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Complex Hyperbolic Geometry

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To
Emily, Evan,
Elizabeth and Michael

PREFACE

This book attempts a fairly comprehensive treatment of the geometry of complex hyperbolic space and its boundary. This subject's richness is enhanced by the confluence of many fields of mathematics: Riemannian geometry, complex analysis, symplectic and contact geometry, Lie theory, harmonic analysis and ergodic theory. The boundary of complex hyperbolic geometry is spherical CR geometry or *Heisenberg geometry*. Many treatments of analysis on bounded domains, Kähler manifolds and analysis on the Heisenberg group currently exist in the literature, but there does not seem to be a comprehensive treatment of the *geometry* of complex hyperbolic space or its boundary.

Largely motivated by applications to geometric structures, moduli spaces and discrete groups, this book does not attempt a thorough discussion of any of these topics. Nor does it attempt a thorough treatment of the analytic aspects listed above. Instead, this book is a "user's guide" to complex hyperbolic geometry, which I hope will stimulate research in this fascinating and important geometry.

This project began as the twin sibling of a computer program. In the early summer of 1988, Mark Phillips, Robert Miner and I began writing an interactive graphics program (called "HEISENBERG") for investigating discrete subgroups acting on complex hyperbolic 2-space. Quickly we discovered that the literature contained many different conventions concerning coordinates on complex hyperbolic space. (For example, we normalize the holomorphic sectional curvature of complex hyperbolic space to be -1, in which case the sectional curvatures κ range in the interval

$$-1 \leq \kappa \leq -1/4.$$

This differs from Epstein [48] where the sectional curvature lies between -1 and -4 and from Mostow [128] where the curvature lies between $-1/2$ and -2. Unfortunately, the range $-4 \leq \kappa \leq -1$ seems to be the most popular.) Computers, like humans, are not fond of inconsistent mathematical formulas. Therefore we must establish all of the formulas correctly once and for all. With an internally consistent exposition, we rest assured that the bugs in our programs are caused by our own stupidity and not by inconsistent formulas from the literature.

Several papers influenced my thinking at an early stage: G.D. Mostow's paper "On a remarkable class of polyhedra in complex hyperbolic space" ([128]) contains the first geometric construction of nonarithmetic lattices acting on complex hyperbolic space. The direct geometric techniques of this paper are very much in the spirit of this book. Élie Cartan's paper "Sur le groupe de la géométrie hypersphérique" ([21]) seems to be the first source on the synthetic Heisenberg geometry. Domingo Toledo's "Representations of surface groups on complex hyperbolic space" ([163]) applies complex hyperbolic geometry to representation theory of discrete groups and flat bundles over Riemann surfaces. As I became more familiar with these papers, the challenge of proving the results of one paper using techniques from the others became irresistible. For example, Mostow's paper described the geometric structure of bisectors (formerly called "equidistant" or "spinal" hypersurfaces) in terms of a decomposition into complex hyperplanes; our first computer experiments revealed the other foliation by totally real totally geodesic subspaces. This observation led to a short proof of one of the theorems of Cartan's paper [21] (see Theorem 5.3.4). This theorem beautifully examplifies synthetic Heisenberg geometry. Cartan's book [24] on the geometry of complex projective 3-space also influenced the evolution of the viewpoint here, as did the paper [19] of Burns and Shnider on spherical CR-structures and the paper [105] of Korányi–Reimann on the complex cross-ratio.

In the summer of 1993, my thinking on this subject changed significantly when Ossip Shvartsman introduced me to the (apparently neglected) work of Georges Giraud from 1915 to 1921. Strongly influenced by Picard, Giraud developed much of the theory of bisectors from a point of view very close to the present one. Indeed, Mostow's decomposition cited above may be found in Giraud's 1921 paper [65]. Furthermore Giraud proved a strong uniqueness theorem for intersections of bisectors (Theorem 8.3.3) which has strong consequences (as Giraud observes) for the structure of Dirichlet fundamental domains. Giraud's results are reproved and extended here. The complex-projective theory of *extors*, or "extended bisectors," is directly motivated by Giraud's ideas.

I have *tried* to present here a consistent set of formulas so that workers in this subject can perform calculations, both by machine and by hand. I have emphasized the interaction between the Kähler geometry on complex hyperbolic space and its degeneration to conformal Heisenberg geometry on the boundary. This is analogous to the more familiar degeneration of the Riemannian geometry of real hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$ to Euclidean conformal geometry on its ideal boundary. We explore various aspects of the geometry: the structure of the automorphism group, the geometry of totally geodesic subspaces, chains, bisectors, etc. Bisectors play a central role here as do the spheres bounding them, which we call *spinal spheres*. Spinal spheres are ubiquitous: they are the analogue of isometric spheres for automorphisms of complex hyperbolic space, they arise as level sets of the Poisson kernel function, they are unions of boundaries of totally real geodesic subspaces level sets of the 3-point invariants defined by Cartan and Toledo and form faces of Dirichlet–Ford fundamental polyhedra of discrete groups. A key notion in the geometry at infinity is a "calibration" of the CR-structure (more traditionally called "pseudo-Hermitian structure" or "contact 1-form"). Playing a role analogous to conformal Riemannian metrics on the boundary of real hyperbolic space, calibrations of the CR-structure form a unifying bridge between spinal spheres at infinity and the potential theory and the symplectic geometry inside. (For the most part we have followed the conventions of Kobayashi–Nomizu [100] for differential geometry and Korányi–Reimann [106] for Heisenberg CR geometry.)

Although many of the results here are included in more general theorems about Lie groups or manifolds of nonpositive curvature, complex hyperbolic space is such an important special case that it merits individual attention. Complex hyperbolic space is the simplest example of a negatively curved Riemannian manifold not having constant curvature. It and its quotients therefore provide basic examples in Riemannian geometry and dynamical systems. The simplest examples of negatively curved Kähler manifolds, complex hyperbolic manifolds, provide important examples in complex analysis and algebraic geometry. General theories should be understood in terms of nontrivial examples. Therefore I have adopted the philosophy that explicit calculations are more valuable than quoting general theorems. For example, the root-space, Cartan and Bruhat decompositions of $\mathbf{PU}(n,1)$ are worked out explicitly, rather than simply quoted from Lie theory. Similarly, we have worked out detailed formulas for geodesics and orthogonal projections onto totally geodesic subspaces, although their qualitative properties are special cases of general results about Hadamard manifolds. This has led to a rather algebraic approach, since I succumbed to the temptation of trying to write everything down explicitly. This luxury is unavailable in a more general setting. The result is a rather long preliminary section, including such nonstandard prerequisites as real structures on complex vector spaces, Hermitian vector "cross" products, outer products and triple Hermitian products.

I have adopted the general point of view of symplectic topology, emphasizing the symplectic properties of complex hyperbolic space as they relate to its Riemannian and holomorphic properties. For example, complex hyperbolic space itself is constructed as a symplectic quotient. A whole section is devoted to the Hamiltonian functions on $\mathbf{H}^n_{\mathbb{C}}$ determining 1-parameter groups of automorphisms. Another section deals with the closely related functions arising as contact Hamiltonians which calibrate the CR-structure on $\partial\mathbf{H}^n_{\mathbb{C}}$ and other geometric entities (for example, totally geodesic submanifolds) on $\mathbf{H}^n_{\mathbb{C}}$. Both of these constructions relate to potential theory on $\mathbf{H}^n_{\mathbb{C}}$ as well as metric properties such as totally geodesic subspaces, orthogonal projections and involutions. The rich interplay between function theory and geometry is one unifying theme of this work.

