infinite disjunctions. In fact, it will be demonstrated later that Archimedes' axiom cannot be formalised with the restricted calculus.

We formulate

$$
\begin{aligned}
4.2.5. \quad (x)(y)[[(\exists z)[S(z, z, z) \wedge [E(x, z) \vee Q(x, z)]]] &\vee \\
[(\exists w_1)[S_1(x, w_1) \wedge Q(y, w_1)]] &\vee \\
[(\exists w_2)[S_2(x, w_2) \wedge Q(y, w_2)]] &\vee \\
[(\exists w_3)[S_3(x, w_3) \wedge Q(y, w_3)]] &\vee \ldots]
\end{aligned}
$$

Adding this axiom to A_Q we obtain A_Q^* as an axiomatic system for the concept of a commutative Archimedean ordered field.

The axiomatic systems defined so far form an adequate background for our work. Other systems will be considered as we proceed. The specified systems are not the only possible ones for the concepts in question, and later we shall discuss what is meant by saying that two axiomatic systems describe the same concept.

We shall not be concerned with questions of independence concerning the axiomatic systems under consideration.

V

SOME METAMATHEMATICAL THEOREMS ON ALGEBRAIC FIELDS

5.1. *Chains of statements.* In this chapter we shall follow the first line of attack indicated in section 1.1. Thus we shall prove a number of metamathematical theorems relating to various types of algebraic fields, some of which are quoted in section 1.4. of the introduction.

We shall say that an infinite sequence of statements X_n, $n = 1, 2, \ldots$ in a language L constitutes an increasing chain if $[X_n \supset X_m]$ is valid whenever $n \geqslant m$. This may be indicated concisely by the formula

$$\ldots \supset X_{n+1} \supset X_n \supset \ldots \supset X_2 \supset X_1$$

which is not however included in L.

The chain will be called strictly, or monotonic, increasing if, conversely, the validity of $[X_n \supset X_m]$ entails $n \geqslant m$.

5.1.1. *Theorem.* Let K be a set of statements $X_1, X_2, \ldots$ which constitute a monotonic increasing chain. Then there is no single statement Y in L which is equivalent to K in the sense that it can be deduced from K while, conversely, all the statements of K can be deduced from it.

Assume contrary to our assertion that Y is such a statement. Then it can be deduced from a finite number of the elements of K, $X_{n_1}, \ldots, X_{n_k}$ say. Now let m be the greatest of the numbers $n_1, \ldots, n_k$. Then, by the definition of an increasing chain, the statements $X_{n_1}, \ldots, X_{n_k}$ can all be deduced from X_m. Hence also, Y can be deduced from X_m, $[X_m \supset Y]$ is valid. On the other hand, since all the statements of K can be deduced from Y, by assumption, this also applies to X_{m+1}, so that $[Y \supset X_{m+1}]$ is valid. But this signifies that X_{m+1} can be deduced from X_m, i.e., that $[X_m \supset X_{m+1}]$ is a valid statement. This is contrary to the definition of a strictly increasing chain and proves that a statement Y as assumed cannot exist.

We have the corollary that a set K which consists of the elements of a strictly increasing sequence cannot be contradictory. In fact, if K were contradictory then any contradictory statement Y (e.g. any statement of the form $[X \wedge \sim X]$, X arbitrary), can be deduced from K, while conversely, all the statements of K can be deduced from it. And this is impossible, as we have seen.

5.2. *Metamathematical theorems on fields with prime characteristic.* To apply the theorem of the preceding section to algebraic field theory, let H_0 be the conjunction of the statements of A_F (taken in any arbitrary order). We then define the statements X_n, $n = 1, 2, \ldots$ by

5.2.1. $X_1 = H_0$

and

$$X_{n+1} = [X_n \wedge \sim H_{p_n}],\ n = 1, 2, \ldots$$

where p_n is the nth prime number ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.). X_n states, in words, that the structures under consideration are commutative algebraic fields, whose characteristic is not p_1, nor p_2, nor $\ldots$, nor p_{n-1}.

The sequence $X_1, X_2, \ldots$ is a strictly increasing chain. The fact that $[X_n \supset X_m]$ is valid if $n \geqslant m$ follows immediately from the definition of the X_n by successive conjunction. On the other hand, since there exist structures which satisfy any given X_m but not any X_n for $n > m$ — viz., the commutative algebraic fields of characteristic p_m — it follows that no such X_n can be deduced from X_m. This shows that the chain is strictly increasing.

Now the set K which consists of the statements $X_1, X_2, \ldots$ is precisely equivalent to A_0 in the sense that K can be deduced from A_0 while conversely A_0 can be deduced from K. In other words, K also constitutes an axiomatic system for the concept of a commutative algebraic field of characteristic 0. It then follows from the theorem of the preceding section that there is no single statement — and hence that there is no finite set of statements — in L which can replace K. We therefore have the theorem,

5.2.2. The concept of a commutative algebraic field of characteristic 0 cannot be formalised by a finite number of axioms within the restricted calculus of predicates.

An allied theorem is

5.2.3. Every theorem of the restricted calculus (formulated in terms of equality, addition, and multiplication) which holds for all commutative fields of characteristic 0 also holds for all other commutative fields of sufficiently high characteristic p ($p > p_0$ where p_0 depends on the theorem).

In fact, let K be the set used in the proof of theorem 5.2.1., and let Y be a statement which holds for all commutative fields of characteristic 0, i.e., whenever all the statements of K hold. Then Y can be deduced from K (3.2.2.) and therefore can be deduced from a finite number of the elements of K, $X_{n_1}, X_{n_2}, \ldots, X_{n_k}$, say. Let m be the greatest of the numbers $n_1, n_2, \ldots, n_k$. Then all the statements $X_{n_1}, \ldots, X_{n_k}$ can be deduced from X_m as before. But by its definition, X_m holds in all commutative fields of characteristic not smaller than p_m, so that Y also holds in all those fields. It follows that p_m may be chosen as the p_0 of theorem 5.2.3.

5.3. *Applications.* We shall now apply 5.2.3. to the proof of some non-trivial theorems of Mathematics proper. We do not intend, now or later, to exhaust all the mathematical potentialities of our metamathematical theorems. Instead, we shall be content to give various interesting applications as we go along. The construction of additional examples presents no difficulty. Let

$$5.3.1. \quad q(x_1,\ldots,x_n) = a_1\, x_1^{i_1^{(1)}}\, x_2^{i_2^{(1)}} \ldots x_n^{i_n^{(1)}} + \\ + a_2\, x_1^{i_1^{(2)}}\, x_2^{i_2^{(2)}} \ldots x_n^{i_n^{(2)}} + \ldots + a_k\, x_1^{i_1^{(k)}}\, x_2^{i_2^{(k)}} \ldots x_n^{i_n^{(k)}}$$

be any polynomial of the variables $x_1, \ldots, x_n$, with integral coefficients. We wish to formalise the predicate '$y = q(x_1, \ldots, x_n)$' within a language which contains the relative symbols $S(,,)$ and $P(,,)$.

We first define the predicates

$$Q_{i_1}(x_1, y), Q_{i_1 i_2}(x_1\, x_2, y), \ldots Q_{i_1 i_2 \ldots i_m}(x_1, x_2, \ldots, y), \ldots,$$

where the i_k are arbitrary non-negative integers, successively by

$$5.3.2.\quad \begin{aligned} &Q_{i_1}(x_1, y) = P_{i_1}(x_1, y) \\ &Q_{i_1 i_2}(x_1, x_2, y) = (\exists z_1)(\exists z_2)\left[Q_{i_1}(x_1, z_1) \wedge P_{i_2}(x_2, z_2) \wedge P(z_1, z_2, y)\right] \\ &\vdots \\ &Q_{i_1 i_2 \ldots i_m}(x_1, x_2, \ldots, x_m, y) = (\exists z_1)(\exists z_2) \\ &\qquad [Q_{i_1 i_2 \ldots i_{m-1}}(x_1, x_2, \ldots x_{m-1}, z_1) \wedge \\ &\qquad P_{i_m}(x_m, z_2) \wedge P(z_1, z_2, y)]. \end{aligned}$$

In this connection the predicates $P_k(x, y)$ are defined as in 4.2.2. Next we define

$$5.3.3.\quad Q_{i_1 i_2 \ldots i_m : a}(x_1, \ldots, x_m, y) = (\exists z)\left[Q_{i_1 i_2 \ldots i_m}(x_1, \ldots, x_m, z) \wedge S_a(z, y)\right]$$

if a is a positive integer, and

$$5.3.4.\quad \begin{aligned} &Q_{i_1 i_2 \ldots i_m : a}(x_1, \ldots, x_m, y) = (\exists z_1)(\exists z_2) \\ &\qquad [Q_{i_1 i_2 \ldots i_m : -a}(x_1, \ldots, x_m, z_1) \wedge S(z_2, z_2, z_2) \\ &\qquad \wedge S(z_1, y, z_2)] \end{aligned}$$

if a is a negative integer. The predicate $S_a(x, y)$ is defined as in 4.2.1. 5.3.3. and 5.3.4. may be regarded as formalised expressions for monomials with integral coefficients, '$y = a x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m}$'. Corresponding to a general polynomial $q(x_1, \ldots, x_n)$ with integral coefficients as given by 5.3.1. we may now define the predicate $R_q(x_1, \ldots, x_n, y)$ by

$$5.3.5.\quad \begin{aligned} &R_q(x_1, \ldots, x_n, y) = (\exists y_1) \ldots (\exists y_k)(\exists z_2) \ldots (\exists z_{k-1}) \\ &\qquad [Q_{i_1^{(1)} \ldots i_n^{(1)}; a_1}(x_1, \ldots, x_n, y_1) \\ &\wedge \ldots \wedge Q_{i_1^{(k)} \ldots i_n^{(k)}; a_k}(x_1, \ldots, x_n, y_k) \wedge \\ &\qquad S(y_1, y_2, z_2) \wedge S(z_2, y_3, z_3) \wedge \ldots \wedge S(z_{k-1}, y_{k-1}, y)] \end{aligned}$$

for $k = 2, 3, \ldots$, with a slight modification of the definiens when $k = 2$. If $k = 1$, $q(x_1, \ldots, x_n) = a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, we write $R_q = Q_{i_1 i_2 \ldots i_m : a}$, while if the coefficients of $q(x_1, \ldots, x_n)$ all vanish, $q(x_1, \ldots, x_n) \equiv 0$, we may choose

5.3.6. $R_q(x_1, \ldots, x_n, y) = [S(x_1, y, x_1) \wedge S(x_2, y, x_2) \wedge \ldots \wedge S(x_n, y, x_n) \wedge S(y, y, y)]$

If we replace the predicates $Q_{i_1(m)\ldots i_n(m):a_m}$, and $Q_{i_1\ldots i_m}$ successively by the expressions by which they are defined, we see that the predicates $R_q(x_1, \ldots, x_n, y)$ are all formulated within a language which contains only the relative symbols S and P. The coefficients of the polynomials are regarded as operators indicating continued addition, not as elements of the system under consideration.

We are now in a position to prove

5.3.7. *Theorem.* If a set of polynomial equations with integral coefficients

5.3.8. $q_1(x_1,\ldots,x_n) = 0,\ q_2(x_1,\ldots,x_n) = 0,\ldots,q_k(x_1,\ldots,x_n) = 0$

has no solution in any extension of the field of rational numbers then it has no solution in any field of characteristic $p > p_0$, if the coefficients are taken modulo p. p_0 is a constant depending on the polynomials $q_1, \ldots, q_k$.

In fact the statement that such a solution exists can be formalised as

5.3.9. $X = (\exists x_1)\ldots(\exists x_n)\,(\exists y_1)\ldots(\exists y_k)\,[R_{q_1}(x_1,\ldots,x_n,y_1) \wedge \ldots \wedge R_{q_k}(x_1,\ldots,x_n,y_k) \wedge S(y_1,y_1,y_1) \wedge \ldots \wedge S(y_k,y_k,y_k)]$

Now $[\sim X]$ is a statement, or theorem in the sense of 5.2.3., and if it holds for all commutative fields of characteristic 0 — which is the hypothesis of 5.3.7. — then it must also hold in all fields of characteristic $p > p_0$, by 5.2.3.

A theorem of a slightly different nature is

5.3.10. If a polynomial $q(x_1, \ldots, x_n)$ with integral coefficients cannot be represented as the product of any two polynomials $q_1(x_1, \ldots, x_n)$ and $q_2(x_1, \ldots, x_n)$ of specified degrees m_1 and m_2, then it cannot be represented as the product of two such polynomials in any field of characteristic $p > p_0$, where the coefficients of $q(x_1, \ldots, x_n)$ are taken modulo p, and the constant p_0 depends on $q(x_1, \ldots, x_n)$.

In fact, it is easy to replace the condition that $q(x_1, \ldots, x_n)$

equals the product of two polynomials q_1 and q_2 of specified degrees by a set of polynomial equations such as 5.3.8., involving the coefficients of q_1 and q_2 as unknowns. 5.3.10 then follows directly from 5.3.7.

As a corollary we have

5.3.11. Let $q(x_1, \ldots, x_n)$ be a polynomial with integral coefficients which is irreducible in all extensions of the field of rational numbers. Then $q(x_1, \ldots, x_n)$, taken modulo p, is irreducible in all fields of characteristic p greater than some constant depending on q.

A mathematical proof of 5.3.7. is as follows.

If 5.3.8. has no solution in any extension of the field of rational numbers, then there exist polynomials $f_1(x_1, \ldots, x_n), \ldots f_k(x_1, \ldots, x_n)$, with rational coefficients such that

$$f_1 q_1 + f_2 q_2 + \ldots + f_k q_k = 1.$$

Multiplying this equation by a suitable integer, we see that there exists polynomials $g_1, \ldots, g_k$ with *integral* coefficients such that

$$g_1 q_1 + g_2 q_2 + \ldots + g_k q_k = a \tag{5.3.12.}$$

where a is an integer. Now let p_0 be the greatest prime divisor of a. Then if we take any $p > p_0$, we have

$$g_1 q_1 + g_2 q_2 + \ldots + g_k q_k \not\equiv 0(p)$$

and this proves that 5.3.8., taken modulo p cannot have any solution in any field of characteristic p.

We see that in the present case it is still easy to provide a mathematical proof, although of an entirely different nature. In the sequel we shall give various examples of theorems for which the elaboration of apurely mathematical proof should be rather more difficult.

We shall say that two solutions of 5.3.8. are different if they differ with respect to at least one of the x_k, $k = 1, 2, \ldots, n$. We then have the theorem,

5.3.13. If the system of polynomial equations 5.3.8. has no more than m different solutions in any commutative field of characteristic 0, then it cannot have more than m solutions in any field of characteristic $p > p_0$, where p_0 depends on the set of polynomials, and the coefficients of the latter are taken modulo p.

To prove the theorem, we formalise the statement '5.3.8. has (at least) m different solutions'.

5.3.14. $$X_m = (\exists y)(\exists x_1^{(1)}) \dots (\exists x_n^{(1)})(\exists x_1^{(2)}) \dots (\exists x_n^{(2)}) \dots (\exists x_n^{(m)})$$
$$[Q_{q_1}(x_1^{(1)}, \dots, x_n^{(1)}, y) \wedge \dots \wedge Q_{q_k}(x_1^{(1)}, \dots, x_n^{(1)}, y)$$
$$\wedge \dots \wedge Q_{q_1}(x_1^{(m)} \dots, x_2^{(m)}, y)$$
$$\wedge Q_{q_k}(x_1^{(m)}, \dots, x_n^{(m)}, y) \wedge S(y, y, y) \wedge$$
$$[\sim E(x_1^{(1)}, x_1^{(2)}) \vee \sim E(x_2^{(1)}, x_2^{(2)})$$
$$\vee \dots \vee \sim E(x_n^{(1)}, x_n^{(2)})] \wedge \dots \wedge [\sim E(x_1^{(m-1)}, x_1^{(m)}) \vee$$
$$\sim E(x_2^{(m-1)}, x_2^{(m)}) \vee \dots \vee \sim E(x_n^{(m-1)}, x_n^{(m)})]] \qquad m = 1, 2, \dots$$

Now if $[\sim X_m]$ holds in all fields of characteristic 0, then by 5.2.3. it holds in all fields of characteristic p where p is greater than some p_0. This proves 5.3.13.

Again we may consider skew fields of specified characteristic, whose axiomatic systems, A_p' or A_0', can be obtained by removing axiom 4.1.15. from the respective A_p or A_0. We may again define predicates $Q_q(x_1, \dots, x_n)$ as in 5.3.5., etc., although $q(x_1, \dots, x_n)$ may now contain terms in which the same variable occurs more than once, e.g., $x_1^2 x_2 x_1^3$ since the variables need be commutable no longer. Theorems corresponding to 5.3.7. and 5.3.13. can then be stated for skew fields, the proof being based on a modified version of 5.2.3.

5.4. *Properties of infinite disjunctions.* Let Y be an infinite conjunction in an extended language L^*, where the X_n are statements within the restricted language L,

5.4.1. $$Y = [X_1 \wedge X_2 \wedge X_3 \wedge \dots \wedge X_n \wedge \dots]$$

We shall call Y 'effectively finite' if one of its partial conjunctions, Y_k,

5.4.2. $$Y_k = [X_1 \wedge [X_2 \wedge [\dots \wedge X_k] \dots]]$$

is such that $[Y_k \supset Y]$ holds in all models in which it is defined.

We then have, as a consequence of the result of section 5.1,

5.4.3. *Theorem.* An infinite conjunction Y is equivalent to a statement Z of L (in the sense that $[[Y \supset Z] \wedge [Z \supset Y]]$ holds in all models in which it is defined) if and only if it is effectively finite.

In fact, clearly $[Y \supset Y_k]$ holds in all models in which it is defined, $k = 1, 2, \ldots$. Hence if, in addition $[Y_k \supset Y]$, for some definite k, holds in all these models, then Y is equivalent to Y_k in the sense of the theorem, showing that the condition is sufficient.

To prove that the condition is necessary, consider the sequence

$$Y_1, Y_2, Y_3 \ldots$$

This sequence forms an increasing chain, $[Y_n \supset Y_m]$ is valid for $n \geqslant m$. Now assume first that following every Y_m there is a Y_n in the chain, $n > m$, such that $[Y_m \supset Y_n]$ is not valid. In that case, we may select a strictly increasing chain from the sequence $Y_1, Y_2, Y_3, \ldots$

$$\ldots \supset Y_{k_n} \supset \ldots Y_{k_3} \supset Y_{k_2} \supset Y_{k_1}$$

The set K of the elements of this strictly increasing sequence holds if and only if Y holds. But according to theorem 5.1.1. there can be no single statement in L which holds if and only if all the elements of K hold. We conclude that if a statement Z as assumed in the theorem exists, then there is a Y_k in the chain such that $[Y_k \supset Y_n]$ is valid for all $n \geqslant k$. But in that case, $[Y_k \supset Y]$ holds in all models in which it is defined. This shows that Y is effectively finite.

Now let Y be an infinite disjunction in an extended language L^* whose components are statements within the restricted language L,

5.4.4. $$Y = [X_1 \vee X_2 \vee X_3 \vee \ldots \vee X_n \vee \ldots]$$

Let

5.4.5. $$Y_k = [X_1 \vee [X_2 \vee [\ldots \vee X_k] \ldots]]$$

Clearly $[Y_k \supset Y]$ holds in all models in which Y is defined. Similarly $[Y_n \supset Y_m]$, $n \leqslant m$, holds in all models in which it is defined, and is valid in L. We shall call Y 'effectively finite' if $[Y \supset Y_n]$, $n \geqslant k$, holds in all models in which Y is defined, where k is a constant which depends on Y, but not on the particular model under consideration. In that case, also, $[Y_m \supset Y_n]$ is a valid statement for all m, $m = 1, 2, \ldots$ and for $n \geqslant k$ (i.e. even when $m > n$). In particular therefore, $[Y_m \supset Y_k]$ is valid for all m.

Conversely, if there exists a positive integer k such that $[Y_m \supset Y_k]$

is valid for all m, then $[Y \supset Y_k]$ (and therefore $[Y \supset Y_n]$ for $n \geqslant k$) will be seen to hold in all models in which Y is defined. In fact $[Y \supset Y_k]$ does not hold, for given model M and correspondence C, only if Y holds, but Y_k does not hold. This can be the case only if none of the X_m, $1 \leqslant m \leqslant k$, hold, but some X_l, $l > k$, holds in M. This in turn implies that Y_l holds, and hence that $[Y_l \supset Y_k]$ does not hold in M. We therefore have the theorem,

5.4.6. Y is effectively finite if and only if there exists a positive integer k such that $[Y_m \supset Y_k]$ is valid for all m.

5.5. *Properties of infinite disjunctions*, continued. Let Y be a statement of the form

5.5.1. $$Y = [(u)(v)(w) \ldots (z)Z(u, v, w, \ldots, z)]$$

in an extended language L^*, where Z is an infinite disjunction of incomplete formulae X_n of a restricted language L, which involve the dummy symbols $u, v, w, \ldots, z$. (Not all the dummy symbols need be involved in every X_n, provided every dummy symbol which occurs as a quantifier preceding Z, occurs in at least one of the X_n). Thus,

$$Z(u, v, w, \ldots, z) = [X_1(u, v, w, \ldots) \vee X_2(u, v, w, \ldots) \vee \ldots \vee X_n(u, v, w, \ldots) \vee \ldots]$$

We define the incomplete formulae Z_k, $k = 1, 2, \ldots$ by

5.5.2. $$Z_k(u, v, w, \ldots) = [X_1(u, v, w, \ldots) \vee [X_2(u, v, w, \ldots) \vee [\ldots \vee X_k(u, v, w, \ldots)] \ldots]],$$

and the statements Y_k by

$$Y_k = [(u)(v) \ldots (z)\, Z_k]$$

where the dummy symbols which do not appear in any X_n, $1 \leqslant n \leqslant k$, do not appear in the quantification. For sufficiently high k, however, all the dummy symbols are contained in at least one X_n, $1 \leqslant n \leqslant k$, by assumption.

Assume that Y is equivalent to a set K (which may be finite or infinite) of statements $W_1, W_2, \ldots$ in L in the sense that Y on one hand, and all the statements of K on the other, either hold, or do not hold, simultaneously, in all models in which Y and the statements of K are defined. In that case, the set K' obtained by adding $[\sim Y]$ to K cannot hold in any model. By the rules of section 2.9., $[\sim Y]$ holds in a given model M if and only if the statement Y' defined by

$$Y' = [(\exists u)(\exists v)(\exists w) \dots (\exists z)[[\sim X_1(u, v, w, \dots)] \wedge [\sim X_2(u, v, w, \dots)] \wedge \dots]]$$

holds in M, under the given correspondence.

Now let $a, b, c, \dots, e$ be object symbols in L, equal in number to the dummy symbols $u, v, w, \dots, z$, which do not appear in either the X or the W. (If necessary, we may extend the set of object symbols of L to ensure the existence of such $a, b, c, \dots, e$). Then if

$$[[\sim X_1(a, b, c, \dots)] \wedge [\sim X_2(a, b, c, \dots)] \wedge \dots]$$

holds in a model M, it follows that Y' and therefore $[\sim Y]$ also hold in it. Hence if K' cannot hold in any model, the set K'' consisting of the statements

$$[\sim X_1(a, b, c, \dots)],\ [\sim X_2(a, b, c, \dots)], \dots$$

together with the statements $W_1, W_2, \dots$ cannot be satisfied in any model either. It is therefore contradictory, and it follows, by 3.2.3., that a finite subset K''' of K'' is also contradictory. Let $[\sim X_k(a, b, c, \dots)]$ be the statement with greatest suffix k contained in K'''. Then by adding to K''' all the statements $[\sim X_m(a, b, c, \dots)]$ with $m < k$, as well as all the statements W_n which are not already contained in K''', we obtain a set H which cannot hold in any model either. The same then applies to the set H' which is obtained from H by replacing the statements

$$[\sim X_m(a, b, c, \dots)],\ m = 1, 2, \dots k,$$

by the single finite conjunction

$$[[\sim X_1(a, b, c, \dots)] \wedge [[\sim X_2(a, b, c, \dots)] \wedge [[\dots \wedge [\sim X_k(a, b, c, \dots)] \dots]$$

This shows that if all the statements $W_1, W_2, \ldots$ hold in a model M, then the statement

$$(u)(v)(w) \ldots (z)[X_1(u, v, w, \ldots) \vee [X_2(u, v, w, \ldots) \vee [\ldots \vee X_k(u, v, w, \ldots)] \ldots]$$

also holds in M. But this is precisely the statement Y_k, and we therefore find that if Y holds whenever the statements of a set K within L hold, then at least one of the statements Y_k also holds in all models M in which the statements of K hold (where k is independent of the particular M under consideration). This conclusion will be used to prove

5.5.3. *Theorem.* A statement Y, as defined by 5.5.1. is 'effectively finite' if and only if there exists a set K of statements in L which contain only relative and object symbols contained in Y, such that Y holds in a model M, (under a correspondence C) if and only if all the statements of K hold in M.

We shall call Y 'effectively finite' if there exists a positive integer k such that $[Y \supset Y_k]$ holds in all the models in which Y is defined. It is clear that, conversely, $[Y_k \supset Y]$ holds for all k, in all models in which Y is defined.

The condition of the theorem is clearly necessary; if Y is 'effectively finite' then the set K containing the single statement Y_k used in the definition of that concept, satisfies the condition of the theorem. Conversely, assume that there exists a set K with the specified properties. We notice that if Y is defined in a model M, then K is defined in M, by the same correspondence. Since Y holds whenever all the statements of K hold, at least one of the statements Y_k also holds whenever K holds, i.e., whenever Y holds, as proved earlier in this section. Thus $[Y \supset Y_k]$ holds in all structures in which Y is defined, showing that Y is effectively finite, and hence that the condition is sufficient.

Let K be a set of statements (finite, countable, or non-countable) all of which are formulated within L and let K' be determined from K by the inclusion of a statement of the form $[\sim Y]$, where Y is defined by 5.5.1. and such that $[\sim Y_k]$ can be deduced from K for all k. We then have the theorem

5.5.4. If a statement V in L is defined and holds in all the structures in which all the statements of K' hold, then V can be deduced from K.

In fact, the set K'' obtained from K' by the addition of $[\sim V]$ does not hold in any model, by assumption. It follows, as before that the set K''' which is obtained from K'' by replacing $[\sim Y]$ by the set of statements $[\sim X_1(a, b, c, \ldots)]$, $[\sim X_2(a, b, c, \ldots)]$, ... is contradictory. And again this implies that the set H which is obtained from K'' by replacing $[\sim Y]$ by $[\sim Y_k]$ is contradictory for at least one k. It follows that the statement $[\sim[[\sim V] \wedge [\sim Y_k]]]$, and thence $[V \vee Y_k]$ can be deduced from K. But since $[\sim Y_k]$ holds in all the models of K, by assumption, V holds in all the models of K, as asserted.

Again, let K be a set of statements formulated within L, and Y a statement of the form $(u)(v)(w) \ldots (z)\, Z$, where Z is an infinite disjunction, as before. We again denote by Y_k the corresponding partial disjunctions, and we assume that $[\sim Y_k]$ can be deduced from K for all k. Then,

5.5.5. *Theorem.* If there exists a structure in which all the statements of K hold, as well as Y, then there also exists a structure in which all the statements of K hold, as well as $[\sim Y]$.

If there were no structure in which all the statements of K hold, as well as $[\sim Y]$, i.e. if Y held in all the structures in which all the statements of K hold then, as before, at least one statement Y_k would hold in such a structure. But we have assumed that $[\sim Y_k]$ can be deduced from K for all k, which proves our theorem.

5.6. *Archimedean and non-Archimedean fields.* Some metamathematical theorems regarding the position of Archimedes' axiom are readily obtained from the preceding section.

5.6.1. *Theorem.* The set of axioms A_Q^* given in section 4.2. for the concept of an Archimedean ordered field cannot be replaced by any set, finite or infinite, of statements within L, formulated in terms of equality, order, addition, and multiplication.

Let X be the conjunction of all the elements of the (finite) set A_Q (see section 4.1) taken in any arbitrary but definite order. We denote the terms in the infinite disjunction for Archimedes' axiom (4.2.5.) by $X_1(x, y)$, $X_2(x, y)$, ... respectively, so that Archimedes' axiom becomes

$$(x)(y)[X_1(x, y) \vee X_2(x, y) \vee \ldots]$$

Then A_Q^* can be replaced by the following statement in L^*

$$Y = (x)(y)[[X \wedge X_1(x, y)] \vee [X \wedge X_2(x, y)] \vee [X \wedge X_3(x, y) \vee \ldots]$$

Now if there existed a set K of statements in L which holds in any structure if and only if Y holds in it, then by theorem 5.5.3. Y must be effectively finite, $[Y \supset Y_k]$ for some positive integer k, where

$$Y_k = (x)(y)[[X \wedge X_1(x, y)] \vee [X \wedge X_2(x, y)] \vee \ldots \vee [X \wedge X_k(x,y)]\ldots]$$

Y_k holds in any given model if and only if

$$Y_k' = X \wedge [(x)(y)[X_1(x, y) \vee [X_2(x, y) \vee \ldots \vee X_k(x, y)] \ldots]$$

holds in it. Now Y_k' signifies, in ordinary language, that all the axioms of an ordered field hold, while for every x and y, x positive, at least one of the numbers $x, 2x, 3x, \ldots, kx$ is greater than y, where k is a positive integer which is independent of x and y. But this condition is not satisfied in any ordered field (take e.g., $x = 1$, $y = k + 1$). This proves the theorem.

5.6.2. *Theorem.* If a theorem V formulated in the restricted calculus in terms of the relations of equality, order, addition, and multiplication, holds for all non-Archimedean ordered fields, then it holds for all Archimedean fields.

Discarding the notation used for the proof of the preceding theorem, we identify V with the statement V of 5.5.4., the set A_Q with the set K of that theorem, and Y with Archimedes' axiom.

Since none of the statements

$$Y_k = (x)(y)[X_1(x, y) \vee [X_2(x, y) \vee \ldots \vee X_k(x, y)] \ldots]$$

holds in any ordered field $[\sim Y_k]$ can be deduced from A_Q for all k. Hence the conditions of theorem 5.5.4. are satisfied, and it is proved that V holds in all ordered fields, and in particular, in all Archimedean ordered fields.

5.7. *Examples.* The metamathematical results of the preceding sections can be used to establish the existence of non-Archimedean fields with certain specified properties. As an example, we shall prove —

5.7.1. *Theorem.* There exists a non-Archimedean ordered field

F such that every polynomial in F possesses a zero in the field between any two values of the argument between which it changes sign.

By theorem 5.5.5. we only have to show that there exists a set of axioms in L representing the concept of 'an ordered field such that every polynomial in it possesses a zero between any two values of the argument between which it changes sign'. If we use this set as the set K of theorem 5.5.5., while Y is Archimedes' axiom, then that theorem states, that if there exists an Archimedean ordered field as described by K, then there also exists a non-Archimedean ordered field of the same description. Examples of such Archimedean ordered fields are provided by the field of all real numbers and by the field of all algebraic real numbers. The theorem now follows immediately.

It may be pointed out that the proof that the field of all real numbers has the property in question requires only a knowledge of the facts that polynomials are continuous functions, and that continuous functions have zeros in every interval in which they change sign. From a logical point of view, this of course implies an excursion into the elements of transcendental theory.

A set K which describes the concept of 'an ordered field such that every polynomial in it possesses a zero between any two values of the argument between which it changes sign' can be constructed as the union of A_Q and of a countable set of statements P_n, $n = 2, 3, \ldots$ which state, in ordinary language,

'Every polynomial of order n possesses a zero between any two values of the argument between which it changes sign'.

We first show, similarly as in section 5.3., that the predicate '$y = q(x)$, where $q(x)$ is a polynomial with coefficients $t_0, t_1, \ldots, t_n$' can be represented within L, by $R(x, t_0, t_1, \ldots, t_n, y)$, say. (Notice that $t_0, t_1, \ldots, t_n$ now denote objects within the structure under consideration). The statement P_n is then given in terms of Q, by

$$\begin{aligned} P_n = (t_0)(t_1) \ldots (t_n)(x_1)(x_2)(y_1)(y_2)(z)[[S(z, z, z) \wedge [Q(x_1, x_2) \\ \wedge [Q(y_1, z) \wedge [Q(z, y_1) \wedge [R(x_1, t_0, \ldots, t_n, y_1) \wedge \\ R(x_2, t_0, \ldots, t_n, y_2)]]]]] \\ \supset [(\exists x_3)[R(x_3, t_0, \ldots, t_n, z) \wedge [Q(x_1, x_3) \wedge Q(x_3, x_2)]]]] \end{aligned}$$

We have shown that the concept of an ordered field in which every polynomial has a zero in an interval in which it changes sign, can be formalised within a restricted language L. This establishes the theorem.

A non-Archimedean field of the type under consideration has a number of important properties in common with the field of real numbers. For instance, the number of the zeros of a polynomial in any given interval within such a field can be determined by means of Sturm's chain.

5.8. *Algebraically closed fields.* We conclude this chapter by proving that the sets $C_2, C_3, C_5, \ldots, C_p, C_0$ defined in section 4.2. are complete, in Tarski's sense (section 3.8.), in a language L which contains the relative symbols $E(,)$, $S(,,)$, and $P(,,)$, but no other relative or object symbols. In ordinary language —

5.8.1. Any theorem formulated within the restricted calculus of predicates in terms of equality, addition, and multiplication, which is true in one particular algebraically closed commutative field M, is true in all algebraically closed commutative fields of the same characteristic.

For instance, the properties of the field of all complex numbers which can be formulated within the restricted calculus in terms of equality, addition, and multiplication, are shared by all other commutative fields of characteristic 0 which are algebraically closed.

In the proof of this theorem, we shall make use of some results which presuppose a certain amount of Algebraic field theory (refs. 14, 15). We shall use the following lemma,

5.8.2. For every polynomial $q(x_1, \ldots, x_n)$ whose coefficients belong to a field F and do not all vanish, there exist numbers $a_1, \ldots, a_n$ which are algebraic with respect to F such that $q(a_1, \ldots, a_n) \neq 0$. (The phrase '$a_1, \ldots, a_n, \ldots$ are algebraic with respect to F' can be replaced by '$a_1, \ldots, a_n, \ldots$ are all in F' if F has characteristic 0 but not in all other cases).

As a corollary we have,

5.8.3. Let $q_1(x_1, \ldots, x_n), q_2(x_1, \ldots, x_n), \ldots, q_k(x_1, \ldots, x_n)$ be a number of polynomials with coefficients in a field F such that at least one coefficient of each polynomial differs from zero. Then there exist numbers $a_1, \ldots, a_n$ which are algebraic with respect to F, such that $q_1(a_1, \ldots, a_n) \neq 0$, $q_2(a_1, \ldots, a_n) \neq 0, \ldots$ $q_n(a_1, \ldots, a_n) \neq 0$.

To reduce this to 5.8.2., we only have to put $q(x_1, \ldots, x_n) = q_1 q_2 \ldots q_k$, and apply 5.8.2. to this polynomial.

To prove theorem 5.8.1., we first assume that the degree of transcendence of M (the maximum number of algebraically independent elements in M) is transfinite. Then M includes an algebraically closed field M_1 whose degree of transcendence is $\aleph_0$. The cardinal number of M_1 also is $\aleph_0$.

We shall make use of the fact that if a theorem of the type under consideration holds in any given field, it also holds in all the fields isomorphic to it. This fact is strictly speaking a direct consequence of the semantic interpretation of a statement, and will be discussed later in greater detail. Now, by a theorem of Steinitz, any two algebraically closed fields whose characteristics and degrees of transcendence are equal, are isomorphic. It follows that the theorems which hold in M_1 also hold in all other algebraically closed fields whose degree of transcendence is $\aleph_0$ and which possess the same characteristic as M_1.

Let X be a statement which holds in M, as above: we propose to show that it also holds in M_1. In fact, let L' be an extension of L, which in addition to the relative symbols E, S, and P, contains object symbols $a_1, a_2, \ldots$ for every object in M_1, for a given correspondence C. Let K be the set of statements in L' which contains the statements of C_p (or C_0), for the given characteristic, and in addition contains either $[S(a_1, a_2, a_3)]$ or $[\sim[S(a_1, a_2, a_3)]]$, either $[P(a_1, a_2, a_3)]$ or $[\sim[P(a_1, a_2, a_3)]]$ and either $[E(a_1, a_2)]$ or $[\sim[E(a_1, a_2)]]$, according as the corresponding relations hold in M_1. Then M_1 is a model of K under the correspondence C, and so is M. Now let K' be the set obtained from K by the inclusion of X. K' cannot be contradictory since all its statements hold in M. By the results of section 3.7., K' therefore has a model M_2 which includes a field isomorphic to M_1, and which is itself of cardinal number $\aleph_0$. This implies that the degree of transcendence of M_2 is $\aleph_0$, and so M_2 is isomorphic to M_1. But M_2 satisfies X, and so therefore does M_1, as asserted.

We have shown that all closed algebraic fields of given characteristic, whose degree of transcendence is at least $\aleph_0$ satisfy the same statements within L. To continue, we formalise the concept of 'a field whose degree of transcendentality is at least $\aleph_0$' in the language L', which contains the relative symbols E, S, and P, and a countable number of object symbols, $a_1, a_2, a_3, \ldots$. A set of statements K_p (or K_0) which corresponds to this concept is obtained as the union of C_p (or C_0), and of the countable set of statements $T_q(a_1, \ldots, a_n)$, where $T_q(a_1, \ldots, a_n)$ stands for 'the numbers $a_1, \ldots, a_n$ do not satisfy the polynomial equation $q(x_1. \ldots, x_n) = 0$', q being any polynomial with integral coefficients, not all zero. These statements can be constructed within L' by the continuous addition, subtraction and multiplication of powers of the a_k, after the manner of section 5.3. They signify that there is no algebraic relation, within the field of rational numbers or within the prime field of characteristic p, which holds between the different a_k, i.e., the a_k are algebraically independent.

Now let X be any statement within L which holds in all the algebraically closed fields of given characteristic and of transfinite degree of transcendence. Then X holds in all the models of K_p (or, of K_0) and can therefore be deduced from it. It follows that X can be deduced from a finite subset K' of K_p (or K_0). K' contains statements of C_p (or C_0), and in addition contains a finite number of statements $T_q(a_1, \ldots; a_k)$ which signify '$q(a_1, \ldots, a_k) \neq 0$'. These statements jointly include only a finite number of object symbols $a_j, a_1, \ldots, a_n$, say. It then follows from 5.8.3. above that every algebraically closed field M of the given characteristic, contains elements $a_1', a_2', \ldots, a_n'$ such that $q(a_1', a_2', \ldots, a_n') \neq 0$ for all the polynomials q which belong to statements T_q of K'. We now establish a correspondence C'' between M and the object and relative symbols contained in K' by interpreting E, S, and P as usual while $a_j' \leftrightarrow a_j$, $j = 1, 2, \ldots n$. Then M is a model of K' under the correspondence C'' (compare section 2.8.), so that X also holds in M. But X does not contain $a_1, \ldots, a_n$ so that X holds in M under any correspondence other than C'' providing E, S, and P still correspond to the relations of equality, addition and multiplication in M. This completes the proof of theorem 5.8.1.

VI

ALGEBRAIC SYSTEMS

6.1. *Algebras of axioms.* In this and the subsequent chapters, we shall follow up the second line of attack mentioned in section 1.1. Thus, we shall investigate certain properties of the familiar systems of Algebra-groups, rings, etc., which can be abstracted from the specific arithmetical operations with which they are normally associated.

A survey of the axiomatic systems defined in Chapter 4, shows that they have certain important features in common. In particular, they all contain a relative symbol of order 2, $E(,)$ such that the following axioms are satisfied.

6.1.1. $(x)[E(x, x)]$ (reflexivity)
6.1.2. $(x)(y)[E(x, y) \supset E(y, x)]$ (symmetry)
6.1.3. $(x)(y)(z)[E(x, y) \wedge E(y, z) \supset E(x, z)]$ (transitivity)

and such that for every relative symbol $A(, \ldots,)$ of order $n = 1, 2, \ldots,$ contained in the axiomatic system in question,

6.1.4. $(x_1) \ldots (x_n)(y_1) \ldots y_n)[E(x_1, y_1) \wedge [E(x_2, y_2) \wedge [\ldots E(x_n, y_n)] \ldots] \supset [A(x_1, \ldots, x_n) \supset A(y_1, \ldots, y_n)]]$ (substitutivity)

A relative symbol $E(,)$, of order 2, will be called an equality with respect to a set of statements K (within a language L, or within an extended language L^*) if the statements 6.1.1.—6.1.4. can be deduced from K, for all relative symbols A contained in K.

There may be more than one equality contained in K.

Let $E_1(,)$ and $E_2(,)$ be two such relative symbols. In that case, it is not difficult to show that the statements

6.1.5. $(x)(y)[[E_1(x, y) \supset E_2(x, y)] \wedge [E_2(x, y) \supset E_1(x, y)]]$

can be deduced from K.

In fact since 6.1.4. can be deduced from K, by assumption, it follows, in particular, that

$$(x)(y)[E_1(x, x) \wedge E_1(x, y) \supset [E_2(x, x) \supset E_2(x, y)]]$$

is deducible from K. (Write E_1 for E, E_2 for A, x for x_1, x_2, and y_1, and y for y_2 in 6.1.4.) But since $(x)[E_1(x, x)]$ and $(x)[E_2(x, x)]$ can be deduced from K, it follows that

$$(x)(y)[E_1(x, y) \supset E_2(x, y)]$$

and similarly

$$(x)(y)[E_2(x, y) \supset E_1(x, y)]$$

and so therefore 6.1.5., are all deducible from K.

In terms of the definition of section 2.6., 6.1.5. shows that any two relative symbols of equality which are contained in a set of statements K, are co-extensive with respect to K.

A set of statements K which contains just one relative symbol of equality, as defined above, will be called an algebra of axioms.

It will be observed that the above definition of equality does not involve any generic distinction between the relative symbol of equality E, and the remaining relative symbols. An alternative possibility would be to assume that $E(a, b)$ indicates that the same object corresponds to the object symbols a and b. This however would require a far reaching modification of the definitions regarding the semantic interpretation of a set of statements (section 2.7 and 2.8.). The definition of the concept of equality adopted here, while it does not exhaust the full meaning of identity, is adequate for our present purpose.

Nevertheless, it will simplify some of the subsequent discussion to make an additional assumption on the character of the relative symbol of equality under consideration. Thus, we shall say that an algebra of axioms is normal if for any two object symbols a and b contained in K, $[\sim E(a, b)]$ can be deduced from K. (The statements $E(a, a)$ and $E(b, b)$ are deducible from K, by 6.1.1.). It will be seen that the condition involves the identity or non-identity of the symbols a and b implicitly; however, this condition is not formulated within L at all. An algebra of axioms which does not involve any object symbols, or which involves no more than one object symbol will be seen to be normal.

6.2. *Algebraic structures.* In agreement with the definition of a relative symbol of equality, we shall say that a relation, of order 2, $E(,)$ is a relation of equality within a structure M, if $E(a, a)$ holds for all objects of M, if $E(b, a)$ holds wherever $E(a, b)$ holds for all objects a and b of M, if $E(a, c)$ holds wherever $E(a, b)$ and $E(b, c)$ both hold for arbitrary objects a, b, and c of M, and if finally $A(a_1, \ldots, a_n)$ entails $A(b_1, \ldots, b_n)$ for all the relations A, and for arbitrary a_k and b_k of M, provided the relations $E(a_1, b_1), E(a_2, b_2) \ldots E(a_n, b_n)$ all hold in M. Two relations of equality defined in a structure M are necessarily co-extensive. A structure in which a relation of equality is defined will be said to be algebraic. A model M of an algebra of axioms K is algebraic, the relative symbol of equality in K corresponding to a relation of equality in M. An algebraic structure will be said to be normal if the relation(s) of equality defined in it do(es) not hold between any two 'different' (i.e. non-identical) objects. We shall also say that the corresponding relations of equality are normal. Since the relations of equality of a structure are co-extensive, this condition either applies to all or to none of them. It will be convenient to assume in the sequel that the algebraic structures under consideration include a single relation of equality only. This limitation is inessential.