A geometric theory of discrete groups requires appropriate notions of polyhedra, which becomes nontrivial in the absence of totally geodesic hypersurfaces from which to form a polyhedron's "faces." Owing to their ubiquity and many nice properties, bisectors (formerly called "equidistant hypersurfaces" or "spinal surfaces") form a reasonable substitute for totally geodesic hypersurfaces. Since the faces of fundamental polyhedra arising from the general constructions of Dirichlet–Poincaré and Ford are bisectors, it seems most natural to use bisectors as the building blocks for polyhedra in $\mathbf{H}^n_{\mathbb{C}}$. However, bisectors may be too rigid —Schwartz's very recent notion of "hybrid cones" ([153]) may be a much better class of submanifolds from which to build fundamental polyhedra.

For such a theory to be workable, it is necessary to understand how such hypersurfaces intersect. A large part of this text is devoted to this question. We prove that the intersection of two bisectors has two, one or none components. However, in an important case, bisector intersections are connected: if there is a single point in $\mathbf{H}^n_{\mathbb{C}}$ (respectively $\partial\mathbf{H}^n_{\mathbb{C}}$) from which two bisectors $\mathfrak{E}_1, \mathfrak{E}_2$ are "equidistant," we say that $\mathfrak{E}_1, \mathfrak{E}_2$ are *coequidistant* (respectively *covertical*). For example, if $\mathfrak{E}_1, \mathfrak{E}_2$ are the bisectors extending faces of a Dirichlet (respectively Ford) fundamental polyhedron, then they are coequidistant (respectively covertical). We prove that if $\mathfrak{E}_1, \mathfrak{E}_2$ are a pair of coequidistant or covertical bisectors, then $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is connected. A statement equivalent to this is stated in [128], although the proof is flawed.

Bisectors are generalized and extended to real hypersurfaces in complex projective space, called *extors*. Bisectors are metric objects, while extors are projective objects. Furthermore intersections of extors are considerably simpler to classify than bisector intersections, and the projective theory illuminates the metric theory. We prove Giraud's theorem that at most one other bisector contains the intersection of a generic pair of bisectors in the context of extors. As noted by Giraud, this implies that, under very general conditions, cycles of side-pairing transformations of Dirichlet–Ford fundamental domains have length at most 3. This phenomenon seems "generic," unlike the familiar cases of constant curvature, when side-pairing cycles can be arbitrarily long.

Summary

Chapter 1 reviews complex 1-dimensional geometry from a point of view which generalizes to higher dimensions. This material—complex projective, elliptic and hyperbolic geometry—should be quite familiar to most readers. Most of the proofs are given as exercises. However, this introductory chapter sets the stage for the more complicated general theory and hopefully motivates some of the later constructions.

Chapter 2 develops the algebraic, analytic and geometric prerequisites needed later. The linear algebra needed in the sequel requires particular attention to the relation between complex and real vector spaces (§2.1). Hermitian structures on complex vector spaces relate to orthogonal and symplectic structures on the underlying real vector spaces (§2.2). In particular real subspaces are studied, and the principal invariant of a real subspace of a Hermitian vector space—the angle of holomorphy—is developed in detail (§2.2.2). Hermitian triple products—an invariant of triples of vectors in a Hermitian product which leads to Cartan's "invariant angulaire"—are introduced (§2.2.5). Hermitian outer products (§2.2.6) and cross-products (§2.2.7) are useful calculational tools and are introduced in this section.

The general point of view is heavily influenced by symplectic geometry (§2.3), since one obtains much of the structure of complex projective space by viewing it as a symplectic (or Kähler) quotient. Its momentum map is explicitly described by Hermitian outer products. The basic differential operators ∂ and $\bar{\partial}$ on a complex manifold (§2.4) are used to construct Kähler potentials (§2.4.1), and also to deduce special properties of curvature in the presence of a Kähler structure

(§2.4.2). The boundary of a symplectic manifold enjoys *contact geometry* (§2.5) and the boundary of a complex manifold enjoys *CR geometry* (§2.5.4). A *contact structure* is a field of tangent hyperplanes and is defined by 1-forms, which *calibrate* the contact structure. Geometric objects in $\mathbf{H}^n_{\mathbb{C}}$ correspond to special families of calibrations of the CR-structure on $\partial\mathbf{H}^n_{\mathbb{C}}$, and in general calibrations relate to defining functions of real hypersurfaces in Kähler manifolds (§2.5.5). An abstract notion of a *Heisenberg space*—a space with a simply transitive action of the Heisenberg group—is defined (§2.6). Explicit contact flows on Heisenberg space are described, and the section concludes (§2.6.4) with a differential operator reminiscent of the "curl" due to Rumin [148, 149].

Chapter 3 begins the study of complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ with its model as the unit ball in $\mathbb{C}^n$. However, the full symmetry of $\mathbf{H}^n_{\mathbb{C}}$ is not apparent until $\mathbb{C}^n$ is completed to complex projective space. Complex reflections are defined, from which the homogeneity of $\mathbf{H}^n_{\mathbb{C}}$ follows. We derive a formula for the midpoint of two points (§3.1.2), a momentum mapping expressing $\mathbf{H}^n_{\mathbb{C}}$ as a symplectic quotient (§3.1.3), a synthetic description of the Bergman metric (§3.1.6), a formula for the Bergman distance function (§3.1.7) and a Kähler potential for the Bergman metric and its relation to the Bergman kernel (§3.1.8).

Complex projective subspaces of $\mathbb{P}^n_{\mathbb{C}}$ meet $\mathbf{H}^n_{\mathbb{C}}$ in *complex-linear* totally geodesic submanifolds (§3.1.4). Such a totally geodesic submanifold is an isometrically embedded copy of a lower-dimensional complex hyperbolic space. In particular a *complex geodesic*—a complex-linear totally geodesic subspace of complex dimension 1—has the Poincaré model of hyperbolic geometry. (For example, it is a round disc with geodesics given by semicircular arcs orthogonal to the bounding circle.) The projection of $\mathbb{P}^n_{\mathbb{C}}$ onto a complex-linear totally geodesic subspace defines the *orthogonal projection* in $\mathbf{H}^n_{\mathbb{C}}$ (§3.1.5). A basic property of $\mathbf{H}^n_{\mathbb{C}}$ is that two distinct points span a unique *complex geodesic;* that is, a complex-linear totally geodesic subspace of complex dimension 1.

The other totally geodesic submanifolds are totally real (§3.1.9). These are isometrically embedded copies of real hyperbolic spaces. In contrast to a complex-linear totally geodesic submanifold, a totally real totally geodesic subspace has the Klein model of hyperbolic geometry. For each k, the automorphism group $\mathbf{PU}(n, 1)$ acts transitively on totally real totally geodesic subspaces of dimension k. Figures 3.4 and 3.3 depict Fermi coordinates on the two types of totally geodesic suspaces. The proof that the only totally geodesic submanifolds of $\mathbf{H}^n_{\mathbb{C}}$ are either totally real or complex-linear is sketched (§3.1.11).

Trigonometry in complex hyperbolic space is developed (§3.2), following Giraud [65], Hsiang [89], Brehm [17], and Leuzinger [111, 112], and earlier work by Blaschke and Terheggen [12, 158]. The cosine and sine laws, as well as the Pythagorean theorem and the formulas for the angle of parallelism, are generalized from real hyperbolic geometry to complex hyperbolic geometry. A new feature here is that associated to a vertex of a triangle are two "angles"—the ordinary Riemannian angle between the geodesic sides—as well as a "complex angle" defined as the angle between the complex geodesics containing the sides.

We view trigonometric formulas in $\mathbf{H}^n_{\mathbb{C}}$ as *deformations* of the trigonometric formulas in real hyperbolic space, where triangles in real hyperbolic space are those triangles which lie in totally geodesic subspaces. All of these formulas have analogues in complex elliptic space. The trigonometric formulas are used to compare real hyperbolic space with complex hyperbolic space (§3.2.5). The effect of the varying sectional curvature is explicitly related to distortion of distance. The angle of holomorphy is interpreted in terms of large triangles (§3.2.6). The *complex altitudes* of a triangle are defined (§3.2.7), and Cartan's theorem ([24]) that the complex altitudes of a triangle are concurrent if and only if the triangle lies in a totally geodesic subspace is proved.

We derive explicit formulas for some of the objects in $\mathbf{H}^n_{\mathbb{C}}$ (§3.3: in particular, formulas for polar vectors of complex hyperplanes (§3.3.1), orthogonal projections (§3.3.2), the exponential map (§3.3.3), the growth rate of volume (§3.3.4) and the geodesic between two points (§3.3.5). While orthogonal projection onto complex-linear totally geodesic subspaces is given by a linear projection, orthogonal projection onto totally real totally geodesic subspaces is considerably more complicated (§3.3.6). Figure 3.8 depicts the image of a real hypersurace (a spinal sphere) under this projection.

Chapter 4 introduces the second projective model of $\mathbf{H}^n_{\mathbb{C}}$, the *paraboloid* or *Siegel domain* model. This model is analogous the upper half plane model of the hyperbolic plane. While the ball model is what $\mathbf{H}^n_{\mathbb{C}}$ looks like "from inside," the paraboloid model is the view "from infinity." Just as the stabilizer of the origin in $\mathbf{H}^n_{\mathbb{C}}$ is represented by the linear unitary group $\mathbf{U}(n)$, the stabilizer of ∞ in the Siegel domain is represented by a group of affine transformations of $\mathbb{C}^n$. The Cayley transform (§4.1.1) relates these two models. Horospheres are introduced as level sets of Busemann functions (§4.1.2). The root-space decomposition of $\mathfrak{su}(n,1)$ leads to a description of the automorphisms of $\mathbf{H}^n_{\mathbb{C}}$ in the Siegel model (§4.1.3) and a Hermitian algebraic description (§4.1.4). From the root-space decomposition derive a very useful set of coordinates—*horospherical coordinates*—on $\mathbf{H}^n_{\mathbb{C}}$ (§4.2.1).

The resulting *Heisenberg geometry* is the conformal CR geometry of the Heisenberg group, the natural geometry on the boundary of $\mathbf{H}^n_{\mathbb{C}}$ (§4.2, §2.6).

Chains—the boundaries of complex geodesics—are described in Heisenberg coordinates as certain ellipses whose vertical projections are Euclidean circles (§4.3.1). We derive a formula for the inversion in a chain (§4.3.2), and use this to describe a moduli space of chains (§4.3.4). Orthogonal projection is described in Heisenberg geometry (§4.3.6) and is used to analyze the complex hyperbolic surfaces which are quotients of Fuchsian groups preserving a chain (§4.3.7). A synthetic geometry of chains "lifts" the synthetic geometry of circles in the Euclidean plane (§4.3.8). Using Heisenberg geometry, we reprove Cartan's theorem ([21]) that a chain-preserving transformation of Heisenberg space is an automorphism (§4.3.9).

The other 1-dimensional objects in Heisenberg geometry are $\mathbb{R}$-circles, the boundaries of totally real totally geodesic 2-planes (§4.4). In Heisenberg geometry, $\mathbb{R}$-circles are either CR-horizontal straight lines (§4.4.3) or CR-horizontal lifts of lemniscates (§4.4.4). Parameters for $\mathbb{R}$-circles analogous to center-radius coordinates for circles and chains are defined, obtaining a moduli space for $\mathbb{R}$-circle(§4.4.8). Unlike chains, infinitely many $\mathbb{R}$-circles pass through a pair of points (or through a point with given horizontal direction) (§4.4.11). Discrete groups whose limit set is an $\mathbb{R}$-circle are described (§4.4.13) and depicted in Figs 4.10–4.12.

Chapter 5 develops the theory of bisectors and spinal spheres. Bisectors enjoy two foliations by totally geodesic submanifolds (§5.1) and their boundaries—*spinal spheres*—enjoy corresponding foliations by hyperchains and $\mathbb{R}$-spheres. Any two bisectors in $\mathbf{H}^n_{\mathbb{C}}$ are equivalent by an element of $\mathbf{PU}(n,1)$, and in a very precise sense bisectors (real codimension 1) are *dual* to geodesics (real dimension 1) (§5.1.4). Following Mostow [128], the geodesic associated to a bisector or spinal sphere is called its *spine.* Since a geodesic is determined by its pair of endpoints on $\partial\mathbf{H}^n_{\mathbb{C}}$, a bisector or spinal sphere is determined by its spine's endpoints, which we call its *vertices.*

Various examples of spinal spheres are computed in Heisenberg coordinates. The horizontal plane $v = 0$ is perhaps the simplest spinal sphere (§5.1.7) and has vertices at the origin and ∞. More generally, spinal spheres with one vertex at ∞ are the The *unit spinal sphere* defined by

$$\|\zeta\|^4 + v^2 = 1$$

(and depicted in Fig. 5.3) does not contain ∞ (§5.1.8). Applying Heisenberg similarities to this spinal sphere, one obtains the class of *vertical spinal spheres,* characterized as those whose complex spine is a vertical chain (§5.1.9). These spinal spheres are metric spheres with respect to a useful metric on Heisenberg space first defined by Cygan [37] (§5.1.9). *Great spinal spheres* play a role analogous to great circles in 1-dimensional geometry (§5.1.10). *Infinite spinal spheres* are spinal spheres passing through ∞ (but not having ∞ as a vertex) (§5.1.12); such spinal spheres are depicted in Figs 5.5–5.8 and Fig. 5.4.

The group of automorphisms of a given bisector $\mathfrak{E}$ one has principal orbits of codimension one (§5.2). The inversion in each slice preserves $\mathfrak{E}$, interchanging its vertices (§5.2.2). Similary $\mathfrak{E}$ is invariant under inversion in each meridian (§5.2.3). Analogous to the fact that two distinct points in $\mathbf{H}^n_{\mathbb{C}}$ determine a unique geodesic, two ultraparallel hyperplanes are slices of a unique bisector (§5.2.4). The slices of $\mathfrak{E}$ are characterized as those hyperplanes whose inversions interchange the vertices of $\mathfrak{E}$ (§5.2.5). The meridians of a spinal sphere are also characterized in terms of inversions in slices, leading to lemmas on tangent $\mathbb{R}$-circles (§5.2.6) which are used in the proof that some bisector intersections are connected.

How bisectors intersect is a principal theme in this work. The simplest case arises when the spines lie in the same complex geodesic (§5.3.1). Another important case arises when two bisectors contain a common meridian (§5.3.5). Particularly interesting is when the spines are ultraparallel geodesics in a totally real totally geodesic subspace P; then the bisectors intersect in a union of P with a complex geodesic. These pairs of bisectors are used in a new proof of Cartan's theorem ([21]) on a remarkable configuration of 7 $\mathbb{R}$-circles and 1 chain (§5.3.4). Following Cartan, this result implies that an $\mathbb{R}$-circle-preserving transformation of Heisenberg space must be an automorphism (§5.3.3).