With every algebraic structure M which is not normal we may associate a normal algebraic structure M in the following way. We divide M into equivalence classes with respect to the relation(s) of equality, i.e. we define that objects a and b of M shall belong to the same class if and only if $E(a, b)$ holds. We then take these equivalence classes as objects of the new structure M', and, corresponding to every relation A of order n in M we define a relation A' in M' which holds between objects of M' (i.e. equivalence classes of M) if and only if A holds in M between corresponding elements of these equivalence classes. This definition is independent of the particular choice of representative elements. Corresponding to the equality E in M we obtain a relation of equality E' in M', which holds only for identical arguments, $E'(a, a)$.

On the other hand, a relation F', of order 2, in M' can be an equality only if the corresponding relation F is an equality in M. In fact, as shown by 6.1.5., E' and F' are co-extensive, and so $F'(a, a)$ holds for all the objects a of M', while F' does not hold between any two different objects of M'. It follows that the corresponding relation F in M holds between all objects of the same equivalence class, and not between any two objects belonging to different equivalence classes. Hence F is co-extensive with E, and so is a relation of equality itself. And if we assume that E is the only relation of equality in M, then E' is the only relation of equality in M'. The process just described will be referred to as normalisation.

Algebraic structures correspond to algebras of axioms, as detailed above. Also, if the consistent set K is a normal algebra (of axioms) then there exists a model of K which is a normal algebraic structure. The proof of the last assertion may be omitted. Finally it is easy to establish that in an algebra of axioms, the statement 6.1.4. still holds if the relative symbol A is replaced by an arbitrary predicate X which involves n dummy symbols,

$$(x_1) \ldots (x_n)(y_1) \ldots (y_n)[[E(x_1, y_1) \vee \ldots \wedge [E(x_2, y_2)[\ldots \wedge E(x_n, y_n)] \ldots]] \supset [X(x_1, \ldots, x_n) \supset X(y_1, \ldots, y_n)]]$$

6.3. *Isomorphisms and homomorphisms.* Concepts of subsumption and of homomorphism will now be discussed in the light of the preceeding definitions. These concepts will be considered primarily in connection with structures, although indirectly they are relevant also to the sets of statements which describe these structures.

We shall say that a structure M is a partial structure of a structure M', $M \lessdot M'$, or that M' is an extension of M, $M' \gtrdot M$, if all the objects of M are objects of M', and if any relation holds between objects of M if and only if it is defined and holds between the same objects, taken as objects of M'. (All the relations of order 0 in M' are then defined and hold in M, vacuously). We shall say that M is a proper partial structure of M' (M' is a proper extension of M) if at the same time $M' \neq M$.

Now let M be a partial structure of an algebraic structure M', and let E be the (only) relation of equality in M'. Then E also is a relation of equality in M. It follows that M is itself an algebraic structure. However, it is quite possible that there is another relation of order 2 defined in M' which is an equality in M though not in M'. Now if M, together with any object a also contains every other object b such that $E(a, b)$ holds in M', then M will be called a substructure of M', $M \Subset M'$, and M' will be called a superstructure of M, $M' \Supset M$. Again we say that M is a proper substructure of M' (M' is a proper superstructure of M), if at the same time $M' \neq M$.

If $M \Subset M'$, clearly $M \lessdot M'$. On the other hand if $M \lessdot M'$ it does not follow that M is also a substructure of M'. However, in that case we may associate with M a certain substructure M^* of M' which is defined as the substructure of all objects of M' which are equal to objects of M. M^* is the meet of all substructures of M' which include M.

A one-to-one correspondence C between the objects and relations of a structure M and the objects and relations of a structure M', relations of order n corresponding to relations of order n, will be called an isomorphism if, whenever a relation holds between the objects of M, the corresponding relation, under C, holds between the corresponding objects of M', and vice-versa. If M and M' coincide, the isomorphism will be called an automorphism. An automorphism C will be called absolute if every relation in M corresponds to itself under C, otherwise C will be called relative.

A correspondence C between the objects and relations of a structure M and the objects and relations of a structure M' will be called a homomorphism, if to every relation of M' there corresponds just one relation of M, and vice-versa, and if to every object of M' there corresponds just one object of M, while to every object of M there corresponds at least one object of M', and such that whenever a relation holds between elements of M', the corresponding relation holds between the corresponding elements of M. If $M \lessdot M'$, then the homomorphism will be called an endomorphism. If all relations correspond to themselves under the endomorphism, then it will be called absolute. If every object of M' which is also contained in M, corresponds to itself when taken as an element of M, then the endomorphism will be called reflexive. Isomorphisms and automorphisms are special cases of homomorphisms and endomorphisms respectively. The only absolute reflexive endomorphism which is also an automorphism is that in which every object and relation corresponds to itself.

Two structures M and M' will be called isomorphic if there exists an isomorphism between them; two algebraic structures will be called quasi-isomorphic if the corresponding normal algebraic structures, defined in the preceeding section, are isomorphic (A correspondence between the elements of the corresponding normal algebraic structures is essentially a correspondence between equivalence classes of the objects of the original structures). Two normal algebraic structures are isomorphic if and only if they are quasi-isomorphic. Two algebraic structures M and M' are quasi-isomorphic if and only if there exists a one-to-one correspondence between the relations of M and M', and a (many-many) correspondence in which to every object of M there corresponds at least one object of M', and to every object of M' there corresponds at least one object of M, such that if a relation holds in M' then the corresponding relation holds between all the corresponding objects of M, and vice-versa, and such that two objects of M' may correspond to the same object of M if and only if they are equal in M' while two objects of M may correspond to the same object of M' if and only if they are equal in M. Similarly, a 'quasi-homomorphism' between two structures is given by a homomorphism between the corresponding normal structures. If a structure M is a model of a set of statements K (in a language L or L^*), then any other structure M' which is isomorphic to M also is a model of K. In fact the complete correspondence which exists between the objects and relation of M and the object and relative symbols of L' (see section 2.8.) and in terms of which M is a model of K, combined with the isomorphic correspondence between M and M', sets up a correspondence between the objects and relations of M' and the object and relative symbols of L', in terms of which M' is a model of K.

This fact was used in the proof of theorem 5.8.1. where the concept of isomorphism was accepted from Algebra. This coincides with the concept of isomorphism as defined above, for normal algebraic structures, while for other types of algebraic structures the 'isomorphism' of Algebra is covered by the concept of 'quasi-isomorphism' defined above. However, if we assume that all the models considered in section 5.8. are normal algebraic, then the distinction between isomorphism and quasi-isomorphism vanishes.

It will be seen that many of the complications of this section were introduced by the fact that the relation of identity is replaced within the object language by the weaker concept of equality. However, the assumption that $E(,)$ is a relative symbol which does not differ essentially from the remaining relative symbols is vital for the development of our analysis in other respects.

6.4. *Similarity of algebras.* A counterpart to one of the questions regarding structures considered in the preceeding section is provided by the following problem which relates to algebras of axioms. To introduce the problem, we revert to the system of axioms A_G specified in Chapter 2. It will be agreed that A_G is an adequate set of axioms for the concept of a group and similarly that A_F is an adequate set of axioms for the concept of a commutative algebraic field, etc. On the other hand these are certainly not the only sets of axioms for the concepts in question. For instance, an alternative set of axioms A'_G for the concept of a group is as follows.

The axioms are formulated in terms of relative sqmbols E and S as before, and in terms of an object symbol, e. They are 4.1.1 — 4.1.4 together with

6.4.1. $(x)(y)(\exists z)[S(x, y, z)]$

6.4.2. $(x)(y)(z)(w)[S(x, y, z) \wedge S(x, y, w) \supset E(z, w)]$

6.4.3. $(x)(y)(z)(t)(u)(w)[S(x, y, z) \wedge [S(z, t, u) \wedge [S(y, t, v) \wedge S(x, v, w)]] \supset E(u, w)]$

6.4.4. $(x)[S(x, e, x)]$

6.4.5. $(x)(\exists y)[S(x, y, e)]$

Although this also is a system of axioms for a group, it certainly differs from the system of axioms given earlier. We may therefore ask under what conditions do two systems represent the same concept. In other words, we may try to give a reasonable formal interpretation of this idea.

Two sets of statements, K and K', formulated within the same language L (restricted or extended), or within different languages, will be called equivalent if there exists a one-to-one correspondence between the object and relative symbols of K and of K' respectively (object symbols corresponding to object symbols and relative symbols corresponding to relative symbols of the same order) such that if in any statement of K we replace the object and relative symbols by the corresponding symbols of K', the transformed statement is deducible from K', and conversely, such that every statement of K' is similarly transformed into a statement which is deducible from K. It is then easy to see the truth of

6.4.6. *Theorem.* If K and K' are two equivalent sets of statements then every model of K also is a model of K', and vice versa.

An algebra of axioms K in a language L (restricted or extended) will be called a contraction of an algebra of axioms K' formulated in the same language if the two algebras contain the same relative symbols, and if the set of object symbols of K is a subset of the set of object symbols of K' such that the following conditions are satisfied (6.4.7.—6.4.14.).

All the statements of K can be deduced from K', in symbols,

6.4.7. $$K \subseteq S(K')$$

For every object symbol a which is contained in K' but not in K, there exists a predicate of order one, $Q_a(\ )$, formulated in terms of the object and relative symbols of K such that

6.4.8. $$Q_a(a)$$

is deducible from K', while the statements

6.4.9. $$(\exists x)Q_a(x)$$

6.4.10. $$(x)(y)[Q_a(x) \wedge Q_a(y) \supset E(x, y)]$$

6.4.11. $$\sim Q_a(a')$$

and

6.4.12. $$(x)[\sim Q_a(x) \vee \sim Q_{a*}(x)]$$

are all deducible from K. In this connection, Q_a and Q_{a*} are defined as above for object symbols a and a^* which do not co-incide, while a' may be any object symbol in K.

6.4.13. If a statement $X \varepsilon K'$ does not contain any object symbols not included in K, then it is deducible from K.

6.4.14. If a statement $X = X(a_1, \ldots, a_n)$ belongs to K', where the a_k, $k = 1, 2, \ldots, n$ are object symbols of X which belong to K' but not to K, then the statement

6.4.15. $$(\exists u_1) \ldots (\exists u_n)[X(u_1, \ldots, u_n) \wedge Q_{a_1}(u_1) \wedge Q_{a_2}(u_2) \wedge \ldots \wedge Q_{a_n}(u_n)]$$

can be deduced from K.

We may establish the theorem,

6.4.16. If K is a contraction of K', then every model M of K' is a model of K; and every model M of K is a model of K'.

The proof will be omitted (see preface).

Two algebras of axioms, K and K', will be called similar if they possess equivalent contractions. By 6.7.6. and 6.4.16. any model of an algebra of axioms K also is a model of any algebra of axioms similar to K.

The above concept of similarity for algebras of axioms is quite simple and in general there should be no difficulty in establishing equivalence in practical cases. For instance, the set A_G is a contraction of the set A_G', as defined by 4.1.1.—6.4.5., and is therefore similar to it. The predicate $Q_e(x)$ in A_G is defined by $Q_e(x) = [(y)[S(y, x, y)]]$. We notice that both A_G and A_G' are normal.

There is, however, a deeper and somewhat more involved concept of similarity which will now be considered.

Two algebras, K and K', will be called related if there exists a one-to-one correspondence between the object symbols of K and of K' respectively, and correspondence between the relative symbols of K and certain predicates of K', $A \leftrightarrow Q'$, and between the relative symbols of K' and certain predicates of K, $A' \leftrightarrow Q$, such that the following conditions are satisfied.

6.4.17. If in a statement X of K, the object symbols are replaced by the corresponding object symbols of K', and the relative symbols (rather, the predicates of order one obtained by bracketing these relative symbols) are replaced by the corresponding predicates of K', while the dummy symbols, which belong to the language of L of K are, if necessary, replaced by dummy symbols belonging to the language L' of K' — then the resulting statement X' can be deduced from K'. If similar exchanges are carried out in a statement X' of K', then the resulting statement X can be deduced from K.

6.4.18. Let $Q'(\ldots,)$ be a predicate in K' corresponding to a relative symbol $A(,\ \ldots)$ in K. If in Q' we replace the object symbols of K' by the corresponding object symbols of K, etc., as in **6.4.17.**, we obtain a predicate Q'' in L, formulated in terms of the relative and object symbols of K. Then the condition states that

$$(u_1) \ldots (u_n)[[A(u_1, \ldots, u_n) \supset Q''(u_1, \ldots, u_n)] \wedge [Q''(u_1, \ldots, u_n) \supset A(u_1, \ldots, u_n)]]$$

can be deduced from K, and that similar statements, mutatis mutandis can be deduced from K' for the relative symbols A' of K'.

Two algebras of axioms will be called relatively similar if they possess related contractions. It may be more difficult than before to establish the existence of this type of similarity in particular cases and to investigate its properties. We shall be content to give a simple example.

To formalise the concept of an ordered field, we may replace the relative symbol $Q(x, y)$, 'x is smaller than y' by the first order relative symbol $T(x)$, 'x is positive'. Thus we may define the axiomatic system A_T as the union of A_F and of the following set.

$$(x)(y)(z)[[S(x, x, x) \wedge S(y, z, x)] \supset [E(x, y) \vee T(y) \vee T(z)]]$$
$$(x)(y)(z)[[S(x, x, x) \wedge S(y, z, x)] \supset [T(y) \supset \sim T(z)]]$$
$$(x)(y)(z)[[T(x) \wedge T(y)] \supset [S(x, y, z) \supset T(z)]]$$
$$(x)(y)(z)[[T(x) \wedge T(y)] \supset [P(x, y, z) \supset T(z)]]$$

The set A_T is an algebra of axioms which is related (and therefore relatively similar) to A_Q by the correspondence under which E, S, and P correspond to themselves, while $Q(x, y)$ in A_Q corresponds to

$$(\exists z)(\exists w)[S(z, z, z) \wedge [S(x, w, y) \wedge T(w)]]$$

in A_T. Conversely, $T(x)$ corresponds to

$$(\exists y)[S(y, y, y) \wedge Q(y, x)]$$

in A_Q.

6.5. *Extension of general structures.* Let M be a model of a set of statements K in a restricted language L such that M contains at least one object. (The condition that L be restricted is important for the subsequent argument). We propose to show that there exists an extension M' of M which contains a set of objects of arbitrary cardinal in addition to the objects of M, such that M' also is a model of K, under the correspondence under which M is a model of K.

The construction of M' is on familiar lines. Let N be any set of objects, not contained in M, and let a be an arbitrary object of M. We then define M' as the structure whose objects comprise the objects of M as well as of N, while the relations of M' are identical with the relations of M and satisfy the following rules.

A relation shall hold between the objects of any given set which does not include objects of N according as the same relation holds between the same objects when regarded as objects of M. A relation shall hold between the objects of any given set which includes objects of N, according as it holds in M, when the objects of N are replaced by a.

It will now be shown that if C is any complete correspondence, under which the relations and objects of M' correspond to relative symbols and object symbols in a language L, then the set of statements in L which hold in M' but which do not include object symbols corresponding to objects of N, coincides with the set of statements which hold in M, under C. It then follows, in particular, that if M is a model of K, so is M', as asserted above. In fact, the assertion clearly holds for all the statements in L which are of order one, provided they do not include object symbols corresponding to objects of N. Now, unless the assertion is true in general, there exists a statement of lowest order, of the type under consideration, which holds in M' but not in M or else which holds in M but not in M'. Let X be such a statement. A survey of rules 2.7.1.—2.7.7. then shows immediately that X cannot have been obtained by conjunction, disjunction or implication (i.e., cannot be of the form $[Y \wedge Z]$, $[Y \vee Z]$ or $[Y \supset Z]$). Also, it cannot have been obtained by negation, for if $[\sim Y]$ holds in M but not in M', then Y which is of lower order holds in M', but not in M. On the other hand if $[\sim Y]$ holds in M' but not in M, then Y holds in M but not in M'. It follows that X is obtained by quantification.

If X has been obtained by universal quantification, i.e., if it is of the form $[(x)\ Y(x)]$, and holds in M', then it certainly holds in M, which is a partial system of M'. Conversely if X holds in M, i.e., if $Y(b)$ holds for all the object symbols b corresponding to objects in M, then since X is minimal, $Y(b)$ also holds in M' and so in particular, $Y(a)$ also holds in M'. Now if a' is any object symbol corresponding to an object in N, then we may replace a in $Y(a)$ by a' and in this way obtain a statement $Y(a')$ which holds in M'. (The proof of this is still not quite trivial. It starts from the fact that we may replace a by a' in all statements of order one, by the definition of M'). Hence, finally, $[(x)\ Y\ (x)]$ holds in M'. On the other hand, if X is of the form $[(\exists x)\ Y\ (x)]$, and it holds in M, then $Y(b)$ holds in M for some object symbol b corresponding to an object in M. Hence, since X is minimal, $Y(b)$ also holds in M', and so $[(\exists x)\ Y\ (x)]$ holds in M'. Conversely, if $[(\exists x)\ Y\ (x)]$ holds in M', then $Y(b)$ holds in M' for some b corresponding to an object in M'. If b corresponds to an object which also is an object of M, then $Y(b)$ holds in M, since X is minimal and so $[(\exists x)\ Y\ (x)]$ holds in M. On the other hand if b belongs to N, then by definition of M' (see above) $Y(a)$ also holds in M' and M, and so again $[(\exists x)\ Y\ (x)]$ holds in M.

This shows that there can be no minimal X as assumed, and proves the assertion.

6.6. *Extension of algebraic structures.* We now come to a type of extension which is of rather greater mathematical interest. It concerns algebraic systems only.

Let K be an algebra of axioms, and M a model in which it holds. The problem under investigation is whether there exists a true superstructure of M in which K holds.

If M is finite, then it is quite possible that no such superstructure exists. For instance, K may contain object symbols $a_1, \ldots, a_n$, and a statement

6.6.1. $(x)[E(x, a_1) \vee E(x, a_2) \vee \ldots \vee E(x, a_{n-1}) \vee E(x, a_n)]$

In that case it is clear that if a model M of K contains n objects no two of which are equal to one another, then M cannot possess a proper superstructure which is a model of K. For such a superstructure would have to contain more than n unequal objects and this is impossible by 6.6.1.

On the other hand, if K has an infinite model M, i.e., a model which contains an infinite number of unequal objects, then we propose to show that it possesses a proper superstructure in which K holds. Before formulating a more precise theorem, we introduce the following definitions which will also be useful in the sequel.

6.6.2. The reduced cardinal number of an algebraic structure M is the cardinal number of the set of equivalence classes in M with respect to the relation(s) of equality, i.e., it is the cardinal number of the set of objects of the normal algebraic structure associated with M.

6.6.3. Given any model M, a set of statements K within a language L will be called a positive diagram of M, if the relations and objects of M are in one-to-one correspondence C with the relative and object symbols contained in K (relations of order n corresponding to relative symbols of order n), such that K consists of all the statements of order one which hold in M. That is to say, any statement of order one $[A(a_1, \ldots, a_n)]$ belongs to K if and only if it holds in M, by the given correspondence.

6.6.4. A set of statements K' will be called a complete diagram of a model M if the relations and objects of M are in one-to-one correspondence with the relative symbols and object symbols of K', as before, and if K' includes a positive diagram K of M, and in addition contains all statements of the form $[\sim[A(a_1, \ldots, a_n)]]$ where A, $a_1, \ldots, a_n$ are arbitrary relative and object symbols belonging to the correspondence such that the corresponding relation does not hold in M.

It will be seen that both positive and complete diagrams hold in the structures to which they belong. An example of a positive diagram will be found in the proof of theorem 5.8.1. If K is a complete diagram of M, then it is easy to show that every model of K contains a partial structure isomorphic to M.

We now come to the main theorem of this section.

6.6.5. Let M be an algebraic structure which is a model of an algebra of axioms K in a language L, and whose reduced cardinal number n is transfinite. Then there exists a superstructure M' of M which is a model of K, and which in addition to the objects of M contains at least one other object. Also, given any cardinal $n' \geqslant n$, there exists a true superstructure of M whose reduced cardinal number is n', and which is a model of K.

Assume first that M is normal and that K does not contain any two relative symbols such that the corresponding relations are co-extensive in M. By assumption, there exists a one-to-one correspondence between the relative and object symbols of K and between the relations and some of the objects of M which can be extended (within L or within an extension of L, L') in such a way that in the extended correspondence C' all the objects of M are in correspondence with object symbols. Let D be a complete diagram of M in L (or L') formulated in terms of the objects and relative symbols of the extended correspondence C'.

Let T be any set of object symbols of L (or L') which are not contained in D, and let D' be the set of statements of the form $[\sim [E(a, b)]]$ where a varies over the objects of T, while for given a, b varies over the objects of T and of D, omitting a. Finally, let K' be the union of K and D and K'' the union of K' and D'. K' is consistent because M is a model of both K and D under the same correspondence C'. We propose to show that K'' also is consistent.

Assume on the contrary that K'' is contradictory. In that case, K'' contains a finite subset H which is also contradictory. H must contain at least one element of T, otherwise H would be a subset of K' and therefore consistent. On the other hand, since H is finite it can contain only a finite number of object symbols, $a_1, \ldots, a_l$ of D and $b_1, \ldots, b_m$ of T, say. Since the reduced cardinal number of M is transfinite, there are object symbols in D, $b'_1, \ldots, b'_m$, say, which are not equal to one another and not equal to any of the $a_1, \ldots, a_l$. Thus all the statements

$$[\sim [E(a_i, b'_j)]] \quad \text{and} \quad [\sim [E(b'_j, b'_k)]] \ , \ i = 1, 2, \ldots l,$$

$j, k, = 1, 2, \ldots m$ hold in M. Now the statements of H which belong to D' are all of the form $[\sim[E(b_j, a_i)]]$ or of the form $[\sim[E(b_j, b_k)]]$. If we replace all the b_j in these statements by the corresponding b'_j, then we obtain statements which hold in M, so that these statements together with the remaining statements of H (which do not contain any b_j) hold in M and are therefore consistent. But if in a consistent set we replace any object symbol throughout by another object symbol, then we obtain a consistent set, and conversely a contradictory set modified in this way yields a contradictory set. Hence H cannot be contradictory, and since all the finite subsets of K'' are consistent, K'' is itself consistent.

It follows that K'' possesses a model M'', and since K'' includes D, M'' includes a structure M^* isomorphic to M, as a partial structure. Also, we may assume that M^* is not only a partial structure, but indeed a sub-structure of M'': in other words, we may assume that if a is an object of M'' which is equal to an object b of M^* then a is itself contained in M^*. In fact, any such object a cannot correspond to any object symbol of T, since all the object symbols of T are unequal to all the object symbols of D corresponding to objects of M. We now define M^{**} as the structure which is obtained from M'' by removing from it all the objects which are equal to objects of M^* although they are not contained in M^*, while keeping the relations between the remaining objects of M' as in M''. It is not difficult to show that M^{**} is a model of K''. Also by the construction of M^{**}, M' is a true substructure of M^*, containing, in addition to the objects of M^*, objects which correspond to the object symbols of T, and no two of which are equal in M^{**}. Thus, provided we take L (or if necessary L') to be sufficiently comprehensive, we may ensure that M' contains, in addition to the objects of M^*, a set of unequal objects of cardinal number n', the cardinal of T. It only remains to replace the objects and relations of M^* in M^{**} by the objects and relations of M, to which M^* is isomorphic, and we obtain a superstructure M' of M whose reduced cardinal is not smaller than $n + n'$.

Now we know from section 3.7., that the cardinal of M' may be assumed not to exceed the cardinal of K''. Also since M is normal, without co-extensive relations, and of reduced transfinite cardinal n, it is not difficult to show that the cardinal of D also is n, so that the cardinal of K'' equals the greatest of the cardinals of K, D, and T. Again, since no two relative symbols of K correspond to co-extensive relations in M, it follows that neither the set of object symbols of K nor the set of relative symbols of K exceeds n, so that the cardinal of K cannot exceed n either. Thus the cardinal of K'' is max. $(n, n') = n + n'$, so that the cardinal of M' may,be assumed not to exceed $n + n'$. The same then applies to the reduced cardinal of M'. On the other hand it was shown earlier that the reduced cardinal of M' is not smaller than $n + n'$, and so finally, the reduced cardinal of M' equals n' or n according as n' is or is not greater than n.

Assume next that M is not necessarily normal, while K still does not contain any two relative symbols such that the corresponding relations are co-extensive in M. The method of construction given above can still be applied directly, but in that case it is no longer certain that the reduced cardinal of the superstructure M' equals the specified n'. However, even in this case it is not difficult to suggest a modified construction which achieves the required result. Broadly speaking, it is based on the temporary removal of all but one of the elements of any set of equal objects. Finally, if M contains co-extensive relations corresponding to relative symbols of K, we may again ensure the existence of a superstructure of M of specified reduced cardinal by removing, temporarily, all the relative symbols but one from every set of relative symbols corresponding to co-extensive relations in M.

From a mathematical point of view the first case is by far the most important.

6.7. *Applications.* As an example of the way in which we may draw mathematical conclusions from theorem 6.6.5., we may prove that there exists algebraically closed fields of characteristic 0, of arbitrary (reduced) cardinal number. Such fields can of course be constructed by purely algebraic means. Alternatively, however, we may start from the fact that, at any rate, the axiomatic system of the concept of an algebraically closed field C_0 is consistent. In fact, C_0 possesses a model, viz., the field of all complex numbers. The fact that this field is algebraically closed is, it is true, proved by transcendental methods but these are in some respects simpler and at any rate fundamentally different from Steinitz' construction of an algebraically closed field. Since C_0 is consistent and countable, it has a countable model M^* whose reduced cardinal is also countable. Theorem 6.6.5. then shows that M has superstructures of arbitrary reduced cardinals, which are models of C_0, as required.

As another application of the theorem, we proceed to give a second proof of the fact that the concept of an Archimedean field cannot be formulated within the restricted calculus of predicates (see 5.6.1.). In fact, it is known that there exists an Archimedean ordered field which does not possess a true superstructure, viz., the field of all real numbers. (This fact, stated as an 'axiom of completeness' was used by Hilbert to formulate a particularly elegant system of axioms for the field of all real numbers (ref. 1)). On the other hand every algebra of axioms which is formulated within the restricted calculus of predicates possesses a true superstructure, as shown above. It follows that the concept of an Archimedean field cannot be formulated within the restricted calculus.

The second theorem on Archimedean fields (5.6.2.) can be proved in a similar way. Assume contrary to the assertion of the theorem that a statement X holds in all non-Archimedean ordered fields but not in one particular Archimedean field M. Let K be the set of statements obtained by adding $[\sim X]$ to the axiomatic system of an ordered field, A_Q. Now all Archimedean fields are isomorphic to subfields of the field of real numbers (isomorphic in the mathematical sense, i.e., quasi-isomorphic according to the terminology of section 6.3.). It follows that all fields of (reduced) cardinal number greater than $2^{\aleph_0}$ are non-Archimedean, so that K cannot hold in such fields. But this is impossible since K is consistent and possesses an infinite model M of reduced cardinal not exceeding $2^{\aleph_0}$, and so must possess a model of reduced cardinal n greater than $2^{\aleph_0}$ (and including M), as we have seen.

Theorem 6.6.5. also throws some light on the position of an axiom of completeness in axiomatic set theory. Thus if in the system of Zermelo–Fraenkel (refs. 16, 17, 18) we regard $\varepsilon(,)$ and the equality $E(,)$ as fundamental concepts, and interpret the idea of a definite property in Skolem's (ref. 19) sense, then the system can be formalised by an algebra of axioms within the restricted calculus. It follows that any model of the system can always be extended in the sense of the theorem, so that this axiomatic system cannot be subjected to an axiom of completeness.

VII

POLYNOMIALS IN GENERAL ALGEBRAS

7.1. *Introduction.* The first concept which we shall attempt to transfer to a general algebraic system is that of a polynomial. The character of a polynomial in Algebra, e.g., of a polynomial in one variable $p(x)$ in a commutative ring R is fundamentally different from the corresponding notion in Analysis. In Analysis it is natural to begin with the most general concept of a function as a correspondence between the independent and the dependent variable. By narrowing the classes of functions under consideration gradually, we finally arrive at the class of polynomials as integral analytic functions whose singularity at infinity, if any, is a pole. It is quite true that even in Analysis, this characterisation of polynomials is normally preceded by a defining equation of the form,

7.1.1. $$y = p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

Nevertheless, the fundamental concept is that of a function as a correspondence, and equation 7.1.1. is only ancillary to it. It may be looked upon as a rule for finding the value of the dependent variable, given the independent variable.

On the other hand in Algebra, the polynomial is taken to be given by an expression such as the right hand side of 7.1.1., and the expression as such is of cardinal importance, while the function defined by 7.1.1. carries less weight. For instance, the two polynomials $p_1(x) = 1 + x^2$ and $p_2(x) = 1 - x$ have identical functional values for all argument values in the field of integers modulo 2. Nevertheless they are considered as different polynomials.

The question then arises what sort of reality can be attributed to such a concept. One method for bringing polynomials within the accepted order of mathematical objects is to conceive them as vectors (i.e., ordered sets) of the elements of the original ring, of arbitrary finite length (compare e.g. ref. 20). Thus $p(x)$ is represented by $(a_0, a_1, \ldots, a_n)$, the 'indeterminate' x is the particular vector

(0, 1) and there are suitable rules for operating with these vectors. This method, inasmuch as it is successful, is of course unexceptionable, but nevertheless it would appear to be a deviation from the original algebraic idea of a polynomial as a rule of construction, or operation. Such a rule of construction belongs, properly speaking, to the language in which the ring is discussed. In fact, it is expressed most appropriately in a language which includes functors (i.e. certain types of descriptions) as fundamental concepts. In our language, we shall represent them instead by predicates which give rise to descriptions. Such a predicate is $R(x, y)$ if we can deduce $(x)\, E!\, (\supset y)\, R(x, y)$ using the notation of Principia Mathematica, i.e., in our notation, if we can deduce $(x)(\exists y)(z)[R(x, y) \wedge [R(x, z) \supset E(y, z)]]$ from the given algebra of axioms. Thus our fundamental idea is that polynomials are merely certain types of predicates formulated in the object language.

7.2. *Prepolynomials and polynomials.* Let K be an algebra of axioms in a language L. We consider predicates in L which involve $n + 1$ dummy symbols, $n \geqslant 0$, $x_1, x_2, \ldots, x_n, y$, $R(x_1, x_2, \ldots, x_n, y)$. The particular position of y is important: thus we may select every one of the dummy variables of R as y, obtaining a different predicate every time. Then $R(x_1, x_2, \ldots, x_n, y)$ will be called a prepolynomial in K, if the two statements

7.2.1. $(x_1) \ldots (x_n)(\exists y)R(x_1, \ldots, x_n, y)$

and

7.2.2. $$(x_1) \ldots (x_n)(y)(z)[R(x_1, \ldots, x_n, y) \wedge R(x_1, \ldots, x_n, z) \supset E(y, z)]$$

are deducible from K. For $n = 0$, there are no quantifiers preceding that which involves y.

Amongst the prepolynomials of K, we select a subset whose elements will be called polynomials. They are defined as being of the form $(\exists z_1) \ldots (\exists z_m)Q(z_1, \ldots, z_m, x_1, \ldots, x_n, y)$ where Q is obtained by the repeated conjunction of formulae of order one in L. That is to say, Q contains no quantifiers and no copulae except $\wedge$, although it may contain any number of relative symbols and object symbols, as well as the specified $m + n + 1$ dummy symbols.

It is understood that while $x_1, \ldots, x_n, y$ are specific dummy symbols the $z_1, \ldots, z_m$ stand for any arbitrary dummy symbols other than $x_1, \ldots, x_n, y$. For $n \geqslant 1$, the set of polynomials (and thence, the set of prepolynomials) is not empty since it contains the elements

$$R(x_1, \ldots, x_n, y) = [E(x_1, x_1) \wedge E(x_2, x_2) \wedge \ldots \wedge E(x_i, y) \wedge \ldots \wedge E(x_n, x_n)] \; 1 \leqslant i \leqslant n$$

In fact the statements 7.2.1. and 7.2.2. for this predicate can be deduced from any algebra of axioms. Also R is of the form stipulated for polynomials ($m = 0$).

For given n, $n = 0, 1, \ldots$ we now define a structure M_n whose objects are the prepolynomials of order n, $R(x_1, \ldots, x_n, y)$ in K, while its relations are defined as follows.

Corresponding to any relative symbol A of order k, $k = 0, 1, 2, \ldots$ in L, we determine a relation A^* in M by defining that A^* shall hold between any prepolynomials $R_1, \ldots, R_k$, if

7.2.3. $(x_1) \ldots (x_n)(y_1) \ldots (y_k)[R_1(x_1, \ldots, x_n, y_1) \wedge R_2(x_1, \ldots, x_n, y_2) \wedge \ldots \wedge R_k(x_1, \ldots, x_n, y_k) \supset A(y_1, \ldots, y_k)]$

is deducible from K.

The set M_n' of all polynomials of order n, $R(x_1, \ldots, x_n, y)$ is a partial structure of M according to the above definition. M_n and M_n' will be known as the polynomial and prepolynomial structures of order n over K, respectively. We may also define the structure M_∞ as the structure whose objects are $R(x_1, x_2, \ldots, x_n, y)$ (n arbitrary) such that any relation A^* holds between the elements $R_1, \ldots, R_k$ of M_∞, if the statement

7.2.4. $q_1 \ldots q_m(y_1) \ldots (y_n)[R_1(x_1 \ldots x_{l_1}, y_1) \wedge \ldots \wedge R_k(x_1, \ldots, x_{l_k}, y_k) \supset A(y_1, \ldots, y_k)]$

is deducible from K, where the $q_1, \ldots, q_m$ are universal quantifiers comprising all the x_k included in the brackets. M_∞ again includes a substructure M_∞' whose objects are all polynomials in K. M_∞ also includes all the structures M_n as partial structures, while M_∞' includes all M_n'. According to the definitions which we have adopted, any given M_n does not include any M_l, $l < n$ (although this would not be difficult to achieve by a slight modification of our definitions), but it includes a substructure isomorphic to M. To obtain this substructure we select for every $R(x_1, \ldots, x_l, y)$ in M_l the prepolynomial $[E(x_{l+1}, x_{l+1}) \wedge E(x_{l+2}, x_{l+2}) \wedge \ldots \wedge E(x_n, x_n) \wedge R(x_1, \ldots, x_l, y)$ which is an element of the set of objects of M_n.

Finally, it will be seen that if K and K' are two equivalent algebras within the same language L then their prepolynomial and polynomial structures are identical.

7.3. *Properties of prepolynomial structures.* A natural correspondence between the relative symbols of K and the relations of a specific M_n is given by $A \leftrightarrow A^*$ for all the relations A of K. Also, corresponding to any object symbol a of K (if any), we select the polynomial

7.3.1. $R_a(x_1, \ldots, x_n, y) = [E(x_1, x_1) \wedge \ldots \wedge E(x_n, x_n) \wedge E(a,y)]$.

In this way we obtain a natural correspondence C between the object and relative symbols of K and some of the objects and relations of M_n (and of M'_n). The question now arises whether M_n is a model of K under the particular correspondence C. If this is the case then K will be called pre-transitive of order n. Similarly, if M'_n is a model of K under the correspondence C then K will be called transitive of order n. It is not difficult to construct examples which show that not every K is transitive or pre-transitive of specified order n. In other words a statement which belongs to K, or which is deducible from K, does not necessarily hold in M_n, or M'_n.

Thus, let K be the set A_F which characterises the concept of an algebraic field. Then 4.1.10. is not satisfied in the corresponding structures M_n and M'_n. For instance, consider the two polynomials $R(x_1, y) = [P(x_1, x_1, y)]$ and $S(x_1, y) = [E(x_1, x_1) \wedge P(y, y, y)]$. These are elements of M'_1 and hence of M_1. Now if 4.1.10. held in M_1 there would exist a prepolynomial T in M_1 such that $P^*(R, T, S)$ holds in M_1, i.e. by 7.2.3., such that

$$(x_1)(y_1)(y_2)(y_3)[R(x_1, y_1) \wedge T(x_1, y_2) \wedge S(x_1, y_3) \supset P(y_1, y_2, y_3)]$$

is deducible from A_F. However, interpreting this statement in a particular field we see, informally speaking, that y_3 always takes the value 1, while y_1 takes the value 0 provided x_1 takes that value. But since the equation $0 \cdot y = 1$ is insoluble, it follows that the statement cannot be satisfied for any prepolynomial T in M_1 (and less so for any polynomial T in M'_1).

However, it is possible to specify important classes of statements, which hold in M_n provided they are deducible from K.

We shall prove

7.3.2. *Theorem.* A statement X which is deducible from K holds in M_n if it is of one of the following three forms,

7.3.3. $X = [A(a_1, \ldots, a_m)]$

7.3.4. $X = [A_1(a_1, \ldots, a_m) \wedge A_2(a_1, \ldots, a_m) \wedge \ldots \wedge A_k(a_1, \ldots, a_m)]$

7.3.5. $X = [\sim [A(a_1, \ldots, a_m)]]$

where A, $A_1, \ldots, A_k$, $a_1, \ldots, a_m$ are relative symbols and object symbols in K, respectively.

In fact, since K is an algebra of axioms, $X \varepsilon S(K)$, where $X = X(a_1, \ldots, a_m)$ implies in all three cases (7.3.3.—7.3.5.) that the following statement also is deducible from K.

7.3.6. $$(x_1) \ldots (x_n)(y_1) \ldots (y_m)[[E(x_1, x_1) \wedge \ldots \wedge E(x_n, x_n) \wedge E(a_1, y_1)] \wedge [E(x_1, x_1) \wedge \ldots \wedge E(x_n, x_n) \wedge E(a_2, y_2)] \wedge \ldots \wedge [E(x_1, x_1) \wedge \ldots \wedge E(x_n, x_n) \wedge E(a_m, y_m)] \supset X(y_1, \ldots, y_m)]$$

Taking the three cases in turn, if X is given by 7.3.3. then a comparison of 7.3.6. and 7.2.3. shows, taking into account 7.3.1., that $A^*(R_{a_1}, \ldots, R_{a_m})$ is true in M_n, and hence that X holds in M_n under C.

Again, if X is of the form given by 7.3.4., then 7.3.6. shows that the relations $A_1^*(R_{a_1}, \ldots, R_{a_m}), \ldots A_k(R_{a_1}, \ldots, R_{a_m})$ are all true in M_n, and hence that X holds in M_n under C.

Finally, if X is given by 7.3.5., then 7.3.6. shows that at any rate

$$(x_1) \ldots (x_n)(y_1) \ldots (y_n)[R_{a_1}(x_1, \ldots, x_n, y_1) \wedge \ldots \wedge R_{a_m}(x_1, \ldots, x_n, y_m \supset A(y_1, \ldots, y_m)]$$

cannot be deduced from K, so that $A(a_1, \ldots, a_m)$ does not hold in M_n, i.e., $[\sim A(a_1, \ldots, a_m)]$ holds in M_n.

Other classes of statements which have the same property are mentioned in the following theorem, 7.3.7. The two theorems overlap.

7.3.7. A statement X which is deducible from K holds in M_n if it is given by

7.3.8. $X = q_1 q_2 \ldots q_m Q(z_1, \ldots, z_m)$, $m = 0, 1, 2, \ldots$

where q_k quantifies z_k, $k = 1, 2, \ldots, m$, and where Q does not involve any quantifiers, and is either a formula of order one, or a repeated conjunction of such formulae, or an implication in which both implicans and implicate are formulae of order one or are conjunctions of such formulae. Moreover, for any one of the q_k which is an existantial quantifier, q_l say, the following statement also is supposed to be deducible from K.

7.3.9. $$q_1' q_2' \ldots q_{l-1}' (z_l')(z_l'') q_{l+1} \ldots q_m [Q(z_1, \ldots, z_{l-1}, z_l', z_{l+1} \ldots, z_m) \wedge Q(z_1, \ldots, z_{l-1}, z_l'', z_{l+1}, \ldots, z_m) \supset E(z_l', z_l'')].$$

In this statement, the quantifiers $q_{l+1}, \ldots, q_m$ are defined as before, while the q_i', $i = 1, 2, \ldots, l-1$, though still quantifying the z_i, are all universal quantifiers.

The proof will be omitted (see preface).

Similarly we may search for statements which, if deducible from K, necessarily hold in the polynomial structure M_n'. Exactly corresponding to theorem 7.3.2., and demonstrable in the same way, we have

7.3.10. *Theorem.* A statement X which is deducible from K holds in M_n' if it is of one of the following three forms,

$$X = [A(a_1, \ldots, a_m)]$$

or

$$X = [A_1(a_1, \ldots, a_m) \wedge A_2(a_1, \ldots, a_m) \wedge \ldots \wedge A_k(a_1, \ldots, a_m)]$$

or

$$X = [\sim [A(a_1, \ldots, a_m)]]$$

where $A, A_1, \ldots, A_k, a_1, \ldots, a_m$ are relative and object symbols in K, respectively.

However, instead of 7.3.7. we now have the less comprehensive theorem

7.3.11. A sufficient condition for a statement X which is deducible from K to hold in M_n', is that it be of one of the following two forms,

7.3.12. $X = (z_1) \ldots (z_m) Q(z_1, \ldots, z_m)$

where Q is of the same description as in 7.3.8.

7.3.13. $X = (z_1) \ldots (z_{m-1})(\exists z_m) Q(z_1, \ldots, z_m)$, $m = 1, 2, \ldots$

where Q is a formula of order one, or a repeated conjunction, of such formulae. When $m = 1$, there will be no universal quantifier preceding the existential quantifier $(\exists z_1)$. Moreover, in this case it is assumed that the ancillary statement,

$$7.3.14. \quad (z_1)\ldots(z_{m-1})(z'_m)(z''_m)[Q(z_1,\ldots,z'_m) \wedge Q(z_1,\ldots,z''_m) \supset E(z'_m, z''_m)]$$

can also be deduced from K.

Although the class of statements defined by 7.3.18. is fairly restricted, it nevertheless includes a number of important cases, as will be shown presently.

A simple check shows that the axioms defining equality (6.1.1.—6.1.4.) all are of the form specified in 7.3.7. and 7.3.11. Thus the relation E^* corresponding to the relative symbol of equality in K, is a relation of equality in M_n and M'_n. This is a first application of the results of the preceding section. Next we shall prove

7.3.15. *Theorem.* Assume that K is an algebra of axioms which includes relative symbols of order 3, S and P, such that the axioms of A_{CR} are deducible from K, i.e., such that all the models of K are commutative rings. Then the polynomial and prepolynomial structures over K also are commutative rings.

In fact a survey of the axioms included in A_{CR} (4.1.1.—4.1.5., 4.1.7.—4.1.10., 4.1.12.—4.1.18.) shows that — with the single exception of 4.1.7. — they all come under the classes of statements mentioned in 7.3.7. and 7.3.11. Being deducible from K by assumption, they therefore hold in M_n and M'_n. Since, as has just been shown, the relation E^* constitutes an equality in M_n and M'_n, 4.1.7. necessarily holds in these structures provided they are not empty. And indeed, for $n \geqslant 1$, they both contain the polynomial

$$[E(x_1, x_1) \wedge \ldots \wedge E(x_{n-1}, x_{n-1}) \wedge E(x_n, y)]$$

while for $n = 0$, the predicate $[S(x, x, x)]$ is of the required description.

We have therefore shown that the polynomial and prepolynomial structures over K are all commutative rings. Corresponding facts hold for A_G and A_R, though not for A_F.

7.4. *Augmented systems of axioms.* We are now in a position to establish the connection between the polynomial and prepolynomial structures of the present theory, and the familiar polynomial rings of algebra.

We first introduce a concept which is of interest apart from the specific purpose for which it is required here. Let K be a consistent set of axioms, and M one of it's models, and let D be a complete diagram of M (relative to the relations of M which correspond to relative symbols in K), such that all the object symbols and relative symbols of K are contained in D, M being a model of K and D under the same correspondence. The union H of K and D will be said to be 'K augmented by D', $H = (K; D)$. The concept is of considerable practical importance because it is algebraic procedure, when studying the extensions of a specific system (ring, field, etc.,) to assume that the structure of the original system is 'completely known', and this may be interpreted formally as meaning that there are object symbols available for all the objects of the original system, and that the logical premisses from which we argue include a complete diagram of the original system. We note that D is an internal model of K (compare 2.8.).

Let K be an algebra of axioms. We consider polynomial and prepolynomial structures M_n, M'_n, $n = 0, 1, 2, \ldots$ over $H = (K; D)$, where D is a complete diagram of a model M of K. To begin with, we shall show that all the M'_n (and therefore all the M_n, which are extensions of the M'_n) contain partial structures which are isomorphic to M. For this purpose, we only have to prove that all the statements of D hold in M'_n (compare section 6.6.).

In fact, corresponding to the object symbols a of D, we have the polynomials $[E(x_1, x_1) \wedge E(x_2, x_2) \wedge \ldots \wedge E(x_n, x_n) \wedge E(a, y)]$ of M'_n and the totality of these polynomials constitutes a partial structure M''_n of M'_n. Also, any statement of D is either of the form $[A(a_1, \ldots, a_m)$ or it is of the form $[\sim [A(a_1, \ldots, a_m)]]$, and so holds in M'_n, by 7.3.17. Thus M''_n is a model of D and is therefore isomorphic to M.

7.5. *Applications to ring extensions.* Now assume in particular that the set K mentioned in the preceding paragraph coincides with A_{CR}, so that M is a commutative ring. Since A_{CR} is transitive and pretransitive (see 7.3.21.) it follows that the polynomial and prepolynomial structures M'_n and M_n over $H = (K; D)$ are commutative rings: also, by the result of section 7.4., they contain partial structures isomorphic to M. We proceed to show that M'_n is quasi-isomorphic to the polynomial ring of n variables adjoined to M, $M(x_1, \ldots, x_n)$. (In particular, M'_0 is quasi-isomorphic to M). It will be observed that the concept of a polynomial ring has not been defined within the present scheme; instead we have to accept it from Algebra. It will simplify the discussion without limiting its scope essentially, to assume that M and $M(x_1, \ldots, x_n)$ are both normal. $M(x_1, \ldots, x_n)$ is a model of H under a correspondence under which to every object in M, $q(x_1, \ldots, x_n)$, there corresponds an object symbol in D.

Also to every object symbol a' of D there corresponds the polynomial

$$7.5.1.\quad R_{a'}(x_1, \ldots, x_n, y) = [E(x_1, x_1) \wedge E(x_2, x_2) \wedge \ldots \wedge E(x_n, x_n) \wedge E(a', y)]$$

of M'_n and in this way we obtain a correspondence between M and a partial structure of M'_n, as detailed in section 7.4. More generally let $q(x_1, \ldots, x_n)$ be an arbitrary element of $M(x_1, \ldots, x_n)$. Corresponding to q, we now select a specific element of M'_n, R_q, such that the statement $(x_1) \ldots (x_n)(\exists y)R_q(x_1, \ldots, x_n, y)$ is a formal expression of 'There exists a y such that $y = q(x_1, \ldots, x_n)$'.

The definition of 5.3.5. is not suitable for our present purpose since it presumes the existence of a unit element (for the definition of $P_k(x, y)$), and refers only to polynomials with integral coefficients, indicating repeated addition.

Instead, we now define

$$7.5.2.\quad P_1(x, y) = E(x, y),\quad P_{n+1}(x, y) = (\exists z)[P_n(x, z) \wedge P(z, x, y)]\ n = 1, 2, \ldots$$

and then, for arbitrary positive integer i_k.

$$7.5.3.\quad Q_{i_1}(x_1, y) = P_{i_1}(x_1, y)$$
$$Q_{i_1 i_2}(x_1, x_2, y) = (\exists z_1(\exists z_2)[Q_{i_1}(x_1, z_1) \wedge P_{i_2}(x_2, z_2) \wedge P(z_1, z_2, y)]$$
$$\vdots$$
$$Q_{i_1 i_2 \ldots i_m}(x_1, x_2, \ldots, x_m, y) = (\exists z_1)(\exists z_2)[Q_{i_1 i_2 \ldots i_{m-1}}(x_1, x_2, \ldots, x_{m-1}, z_1) \wedge P_{i_m}(x_m, z_2) \wedge P(z_1, z_2, y)].$$

Next, we define for any object a of M,

$$7.5.4.\quad Q_{i_1 i_2 \ldots i_m; a}(x_1, \ldots, x_m, y) = (\exists z)\,[Q_{i_1 i_2 \ldots i_m}(x_1, \ldots, x_m, z) \wedge P(a', z, y)]$$

where a' is the object symbol corresponding to a in D and

$$7.5.5.\quad Q_{i_1 i_2 \ldots i_m; a}(x_1, \ldots, x_m, y) = (\exists z)\,[Q_{i_1 i_2 \ldots i_m}(x_1, \ldots, x_m, z) \wedge S_Q(z, y)]$$

if a is a positive integer indicating repeated addition, and

$$7.5.6.\quad Q_{i_1 i_2 \ldots i_m; a}(x_1, \ldots, x_m, y) = (\exists z_1)(\exists z_2)\,[Q_{i_1 i_2 \ldots i_m; -a}(x_1, \ldots, x_m, z_1) \wedge S(z_2, z_2, z_2) \wedge S(z_1, y, z_2)]$$

if a is a negative integer indicating repeated subtraction.

Now let $q(x_1, \ldots, x_n)$ be any polynomial in $M(x_1, \ldots, x_n)$

$$q(x_1, \ldots, x_n) = a_0 + a_1 x_{k_1^{(1)}}^{i_1^{(1)}} x_{k_2^{(1)}}^{i_2^{(1)}} \ldots x_{k_{m_1}^{(1)}}^{i_{m_1}^{(1)}} + \ldots + a_s x_{k_1^{(s)}}^{i_1^{(s)}} \ldots x_{k_{m_s}^{(s)}}^{i_{m_s}^{s}}$$

where the k are all integers between 1 and n, the i are positive integers, and the a_k, $k = 1, 2, \ldots, s$ are either elements of M, supposed different from 0, or positive integers, indicating repeated addition, or negative integers indicating repeated subtraction. a_0 is an element of M; if $q(x_1, \ldots, x_n)$ does not include a constant term we still write $a_0 = 0$.

We then define the corresponding predicate R_q by

$$\begin{aligned}7.5.7.\quad R_q(x_1, \ldots, x_n, y) = {} & (\exists y_1) \ldots (\exists y_s)(\exists z_1) \ldots (\exists z_{s-1}) \\ & [Q_{i_1^{(1)} i_2^{(1)} \ldots i_{m_1}^{(1)}; a_1}(x_{k_1^{(1)}}, x_{k_2^{(1)}}, \ldots, x_{k_{m_1}^{(1)}}, y_1) \\ & \wedge \ldots \wedge Q_{i_1^{(s)} i_2^{(s)} \ldots i_{m_s}^{(s)}; a_s}(x_{k_1^{(s)}}, x_{k_2^{(s)}} \ldots x_{k_{m_s}^{(s)}}, y_s) \\ & \wedge S(y_1, y_2, z_1) \wedge S(z_1, y_3, z_2) \wedge \ldots \wedge S(z_{s-2}, y_s, z_{s-1}) \\ & \wedge S(a_0', z_{s-1}, y)]\end{aligned}$$

where a_0' is the object symbol corresponding to a_0 in D. If the right hand side of 7.5.7. does not include all the x_n, then we add terms of the form $E(x_k, x_k)$, in conjunction, for those x_k which are not included. Again, if $q(x_1, \ldots, x_n)$ reduces to a constant a, then we define $R_q = R_a$, as given by 7.5.1.

Assume then that we have selected an object R_q of M_n' for every

$q(x_1, \ldots, x_n)$. We may then show that every other object of M_n' is equal, within M_n', to at least one R_q. In fact, let R be any object of M_n',

$$R(x_1, \ldots, x_n, y) = (\exists z_1) \ldots (\exists z_m)\, Q(z_1, \ldots, z_m, x_1, \ldots, x_n, y)$$

where Q is obtained by the repeated conjunction of formulae of order one,

$$Q = [A_1(z_1, \ldots, x_n, y) \wedge A_2(z_1, \ldots, x_n, y) \wedge \ldots \wedge A_s(z_1, \ldots, x_n, y)].$$

Since R is a (metamathematical) polynomial in H, it follows that the statement $(x_1) \ldots (x_n)(\exists y)R(x_1, \ldots, x_n, y)$ i.e.,

7.5.8. $(x_1) \ldots (x_n)(\exists y)(\exists z_1) \ldots (\exists z_m)\, Q(z_1, \ldots, z_m, x_1, \ldots, x_n, y)$

can be deduced from H. It therefore holds in all the models of H, e.g., in the ring $M(a_1, \ldots, a_n)$ obtained from M by the adjunction of the n indeterminate quantities $a_1, \ldots, a_n$. (This ring is isomorphic to $M(x_1, \ldots, x_n)$ but in algebraic considerations may be considered to be a separate entity). Now interpreting 7.5.8. semantically in $M(a_1, \ldots, a_n)$, it states that in the particular case where the object symbols standing for $x_1, \ldots, x_n$ correspond to $a_1, \ldots, a_n$, there exist elements of $M'(a_1, \ldots, a_n)$, i.e., polynomials of $a_1, \ldots, a_n$, $y = q_0(a_1, \ldots, a_n)$, $z_i = g_i(a_1, \ldots, a_n)$ such that all the relations corresponding to the relative symbols in Q hold in $M(a_1, \ldots, a_n)$ for these arguments,

7.5.9. $A_i(g_1(a_1, \ldots, a_n), \ldots, g_m(a_1, \ldots, a_n), a_1, \ldots, a_n, q_0(a_1, \ldots, a_n))$

holds in $M(a_1, \ldots, a_n)$ for $i = 1, 2, \ldots s$. A_i is one of the relations corresponding to relative symbols of A_{CR}, i.e., corresponding to E, S, or P. In addition to $a_1, \ldots, a_n, g_1, \ldots, g_n, q_0$, it may involve any objects of M.

We wish to prove that R equals R_{q_0} according to the relation of equality defined in M_n'. By definition this is the case, if

7.5.10. $(x_1) \ldots (x_n)(y)(z)[R(x_1, \ldots, x_n, y) \wedge R_{q_0}(x_1, \ldots, x_n, z) \supset E(y, z)]$

can be deduced from H. Since R and R_{q_0} are both (metamathematical) polynomials, it is sufficient to prove instead that

7.5.11. $(x_1) \ldots (x_n)(\exists y)[R(x_1, \ldots, x_n, y) \wedge R_{q_0}(x_1, \ldots, x_n, y)]$

can be deduced from H. Alternatively again, it is sufficient to show that 7.5.11. holds in all models of H. One model of H in which 7.5.11. certainly does hold is M itself. For, given objects $\beta_1, \ldots, \beta_n$, we choose the object corresponding to y as $q_0(\beta_1, \ldots, \beta_n)$. Then the principle of substitution in Algebra states that every set of relations which hold between the polynomials of a ring, still holds if we replace the variables or indeterminate quantities, by arbitrary elements of the ring. (It is quite easy to verify directly the special cases of this principle which are required here). Hence choosing $g_i(\beta_1, \ldots, \beta_n)$ for the remaining dummy symbols z_i involved in $(\exists z_1) \ldots (\exists z_n)Q(z_1, \ldots, z_n, x_1, \ldots, x_n, y)$, we see that all the relations

$$7.5.12. \quad A_i(g_1(\beta_1, \ldots, \beta_n), \ldots, g_m(\beta_1, \ldots, \beta_n), \beta_1, \ldots, \beta_n q_0 \\ \beta_1, \ldots, \beta_n))$$

hold in M. Also, R_{q_0} holds in M, for the object symbols corresponding to $\beta_1, \ldots, \beta_n$ and to $y = q_0(\beta_1, \ldots, \beta_n)$ by its definition, and so 7.5.11. holds in M.

To show that 7.5.11. also holds in all other models of H, we recall that any model of H contains a partial system isomorphic to M. It is therefore sufficient to show that 7.5.11. holds in any extension M^* of M. Then the ring $M^*(\alpha_1, \ldots, \alpha_n)$ obtained from M^* by the adjunction of the n indeterminate qualities $\alpha_1, \ldots, \alpha_n$, includes $M(\alpha_1, \ldots, \alpha_n)$. Thus, all the relations of 7.5.9. hold in M^* $(\alpha_1, \ldots, \alpha_n)$ for the arguments $g_1, \ldots, g_m, \alpha_1, \ldots, \alpha_n, q_0$, as before. Hence also, it can be shown by means of the principle of substitution that 7.5.11. holds in M^*, as required.

We have therefore shown that any metamathematical polynomial of M_n' is equal to a specific polynomial R_q for one $q(x_1, \ldots, x_n)$. Also, it is not difficult to show that, for different $q_1, q_2, E^*(R_{q_1}, R_{q_2})$ does not hold in M_n', while, for arbitrary q_1 and q_2 the relations $S(R_{q_1}, R_{q_2}, R_{q_1+q_2})$ and $P(R_{q_1}, R_{q_2}, R_{q_1 q_2})$ both hold in M_n'. This shows that the partial structure of M_n' which consists of the polynomials R_q is quasi-isomorphic to M_n'. Hence, taking into account that E^* is a relation of equality in M_n', M_n' is quasi-isomorphic to

$$M(x_1, x_2, \ldots, x_n).$$

By contradistinction, M_n is not related directly to the polynomial ring $M(x_1, \ldots, x_n)$. The investigation of its structure, for given M, may be of considerable mathematical interest.

If H is given by the algebra of axioms for a general (non-commutative) ring. A_r, augmented by a complete diagram D of a model M of A_R, $H = (A_R; D)$, then both M_n and M_n' are still rings. However, the ring of polynomials $M(x_1, \ldots, x_n)$ over a general ring is usually defined as consisting of the terms

$$\sum a_k x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n},$$

where it is assumed that the variables $x_1, \ldots, x_n$ are commutable with the elements of M as well as with each other. The ring M_n' is not subject to the corresponding assumption; in fact it may now be quite difficult to characterise M_n' by purely algebraic means. The assumption that all the variables $x_1, \ldots, x_n$ belong to the centre of the ring is certainly not satisfied in all cases. Thus, for example, whether two polynomials ax_1 and x_1a are properly to be regarded as equal or different in $M(x_1)$ may well depend on the individual a. The corresponding objects of M_1', $P(a, x_1, y)$ and $P(x_1, a, y)$ are regarded as equal or unequal, according as

$$(x_1)(y)(z)[P(a, x_1, y) \wedge P(x_1, a, z) \supset E(y, z)]$$

is deducible from H. This is not necessarily the same as $ax_1 = x_1a$ identically in M, and it may be difficult to decide by purely algebraic means whether it does or does not apply. Nevertheless, it may be argued that M_n' has more claim to the name of a polynomial extension than the algebraic construct which is based on the assumption that all the variables $x_1, \ldots, x_n$ belong to the centre of the ring.

7.6. *Applications to ring extensions, continued.* As a complementary investigation, we may consider the structures M_n and M_n' over the algebra of axioms A_{CR}, without reference to any particular model. We propose to show that for $n \geqslant 1$, M_n' is quasi-isomorphic to the ideal $(x_1, \ldots, x_n)$ in the polynomial ring of n variables adjoined to the ring of integral numbers. In other words, M_n' is isomorphic to the ring of polynomials of n variables, whose coefficients are integers, the constant being equal to 0. We denote this ring by M_n^*.

As in the preceding section, we select a specific element of M_n', R_q, for every given polynomial $q(x_1, \ldots, x_n)$ belonging to M_n^*. The polynomials now possess only integral coefficients, indicating repeated addition or subtraction. The definition 7.5.7. can still be used, but when $q(x_1, \ldots, x_n) \equiv 0$ identically, we may now choose

7.6.1. $R_q(x_1, \ldots, x_n, y) = [E(x_1, x_1) \wedge \ldots \wedge E(x_n, x_n) \wedge S(y,y,y)]$

Since the ring M_n^* is a model of A_{CR}, it can be shown that all the R_q are different in M_n' for different q, and that the operations with R_q are isomorphic to the corresponding operations in M_n^*, $S(R_{q_1}, R_{q_2}, R_{q_1+q_2})$ and $P(R_{q_1}, R_{q_2}, R_{q_1 q_2})$ holds in M_n'. Thus, a partial structure of M_n' is isomorphic to M_n^*, and it only remains to show that every object R of M_n' is equal to an R_q. That is to say that, given R, we can find R_q, such that

7.6.2. $(x_1) \ldots (x_n)(\exists y)[R(x_1, \ldots, x_n, y) \wedge R_q(x_1, \ldots, x_n, y)]$

is deducible from A_{CR}. It is in fact not difficult to determine $q(x_1, \ldots, x_n)$ in such a way that this condition is satisfied, by the method which was employed in the preceding section for a very similar purpose.

We may consider the corresponding problem for the algebra of axioms corresponding to a commutative ring with unit element, A_{CU}, which is obtained from A_{CR} by the addition of the axiom,

7.6.3. $(x)[P(x, e, x,)]$

where e is an object symbol. A_{CU} is transitive and pre-transitive of all orders (so that M_n and M_n' are again rings with unit elements). The methods of this and the preceding section show that M_n' is quasi-isomorphic to the polynomial ring of n variables adjoined to the ring of rational integers. M_0' is quasi-isomorphic to the ring of rational integers. Thus, the equivalence classes of M_0' are isomorphic to the ring of rational integers.

Alternatively, we might *define* the ring of integers as the ring of equivalence classes of M_0' over the (finite) algebra of axioms A_{CU}. This adds one more to the various methods available for the introduction of integers — the definition of integers as sets of statements.

7.7. *Homomorphisms of prepolynomial structures.* Let K be an algebra of axioms, M one of its models, and M_n and M_n' prepolynomial and polynomial structures over K, supposed not empty. We may then specify partial structures of M which are quasi-homomorphic to M_n or to M_n', in the following way. Since M is a model of K, there exists a correspondence C' (under which M is a model of K) such that to every object in M, there corresponds an object symbol in L (or L'). Now for every prepolynomial $R(x_1, \ldots, x_n, y)$, the two statements

7.7.1. $(x_1) \ldots (x_n)(\exists y)R(x_1, \ldots, x_n, y)$

and

7.7.2. $(x_1) \ldots (x_n)(y)(z)[R(x_1, \ldots, x_n, y) \wedge R(x_1, \ldots, x_n, z) \supset E(y, z)]$

are deducible from K, and therefore hold in M. We now select n objects $a_1, \ldots, a_n$ of M, some of which may coincide. Let $a_1', \ldots, a_n'$ be the object symbols corresponding to $a_1, \ldots, a_n$. Then the statements $(\exists y)R(a_1', \ldots, a_n', y)$ and $(y)(z)[R(a_1', \ldots, a_n', y) \wedge R(a_1', \ldots, a_n', z) \supset E(y, z)]$ hold in M. Thus, $R(a_1', \ldots, a_n', a')$ holds for certain object symbols, corresponding to objects a in M. In particular, if $R(a_1', \ldots, a_n', a')$ holds for a specific a' corresponding to a in M, then it holds for all the object symbols corresponding to objects which are equal to a in M, and only for these. This defines a correspondence C^* between the prepolynomials of M_n and the objects of equivalence classes with respect to the relation of equality in M. Also, it follows from 7.2.3. that if a relation A^* holds between certain prepolynomials, then the relation A holds between the corresponding elements of M under C^*. Thus, M_n is quasi-homomorphic to the partial structure $M^{(n)}$ of M which consists of the objects of M which appear in C^*. If $n = 0$, the preliminary choice of objects $a_1, \ldots, a_n$ is redundant. If M is normal, then M_n is homomorphic to $M^{(n)}$.

Also, M_n' which is a partial structure of M_n is quasi-homomorphic to a partial structure of $M^{(n)}$: it is homomorphic to that partial structure if M is normal.

Now assume that K is transitive of order n, so that M_n' is a model of K. (If $n = 0$, then we have to add the assumption that M_0' is not empty). We consider M_1' as the model M of the procedure outlined above, and define the prepolynomial $R^*_{x_k}(x_1, \ldots, x_n, y)$ as the a_k of that procedure, $k = 1, 2, \ldots, n$, where $R^*_{x_k}$ is defined by

$$7.7.3. \quad R^*_{x_k}(x_1, \ldots, x_n, y) = [E(x_1, x_1) \wedge E(x_2, x_2) \wedge \ldots \wedge E(x_k, y) \wedge \ldots \wedge E(x_n, x_n)]$$

(We note that if M_n has the same significance as in sections 7.5. and 7.6., then the polynomial $R^*_{x_k}$, $k = 1, 2, \ldots, n$, equals, within M_n, the polynomial R_{x_k} as defined in those sections, i.e., it equals the polynomial R_q for $q(x_1, \ldots, x_n) = x_k$). In this way we obtain a quasi-homomorphism between M_n and M'_n which is a subset of M_n. The correspondence is reflexive, i.e., any object of M'_n taken as an object of M_n, corresponds to itself (and to the polynomials equal to itself) as an object of M'_n.

In the particular case when K includes A_{CR}, so that M_n, and M'_n are commutative rings, it is instructive to consider the normal algebraic structures associated with M_n and M'_n, say N_n and N'_n. N'_n is a subring of N_n and the quasi-homomorphism constructed above includes a reflexive endomorphism between N_n and N'_n. This endomorphism is completely determined by the set J of objects of N_n which correspond to the zero of N'_n. J is an ideal in N_n. If we regard M'_n, and therefore N'_n as known, we may therefore reduce the study of M_n to the study of a particular ideal in N_n.

Our reason for selecting $R_{x_1}, R_{x_2}, \ldots, R_{x_n}$ as the $a_1, a_2, \ldots, a_n$ required for the construction of the correspondence becomes obvious when K equals A_{CR}, or when it equals A_{CR} augmented by a complete diagram D. For in that case M'_n is quasi-isomorphic to a polynomial ring $M(x_1, \ldots, x_n)$, under a correspondence under which R_{x_1}, R_{x_2} etc., correspond to x_1, x_2, etc., respectively. Combining this with the correspondence between M'_n and M_n, we obtain a homomorphism between M_n and $M(x_1, \ldots, x_n)$.

VIII

ALGEBRAIC PREDICATES

8.1. *Introduction.* In this and the following chapter we shall study problems concerning the relation between a given structure and its extensions and partial structures, all of which are supposed to be models of specified sets of axioms.

Let K be a consistent set of axioms within a restricted language L, M one of its models, and H the set K augmented by a full diagram D of M, $H = (K; D)$. Then every model M' of H contains a partial model isomorphic to M. Conversely we may then construct an extension of M which is isomorphic to M'.

In the algebraic case, the original reason for the interest in the extensions of a structure M was that the extension might contain elements with properties which are not shared by any element of M, in particular, the property of satisfying a specific algebraic equation. More generally, we may be concerned with the construction of an extension M' of M in which a set of statements K' holds in addition to the statements of K. Thus the union H' of K, K' and D holds in M'. The question whether a structure M' of this description exists is equivalent to the other question, whether H' is consistent. Since H' is consistent if and only if all its finite subsets are consistent, we have the following theorem.

8.1.1. There exists a model of K which is an extension of M, and in which K' also holds, if and only if, for every finite subset K'' of K', there exists an extension of M which satisfies both K and K''.

This simple theorem is already quite effective in shortening some of the more abstract arguments of Steinitz' theory (refs. 14, 15). For instance, let us try to prove that with the aid of 8.1.1. that there exists an algebraically closed extension F' of any given commutative field F. We put $K = A_F$, $M = F$ and we define statements X_q within L as stating that 'the polynomial $q(x)$ can be decomposed into linear factors', where $q(x)$ is any polynomial with coefficients in M, the coefficient of the highest power being one. Such a statement can be formalised, for instance, as 'There exist $y_1, y_2, \ldots, y_n$ whose fundamental symmetrical functions equal $-a_1, +a_2, \ldots, (-1)^n a_n$, where $a_1, a_2, \ldots, a_n$ are the coefficients of

$$q(x) = x^n + a_1 x^{n-1} + \ldots + a_n.$$

Assume now that we have already proved that for every polynomial $q(x)$ with coefficients in F, there exists an extension of F in which $q(x)$ can be decomposed into linear factors. By putting $q(x) = q_1(x)q_2(x) \ldots q_n(x)$, we see that for every finite set of polynomials with coefficients in F there exists an extension of F in which all the polynomials $q_1(x), q_2(x), \ldots, q_n(x)$, can be decomposed into linear factors. It then follows from 8.1.1. that there exists an extension F^* of F in which all the statements X_q hold i.e., in which all the polynomials with coefficients in F can be decomposed into linear factors. The set of elements of F^* which satisfy polynomials with coefficients in F is the required algebraically closed field F'.

To prove this we have to accept the following additional theorems.

8.1.2. The sum, difference, product and quotient of elements which are algebraic with respect to F, are algebraic with respect to F. This proves that F' is a field.

8.1.3. A polynomial in F^* whose coefficients are algebraic with respect to F possesses a root in F^* which is algebraic with respect to F.

Thus we see that theorem 8.1.1. while not very helpful in so far as the more specifically algebraic parts of the problem are concerned, enables us to omit the 'set theoretical' argument by which Steinitz passes from the finite to the infinite extensions. A more effective way for coping with the same problem will be given in section 8.7. below.

We shall find it convenient sometimes in the sequel to employ geometrical language. Let M be any structure which contains at least one object. For given $n \geqslant 1$, we consider the set of (ordered) n-ples of equal or different objects in M, $P = (a_1, \ldots, a_n)$. Every such n-ple will be called a point, and the totality of these points will be said to constitute the n-dimensional space S_n over M. The objects $a_1, \ldots, a_n$ will be called the coordinates of P. If M is an algebraic structure then the points $P_1 = (a_1', \ldots, a_n')$ and $P_2 =$

$(a_1'', \ldots, a_n'')$ will be called equal if and only if their respective coordinates are equal within M, $E(a_1', a_1''), \ldots, E(a_n', a_n'')$.

Now let C be a (complete or incomplete) correspondence between M and a language L, and let $Q(x_1, \ldots, x_n)$ be a predicate in L whose object and relative symbols all correspond to objects and relations in M under C. Further let C' be any complete correspondence which includes C (see section 2.8.). Then $Q(x_1, \ldots, x_n)$ will be said to hold at the point $P = (a_1, \ldots, a_n)$ if $Q(a_1', \ldots, a_n')$ holds in M under C', where $a_1', \ldots, a_n'$ are the object symbols corresponding to $a_1, \ldots, a_n$ respectively under C'. If $n = 1$, we shall also write simply that $Q(x)$ holds at, or for, a_1 (instead of (a_1)).

The question whether or not $Q(x_1, \ldots, x_n)$ holds at a point P depends only on the correspondence C and not on the additional choice of the object symbols corresponding to objects of M under C'. Unless stated otherwise the correspondence C will be assumed to be given once and for all.

Let J_Q be a set of predicates of order n, $Q(x_1, \ldots, x_n)$, such that all the object and relative symbols of the elements of J_Q are included in the given correspondence C. Then the variety V of J_Q in S_n is defined as the set of all points in S_n, at which all the predicates of J_Q hold.

A set J_Q of predicates of order n will be called conjunctive if together with any two elements, it also contains their conjunction; it will be called disjunctive if together with any two elements it also contains their disjunction.

8.2. *Bounded predicates.* The elements of an extension F' of a field F can be classified as algebraic or transcendental with respect to F. Algebraic elements are elements which satisfy a polynomial with coefficients in F. Thus we are again led to the consideration of predicates as a preliminary to the consideration of specific elements. An important property of polynomials is that they can be satisfied only by a finite number of different ('unequal') elements. This property will now be generalised.

Let K be an algebra of axioms. A predicate $Q(x)$ will be called bounded in K if at least one of the statements

8.2.1. $X_m = (x_1)(x_2) \ldots (x_{m+1})[Q(x_1) \wedge \ldots \wedge Q(x_{m+1}) \supset E(x_1, x_2) \vee E(x_1, x_3) \vee \ldots \vee E(x_1, x_{m+1}) \vee E(x_2, x_3) \vee \ldots \vee E(x_m, x_{m+1})]$, $m = 1, 2, \ldots$

can be deduced from K. The smallest m for which X_m is deducible from K will be called the degree of the predicate, $\deg Q(x) = m$. However, if $(x)[\sim Q(x)]$ is deducible from K then we put $\deg Q(x) = 0$.

Similarly, we may define the concept of a bounded predicate of higher order as follows. $Q(x_1, \ldots, x_n)$ will be said to be bounded in the algebra of axioms K if at least one of the statements

$$8.2.2.\quad \begin{aligned} X_m = {} & (x_1^{(1)}) \ldots (x_n^{(1)})\,(x_1^{(2)}) \ldots (x_n^{(2)}) \ldots (x_1^{(m+1)}) \ldots (x_n^{(m+1)}) \\ & [Q(x_1^{(1)}, \ldots, x_n^{(1)}) \wedge \ldots \wedge Q(x_1^{(m+1)}, \ldots, x_n^{(m+1)}) \supset \\ & [E(x_1^{(1)}, x_1^{(2)}) \wedge E(x_2^{(1)}, x_2^{(2)}) \wedge \ldots \wedge E(x_n^{(1)}, x_n^{(2)})] \vee \\ & [E(x_1^{(1)}, x_1^{(3)}) \wedge E(x_2^{(1)}, x_2^{(3)}) \wedge \ldots \wedge E(x_n^{(1)}, x_n^{(3)})] \vee \ldots \vee \\ & [E(x_1^{(m)}, x_1^{(m+1)}) \wedge E(x_2^{(m)}, x_2^{(m+1)}) \wedge \ldots \wedge E(x_n^{(m)}, x_n^{(m+1)})]] \\ & \qquad\qquad m = 1, 2, \ldots \end{aligned}$$

can be deduced from K. The smallest m for which X_m is deducible from K will again be called the degree of Q, $\deg Q(x_1, \ldots, x_n) = m$, except when $(x_1) \ldots (x_n)[\sim Q(x_1, \ldots, x_n)]$ is deducible from K in which case we put $\deg Q(x_1, \ldots, x_n) = 0$.

It is easy to see that the conjunction of bounded predicates of the same order is again a bounded predicate such that

$$8.2.3.\quad \deg [Q_1(x_1, \ldots, x_n) \wedge Q_2(x_1, \ldots, x_n)] \leqslant \min (\deg Q_1 (x_1, \ldots, x_n), \deg Q_2 (x_1, \ldots, x_n)).$$

Also, the conjunction of a bounded predicate Q_1 and a predicate Q_2 of the same order which is unbounded is again bounded, and its degree cannot exceed the degree of Q.

The disjunction of two bounded predicates Q_1 and Q_2 also is bounded, and its degree cannot exceed the sum of the degrees of Q_1 and Q_2.

$$8.2.4.\quad \deg [Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n)] \leqslant \deg Q_1(x_1, \ldots, x_n) + \deg Q_2(x_1, \ldots, x_n).$$

It will simplify the proof to assume $n = 1$ although it is quite similar for arbitrary n. Let m_1 and m_2 be the degrees of Q_1 and Q_2 respectively, so that the statement 8.2.1. can be deduced from K, for $Q = Q_1$, $m = m_1$ and $Q = Q_2$, $m = m_2$, respectively. We want to show that 8.2.1. also can be deduced from K for $Q = Q_1 \vee Q_2$, $m = m_1 + m_2$. For this purpose we only have to establish that for the case in question, 8.2.1. holds in all the models M of K. In other words we have to verify that

8.2.5. $$[Q_1(a_1) \vee Q_2(a_1)] \wedge [Q_1(a_2) \vee Q_2(a_2)] \wedge \ldots \wedge [Q_1(a_{m_1+m_2}) \vee Q_2(a_{m_1+m_2})] \wedge [Q_1(b) \vee Q_2(b)] \wedge \sim E(a_1, a_2) \wedge \ldots \wedge \sim E(a_{m_1+m_2-1}, a_{m_1+m_2}) \supset E(b, a_1) \vee \ldots \vee E(b, a_{m_1+m_2})]$$

holds in M for arbitrary object symbols corresponding to objects in M within a language which includes K. If the implicans of 8.2.5. does not hold in M, then 8.2.5. certainly does. On the other hand, if the implicans holds in M, then for every a_i, $i = 1, 2, \ldots, m_1 + m_2$, at least one of the statements $Q_1(a_i)$, or $Q_2(a_i)$ holds in M. Thus the number of suffixes i for which $Q_1(a_i)$ holds together with the number of suffixes for which $Q_2(a_i)$ holds, cannot be less than $m_1 + m_2$. But $Q_1(a_i)$ cannot hold for more than m_1 different i, by 8.2.1. since $E(a_i, a_k)$ holds for all $i \neq m$. Similarly $Q_2(a_i)$ cannot hold for more than m_2 different i. Hence $Q_1(a_i)$ holds for exactly m_1 different i, while $Q_2(a_i)$ holds for exactly m_2 different i. Also, either $Q_1(b)$ holds in M, or $Q_2(b)$ or both. In the first case b equals one of the a_i for which $Q_1(a_i)$ holds, in the second case b equals one of the a_i for which $Q_2(a_i)$ holds. Thus in any case the implicate of 8.2.5. holds in M, and so therefore does 8.2.5.

Instead of defining the properties of a predicate in terms of deducibility from an algebra of axioms K we may consider the question in connection with the models of K. Thus, we shall say that a predicate $Q(x)$ is saturable in an algebra of axioms K if for any existing chain of models of K,

$$M_1 \lessdot M_2 \lessdot M_3 \lessdot M_4 \lessdot \ldots$$

and a sequence of objects a_n, $a_n \varepsilon M_n$, such that $Q(a'_k)$ holds in M_n for all $k \leqslant n$, there exists an n_0 such that for all $n > n_0$, a_n equals some a_m, $m \leqslant n_0$. In this connection a'_n is the object symbol corresponding to the object a_n under a correspondence under which all the M_n are models of K, and $Q(x)$ is supposed to be formulated in terms of the relative and object symbols of K.

We shall prove the theorem

8.2.6. A necessary and sufficient condition for a predicate $Q(x)$ to be saturable in an algebra of axioms K is that it be bounded in K.

It is not difficult to see that the condition is sufficient. For, let m be the degree of $Q(x)$. If $Q(x)$ is not saturable then there exists a chain $M_1 \lessdot M_2 \lessdot M_3 \lessdot \ldots$ as above such that for sufficiently high k, M_k contains more than m different objects at which $Q(x)$ holds. This is impossible.

To prove that the condition is necessary, we require the following lemma.

8.2.7. If a predicate $Q(x)$ holds at an infinite number of different objects, a_n, of a model M of an algebra of axioms K (by the same correspondence by which $Q(x)$ holds at the a_n), then there exists an extension M' of M in which $Q(x)$ holds at the same a_n, and at one additional object b, $b \varepsilon M—M'$ at least.

Let H be the algebra of axioms K augmented by a full diagram D of M, and let D' be the set of statements $[\sim E(b', a')]$ where a' varies over all the object symbols of D while b' is an object symbol not contained in D. Further let Y_0 be the statement $Q(b')$, while Y_n is defined as the statement $Q(a_n')$, $n = 1, 2, \ldots$ for all the object symbols a_n' corresponding to the objects a_n which are mentioned in the lemma. Let K' be the set of all Y_n $n = 0, 1, 2, \ldots$, and let H' be the union of H, D', and K'. H' is consistent. To prove this we only have to show that all the finite subsets of H' are consistent. Let H'' be such a finite subset. Since H'' is finite, there exists a positive integer n_0 such that none of the object symbols included in H'' is equal to any a_n', $n \geqslant n_0$. We now establish a correspondence between the object and relative symbols of H'' and some of the objects and relations of M in the following way. All the object and relative symbols of H'' are included in H except b'. We define that these shall correspond to the same objects and relations in M, as in the given correspondence which establishes M as a model of H. The only other object symbol in H'', b', shall correspond to a_{n_0}. We maintain that M is a model of H'' under this correspondence. This is clear in so far as the statements of H'' which belong also to H are concerned. Furthermore, all the statements of $H'' \cap D'$ are satisfied, by the definition of n_0. Finally the statements Y_n, $n = 1, 2, \ldots$ hold in M by assumption, and $Y_0 = Q(b')$ holds in M

since b' corresponds to a_{n_0} in M. Thus H'' is consistent, and it follows that H' also is consistent. Let M' be a model of H. M' contains a partial model corresponding to D which is isomorphic to M. We may therefore assume that M' is actually an extension of M. Then $Q(x)$ holds at all the a_n in M' since $Q(a_n')$ holds in M', and it also holds at an object b, corresponding to b', which is not included in M.

Having established the lemma, we are now in a position to prove the necessity of the condition in theorem 8.2.6. If $Q(x)$ is not bounded in K then none of the statements X_n defined in 8.2.1. above can be deduced from K. It follows that the set K^* of statements $[\sim X_n]$, $n = 1, 2, \ldots$, is consistent with K. For if the union of K and K^* were contradictory, then a finite subset K'' of K^* would be inconsistent with K. But clearly $\sim X_n \supset \sim X_m$ is deducible from K for all $n \geqslant m$. Hence if K'' is inconsistent with K, then the statement $[\sim X_n]$ with the highest suffix n included in K'', is inconsistent with K. It follows that X_n is deducible from K, which is contrary to the assumption that $Q(x)$ is not bounded in K. Thus the union H^* of K and K^* is consistent.

Let M_1 be a model of H^*. Since H^* includes K^* it follows thatM_1 contains an infinite number of different (unequal) objects at which $Q(x)$ holds. Let a_1 be one of them. By lemma 8.2.7. there exists an extension M_2 of M_1 which is a model of H^* and which contains an object a_2 not contained in M_1, and not equal to a_1, such that $Q(x)$ holds at a_1 and a_2. By the same lemma, there exists an extension M_3 of M_2 which is a model of H^* and which contains an object a_3 not contained in M_2, and not equal to either a_1 or a_2, such that $Q(x)$ holds at a_1, a_2 and a_3. In this way we may construct an ascending chain of models of H^* which are therefore models of K, $M_1 \lessdot M_2 \lessdot \lessdot M_3 \lessdot \ldots$ such that $Q(x)$ holds at elements objects a_n, $a_n \varepsilon M_n$, in all the M_k, $k \geqslant n$. But all these a_n are different from one another, and so $Q(x)$ cannot be saturable in K. This shows that the condition is necessary.

Regarding two points P_1 and P_2 in the space S_n over an algebraic structure M as 'equal' if all their coordinates are equal in M, and as 'different' if they are not equal, we may reword the definition of a bounded predicate as follows: A predicate $Q(x_1, \ldots, x_n)$ is called bounded of degree n in an algebra of axioms K, if no variety

V of Q in the space S_n over a model M of K includes more than n different points, although there exists a model M of K such that the corresponding variety includes just n different points.

We may also generalise the concept of a saturable predicate $Q(x_1, \ldots, x_n)$ for arbitrary n in the following way. Let $M_1 \lessdot M_2 \lessdot \lessdot M_3 \lessdot \ldots$ be any ascending chain of models of K, and let $S_n^{(k)}$ be the sequence of n dimensional spaces over M_k, $S_n^{(1)} \subseteq S_n^{(2)} \subseteq \ldots$. Further let V_m be the variety of Q in $S_n^{(m)}$, and define W_m as the set of points $W_m = V_m \cap V_{m+1} \cap V_{m+2} \cap \ldots$. Then $Q(x_1, \ldots, x_n)$ is called saturable if, for any such chain of models, $W_m = W_{m+1} = W_{m+2} = \ldots$ for sufficiently high m. The definition reduces to that given above for the special case $n = 1$. As before we may then prove the theorem,

8.2.8. A predicate is bounded if and only if it is saturable.
A theorem of similar type is

8.2.9. If a predicate $Q(x_1, \ldots, x_n)$ in K is such that any variety V of Q in S_n over a model M of K is finite, then Q is bounded.

For if Q is unbounded, then for any positive m there exist models M such that the variety V of Q in S_n over M contains at least m different points. Hence, as in the proof of 8.2.6., K is consistent with the set K^* of statements $[\sim X_m]$, $m = 1, 2, \ldots$. And any model M' of $K \cup K^*$ then has the property that the variety V of Q in S_n over M contains an infinite number of different points, contrary to assumption.

8.3. *Decomposition of bounded predicates.* Let J_Q be a set of predicates of order n, $Q(x_1, \ldots, x_n)$ which are bounded in an algebra of axioms K, $n \geqslant 1$. A predicate $Q_2(x_1, \ldots, x_n)$ will be called a divisor of a predicate $Q_1(x_1, \ldots, x_n)$, $Q_2 \mid Q_1$, if

8.3.1. $(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \supset Q_2(x_1, \ldots, x_n)]$

is deducible from K. Two predicates Q_1 and Q_2 will be called associated if they divide each other, $Q_1 \equiv Q_2$. If Q_1 divides Q_2, then $\deg Q_1 \leqslant \deg Q_2$; if Q_1 and Q_2 are associated then $\deg Q_1 = \deg Q_2$.

A predicate Q whose degree is greater than 0 is called prime if it is associated with all its divisors except those whose degree is 0. Thus the degree of a divisor of a prime predicate Q is either 0 or it equals the degree of Q. But conversely, this condition is not sufficient to ensure that Q is prime.

All the predicates whose degree is 0 are associated. Let $Q_1(x_1, \ldots, x_n)$ and $Q_2(x_1, \ldots, x_n)$ be two prime predicates in a conjunctive domain J_Q. Then if Q_1 and Q_2 both hold at a point P in S_n over a model M of K (i.e., if their varieties have a point in common), it follows that they are associated. More generally, if Q_1 is a prime predicate, whose variety in S_n over a model M has a point in common with the variety of a predicate Q_2, then Q_1 divides Q_2. In fact, consider the predicate $Q_3 = Q_1 \wedge Q_2$. Q_3 is bounded in K. Its degree is greater than 0 since it holds at a point P in S_n. Also, Q_3 divides Q_1, and since Q_1 is prime, it then follows that Q_1 also divides Q_3; $(x_1) \ldots (x_n)[Q_1 \supset Q_1 \wedge Q_2]$ is deducible from K. Hence also, $(x_1) \ldots (x_n)[Q_1 \supset Q_2]$, showing that Q_1 divides Q_2 as asserted. Moreover, if Q_1 and Q_2 are both prime, then they divide each other and are therefore associated.

Assume now that J_Q is also disjunctive. A predicate Q of J_Q will be called reducible if it is associated with another predicate which is of the form $Q_1 \vee Q_2$, such that the degrees of Q_1 and Q_2 are both positive, but both smaller than the degree of Q. A predicate of positive degree which is not reducible is called irreducible. A prime predicate is irreducible. It is readily shown by induction that every predicate of positive degree in J_Q is associated with a disjunction of a finite number of irreducible predicates in J_Q.

8.3.2. $$Q \equiv Q_1 \vee Q_2 \vee \ldots \vee Q_m$$

The representation will be called irredundant if by omitting any one of the terms on the right hand side we destroy the validity of the formula. The representation of a predicate by an irredundant disjunction is not necessarily unique. However, essential uniqueness is ensured if the following additional condition is satisfied in J_Q.