Spinal spheres also arise analytically as level sets of functions related to calibrations of the CR-structure (§5.4). If ω is a 1-form calibrating the CR-structure at ∞ and g is an automorphism, then the set $\mathcal{S}$ of points for which $g^*\omega = \omega$ is defined as the *isometric sphere* of g with respect to ω. When ω is the calibration associated to a point O in $\mathbf{H}^n_{\mathbb{C}}$ (corresponding to the Kähler potential associated to O (§3.1.8)) then $\mathcal{S}$ is the spinal sphere bounding the bisector equidistant from O and $g^{-1}(O)$. Since such bisectors arise in the construction of Dirichlet fundamental domains, we call such spinal spheres *Dirichlet isometric spheres* (§5.4.2). These spinal spheres arise as the level sets of the Poisson kernel function (§5.4.3). As O tends to ∞, the Dirichlet isometric spheres converge to spinal spheres analogous to the isometric circles defined classically by Ford, and we call these spinal spheres *Ford isometric spheres* (§5.4.4). The association of a Ford isometric sphere to an automorphism closely relates to the Bruhat decomposition of $\mathbf{SU}(n,1)$ (§5.4.5). Just as for interior points and boundary points, complex-linear totally geodesic subspaces (§5.4.6) and totally real totally subspaces (§5.4.7) determine calibrations.

Although bisectors are not totally geodesic, their two foliations by totally geodesic submanifolds severely constrain their local geometry. The horospherical coordinates developed in §4.2.1 define "geographical coordinates" on a bisector, in which many of the components of the second fundamental form vanish (§5.5). All of these results have analogues in complex elliptic space (§5.5.4). Finally, a bisector is highly nonconvex, and its geographical coordinates describe qualitatively the regions containing geodesics with endpoints on it (§5.5.5).

Chapter 6 further pursues the automorphisms of $\mathbf{H}^n_{\mathbb{C}}$ and divides into two independent parts. The first part (§6.1) studies $\mathfrak{su}(n,1)$ from the point of view of the symplectic geometry of $\mathbf{H}^n_{\mathbb{C}}$. In particular, one-parameter subgroups of $\mathbf{PU}(n,1)$ define Hamiltonian flows (using the momentum map calculations from §2.3.1). The prototype of this is the explicit symplectic duality between bisectors and geodesics (§6.1.3), which leads to distance formulas for geodesics, totally geodesic submanifolds and bisectors (§6.1.4, §6.1.5).

The second part of Chapter 6 concerns conjugacy classes in $\mathbf{SU}(2,1)$. Algebraic calculations in $\mathbf{SL}(2,\mathbb{C})$ involve the trace function $\mathbf{SL}(2,\mathbb{C}) \to \mathbb{C}$; generically two elements of $\mathbf{SL}(2,\mathbb{C})$ are conjugate if and only if their traces agree. The trace function $\mathbf{SU}(n,1) \to \mathbb{C}$ satisfies analogous properties (§6.2.3), and provides an analogous trichotomy into elliptic, parabolic and hyperbolic conjugacy classes (§6.2.1). As for $\mathbf{SL}(2,\mathbb{R})$—but not $\mathbf{SL}(2,\mathbb{C})$—nonelementary groups containing no elliptic elements are discrete (§6.2.2).

Chapter 7 discusses three important numerical invariants. The angular invariant of Cartan [21] parametrizes triples of ideal points (§7.1) and admits a geometric interpretation as an area of an orthogonal projection of a triangle

(§7.1.2). Ideal triples are fully symmetric and have a unique *barycenter* (§7.1.3). Cartan's invariant relates to the characteristic class of flat bundles (§7.1.4) studied by Toledo [163]. This invariant also relates to a notion of "parallel transport" in Heisenberg geometry (§7.1.5) and geodesic projections (§7.1.6). Ideal triangle groups are also parametrized by the angular invariant (§7.1.7).

Generalizing the ordinary cross-ratio, Korányi and Reimann introduced in [105] a complex number associated to an ordered 4-tuple in Heisenberg space. We describe this invariant (§7.2) from a somewhat different point of view, interpret it geometrically (§7.2.1), determine the meaning of its reality (§7.2.2), and describe a general setting for invariants of its type (§7.2.3).

Pairs consisting of a real geodesic γ and a complex hyperplane c are parametrized in terms of a similarly definied invariant denoted $\eta(q_1, q_1; c)$ where q_1 and q_2 are the endpoints of γ (§7.3). This invariant relates to Cartan's invariant (§7.3.2), to the distance $\rho(\gamma, c)$ (§7.3.3) and to the distance $\rho(\gamma_{\mathbb{C}}, c)$ where $\gamma_{\mathbb{C}}$ is the complex geodesic containing γ (§7.3.4). The orthogonal projection $\Pi_c(\gamma)$ of γ onto c is an arc of a hypercycle, whose length and curvature are functions of $\eta(q_1, q_1; c)$ (§7.3.5). Similarly, as observed by Mostow [128], bisectors meet complex geodesics in hypercycles, whose geodesic curvature can be computed in terms of this invariant (§7.3.6). The condition that the bisector $\mathfrak{E}$ with spine γ meets c is equivalent to the condition that the complex number $\eta = \eta(q_1, q_1; c)$ lies inside the parabolic region $\mathfrak{P}$ of $\mathbb{C}$ defined by

$$\mathrm{Im}(\eta)^2 + 2\mathrm{Re}(\eta) < 1.$$

η is real precisely when $\Pi_c(\gamma)$ is a geodesic, or equivalently when $\gamma \cap c$ is a geodesic (§7.3.7). The distance $\rho(\mathfrak{E}, c)$ is a function of η (§7.3.8) and η admits an expression by complex cross-ratios (§7.3.9).

The invariant η was developed to study intersections of bisectors. Convexity of orthogonal projections of bisectors onto complex geodesics closely relates to the the connectedness of bisector intersections (§7.3.10). Examples of these projections are depicted in Fig. 7.4–7.9.

Chapter 8 begins the general theory of *extors* in $\mathbb{P}^n_{\mathbb{C}}$. Extors extend and generalize metric bisectors in both complex hyperbolic and elliptic geometry just as circles in $\mathbb{P}^1_{\mathbb{C}}$ extend and generalize geodesics in $\mathbf{H}^1_{\mathbb{C}}$ and $\mathbf{E}^1_{\mathbb{C}}$ (§8.1). Extors are not smooth, but their singular strata are homogeneous spaces (§8.2) and are thus smooth. Although defined in terms of a "slice decomposition," they also admit a "meridian decomposition," generalizing the two decompositions of bisectors in $\mathbf{H}^n_{\mathbb{C}}$ and $\mathbf{E}^n_{\mathbb{C}}$ (§8.2.3). Pairs of extors are classified into 4 basic types (§8.3) which organize the more complicated intersection phenomena of metric bisectors. In complex dimension 2, extors generically intersect in *Clifford tori* (§8.3.4). We reprove Giraud's theorem [65] that for such a generic pair of extors $\mathfrak{E}_1, \mathfrak{E}_2$, there is a unique extor $\mathfrak{E}_3 \neq \mathfrak{E}_1, \mathfrak{E}_2$ containing $\mathfrak{E}_1 \cap \mathfrak{E}_2$ (§8.3.5). In higher dimensions, a generic pair of extors will have transverse foci (§8.3.7) but such pairs will not arise from Dirichlet–Ford constructions.