8.3.3. If the predicate Q_1 divides the predicate Q_2, and $\deg Q_1 = \deg Q_2 > 0$, then Q_1 is associated with Q_2.

Assuming that 8.3.3. is satisfied let $Q_1 \vee Q_2 \vee \ldots \vee Q_m$ and $Q_1' \vee Q_2' \vee \ldots \vee Q_{m'}'$ be two irredundant representations of a predicate of positive degree, Q, by irreducible predicates $Q_1, \ldots, Q_m$, $Q_1', \ldots, Q_{m'}'$,

$$Q \equiv Q_1 \vee Q_2 \vee \ldots \vee Q_m \equiv Q_1' \vee Q_2' \vee \ldots \vee Q_{m'}',$$

then, for $i = 1, 2, \ldots, m$

$$Q_i \equiv Q_i \wedge Q \equiv Q_i \wedge [Q_1' \vee Q_2' \vee \ldots \vee Q_{m'}'] \equiv [Q_i \wedge Q_1'] \vee [Q_i \wedge [Q_2' \vee \ldots \vee Q_{m'}']]$$

so that

$$\deg [Q_i \wedge Q_1'] \leqslant \deg Q_i \text{ and } \deg [Q_i \wedge [Q_2' \vee \ldots \vee Q_{m'}']] \leqslant \deg Q_i.$$

Since Q_i is irreducible, it follows that one of the following four conditions is satisfied.

$$\begin{aligned}
&\deg [Q_i \wedge Q_1'] = \deg [Q_i \wedge [Q_2' \vee \ldots \vee Q_{m'}']] = 0;\\
&\deg [Q_i \wedge Q_1'] = \deg [Q_i \wedge [Q_2' \vee \ldots \vee Q_{m'}'] = \deg Q_i;\\
&\deg [Q_i \wedge Q_1'] = 0, \ \deg [Q_i \vee [Q_2' \wedge \ldots \vee Q_{m'}']] = \deg Q_i;\\
&\deg [Q_i \wedge Q_1'] = \deg Q_i, \ \deg [Q_i \wedge [Q_2' \vee \ldots \vee Q_m']] = 0
\end{aligned}$$

The first possibility can be ruled out, since

$$\deg Q_i \leqslant \deg [Q_i \wedge Q_1'] + \deg [Q_i \wedge [Q_2' \vee \ldots \vee Q_m']]$$

In either the second or the fourth case, we have $\deg [Q_i \wedge Q_1'] = \deg Q_i$. Then by 8.3.3., since $Q_i \wedge Q_1' \mid Q_i$ it follows that Q_i is associated with $Q_i \wedge Q_1'$. Thus $Q_i \mid Q_i \wedge Q_1'$ and so finally $Q_i \mid Q_1'$. In the third case we may show in a similar way that Q_i is associated with $Q_i \wedge [Q_2' \vee \ldots \vee Q_{m'}']$ (and thence that Q_i divides $[Q_2' \vee \ldots \vee Q_{m'}']$). In that case we consider

$$Q_i \equiv Q_i \wedge [Q_2' \vee \ldots \vee Q_{m'}'] \equiv [Q_i \wedge Q_2'] \vee [Q_i \wedge [Q_3' \vee \ldots \vee Q_{m'}']]$$

As before, this leads to the conclusion that either $Q_i \mid Q_2'$ or $Q_i \equiv Q_3' \vee \ldots \vee Q_m'$. Continuing in this way, we find that Q_i divides at least one Q_k', $1 \leqslant k \leqslant m'$. Similarly, every Q_k' divides at least one Q_i. Thus, for any specific i, $1 \leqslant i \leqslant m$ we have $Q_i \mid Q_k'$ and $Q_k' \mid Q_{i'}$ for some i', $1 \leqslant i' \leqslant m$. But if so $Q_i \vee Q_{i'} \equiv Q_i$, and this shows that $Q_{i'}$ is redundant in the representation of Q, unless $i = i'$. Since the representation is irredundant, by assumption, this establishes that $i = i'$, and hence $Q_i \equiv Q_k'$. Thus, for every Q_i there is a Q_k' with which it is associated, and vice versa. But since the representations are irredundant, no two terms of the same disjunction can be associated with each other. Hence, finally, 8.3.4. *Theorem.* Given two representations of the same predicate by irredundant disjunctions of irreducible predicates,

$$Q \equiv Q_1 \vee Q_2 \vee \ldots \vee Q_m \equiv Q'_1 \vee Q'_2 \vee \ldots \vee Q'_{m'}$$

we have $m = m'$ and, after a suitable rearrangement of the second representation, $Q_i \equiv Q'_i$, $i = 1, 2, \ldots, m$.

Theorem 8.3.4. can still be proved if J_Q, while disjunctive, and satisfying 8.3.3., is not necessarily conjunctive but instead fulfils the following condition.

8.3.5. If Q_1 and Q_2 belong to J_Q there exists an element Q of J_Q such that $Q \equiv Q_1 \wedge Q_2$.

A set of predicates J_Q which satisfies 8.3.5. may be called quasi-conjunctive.

8.4. *Algebraic predicates.* Although the concept of a bounded polynomial of order one shows certain similarities with the concept of a polynomial equation, $q(x) = 0$, the former is still too wide to be a genuine generalisation of the latter. To illustrate this, let K be the algebra of axioms of a commutative field, $K = A_F$, while M is the field obtained by the adjunction of a real root of the polynomial

8.4.1. $$x^4 - 2x^2 - 1$$

to the field of rational numbers. The roots of 8.4.1. are, $\pm\sqrt{1 + \sqrt{2}}$ (real) and $\pm\sqrt{1 - \sqrt{2}}$ (imaginary). Thus M contains the two real roots of 8.4.1. and does not contain the two imaginary roots of the same polynomial. Now let $q(x)$ be the polynomial

8.4.2. $$q(x) = x^2 - 2x - 1$$

and let $Q_q(x)$ be a predicate which states 'x is a root of the polynomial $q(x)$'. Q_q can be constructed as in earlier chapters. We then define another predicate $Q(x)$ by

8.4.3. $$Q(x) = [Q_q(x) \wedge [(\exists y)P(y, y, x) \wedge [(z)[Q_q(z) \wedge [(\exists w)P(w, w, z)] \supset E(x, z)] \ldots]$$

In words, 'x is a root of $q(x)$, and there exists an element whose square x is; moreover, x is the only element of that description'.

It is not difficult to verify that $Q(x)$ is a bounded predicate of degree one in A_p. Also, it holds at the number $a = 1 + \sqrt{2}$ of M although the algebraic degree of that number is 2.

It will be seen that a no longer satisfies the predicate if we extend M^* by the adjunction of $\sqrt{1 - \sqrt{2}}$. Thus one might say that $Q(x)$ is not truly an inherent but only an accidental property of a. Accordingly, we may tend to confine ourselves to predicates which are more genuinely inherent in the objects by which they are satisfied. Nevertheless, it is unavoidable that in considering whether an object a does or does not satisfy a certain predicate, we should refer to a model in which a is contained. In fact, an object a as such has no predicates at all attached to it, except with reference to a model or models in which it is contained.

A predicate $Q(x_1, \ldots, x_n)$ (bounded or unbounded) will be called persistent at a point P in S_n over a model M of a set of axioms K, if Q holds at P in S_n and in all the spaces $S'_n \supseteq S_n$ over extensions $M' \succ M$ which are models of K. The predicate will be called persistent if it is persistent at all the points at which it holds. It will be called invariant if it is persistent and if $\sim Q(x_1, \ldots, x_n)$ also is persistent. A predicate which is persistent and bounded will be called algebraic; a predicate which is invarient and bounded will be called strongly algebraic. It is not difficult to verify

8.4.4. *Theorem.* The conjunction of two persistent predicates is persistent. It follows that the conjunction of two algebraic predicates is algebraic.

8.4.5. *Theorem.* The disjunction of two persistent predicates is persistent; the disjunction of two algebraic predicates is algebraic.

Let the two persistent predicates be $Q_1(x_1, \ldots, x_n)$ and $Q_2(x_1, \ldots, x_n)$. If $Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n)$ holds at a point P in S_n over some model M of K, then either Q_1 or Q_2 or both hold at P, Q_1, say. Hence Q_1 holds at P in all $S'_n \supseteq S_n$ as above, and so therefore does $Q_1 \vee Q_2$. This shows that $Q_1 \vee Q_2$ is persistent.

8.4.6. *Theorem.* The conjunction of two invariant predicates is invariant. It follows that the conjunction of two strongly algebraic predicates is strongly algebraic.

For Q_1 and Q_2 are invariant by definition if and only if Q_1, $\sim Q_1$, Q_2 and $\sim Q_2$ are all persistent. Hence, by 8.4.4., $Q_1 \wedge Q_2$ is persistent, while by 8.4.5., $\sim Q_1 \vee \sim Q_2$ is persistent. But $\sim Q_1 \vee \sim Q_2$ is associated with $\sim [Q_1 \wedge Q_2]$ and so as is easily seen $\sim [Q_1 \wedge Q_2]$ also is persistent and $[Q_1 \wedge Q_2]$ is invariant, as asserted. Similarly, we can prove,

8.4.7. *Theorem.* The disjunction of two invariant predicates is invariant; the disjunction of two strongly algebraic predicates is strongly algebraic.

Thus the set of algebraic predicates of order n forms a conjunctive disjunctive subset of the set of bounded predicates of the type considered in the preceding section. It will be denoted by J_A. Similarly, the set of all strongly algebraic predicates of order n constitutes a conjunctive subset of J_A which will be denoted by J_S. In particular if condition 8.3.3. applies to the predicates of J_A, or of J_S, then there is an essentially unique representation of every (strongly) algebraic predicate as the irredundant disjunction of irreducible (strongly) algebraic predicates.

The following condition is sufficient to ensure that conditions 8.3.3. and its consequences apply to the domain of strongly algebraic predicates of specified order n.

8.4.8. If the degree of a strongly algebraic predicate $Q(x_1, \ldots, x_n)$ is m, and if the variety V of Q in S_n over a model M contains exactly $k < m$ different points, then there exists an extension M' of M which is a model of K such that the variety V' of Q in S'_n over M' contains m different points.

To prove that 8.3.3. is a consequence of 8.4.8. let Q_1 and Q_2 be two strongly algebraic predicates such that $Q_1 \mid Q_2$ and deg $Q_1 =$ deg $Q_2 = m$. We have to show that Q_1 is associated with Q_2, i.e., that $Q_2 \mid Q_1$. In other words, we have to show that if Q_2 holds at a point P in S_n over a model M of K, then Q_1 also holds at P.

Let M be any model of K and assume that the variety V_1 of Q_1 in S_n over M contains exactly k different points while the variety V_2 of Q_2 contains exactly l different points, $l \leqslant m$. Since $Q_1 \mid Q_2$, $V_1 \subseteq V_2$, so that $k \leqslant l$. To prove our assertion it is sufficient to show that $k = l$.

Assume on the contrary that $k < l \leqslant m$. Then, by 8.4.8. there exists an extension M' of M such that the variety V'_1 of Q_1 in S'_n over M' contains m different points. Also, since Q_1 is persistent, $V_1 \subseteq V'_1$, so that V'_1 contains at least $m - k$ different points

$P_{k+1}, \ldots, P_m$ in addition to the points of V_1. But since Q_1 divides Q_2, these points also belong to the variety V_2' of Q_2 in S_n'. On the other hand, since Q_1 is invariant these points cannot belong to $V_2 - V_1$. Thus V_2' contains at least $m - k + l > m$ different points, contrary to assumption. This proves that in fact $k = l$, $Q_1 \mid Q_2$, and so $Q_1 \equiv Q_2$.

8.5. *Properties of algebraic predicates.*

8.5.1. *Theorem.* Let $Q_1(x)$, $Q_2(x)$, ... $Q_m(x)$ be algebraic polynomials of order one in an algebra of axioms K, and let $R(x_1, \ldots, x_n, y)$ be a polynomial predicate in K. Then the predicate

$$Q'(x) = (\exists x_1)(\exists x_2) \ldots (\exists x_n)[Q_1(x_1) \wedge Q_2(x_2) \wedge \ldots \wedge Q_m(x_m) \wedge R(x_1, \ldots, x_n, x)]$$

is algebraic in K and $\deg Q'(x) \leqslant \deg Q_1(x) \deg Q_2(x) \ldots \deg Q_m(x)$.

Let $\deg Q_i(x) = m_i$, $i = 1, 2, \ldots, m$. Given any model M of K, assume that $Q_i(x)$ holds for k_i different objects of M, so that $k_i \leqslant m_i$. Since R is a prepolynomial $Q'(x)$ can hold only for not more than $k_1 . k_2 \ldots k_m \leqslant m_1 . m_2 \ldots m_n$ different objects in M. Hence Q' is bounded in K and its degree does not exceed $m_1 m_2 \ldots m_n$. (This part of the proof applies to all prepolynomials $R(x_1, \ldots, x_n, y)$.

To prove that $Q'(x)$ is also persistent, we recall that by definition a polynomial predicate is a predicate of the form

$$R(x_1, \ldots, x_n, y) = (\exists z_1) \ldots (\exists z_j)[A_1 \wedge A_2 \wedge \ldots \wedge A_l]$$

where $A_1, A_2, \ldots, A_l$ are relative symbols of K; (besides, the predicate may also involve object symbols of K). This shows that if R holds at a point $P = (a_1, \ldots, a_n, a)$ in S_{n+1} over a model M_1 then R also holds at P in S_{n+1}' over an extension $M' \succ M$. Also, since the predicates $Q_1(x)$, $Q_2(x)$, $\ldots$, $Q_n(x)$ are persistent, it follows that if they hold at $a_1, a_2, \ldots, a_n$ respectively in M they hold at these objects in $M' \succ M$. Thus if $Q'(x)$ holds at a in M, it also holds at a in $M' \succ M$. This proves that $Q'(x)$ is persistent.

An object a in a model M of an algebra of axioms K will be called (strongly) algebraic if it satisfies a (strongly) algebraic predicate in K. The lowest degree of any strongly algebraic predicate which satisfies a will be called the degree of a (we shall not consider the corresponding concepts for general algebraic predicates). If objects $a_1, a_2, \ldots a_n$ in M are persistently algebraic of degrees $m_1, m_2, \ldots, m_n$ respectively, and if $a \varepsilon M$ is an object determined by a polynomial predicate $R(x_1, x_2, \ldots, x_n, y)$ — (i.e., R holds at $(a_1, a_2, \ldots, a_n, a)$ — then by 8.5.1. above, a is algebraic of degree $m \leqslant m_1 \cdot m_2 \ldots m_n$. Thus if it is known that all the objects of M can be obtained in this way from a set of algebraic objects, we may infer that all the objects of M are algebraic. A model, all of whose objects are algebraic, will be called algebraic (in the given algebra of axioms, K).

A model M of an algebra of axioms K will be called algebraically complete if any algebraic predicate $Q(x)$ of order one in K is either satisfied by objects of M, or if there is no model M' of K, $M' > M$, in which it is satisfied. An equivalent condition is that every algebraic predicate in K is either satisfied in M or that it is inconsistent with K' where K' equals K augmented by a diagram D of M, $K' = (K; D)$.

A model M of an algebra of axioms K will be called algebraically closed if every algebraic predicate in $K' = (K; D)$ which is of positive degree, is satisfied in M. It is clear that the definition is independent of the particular choice of D.

A model M of K which is algebraically closed, also is algebraically complete.

8.5.2. *Theorem.* A model M of an algebra of axioms K which is complete in Tarski's sense, (section 3.8) is algebraically complete.

For let $Q(x)$ be any algebraic predicate of order one. Then the statement $(\exists x)Q(x)$ can either be deduced from K — so that $Q(x)$ holds at an object of M — or it is inconsistent with K, so that $Q(x)$ cannot hold at any object in any model of K.

8.5.3. *Theorem.* Let M be an algebraically closed model of an algebra of axioms K, and let M' be an extension of M, which is also a model of K. Then any object a of M', which satisfies an algebraic predicate in $K' = (K; D)$ must be equal to an object of M.

For assume that the algebraic predicate $Q(x)$ in K' holds at a. Since M is algebraically closed, $Q(x)$ also holds at a number of objects of M. Let $a_1, a_2, \ldots, a_n$ be a set of these objects, such that every other object at which $Q(x)$ holds is equal to one of them, and let $a_1', a_2', \ldots, a_n'$ be the corresponding object symbols. Consider the predicate

$$Q'(x) = [Q(x) \wedge \sim E(x, a_1') \wedge \sim E(x, a_2') \wedge \dots \wedge \sim E(x, a_n')]$$

This predicate is algebraic, and since it is not satisfied in M it must be inconsistent with K'. Thus any object of M' which satisfies Q must be equal to one of the objects $a_1, a_2, \dots, a_n$.

8.5.4. *Theorem.* Every model M of an algebra of axioms K has an extension M' which is a model of K and which is algebraically complete.

This theorem has some affinity with Lindenbaum's theorem on complete systems (section 3.9).

Let $\{Q_\nu\}$, $\nu = 1, 2, \dots$ be the well ordered set of all predicates of order one formulated in terms of the object and relative symbols of K. To shorten the proof we shall assume (as we may) that the set $\{Q_\nu\}$ has a last element. We then define a well ordered series of models M_ν of K, in the following way.

We first set $M_0 = M$, and define D_0 as a complete diagram of M_0, such that $(K; D_0)$ is an augmented set of axioms. Secondly, if ν is not a limit number, then we define M_ν as an extension of $M_{\nu-1}$ which satisfies $(K; D_{\nu-1})$ as well as the statement $(\exists x)Q_\nu(x)$, if $(\exists x)Q_\nu(x)$ is consistent with $(K; D_{\nu-1})$ and $M_\nu = M_{\nu-1}$ if $(\exists x)Q_\nu(x)$ is inconsistent with $(K; D_{\nu-1})$. We also define D_ν as a complete diagram of M_ν such that it includes $D_{\nu-1}$.

On the other hand, if ν is a limit number, then we first define M_ν as the union of all M_λ, $\lambda < \nu$ and D_ν' as the union of all D, $\lambda < \nu$. D_ν' is a complete diagram of M_ν'. Also, it is not difficult to see that the correspondence between the object and relative symbols of D_ν and the objects and relations of M_ν' is compatible with the correspondence between the object and relative symbols of K and the objects and relations of M_ν'. Also, $K \cup D_\nu'$ must be consistent, since all the D_λ, $\lambda < \nu$, are consistent with K, being complete diagrams of models of K. It follows that (although M_ν' need not be a model of K there exists an extension M_ν'' of M_ν' which is a model of K. Let D_ν'' be a complete diagram of M_ν'' which includes D_ν'. We now define M_ν as an extension of M_ν'' which satisfies $(\exists x)Q_\nu(x)$, according as the statement $(\exists x)Q_\nu(x)$ is, or is not, consistent with $K \cup D''$. We also define $D_\nu \subseteq D_\nu''$ as a complete diagram of M_ν.

In this way we define a model M_ν of K for every ν for which there exists a predicate $Q_\nu(x)$, $\nu \leqslant \mu$, where $Q_\mu(x)$ is the last element of Q_ν. Then $M' = M_\mu$ is a model of K and includes M. Also, let $Q_\nu(x)$ be any persistent predicate in K. If $(\exists x)Q_\nu(x)$ is consistent with $K \cup D_\mu$, then it is consistent with $K \cup D_{\nu-1}$, or $K \cup D_\nu''$ as the case may be. It follows that $Q_\nu(x)$ holds for some object a in M_ν. But since $Q_\nu(x)$ is persistent it also holds for a in M'. Since all algebraic predicates are persistent, the theorem then follows.

8.6. *Applications to algebraic fields.* The following theorem justifies the adjective 'strongly algebraic' for an invariant bounded predicate.

8.6.1. Let $K = (A_F; D)$ where A_F is the algebra of axioms of a commutative field, and D is a complete diagram of a particular commutative field F of arbitrary characteristic. For every polynomial $q(x) \not\equiv 0$ with coefficients in F, there then exists a strongly algebraic predicate $Q(x)$ in K such that $Q(x)$ holds at any element a of any extension F' of F, if and only if a is a root of $q(x)$, $q(a) = 0$. Conversely, for every strongly algebraic predicate $Q(x)$ in K there exists a polynomial $q(x)$ with coefficients in F such that $Q(x)$ holds at any element a in any extension F' of F if and only if a is root of $q(x)$, $q(a) = 0$.

To prove the first part of the theorem, let

$$q(x) \equiv a_0 + a_1 x + \ldots + a_n x^n,\ n \geqslant 1,\ a_n \neq 0,\ a_i \varepsilon F,$$
$$i = 0, 1, 2, \ldots, n.$$

Then a strongly algebraic predicate as required is given by

8.6.2. $$Q_q(x) = (\exists y_1)(\exists y_2) \ldots (\exists y_n)(\exists z_1)(\exists z_2) \ldots (\exists z_n)(\exists w_1)(\exists w_2) \ldots (\exists w_{n-1})$$
$$[P_1(x, y_1) \wedge P_2(x, y_2) \wedge \ldots \wedge P_n(x, y_n) \wedge P(a_1', y_1, z_1) \wedge P(a_2', y_2, z_2)$$
$$\wedge \ldots \wedge P(a_n', y_n, z_n) \wedge S(a_0', z_1, w_1) \wedge S(w_1, z_2, w_2) \wedge \ldots \wedge S(w_{n-1}, z_n, 0)]$$

In this definition $a_0', a_1', \ldots, a_n'$ are the object symbols corresponding to $a_0, a_1, \ldots, a_n$ respectively. The object symbol 0 corresponds to a ('the', if F is normal) neutral element with respect to addition. The relative symbols $P_1, P_2, \ldots, P_n$ were defined in 5.3.2. and indicate powers of x. It is clear that $Q_q(x)$ holds if and only if $q(x)$ vanishes, and this alone shows that Q_q is persistent and bounded.

But $Q_q(x)$ also is persistent and so $Q_q(x)$ is strongly algebraic. Again if $q(x) \equiv a_0 = \text{const.}$, then $q(x)$ is not satisfied by any element $a \,\varepsilon\, F'$, so that a predicate as required is given by $Q(x) = \sim E(x, x)$ which is strongly algebraic of degree 0.

Conversely, let $Q(x)$ be any strongly algebraic predicate in K. We first show that $Q(x)$ cannot hold for any transcendental element a of any extension F' of F. Assume on the contrary that $Q(a)$ holds in F' where a' is the object symbol corresponding to any object $a \,\varepsilon\, F'$ which is transcendental with respect to F. Let F'' be the subfield of F' obtained by the adjunction of the transcendental element a to F, $F'' = F(a)$. Since $Q(x)$ is invariant it also holds at a in F''. But F'' is quasi-isomorphic (or isomorphic if F' is normal) to the simple transcendental extension of F, $F(b)$, by any other element $b \,\varepsilon\, F'$ which is transcendental with respect to F. It then follows that $Q(x)$ holds at b in $F(b)$ and therefore F'. But this is impossible since F' includes an infinite number of different elements which are transcendental with respect to F.

Now assume that $Q(x)$ satisfies an element $a \,\varepsilon\, F'$ where a is algebraic with respect to F and is a root of the irreducible polynomial $q_1(x)$ with coefficients in F. Let F'' be the field obtained from F by the adjunction of a, $F'' = F(a)$, $F'' \lessdot F'$. Since F'' is quasi-isomorphic to its conjugate fields (which are obtained by the adjunction of any other root of $q(x)$ to F), it follows that $Q(x)$ holds for all the roots of $q(x)$ in any extension of F. Since $Q(x)$ is bounded, it follows that the polynomial $q(x)$ required by the theorem is the product of a finite number of irreducible polynomials $q(x) = q_1(x) \ldots q_k(x)$ obtained as above.

Thus, every invariant predicate in K is associated with a predicate $Q_q(x)$ defined as above. It will be seen that the degree of $Q_q(x)$ and of its associated predicates equals the number of different roots of $q(x)$. It follows that the degree of an element a of $F' \gtrdot F$ as defined in 8.5. is identical with the algebraic degree of a if F is of characteristic 0, or if at least a is separable, but that it equals the reduced algebraic degree of a if a is not separable.

8.6.3. If $Q(x)$ is algebraic (persistent and bounded but not necessarily strongly algebraic) then $Q(x)$ cannot hold for any transcendental element a of any extension F' of F.

Assume on the contrary that $Q(a')$ holds in F' where a' is the object $a \varepsilon F'$ which is transcendental with respect to F. Let F'' be the algebraic closure of F'. $Q(a')$ holds in F'' since $Q(x)$ is persistent. Now let b be any element of F'' which is transcendental with respect to F. It then follows from Steinitz' theory (ref. 21) that there exists a quasi-automorphism of F'' (an automorphism, if F' is normal) which carries a into b while carrying every element of F into itself. It follows that $Q(x)$ holds for all the transcendental elements of F'' with respect to F, although there is an infinite number of different elements of that type. And this is impossible since $Q(x)$, is bounded. Thus, any element $a \varepsilon F'$ which satisfies an algebraic predicate is algebraic with respect to F in the accepted sense, in agreement with a statement made in the introduction (section 1.6.).

It will be noticed that the proof of 8.6.3. makes use of algebraic results which are considerably more recondite than those required for the proof of 8.6.1. Only the strongly algebraic predicates can be regarded as genuine counterparts of polynomials. Thus, the persistent algebraic, but not strongly algebraic, predicate $P(x, x, 2) \wedge [(\exists y) P(y, y, x)]$ is satisfied in $F' = F(\sqrt[4]{2})$, where F is the field of rational numbers, only by $+\sqrt{2}$ although the conjugate of $+\sqrt{2}$, $-\sqrt{2}$ is also contained in F'. It follows that there is no polynomial $q(x)$ as required by the second part of theorem 8.6.1. for the predicate in question.

8.7. *Applications to algebraic fields, continued.* We are now in a position to compare the results obtained in the preceding sections, particularly those which apply to strongly algebraic predicates, with the familiar results concerning algebraic polynomials of one variable and their roots.

Referring to the arithmetic of predicates developed in 8.3., we observe that two strongly algebraic predicates $Q_{q_1}(x)$ and $Q_{q_2}(x)$ corresponding to polynomials $q_1(x)$ and $q_2(x)$ are associated if and only if the irreducible factors of $q_1(x)$ and $q_2(x)$ are equal. Condition 8.4.8., and hence condition 8.3.3., are satisfied for strongly algebraic predicates, although 8.3.3. is not satisfied for the domain of all algebraic predicates. Thus the algebraic predicate $P(x, x, 2) \wedge [(\exists y) P(y, y, x)]$ mentioned above divides the strongly algebraic predicate $P(x, x, 2)$, but is not associated with it, although both predicates are of degree 2.

Again, for strongly algebraic predicates $Q_{q_1}(x)$ and $Q_{q_2}(x)$ we have $Q_{q_1} \wedge Q_{q_2} \equiv Q_{(q_1, q_2)}$, where (q_1, q_2) is the greatest common divisor of q_1 and q_2, and $Q_{q_1} \vee Q_{q_2} \equiv Q_{q_1 q_2}$. Irreducible strongly algebraic predicates correspond to powers of irreducible polynomials.

Coming to 8.5., we may use theorem 8.5.4. to construct a field F' in which all polynomials with coefficients in the given field F can be decomposed into linear factors. (Compare 8.1.). Since we wish to test the extent to which the mathematical procedure can be eliminated, we have to examine carefully what algebraic tools are involved in the application. Thus, we accept from Algebra that every polynomial with coefficients in F possesses an irreducible factor; we also accept the construction of any simple algebraic extension of a field F by the adjunction of a root of an irreducible polynomial, i.e., the construction of a quotient ring in the polynomial ring of one variable adjoined to F. We may therefore use the fact that for every polynomial $q(x)$ with coefficients in F, which does not reduce to a constant, there exists an extension F' of F in which $q(x)$ has at least one root, a. We shall also make use of the very simple remainder theorem, according to which there is a polynomial $q_1(x)$ with coefficients in F', whose degree is one less than the degree of $q(x)$, such that $q(x) \equiv (x - a)\, q_1(x)$. Finally, we observe that the construction of the strongly algebraic predicate $Q_q(x)$ for a given polynomial $q(x)$ (8.6.2.) does not require any of the slightly more advanced methods required for the proof of the second part of the theorem 8.6.1.

Now let $K = (A_F; D)$ be the set of axioms of a commutative field augmented by a complete diagram D of the field F, as before. By theorem 8.5.4. we may then construct a field $F_1 > F$, such that every persistent predicate (of order one) in K holds for one object of F_1, at least, or else $Q(x)$ does not hold for any object in any extension of F_1 either. Let $K_1 = (A_F; D_1)$ be the set of field axioms augmented by a complete diagram D_1 of F_1. Again by theorem 8.5.4. we construct a field $F_2 > F_1$ such that every persistent predicate $Q(x)$ (of order one) in K, holds for one object of F_2 at least, or else $Q(x)$ does not hold for any object in any extension of F_2, either. Let $K_2 = (A_F, D_2)$ be the set of field axioms augmented by a complete diagram D_2 of F_2. Continuing in this way, we obtain an ascending chain of fields, $F \lessdot F_1 \lessdot F_2 \lessdot \ldots$.

Let F' be the union of all F_n, n = 1, 2, 3, ... Then every polynomial with coefficients in F can be decomposed into linear factors within F'. Let $q(x)$ be any polynomial of positive degree with coefficients in F. Then $Q_q(x)$ is a strongly algebraic predicate in K and is therefore persistent in K. Now if $q(x)$ has no root in F', we may construct an extension F'' of F' in which $q(x)$ has a root. But both F' and F'' are extensions of F_1, and so since F_1 is algebraically complete, $Q_q(x)$ must be satisfied at an object within F_1, i.e., $q(x)$ must have a root a within F_1. Putting $q(x) = (x - a)$ $q_1(x)$, we may then construct the strongly algebraic predicate $Q_{q_1}(x)$ in K, which must be satisfied at an object a in F_2, so that a is a root of $q_1(x)$. Continuing in this way we see that $q(x)$ can actually be decomposed into linear factors within F_n, where n is the degree of $q(x)$.

The elements of F' are not necessarily all algebraic with respect to F, nor is it true that F' must be algebraically closed. If we want to show that F' possesses a subfield with these two properties we still have to accept additional theorems from Algebra (e.g. 8.1.1., 8.1.2.).

Instead of considering the set of field axioms A_F in conjunction with a complete diagram D, we may put $K = A_p$, where A_p is the set of axioms which defines commutative fields of characteristic p or 0 (see section 4.2.). In that case we may still construct a strongly algebraic predicate $Q_q(x)$ for any polynomial $q(x)$ with integral coefficients, where the coefficients are now regarded as operators indicating repeated addition or subtraction. And by considering the objects for which any given strongly algebraic predicate holds in any field of the given characteristic, we may again associate $Q(x)$ with a predicate of the particular form $Q_q(x)$. Thus, the set of strongly algebraic predicates of order one in A_p is essentially equivalent to the set of polynomials with coefficients in the prime field of characteristic p (or 0).

Before leaving this subject, we may point out that the algebraic predicates of order $n > 1$ are associated with specific sets of polynomials of n variables, not with single polynomials, which correspond to unbounded predicates.

IX

CONVEX SYSTEMS

9.1. *Definition and properties of convex systems.* In this chapter we will discuss the implications of another property which is common to all the systems of axioms defined in Chapter 4. This property is, broadly speaking, that the meet (intersection) of any number of models of such a system of axioms is again a model. More precisely, we introduce this property by the following definition.

9.1.1. An algebra of axioms K will be called convex if the meet of any set of substructures of a model M of K — all of which are models of K under the correspondence under which M is a model of K — is not empty and is again of model of K.

A survey of the systems defined in Chapter 4 shows that they all satisfy condition 9.1.1. However, this is no longer true if we replace the word 'substructures' in that condition by 'partial structures'. In fact it is quite possible that a group G, i.e., a model of A_g contains two equal, but not identical, neutral elements and that two partial structures G' and G'' of G which are also groups, contain just one of the two neutral elements each so that their meet does not contain a neutral element, and cannot therefore be a group.

An example of an algebra of axioms which is not convex is obtained by adding the statement

9.1.2. $$(\exists x)(y)[\sim P(y, y, x)]$$

'There exists an x which is not a square' — to the set of axioms of a commutative field A_F. For the proof that this system is not convex, see below.

Keeping K fixed we shall refer to models of K simply as models. Also, as above when a model M is stated to be a submodel of a model M', it will be understood that the correspondence by which M' is a model of K includes a correspondence by which M is a model of K. Similarly, when we consider sets of models simultaneously, some of which may have objects in common, it will be assumed that they can be regarded as models of K by correspondences which are all included in one and the same correspondence.

Let M' be a model of a convex algebra of axioms K, and let M be a submodel of M'. Given any object a of M' which does not belong to M, we consider the set of submodels of M' which include M and a. This set is not empty, since it includes M'. The meet of its elements will be said 'to be obtained by the adjunction of a to M,' or 'to be generated by a over M,' and will be denoted by $M(a)$. A submodel of M' which is generated by one of its elements over M will be called a simple supermodel of M. The supermodel $M(S)$ of M obtained by the adjunction to M of all the objects of a set S is defined in a similar way.

It is the systematic procedure of algebraic field theory and of its extensions and related theories, to gain insight into the structure of all possible supermodels of a given model M by first studying all the simple supermodels of any model. All other supermodels of M are then obtained by the successive adjunction of elements. The possibility of this course of action is based on the following condition which holds, for instance, for algebraic fields.

9.1.3. Let S be a non-empty set of models such that between any two elements M_1 and M_2 of S, we have either $M_1 \Subset M_2$ or $M_2 \Subset M_1$. Then the union M^* of all elements of S — relations between the objects of M^* holding whenever they hold in a model $M_k \varepsilon S$ in which they are all contained — is a model of the given algebra of axioms.

We propose to show that all convex algebras of axioms satisfy condition 9.1.3. The proof requires some preliminary considerations.

We observe that so far our language L has been developed in such a way, that we have no actual statement within L which would affirm or deny that a specific object belongs to a model of a set K. In general, the indication that an object symbol a denotes an object which belongs to a model of K is simply the fact that a is contained in K.

In order to be able to state that a specific object does or does not belong to a specific model, we proceed as follows. Given any specific structure M (which may be a partial structure of another structure), we add a relation $R_M(\ )$, of order one, to the relations of M. R_M will be supposed to hold for all objects of M, by definition. On the other hand, given a set of axioms K, we choose a relative symbol $R_K(\ )$ of order one in L (or if necessary in an extended language L') which is not contained in K. For every statement X in L, we then define a statement $X' = r(X)$ recursively by the following rules.

9.1.4. If X is of order 0, $X' = X$. If X is formed in accordance with 2.3.2., i.e., $X = [\sim Y]$, or $X = [Y \wedge Z]$ or $X = [Y \vee Z]$ or $X = [Y \supset Z]$, then $X' = r(X)$ will be defined as $[\sim r(Y)]$, $[r(Y) \wedge r(Z)]$, $[r(Y) \vee r(Z)]$, or $[r(Y) \supset r(Z)]$ respectively.

If X is obtained by existential quantification, $X = [(\exists u)Y(u)]$ then $X' = [(\exists u)[[R_k(u)] \wedge r(Y(u))]]$. Finally, if $X = [(u)Y(u)]$ then $X' = [(u)[[R_K(u)] \supset r(Y(u))]]$.

The above definition transforms the set of all statements of L, K_L say, into a subset $K_L' \subseteq K_L$. The correspondence is one-to-one: given any statement X we can find the corresponding $X' \varepsilon K_L'$, but given $X' \varepsilon K_L'$ we can also find the corresponding X such that $X' = r(X)$. This, however, will not be necessary.

The correspondence $X \to X'$, $X' = r(X)$, transforms the set of statements K into another set K'. Let M' be the structure which is obtained from a structure M by the addition of the relation R_M as above. Then an appeal to rules 2.7.1.—2.7.7. shows that M is a model of K under a given correspondence C, if and only if M' is a model of K' under the correspondence C' which is obtained from C by the inclusion of $R_K \leftrightarrow R_M$.

Another lemma which will be required for the proof of 9.1.3. and which has been used earlier in special cases is as follows.

9.1.5. Let M be a model of an algebra of axioms K under a correspondence C. Then a partial structure $M' \lessdot M$, such that for every object of M there is an equal object in M', and such that all the objects of M which correspond to object symbols in K are included in M', also is a model of K.

The proof will be omitted.

We are now in a position to prove that 9.1.3. is satisfied by all convex algebras of axioms. Using the notation of that condition, let D^* be a complete diagram of M^*, which is compatible with K, in the sense that the relative and object symbols of K are also contained in D^* and are in correspondence with the same objects and relations. D^* includes complete diagrams D_k for all the elements M_k of S, $D_i \subseteq D_k$ for $M_i \Subset M_k$, and D^* is the union of these D_k.

The union H of K and D^* is consistent. For if it were inconsistent, then K would be inconsistent with a finite number of statements of D^*, $X_1, \ldots, X_n$, say. But since D^* is the union of all D_k, every X_i is contained in one diagram D_k at least. Let then $D_1, \ldots, D_n$ be the diagrams of $M_1, \ldots, M_n$ such that $X_i \varepsilon D_i$. One of these diagrams D_m say, includes all the others, and so D_m must include $X_1, \ldots, X_n$, and is therefore inconsistent with K. And this is impossible since D_m is a complete diagram of the model M_m of K.

It follows that H has a model M' which contains a partial structure isomorphic to M^*. As in earlier cases, we may therefore assume that M^* is a partial structure of M'.

Let $M_k \Subset M^*$ be any element of S. Then all the object symbols contained in K correspond to objects in M_k and cannot therefore correspond to objects in $M' - M^*$. It then follows from 9.1.5. that if we remove from M' all the objects which are equal to objects of M^*, we obtain a structure M'' which is still a model of K. Also M^* is now a substructure of M''.

Let $R_M(\ )$ be a one-dimensional relation which is not already contained in M''. We define that R_M shall hold for all the objects of $M^* \Subset M''$, and only for these. We denote the structure which is obtained from M'' by the addition of R_M by M''_R; also, we denote by $M_k^{(R)}$ the substructures of M''_R which are obtained from the structures M_k by adding R_M as above, and by M_R^* the structure corresponding to M^*. Let D be a complete diagram of M'' compatible with K (see above) and let D' be the set of statements $[R_K(a)]$ and $[\sim [R_K(a)]]$ for all the object symbols a of D according as $R_M(a')$ does or does not hold for the corresponding object a', where R_K is a relative symbol not contained in K. Then $D_R = D \cup D'$ constitutes a complete diagram for M''_R. Also, let K' be the set of statements obtained by transforming the statements of K in accordance with the rules given under 9.1.4., and let H_R be the union of K' and D_R. The correspondence C between the object and relative symbols of D_R and the objects and relations of M''_R includes a correspondence between the object and relative symbols of K' and some of the objects and relations of every $M_k^{(R)}$, and by the discussion following 9.1.4. above, K' holds in every $M_k^{(R)}$ under the correspondence in question.

To ensure that $H_R = K' \cup D_R$ is consistent, it is sufficient to show that every finite subset D'_R of D_R is consistent with K'. Since D'_R is finite it can involve only a finite number of object symbols. Except for the object symbols corresponding to objects in $M'' - M^*$, the object symbols of D'_R correspond to a finite number of objects in M^*, and therefore to the objects of some specific $M_k \varepsilon S$. Now consider the partiol structure **M** of M''_R obtained by removing from M''_R all the objects of $M^* - M_k^{(R)}$ i.e., all the objects of $M^* - M_k$. We are going to show that **M** is a model of $K' \cup D'_R$ under the correspondence C' which is obtained from C by neglecting the objects of $M^* - M_k$. Inspection of the statements of D'_R which are either of order one, or of order 2 and of the form $R_K(a)$, shows immediately that they all hold in **M**. Also, K holds in M_k and so K' holds in $M_k^{(R)}$. Now it is not difficult to verify that if to a model of K', i.e.g., $M_k^{(R)}$ we add any number of objects with the provision that R_M shall not hold at these objects, then we obtain a structure which is still a model of K'. But the objects of $\mathbf{M} - M_k^{(R)}$ which are also the objects of $\mathbf{M} - M_R^*$ have precisely that property. Thus, K', like D'_R, holds in **M** so that $K \cup D'_R$ is consistent, and hence also $K \cup D_R = H_R$ is consistent.

It follows that H_R has a model M_R which, being in particular a model of D_R, contains a partial model isomorphic to M''_R; we may even assume as before that this partial model coincides with M''_R, so that M_R is an extension of M''_R. Again, by removing objects of M_R which are equal to objects of M''_R, as by 9.1.5., we may transform M_R into a supermodel of M''_R. Now let $\mathbf{M}_R$ be the partial structure of M_R which consists of the objects a' for which $R_M(a')$ holds. $\mathbf{M}_R$ is in fact a substructure of M_R since M_R is algebraic. Also, M_R includes M^* and it does not include any object which belongs to $M'' - M^*$. Now it can be shown (compare the end of the last paragraph) that if we remove from a model of K' e.g., M_R, any number of objects for which R_M does not hold we still obtain a model of K'. Thus M_R is a model of K' and so the structures M^{**} and $\mathbf{M}^*$ obtained from M_R and $\mathbf{M}_R$ by neglecting the relation R_M, are both models of K. Since $\mathbf{M}^*$ and M'' are both submodels of the model M^{**} of the convex algebra of axioms K, their meet must be a model of K. But $M'' \Supset M^*$ and $\mathbf{M}^* \Supset M^*$, while $\mathbf{M}^*$ does not contain any objects of $M'' - M^*$, and so finally $\mathbf{M}^* \cap M'' = M^*$ is a model of K. This proves 9.1.3.

9.1.6. *Theorem.* Let M be a true submodel of a model M' of a convex algebra of axioms K. Then there exists a well-ordered series $\{M_\nu\}$ of models of K such that: $M = M_0$; if ν is not a limit number then M_ν is a simple supermodel of $M_{\nu-1}$; if ν is a limit number then M_ν is a simple supermodel of M'_ν, where M'_ν is the union of all M_λ, $\lambda < \nu$; and M' is the union of all M_ν.

This theorem shows that every supermodel of a given model can be obtained by the successive adjunction of single elements. To prove it, we select for every true submodel M'' of M' which includes M an object $a \varepsilon M' - M''$, and we write $a = f(M'')$. We also denote by $F(M'')$ the simple supermodel of M'' which is generated by $f(M')$. $\{M''\}$ may now be defined by transfinite induction, as follows:

$M_0 = M$; if ν is not a limit number then $M_\nu = F(M_{\nu-1})$; if ν is a limit number, then $M_\nu = F(M'_\nu)$ where M'_ν is the union of all M_λ, $\lambda < \nu$. M'_ν is a model of K, by 9.1.3. This procedure can only fail if ν is not a limit number, and $M_{\nu-1} = M'$, in which case the series terminates with $M_{\nu-1}$, or if ν is a limit number and $M'_\nu = M'$. In that case again, we only define M_λ for all $\lambda < \nu$. On the other hand, the procedure must terminate at some stage because the cardinal of $\{M_\nu\}$ cannot exceed the cardinal of M'.

In this way we obtain all the possible supermodels of a given model.

We may use condition 9.1.3. to show that certain algebras of axioms are not convex. Thus let K be the set obtained by the addition of 9.1.2. above to A_F. We define the model F_1 as the field of rational numbers, and the model F'_1 as the field obtained from F_1 by the adjunction of a transcendental element x. For $n > 1$ we define F_n as the algebraic closure of F'_{n-1} and F'_n as the field obtained from F_n by the adjunction of an element X_n which is transcendental with respect to F_n. We shall suppose that all the F_n and F'_n are normal so that $F'_1 \Subset F'_2 \Subset \ldots$. Let F be the union of the elements of this chain. F is algebraically closed, and therefore does not satisfy 9.1.2. On the other hand every F'_n satisfies 9.1.2. since $x_n \varepsilon F'_n$ has no square root within F'_n. It follows that the algebra K cannot be convex.

9.2. *Convex systems and algebraic predicates.* A model M of an algebra of axioms K is called a prime model of K if it possesses no true substructure which is again a model of K.

This definition still depends on the correspondence between K and M since, as stated earlier, we always assume implicitly that the correspondence under which any substructure of M is a model of K is supposed to be included in the correspondence under which M is a model of K.

Every model M of a convex algebra of axioms contains exactly one prime model. This is the meet of all submodels of M.

Thus all the possible models of a convex algebra of axioms K can be obtained by the procedure described above as supermodels of the prime models of K.

Let $H = (K; D)$ be defined by the convex algebra of axioms K augmented by a complete diagram D of a model M of K. Then H also is convex and the prime model contained in any model M' of H is isomorphic to M.

Skolem's derivation of the well-known theorem named after him and Löwenheim (ref. 22) shows, in our terminology, that every model of a finite or countable algebra of axioms contains a submodel whose reduced cardinal is smaller than or equal to $\aleph_0$. It follows that the reduced cardinals of the prime models of a finite or countable algebra of axioms are smaller than or equal to $\aleph_0$.

There is an intimate connection between the concept of convexity and some of the concepts of the preceding chapter. This is indicated by the following theorem.

9.2.1. Every object of a prime model M of a convex algebra of axioms K satisfies an algebraic predicate in K.

Proof. Some of the objects of the prime model M may correspond to object symbols contained in K. These objects constitute a set S, which may be empty, and for them it is not difficult to specify a strongly algebraic predicate of the first degree as required by the theorem. In fact let $a \varepsilon S$, and let a' be the object symbol corresponding to a. Then $Q(x) = E(a', x)$ is the required predicate. To prove that the theorem also holds for objects of M outside S, let M' be any model which is isomorphic to M such that the relations of M and M' coincide, as well as those objects which, within M, belong to S. On the other hand, for every object of M which does not belong to S, it will be supposed that the corresponding object of M' does not belong to M. Thus, M' is obtained from M by replacing all the objects of M which do not belong to S by different objects which are outside M. Let D and D' be complete diagrams of M and M' respectively such that the relative symbols of D, D' and K which correspond to the same relations coincide, as well as the object symbols which correspond to the same objects in S, while differing for different objects of M and M'. We maintain that the union of D, D' and K, $H = K \cup D \cup D'$, is consistent. Indeed, consider the structure M'' whose objects are the objects of M and of M', such that a relation A of M or M' holds between objects of M'', $a_1, \ldots, a_m$, say, where $a_1, \ldots, a_k \varepsilon M$, $a'_{k+1}, \ldots, a'_m \varepsilon M'$, if and only if $A(a_1, \ldots, a_k, a_{k+1}, \ldots, a_m)$ holds in M, where the a_i are the objects of M corresponding to the a'_i in M', $i = k+1, \ldots, m$. In that case also, $A(a'_1, \ldots, a'_k, a'_{k+1}, \ldots, a'_m)$ holds in M'. For instance, if a and a' are two corresponding objects of M and M' respectively, then $E'(a, a')$ holds in M'', since $E'(a, a)$ holds in M, where E' is the relation of equality in M, M' and M''. Thus, corresponding objects in M and M' become equal within M''. The correspondences between D and M, and D' and M' respectively, establish a correspondence between $D \cup D'$ and M'', and hence between the object and relative symbols of H and some of the objects, and the relations of M''. It is not difficult to see that D and D' hold in M'' under this correspondence. But we may regard M'' as having been obtained from M by the addition of objects which are equal to objects already included in M. Since M is a model of K, it then follows from an argument used in the proof of 9.1.3. that M'' also is a model of K. Thus M'' is a model of the set $H = K \cup D \cup D'$, which is therefore consistent.

Now let a be an object symbol of D which is not contained in K, and let D'' be the set of statements $[\sim E(a, a')]$ where a' varies over all the objects symbols of D'. We propose to show that the set $H' = H \cup D''$ is inconsistent. In the alternative case, let M^* be a model of H'. M^* contains structures isomorphic to M and M', corresponding to D and D', and we may therefore assume that these structures are actually included in M^*. Let M'' be the substructure of M^* obtained by adding to M' all the objects of M^* which are equal to objects of M' without being included in M'.

M'', like M', is a model of K. We now modify M^* by removing from it all the objects which are equal to objects of M without being contained in it. The resultant structure $\mathbf{M}$ is a model of K, as before, and M is a submodel of $\mathbf{M}$. Denoting by $\mathbf{M}'$ the join of M'' and $\mathbf{M}$, we see that $\mathbf{M}'$ is a substructure of $\mathbf{M}$: also it still is a model of K although, owing to the removal of some of the elements of M'', $\mathbf{M}'$ may not now be a model of D' by the same correspondence. On the other hand, the object a^* of M which corresponds to the object symbol a of D is not contained in $\mathbf{M}'$ nor is it contained in the structure M'' which includes $\mathbf{M}'$. For a is not included in D', by assumption nor, by the definition of D'', can it be equal to any object symbol of D'. Hence also, the corresponding a^* in M cannot be equal to any object of M', and so is not included in M''. It follows that the meet $\mathbf{M}''$ of $\mathbf{M}'$ and M does not include a^*, although $\mathbf{M}''$ is a model of the convex algebra of axioms K. And this shows that $\mathbf{M}''$ is a true submodel of M, contrary to the assumption that M is prime.

We have shown that D'' is inconsistent with H, so that a finite subset of D'', consisting of statements $[\sim E(a, a_1')]$, $[\sim E(a, a_2')], \ldots,$ $[\sim E(a, a_n')]$, say, is inconsistent with H. In other words, the negation of the conjunction of these statements is deducible from H. If we recall that $H = K \cup D \cup D'$, we see therefore that there are statements $X_1, X_2, \ldots, X_j \,\varepsilon\, K$, $Y_1, Y_2, \ldots, Y_m \,\varepsilon\, D$ and $Y_1', Y_2', \ldots, Y_{m'}' \,\varepsilon\, D'$ such that

$$9.2.2. \quad [X_1 \wedge X_2 \wedge \ldots \wedge X_j] \wedge [Y_1 \wedge Y_2 \wedge \ldots \wedge Y_m] \wedge [Y_1' \wedge Y_2' \wedge \ldots \wedge Y_{m'}'] \supset [E(a, a_1') \vee E(a, a_2') \vee \ldots \vee E(a, a_n')]$$

is valid. We write $X = [X_1 \wedge X_2 \wedge \ldots \wedge X_j]$, $Y = [Y_1 \wedge \ldots \wedge Y_m]$ and $Y' = [Y_1' \wedge \ldots \wedge Y_{m'}']$. Then Y is a conjunction of statements of order 1 or 2 and, apart from a, involves object symbols common to K, D, and D', $b_1, \ldots, b_k$ say, and object symbols of D which are not contained in K or D', $c_1, \ldots, c_l$, say. Similarly X' is a conjunction of statements of order 1 or 2, and involves object symbols common to K, D, and D', $b_1', \ldots, b_{k'}'$, and object symbols of D' which are not contained in K or D', $c_1', \ldots, c_{l'}'$, in addition to $a_1', \ldots, a_n'$ which may belong to either type.

Writing $Y = Y(a, b, \ldots, b_k, c_1, \ldots, c_l)$, $Y' = Y'(a_1', \ldots, a_n',$

$b'_1, \ldots, b'_{k'}, c'_{1'}, \ldots, c'_l)$, 9.2.2. may now be restated as follows —

$$9.2.3.\quad Y(a, b_1, \ldots, b_k, c_1, \ldots, c_l) \wedge Y'(a'_1, \ldots, a'_n, b'_1, \ldots, b'_{k'}, c'_1, \ldots, c'_{l'}) \supset [E(a, a'_1) \vee E(a, a'_2) \vee \ldots \vee E(a, a'_n)] \vee [\sim X].$$

Since 9.2.3. is a valid statement, it follows from the repeated application of 2.5.8. that

$$9.2.4.\quad [(\exists z_1) \ldots (\exists z_l)(\exists z'_1) \ldots (\exists z'_{l'})[Y(a_1, b_1, \ldots, b_k, z_1, \ldots, z_l) \wedge Y'(a'_1, \ldots, a'_n, b'_1, \ldots, b'_{k'}, z'_1, \ldots, z'_{l'})]] \supset [E(a, a'_1) \vee \ldots \vee E(a, a'_n)] \vee [\sim X]$$

is valid. Modifying this expression slightly, we see that

$$9.2.5.\quad X \supset [[(\exists z_1) \ldots (\exists z_l)Y(a, b_1, \ldots, b_k, z_1, \ldots, z_l)] \wedge [(\exists z'_1) \ldots (\exists z'_{l'})Y'(a'_1, \ldots, a'_n, \ldots z'_1, \ldots z'_{l'})] \supset [E(a, a'_1) \vee \ldots \vee E(a, a'_n)]]$$

is valid.

Now let $a'_1, \ldots, a'_i$ be included in K, and hence possibly in X, while the object symbols $a'_{i+1}, \ldots, a'_n$ are not included in K and cannot therefore be included in X. Then the repeated application of 2.5.7. shows that

$$9.2.6.\quad X \supset [(x)(x'_{i+1}) \ldots (x'_n)[[(\exists z_1) \ldots (\exists z_l)Y(x, b_1, \ldots, b_k, z_1, \ldots, z_l)] \wedge [(\exists z'_1) \ldots (\exists z'_{l'}) Y'(a'_1, \ldots, a'_i, x'_{i+1}, \ldots, x'_n, b'_1, \ldots, b'_{k'}, z'_1, \ldots, z'_{l'})] \supset [E(x, a'_1) \vee \ldots \vee E(x, a'_i) \vee E(x, x'_{i+1}) \vee \ldots \vee E(x,x'_n)]]]$$

is a valid statement. Thus the expression which is obtained from 9.2.6. by omitting $X \supset$ is deducible from K. Since the relative symbol of equality is an equivalence relation, it is then easy to see that the statement

$$9.2.7.\quad (x)(x'_1) \ldots (x'_i)(x'_{i+1}) \ldots (x'_n)[Q'(x) \wedge Q''(x'_1, \ldots, x'_i, x'_{i+1}, \ldots, x'_n) \supset E(x, x'_1) \vee \ldots \vee E(x, x'_i) \vee E(x, x'_{i+1}) \vee \ldots E(x, x'_n)]$$

is deducible from K where

$$9.2.8.\quad Q'(x) = (\exists z_1) \ldots (\exists z_l)Y(x, b_1, \ldots, b_k, z_1, \ldots, z_l)$$

and

$$Q''(x_1', \ldots, x_n') = (\exists z_1') \ldots (\exists z_{l'}')[Y'(x_1', \ldots, x_n', b_1, \ldots, z_{l'}') \wedge E(a_1', x_1') \wedge \ldots \wedge E(a_i', x_i')]$$

are both predicates in K. Substituting $x_1, \ldots, x_n$ in turn for the dummy symbol x in 9.2.7. it is then not difficult to show that the following statement also is deducible from K.

9.2.9. $(x_1) \ldots (x_n)(x)(x_1') \ldots (x_n')[Q'(x_1) \wedge Q'(x_2) \wedge \ldots \wedge Q'(x_n) \wedge Q'(x) \wedge Q''(x_1', \ldots, x_n') \supset [E(x_1, x_1') \vee \ldots \vee E(x_1, x_n')] \wedge [E(x_2, x_1') \vee \ldots \vee E(x_2, x_n')] \wedge \ldots \wedge [E(x_n, x_1') \vee \ldots \vee E(x_n, x_n')] \wedge [E(x, x_1') \vee \ldots \vee E(x, x_n')]]$

The semantic interpretation of 9.2.9. shows immediately that if we can ensure that 'there are $y_1, \ldots, y_n$ such that $Q''(y_1, \ldots, y_n)$ holds' in any particular model of K, then the statement

9.2.10. $(x_1) \ldots (x_n)(x)[Q'(x_1) \wedge Q'(x_2) \wedge \ldots \wedge Q'(x_n) \wedge Q'(x) \supset [E(x_1, x_2) \vee E(x_1, x_3) \vee \ldots \vee E(x_1, x_n) \vee \ldots \vee E(x_{n-1}, x_n) \vee E(x, x_1) \vee \ldots \vee E(x, x_n)]$

also holds in that model. In other words, the statement

9.2.11. $[(\exists y_1) \ldots (\exists y_n)Q''(y_1, \ldots, y_n)] \supset [(x_1) \ldots (x_n)(x) [Q'(x_1) \wedge \ldots \wedge Q'(x)] \supset Z]$

is deducible from K, where Z is the implicate of 9.2.10. The same therefore applies to the statement

9.2.12. $(y_1) \ldots (y_n)(x_1) \ldots (x_n)(x)[\sim Q''(y_1, \ldots, y_n) \vee [Q'(x_1) \wedge \ldots \wedge Q'(x) \supset Z]]$

and to

$$(y_1) \ldots (y_n)(x_1) \ldots (x_n)(x)[\sim Q''(y_1, \ldots, y_n) \vee \sim Q'(x_1) \vee \ldots \vee \sim Q'(x) \vee Z]$$

and so to

$$(x_1) \ldots (x_n)(x)[[(y_1) \ldots (y_n)Q''(y_1, \ldots, y_n)] \wedge Q'(x_1) \wedge \ldots \wedge Q'(x_n) \wedge Q'(x) \supset Z]$$

since Z does not include any of the dummy symbols y_i. Hence, finally, the statement

$$9.2.13. \quad (x_1) \ldots (x_n)(x)[Q(x_1) \wedge \ldots \wedge Q(x_n) \wedge Q(x) \supset E(x_1, x_2) \vee \ldots \vee E(x_{n-1}, x_n) \vee E(x, x_1) \vee \ldots \vee E(x, x_n)]$$

is deducible from K where $Q(x)$ is defined by

$$9.2.14. \quad Q(x) = (\exists y_1) \ldots (\exists y_n)[Q''(y_1, \ldots, y_n) \wedge Q'(x)]$$

9.2.13. shows that $Q(x)$ is bounded and of a degree not exceeding n. Moreover, $Q(x)$ is peristent, as is easily verified directly or by means of the fact that it is associated with the predicate,

$$9.2.15. \quad Q^*(x) = (\exists y_1) \ldots (\exists y_n)(\exists z_1) \ldots (\exists z_l)(\exists z_1') \ldots (\exists z_{l'}')[Y_1 \wedge \ldots \wedge Y_m \wedge Y_1' \wedge \ldots \wedge Y_{m'}']$$

where the statements Y_i and Y_i' are of order one, or are negations of statements of order one. It follows that if $Q^*(x)$ holds for any particular object in a model M' of K it also holds for that object in any extension of M' which is a model of K. Finally both $Q(x)$ and $Q^*(x)$ will be seen to hold for a and this proves the theorem.

9.3. *Definite algebras.* Now let M and M' be models of a convex algebra of axioms K, $M \Subset M'$ and let $K' = (K; D)$ be the set K augmented by a full diagram of M. Let K'' be a set of statements of the form $E(a_1, a_1)$, $E(a_2, a_2)$, $\ldots$ where $a_1, a_2, \ldots$ are arbitrary object symbols not contained in D. Clearly K'' holds in M' if we let $a_1, a_2, \ldots$ correspond to any set S of objects of M' not contained in M. Then a simple application of the preceding theorem 9.2.1. shows that every object of $M(S) \Subset M'$ obtained from M by the adjunction of S, satisfies an algebraic predicate in $K' \cup K''$. In fact, $M(S)$ is the prime model of $K' \cup K''$ in M'.

The set K'' which clearly does not add anything to our information on M' had to be introduced only for the purpose of including algebraic predicates which contain $a_1, a_2, \ldots$.

In the case of a commutative field M, the elements of the reduced field $M(S)$ can all be expressed as rational functions of the elements of S with coefficients in M. These actually correspond to predicates of degree one. However, examples can be given to show that the lowest degree of any predicate of $K' \cup K''$ which holds for a specified object of $M(S)$, may be greater than one.

For instance, consider the algebra of axioms K which, in addition to the relative symbol of equality $E(,)$, includes another relative symbol of order 2, $F(,)$ such that both E and F satisfy the laws of reflexivity, symmetry, and transitivity, 6.1.1.—6.1.3., while E also satisfies a law of substitutivity

9.3.1. $(x_1)(x_2)(y_1)(y_2)[E(x_1, y_1) \wedge E(x_2, y_2) \wedge F(x_1, x_2) \supset F(y_1, y_2)]$

In addition, K is supposed to contain the following axioms.

9.3.2. $(\exists x)(y)[[F(x, y) \supset E(x, y)] \wedge [(z)[E(x, z) \vee [(\exists v)F(z, v) \wedge \sim E(z, v)] \ldots]$

'There exists an x such that, if x is in the relation F to any y, then x equals y, and any other z which has the same property, equals x.'

9.3.3. $(x)(y)[F(x, y) \wedge \sim E(x, y) \supset [(\exists z)[F(x, z) \wedge \sim E(x, z) \wedge \sim E(y, z)]]]$

'If the relation F holds between two different x and y then it also holds between x and some z which equals neither x nor y.'

9.3.4. $(x)(y)(z)(v)[F(x, y) \wedge F(x, z) \wedge F(x, v) \supset E(x, y) \vee E(x, z) \vee E(x, v) \vee E(y, z) \vee E(y, v) \vee E(z, v)]$

'The equivalence classes of objects defined by F cannot contain more than three different elements.'

K is consistent; to construct a model for it, let N be any set of objects, either infinite, or if finite, including $3n + 1$ elements. We divide N into classes $N_0, N_1, N_2, \ldots$ (finite or transfinite in number) such that N_0 contains just one element, whereas all the other N_k contain three elements each. We now define that the relation of equality $E'(,)$ shall hold only when the same object occupies both places of the relation, so that E' is normal, while $F'(,)$ corresponding to F shall hold between any two objects of the same class N_n. It will be seen that the structure defined in this way is a model of K. Also, it is now evident that K is convex.

In particular, consider the case in which N includes exactly four objects a_0', a_1', a_2', a_3'. We divide N into classes $N_0 = \{a_0'\}$ and

$N_1 = \{a_1', a_2', a_3'\}$. The corresponding structure M' is given by the relations

9.3.5. $E'(a_i', a_i')$, $i = 0, 1, 2, 3$
$F'(a_0', a_0')$, $F(a_i', a_j')$ $i, j = 1, 2, 3$

A substructure M of M' is given by

9.3.6. $E'(a_0', a_0')$, $F'(a_0', a_0')$

M is again a model of K. A complete diagram of M is

$$D = \{E(a_0, a_0),\ F(a_0, a_0)\}$$

Let $K' = K \cup D = (K; D)$, and let K'' be the set whose only element is the statement $E(a_1, a_1)$, where a_1 is any object symbol of the language except a_0. Letting a_1 correspond to a_1' in M_1' we have $S = \{a_1'\}$ in the notation used at the beginning of this section. Then $M(S)$, the submodel of M' which is obtained by the adjunction of S to M, coincides with M'. In agreement with 9.2.1. all the objects of M' satisfy algebraic predicates in $K' \cup K''$; a_0' and a_1' satisfy the algebraic predicates $E(a_0, x)$ and $E(a_1, x)$ which are of the first degree; a_2' and a_3' both satisfy the algebraic predicate $F(a_1, x)$ which is of degree Three. They also satisfy the algebraic predicate $F(a_1, x) \wedge \sim E(a_1, x)$ which is of the second degree, but they do not satisfy any algebraic predicate of the first degree in $K' \cup K''$. This example shows that we cannot expect all the objects of $M(S)$ to satisfy algebraic predicates of the first degree: more generally it shows that we cannot qualify the algebraic predicate which is mentioned in 9.2.1. as being of the first degree. We now define,

9.3.7. A model M of a convex algebra of axioms K will be called definite if every object of any submodel M' of M which is generated by a set S of objects of M satisfies an algebraic predicate of degree one in $K \cup K''$.

In this definition K'' is the set of all statements of the form $E(a, a)$ where a varies over a set of object symbols not contained in K, corresponding to those objects of S which do not correspond to object symbols of K. Here again, the set K'' is introduced merely in order to include predicates which contain object symbols corresponding to arbitrary objects of S.

A convex algebra of axioms will be called definite if all its models are definite with respect to it.

It is important to bear in mind the difference between an algebraic predicate of degree one, and a polynomial or prepolynomial. If $R(x_1, \ldots, x_n, y)$ is a polynomial in an algebra of axioms K (see chapter 7) and $a_1, \ldots, a_n$ are object symbols included in K, then $Q^*(x) = R(a_1, \ldots, a_n, x)$ is an algebraic predicate in K. But this does not hold for general prepolynomials nor can every algebraic predicate of degree one be represented in this way.

9.4. *Properties of definite algebras.* Let M be a model of a convex algebra of axioms K, and let S' be the set of all those objects of M which correspond to object symbols of K; S' may be empty. An automorphism of M which centralises the objects of S' (i.e., carries every object of S' into itself) will be called admissible. The admissible automorphisms of M constitute a group G. We may study the connection between the properties of M and of G, on the assumption that M is a definite model of K, and that it is normal.

Since an understanding of the part played by the correspondence between the objects and relations of M and the object and relative symbols of the language L of K is of cardinal importance in connection with the present subject, we may be justified in recalling some of its salient features.

A complete correspondence C between L and M is a correspondence in which to every object and relation in M, there corresponds just one object or relative symbol in L. Assuming that these object and relative symbols comprise all those which are contained in K, we then say that M is a model of K if the statements of K hold in M under C, according to the rules laid down in section 2.7. Also, in that case K is a model of M under any other correspondence such that the object and relative symbols of K correspond to the same objects and relations of M as under C. Thus when we say that 'M is a model of K' we assume implicitly that the correspondence between the object and relative symbols of K on one hand, and the objects and relations of M on the other, is fixed once and for all. The particular correspondence chosen for the remaining objects and relations is irrelevant.

Now if $K' = (K; D')$ is the algebra K augmented by a complete diagram D of M under a correspondence C as above, then any automorphism $g \varepsilon G$ also establishes a new correspondence C' between the object symbols a' of D' and the objects a of M. This correspondence is obtained by replacing any $a \leftrightarrow a'$ in C by $a' \leftrightarrow ga$ where ga is the object into which a is carried by g. Thus M also is a model of D' under C' and it now appears that submodels of M (substructures of M which are models of K) are carried into submodels of M. Also, let $Q(x)$ be any predicate which involves only object symbols of K. Then if $Q(x)$ holds for a in M, i.e., $Q(a')$ holds in M, where $a' \leftrightarrow a$ under C, then $Q(a')$ also holds in M under C', under which $a' \leftrightarrow ga$. Thus $Q(x)$ also holds for ga. Hence in particular, if $Q(x)$ is bounded in K of degree one, then ga must coincide with a so that a is centralised by g.

Condition 9.3.7. applied to the case that S is empty, tells us that all the objects of the prime model M' in M satisfy algebraic predicates of degree one in K, so that they are centralised by every $g \varepsilon G$. Thus, the objects of M' are centralised by all the elements of G.

9.4.1. *Theorem.* The set S'' of objects of S which are centralised by any given $g \varepsilon G$ constitutes the totality of objects contained in a submodel M'' of S.

For let M'' be the submodel generated by the set S'' of objects which are centralised by a given $g \varepsilon G$, and let a be any object of M''. Since M is definite, a satisfies an algebraic predicate of order one in $K \cup K''$, $Q(x)$, say, K'' being the set of objects of the form $E(a'', a'')$, where a'' varies over the elements of S''. Now since g centralises the objects of S'', $Q(x)$ also holds for ga, and so ga equals a. This shows that a is centralised by g and therefore belongs to S''.

As a corollary, we obtain the result that the set S'' of objects of S which are centralised by any given subgroup $G' \subseteq G$ constitutes the totality of objects contained in a submodel M'' of S. Conversely, it is clear that the totality of admissible automorphisms of M which centralise all the objects of a submodel of M constitutes a subgroup of G.

9.4.2. *Theorem.* Assume that there exist n different objects in M, $a_1, a_2, \ldots, a_n$, such that M is generated by these objects (i.e. is the only submodel of itself which contains all of them) and such that they all satisfy an algebraic predicate $Q(x)$ of degree n in k. Then G is finite.

To prove this, we observe first of all that the only element of G which centralises all the objects $a_1, a_2, \ldots, a_n$ is the neutral element. For since M is definite, every object a of M satisfies an algebraic predicate of degree one in $K \cup K''$, $Q'(x)$, say, where $K'' = \{E(a_1', a_1'), E(a_2', a_2') \ldots, E(a_n', a_n')\}$, a_k' being the object symbol corresponding to a_k, $k = 1, 2, \ldots, n$. But if an element $g \varepsilon G$ centralises $a_1, a_2, \ldots, a_n$ then it follows as above that $Q'(x)$ holds also for ga. Hence ga equals a and so coincides with it since M is normal. Thus $g = e$, where e is the neutral element of G.

On the other hand, any $g \varepsilon G$ transforms the element of the set $a_1, a_2, \ldots, a_n$ into elements of the same set since, as before, ga_i must satisfy $Q(x)$ for all $g \varepsilon G$ and for all a_i, $1 \leqslant i \leqslant n$. In other words, g effects a permutation of the set $a_1, a_2, \ldots, a_n$. Assume now that two elements of G, g_1 and g_2 both transform $a_1, a_2, \ldots, a_n$ into $a_{k_1}, a_{k_2}, \ldots, a_{k_n}$. Then $g_1^{-1} g_2$ transforms $a_1, a_2, \ldots, a_n$ into $a_1, a_2, \ldots, a_n$ i.e., it centralises the elements of the set. Hence $g_1^{-1} g_2 = e$, $g_2 = g_1$. This shows that any element of G is determined completely by the permutation which it effects on the set $a_1, a_2, \ldots, a_n$. Thus G is isomorphic to a subgroup of the symmetrical group of order n, so that the number of elements of G cannot exceed $n!$.

The results of this section constitute the foundations of a Galois theory for definite algebras. If $K = (A_F; D)$ where D is the complete diagram of a field F, and M is an extension of F which is a normal structure, then F is the prime model M' mentioned above. And in the particular case where $Q(x)$ in 9.4.2. is a strongly algebraic predicate $Q_q(x)$, the assumption of 9.4.2. is that M is obtained by the adjunction of some or all of the roots of $q(x)$ to F.

The results obtained so far represent only a small proportion of the main propositions of Galois theory, and any further developments require additional assumptions. Even in Algebra, Galois' theorems can be stated without modifications only for separable fields. We do not propose to develop a corresponding general theory in the present work. Instead, we shall confine ourselves to the metamathematical interpretation of the condition of separability.

9.5. *Separable and perfect systems.* To lead up to a metamathematical concept of separability we may ask a question which is in some ways complementary to the considerations which gave rise to the definition of a definite model. Let K be any algebra of axioms, M one of its models, and K' the set K augmented by a complete diagram D of M. The question is whether every (strongly) algebraic predicate of degree one in K is satisfied by some object of M. We may also ask, whether every (strongly) algebraic predicate of degree one in $K' = (K; D)$ holds for some object of M. Selecting one of the various possibilities implied by these questions, we now define,

9.5.1. A model M of an algebra of axioms K will be called separable, if every algebraic predicate of degree one in $K' = (K; D')$, where D' is a complete diagram of a submodel M' of M, holds for some object of M' if it holds for any object of M.

9.5.2. An algebra of axioms will be called perfect if all its models are separable with respect to it.

An equivalent definition is

9.5.3. An algebra of axioms K will be called perfect if every algebraic predicate of degree one in $K' = (K; D')$ holds for some object of M', where M' is any model of K, and D' is a complete diagram of M'.

It is easy to see that an algebra of axioms K which satisfies 9.5.3., satisfies 9.5.2. If K does not satisfy 9.5.3., then there exists a model M' of K and an algebraic predicate $Q(x)$ of degree one in $K' = (K; D')$ — where D' is a complete diagram of M' — such that $Q(x)$ does not hold for any object in M'. Since $Q(x)$ is of degree one in K', there exists a model M of K' in which $Q(x)$ holds for some object. M includes a partial model isomorphic to M', and so as in previous arguments, we may assume that M' is actually a submodel of M. Then $Q(x)$ holds for some a' in M but not in M'. This is contrary to the condition 9.5.2.

Notice that we require not only that the predicate holds in M for an object a' which is contained in M', but more definitely that the predicate holds for a' in M'.

We shall now establish the connection with the corresponding standard algebraic terms.

9.5.4. An extension M^* of a commutative algebraic field F

is separable (ref. 23), if all the elements of M^* which are algebraic with respect to F satisfy irreducible polynomials in F which do not possess multiple factors. According to this definition M may contain elements which are transcendental with respect to F.

9.5.5. *Theorem.* Let $K = (A_F; D)$ so that K is the set of axioms of a commutative field augmented by a complete diagram of a field F, and let M be any superfield of F. Then M is separable as a model of K in the sense defined in 9.5.1., if and only if M is a separable extension of F in the algebraic sense (9.5.4.).

Assume first that M is an inseparable extension of F in the algebraic sense, and let a be an element of M which is inseparable with respect to F. Then a is a root of an irreducible polynomial $f(x)$ with coefficients in F which is of the form $f(x) = \varphi(x^p)$ where p is the characteristic of F and M, $p \neq 0$ and

9.5.6. $\varphi(y) = a_0 + a_1 y + a_2 y^2 + \ldots + a_n y^n,\ a_n \neq 0,\ n \geqslant 1$

Now let M' be the subfield of M which is obtained from F by the adjunction of the element $a^* = a^p$. a^* is a root of $\varphi(y)$ so that M' is of degree n with respect to F. It follows that M' does not include the element a which is of degree $n.p$ with respect to F. Let $K' = (K; D')$ where D' is a complete diagram of M' and let a'' be the object symbol of D' which corresponds to a^*. Consider the predicate $Q(x) = P_p(x, a'')$ where P_p is defined as in 4.3.2., 'a'' is the pth power of $x, x^p = a''$'. $Q(x)$ is evidently persistent (and indeed invariant) in K', and it holds for a in M. Also by standard field theory, the equation $x^p = b$, b arbitrary, possesses just one root which is of multiplicity p. Thus the statement

9.5.7. $(x_1)(x_2)[P_p(x_1, a') \wedge P_p(x_2, a') \supset E(x_1, x_2)]$

is deducible from K', where a' is the object symbol corresponding to a. It follows that $Q(x)$ is bounded, of degree one. Thus, a satisfies an algebraic predicate of degree one in K', although it does not belong to the field M' which is generated by a^*. This shows that 9.5.1. is not satisfied.

Conversely, let M be a separable extension of F in the algebraic sense, and let a be an element of M which satisfies an algebraic predicate $Q(X)$ of degree one in K', where $K' = (K; D')$, D' being a complete diagram of a subfield M' of M which is an extension of F. We have to prove that a satisfies $Q(x)$ within M'. Since a satisfies an algebraic predicate with respect to K, it must be algebraic with respect to F (see 8.6.3.).

Assume then that a satisfies the irreducible polynomial $f(x)$, of degree n, with coefficients in M'. $f(x)$ has no multiple roots since a is separable, by assumption. Let M^* be the algebraic closure of M. M^* contains n different roots of $f(x)$, including a. $Q(x)$ holds for a in M^* since it is persistent. But if $n > 1$, then $Q(x)$ must hold also for all the other roots of $f(x)$ in M^*. For in that case there exists an isomorphism of M^* which carries a into any other given root of $f(x)$, while centralising all the objects of M'. But $Q(x)$ was supposed to be of degree one, and so we may conclude that $n = 1$, showing that a belongs to M'.

9.5.8. Let K be the set of axioms of a commutative field augmented by a complete diagram of a field F, $K = (A_F : D)$ as before. Then K is perfect in the sense defined by 9.5.2. if and only if F is a perfect field in the algebraic sense.

For a perfect field in the algebraic sense is a field which has no separable extensions, and every model of K is isomorphic to an extension of F.

The set of axioms A_F taken by itself is not perfect according to our definition, nor are the sets $A_2, A_3, A_5, \ldots$ corresponding to the concepts of fields of specified prime characteristic. On the other hand A_0, the algebra of axioms of fields of characteristic 0, is perfect. Also, if we add to a given A_p, $p \neq 0$ the axiom

9.5.9. $$(x)(\exists y)P_p(y, x)$$

then the resultant set A_p' is perfect. (Models of A_p' are of importance in connection with a modified Galois theory for inseparable fields, compare ref. 24). Again, the set of axioms A_0^* of an integral domain of characteristic 0 obtained, for example, by replacing 4.1.11. in A_0 by the pair of axioms

9.5.10. $$(\exists x)(y)[P(y, x, y)]$$

and

9.5.11. $$(x)(y)(z)(w)[S(x, x, x) \vee [P(x, y, z) \wedge P(x, w, z) \supset E(y, w)]]$$

is not perfect. In fact, if M' is the ring of integers, then the predicate

9.5.12. $Q(x) = (\exists y)(\exists z)[S(y, y, z) \wedge P(x, z, y) \wedge \sim S(y, y, y)]$

is algebraic of degree One in $(A_0^*; D')$ where D' is a complete diagram of M', but it is not satisfied by any object in M'.

9.6. *Symmetrical predicates.* The main theorem of the present section (9.6.3. below) may serve as an illustration of the implications of the concept of separability. Although the conclusions of the theorem hold in algebraic field theory, their derivation in that theory is quite different, being in fact equally valid for inseparable fields.

Given any predicate $R(x_1, \ldots, x_n, y)$, $n \geqslant 1$, we consider the $n!$ predicates (including R itself) which are obtained from Q by permuting the suffixes of the dummy symbols $x_1, x_2. \ldots, x_n$. Thus, let $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots n \\ k_1 & k_2 & k_3 & \ldots k_n \end{pmatrix}$ be any permutation of order n. We then define the predicate $R_\pi(x_1, \ldots, x_n, y)$ by

9.6.1. $R_\pi(x_1, x_2, \ldots, x_n, y) = R(x_{k_1}, x_{k_2}, \ldots, x_{k_n}, y)$

for instance, if $R(x_1, x_2, y) = (\exists z)[S(x_1, y, z) \wedge P(x_2, x_1, z)]$ and $\pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ then $R_\pi(x_1, x_2, y) = (\exists z)[S(x_2, y, z) \wedge P(x_1, x_2, z)]$. Clearly $R_\iota = R$, where ι is the identical permutation.

A predicate $R(x_1, \ldots, x_n, y)$, $n \geqslant 1$ will be called absolutely symmetrical if

9.6.2. $(x_1) \ldots (x_n)(y)[R_{\pi_1}(x_1, \ldots, x_n, y) \supset R_{\pi_2}(x_1, \ldots, x_n, y)]$

is valid for all permutations π_1, and π_2, of order n. A sufficient condition for this to happen is that 9.6.2. be valid for $\pi_1 = \iota$ and arbitrary π_2 and for $\pi_2 = \iota$ and arbitrary π_1.

$R(x_1, \ldots, x_n, y)$ will be called symmetrical relative to a set of axioms K, if 6.2.2. can be deduced from K for all π_1, π_2 of order n.

9.6.3. *Theorem.* Let M be a separable model of a convex algebra of axioms K, and let $Q(x)$ be an algebraic predicate of nth degree in K which is satisfied by n different objects $a_1, a_2, \ldots, a_n$ in M. Let M' be the submodel of M which is generated by the set $S = \{a_1, a_2, \ldots, a_n\}$. M' may also be described as the submodel of M which is obtained by the adjunction of the set S to the prime model M'' in M. Finally, let $R(x_1, x_2, \ldots, x_n, y)$ be a predicate which is persistent and symmetrical relative to the set K such that $Q'(y) = R(a_1', \ldots, a_n', y)$ is an algebraic predicate of degree one in $K \cup K'$, $K' = \{E(a_1', a_1'), E(a_2', a_2'), \ldots\}$ where the object symbols $a_1', a_2', \ldots, a_n'$ correspond to the objects $a_1, a_2, \ldots a_n$ respectively. Then if $Q'(y)$ holds for an object a in M', a must belong to the prime model M''.

If all the object symbols $a_1', a_2', \ldots, a_n'$ are contained in K then $Q'(y)$ is an algebraic predicate of degree one in K, and so a belongs to M'' since M is perfect. Again, if some but not all the a_k' are contained in K, $a_{m+1}', \ldots, a_n'$ say, $1 \leqslant m < n$, then we replace $Q(x)$ by $\mathbf{Q}(x) = Q(x) \wedge \sim E(a_{m+1}', x) \wedge \sim E(a_{m+2}', x) \wedge \ldots \wedge \sim E(a_n', x)$, $R(x_1, \ldots, x_n, y)$ by $\mathbf{R}(x_1, \ldots, x_m, y) = R(x_1, \ldots, x_m, a_{m+1}', \ldots, a_n', y)$ and $Q'(y)$ by $\mathbf{Q}'(y) = R(a_1', \ldots, a_m', y)$. Then $\mathbf{Q}$, $\mathbf{Q}'$, $\mathbf{R}$, and m can take the place of Q, Q', R, and n in the formulation of 9.6.3., where it is now known that the objects $a_1, a_2, \ldots, a_m$ do not correspond to object symbols of K. Having effected this reduction, we return to the original notation used in 9.6.3.

To prove 9.6.3. — where we may now assume that $a_1', \ldots, a_n'$ are not contained in K — we construct an algebraic predicate of degree one in K which holds for a in M. This predicate is

9.6.4. $$Q^*(y) = (\exists x_1) \ldots (\exists x_n)[\sim E(x_1, x_2) \wedge \sim E(x_1, x_3) \wedge \ldots \wedge \sim E(x_{n-1}, x_n) \wedge Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n) \wedge R(x_1, x_2, \ldots, x_n, y)]$$

Taking into account that the statements $E(a_i', a_k')$, $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant n$, $i \neq k$; $Q(a_i')$, $1 \leqslant i \leqslant n$; and $R(a_1', a_2', \ldots, a_n', a')$ all hold in M, where a' is an object symbol corresponding to a, we see that $Q^*(y)$ holds for a. To show that $Q^*(y)$ is bounded of degree one, we have to verify that $(y_1)(y_2)[Q^*(y_1) \wedge Q^*(y_2) \supset E(y_1, y_2)]$ can be deduced from K. Now since $Q'(y)$ is an algebraic predicate of degree one in $K \cup K'$ and K' can be deduced from K, it follows that

9.6.5. $$(y_1)(y_2)[Q'(a_1', \ldots, a_n', y_1) \wedge Q'(a_1', a_2', \ldots, a_n', y_2) \supset E(y_1, y_2)]$$

can be deduced from K. Let M^* be any model of K such that $Q^*(y)$ holds for objects b_1 and b_2 in M^*. By 9.6.4., $Q(x)$ then holds for n different objects in M^*, $c_1, c_2, \ldots, c_n$, say. Also, if $b_1', b_2', c_1', c_2',$

$\ldots, c'_n$ are object symbols corresponding to $b_1, b_2, c_1, c_2, \ldots, c_n$ (some of which may be contained in K), 9.6.4. tells us that the statements $R_{\pi_1}(c'_1, \ldots, c'_n, b'_1)$ and $R_{\pi_2}(c'_1, \ldots, c'_n, b'_2)$ hold in M^* for certain permutations π_1 and π_2. Since $R(x_1, \ldots, x_n, y)$ is symmetrical relative to K it follows that the statements $R_\pi(c'_1, \ldots, c'_n, b'_1)$ and $R_\pi(c'_1, \ldots, c'_n, b'_2)$ hold in M for all permutations π, e.g., $\pi = \iota$. Now, in view of the fact that the object symbols $a_1, a_2, \ldots, a_n$ are not contained in K, 9.6.5. above, in conjunction with 2.5.7., shows that the statement

9.6.6. $$(x_1) \ldots (x_n)(y_1)(y_2)[R(x_1, \ldots, x_n, y_1) \wedge R(x_1, \ldots, x_n, y_2) \supset E(y_1, y_2)]$$

is deducible from K. But we have just established that the statements $R(c'_1, \ldots, c'_n, b'_1)$ and $R(c'_1, \ldots, c'_n, b'_2)$ hold in M^* and so, by 9.6.6., $E(b'_1, b'_2)$ also holds in M^*. Having proved this relation for an arbitrary model M^* of K — and not only for M — we may now infer that $Q^*(y)$ is bounded of degree one in K.

Finally, $Q(x)$ and $R(x_1, \ldots, x_n, y)$ are persistent predicates in K, by assumption; hence, as the inspection of 9.6.4. shows, $Q^*(y)$ also is persistent in K. It is therefore algebraic of degree one in K. This completes the proof of theorem 9.6.3.

X

IDEALS

10.1. *Introduction.* If we consider the properties of an ideal J in a commutative ring M, which can be formulated without reference to the particular operations (addition and multiplication) defined in the ring, the most characteristic fact which emerges is undoubtedly that J gives rise to a quotient ring (difference ring) M'. M' consists of sets of elements of M, every element of M belonging to just one element of M'. Moreover, there is a homomorphism in between M and M' under which every element of M corresponds to the element of M' in which it is contained. The ideal J then coincides with the element of M' which corresponds to the zero (neutral element with respect to addition) in M. In a non-commutative ring M, a (left and right hand) ideal is still similarly associated with a homomorphic ring M', and the corresponding fact applies to normal sub-groups within general groups.

Accordingly, when trying to dissociate the concept of an ideal from the specific operations defined in a ring, one might be inclined to identify it with a homomorphism, as described above. This has the advantage of great simplicity, and does not require the extensive use of formal language. On the other hand, it is not easy to express all the familiar concepts of ideal theory in terms of the homomorphism associated with it. As an example of such a concept we may mention that of an ideal basis, which is of considerable importance in ideal theory.

However, one way of looking at a homomorphism as described above is expressed by the following procedure. Let D be a *positive* diagram of a commutative ring M, J an ideal in M, and M^* the quotient ring M/J. We now obtain a set D' by adding $E(a, a')$ to D for any object symbols a, a' of D which correspond to objects of M which belong to the same elements of M^* under the homomorphism. In other words, we add to D an additional set of statements which 'identify' (make equal) any two elements of M whose difference belongs to J. Thus an ideal is associated with a certain set of statements (i.e. statements of the forms $E(a, a')$) which are additional to a given set (in the present case A_{CR} and D). In the following sections we shall elaborate this idea and show how certain branches of ideal theory can be developed in terms of it.

10.2. *Metamathematical ideals.* Let K and J_0 be two sets of statements within a restricted language L. A subset J of J_0 will be called an ideal in J_0 over K if $J_0 \cap S(K \cup J) \subseteq J$, i.e. if all the statements of J_0 which can be deduced from the union of J and K are included in J. We shall refer to K as the axiomatic system of the theory, and to J_0 as its domain. If J_0 consists only of statements which do not include any dummy symbols then it will be said to be a 'pure domain'.

The concept of an ideal as given above is essentially similar to the concept of a 'relative system' defined by Tarski for different purposes in his theory of systems (ref. 13), if we regard K as the set of 'logical' or tautological statements of this theory. On the other hand, it should be noted that the concept of an ideal in Boolean systems as used by Stone and Tarski, is quite different from the present concept.

A basis of an ideal J (in J_0 over K) is any subset B of J such that $J \subseteq S(K \cup B)$, i.e. such that all the statements of J can be deduced from the union of K and B.

Before identifying certain general ideals and ideal bases as defined above with the ideals and ideal bases of Algebra, we shall develop some of the general theory corresponding to them.

We shall assume that the sets K and J_0 are fixed throughout the remainder of this section so that the term ideals will always indicate ideals in J_0 over K. Let C be the set of all these ideals; C includes J_0.

It can be verified without difficulty that the meet of any number of ideals is again an ideal. Following the conventions of algebra we denote the meet of any finite number of ideals $J_1, \ldots, J_m$ by $[J_1, \ldots, J_m]$. There will be no occasion for confusing this use of square brackets with their use within our object language.

The sum of any number of ideals is defined as the meet of all the ideals which include the given ideals. The sum of any finite number of ideals $J_1, \ldots, J_m$ will be denoted by $(J_1, \ldots, J_m)$. More generally, if $A_1, \ldots, A_m$ are sets of statements in J_0, we denote by $(A_1, \ldots, A_m)$ the meet of all the ideals including all the A_k, $k = 1, 2, \ldots, m$, and we say that $(A_1, \ldots, A_m)$ is generated by $A_1, \ldots, A_m$. Similarly, if $X_1, \ldots, X_m$ are statements in J_0, we denote by $(X_1, \ldots, X_m)$ the meet of all the ideals which include $X_1, \ldots, X_m$, and we say that $(X_1, \ldots, X_m)$ is generated by $X_1, \ldots, X_m$. If B is a basis of an ideal J, then J is generated by B.

If J and J' are two ideals such that $J' \supseteq J$ then we shall say that J' divides J or is a divisor of J: if, in addition $J' \neq J$, we shall call J a proper divisor of J.

An ideal J will be called irreducible if $[J_1, J_2] = J$ does not hold unless either $J_1 = J$ or $J_2 = J$, for any two ideals J_1 and J_2. Similarly, an ideal J will be called indecomposable if for any two ideals J_1 and J_2, $(J_1, J_2) = J$ cannot hold unless either $J_1 = J$ or $J_2 = J$. In the branch of ideal theory which we have selected for consideration, we shall be concerned with irreducibility rather than with indecomposability.

We shall say that the ideals (in J_0 over K) satisfy the maximum condition if any non-empty set of ideals, $C' \subseteq C$, includes at least one ideal which is not a true subset of any other ideal of C'.