Chapter 9 develops the theory of bisector intersections in $\mathbf{H}^n_{\mathbb{C}}$. Using ideas of Mostow [128], each component of a bisector intersection is contractible. Examples of disconnected bisector intersections are constructed (§9.1.2) and interpreted in terms of orthogonal projections of bisectors (§9.1.3). Associated to a bisector and a complex geodesic is a component of a hyperbola of invariants η. The intersection of this curve with $\mathfrak{P}$ is the invariant used to prove that bisector intersections have at most two components (§9.1.4).

The proof that intersections of coequidistant/covertical pairs are connected is a continuity argument (§9.2). First, tangencies between spinal spheres are analyzed using Heisenberg geometry (§9.2.1). Then a coequidistant or covertical pair is deformed to a pair of bisectors with orthogonal complex spines, where the intersection can be analyzed explicitly (§9.2.4). The basic transversality follows from the the bisectors not having a slice in common. The "cotranchal" case— when two bisectors possess a common slice— must be handled separately (§9.2.5). These properties of bisectors constrain the combinatorics of Dirichlet–Ford fundamental polyhedra (§9.3). Giraud's theorem implies that cycles of side-pairing transformations have length at most 3, and the above connectedness theorem implies that for some discrete subgroups of $\mathbf{PO}(2,1)$ the combinatorics of the Dirichlet–Ford polyhedra in $\mathbf{H}^2_{\mathbb{C}}$ correspond to the combinatorics in $\mathbf{H}^2_{\mathbb{R}}$. A summary of Giraud's paper [65] is appended for the reader's convenience.

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W.M.G.

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CONTENTS

1

THE COMPLEX PROJECTIVE LINE

This preliminary section discusses 1-dimensional complex geometry from the point of view adopted in this book. We begin with projective geometry, from which both complex elliptic and hyperbolic 1-dimensional geometry derive. We base our treatment on the books of Cartan [24] and Coolidge [31]. For an excellent treatment of projective geometry in the real case, see Coxeter [34]. A principal geometric object is a *circle* (which Cartan calls a "chain" in [24]) which is just a Euclidean circle or straight line. We find several algebraic ways of describing circles, which will be useful in the sequel. In particular we describe circles in terms of 2×2 Hermitian matrices. We shall make essential use of the *null polarity* (the unique object in projective geometry corresponding to the symplectic structure on $\mathbb{C}^2$) to identify anti-polarities with anti-involutions. Furthermore, elliptic and hyperbolic geometry are developed in terms of *anti-polarities* which are the projective counterpart of Hermitian forms.

1.1 Projective geometry

1.1.1 *Linear algebra in* $\mathbb{C}^2$

If $\mathbb{V}$ is a vector space, a *line* in $\mathbb{V}$ is a 1-dimensional linear subspace. The *projective space* $\mathbb{P}(\mathbb{V})$ *associated to* $\mathbb{V}$ is the space of all lines in $\mathbb{V}$. Since a nonzero vector spans a unique line, and two nonzero vectors span the same line, $\mathbb{P}(\mathbb{V})$ consists of the orbits of the group of nonzero scalars acting by scalar multiplication on the set of nonzero vectors in $\mathbb{V}$. We denote the point in $\mathbb{P}(\mathbb{V})$ corresponding to the line spanned by a nonzero vector $v \in \mathbb{V}$ by $[v]$. In this way, we may give $\mathbb{P}(\mathbb{V})$ the quotient topology.

The *complex projective line* $\mathbb{P}^1_{\mathbb{C}}$ is the projective space $\mathbb{P}(\mathbb{C}^2)$ associated to a 2-dimensional complex vector space. Thus a point in $\mathbb{P}^1_{\mathbb{C}}$ corresponds to a (complex) line in $\mathbb{V}$. We shall work in the standard affine chart whereby

$$\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

via the embedding

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{P}^1_{\mathbb{C}} \\ z &\longmapsto \begin{bmatrix} z \\ 1 \end{bmatrix} \\ \infty &\longmapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

This is the classical *Gaussian representation* of $\mathbb{P}^1_{\mathbb{C}}$.

1.1.2 *Collineations*

The projective automorphism group of $\mathbb{P}^1_{\mathbb{C}}$ is the group of transformations of $\mathbb{P}^1_{\mathbb{C}}$ induced by the linear automorphism group $\mathbf{GL}(2,\mathbb{C})$ of $\mathbb{V}$. Projective transformations are sometimes also called *collineations* or *homographies.* For example, a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}(2,\mathbb{C})$$

acts on points in the Gaussian representation by the *linear fractional transformation*

$$z \longmapsto \frac{az+b}{cz+d}.$$

This action is not faithful; its kernel consists of all scalar matrices

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \neq 0$$

and its image is denoted by $\mathbf{PGL}(2,\mathbb{C})$.

Exercise 1.1.1 *The Jordan normal form for* 2×2 *matrices furnishes a normal form for elements of* $\mathbf{SL}(2,\mathbb{C})$. *Let* $A \in \mathbf{SL}(2,\mathbb{C})$. *Then there are the following possibilities:*

1. $A = \pm I$. *In that case* A *acts trivially on* $\mathbb{P}^1_{\mathbb{C}}$.
2. $A \neq \pm I$ *but* A *has only one eigenvalue. In that case the eigenvalue must be* ± 1. *Then* A *has exactly one fixed point on* $\mathbb{P}^1_{\mathbb{C}}$ *and is conjugate to the matrix*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which represents the translation $z \mapsto z+1$ *in affine coordinates. In this case* A *is said to be* parabolic.

3. A *has two distinct unreal eigenvalues of unit length. In that case the eigenvalues are complex conjugates* $e^{\pm i\theta}$ *and* A *has exactly two fixed points on* $\mathbb{P}^1_{\mathbb{C}}$ *and is conjugate to the diagonal matrix*

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

representing a rotation $z \mapsto e^{2i\theta}z$ *in affine coordinates. In this case* A *is said to be* elliptic.

4. *A has two distinct real eigenvalues, necessarily reciprocals. Then A has exactly two fixed points on* $\mathbb{P}^1_{\mathbb{C}}$ *and is conjugate to the diagonal matrix*

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

representing a dilation $z \mapsto \lambda^2 z$ *in affine coordinates. In this case A is said to be* (purely) hyperbolic.

5. *A has two distinct eigenvalues, neither real nor of unit length. Then A has exactly two fixed points on* $\mathbb{P}^1_{\mathbb{C}}$ *and is conjugate to the diagonal matrix*

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

representing a dilation $z \mapsto \lambda^2 z$ *in affine coordinates. In this case A is said to be* loxodromic.