An equivalent condition is that every ascending infinite chain of ideals,

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots \subseteq J_n \subseteq \ldots$$

is such that from some positive integer n onwards all the ideals are equal, $J_n = J_{n+1} = J_{n+2} = \ldots$.

Yet another equivalent condition is that every ideal have a finite basis. The proof that the three conditions are equivalent is precisely the same as in standard ideal theory.

For classes of ideals which satisfy this condition we have the following principle of induction.

If a property applies to J_0 and if from the fact that it applies to all the divisors of an ideal J we can deduce that it applies to J then the property applies to all ideals (of C, i.e. all ideals in J_0 over K).

We shall say that a minimum condition is satisfied if every non-empty set of ideals, $C' \subseteq C$, includes at least one ideal which does not include any other ideal of C' as a true subset. An equivalent condition is, that every descending chain of ideals,

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots \supseteq J_n \supseteq \ldots$$

be such that from some n onwards all the ideals are equal,

$$J_n = J_{n+1} = J_{n+2} = \ldots\ldots$$

For classes C of ideals which satisfy the maximum condition — and these are the classes with which we shall be largely concerned — we have the fundamental theorem.

10.2.1. Every ideal is the meet of a finite number of irreducible ideals (in C).

The proof is based on the principle of induction and again can be taken from Algebra almost in its entirety. The theorem clearly holds for all irreducible ideals, and in particular for J_0. Assume then that it applies to all the divisors of an ideal J. If J is irreducible then we have finished. If not, it can be written in the form $J = [J', J'']$ where J' and J'' are proper divisors of J. Since we assume that the theorem is true for J' and J'', we have,

$$J' = [J'_1, J'_2, \ldots, J'_n] \quad , \quad n' \geqslant 1$$

and

$$J'' = [J''_1, J''_2, \ldots, J''_n] \quad , \quad n'' \geqslant 1$$

where the J'_n, J''_n are all irreducible. Hence also,

$$J = [J'_1, J'_2, \ldots, J'_n, J''_1, J''_2, \ldots, J''_n]$$

can be represented as the meet of a finite number of irreducible ideals. By the principle of induction, this proves the theorem.

10.3. *Connection between ideals in different domains.* Let K, K', J_0 be three sets of statements in a language L, such that $K \subseteq K'$. Then all the ideals in J_0 over K' will be seen to be ideals in J_0 over K, while the converse is not generally true.

Again, let K, J_0, J_0^* be three sets of statements in a language L such that $J_0 \subseteq J_0^*$. We propose to investigate the relation between the class C of ideals in J_0 (over K) and the class C^* of ideals in J_0^* (over K).

Let J be any ideal in J_0, then we shall denote by J^* the set of statements of J_0^* which can be deduced from the union of K and J.

J^* is an ideal in J_0^*; it will be called the closure of J in J_0^*. Conversely, let J' be any ideal in C^*, then the meet of J' and J_0 is an ideal J in J_0. If J_1' and J_2' are two different ideals in J_0^*, we may find that nevertheless the corresponding ideals J_1 and J_2 in J_0 are equal, $J_1' \cap J_0 = J_2' \cap J_0$. On the other hand, if J^* is the closure of $J \varepsilon C$ in J_0^*, then $J = J^* \cap J_0$.

If C^* satisfies the maximum or minimum condition, so does C. In fact, the existence of an infinite strictly ascending chain of ideals in J_0, $J_1 \subset J_2 \subset J_3 \subset \ldots \subset J_n \subset \ldots$ implies that the chain of the closures of these ideals in C^* is also strictly ascending $J_1^* \subset J_2^* \subset J_3^* \subset \ldots$. Thus, if C^* satisfies the maximum condition, so does C. A similar proof applies to the corresponding theorem for the minimum condition.

It is not generally true that if the maximum (or minimum) condition is satisfied in C, then the corresponding condition is satisfied in C^*. There are however certain cases in which the fact that one or the other of these conditions is satisfied in C implies that the same condition is satisfied in C^*. Some cases, referring to the maximum condition, will now be considered.

Given J_0, we denote by $J_0^\wedge$ the set of statements obtained from the statements of J_0 by the exclusive use of the operation of conjunction (including the statements of J_0). Thus, if X, Y, Z are elements of J_0, then the statements $[X \wedge Y]$, $[X \wedge [Y \wedge Z]]$, $[[X \wedge Y] \wedge Z]$, are all elements of J_0. We again write $[X \wedge Y \wedge Z]$ indiscriminately for the two last mentioned conjunctions. Similarly, we denote by $J_0^\vee$ the set of statements obtained from J_0 by the exclusive use of the operation of disjunction. We denote the classes of ideals in J_0, $J_0^\wedge$, and $J_0^\vee$ over a fixed set of axioms K, by C, $C^\wedge$ and $C^\vee$, respectively.

The connection between $C^\wedge$ and C is quite simple. Let J be any ideal in C and let J be the meet of J_0 and $J^\wedge$, $J = J_0 \cap J^\wedge$. We propose to show that $J^* = J^\wedge$. In fact, J^*, the closure of J contains all the statements obtained by successive conjunction from statements of J, since all such statements can be deduced from J. Now assume that J^* also contains a statement X which cannot be obtained by successive conjunction from J (although, by the definition of $J^\wedge$, it can be obtained by successive conjunction from the statements of J_0). X cannot belong to J_0, since in that case it would also belong to J. Assume in particular that X is a statement of lowest order, of this description. Since X does not belong to J_0, it can be written in the form $X = [X_1 \wedge X_2]$ where X_1 and X_2 are both of lower order than X. Since both X_1 and X_2 can be deduced from $[X_1 \wedge X_2]$ it follows that these statements belong to J^*. Since X is a statement of lowest order amongst the statements which belong to J^*, though they cannot be obtained by the successive conjunction of elements of J, nor belong to J, it follows that X_1 and X_2 can both be obtained by the successive conjunction of such elements, e.g. $X_1 = [X'_1 \wedge \ldots \wedge X'_{n'}]$, $X_2 = [X''_1 \wedge \ldots \wedge X''_{n''}]$ where the X'_k, X''_k belong to J. Thus also $X = [X_1 \wedge \ldots \wedge X'_{n'} \wedge X''_1 \wedge \ldots \wedge X''_{n''}]$, so that X can in fact be obtained by the successive conjunction of elements of J, and the assumption that there are elements of J^* which cannot be so obtained is untenable. We see therefore that J^* is given by the totality of statements which can be obtained by successive conjunction from J (including the statements of J). On the other hand, let now X be any element of $J^\wedge$. It can be written as the successive conjunction of elements of J_0, $X_1, \ldots, X_m$, say, $X = [X_1 \wedge \ldots \wedge X_m]$. Also, all these statements can be deduced from X, and so they belong to $J^\wedge$ and hence to J. Conversely, every conjunction of statements of J belongs to $J^\wedge$. Thus $J^\wedge$ also is given by the totality of statements which can be obtained by successive conjunction from the statements of J, $J^* = J^\wedge$.

It has been shown that any ideal of $C^\wedge$ is the closure of its meet with J_0. This establishes a one-to-one correspondence between the ideals of C and the ideals of $C^\wedge$, $J \leftrightarrow J^*$, such that $J_1 \subseteq J_2$ for any two ideals if and only if $J_1 \subseteq J_2^*$ for the corresponding ideals in C. Hence C satisfies the maximum (or minimum) condition if and only if $C^\wedge$ satisfies the same condition.

10.4. *Maximum conditions for disjunctive domains.* The connection between C and $C^\vee$ is less simple. However, we shall prove the following theorem.

10.4.1. If the maximum condition holds in C then it also holds in $C^\vee$.

Assume that $C^\vee$ does not satisfy the maximum condition. We propose to show that in that case, C cannot satisfy the maximum condition either.

Since $C^{\vee}$ does not satisfy the maximum condition, it contains an ideal J without finite base. From this ideal J we may select a sequence of elements, $\{Y_1, Y_2, Y_3, \ldots\}$, such that the ideals $J_k = (Y_1, \ldots, Y_k)$ constitute an ascending chain of ideals in $J_0^{\vee}$, where for any given J_k there exists an J_m, $m > k$, such that J_k is a proper subset of J_m.

Now all the Y are disjunctions of statements in J_0, thus

$$\begin{aligned} Y_1 &= [X_1^{(1)} \vee X_2^{(1)} \vee \ldots \vee X_{m_1}^{(1)}] \\ Y_2 &= [X_1^{(2)} \vee X_2^{(2)} \vee \ldots \vee X_{m_2}^{(2)}] \\ &\vdots \\ Y_n &= [X_1^{(n)} \vee X_2^{(n)} \vee \ldots \vee X_{m_n}^{(n)}] \end{aligned}$$

where the $X_i^{(k)}$ belong to J_0.

We are going to show how to select from the set of $X_i^{(k)}$ an infinite subset $Y_1^* = X_{i_1}^{k_1}$, $Y_2^* = X_{i_2}^{k_2}$, ..., $Y_n^* = X_{i_n}^{k_n}$, ... such that in the ascending chain of ideals $J_1 \subseteq J_2 \subseteq \ldots \subseteq J_n \subseteq \ldots$ an infinite number are different from each other, where $J_n = (Y_1^*, Y_2^*, \ldots Y_n^*)$.

If $m_1 = 1$, so that Y itself belongs to J_0, then we define simply $Y_1^* = Y_1$. On the other hand, if $m_1 > 1$, we propose to show that we can replace Y_1 by a statement Y_1' which contains less than m_1 components, all selected from $X_1^{(i)}, \ldots, X_{m_1}^{(i)}$ such that the resulting sequence $\{Y_1', Y_2, Y_3, \ldots, Y_n, \ldots\}$ still gives rise to an ascending sequence of ideals an infinite number of which are different from each other. In fact, put $Z_1' = X_1^{(1)}$, $Z_1'' = [X_2^{(1)} \vee \ldots \vee X_m^{(1)}]$ then $Y_1 = [Z_1' \vee Z_1'']$. Consider now the two sequences of ideals, $\{J_n'\}$ and $\{J_n''\}$ where $J_n' = (Z_1', Y_2, \ldots, Y_n)$ and $J_n'' = (Z_1'', Y_2, \ldots, Y_n)$.

We have $J_n = [J_n', J_n'']$. In fact, $[Z_1' \vee Z_1'']$ can be deduced from both Z_1' and from Z_1'', and therefore belongs to $[J_n', J_n'']$, so that $J_n \subseteq [J_n', J_n'']$. Conversely, any statement that can be deduced from K together with $Y_2, Y_3, \ldots Y_n$, and Z_1', and also from K together with $Y_2, \ldots, Y_n$, and Z_1'', can also be deduced from K together with $Y_2, \ldots, Y_n$, and $[Z_1' \vee Z_1''] = Y_1$ by a simple application of the calculus of propositions. Hence $[J_n', J_n''] \subseteq J_n$, and so $J_n = [J_n', J_n'']$, as asserted.

It follows that at least one of the two sequences

$$J_1' \subseteq J_2' \subseteq J_3' \subseteq \ldots$$

and

$$J_1'' \subseteq J_2'' \subseteq J_3'' \subseteq \ldots$$

must contain an infinite number of different ideals. Otherwise, we should have $J'_n = J'_{n'+1} = J'_{n'+2} = \ldots$ for some n', and $J''_{n''} = J''_{n''+1} = J''_{n''+2} = \ldots$ for some n''. Hence, for $n = \max(n', n'')$,

$$[J'_n, J''_n] = [J'_{n+1}, J''_{n+1}] = [J'_{n+2}, J''_{n+2}] = \ldots$$

i.e. $J_n = J_{n+1} = J_{n+2} = \ldots$, contrary to the assumption that an infinite number of J_n are different from each other. Thus either the sequence $\{Z'_1, Y_2, Y_3 \ldots\}$ or the sequence $\{Z''_1, Y_2, Y_3, \ldots\}$, or both, give rise to an infinite sequence of ideals. If the first assumption applies, we put $Z'_1 = Y'_1$, if it does not apply, we put $Z''_1 = Y'_1$. In either case, Y'_1 contains less than k of its original components. In the first case, we put $Y^{(1)}_1 = X^{(1)}_1$, in the second we apply the same procedure to Y'_1 in order to reduce the number of components still further. Thus we finally replace Y_1 by one of its components $Y^{(1)}_1 = X^{(1)}_{i_1}$, say, such that $\{Y^{(1)}_1, Y_2, Y_3, \ldots\}$ still gives rise to a chain of ideals $\mathbf{J}^{(1)}_1, \mathbf{J}^{(2)}_1, \ldots$, an infinite number of which are different from each other, where $\mathbf{J}^{(2)}_1 = (Y^{(1)}_1)$, $\mathbf{J}^{(1)}_2 = (Y^{(1)}_1, Y_2)$, $\mathbf{J}^{(1)}_3 = (Y^{(1)}_1, Y_2, Y_3)$, etc. From the sequence $\{Y^{(1)}_1, Y_2, Y_3, Y_4, \ldots\}$, we now remove all the statements which are included in $\mathbf{J}^{(1)}_1 = (Y^{(1)}_1)$, except $Y^{(1)}_1$. The resulting sequence $\{Y^{(1)}_1, Y^{(1)}_2, Y^{(1)}_3, \ldots\}$ still gives rise to an ascending chain of ideals $J^{(1)}_1 \subset J^{(1)}_2 \subseteq J^{(1)}_3 \subseteq \ldots \subseteq J^{(1)}_n = (Y^{(1)}_1, Y^{(1)}_2, \ldots Y^{(1)}_n) \ldots$, an infinite number of which are different from each other.

Now $Y^{(1)}_2$ is of the form

$$Y^{(1)}_2 = [X^{(k_2)}_1 \vee X^{(k_2)}_2 \ldots X^{(k_2)}_{m_{k_2}}].$$

If $m_{k_2} = 1$, so that $Y^{(1)}_2$ belongs to J_0, then we define simply $Y^{(2)}_2 = Y^{(1)}_2$. On the other hand, if $m_{k_2} > 1$, we put

$$Z'_2 = X^{(k_2)}_1, Z''_2 = [X^{(k_2)}_2 \vee \ldots \vee X^{(k_2)}_{k_{m_2}}].$$

Then $Y^{(1)}_2 = [Z'_2 \vee Z''_2]$. Similarly as before, we now consider the two sequences $\{{}^2J'_n\}$ and $\{{}^2J''_2\}$, where

$${}^2J'_1 = {}^2J''_1 = (Y^{(1)}_1), {}^2J'_2 = (Y^{(1)}_1, Z'_2), {}^2J''_2 = (Y^{(1)}_1, Z''_2)$$

while for general $n > 2$,

$${}^2J'_n = (Y^{(1)}_1, Z'_2, Y^{(1)}_3, Y^{(1)}_4, \ldots), {}^2J''_n = [Y^{(1)}_1, Z''_2, Y^{(1)}_3, Y^{(1)}_4, \ldots].$$

Again we have $J^{(1)}_n = [{}^2J'_n, {}^2J''_n]$, and again we conclude that at least one of the two sequences must contain an infinite number of ideals which are different from each other. Repeating this procedure if necessary, we finally select a $Y_n^{(2)} = X_{i_2}^{(k_1)}$, such that the sequence

$$\{Y_1^{(1)}, Y_2^{(2)}, Y_3^{(1)}, Y_4^{(1)}, \ldots\}$$

gives rise to a chain of ideals

$$\mathbf{J}_1^{(2)} \subseteq \mathbf{J}_2^2 \subseteq \mathbf{J}_3^{(2)} \subseteq \ldots$$

an infinite number of which are different from each other, where

$$\mathbf{J}_1^{(2)} = (Y_1^{(1)}), \mathbf{J}_2^{(2)} = (Y_1^{(1)}, Y_2^{(2)}), \text{ and } \mathbf{J}_n^{(2)} = (Y_1^{(1)}, Y_2^{(2)}, \ldots Y_n^{(1)})$$

for $n \geqslant 2$. Moreover, $\mathbf{J}_1^{(2)}$ is still a true subset of $\mathbf{J}_2^2$. For since $Y_2^{(1)}$ can be deduced from $Y_2^{(2)}$, $Y_2^{(2)} \varepsilon \mathbf{J}_2^{(1)}$ would imply $Y_2^{(1)} \varepsilon \mathbf{J}_2^{(1)}$, contrary to construction.

Once again, we remove all the $Y_k^{(1)}$, $k \geqslant 3$, which are contained in $\mathbf{J}_2^{(2)}$. We obtain an infinite sequence $\{Y_1^{(1)}, Y_2^{(2)}, Y_3^{(2)}, Y_4^{(2)}, \ldots\}$ where $Y_k^{(2)} = Y_{l_k}^{(1)}, l_1 = 2 < l_2 < l_3 < \ldots$, and which still gives rise to a chain of ideals $J_n^{(2)}$ an infinite number of which are different from each other, $J_1^{(2)} = (Y_1^{(1)})$, $J_2^{(2)} = (Y_1^{(1)}, Y_2^{(2)}), J_n^{(2)} = (Y_1^{(1)}, Y_2^{(2)}, Y_3^{(2)}, \ldots, Y_n^{(2)})$ and such that

$$J_1^{(2)} \subset J_2^{(2)} \subset J_3^{(2)} \subset J_4^{(2)} \subseteq \ldots$$

In this way we may continue indefinitely, and so finally obtain a sequence of statements in J_0,

$$\{Y_1^{(1)}, Y_2^{(2)}, \ldots Y_n^{(n)}, \ldots\}$$

such that no $Y_n^{(n)}$, $n = 2, 3, 4, \ldots$ can be deduced from the union of K and of the statements $Y_1^{(1)}, Y_2^{(2)}, \ldots Y_{n-1}^{(n-1)}$. Thus the set of ideals L_n in J_0, $L_n = (Y_1^{(1)}, \ldots Y_n^{(n)})$ is strictly ascending,

$$L_1 \subset L_2 \subset L_3 \subset L_4 \subset \ldots$$

This is contrary to the assumption that the maximum condition is satisfied in C, and proves theorem 10.4.1.

Two domains J_0 and J_0' will be called equivalent over a set of axioms K if there is a one-to-one correspondence between the ideals, over K, of J_0 and J_0' respectively, $J \leftrightarrow J'$, such that $J \subseteq S(K \cup J')$ and $J' \subseteq S(K \cup J)$.

If ideals J_1, J_2 in J_0 correspond to ideals J_1', J_2' in the equivalent domain J'_0, for given K, then $J_1 \subseteq J_2$ implies $J'_1 \subseteq J'_2$, and vice versa. If the maximum (minimum) condition is satisfied in a domain J_0 then it is also satisfied in any equivalent domain. As we have seen, the domains J_0 and $J_0^{\wedge}$ are equivalent, for given J_0 and K.

10.5. *Properties of disjunctive ideals.* If the domain J_0 is such that, together with any two statements X_1 and X_2 which belong to it, the statement $[X_1 \vee X_2]$ also belongs to it, then J_0 will be said to constitute a disjunctive domain (and its ideals will be called 'disjunctive ideals'). This condition is equivalent to $J_0^{\vee} = J_0$. Given any J_0, if $G_0 = J_0^{\vee}$, then $G_0^{\vee} = G_0$, so that G_0 is a disjunctive domain. The theory of ideals in disjunctive domains which is given below culminates in a theorem for the representation by irreducible ideals in domains which satisfy the maximum condition. The theory does not apply to ideals in general, and in particular has no counterpart in the theory of 'ordinary' ideals in algebraic rings.

The following two distributive laws apply in all disjunctive domains J_0, for arbitrary ideals, J, J_1, J_2.

10.5.1. $[(J, J_1),\ (J, J_2)] = (J,\ [J_1, J_2])$

10.5.2. $([J, J_1],\ [J, J_2]) = [J,\ (J_1, J_2)]$

Proof of 10.5.1. An element of (J, J_1) is any statement X of J_0 which can be deduced from the union of K, J, and J_1. Thus X can be deduced from a finite number of statements of K, $X_1^{(1)}, X_2^{(1)}, \ldots, X_p^{(1)}$, say, together with a finite number of statements of J, $Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_q^{(1)}$, and a finite number of statements of J_1, $Z_1^{(1)}, Z_2^{(1)}, \ldots, Z_r^{(1)}$. If X is a statement which belongs to the left hand side of 10.5.1., then it is at the same time deducible from the union of K, J, and J_2, and therefore from a set of statements consisting of

$$X_1^{(2)}, X_2^{(2)}, \ldots, X_{p'}^{(2)}, \varepsilon\, K;\ \ Y_1^{(2)}, Y_2^{(2)}, \ldots, Y_{q'}^{(2)}, \varepsilon\, J;\ \ Z_1^{(2)}, Z_2^{(2)}, \ldots, Z_{r'}^{(2)}, \varepsilon\, J_2.$$

This implies that both

$$[Z_1^{(1)} \wedge Z_2^{(1)} \wedge \ldots \wedge Z_r^{(1)} \supset X] \text{ and } [Z_1^{(2)} \wedge Z_2^{(2)} \wedge \ldots \wedge Z_{r'}^{(2)} \supset X]$$

are deducible from the union of J and K. But, by the rules of the calculus of propositions, $[[U \supset W] \wedge [V \supset W] \supset [U \vee V \supset W]]$ is a valid statement, and so the statement

$$[[Z_1^{(1)} \wedge Z_2^{(1)} \wedge \ldots \wedge Z_r^{(1)}] \vee [Z_1^{(2)} \wedge Z_2^{(2)} \wedge \ldots \wedge Z_{r'}^{(2)}] \supset X]$$

is deducible from the union of J and K. Hence also, by one of the distributive rules of the calculus of propositions,

$$[[[Z_1^{(1)} \vee Z_1^{(2)}] \wedge [Z_1^{(1)} \vee Z_2^{(2)}] \wedge \ldots \wedge [Z_2^{(1)} \vee Z_1^{(2)}] \wedge [Z_2^{(1)} \vee Z_2^{(2)}] \wedge \ldots \wedge [Z_r^{(1)} \vee Z_{r'}^{(2)}]] \supset X]$$

is deducible from the union of J and K. But all the statements $W_{ik} = [Z_i^{(1)} \vee Z_k^{(2)}]$ belong to $[J_1, J_2)$. Writing the last statement in the form

$$[W_{11} \wedge W_{12} \wedge \ldots \wedge W_{rr'}, \supset X]$$

we see that X is deducible from the union of K, J, and $[J_1, J_2]$, so that it belongs to the right hand side of 10.5.1., $[(J, J_1), (J, J_2)] \subseteq (J, [J_1, J_2])$. On the other hand the relation $[(J, J_1), (J, J_2)] \supseteq (J, [J_1, J_2])$ is true for both disjunctive and non-disjunctive domains. In fact, any statement which belongs to the right hand side of 10.5.1. can be deduced from the union of J, of K, and of the meet of J_1 and J_2. It can therefore be deduced both from the union of J, K, and J_1, and from the union of J, K, and J_2, i.e., it belongs to the left hand side of 10.5.1. This completes the proof.

From 10.5.1. we have more generally

$$[(J, J_1), (J, J_2), \ldots, (J, J_n)] = (J, [J_1, J_2, \ldots, J_n])$$

The proof is by induction. Let $n > 2$ and assume that we have proved the formula for $n - 1$. Then

$$[(J, J_1), (J, J_2), \ldots, (J, J_n)] = [[(J, J_1), (J, J_2), \ldots, (J, J_{n-1})], (J, J_n)]$$
$$= [(J, [J_1, \ldots, J_{n-1})], (J, J_n)] = (J, [[J_1, \ldots, J_{n-1}], J_n]) = (J, [J_1, \ldots, J_n])$$

The proof of 10.5.2. will be omitted (see preface).

10.5.3. *Theorem.* If the statements $X_1, \ldots, X_n$ constitutes a basis for an ideal J in a disjunctive domain, and the statements $Y_1, Y_2, \ldots, Y_m$ constitute a basis for an ideal J_2 then the set of nm disjunctions $[X_1 \vee Y_1]$, $[X_1 \vee Y_2]$, $\ldots$, $[X_1 \vee Y_n]$, $[X_2 \vee Y_1]$, $\ldots$, $[X_n \vee Y_m]$ constitutes a basis for $J = [J_1, J_2]$.

In fact since the statements $[X_i \supset X_i \vee Y_k]$ and $[Y_k \supset X_i \vee Y_k]$ are valid, it follows that all the disjunctions mentioned belong to $[J_1, J_2]$. Conversely any statement Z of $[J_1, J_2]$ is such that both $[[X_1 \wedge X_2 \wedge \ldots \wedge X_n] \supset Z]$ and $[[Y_1 \wedge Y_2 \wedge \ldots \wedge Y_m] \supset Z]$ can be deduced from K. It follows that

$$[[X_1 \wedge X_2 \wedge \ldots \wedge X_m] \vee [Y_1 \wedge Y_2 \wedge \ldots \wedge Y_m] \supset Z]$$

also is deducible from K. This shows that

$$[[X_1 \vee Y_1] \wedge [X_1 \vee Y_2] \wedge \ldots \wedge [X_n \vee Y_m] \supset X]$$

is deducible from K, so that the set $[X_1 \vee Y_1], \ldots, [X_n \vee Y_m]$ does in fact constitute a basis for J_1, J_2.

10.5.4. *Theorem.* A necessary and sufficient condition for an ideal J in a disjunctive domain J_0 to be irreducible is that for all X_1, $X_2 \,\varepsilon\, J_0$, $[X_1 \vee X_2] \,\varepsilon\, J$ only if either $X_1 \,\varepsilon\, J$ or $X_2 \,\varepsilon\, J$.

The condition is necessary. If it is not satisfied, then there exist X_1, $X_2 \,\varepsilon\, J_0$ such that neither $X_1 \,\varepsilon\, J$ nor $X_2 \,\varepsilon\, J$ although $[X_1 \vee X_2] \,\varepsilon\, J$. Let $J_1 = (J,(X_1))$, $J_2 = (J,(X_2))$, so that both J_1 and J_2 are proper divisors of J. By 10.5.1. and 10.5.3.

$$[J_1, J_2] = [(J, X_1),\ (J, X_2)] = (J,[X_1, X_2]) = (J,([X_1 \vee X_2])) = J$$

so that J is reducible, contrary to assumption.

The condition is sufficient. For assume that an ideal J is reducible $J = [J_1, J_2]$ where $J \subset J_1$, $J \subset J_2$. Let X_1 be an element of J_1 which does not belong to J and X_2 an element of J_2 which does not belong to J. Since $[X_1 \vee X_2]$ is deducible from both X_1 and X_2, it belongs both to J_1 and J_2, and therefore to J. Hence X_1 and X_2 violate the condition.

We now come to the main theorem of this section. The representation of an ideal as the meet of a number of ideals $J = [J_1, \ldots J_n]$ will be called irredundant, if the omission of any one of the J_k destroys the equality. Any representation of an ideal as the meet of a finite number can be transformed into an irredundant representation by the omission of some of the components of the representation. We propose to show that any two irredundant representations of an ideal J as the meet of irreducible ideals can differ only as to order. Thus,

10.5.5. *Theorem.* If $J = [J_1, \ldots, J_n] = [J'_1, \ldots, J'_m]$ where the representations are irredundant and the ideals J_k, J'_k all are irreducible, then $n = m$, and every J_k equals some J'_i and vice-versa.

For the proof, consider the expressions (J_i, J'_1) and $(J_i, [J'_2, J'_3, \ldots, J'_m])$ $i = 1, 2, \ldots, n$. By 10.5.1.,

$$\begin{aligned}[(J_i, J'_1), (J_i, [J'_2, J'_3, \ldots, J'_m])] &= (J_i, [J'_1, [J'_2, \ldots, J'_m]]) \\ &= (J_i, [J'_1, \ldots, J'_m]) = (J_i, J) = J_i\end{aligned}$$

and so, since J_i is irreducible, either

$$(J_i, J'_1) = J_i, J_i \supseteq J'_1 \text{ or } (J_i, [J'_2, J'_2, \ldots, J'_m]) = J_i, J_i \supseteq [J'_2, J'_3, \ldots, J'_m].$$

If the first alternative does not hold so that the second alternative holds, consider

$$\begin{aligned}&[(J_i, J'_2), (J_i, [J'_3, J'_4, \ldots, J'_m])] = \\ &(J_i, [J'_2, [J'_3, \ldots, J'_m]]) = \\ &(J_i, [J'_2, J'_3, \ldots, J'_m]) = J_i.\end{aligned}$$

Since J_i is irreducible, this again leads to the conclusion that $J_i \supseteq J'_2$ or else $J_i \supseteq [J'_3, \ldots, J'_m]$. Continuing in this way, we finally infer that J_i divides at least one J'_k, $1 \leqslant k \leqslant m$. Conversely, we can show that every J'_k divides at least one J_i. Now, for a specific J_i, assume that we have found an appropriate J'_k such that $J_i \supseteq J'_k$. For J'_k in turn, there exists an $J_{i'}$, such that $J'_k \supseteq J_{i'}$. Hence $J_i \supseteq J_{i'}$. But if $i \neq i'$, then we can omit J_i in the representation $J = [J_1, J_2, \ldots, J_m]$, without affecting the equality. This is contrary to the assumption that the representation is irredundant, and so $J_i = J_{i'} = J'_k$. Thus, for every i there exists a $k = k(i)$ such that $J_i = J'_{k(i)}$, $i = 1, 2, \ldots, n$, $1 \leqslant k(i) \leqslant m$. But this means that all the ideals J_i which appear in the first representation, appear also in the second representation. Since the second representation is irredundant, it follows that it cannot contain any other ideals. This proves the theorem.

10.5.6. *Corollary.* In a disjunctive domain in which the maximum condition is satisfied, every ideal can be represented in one way and — except for order — in one way only as the irredundant meet of a finite number of irreducible ideals.

10.5.7. *Theorem.* Let $J_0^\vee$ be the disjunctive domain corresponding to a domain J_0, and let $J^\vee$ be an irreducible ideal in J_0.

Further let J be the meet of J_0 and $J^{\vee}$. Then $J^{\vee}$ is the disjunctive closure of J.

Proof. Let $X = [Y_1 \vee \ldots \vee Y_n]$ be any element of $J^{\vee}$, $Y_i \varepsilon J_0$, $i = 1, 2, \ldots, n$. We then infer by the repeated application of 10.5.4. that at least one of the statements Y_i belongs to $J^{\vee}$, $Y_k \varepsilon J^{\vee}$ say. Hence also $Y_k \varepsilon J$, and so, since $Y_k \supset X$ is valid, X can be deduced from a statement of J, and therefore belongs to the closure of J, J^*. Thus, $J^{\vee} \subseteq J^*$. On the other hand, a statement which can be deduced from $J \cup K$, can certainly be deduced from $J^{\vee} \cup K$, and so $J^* \subseteq J^{\vee}$.

It follows that every irreducible ideal in J_0 is the closure of an ideal in J_0.

10.5.8. *Theorem.* If J_0 satisfies the maximum condition, then the disjunctive closure $J^{\vee}$ of an irreducible ideal, $J \subseteq J_0$ is irreducible.

If J is not irreducible then it can be represented as the irredundant meet of a finite number of irreducible disjunctive ideals, $J^{\vee} = [J_1^{\vee}, \ldots, J_m^{\vee}]$. Since the $J_i^{\vee}$ are irreducible they are by 10.5.7. the respective disjunctive closures of the ideals $J_i = J_0 \wedge J_i^{\vee}$. Hence, if $J_i = J$ for some i, then we should have $J_i^{\vee} = J^{\vee}$, contrary to the assumption that the representation of J is irredundant. Thus $J_i \neq J$, $i = 1, \ldots, m$. On the other hand

$$J_0 \cap J^{\vee} = J_0 \cap [J_1^{\vee}, \ldots, J_m] = [J_0 \cap J_1^{\vee}, \ldots, J_0 \cap J_m^{\vee}]$$

or $J = [J_1, \ldots, J_m]$, since $J_0 \cap J^{\vee} = J$. This implies that J is reducible, contrary to assumption.

10.6. *Ideals and homomorphisms.* Let H be a consistent algebra of axioms and M one of its models; let D be a positive diagram of M, such that object symbols of H and of D which correspond to the same objects in M coincide. The union of H and D will be denoted by K. Further let J_0 be the domain of all statements of order one, which contain only relative or object symbols included in K. We propose to establish a correlation between some of the ideals in J_0 over K and the homomorphisms of the model M with structures M' which are also models of H.

Let M' be such a structure. We define J as the ideal in J_0 which is generated by the statements $[E(a, b)]$ such that a and b correspond to objects a' and b' in M, which correspond to equal objects in M'. It follows from this definition that if a statement of the form $[E(a, b)]$ belongs to J, then the corresponding objects a and b in M correspond to equal objects in M'.

To see this, we define a structure M^* which contains the same objects and relations as M, such that any relation holds in M^*, whenever the corresponding relation holds between corresponding objects in M'. Then it follows from the homomorphism between M and M' that whenever a relation holds between certain objects as objects of M, it also holds between them as objects of M^*. Similarly, since M' is a model of H, M^* also is a model of H, and since M is a model of D, M^* also is a model of D. Thus M^* is a model of K. Now assume that some statement $[E(a, b)]$ belongs to J. We shall prove that in that case $E^*(a', b')$ holds in M^*, where a', b' are the objects of M which correspond to a and b, and E^* is the relation of equality in M^*. (By the definition of M^*, this in turn implies that a' and b' correspond to equal objects in M'). Assume on the contrary, that $E^*(a', b')$ does not hold in M^*, so that the statement $[\sim E(a, b)]$ holds in M^*. Comparing the definition of J with the definition of M^*, we see that if a statement $[E(c, d)]$ belongs to J then it holds in M^*. And since these statements form a basis of J over K, it follows that all the statements of J hold in M^*. But this means that M^* is a structure in which the statements of K and of J, as well as the statement $[\sim E(a, b)]$ all hold, so that the union of K and of J is consistent with $[\sim E(a, b)]$. This is contrary to the assumption that $[E(a, b)]$ belongs to J. In this sense, therefore, there exists an ideal J in J_0 for every homomorphism of H. On the other hand, it will be seen that M^* is quasi-isomorphic to M'. From this, it is not difficult to deduce that if two structures M' and M'' are homomorphic to M, and are models of H, as above, such that they correspond to the same ideal J in J_0 over K, then M' and M'' are quasi-isomorphic.

When the place of H is taken by A_R, so that M is a general ring, then we can not only find an ideal J in J_0 as defined above, for every homomorphism of J, but conversely, for every ideal J in J_0, we can construct a model M' which is isomorphic to M, and which bears the established relation to J. This follows from a one-to-one correspondence which will now be established between the ideals J

of J_0 and the 'ordinary' (left and right hand) ideals in M. To distinguish them from our metamathematical ideals, we shall refer to them briefly as arithmetical ideals and shall denote them by J_a.

Given J, we define the corresponding J_a as the set of all the objects a' of M such that $E(a, 0) \varepsilon J$, where a is the object symbol corresponding to a' in D and 0 is the object symbol corresponding to a neutral element with respect to addition in M. (There may be more than one neutral element with respect to addition in M, unless M is normal, but all these elements are equal in M).

J_a is an arithmetical ideal. In fact if $E(a, 0) \varepsilon J$ and $E(b, 0) \varepsilon J$, then $E(c, 0) \varepsilon J$ if $S(a, b, c)$ holds in M, (and so belongs to D), since $E(c, 0)$ can be deduced from $E(a, 0)$, $E(b, 0)$, $S(a, b, c)$ and A_R, and so can be deduced from the union of J, D, and A_R. Similarly $E(d, 0) \varepsilon J$, if $S(a, d, b)$ holds in M, and if $E(a, 0) \varepsilon J$ and $E(b, 0) \varepsilon J$, as before. Finally, $E(f, 0) \varepsilon J$ and $E(g, 0) \varepsilon J$ if $E(a, 0) \varepsilon J$ and $P(h, a, f)$ and $P(a, h, g)$ belong to D, for an arbitrary object symbol h, since $E(f, 0)$ and $E(g, 0)$ can be deduced from $E(a, 0)$, $P(h, a, f)$, $P(a, h, g)$, and A_R.

We observe that J_a cannot be empty, for since J contains at least one statement, viz., $E(0, 0)$, J_a includes a neutral element with respect to addition.

On the other hand, let J_a be an arithmetical ideal. We then define J as the set of all statements of J_0 which hold in M'. Clearly J is an ideal, and the relation between J and J' is reciprocal, $J \leftrightarrow J_a$.

Moreover, under this correspondence, if $J_1 \leftrightarrow J_{a.1}$ and $J_2 \leftrightarrow J_{a.2}$ and $J_{a.1} \subseteq J_{a.2}$, then $J_1 \subseteq J_2$ and vice versa.

Let $J \leftrightarrow J_a$. The set $a_1', \ldots, a_n', \ldots$ is an ideal basis of J_a, if and only if the statements $[E(a_1, 0)]$, $[E(a_2, 0)]$, $\ldots$, $[E(a_n, 0)]$, $\ldots$ form a basis for J. In fact, let J_a be an arithmetical ideal generated by a single element a_1', $J = (a_1')$. Then the corresponding metamathematical ideal J contains the statement $E(a_1, 0)$. Now let J' be the metamathematical ideal generated by $E(a_1, 0)$. Then $J' \subseteq J$, and so the corresponding J_a' is a subset of J_a. But J_a' contains a_1, and so coincides with J_a. Hence also $J' = J$. Similarly, if a metamathematical ideal is generated by a statement $[E(a_1, 0)]$ then the corresponding arithmetical ideal is generated by a_1'. Again, if the set $a_1', a_2', \ldots, a_n', \ldots$ form a basis of J_a, then J_a is the meet of all the arithmetical ideals including the ideals (a_1'), (a_2'), ... (a_n') ... and so corresponds to the ideal J in J_0 which is the meet of all the ideals in J_0, which include $([E(a_1, 0)])$, $([E(a_2, 0)])$, ..., $([E(a_n, 0)])$, ..., i.e., which is generated by these statements and vice versa.

We have seen how the present generalised concept of an ideal preserves the idea of the basis. Similarly, a maximum condition in J_0 corresponds to a maximum condition in the related set of arithmetical ideals, irreducible ideals correspond to irreducible arithmetical ideals, and the representations by meets of such ideals correspond. There the generalisation ends. In particular, it is not possible to generalise the concepts of prime ideals and of primary ideals, which are linked with the particular operation of multiplication. (See, however, Chapter 11.)

10.7. *Disjunctive ideals in rings.* We shall now consider ideals in the disjunctive domain $J_0^{\vee}$ obtained from the domain J_0 defined in the preceding section for $H = A_{CR}$. Thus, the elements of J_0 are the statements

10.7.1. $X = A_1 \vee A_2 \vee \ldots \vee A_n \qquad n = 1, 2, \ldots$

where A_k, $k = 1, 2, \ldots, n$ stands for a term of the form $E(a, b)$ or $S(a, b, c)$ or $P(a, b, c)$. It will simplify the discussion, without restricting it in any essential way, if we assume that M is normal. We may then denote by $a + b$, $a - b$, ab, the object symbols corresponding to the sum, difference, and product of the objects a' and b' corresponding to a and b.

We wish to characterise the sets of disjunctions X which constitute ideals J in J_0 by purely arithmetical conditions. To begin with, it is clear that a statement $X = A_1 \vee \ldots \vee A_n$ in which A_k, say, is of the form $E(a, b)$ is contained in an ideal J if and only if the statement X' obtained from X by the replacement of A_k by $E(a - b, 0)$, is also contained in J. Similarly, if A_k is of the form $S(a, b, c)$, then it belongs to J if and only if the statement X' obtained from X by replacing A_k by $E(a + b - c, 0)$ belongs to J, and finally if A_k is of the form $P(a, b, c)$, then it belongs to J if and only if the statement X' obtained from X by replacing A_k by $E(ab - c, 0)$ also belongs to J. It will therefore be seen that we may confine ourselves to statements of the form

10.7.2. $X = E(a, 0) \vee E(b, 0) \vee \ldots \vee E(c, 0)$

Thus, if we denote by G_0 the set of statements of J_0 which are of the form 10.7.2., there is a one-to-one correspondence between the ideals G in G_0 over K and their closures J in $J_0^{\vee}$ over K.

The main theorem of this section is,

10.7.3. In order that a non-empty set G of statements as given by 10.7.2. should constitute an ideal in G_0, it is necessary and sufficient that the following conditions be satisfied.

10.7.4. If we change the order of terms in a statement of G in any way we obtain a statement of G.

10.7.5. If $X \varepsilon G$ and $Y \varepsilon G_0$, then $[X \vee Y] \varepsilon G$.

10.7.6. If $X \varepsilon G$ contains equal components, then by omitting all of them except one, we obtain a statement $X' \varepsilon G$. For instance, if $[E(a, 0) \vee E(a, o) \vee E(b, 0)] \varepsilon G$, then $[E(a, 0) \vee E(b, 0)] \varepsilon G$.

10.7.7. If $X = [E(a_1, 0) \vee E(a_2, 0) \vee \ldots \vee E(a_n, 0)] \varepsilon G, \ n \geqslant 1,$

and

$$Y = [E(b_1, 0) \vee E(b_2, 0) \vee \ldots \vee E(b_m, 0)] \varepsilon G, \ m \geqslant 1,$$

then $Z \varepsilon G$, Z being the disjunction of the nm expressions

$$E(c_{i,k}\, a_i \pm P_{i,k}\, a_i + d_{i,k}\, b_k \pm q_{i,k}\, b_k, 0) \quad \begin{array}{l} i = 1, 2, \ldots n \\ k = 1, 2, \ldots m \end{array}$$

where the $c_{i,k}$, $d_{i,k}$ correspond to arbitrary objects of M while the $p_{i,k}$ and $q_{i,k}$ are non-negative integers indicating continued addition, $0a_i = 0$, $1a_i = a_i$, $2a_i = a_i + a_i$, etc.

In particular by 10.7.4. and 10.7.5., $E(0, 0) \varepsilon G$. Other special cases of 10.7.7. are the following

10.7.8. If $[E(a_1, 0) \vee E(a_2, 0) \vee \ldots \vee E(a_n, 0)] \varepsilon G$, then
$$[E(c_1\, a_1, 0) \vee E(c_2\, a_2, 0) \vee \ldots \vee E(c_n\, a_n, 0)] \varepsilon G$$

where the c_k correspond to arbitrary ring elements or are non-negative integers regarded as operators as before.

10.7.9. If $[E(a_1, 0) \vee E(a_2, 0) \vee \ldots \vee E(a_n, 0)] \varepsilon G$ then all the statements $[E(\pm\, a_1, 0) \vee E(\pm\, a_2, 0) \vee \ldots \vee E(\pm\, a_n, 0)]$ also belong to G for arbitrary distributions of $+$ and $-$.

Conditions 10.7.4. — 10.7.7. are necessary: 10.7.4. is obvious, 10.7.5. and 10.7.6. follow immediately from the fact that $[X \supset X \vee Y]$ and $[X \vee X \supset X]$ are valid for arbitrary statements X and Y. To show that 10.7.7. is necessary, we have to prove that Z can be deduced from A_{CR} and D together with X and Y. In other words, we have to show that if X and Y hold in a commutative ring M' of which D is a positive diagram, then Z also holds in it. But X and Y respectively only hold in M' if $E(a_i, 0)$ holds in M' for some i, $1 \leqslant i \leqslant n$, and $E(b_k, 0)$ holds in M' for some k, $1 \leqslant k \leqslant m$. Hence,

$$E(c_{i,k}\, a_i \pm p_{i,k}\, a_i + d_{i,k}\, b_k \pm q_{i,k}\, b_k, 0)$$

holds in M' for these specific i and k, and this entails that Z also holds in M'.

Conditions 10.7.4. — 10.7.7. are sufficient. Given a non-empty set $G \subseteq G_0$ which satisfies these conditions, we have to show that any statement $X = [E(b_1, 0) \vee E(b_2, 0) \vee \ldots \vee E(b_n, 0)]$ which can be deduced from G in conjunction with A_{CR} and the positive diagram D, is contained in G.