The collineation associated to A is parabolic or the identity if and only if $\operatorname{trace}(A) = \pm 2$*; it is elliptic if and only if* $-2 < \operatorname{trace}(A) < 2$*; it is purely hyperbolic if and only if* $\operatorname{trace}(A) \in \mathbb{R}$ *and* $|\operatorname{trace}(A)| > 2$*. Otherwise it is loxodromic.*

1.1.3 *Anti-collineations*

Projective transformations of $\mathbb{P}^1_{\mathbb{C}}$ are necessarily complex analytic—their differentials preserve the complex structure on the tangent spaces to projective space. In addition to the projective transformations are the *anti-projective transformations* (sometimes called *anti-collineations* or *anti-homographies*), which are the anti-holomorphic transformations induced by conjugate-linear mappings of $\mathbb{V}$. An anti-collineation of order 2 is called an *anti-involution*. The simplest anti-involution is defined by complex conjugation on $\mathbb{C}^2$:

$$\rho : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$$
$$z \longmapsto \bar{z}$$

in inhomogeneous coordinates and by

$$\rho : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$
$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{Z}_1 \\ \bar{Z}_2 \end{bmatrix} \tag{1.1}$$

in homogeneous coordinates. The group of transformations of $\mathbb{P}^1_{\mathbb{C}}$ consisting of collineations and anti-collineations is generated by ρ and $\mathbf{PGL}(2, \mathbb{C})$. Another anti-involution is *inversion* in the unit circle, introduced by Steiner. In affine coordinates this inversion

$$z \longmapsto 1/\bar{z} = \frac{z}{|z|^2}$$

is induced by the anti-linear involution of $\mathbb{C}^2$:

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{Z}_2 \\ \bar{Z}_1 \end{bmatrix}. \tag{1.2}$$

Both these anti-involutions fix points in $\mathbb{P}^1_{\mathbb{C}}$.

Exercise 1.1.2 *Find an anti-involution of $\mathbb{P}^1_{\mathbb{C}}$ which has no fixed points.*

Just as collineations are represented by matrices, an anti-linear map of $\mathbb{C}^2$ is determined by a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

acting on $Z \in \mathbb{C}^2$ by

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \longmapsto \begin{bmatrix} a\bar{Z}_1 + b\bar{Z}_2 \\ c\bar{Z}_1 + d\bar{Z}_2 \end{bmatrix}.$$

Exercise 1.1.3 *Determine, in terms of algebraic properties of A, when the anti-collineation induced by A*

1. *is an anti-involution with fixed points;*
2. *is an anti-involution without fixed points.*

Anti-involutions with fixed points we call *hyperbolic* and one without fixed points we call *elliptic*. (Cartan [24] calls these anti-involutions "de première espèce" and "de seconde espèce" respectively.)

1.1.4 *Dual projective space*

Let $\mathbb{V}$ be a vector space. The vector space $\mathbb{V}^*$ *dual to* $\mathbb{V}$ consists of linear functionals $\mathbb{V} \longrightarrow \mathbb{C}$. We interpret the projective space $\mathbb{P}(\mathbb{V}^*)$ in terms of *hyperplanes* in $\mathbb{V}$ as follows. A *hyperplane* in a vector space $\mathbb{V}$ is a linear subspace of codimension 1. If $\psi \in \mathbb{V}^*$ is nonzero, then $\mathrm{Ker}(\psi)$ is a hyperplane and two linear functionals define the same hyperplane if and only if they are nonzero scalar multiples of one another. Thus the lines in $\mathbb{V}^*$ correspond bijectively to the hyperplanes in $\mathbb{V}$.

We also consider isomorphisms of a projective space with its dual. Consider a map $f : \mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\mathbb{V}^*)$, a rule which associates to every point in $\mathbb{P}(\mathbb{V})$ a hyperplane in $\mathbb{P}(\mathbb{V})$. Such a map is required to be *projective* in the sense that it preserves incidences: if $l_1, l_2, l_3 \in \mathbb{P}(\mathbb{V})$ are collinear points (that is, if $l_i = [v_i]$ for $v \in \mathbb{V}$, then v_1, v_2, v_3 are linearly dependent), then the hyperplanes $f(l_1), f(l_2), f(l_3)$ intersect in a projective subspace of codimension 2. It is a remarkable fact that such projective isomorphisms (called *correlations*) correspond to linear isomorphisms $\mathbb{V} \longrightarrow \mathbb{V}^*$. (See Coxeter [35].)

When $\mathbb{V}$ has dimension 2, lines and hyperplanes coincide. There results a *canonical* identification of a 1-dimensional projective space with its dual.

This correlation is called the *null polarity* and is defined as follows. If $\mathbb{V}$ is 2-dimensional, then a *symplectic structure* on $\mathbb{V}$, that is, a nondegenerate skew-symmetric pairing

$$\nu : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{C},$$

induces an isomorphism

$$\hat{\nu} : \mathbb{V} \longrightarrow \mathbb{V}^*$$

having the property that

$$\hat{\nu}(v)(v) = 0.$$

Furthermore since $\dim(\mathbb{V}) = 2$, the space of skew-symmetric pairings is itself 1-dimensional, so that any nondegenerate skew-symmetric pairing ν is a nonzero multiple of a fixed one ν_0.

A linear map $f : \mathbb{V} \longrightarrow \mathbb{V}^*$ identifies with a bilinear form $\tilde{f} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{C}$ via the rule

$$\tilde{f}(v_1, v_2) = f(v_1)(v_2).$$

Such bilinear forms correspond to square matrices F by

$$\tilde{f}(v_1, v_2) = v_2^{\dagger} F v_1.$$

The correlation is an isomorphism if and only if the corresponding bilinear form is nondegenerate.

For example the null polarity corresponds to the skew-symmetric matrix

$$J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1.3}$$

(or, more accurately, any nonzero scalar multiple of J_0). The correlation $\mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\mathbb{V}^*)$ corresponding to a bilinear form $\tilde{f}$ defines (by composition with the inverse of the null polarity $\mathbb{P}(\mathbb{V}^*) \longrightarrow \mathbb{P}(\mathbb{V})$) a collineation $\mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\mathbb{V})$ corresponding to the linear transformation defined by the matrix $F \circ J_0$.

Definition 1.1.4 *$p \in \mathbb{P}(\mathbb{V})$ and $H \in \mathbb{P}(\mathbb{V}^*)$ are* incident *if $p \in H$.*

Exercise 1.1.5 *A* polarity *is a correlation $c : \mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\mathbb{V}^*)$ whose inverse $c^{-1} : \mathbb{P}(\mathbb{V}^*) \longrightarrow \mathbb{P}(\mathbb{V})$ "corresponds to itself;" that is, $p \in \mathbb{P}(\mathbb{V})$ and $H \in \mathbb{P}(\mathbb{V}^*)$ are incident if and only if $c^{-1}(H) \in \mathbb{P}(V)$ and $c(p) \in \mathbb{P}(V^*)$ are incident. Then a nondegenerate bilinear form B defines a polarity if and only if B is symmetric or skew-symmetric. Compare [35].*

Exercise 1.1.6 *Let $A \in \mathbf{GL}(2, \mathbb{C})$. Then*

$$A^{\dagger} J_0 A = \det(A) J_0.$$

1.1.5 *The conjugate projective space*

One may also define the *conjugate* projective space $\overline{\mathbb{P}^1_{\mathbb{C}}}$, which has the same underlying set as $\mathbb{P}^1_{\mathbb{C}}$ but enjoys the opposite complex structure. In other words, the identity mapping $\overline{\mathbb{P}^1_{\mathbb{C}}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ induces an anti-projective isomorphism. In this way an anti-projective mapping is defined as a projective mapping defined on the *conjugate* projective space.

An *anti-correlation* of a projective space $\mathbb{P}(\mathbb{V})$ is thus a projective isomorphism from $\mathbb{P}(\mathbb{V})$ to its conjugate dual $\overline{\mathbb{P}(\mathbb{V}^*)}$. An anti-correlation $\mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\bar{\mathbb{V}}^*)$ corresponds to a *sesquilinear form;* that is, a form $H : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{C}$ which is bilinear over $\mathbb{R}$ and is *Hermitian-symmetric:*

$$H(\lambda_1 v_1, \lambda_2 v_2) = \lambda_1 \overline{\lambda_2} H(v_1, v_2)$$

for $v_1, v_2 \in \mathbb{V}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Such a form is described by a matrix F as follows:

$$H(v_1, v_2) = (\bar{v}_2)^\dagger F v_1.$$

Composition of an anti-correlation $\phi : \mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\mathbb{V}^*)$ with the inverse of the null polarity $\nu^{-1} : \mathbb{P}(\mathbb{V}^*) \longrightarrow \mathbb{P}(\mathbb{V})$ yields an anti-collineation $\nu^{-1} \circ \phi : \mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\mathbb{V})$.