The assumption is that a statement

10.7.10. $$[[E(a_1^{(1)}, 0) \vee E(a_2^{(1)}, 0) \vee \ldots] \wedge [E(a_1^{(2)}, 0) \vee E(a_2^{(2)}, 0) \vee \ldots] \wedge \ldots] \supset [E(b_1, 0) \vee (b_2, 0) \vee \ldots]$$

can be deduced from $K = A_{CR} \cup D$, where the statements

$$[E(a_1^{(i)}, 0) \vee E(a_2^{(i)}, 0) \vee \ldots]$$

belong to G. This is the case if and only if

$$[[E(a_{k_1}^{(1)}, 0) \wedge E(a_{k_2}^{(2)}, 0) \wedge \ldots] \vee [E(a_{l_1}^{(1)}, 0) \wedge E(a_{l_2}^{(2)}, 0) \wedge \ldots] \vee \ldots] \supset [E(b_1, 0) \vee E(b_2, 0) \vee \ldots]$$

can be deduced from K where the implicans contains all the possible combinations. Now if a statement $X_1 \vee X_2 \vee \ldots \vee X_n \supset Y_1 \vee Y_2 \vee \vee \ldots \vee Y_m$ can be deduced from K, then $X_i \supset Y_1 \vee Y_2 \vee \ldots \vee Y_m$ $i = 1, 2, \ldots n$ also can be deduced from K. This shows that in our case

10.7.11. $[E(a_{k_1}^{(1)}, 0) \wedge E(a_{k_2}^{(2)}, 0) \wedge \ldots] \supset [E(b_1, 0) \vee E(b_2, 0) \vee \ldots]$ can be deduced from K for all the possible combinations $k_1, k_2, \ldots$ 10.7.11. states that X is contained in the ideal G^* in G_0 which is generated by the set $E(a^{(1)}_{k_1}, 0)$, $E(a^{(2)}_{k_2}, 0)$, ... It follows that X is certainly contained in the closure J^* of G^* in $J^{\vee}_0$. On the other hand, let J be the ideal generated by the set $\{E(a^{(1)}_{k_1}, 0), E(a^{(2)}_{k_2}, 0), \ldots\}$ in J_0. By the procedure of section 10.6., we may associate J with a homomorphism of M, and in particular, we may associate it with a structure M^* whose objects coincide with the objects of M, and which is a model of K as well as of J. It follows that $X = [E(b_1, 0) \vee E(b_2, 0) \vee \ldots]$ holds in M^*, i.e. $E(b_i, 0)$ holds in M^* for some i. This shows that the object corresponding to b_i belongs to the arithmetical ideal generated by the objects corresponding to $a^{(1)}_{k_1}, a^{(2)}_{k_2}, \ldots$. Hence, b_i is of the form $\Sigma(c_l\, a^{(l)}_{k_l} \pm d_l\, a^{(l)}_{k_l})$ where the c_l are object symbols of D, and the d_l are non-negative integers. There is such an expression for every combination $a^{(1)}_{k_1}, a^{(2)}_{k_2}, \ldots$ which may occur in the implicans of 10.7.11. Thus X is identical with a disjunction whose terms are

10.7.12. $$E(\sum_l (c_l\, a^{(l)}_{k_l} \pm d_l\, a^{(l)}_{k_l}), 0)$$

varying over all the combinations $a^{(1)}_{k_1}, a^{(2)}_{k_2}, \ldots$ which may occur in 10.7.11., possibly together with some terms $E(b_k, 0)$ which cannot be represented by 10.7.12. It is also quite possible that for the different sets $a^{(1)}_{k_1}, a^{(1)}_{k_2}, \ldots$ and $a^{(1)}_{l_1}, a^{(1)}_{l_2}, \ldots$, 10.7.12. corresponds to the same $E(b_i, 0)$. We denote by X' the disjunction of all the statements 10.7.12. X' belongs to G, by the repeated application of 10.7.7. But X is obtained from X' by the possible omission of some terms which coincide with a surviving term, and by the possible addition to the disjunction of some more terms $E(b_k, 0)$. Hence $X \varepsilon G$, by conditions 10.7.5., and 10.7.6. This proves the theorem.

A slight modification of condition 10.7.7. is required to adapt theorem 10.7.3. to general non-commutative rings, $H = A_R$. The condition becomes

10.7.13. If $[E(a_1, 0) \vee E(a_2, 0) \vee \ldots \vee E(a_n, 0)] \varepsilon G$

and

$$[E(b_1, 0) \vee E(b_2, 0) \vee \ldots \vee E(b_n, 0)] \varepsilon G$$

then $Z \varepsilon G$, Z being the disjunction of the nm expressions,

$$E(c_{i,k}\, a_i + a_i\, c'_{i,k} \pm p_{i,k}\, a_i + d_{i,k}\, b_i + b_i\, d'_{i,k} \pm q_{i,k}\, b_i, 0)$$

where the $c_{i,k}$, $c'_{i,k}$, $d_{i,k}$, $d'_{i,k}$, correspond to arbitrary objects of M, while the $p_{i,k}$ and $q_{i,k}$ are non-negative integers, as before.

We may remove all traces of Symbolic Logic from the concept of a disjunctive ideal, by replacing every statement of the form $[E(b_1, 0) \vee E(b_2, 0) \vee \ldots]$ by a finite set $(b_1, b_2, \ldots)$. Thus, a disjunctive ideal becomes a set of finite sets of the elements of a given ring, obeying certain conditions which correspond to 10.7.5. — 10.7.7., above, for a commutative ring, or to 10.7.5., 10.7.6., and 10.7.13. for a general ring. The theory of disjunctive ideals given in section 10.5. can then be developed without the use of Metamathematics.

Although disjunctive ideals have no direct counterpart in 'ordinary' ideal theory, we shall see in the next chapter that their properties have a bearing on the theory of polynomial ideals.

XI

PRE-IDEALS

11.1. *Properties of varieties.* In the present chapter we shall be concerned with a metamathematical version of the theory of polynomial ideals, and in this connection we may regard general predicates as the counterparts of polynomial conditions.

Let K be a set of axioms and M a model of K. Furthermore, let J_Q be a set of predicates of order n, $n = 1, 2, \ldots$ formulated in terms of the object and relative symbols of K and involving the specific unquantified dummy symbols $x_1, x_2, \ldots, x_n$. As before, two predicates will be regarded as different even if they differ only with respect to the places in which the dummy symbols occur (e.g. $S(a, x_1, x_2)$ and $S(a, x_2, x_1)$). For given n, the set of ordered n-ples of equal or different objects in M, $(a_1, \ldots, a_n)$ has been called a point, and the totality of these points was said to constitute the n-dimensional space S_n over M. Also, we said that a predicate $Q(x_1, \ldots, x_n)$ holds at a point $(a_1, \ldots, a_n)$ in M, if $Q(a_1', \ldots, a_2')$ holds in M, where $a_1', \ldots, a_n'$ are the object symbols corresponding to $a_1, \ldots, a_n$, respectively. (See section 8.1.).

Let J be a subset of J_Q. Then the totality V of points $P = (a_1, \ldots, a_n)$ of S_n at which all the predicates $Q(x_1, \ldots, x_n)$ hold was called the variety of J in S_n. Any subset of S_n will be called 'a variety' in S_n if there is a subset of J_Q of which it is the variety. Thus the set of varieties in S_n as defined above, depends on J_Q and on K, although the space S_n depends only on M. We keep J_Q and K fixed through the discussion.

The meet of two varieties is again a variety: for if the variety V_1 is determined by a subset J_1 of J_Q and V_2 is determined by $J_2 \subseteq J_Q$, then the variety $V_1 \cap V_2$ is determined by the union of J_1 and J_2, $J_1 \cup J_2$. The union of two varieties is not necessarily a variety. Certain conditions under which this is the case, however, will now be considered.

We recall that the set J_Q is called disjunctive if whenever the predicates $Q_1(x_1, \ldots, x_n)$ and $Q_2(x_1, \ldots, x_n)$ belong to J_Q,

$Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n)$ also belongs to J_Q (see 8.1.). J_Q is called quasi-disjunctive if for every $Q_1 \varepsilon J_Q$ and $Q_2 \varepsilon J_Q$ there exists a predicate $Q \varepsilon J_Q$ such that $Q \equiv Q_1 \vee Q_2$, i.e. such that the statements

11.1.1. $$(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n) \supset Q(x_1, \ldots, X_n)]$$

and

$$(x_1) \ldots (x_n)[Q(x_1, \ldots, x_n) \supset Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, X_n)],$$

are deducible from K. A disjunctive set J_Q is quasi-disjunctive.

If J_Q is quasi-disjunctive, then the union of two varieties V_1 and V_2 in S_n is again a variety. In fact, assume that V_1 is the variety of $J_1 \subseteq J_Q$ and V_2 the variety of $J_2 \subseteq J_Q$. Disregarding trivial cases, we may assume that neither J_1 nor J_2 are empty. Now given $Q_1 \varepsilon J_1$ and $Q_2 \varepsilon J_2$, we can find at least one $Q \varepsilon J_Q$ such that 11.1.1. is satisfied. Let $J \subseteq J_Q$ be the set of $Q \varepsilon J_Q$ for which such $Q_1 \varepsilon J_1$ and $Q_2 \varepsilon J_2$ exist. Then the variety of J is $V_1 \cup V_2$. For let $P = (a_1, \ldots, a_n)$ be a point of S. If P does not belong to V_1, then there exists a predicate $Q_1 \varepsilon J_1$, which does not hold at P, $Q_1(a_1', \ldots, a_n')$ does not hold in M. Similarly, if P does not belong to V_2, then $Q_2(a_1', \ldots, a_n')$ does not hold in M for some $Q_2 \varepsilon J_2$. Now, by the definition of J, there exists a predicate $Q \varepsilon J$ such that

$$(x_1) \ldots (x_n)[Q(x_1, \ldots, x_n) \supset Q_1(x_1, \ldots, x_n) \vee Q_2(x_2, \ldots, x_n)]$$

It follows that $Q(a_1', \ldots, a_n') \supset Q_1(a_1', \ldots, a_n') \vee Q_2(a_1', \ldots, a_n')$ is deducible from K and therefore holds in M. But since neither $Q_1(a_1', \ldots, a_n')$ nor $Q_2(a_1', \ldots, a_n')$ holds in M, $Q(a_1', \ldots, a_n')$ cannot hold in M either. In other words, P cannot belong to the variety of J unless it belongs either to V_1 or to V_2. On the other hand, if P belongs to V_1, then $Q_1(a_1', \ldots, a_n')$ holds in M for all $Q_1 \varepsilon J_1$. Hence also, by the first statement in 11.1.1., $Q(a_1', \ldots, a_n')$ holds in M for all $Q \varepsilon J$. It follows that P belongs to the variety of J. A similar argument applies to the elements of V_2. This shows that the variety of J is indeed $V_1 \cup V_2$.

A variety V_1 which is a (proper) subset of another variety V_2 will be called a (proper) subvariety of V_2. A variety will be called reducible if it can be represented as the union of two proper subvarieties, otherwise—irreducible.

11.2. *Definition of pre-ideals.* Let K and J_Q be given as in the preceding section. A subset J of J_Q will be called a pre-ideal (short for 'predicate ideal') in J_Q over K, if a predicate $Q(x_1, \ldots, x_n)$ of J_Q belongs to J, provided there are predicates $Q_1, Q_2, \ldots, Q_m$ in J (m any positive integer) such that the statement

11.2.1. $$(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \wedge Q_2(x_1, \ldots, x_n) \wedge \ldots \wedge Q_m(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$$

is deducible from K.

All the definitions concerning general ideals which are given in Chapter 10 can be adapted to pre-ideals. In particular, a basis of a pre-ideal J in J_Q is any subset B of J, such that for every $Q \varepsilon J_Q$ there exist $Q_1, Q_2, \ldots, Q_m \varepsilon B$ so that 11.2.1. is deducible from K. The various versions of the maximum and minimum conditions can be transferred literally, the same applies to the concept of irreducibility, and it can again be shown that if the maximum condition is satisfied in J_Q then every pre-ideal can be represented as the join of a finite number of irreducible pre-ideals. Moreover, the theory of pre-ideals can be completely reduced to the theory of ideals by the following artifice.

Let $b_1, \ldots, b_n$ be a set of object symbols not contained in K. With every predicate $Q(x_1, \ldots, x_n)$ in J_Q, we then associate a statement $Q(b_1, \ldots, b_n)$. The totality of these statements constitutes a domain J_0 whose elements are in one-to-one correspondence $Q(x_1, \ldots, x_n) \leftrightarrow Q(b_1, \ldots, b_n)$. We are going to show that under this correspondence, the ideals in J_0 over K correspond to the pre-ideals in J_Q over K.

For this purpose, we only have to verify that the statement

11.2.2. $$Q_1(b_1, \ldots, b_n) \wedge Q_2(b_1, \ldots, b_n) \wedge \ldots \wedge Q_m(b_1, \ldots, b_n) \supset Q(b_1, \ldots, b_n)$$

is deducible from K if and only if 11.2.1. is deducible from K. And this follows immediately from 2.5.5. and 2.5.7., taking into account that the object symbols $b_1, \ldots, b_n$ are not contained in K.

If J_Q is disjunctive, so is J_0; also, if J_Q satisfies the maximum condition, so does J_0. If these two conditions are satisfied, then there is a single irredundant representation of any given pre-ideals by irreducible pre-ideals. This can either be proved directly or by reference to section 10.5. If J_Q is only quasi-disjunctive and satisfies the maximum condition, this uniqueness theorem still holds. To see this, let $J_Q^{\vee}$ be the set of predicates obtained from J_Q by the operation of disjunction ($J_Q^{\vee}$ is supposed to include the predicates of J_Q). $J_Q^{\vee}$ is disjunctive. Also, the pre-ideals of J_Q and of J_Q are in one-to-one correspondence such that if $J_1 \leftrightarrow J_1^{\vee}$ and $J_2 \leftrightarrow J_2^{\vee}$, $J_1, J_2 \subseteq J_Q$, $J_1^{\vee}, J_2^{\vee} \subseteq J_Q^{\vee}$, then $J_1 \subseteq J_2$ implies $J_1^{\vee} \subseteq J_2^{\vee}$ and vice versa. That is to say J_Q and $J_Q^{\vee}$ (and similarly, the corresponding J_0 and $J_0^{\vee}$) are equivalent according to the terminology of section 10.4. To establish the correspondence we require the following lemma.

11.2.3. For every predicate

$$Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n)$$

of J_Q, there exists a predicate $Q(x_1, \ldots, x_n) \varepsilon J_Q$ such that the statements

11.2.4. $$(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$$

and

$$(x_1) \ldots (x_n)[Q(x_1, \ldots, x_n) \supset Q_1(x_1, \ldots, x_n) \vee \vee Q_2(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n)]$$

are deducible from K.

The proof is by induction. For $m = 2$, the assertion is identical with the assumption that J_Q is quasi-disjunctive. Assume that we have already proved the lemma for $m - 1$, $m = 3, 4, 5, \ldots$. Given $Q_1(x_1, \ldots, x_n) \vee Q_2(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n) \varepsilon J_Q$ we can then find $Q'(x_1, \ldots, x_n) \varepsilon J_Q$, such that

11.2.5. $$(x_1) \ldots (x_n)[Q_2(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n) \supset Q'(x_1, \ldots, x_n)]$$

and

$$(x_1) \ldots (x_n)[Q'(x_1, \ldots, x_n) \supset Q_2(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n)]$$

are deducible from K. Hence also the statements

$$11.2.6. \quad (x_1) \ldots (x_n)[Q_1 \vee [Q_2 \vee \ldots \vee Q_m] \supset Q_1 \vee Q']$$

and

$$(x_1) \ldots (x_n)[Q_1 \vee Q' \supset Q_1 \vee [Q_2 \vee \ldots \vee Q_m]]$$

are deducible from K. Again, since J_Q is quasi-disjunctive there exists a predicate $Q(x_1, \ldots. x_n) \varepsilon J_Q$ such that the statements

$$(x_1) \ldots (x_n)[Q_1 \vee Q' \supset Q]$$

and

$$(x_1) \ldots (x_2)[Q \supset Q_1 \vee Q']$$

are deducible from K. Combining these with 11.2.6., we obtain 11.2.4.

We define the pre-ideal $J^* \subseteq J_Q^{\vee}$ corresponding to any pre-ideal J in J_Q as the closure of J in $J_Q^{\vee}$, i.e., as the set of all predicates $Q \varepsilon J_Q$ such that $(x_1) \ldots (x_n)[Q_1 \wedge \ldots \wedge Q_m \supset Q]$ can be deduced from K for some $Q_1, \ldots, Q_m \varepsilon J$. To prove that this establishes a one-to-one correspondence between all the pre-ideals of J_Q and all the pre-ideals of $J_Q^{\vee}$, let $J^{\vee}$ be any pre-ideal of $J_Q^{\vee}$. We are going to show that $J^{\vee}$ corresponds to just one pre-ideal J of J_Q under the correspondence defined above. In fact, define J as the meet of $J^{\vee}$ and J_Q, $J = J^{\vee} \cap J_Q$. For any predicate $Q_1(x_1, \ldots, x_n) \vee \ldots \vee Q_m(x_1, \ldots, x_n) \varepsilon J^{\vee}, Q_i \varepsilon J_Q$ we can find a predicate $Q(x_1, \ldots, x_m) \varepsilon J_Q$, such that 11.2.4. is satisfied. Then $Q \varepsilon J_Q^{\vee}$, by the definition of a pre-ideal and by the first statement of 11.2.4. Hence $Q \varepsilon J$, and so, by the second statement of 11.2.4., $Q_1 \vee \ldots \vee Q_m$ belong to the closure J^* of J. On the other hand, any predicate which belongs to the closure of J, certainly belongs to $J^{\vee}$, so that $J^{\vee} = J^*$. Also, it is easily seen that two different pre-ideals in J_Q cannot have the same closure in $J_Q^{\vee}$, so that the correspondence is one-to-one. Finally, $J_1 \subseteq J_2$ implies $J_1^* \subseteq J_2^*$, so that the correspondence has all the required properties. It follows that irreducible pre-ideals in J_Q correspond to irreducible pre-ideals in $J_Q^{\vee}$, and that if a maximum condition is satisfied in J_Q (and therefore in $J_Q^{\vee}$) then every pre-ideal in J_Q can be represented in essentially one way as the irredundant meet of a finite number of irreducible pre-ideals.

11.3. *Pre-ideals and their varieties.* Given M and J_Q as in section 11.1., let V be any variety in S_n. Then the set J of all predicates of J_Q which hold at all points of V is a pre-ideal in J_Q over K. For let $Q(x_1, \ldots, x_n)$ be any predicate of J_Q for which there exist predicates $Q_1, \ldots, Q_m \varepsilon J$, such that

11.3.1.

$$(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \wedge \ldots \wedge Q_m(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$$

can be deduced from K. Then if $P = (a_1, \ldots, a_n)$ is any point in S_n, the statement

11.3.2. $$Q_1(a_1', \ldots, a_n') \wedge \ldots \wedge Q_m(a_1', \ldots, a_n') \supset Q(a_1', \ldots, a_n')$$

can be deduced from K, and so, since the statements $Q_1(a_1', \ldots, a_n')$, $\ldots, Q_m(a_1', \ldots, a_n')$ all hold in M, $Q(a_1', \ldots, a_n')$ also holds in M. This shows that Q belongs to J so that J is an ideal. J will be said 'to belong to V'. We observe that if J belongs to a variety V, then V is the variety of J in S_n. For since V is a variety, it must be the variety of some subset of J_Q, J_1, say. Since the predicates of J_1 hold at all the points of V, $J_1 \subseteq J$. Thus, the variety of J_1, V_1, say, is a subset of V, $V_1 \subseteq V$. But the statements of J all hold at any given point of V and so $V_1 \supseteq V$, $V_1 = V$, as required.

On the other hand, let V be the variety of an ideal $J \subseteq J_Q$, and let $Q(x_1, \ldots, x_n)$ be a predicate of J_Q which holds at all the points of V. If this is true in the space S_n over one particular structure M only, then it does not by any means follow that Q belongs to J. However, if it applies in the spaces S_n over all the models M of K and if J_Q satisfies the maximum condition, then it can be shown that Q belongs to J. In fact, consider the set of statements K, together with the statement

11.3.3. $$X = (\exists x_1) \ldots (\exists x_n)[Q_1(x_1, \ldots, x_n) \wedge \ldots \wedge Q_m(x_1, \ldots, x_n) \wedge \sim Q(x_1, \ldots, x_n)]$$

where the predicates $Q_1, \ldots, Q_m$ form a finite basis of J (such a basis exists since J_Q satisfies the maximum condition). By assumption Q holds at a point of S_n over a model M if all the predicates of J hold at that point, and this is the case if and only if all the elements of the basis $Q_1, \ldots, Q_m$, hold at that point. It follows that

$Q_1(x_1, \ldots, x) \wedge \ldots \wedge Q_m(x_1, \ldots, x) \wedge \sim Q(x_1, \ldots, x_n)$ does not hold at any point of any model of K, so that the addition of X to K produces a contradictory set. Thus, X can be deduced from K and so $(x_1) \ldots (x_n)[Q_1 \wedge \ldots \wedge Q_m \supset Q]$ can be deduced from K: Q belongs to J.

11.3.4. *Theorem.* Let V_1, V_2, V', and V'' be the varieties of the pre-ideals $J_1, J_2, J' = (J_1, J_2), J'' = [J_1, J_2]$. Then $V' = V_1 \cap V_2$. And if the domain is disjunctive or quasi-disjunctive, $V'' = V_1 \cup V_2$.

This theorem follows readily from section 11.1.

11.3.5. *Theorem.* If J_Q is disjunctive then a necessary and sufficient condition for a variety V to be irreducible is that the pre-ideal J which belongs to V is irreducible.

Assume that J is reducible, $J = [J_1, J_2]$, where J_1 and J_2 are proper divisors of J. Then $V = V_1 \cup V_2$ (see 11.3.4.). To prove that V is reducible, we only have to show that both V_1 and V_2 are proper subvarieties of V. In fact, if $V_1 = V$, then all the predicates of J hold at all the points of V so that $J_1 \subseteq J$, which is impossible since J_1 is a proper divisor of J. Hence, $V_1 \neq V$ and, similarly $V_2 \neq V$.

On the other hand, if $V = V_1 \cup V_2$ where $V_1 \neq V$, $V_2 \neq V$, let J_1 and J_2 be the pre-ideals which belong to V_1 and V_2 respectively. Then J is a proper subset of J_1, J_1 contains a predicate Q_1 which is not contained in J. For since V and V_1 are the varieties of J and J_1 respectively, $J = J_1$ would imply $V = V_1$. Similarly, J_2 contains a predicate Q_2 which is not contained in J. Consider the predicate $Q_1 \vee Q_2$. $Q_1 \vee Q_2$ holds at all the points of V since it holds at all the points of both V_1 and V_2. Hence $Q_1 \vee Q_2 \varepsilon J$ although neither $Q_1 \varepsilon J$ nor $Q_2 \varepsilon J$. This shows that J is reducible (see 10.5.4.).

Theorem 11.3.5. still applies if J_Q is only quasi-disjunctive. This can be proved directly, but follows also from the fact that the correspondence between the ideals of J_Q and J given in section 11.2. is such that corresponding pre-ideals in J_Q have the same variety.

The first part of the proof of 11.3.5. can also be used to prove

11.3.6. *Theorem.* If J is a reducible pre-ideal in the disjunctive or quasi-disjunctive domain J_Q, and V its variety, then V is reducible.

Next we have the following theorem which for simplicity will be proved for disjunctive domains, although by the above remark, it applies also to quasi-disjunctive domains.

11.3.7. *Theorem.* If J is an irreducible pre-ideal in a disjunctive or quasi-disjunctive domain J_Q, $J \neq J_Q$, then there exists a model M of K such that J belongs to V where V is the variety of J in S_n over M.

To prove the theorem for the case that J is disjunctive, let $b_1, \ldots, b_n$ be a set of object symbols not contained in K. Define H as the union of K and of all the statements $Q(b_1, \ldots, b_n)$ where $Q(x_1, \ldots, x_n) \varepsilon J$, and of all the statements $\sim Q'(b_1, \ldots, b_n)$ where $Q'(x_1, \ldots, x_n) \varepsilon J_Q - J$. If H is inconsistent then there exist finite numbers of elements of J and of $J_Q - J$, $Q_1, \ldots, Q_m \varepsilon J$ and $Q'_1, \ldots, Q'_m \varepsilon J_Q - J$ such that the statement

$$Q_1(b_1, \ldots, b_n) \wedge \ldots \wedge Q_m(b_1, \ldots, b_n) \wedge \sim Q'_1(b_1, \ldots, b_n) \wedge \ldots \wedge \sim Q'_{m'}(b_1, \ldots, b_n)$$

is inconsistent with K. It follows that the statement

$$\sim [Q_1(b_1, \ldots, b_n) \wedge \ldots \wedge Q_m(b_1, \ldots, b_n) \wedge \sim Q'_1(b_1, \ldots, b_n) \wedge \ldots \wedge \sim Q'_{m'}(b_1, \ldots, b_n)]$$

is deducible from K, and the same therefore applies to

$$\sim [Q_1(b_1, \ldots, b_n) \wedge \ldots \wedge Q_m(b_1, \ldots, b_n)] \vee [Q'_1(b_1, \ldots, b_n) \vee \ldots \vee Q'_{m'}(b_1, \ldots, b_n)]$$

and to

$$[Q_1(b_1, \ldots, b_n) \wedge \ldots \wedge Q_m(b_1, \ldots, b_n)] \supset [Q_1(b_1, \ldots, b_n) \vee \ldots \vee Q'_{m'}(b_1, \ldots, b_n)]$$

and — taking into account 2.5.7. — to

11.3.8.

$$(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \wedge \ldots \wedge Q_m(x_1, \ldots, x_n) \supset Q'_1(x_1, \ldots, x_n) \vee \ldots \vee Q'_{m'}(x_1, \ldots, x_n)].$$

11.3.8. shows that the predicate

$$Q'_1(x_1, \ldots, x_n) \vee \ldots \vee Q'_{m'}(x_1, \ldots, x_n)$$

belongs to J. But since J is irreducible, it then follows from 10.5.4., that one of the $Q'_i(x_1, \ldots, x_n)$ $1 \leqslant i \leqslant m'$, belongs to J, contrary to assumption. We conclude that H is consistent.

Let M be a model of H, and let $b'_1, \ldots, b'_n$ be the objects of M which correspond to the object symbols $b_1, \ldots, b_n$ respectively. The variety V of J in S_n over M includes the point $P = (b'_1, \ldots, b'_n)$. But J belongs to V, because if $Q'(x_1, \ldots, x_n)$ is any predicate of J_Q which does not belong to J, then Q' does not hold at P, by construction. This proves the theorem.

Given the set of statements K and the domain of predicates J_Q, let V be a variety in a space S_n over a model M of K. A point $P \varepsilon V$ will be called a general point of V if any predicate of J which holds at P, holds at all other points of V. We then have the theorem,

11.3.9. Let J be a pre-ideal in the disjunctive or quasi-disjunctive domain J_Q. Then J is irreducible if and only if there is a space S_n over a model M of K in which J belongs to a variety V which possesses a general point P.

The second part of the theorem is established by the preceding construction in which $P = (b'_1, \ldots, b'_n)$ will be seen to be a general point of V. To prove the first part, let $P = (b'_1, \ldots, b'_n)$ be a general point in a variety V in S_n over a model M of K, J belonging to V. Assume that J is reducible, $J = [J_1, J_2]$, $J \neq J_1$, $J \neq J_2$, and let Q_1, Q_2, be elements of $J_1 - J$, and $J_2 - J$ respectively. Then Q_1 cannot hold at P, otherwise it would hold at all points of V and so would belong to J. Similarly, Q_2 cannot hold at P. But since J_Q is quasi-disjunctive, there exists a predicate $Q \varepsilon [J_1, J_2]$ which is associated with $Q_1 \vee Q_2$, $Q \equiv Q_1 \vee Q_2$. As an element of $J = [J_1, J_2]$, Q must hold at P, but this is impossible since Q is associated with the disjunction of two predicates, neither of which holds at P. Thus the assumption that J is reducible leads to a contradiction, and it follows that J is irreducible.

11.3.10. *Theorem.* If the disjunctive or quasi-disjunctive domain J_Q satisfies the maximum condition, then every variety V in S_n can be represented in one and only one way as the irredundant union of a finite number of irreducible varieties, $V = V_1 \cup V_2 \cup \ldots \cup V_q$

The union will be called irredundant, if the omission of one of the components V_k destroys the equality.

Let J be the ideal which belongs to a given V. By 10.2., J can be represented as the meet of a finite number of irreducible pre-ideals

$J = [J_1, \ldots, J_m]$. Let $V_1, \ldots, V_m$ be the varieties of $J_1, \ldots, J_m$ respectively. By 11.3.6. above these varieties are irreducible and by 11.1., $V = V_1 \cup V_2 \cup \ldots \cup V_m$. Now it may be possible to omit a component from the right hand side of this equation, without destroying the equality: if so, we omit the component. In this way we obtain, after a finite number of steps, a representation of V as the irredundant union of a finite number of irreducible varieties, $V = V_1 \cup V_2 \cup \ldots \cup V_q$.

Let $V = V_1 \cup V_2 \cup \ldots \cup V_q = V_1' \cup V_2' \cup \ldots \cup V_{q'}'$, be two representations of V of this description. Consider the sets $V_i \cap V_1'$ and $V_i \cap (V_2' \cup \ldots \cup V_{q'}')$ for arbitrary i, $1 \leqslant i \leqslant n$. The union of these two sets is V_i. Since V_i is irreducible, this shows that $V_i \cap V_1' = V_i$, $V_i \subseteq V_1'$, or else $V_i \cap (V_2' \cup \ldots \cup V_{q'}') = V_i$, $V_i \subseteq V_2' \cup \ldots \cup V_{q'}'$. In the second case, we consider $V_i \cap V_2'$ and $V_i \cap (V_3' \cup \ldots \cup V_{q'}')$, and again find that $V_i \subseteq V_2'$ or else $V_i \subseteq V_3' \cup \ldots \cup V_{q'}'$. In this way we can prove that $V_i \subseteq V_k'$ for some k, and similarly, that $V_k' \subseteq V_{i'}$, for some i'. We may now complete the argument as in the proof of 10.5.5.

It will be seen that the developments of this section are in many respects analogous to the theory of polynomial ideals, although the analogy is by no means complete. For instance, our irreducible pre-ideal corresponds sometimes to an irreducible ideal and sometimes to a prime ideal in the theory of polynomial ideals. Thus, in that theory, an ideal which belongs to an irreducible variety is prime and not only irreducible. In general, it would be true to say that the picture presented above is simpler than the corresponding picture for polynomial ideals. However, it will appear presently that the concept of a pre-ideal is not exactly a generalisation of the concept of a polynomial ideal from which the particular algebraic case can be regained by specialisation. In fact, the logical structure of that concept is somewhat more complicated: it will be considered in the following sections.

There is a close connection between the decomposition of a bounded predicate (see section 8.3) and the representation of a pre-ideal as the meet of irreducible pre-ideals.

Let J_Q be a set of predicates of order n, $Q(x_1, \ldots, x_n)$ which are bounded in an algebra of axioms K. Assume that J_Q is conjunctive and disjunctive and that it satisfies 8.3.3.

11.3.11. *Theorem.* Every pre-ideal in J_Q over K is generated by one of its elements.

Such a pre-ideal may be called a principal pre-ideal.

To prove 11.3.11, we define the degree m of a pre-ideal J in J_Q, $\deg J = m$, as the lowest degree of any element of J. The only pre-ideal in J_Q whose degree may be 0 is J_Q itself.

Now let J be any pre-ideal in J_Q, and let Q be an element of J such that $\deg Q = \deg J = m$. We propose to show that Q divides all elements of J, so that J is generated by Q, proving 11.3.11.. In fact, let Q' be any element of J. Then $\deg Q \wedge Q' \leqslant \min(\deg Q, \deg Q') \leqslant m$. But $Q \wedge Q' \varepsilon J$, and so $\deg Q \wedge Q' \geqslant m$, i.e., $\deg Q \wedge Q' = m$. Also $Q \wedge Q' \mid Q$ and so, by 8.3.3., $Q \equiv Q \wedge Q'$ and hence $Q \mid Q'$. This proves 11.3.11. We also see that all the elements of J whose degrees equal the degree of J are associated with one another.

11.3.11. still holds if J_Q, while still disjunctive, is only quasi-conjunctive.

If $Q \equiv Q_1 \vee Q_2 \vee \ldots \vee Q_m$, where $Q, Q_1, \ldots, Q_m$ are elements of J, then it is not difficult to show that $(Q) = [(Q_1), (Q_2), \ldots, (Q_m)]$ where $(Q), (Q_1), \ldots, (Q_m)$, are the pre-ideals generated by $Q, Q_1, \ldots, Q_m$, respectively. Conversely $(Q) = [(Q_1), (Q_2), \ldots, (Q_m)]$ entails $Q \equiv Q_1 \vee Q_2 \vee \ldots \vee Q_m$.

11.3.12. *Theorem.* The pre-ideals of J_Q (over K) satisfy the finite chain condition.

Let $J_1 = (Q_1)$, $J_2 = (Q_2)$ be two pre-ideals in J_Q, so that $\deg Q_1 = \deg J_1$, $\deg Q_2 = \deg J_2$. Assume now that $J_1 \supset J_2$. Then $Q_1 | Q_2$, and so $\deg Q_1 \leqslant \deg Q_2$. But if $\deg Q_1 = \deg Q_2$ then $Q_1 \equiv Q_2$, by 8.3.3., and so $J_1 = J_2$, contrary to assumption. Hence $J_1 \supset J_2$ entails $\deg J_1 = \deg Q_1 < \deg Q_2 = \deg J_2$. This shows that if there existed a strictly ascending infinite chain of pre-ideals in J_Q,

$$J_1 \subset J_2 \subset J_3 \subset \ldots$$

then we should have

$$\deg J_1 > \deg J_2 > \deg J_3 > \ldots$$

which is impossible.

It follows from 11.13.12. that we can represent any pre-ideal $J = (Q)$ in J_Q as the irredundant meet of pre-ideals

$J_1 = (Q_1), \ldots J_m = (Q_m)$,

11.3.13. $(Q) = [(Q_1), (Q_2), \ldots, (Q_m)]$

Hence

11.3.14. $Q \equiv Q_1 \vee Q_2 \vee \ldots \vee Q_m$

where the disjunction on the right hand side of 11.3.14. is irredundant. And from the fact that the representation given by 11.3.13. is unique except for order, we may then obtain an alternative proof of theorem 8.3.4.

11.4. *Pseudo-ideals and pre-ideals.* Let K be a set of axioms and J_Q a set of predicates $Q(x_1, \ldots, x_n)$, for given $n \geqslant 1$, formulated within the same language as K. Let C' be a class of subsets of J_Q such that the meet of any number of sets of C' again belongs to it. We shall refer to the elements of C' as pseudo-ideals. The meet J^* of all pre-ideals in J_Q over K which include a given pseudo-ideal J will be called the pre-ideal closure of J, and the disjunctive closure of J^* in $J_Q^{\vee}$ will be called the disjunctive pre-ideal closure of J.

Furthermore, we shall assume that the following three conditions are satisfied.

11.4.1. If S_n is the space of n dimensions over a model M of K and V_1 and V_2 are the varieties of two pseudo-ideals J_1 and J_2, then $V_1 \cup V_2$ is the variety of $[J_1, J_2]$.

11.4.2. If a pseudo-ideal in J_Q is irreducible (i.e. if it cannot be represented as the meet of two pseudo-ideals of which it is a proper subset), then its disjunctive pre-ideal closure in $J_Q^{\vee}$ is irreducible.

11.4.3. The class of pseudo-ideals C' satisfies the maximum condition.

Although this condition was formulated originally for ideals (and pre-ideals) only (see section 10.2.) it clearly retains its meaning for the class C', or generally, for arbitrary classes of subsets of a given set. This condition is equivalent, quite generally, to the finite (ascending) chain condition and to the principle of induction. It is also equivalent to the finite basis condition if, as in the present case, the meet of any number of sets of the given class belongs to the class. And we have, as a consequence of 11.4.3.

11.4.4. Every pseudo-ideal J (in C') can be represented as the meet of a finite number of irreducible pseudo-ideals.

$$J = [J_1, \ldots, J_m]$$

To every J_k, $k = 1, 2, \ldots, m$, there belongs a disjunctive ideal closure $J_k^{\vee}$ in $J_Q^{\vee}$. A specific $J_k^{\vee}$ will be called isolated in the representation if no other $J_i^{\vee}$ is a true subset of $J_k^{\vee}$, $1 \leqslant i \leqslant m$, $i \neq k$. Clearly, to every representation of J by irreducible ideals there belongs at least one isolated disjunctive pre-ideal. We propose to show

11.4.5. *Theorem.* The set of isolated disjunctive pre-ideals in the representation of a pseudo-ideal J as the meet of a finite number of irreducible pseudo-ideals depends only on J and not on the particular choice of the representation.

Proof. Assume that we are given two representations of a pseudo-ideal J by irreducible pseudo-ideals

$$J = [J_1, \ldots, J_m] = [J'_1, \ldots, J'_{m'}]$$

Let $J_i^{\vee}$ be the disjunctive pre-ideal closure belonging to a pseudo-ideal in the first representation, J_i, say.

By 11.4.2., $J_i^{\vee}$ is irreducible. Hence, by 11.3.5. and 11.3.7., there exists a model M of K such that the variety V_i of $J_i^{\vee}$ in S_n over M is irreducible. Let V, V_k, V'_k be the varieties of J, J_k, J'_k in this particular space. (Observe that the varieties of a pseudo-ideal and of its disjunctive pre-ideal closure coincide). Then by 11.4.1.,

$$V = V_1 \cup V_2 \cup \ldots \cup V_m = V'_1 \cup V'_2 \cup \ldots \cup V'_{m'}$$

With the exception of V_i these varieties need not be irreducible, but a familiar argument now shows that V_i is a subset of at least one V'_k on the right hand side, $V_i \subseteq V'_k$. But since $J_i^{\vee}$ belongs to V_i, and since all the predicates of J'_k hold at the points to V'_k, and thence at the points of V_i, it follows that $J_i^{\vee} \supseteq J_k'^{\vee}$.

In a similar way, we may show that every disjunctive ideal closure of the second representation divides the disjunctive ideal closure of a pseudo-ideal of the first representation, $J_k'^{\vee} \supseteq J_{i'}^{\vee}$. Thus $J_i^{\vee} \supseteq J_{i'}^{\vee}$, for given i, for some i'. Hence, if $J_i^{\vee}$ is isolated, $J_i^{\vee} = J_{i'}^{\vee} = J_k'^{\vee}$. Thus, any disjunctive ideal which belongs to the pseudo-ideals of the first representation appears also in the second representation. Also, $J_k'^{\vee}$ must be isolated in the second representation. For, as is easily seen, $J_k'^{\vee}$ must include at least one isolated $J_{k'}'^{\vee}$ as a subset, unless it is itself isolated. But $J_{k'}'^{\vee}$ includes a disjunctive pre-ideal $J_{i''}^{\vee}$ and so $J_i^{\vee} \supseteq J_{i''}^{\vee}$. This shows that $J_i^{\vee} = J_{i''}^{\vee} = J_{k'}'^{\vee} = J_k^{\vee}$, since J_i is isolated. Thus $J_i^{\vee} = J_k'^{\vee}$ is also isolated in the second representation.

We have therefore shown that the isolated disjunctive pre-ideals of the first representation are included amongst the isolated disjunctive pre-ideals of the second representation. Similarly, the isolated disjunctive pre-ideals of the second representation are included amongst those of the first. Thus, the two sets coincide. However, it is quite possible that the same disjunctive pre-ideal belongs to more than one pseudo-ideal in a given representation.

Since every disjunctive pre-ideal which is not isolated ('embedded') in any given representation, includes at least one isolated disjunctive pre-ideal as a subset, it follows that if $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m$ are the varieties of the isolated disjunctive pre-ideals which belong to any representation of a pseudo-ideal J by irreducible pseudo-ideals, then

11.4.6. $V = \mathbf{V}_1 \cup \mathbf{V}_2 \cup \ldots \cup \mathbf{V}_m$

where V is the variety of J. By 11.3.6. this is a representation by irreducible varieties. If J_Q is quasi-disjunctive, J_Q is equivalent to $J_Q^{\vee}$; it follows that in that case the pre-ideal closures which occur in any representation of J as the meet of irreducible pseudo-ideals, and which are isolated, in the same sense as before, depend only on J and not on the particular representation under consideration. This is the (incomplete) counterpart of an important result of standard ideal theory, as detailed in section 11.6. below.

It will be appreciated that the above proof of 11.4.5., is based on certain extraneous postulates, in particular on 11.4.1. and 11.4.2. Of these, 11.4.2. is the more critical; the proof that this condition is satisfied in the case considered in section 11.6. below appears to be closely linked with the particular properties of the ring operations.

Condition 11.4.1. is equivalent to

11.4.7. Let (Q_1) and (Q_2) be the pseudo-ideals generated by the predicates $Q_1(x_1, \ldots, x_n)$ and $Q_2(x_1, \ldots, x_n)$ of J_Q, respectively (i.e. the meets of all pseudo-ideals which include Q_1 and Q_2, respectively), and let $J = [(Q_1), (Q_2)]$. Assume further that neither Q_1 nor Q_2 hold at some specific point P in S_n. Then J contains an element $Q(x_1, \ldots, x_n)$ which does not hold at P.

11.4.7. follows from 11.4.1. For if 11.4.7. is not satisfied, then there exist predicates $Q_1, Q_2 \varepsilon J_Q$ and a point P such that neither Q_1 nor Q_2 hold at P although all the predicates of $[(Q_1), (Q_2)]$ hold at P. Now let V_1 be the variety of (Q_1) and V_2 the variety of (Q_2). Then $V_1 \cup V_2$ does not include P although P is included in the variety of $[(Q_1), (Q_2)]$. This shows that unless 11.4.7. holds, 11.4.1. does not hold either.

Conversely, 11.4.1. follows from 11.4.7. In fact, let V_1, V_2 and V be the varieties corresponding to the pseudo-ideals J_1, J_2, and $J = [J_1, J_2]$ respectively. Since $J \subseteq J_1$ and $J \subseteq J_2$, the predicates of J hold at all the points of both V_1 and V_2, and so $V \supseteq V_1 \cup V_2$. On the other hand, if $V \neq V_1 \cup V_2$, there exists a point P which belongs to V but not to $V_1 \cup V_2$. Hence there exist predicates $Q_1 \varepsilon J_1$, $Q_2 \varepsilon J_2$ which do not hold at P, and so by 11.4.7. there exists a predicate $Q \varepsilon J$ which does not hold at P either. This is contrary to the assumption that V is the variety of J, and we conclude that in fact $V = V_1 \cup V_2$.

We did not require the assumption that if the disjunctive pre-ideal closure of a pseudo-ideal is irreducible, then the pseudo-ideal is itself irreducible. This is in fact not necessarily true, e.g. in the case considered in section 11.6. below.

11.5. *Pseudo-ideals and pre-ideals,* continued. Let H and K be two algebras of axioms, $H \subseteq K$. Given $n \geqslant 1$, let M_n and M_n' be the pre-polynomial and polynomial structures of order n over H, and let S_n be the n-dimensional space over a model M of K. Further, let D_n be a positive diagram of M_n within the language L of K (or if necessary, within a more extensive language L'). According to section 7.3., there is a natural correspondence C between the relative and object symbols contained in K, and the relations, and some of the objects of M_n, and we shall assume that the correspondence between the object and relative symbols of D_n, and the objects and relations of M_n, includes C. That is to say, object and relative symbols of K and of D_n which correspond to the same object or relation in M_n are identical. Since M_n' is a partial structure of M_n, D_n includes a subset D_n' which is a positive diagram of M_n'. Let J_0^* be the set of all statements of order one in L, which include only relative and object symbols of D_n. Every statement of J_0^* is of the form $[A(a_1, \ldots, a_m)]$ where A is a relative symbol contained in K, and $a_1, \ldots, a_m$ are object symbols corresponding to pre-polynomials $R_1(x_1, \ldots, x_n, y), \ldots, R_m(x_1, \ldots, x_n, y)$ in M_n. With $[A(a_1, \ldots, a_m)]$ we now associate the predicate

11.5.1. $$Q(x_1, \ldots, x_n) = (\exists z_1) \ldots (\exists z_m)[R_1(x_1, \ldots, x_n, z_1) \wedge \ldots \wedge R_m(x_1, \ldots, x_n, z_m) \wedge A(z_1, \ldots, z_m)].$$

The aggregate of these predicates will be denoted by J_Q^*. There is a one-to-one correspondence between the elements of J_Q^* and J_0^*. Corresponding to the class of ideals in J_0^* over $H' = H \cup D_n$, we therefore obtain a class of subsets of J_Q^*. The latter may be regarded as a class of pseudo-ideals in the sense of section 11.4. Clearly the concept of these pseudo-ideals differs fundamentally from the pre-ideals in J_Q^* over K, even when $H = K$.