Exercise 1.1.7 *Show that the anti-collineation ϕ is an anti-polarity (that is, equals its own inverse) if and only if the anti-collineation $\nu^{-1} \circ \phi$ is an anti-involution.*

The map on projective spaces is an isomorphism if and only if H is a nondegenerate. The anti-correlation is an anti-polarity if and only if H is a Hermitian form. In that case F is a *Hermitian matrix:*

$$\bar{F}^\dagger = F.$$

Exercise 1.1.8 *Let ϕ be the anti-polarity defined by the standard Hermitian form*

$$\langle\langle u, v \rangle\rangle = u_1 \bar{v_1} + u_2 \bar{v_2}$$

on $\mathbb{C}^2$. Show that the corresponding anti-involution on the $\mathbb{P}^1_{\mathbb{C}}$ equals

$$\nu^{-1} \circ \phi : z \longmapsto 1/\bar{z}.$$

If F is the Hermitian matrix, then the matrix A representing the anti-involution is given by

$$F = A^\dagger J_0. \tag{1.4}$$

The condition that $z \mapsto A\bar{z}$ has order 2 is equivalent to $A\bar{A} = I$, which is equivalent to F being Hermitian.

1.1.6 *The cross-ratio*

Let $C_4(\mathbb{P}^1_{\mathbb{C}})$ denote the subset of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ consisting of quadruples of distinct points. The *cross-ratio* is the mapping

$$C_4(\mathbb{P}^1_{\mathbb{C}}) \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

defined in Gaussian coordinates by

$$\mathbf{X}\{z_1, z_2, z_3, z_4\} = \frac{z_4 - z_1}{z_4 - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2} \tag{1.5}$$

and is invariant under the diagonal action of $\mathbf{PGL}(2, \mathbb{C})$ on $C_4(\mathbb{P}^1_{\mathbb{C}})$. (For a geometric description of this classical invariant, see §1.1.7.) In particular,

$$\mathbf{X}\{0, \infty, 1, z\} = z. \tag{1.6}$$

The symmetric group $\mathfrak{S}_4$, consisting of all permutations of $\{1, 2, 3, 4\}$, acts on $C_4(\mathbb{P}^1_{\mathbb{C}})$ by

$$\sigma : (z_1, z_2, z_3, z_4) \longmapsto (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}).$$

The normal subgroup

$$\mathfrak{D} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \lhd \mathfrak{S}_4$$

comprising products of pairs of disjoint transpositions leaves the cross-ratio invariant. $\mathfrak{S}_4$ decomposes as a semidirect product $\mathfrak{S}_4 \cong \mathfrak{S}_3 \ltimes \mathfrak{D}$ where $\mathfrak{S}_3 \subset \mathfrak{S}_4$ is the subgroup fixing 4. In particular $\mathfrak{S}_3$ is a system of coset representatives for $\mathfrak{S}_4/\mathfrak{D}$ so the equivariance of cross-ratio with respect to $\mathfrak{S}_4$ is described by the six formulas:

$$\begin{aligned}
\mathbf{X}\{0, \infty, 1, z\} &= z \\
\mathbf{X}\{\infty, 0, 1, z\} &= 1/z \\
\mathbf{X}\{1, \infty, 0, z\} &= 1 - z \\
\mathbf{X}\{\infty, 1, 0, z\} &= 1/(1 - z) \\
\mathbf{X}\{0, 1, \infty, z\} &= z/(z - 1) \\
\mathbf{X}\{1, 0, \infty, z\} &= (z - 1)/z.
\end{aligned}$$

Exercise 1.1.9 *A transformation $f : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ is*

1. *a collineation if and only if*

$$\mathbf{X}\{f(z_1), f(z_2), f(z_3), f(z_4)\} = \mathbf{X}\{z_1, z_2, z_3, z_4\};$$

2. *an anti-collineation if and only if*

$$\mathbf{X}\{f(z_1), f(z_2), f(z_3), f(z_4)\} = \overline{\mathbf{X}\{z_1, z_2, z_3, z_4\}}.$$

Exercise 1.1.10 *Let $z_0, z_1, z_\infty \in \mathbb{P}^1_{\mathbb{C}}$ be distinct. Show that*

$$z \longmapsto \mathbf{X}\{z, z_0, z_1, z_\infty\}$$

is the unique collineation mapping z_0 to 0, z_1 to 1 and z_∞ to ∞.

1.1.7 *Cross-ratio and slopes*

Here is a geometric description of the cross-ratio. The points z_1, z_2, z_3, z_4 in $\mathbb{P}^1_{\mathbb{C}}$ correspond to distinct lines $L_i \subset \mathbb{C}^2$. Since $L_1 \neq L_2$, the vector space $\mathbb{C}^2$ decomposes as a direct sum

$$\mathbb{C}^2 = L_1 \oplus L_2.$$

Every line in $\mathbb{C}^2$ distinct from L_1, L_2 is the graph of a nonzero linear map $f : L_1 \longrightarrow L_2$:

$$L = \mathrm{graph}(f) = \{(u, f(u)) \mid u \in L_1\} \subset L_1 \oplus L_2.$$

Thus $L_3 = \mathrm{graph}(f_3)$ and $L_4 = \mathrm{graph}(f_4)$ correspond to nonzero linear maps $f_3, f_4 : L_1 \longrightarrow L_2$. Two nonzero linear maps between lines differ by multiplication by a scalar, which will be the cross-ratio above. If

$$f_4(x) = \lambda_{4,3} f_3(x)$$

for $x \in L_1$, then we define

$$\mathbf{X}\{z_1, z_2, z_3, zp_4\} = \lambda_{4,3}.$$

Exercise 1.1.11 *Cross-ratio extends to the larger subset of* $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ *consisting of ordered quadruples where at most two entries are equal.*

Explicitly, suppose that $z_1, z_2 \in \mathbb{C}$ define vectors

$$e_1 = \begin{bmatrix} z_1 \\ 1 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} z_2 \\ 1 \end{bmatrix}$$

spanning lines $L_1, L_2 \subset \mathbb{C}^2$. A linear map $f : L_1 \longrightarrow L_2$ is determined by a scalar λ (called the *multiplier*) such that

$$f(e_1) = \lambda e_2. \tag{1.7}$$

Let L be the line spanned by

$$\begin{bmatrix} z \\ 1 \end{bmatrix} \longleftrightarrow z \in \mathbb{P}^1_{\mathbb{C}}.$$

Then $L = \mathrm{graph}(f)$ where f is defined as in (1.7) and

$$\lambda = \frac{z - z_1}{z - z_2}.$$

Thus if L_3 corresponds to z_3 and L_4 corresponds to z_4 their respective multipliers are

$$\lambda_{4,3} = \frac{\lambda_3}{\lambda_4} = \frac{z_4 - z_1}{z_4 - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}.$$

1.2 Circles

It follows from Exercise 1.1.9 that if four points are fixed by an anti-collineation, then their cross-ratio is real. We define a *circle* in $\mathbb{P}^1_{\mathbb{C}}$ to be the fixed-point set of an anti-collineation. In the usual Gaussian representation a circle is either a Euclidean straight line or a Euclidean circle. A circle passing through ∞ is a straight line, that is a degenerate Euclidean circle. (Compare Coolidge [31] or Young [171].) These ubiquitous geometric entities arise as geodesics, metric circles and bisectors in elliptic and hyperbolic geometry. Points in $\mathbb{P}^1_{\mathbb{C}}$ are limits of degenerating sequences of circles (circles of zero radius). Cartan [24] refers to these objects as *chains,* but since we wish to use the terminology "chain" later on for different objects (following Cartan [21]), we call these objects *circles* rather than chains. We reserve the terminology "Euclidean circle" and "metric circle" for the more specialized (and common) usages of this term.