Instead of considering ideals in the set J_0^* of statements of order One formulated in terms of the object and relative symbols of D_n, we may consider ideals in the subset J_0 of J_0^* whose statements involve only object and relative symbols of D_n'. The corresponding set of predicates defined by 11.5.1. is a subset J_Q of J_Q^*. Subsets of J_Q which correspond to ideals in J_0 over $K' = K \cup D_n'$ will again be regarded as pseudo-ideals in J_Q, while the pre-ideals in J_Q over K will be referred to simply as pre-ideals. It will be shown presently that the concept of an ideal in J_0 coupled with a pseudo-ideal in J_Q, as above, is the genuine counterpart of the concept of a polynomial ideal.

11.5.2. *Theorem.* Every pre-ideal in J_Q^* over K is a pseudo-ideal in J_Q^*, as defined above.

For the proof, we may assume that $K = H$, since any pre-ideal in J_Q^* over $K \supseteq H$ also is a pre-ideal in J_Q^* over H.

Let J be such a pre-ideal. J consists of predicates of the form given by 11.5.1., corresponding to certain statements in J_0^*. Denoting the set of these statements by J', we have to show that J' is an ideal in J_0^* over $H' = H \cup D_n$.

Let $X = [A(a_1, \ldots, a_m)]$ be any statement of J_0^* which can be deduced from the union of J' and H'. That is to say, there exist statements $X_1 = [A_1(a_1^{(1)}, \ldots, a_{m_1}^{(1)})], \ldots, X_k = [A_k(a_1^{(k)}, \ldots, a_{m_k}^{(k)})]$ in J' such that $X_1 \wedge \ldots \wedge X_k \supset X$, i.e.

11.5.3. $$[A_1(a_1^{(1)}, \ldots, a_{m_1}^{(1)})] \wedge \ldots \wedge [A_k(a_1^{(k)}, \ldots, a_{m_k}^{(k)})] \supset [A(a_1, \ldots, a_m)]$$

is deducible from H'. We have to show that the predicate $Q(x_1, \ldots, x_n)$ which corresponds to X by 11.5.1. belongs to J, i.e. that it cannot belong to $J_Q^* - J$.

Assume on the contrary that $Q \varepsilon J_Q^* - J$. In that case the statement

$$(x_1) \ldots (x_n)[Q_1(x_1, \ldots, x_n) \wedge \ldots \wedge Q_k(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$$

cannot be deduced from $K = H$, where Q_i correspond to X_i by 11.5.1., $i = 1, 2, \ldots, k$. It follows that the statement

11.5.4. $$(\exists x_1) \ldots (\exists x_n)[Q_1(x_1, \ldots, x_n) \wedge \ldots \wedge Q_k(x_1, \ldots, x_n) \vee \sim Q(x_1, \ldots, x_n)]$$

is consistent with H.

Let M be a model of H and of 11.5.4., i.e. a model of H in which 11.5.4. holds. Then M contains objects $c_1, \ldots, c_n$ such that the predicate $[Q_1(x_1, \ldots, x_n) \wedge \ldots \wedge Q_k(x_1, \ldots, x_n) \wedge \sim Q(x_1, \ldots, x_n)]$ holds at $(c_1, \ldots, c_n)$. Further let $R_1(x_1, \ldots, x_n, y), \ldots, R_m(x_1, \ldots, x_n, y)$, $R_1^{(1)}(x_1, \ldots, x_n, y), \ldots, R_{m_1}^{(1)}(x_1, \ldots, x_n, y), \ldots, R_{m_k}^{(k)}(x_1, \ldots, x_n, y)$ be the prepolynomials corresponding to the object symbols $a_1, \ldots, a_m$, $a_1^{(1)}, \ldots, a_{m_1}^{(1)}, \ldots, a_{m_k}^{(k)}$, respectively. Then there are objects $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{a}_1^{(1)}, \ldots, \mathbf{a}_{m_k}^{(k)}$ in M such that the predicates

$$R_1, \ldots, R_m, R_1^{(1)}, \ldots, R_{m_k}^{(k)}$$

hold at $(c_1, \ldots, c_n, \mathbf{a}_1), \ldots, (c_1, \ldots, c_n, \mathbf{a}_m), (c_1, \ldots, c_n, \mathbf{a}^{(1)}), \ldots, (c_1, \ldots, c_n, \mathbf{a}_{m_k}^{(k)})$, respectively, (under the given correspondence under which M is a model of K). Taking into account that the $R_1, \ldots, R_{m_k}^{(k)}$ are prepolynomials, we then see from 11.5.1. that the predicates $[A_1(x_1, \ldots, x_{m_1})], \ldots, [A_k(x_1, \ldots, x_{m_k})]$ hold at $(\mathbf{a}_1^{(1)}, \ldots, \mathbf{a}_{m_1}^{(1)}), \ldots, (\mathbf{a}_1^{(k)}, \ldots, \mathbf{a}_{m_k}^{(1)})$, respectively. On the other hand, the fact that $Q(x_1, \ldots, x_n)$ holds at $c_1, \ldots, c_n$ shows, by 11.5.1.

that $\sim[A(x_1, \ldots, x_m)]$ holds at $\mathbf{a}_1, \ldots, \mathbf{a}_m$. The proof of 11.5.2. will be complete if we can show that M is a model of H and of D_n under a correspondence under which the object symbols $a_1, \ldots, a_m$, $a_1^{(1)}, \ldots, a_{m_k}^{(k)}$ correspond to the objects $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{a}_1^{(1)}, \ldots \mathbf{a}_{m_k}^{(k)}$, respectively. For the statements $[A_1(a_1^{(1)}, \ldots, a_{m_1}^{(1)})], \ldots [A_k(a_1^{(k)}, \ldots, a_{m_k}^{(k)})]$ hold in M under such a correspondence, while $[A(a_1, \ldots, a_m)]$ does not hold in it under the same correspondence, and this shows that 11.5.3. is not deducible from H', contrary to assumption.

Let C be a complete correspondence under which the statements of H as well as the statement 11.5.4 hold in M. We may assume that no two of the objects $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{a}_1^{(1)}, \ldots, \mathbf{a}_{m_k}^{(k)}$ coincide. For if this is not the case from the outset, we may modify M by the addition of non-coinciding objects which, by definition, hold in a relation whenever $\mathbf{a}_1, \ldots, \mathbf{a}_{m_k}^{(k)}$, hold, respectively. The new structure will still be a model of H, and 11.5.4. will still hold in it. But by replacing $\mathbf{a}_1^{(1)}, \ldots, \mathbf{a}_{m_k}^{(k)}$ by the additional objects, the requirement that no two of the objects coincide is now satisfied. By a similar modification we may ensure that none of the objects $\mathbf{a}_1^{(1)}, \ldots, \mathbf{a}_{m_k}^{(k)}$ corresponds to an object symbol in H under C.

Corresponding to each object symbol a in D_n we now select an object of M in the following way. If a is included in H, then we select the object $\mathbf{a}$ which corresponds to a under C. If a is any other object symbol in D_n, let $R(x_1, \ldots, x_n, y)$ be the prepolynomial corresponding to a. Then there exists an object $\mathbf{a}$ in M, such that R holds at $(c_1, \ldots, c_n, \mathbf{a})$; there may be more than one such object in M, but for given R, all such objects must be equal in M. We select one of them, subject to the condition that if a is one of the a_i or $a_i^{(j)}$, then we select the corresponding $\mathbf{a}_i$ or $\mathbf{a}_i^{(j)}$. This yields the required correspondence under which the statements of D_n hold in M, and establishes the theorem. In a similar way we may prove

11.5.5. *Theorem.* Every pre-ideal in J_Q over K is a pseudo-ideal in the sense detailed above.

11.6. *Application to the theory of polynomial ideals.* We shall now interpret the results of the preceding sections in terms of the theory of polynomial ideals. Let H be given by A_{CR} augmented by a complete diagram D of a commutative field F, while K is the set of field axioms A_F augmented by D, so that $H \subseteq K$. Using the notation of 10.5., M'_n is then quasi-isomorphic to the polynomial ring of n variables adjoined to F, $R^{(n)}$, and so the ideals in J_0 over $H' = H \cup D'_n$ are correlated with the ideals of $R^{(n)}$ as detailed in section 10.6.

Now let M be a model of K, i.e. a commutative field which is isomorphic to an extension of F. The variety V of an arithmetical ideal J_a in $R^{(n)}$ is then defined as the set of points of S_n at which the polynomials of J_a all vanish. And it is not difficult to show that this coincides with the variety of the pseudo-ideal J' in J_Q corresponding to the ideal J in J_0 over H' which is associated with J_a as in 10.6. Thus algebraic varieties and their decomposition into irreducible varieties can be completely determined by metamathematical methods.

By theorem 11.5.5. the pre-ideals in J_Q over K are all pseudo-ideals in J_Q, and it is of some interest to characterise the corresponding arithmetical ideals. From the definition of a pre-ideal it is in fact not difficult to infer that the following property is characteristic for such arithmetical ideals.

11.6.1. Any polynomial $f(x_1, \ldots, x_n)$ which vanishes whenever all the polynomials of J_a vanish (in all extensions of F), belongs to J_a.

By the theorem of Hilbert–Netto (ref. 25), condition 11.6.1. is equivalent to the other condition, that if a power of any polynomial of $R^{(n)}$ belongs to J_a, then the polynomial itself belongs to J_a; $f^\delta \varepsilon J_a$ entails $f \varepsilon J_a$.

It will now be shown that conditions 11.4.1.—11.4.3. are satisfied in the case under consideration. 11.4.3. holds because the same condition applies to the arithmetical ideals of $R^{(n)}$, by Hilbert's basis theorem, and hence to the ideals in J_0 over H' and to the pseudo-ideals in J_Q. Before proving 11.4.1. it will be convenient to establish

11.6.2. *Theorem.* The set J_Q is quasi-disjunctive with respect to K.

In fact let $Q(x_1, \ldots, x_n)$ be any element of J_Q as given by 11.5.1. By the results of chapter 7 we may confine ourselves to the case where the polynomial predicates $R_i(c_1, \ldots, x_n, y)$ are given by $R_{q_i}(x_1, \ldots, x_n, y)$, where $q_i(x_1, \ldots, x_n)$ is an algebraic polynomial in $R^{(n)}$ (see 7.5.7.). The relative symbol A in 11.5.1. now stands for E, S, or P, and in these respective cases $Q(x_1, \ldots, x_n)$ can be replaced by

$$Q'(x_1, \ldots, x_n) = (\exists y)[R_{q_3-q_2}(x_1, \ldots, x_n, y) \wedge S(y, y, y)],$$

$$Q'(x_1, \ldots, x_n) = (\exists y)[R_{q_3-(q_1+q_2)}(x_1, \ldots, x_n, y) \wedge S(y, y, y)] \text{ or}$$

$$Q'(x_1, \ldots, x_n) = (\exists y)[R_{q_3-q_1q_2}(x_1, \ldots, x_n, y) \wedge S(y, y, y)].$$

That is, to say $Q(x_1, \ldots, x_n)$ is associated with $Q'(x_1, \ldots, x_n)$ in these cases. $Q'(x_1, \ldots, x_n)$ simply corresponds to the vanishing of a polynomial $q'(x_1, \ldots, x_n)$, where q' is given by $q_1—q_2$, $q_3—(q_1 + q_2)$, or $q_3—q_1q_2$, respectively. Let $Q_1'(x_1, \ldots, x_n)$ and $Q_2'(x_1, \ldots, x_n)$ be two predicates of this description corresponding to the vanishing of the polynomials $q_1'(x_1, \ldots, x_n)$ and $q_2'(x_1, \ldots, x_n)$ respectively. Then the predicate $Q_3'(x_1, \ldots, x_n) = (\exists y)[R_{q_1'q_2'}(x_1, \ldots, x_n, y) \wedge S(y, y, y)]$ which corresponds to the vanishing of $q_3'(x_1, \ldots, x_n) = q_1'(x_1, \ldots, \ldots, x_n) \cdot q_2'(x_1, \ldots, x_n)$ is associated with $Q_1' \vee Q_2'$ with respect to K. Observe that this is true only because K includes the field axioms, which ensure that a product vanishes if and *only if* at least one of its factors vanishes; Q_3' and $Q_1' \vee Q_2'$ are not in general associated with respect to H.

To show that J_Q is quasi-disjunctive it only remains now to replace $Q_3'(x_1, \ldots, x_n)$ by an element of J_Q with which it is associated with respect to K. Such a predicate is

$$(\exists z_1)(\exists z_2)[R_{q_1'q_2'}(x_1, \ldots, x_n, z_1) \wedge R_0(x_1, \ldots, x_n, z_2) \supset E(z_1, z_2)]$$

where $R_0(x_1, \ldots, x_n, y) = [E(x_1, x_1) \wedge \ldots \wedge E(x_n, x_n) \wedge S(y, y, y)]$.

Similarly, it is not difficult to see that if three predicates of J_Q, $Q_1(x_1, \ldots, x_n)$, $Q_2(x_1, \ldots, x_n)$, and $Q_3(x_1, \ldots, x_n)$, are such that they can be replaced by predicates $Q_1'(x_1, \ldots, x_n)$, $Q_2'(x_1, \ldots, x_n)$, $Q_3'(x_1, \ldots, x_n)$ as above, indicating the vanishing of polynomials $q_1'(x_1, \ldots, x_n)$, $q_2'(x_1, \ldots, x_n)$, and $q_3'(x_1, \ldots, x_n) = q_1'(x_1, \ldots, x_n) \cdot q_2'(x_1, \ldots, x_n)$ respectively, then $Q_3(x_1, \ldots, x_n)$ belongs to the pseudo-ideal which is the join of the pseudo-ideals (Q_1) and (Q_2). Therefore $Q_3(x_1, \ldots, x_n)$ can take the place of the predicate

$Q(x_1, \ldots, x_n)$ whose existence is required by 11.4.7. Thus 11.4.7. is satisfied, and hence 11.4.1.

Finally, before proving 11.4.2, we shall establish

11.6.3. *Theorem.* If a pseudo-ideal in J_Q is irreducible, then its pre-ideal closure in J_Q is irreducible.

Expressed in terms of the corresponding arithmetical ideals, the assertion is as follows. Given an irreducible arithmetical ideal J_a, let J_r be the set of all polynomials a power of which is included in J_a. J_r is an arithmetical ideal. Taking into account 11.6.1. et seq., we see that if J is the pseudo-ideal which corresponds to J_a, then the pre-ideal closure of J corresponds to J_r. J_r has the property that if $f^\delta \varepsilon J_r$ for any δ, then $f \varepsilon J_r$, and we have to show that it cannot be represented as the meet of any two ideals with the same property. In fact, it can be shown that J_r cannot be represented as the meet of *any* two arithmetical ideals. For by standard ideal theory, J_a is primary since it is irreducible, and J_r is the prime ideal which belongs to it. But a prime ideal is irreducible, as asserted.

Since J_Q is quasi-disjunctive, it is equivalent to J_Q, and so the disjunctive closure of an irreducible pre-ideal in J_Q is irreducible. This proves 11.4.2.

11.6.4. *Theorem.* A pre-ideal J in J_Q is irreducible if and only if the corresponding arithmetical ideal J_a in $R^{(n)}$ is prime.

If J is not irreducible as a pre-ideal then it is certainly not irreducible as a pseudo-ideal. It follows that J_a also is reducible and so cannot be prime. On the other hand, if J is irreducible, then there exists a model M of K (a field which is isomorphic to an extension of the field F) in which J belongs to its variety V (11.3.9.) which is irreducible (11.3.5.). It follows that V belongs to the arithmetical ideal J_a. And since V is irreducible, J_a must be prime.

Theorem 11.4.5. may now be interpreted as stating, for the present case, that all the isolated prime ideals of a given ideal are independent of any particular representation, although the complete mathematical theory shows that this applies to *all* the prime ideals which belong to an irredundant representation of any given ideal (ref. 26).

11.7. *Disjunctive pseudo-ideals and pre-ideals.* The classical theory of ideals in polynomial rings depends largely on the fact that the fundamental structures, to which the variables are adjoined as well as the extensions of these structures over which the algebraic varieties are defined, are commutative fields or, at least, integral domains. Although particular cases of a more general nature have been considered, it is clear that the classical theory cannot be maintained, for instance, if the algebraic varieties are based on rings with zero-divisors. In such systems, in particulaı, it is no longer true that the union of two varieties is a variety. However, this property of algebraic varieties can be saved if we agree to transfer our main attention to the various types of disjunctive ideals. In fact, a survey of the results of this and the preceding chapter shows that the conditions associated with disjunctive domains are in many respects simpler and more transparent than those which prevail in ordinary ideal domains. It would appear that from the mathematical point of view, the investigation of the mutual relations between disjunctive ideals and their varieties is worthy of attention no less than the corresponding problem for ordinary ideals.

Bearing these remarks in mind, we now continue the development of the general theory which was introduced in section 11.5. We shall confine ourselves to the discussion of polynomial structures and of the ideals associated with them, although similar results hold for the corresponding pre-polynomial structures.

Using the notation of 11.5, let $J_0^\vee$ be the disjunctive domain obtained from J_0. With every element of J_0,

$$X = [A_1(a_1^{(1)}, \ldots, a_{m_1}^{(1)})] \vee \ldots \vee [A_k({}^{(}a_1^{k)}, \ldots, a_{m_k}^{(k)})],\ A_i \varepsilon J_0,$$

we associate the predicate

11.7.1. $Q(x_1, \ldots, x_n) = [Q_1(x_1, \ldots, x_n) \vee \ldots \vee Q_k(x_1, \ldots, x_n)]$
where Q_i corresponds to A_i as in 11.5.1. The aggregate of these predicates is the disjunctive domain derived from J_Q and accordingly will be denoted by $J_Q^\vee$. Corresponding to every ideal in $J_0^\vee$ over H' we obtain a set of predicates in J_Q, which will again be called a (disjunctive) pseudo-ideal.

We note that $Q(x_1, \ldots, x_n)$ in 11.7.1. may be replaced by the predicate $Q'(x_1, \ldots, x_n)$ in the sense that

$(x_1) \ldots (x_n)[Q(x_1, \ldots, x_n) \supset Q'(x_1, \ldots, x_n)]$ and $(x_1) \ldots (x_n)$
$[Q'(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$

are both valid statements, where $Q'(x_1, \ldots, x_n)$ is given by

11.7.2. $$Q'(x_1, \ldots, x_n) = (\exists z_1^{(1)}) \ldots (\exists z_{m_1}^{(1)}) \ldots (\exists z_1^{(k)}) \ldots (\exists z_{m_k}^{(k)}) [R_1^{(1)}(x_1, \ldots, x_n, z_1^{(1)}) \wedge \ldots \wedge R_{m_1}^{(1)}(x_1, \ldots, x_n, z_{m_1}^{(1)}) \wedge \ldots \wedge R_1^{(k)}(x_1, \ldots, x_n, z_1^{(k)}) \wedge \ldots \wedge R_{m_k}^{(k)}(x_1, \ldots, x_n, z_{m_k}^{(k)}) \supset A_1(z_1^{(1)}, \ldots, z_{m_1}^{(1)}) \vee \ldots \vee A_k(z_1^{(k)}, \ldots, z_{m_k}^{(k)})]$$

In this formula the polynomial predicates $R_i^{(j)}$ correspond to the object symbols $a_i^{(j)}$ respectively. $Q'(x_1, \ldots, x_n)$ is not, in general, an element of $J_Q^{\vee}$.

Similarly as in 11.5., we may prove the theorem

11.7.3. Every disjunctive pre-ideal in $J_Q^{\vee}$ over K is a disjunctive pseudo-ideal in the sense detailed above.

We shall distinguish a pre-ideal from a pseudo-ideal in general by affixing an asterisk on the left of the symbol denoting it.

11.7.4. *Theorem.* Let $*J_1$ and $*J_2$ be the disjunctive pre-ideal closures of the disjunctive pseudo-ideals J_1 and J_2. Then $*J = (*J_1, *J_2)$ is the disjunctive pre-ideal closure of the disjunctive pseudo-ideal $J = (J_1, J_2)$.

Let $*J'$ be the pre-ideal closure of $J = (J_1, J_2)$. $*J = (*J_1, *J_2)$ includes J_1 and J_2 and so includes $*J'$, $*J \supseteq *J'$. On the other hand $*J' \supseteq J \supseteq J_1$, and so $*J' \supseteq J_1$, and similarly $*J \supseteq J_2$. Hence $*J' \supseteq (*J_1, *J_2) = *J$. This proves 11.7.4.

11.7.5. *Theorem.* Let $*J_1$ and $*J_2$ be the disjunctive pre-ideal closures of the disjunctive pseudo-ideals J_1 and J_2. Then $*J = [*J_1, *J_2]$ is the disjunctive pre-ideal closure of the disjunctive pseudo-ideal $J = [J_1, J_2]$.

Let $*J'$ be the pre-ideal closure of $J = [J_1, J_2]$. Then $*J_1 \supseteq J_1$ and $*J_2 \supseteq J_2$, and so $*J = [*J_1, *J_2] \supseteq [J_1, J_2]$. Hence $*J \supseteq *J'$. Conversely, for every element $Q(x_1, \ldots, x_n)$ of $*J = [*J_1, *J_2]$ there exist predicates $Q_i'(x_1, \ldots, x_n)$, $i = 1, 2, \ldots m$, $Q_i' \varepsilon J_1$, such that the statement

$$(x_1) \ldots (x_n)[Q_1'(x_1, \ldots, x_n) \wedge \ldots \wedge Q_{m_1}'(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$$

can be deduced from K, while at the same time there exist predicates

$Q_i''(x_1, \ldots, x_n)$, $i = 1, 2, \ldots, m_2$, $Q_i'' \varepsilon J_2$ such that

$$(x_1) \ldots (x_n)[Q_1''(x_1, \ldots, x_n) \wedge \ldots \wedge Q_{m_2}''(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n)]$$

can be deduced from K. It follows that the statement

$$11.7.6. \quad (x_1) \ldots (x_n)[[Q_1'(x_1, \ldots, x_n) \vee Q_1''(x_1, \ldots, x_n)] \wedge \ldots \wedge [Q_{m_1}'(x_1, \ldots, x_n) \vee Q_{m_2}''(x_1, \ldots, x_n)] \supset Q(x_1, \ldots, x_n)]$$

also can be deduced from K where the implicans includes all the disjunctions $[Q_i'(x_1, \ldots, x_n) \vee Q_k'(x_1, \ldots, x_n)]$. But all these disjunctions are included both in J_1 and in J_2 and they are therefore included in $[J_1, J_2]$. It then follows from 11.7.6. that $Q(x_1, \ldots, x_n)$ is included in the pre-ideal closure of $[J_1, J_2)$ which is $*J'$. Hence $*J' \supseteq *J$ as required.

11.7.7. A pre-ideal $*J$ in $J_Q^{\vee}$ which is irreducible as a disjunctive pre-ideal is also irreducible as a disjunctive pseudo-ideal.

For assume that $*J = [J_1, J_2]$ where $*J$ is a true subset of the pseudo-ideals J_1 and J_2. Let $*J_1$ and $*J_2$ be the pre-ideal closures of J_1 and J_2. Then by theorem 11.7.5. $*J = [*J_1, *J_2]$, since $*J$ is its own pre-ideal closure. This is impossible since $*J$ is a true subset of both $*J_1$ and $*J_2$ although, by assumption, it is an irreducible pre-ideal.

11.7.8. *Theorem.* A necessary and sufficient condition for a disjunctive ideal J in $J_0^{\vee}$ to be irreducible is that of the elements of any disjunction X which belongs to J,

$$X = X_1 \vee X_2 \vee \ldots \vee X_m, \; X_i \varepsilon J_0, \; i = 1, 2, \ldots, m,$$

one at least should belong to J; $X_k \varepsilon J$, $1 \leqslant k \leqslant m$.

The necessity of the condition follows from the repeated application of 10.5.4. Also, if X is any disjunction, $X = Y \vee Z$, we must have $Y = Y_1 \vee \ldots \vee Y_m$, $Y_i \varepsilon J_0$, $i = 1, 2, \ldots, m$ and $Z = Z_1 \vee \ldots \ldots \vee Z_{m'}$, $Z_i \varepsilon J_0$, $i = 1, 2, \ldots, m'$, since the statements of J_0, which are of order one, are not themselves disjunctions. It follows that if the condition of the theorem is satisfied, then either one of the Y_i belongs to J and so $Y \varepsilon J$, or one of the Z_i belongs to J and so $Z \varepsilon J$. And by 10.5.4. this is sufficient to ensure that J is irreducible.

Thus it appears that if the aggregate of disjunctions $X_0 \vee X_1 \vee X_2 \vee \ldots \vee X_m$, where X_0 belongs to a given ideal J in J_0, and of similar disjunctions obtained by a change of order, is a disjunctive ideal at all, then it is irreducible. It is then the disjunctive closure of J (compare 10.5.7.).

Owing to the correspondence between ideals in J_0 and $J_0^{\vee}$ on one hand, and J_Q and $J_Q^{\vee}$ on the other, theorems similar to 11.7.8. hold for pseudo-ideals and pre-ideals in $J_Q^{\vee}$. Thus

11.7.9. *Theorem.* A necessary and sufficient condition for a disjunctive pre-ideal J in $J_Q^{\vee}$ to be irreducible (as a pre-ideal or as a pseudo-ideal, see 11.7.7.) is that of the elements of any disjunction which belongs to J_Q.

$$Q = [Q_1 \vee Q_2 \vee \ldots \vee Q_m], Q_i \varepsilon J_0, i = 1, 2, \ldots, m,$$

one at least should belong to J; $Q_k \varepsilon J$ for some positive $k \leqslant m$.

Clearly the varieties of a pseudo-ideal and of its pre-ideal closure in any space S_n, coincide. Accordingly the following result can be obtained from 11.3.4., taking into account 11.7.4. and 11.7.5.

11.7.10. *Theorem.* Let V_1, V_2, V', and V'' be the varieties of the disjunctive pseudo-ideals $J_1, J_2, J' = (J_1, J_2)$ and $J'' = [J_1, J_2]$, respectively. Then $V' = V_1 \cap V_2$ and $V'' = V_1 \cup V_2$.

We shall now assume that the maximum condition is satisfied by the ideals of J_0 and, therefore, by the pseudo-ideals of J_Q. By 10.4.1. it is then also satisfied by the disjunctive ideals of $J_0^{\vee}$ and the disjunctive pseudo-ideals of $J_Q^{\vee}$. And it follows from 11.5.5. and 11.7.3. that the same condition holds for the pre-ideals of J_Q and the disjunctive pre-ideals of $J_Q^{\vee}$. We can then represent every disjunctive ideal in $J_0^{\vee}$ in just one way as the irredundant meet of a finite number of irreducible disjunctive ideals (see 10.5.6.). It follows that every disjunctive pseudo-ideal in $J_Q^{\vee}$ can be represented in just one way as the irredundant meet of a finite number of irreducible disjunctive pseudo-ideals,

11.7.11. $$J = [J_1, \ldots, J_m]$$

Similarly, we can represent every disjunctive pre-ideal in $J_Q^{\vee}$ in just one way as the irredundant meet of a finite number of irreducible disjunctive pre-ideals.

11.7.12. $*J = [*J_1, \ldots, *J_m]$.

Let $V, V_1, \ldots, V_m$ be the varieties of the pseudo-ideals $J, J_1, \ldots, J_m$ in 11.7.11. or of the pre-ideals $*J, *J_1, \ldots, *J_m$ in 11.7.12. Then theorems 11.3.4. and 11.7.10. show that in either case $V = V_1 \cup \ldots \cup V_m$. But in neither case can we affirm that the varieties $V_1, \ldots, V_m$ are irreducible, although this is true in the classical case considered in the preceding section. Nevertheless, it will now be shown that we can determine the irreducible component varieties of any given variety V by the analysis of the disjunctive pre-ideals of $J_Q^\vee$.

11.7.13. Let $*J$ be the disjunctive pre-ideal which belongs to a variety V in a space S_n, and let

11.7.14. $*J = [*J_1, \ldots, *J_m]$

be the (unique) representation of $*J$ as an irredundant meet of irreducible disjunctive pre-ideals. Then

11.7.15. $V = V_1 \cup \ldots \cup V_m$

is the (unique) representation of V as the irredundant union of irreducible varieties, where V_i is the variety of $*J_i$, $i = 1, \ldots, m$.

By 11.3.10., V can be represented in just one way as the irredundant union of a finite number of irreducible varieties,

11.7.16. $V = V'_1 \cup \ldots \cup V'_{m'}$.

Now let $*J'_1, \ldots, *J'_{m'}$ be the disjunctive pre-ideals which belong to $V'_1, \ldots, V'_{m'}$, respectively. By 11.3.5. these pre-ideals are irreducible. Define the pre-ideal $*J'$ as their meet,

11.7.17. $*J' = [*J_1, \ldots, *J'_{m'}]$.

Then $V' = V'_1 \cup \ldots \cup V'_{m'}$ where V' is the variety of $*J'$, and comparison with 11.7.16. shows that $V' = V$. This in turn implies that $*J'$ is a subset of $*J$ which belongs to V, $*J' \subseteq *J$. On the other hand, consider any V'_i, $1 \leqslant i \leqslant m'$. We have

$$V'_i = V'_i \cap V = V'_i \cap (V_1 \cup \ldots \cup V_m) = (V'_i \cap V) \cup \ldots \cup (V'_i \cup V_m).$$

But V'_i is irreducible and so $V'_i \cap V_k = V'_i$ for at least one k, $1 \leqslant k \leqslant m$, i.e. $V'_i \subseteq V_k$. This in turn entails $*J'_i \supseteq *J_k$. Thus for every i, $1 \leqslant i \leqslant m'$ there exists a $k = k(i)$, $1 \leqslant k \leqslant m$ such that ${}^*J'_i \supseteq {}^*J_{k(i)}$. Hence

$$^*J = [^*J_1, \ldots, {}^*J_m] \subseteq [^*J_{k(1)}, \ldots, {}^*J_{k(m')}] \subseteq [J'_1, \ldots, {}^*J_{m'}] = {}^*J'.$$

This shows that ${}^*J \subseteq {}^*J'$, which in conjunction with ${}^*J' \subseteq {}^*J$, yields ${}^*J = {}^*J'$. 11.7.17. now becomes

11.7.18. $\quad {}^*J = [^*J'_1, \ldots, {}^*J'_{m'}]$.

This representation of ${}^*J$ cannot be redundant because in that case the representation of V in 11.7.16. also would be redundant, contrary to assumption. It follows that $m = m'$, and that the right hand side of 11.7.18. can differ from the right hand side of 11.7.14. only as to order. Similarly, therefore, the right hand side of 11.7.15. can differ from the right hand side of 11.7.16. only as to order. This shows that 11.7.15. provides the required representation of V, and moreover that the disjunctive pre-ideals ${}^*J_1, \ldots, {}^*J_m$ belong to the varieties $V_1, \ldots, V_m$ respectively.

11.8. *Application to primary rings.* A simple illustration of the results of the preceding section will now be given.

Let $A_{CR}^{(\delta)}$ consist of the axioms of A_{CR} in addition to the following (11.8.1—11.8.3). δ is a fixed positive integer.

11.8.1. $\quad (\exists x)(y)[P(y, x, y)]$

11.8.2. $\quad (x)[[(\exists y)(\exists z)(u)[P(x, y, z) \wedge P(u, z, u)]] \vee [(\exists y)(\exists z)[P(x, y, z) \wedge S(z, z, z)]]]$

11.8.3. $\quad (x_1) \ldots (x_\delta)[[(\exists y_1) \ldots (\exists y_\delta)(\exists z)[P(x_1, y_1, z) \wedge P(x_2, y_2, z) \wedge \ldots \wedge P(x_\delta, y_\delta, z) \wedge S(z, z, z)]] \supset [(\exists z_1) \ldots (\exists z_{\delta-1})[P(x_1, x_2, z_1) \wedge P(z_1, x_3, z_2) \wedge \ldots \wedge P(z_{\delta-2}, x_\delta, z_{\delta-1}) \wedge S(z_{\delta-1}, z_{\delta-1}, z_{\delta-1})$

Axiom 11.8.1. postulates the existence of a unit element, 11.8.2. ensures that every element is either a divisor of zero or a divisor of one, and 11.8.3. states that the product of any δ elements which are zero divisors, equals zero. We may consider that the case $\delta = 1$ is included, although for that case a slight modification of 11.8.3. becomes necessary. The set A_δ represents an axiomatic system for a special type of primary ring (ref. 27) as considered by Krull (ref. 28, compare also Fraenkel ref. 29, 30). Such a ring will be called a primary ring of order δ. It may also be characterised as a commutative ring whose zero divisors constitute an ideal J, such that $J^{\delta} = (0)$, and such that J is the only maximal ideal in the ring. For $\delta = 1$, this concept reduces to that of a commutative field, $A_{CR}^{(1)}$ is equivalent to A_F.

Let M_δ be a model of $A_{CR}^{(\delta)}$ which satisfies the finite basis condition for (arithmetical) ideals. Using the notation of sections 11.5. and 11.7., we define H as A_{CR} augmented by a complete diagram D of M_δ, $H = (A_{CR};D)$ and K as $A_{CR}^{(\delta)}$ augmented by D, $K = (A_{CR}^{(\delta)};D)$. Then $H \subseteq K$ as required. The polynomial structure M'_n as defined in 11.5. is quasi-isomorphic to the polynomial ring $R^{(n)}$ of n variables adjoined to M_δ. The ideals of J_0, and thence the pseudo-ideals of J_Q are correlated whith the arithmetical ideals of $R^{(n)}$ as detailed in 10.6., and the pre-ideals of J_Q therefore correspond to a subset of the set of arithmetical ideals in $R^{(n)}$. Arithmetical ideals which belong to this subset may be called root ideals (or radicals). Also, it was shown in section 10.7. that, for the case of a ring, the disjunctive ideals can be dissociated from their metamathematical origins and can be replaced by specific sets of finite sets of ring elements, which will again be called disjunctive ideals in $R^{(n)}$. Some of these constructs correspond to the disjunctive pre-ideals of J_Q and will be called disjunctive root ideals. Results involving pre-ideals and other pseudo-ideals and disjunctive pre-ideals and other disjunctive pseudo-ideals may then be reformulated in terms of the various types of ideals and disjunctive ideals in $R^{(n)}$. Adopting this procedure, we shall now restate the results of the preceding sections, as applied to the present case, in purely algebraic terms.

For given $\delta \geqslant 1$, let M_δ be a primary ring of order δ which satisfies the finite basis (maximum) condition, and let $R^{(n)} = R(x_1, \ldots, x_n)$ be the ring of polynomials of n variables, $x_1, \ldots, x_n$, adjoined to R. We define ideals in $R^{(n)}$ in the usual way. By Hilbert's basis theorem, $R^{(n)}$ satisfies the maximum condition.

A disjunctive ideal in $R^{(n)}$ is a set J of finite non-empty sets v of polynomials $q = q(x_1, \ldots, x_n)$ in $R^{(n)}$, which satisfies the following conditions.

11.8.4. If a set v belongs to J then any other finite set of polynomials of $R^{(n)}$ which includes v, also belongs to J.

11.8.5. Let $v_1 = (q_1', q_2', \ldots, q_{m'}')$ and $v_2 = (q_1'', q_2'', \ldots, q_{m''}'')$ be two elements of J (the order is irrelevant since the sets are not ordered), and let v be any set of $m' \, m''$ polynomials of the form

$$(c_{11}\, q_1' + d_{11}\, q_1'', c_{11}\, q_1' + d_{12}\, q_2'', \ldots, c_{ik}\, q_i' + d_{ik}\, q_k'', \ldots, c_{m'm''}\, q_{m'}' + d_{m'm''}\, q_{m''})$$

where c_{ik} and d_{ik}, $i = 1, 2, \ldots, m'$, $k = 1, 2, \ldots, m''$ are arbitrary polynomials of $R^{(n)}$. Then v belongs to J.

The meet and sum of ideals and of disjunctive ideals, and the concepts of reducibility and of redundancy, etc., are defined as before. The symbols [ ] and ( ) will also be retained. The results of the preceding section, as applied to the present case, may now be summarised as follows.

The class of disjunctive ideals satisfies the maximum condition (10.4.1.).

The meet of any number of disjunctive ideals is a disjunctive ideal. Every disjunctive ideal can be represented in just one way as the irredundant meet of a finite number of irreducible disjunctive ideals (10.5.6.).

A necessary and sufficient condition for a disjunctive ideal J to be irreducible is that, together with every set $v \,\varepsilon\, J$ which contains more than one polynomial, a proper subset of v also is contained in J. It follows that together with every set v which contains more than one polynomial, a subset of v which contains just one polynomial also is contained in J (10.5.4.).

The set of all elements v of a disjunctive ideal J' is non empty. The union of its elements is an ordinary ideal J which will be said to belong to J'. Conversely, given any ordinary ideal J, we consider the set J'' of unit sets which contain just one element of J. The meet of all disjunctive ideals which include J'' will be called the disjunctive closure of J. The ideal which belongs to the disjunctive closure of J is J itself.

The disjunctive closure of an irreducible ideal is irreducible.

Now let M be any primary ring of order δ which is an extension of M_δ. We define S_n as the n-diemsional space of points $(x_1, \ldots, x_n)$ with coordinates in M, in the usual way.

The variety V of any set J of polynomials in $R^{(n)}$ (in particular, of an ideal) is defined in the usual way as the set of points $(x_1, \ldots, x_n)$ at which all the polynomials of J vanish.

The variety V of any set of finite sets v of polynomials in $R^{(n)}$ (in particular, of a disjunctive ideal) is defined as the set of points $(x_1, \ldots, x_n)$ at which at least one polynomial in every element v of J vanishes. The variety of an ordinary ideal coincides with the variety of its disjunctive closure.

For any variety V, the set J of all finite sets of polynomials v such that at every point of V one polynomial of v at least vanishes, is a disjunctive ideal. J is said to be the disjunctive ideal which belongs to V. In that case also, V is the variety of J.

If V_1 and V_2 are the varieties of the disjunctive ideals J_1 and J_2 respectively, then $V_1 \cap V_2$ and $V_1 \cup V_2$ are the varieties of (J_1, J_2) and $[J_1, J_2]$ respectively (11.7.10.).

Every variety can be represented in just one way as the irredundant union of a finite number of irreducible varieties (11.3.10.).

An 'extension' M of M_δ will be understood to be a primary ring of order δ which includes M_δ.

An ideal J will be called a root ideal if any polynomial $q(x_1, \ldots, x_n) \varepsilon R^{(n)}$ which vanishes at all points of the variety V of J in the space S_n over any extension M of M_δ, belongs to J.

A disjunctive ideal J will be called a disjunctive root ideal if any finite set of polynomials, v, such that at every point of the variety V of J (in every S_n over an extension M of M_δ) at least one polynomial of v vanishes, belongs to J.

These definitions of root ideals correspond to the definition of pre-ideals by concepts of deducibility. Since the definitions involve the idea of arbitrary extensions of the given ring, it is the algebraic tendency to characterise root ideals — or in the classical case, the fact that a polynomial vanishes whenever a given set of polynomials vanish — by purely arithmetical or 'rational' conditions, e.g., the theorem of Hilbert–Netto. From the metamathematical point of view, the concept of deducibility is no less 'rational'.

The disjunctive ideal which belongs to any given variety is a disjunctive root ideal (11.3).

If a disjunctive root ideal is irreducible within the class of disjunctive root ideals then it is irreducible within the class of all disjunctive ideals (11.7.7.).

A necessary and sufficient condition in order that a variety V be irreducible is that the disjunctive ideal which belongs to V be irreducible (11.3.5.).

If J is an irreducible disjunctive root ideal then there exists an extension M of M_δ, such that if V is the variety of J in S_n over M, then J belongs to V (11.3.7.).

A point P which is contained in a variety V will be called a general point of V if any $q(x_1, \ldots, x_n) \varepsilon R^{(n)}$ which vanishes at P vanishes at all other points of V. A necessary and sufficient condition for a disjunctive root ideal J to be irreducible is that there exist an extension M of M_δ, such that the variety V of J in S_n over M contains a general point P (11.3.9.).

The ideal which belongs to a disjunctive root ideal is a root ideal. The disjunctive closure of a root ideal is a disjunctive root ideal.

The irreducible disjunctive ideals which appear in the representation of a disjunctive root ideal as the irredundant meet of a finite number of irreducible disjunctive ideals are all root ideals.

Let J be the disjunctive root ideal which belongs to a variety V, and let $J = [J_1, \ldots, J_m]$ be the representation of J as the irredundant join of a finite number of irreducible disjunctive ideals. Then $V = V_1 \cup \ldots \cup V_m$ is the representation of V as the irredundant union of a finite number of irreducible varieties where $V_1, \ldots, V_m$ are the varieties of $J_1, \ldots, J_m$ respectively. Moreover, in that case the disjunctive ideals, $J_1, \ldots, J_m$ *belong* to the varieties $V_1, \ldots, V_m$, respectively (11.7.13.).

The above propositions have all been derived by metamathematical methods, the only standard result used being Hilbert's basis theorem. However, a considerable proportion of these propositions can no doubt be derived by more conventional methods.

The above theory can now be supplemented by a more specifically arithmetical investigation. For instance, it is not difficult to show that all primary ideals of exponent not exceeding δ which contain no constants except 0 are root ideals in the sense defined above. Further developments along these lines are outside the scope of the present work.

11.9. *Ideals in the theory of algebraic numbers.* The metamathematical ideals which we associated with the arithmetical ideals of a ring were closely related to the homomorphisms produced by the latter. As we have seen this is a natural and fruitful approach, particularly with regard to the theory of polynomial rings. In the theory of algebraic integers the arithmetical ideals are still associated with the homomorphisms of the ring in which they are defined, but their genesis was different. As is well known, these ideals were introduced by Dedekind (ref. 31), following Kummer (ref. 32) in an effort to rehabilitate the multiplicative properties of algebraic integers. Given the algebra of axioms of a ring, together with a positive diagram D of a model M of A_{CR}, let us consider the set J_Q of predicates 'y divides a'' where a' is an arbitrary object of M, symbolically $(\exists z)P(y, z, a)$. We may now ask, given that y divides $a_1', a_2', \ldots, a_n'$, what other objects does y necessarily divide? The answer is, y divides all the objects which belong to the ideal $(a_1', a_2', \ldots, a_n')$, and it is apparent that this consideration led Dedekind to his definition of an ideal in the first place. In our present order of ideas, we may therefore define an ideal in J_Q as a set J of predicates $(\exists z)P(y, z, a)$ such that if $(\exists z)P(y, z, a)$ 'necessarily' whenever $(\exists z)P(y, z, a_i)$, $i = 1, 2, \ldots, n$, then $(\exists z)P(y, z, a)$. Now if we interpret 'necessarily' as meaning that $(y)[[(\exists z)P(y, z, a_1)] \wedge \ldots \wedge [(\exists z)P(y, z, a_n)] \supset [(\exists z)P(y, z, a)]]$, can be deduced from $A_{CR} \cup D$, then we see that this new type of ideal coincides with certain pre-ideals in J_Q. Thus, the aggregate of object symbols which appear in the predicates of such a pre-ideal of J, correspond to an arithmetical ideal in M, and vice-versa. However, the further development of the theory of ideals of algebraic integers appears to be largely outside the scope of the metamathematical methods developed so far. Whether the approach sketched above may eventually lead to a positive contribution to that theory will not be decided here.

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