1.2.1 *Circles and anti-involutions*

Let $\phi : \mathbb{P}(\mathbb{V}) \longrightarrow \mathbb{P}(\bar{\mathbb{V}}^*)$ be an anti-polarity. Its *null locus* $N(\phi)$ consists of all points $x \in \mathbb{P}(\mathbb{V})$ such that x is incident to $\phi(x)$. If ϕ is defined by a nondegenerate form $\Phi : \mathbb{V} \times \bar{\mathbb{V}} \longrightarrow \mathbb{C}$, then

$$N(\phi) = \{[v] \mid v \in \mathbb{V}, \Phi(v, v) = 0\}.$$

An anti-polarity is *hyperbolic* (respectively *elliptic*) if $N(\phi)$ is nonempty (respectively empty).

Exercise 1.2.1 *Suppose* $\dim \mathbb{V} = 2$ *and let* ν *denote the null polarity. The null locus* $N(\phi)$ *equals the fixed-point set of the corresponding anti-involution* $\nu^{-1} \circ \phi$.

Exercise 1.2.2 *Let* $C \subset \mathbb{P}^1_{\mathbb{C}}$ *be a 1-dimensional (real) submanifold of* $\mathbb{P}^1_{\mathbb{C}}$. *Then the following are equivalent:*

1. C *is a circle.*
2. *There is a (necessarily) unique anti-involution* ι *whose fixed-point set is* C.
3. *For all distinct* $z_1, z_2, z_3, z_4 \in C$, *the cross-ratio* $\mathbf{X}\{z_1, z_2, z_3, z_4\} \in \mathbb{R}$.
4. *There exists a collineation* $\phi \in \mathbf{PGL}(2, \mathbb{C})$ *such that* $\phi(C) = \mathbb{P}^1_{\mathbb{R}}$.

Suppose $z_1, z_2, z_3 \in \mathbb{P}^1_{\mathbb{C}}$ are distinct points. Then

$$\{z \in \mathbb{P}^1_{\mathbb{C}} \mid \mathbf{X}\{z_1, z_2, z_3, z\} \in \mathbb{R}\} \cup \{z_1, z_2, z_3\}$$

is the unique circle containing z_1, z_2, z_3.

Exercise 1.2.3 *Derive a formula for the anti-involution fixing this circle.*

Exercise 1.2.4 *Let* $C \in \mathbb{P}^1_{\mathbb{C}}$ *be a circle and let* $c_1, c_2 \in C$ *be distinct points. Let* $v_1, v_2 \in \mathbb{C}^2$ *be vectors such that* $c_i = [v_i]$. *Let* $S = \mathbb{R}v_1 + \mathbb{R}v_2$ *be the* $\mathbb{R}$*-linear 2-plane spanned by* v_1, v_2. *Then* C *consists of all* $[v]$ *where* $v \in S$.

1.2.2 *Circles and Hermitian matrices*

Since circles correspond to anti-polarities (as well as anti-involutions), they can be effectively represented by matrices. A circle in $\mathbb{P}^1_{\mathbb{C}}$ is the null locus of a hyperbolic anti-polarity, which is represented by an *indefinite Hermitian matrix.* For example, the anti-polarity with null locus the real axis corresponds to the Hermitian form

$$H(Z,W) = i(Z_1\bar{W}_2 - Z_2\bar{W}_1)$$

(the corresponding anti-involution is complex conjugation; see (1.1)). The corresponding matrices are

$$F = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The anti-polarity with null locus the unit circle axis corresponds to the Hermitian form

$$H(Z,W) = Z_1\bar{W}_1 - Z_2\bar{W}_2$$

(the corresponding anti-involution is given in (1.2)) and the corresponding matrices are

$$F = \mathbb{I}_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Suppose that F is a Hermitian matrix representing C. That is, $C = N(\phi)$ where ϕ is the anti-polarity defined by F. Suppose that $T \in \mathbf{GL}(2,\mathbb{C})$ is a matrix representing a collineation of $\mathbb{P}^1_{\mathbb{C}}$, which we also denote by T. Then the Hermitian matrix corresponding to $T(C)$ is $\bar{T}^\dagger F T$.

Exercise 1.2.5 *The Hermitian matrix corresponding to the line $\zeta_0 + e^{i\theta}\mathbb{R}$ is*

$$\begin{bmatrix} 1 & 0 \\ -\bar{\zeta}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 & ie^{i\theta} \\ -ie^{-i\theta} & 0 \end{bmatrix} \begin{bmatrix} 1 & -\zeta_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & ie^{i\theta} \\ -ie^{-i\theta} & -2\mathrm{Im}(e^{-i\theta}\zeta_0) \end{bmatrix}$$

and a Hermitian matrix corresponding to the circle centered at ζ_0 with radius r_0 is

$$\begin{bmatrix} 1 & 0 \\ -\bar{\zeta}_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -r_0^2 \end{bmatrix} \begin{bmatrix} 1 & -\zeta_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\zeta_0 \\ -\bar{\zeta}_0 & \zeta_0\bar{\zeta}_0 - r_0^2 \end{bmatrix}.$$

1.2.3 *Pairs of circles*

Two distinct circles may intersect in none, one or two points. The space of equivalence classes of pairs of circles is 1-dimensional. If C_1, C_2 are circles intersecting in two points, their intersection is transverse, and the angles of intersection at each of the two points of $C_1 \cap C_2$ are equal, giving a well-defined invariant. If

$C_1 \cap C_2$ is a single point, then C_1 and C_2 are tangent. If C_1 and C_2 are disjoint, then there exists a circle c which is orthogonal to both C_1 and C_2.

This trichotomy can be readily understood in terms of the hyperbolic anti-involutions (inversions) α_i determined by C_i. The product $\alpha_1 \circ \alpha_2$ is a collineation which fixes $C_1 \cap C_2$. If $\alpha_1 \circ \alpha_2$ is elliptic, then it is represented by an element of $\mathbf{SL}(2, \mathbb{C})$ with trace $\pm 2\cos(\theta)$, where θ is the angle between C_1 and C_2. $\alpha_1 \circ \alpha_2$ is parabolic if and only if $C_1 \cap C_2$ is a single point. $\alpha_1 \circ \alpha_2$ is hyperbolic if and only if $C_1 \cap C_2 = \emptyset$.

The collineation group $\mathbf{PGL}(2, \mathbb{C})$ acts transitively on circles. Circles C_1, C_2 corresponding to Hermitian matrices F_1, F_2 respectively meet at angle θ if and only if

$$\frac{\operatorname{trace}(J_0 \bar{F}_1 \ J_0 F_2)^2}{\det(J_0 \bar{F}_1 J_0 H_2)} = 4\cos^2(\theta) \tag{1.8}$$

(where J_0 is defined in (1.3)). To prove (1.8), consider the product of the anti-involutions corresponding to the circles C_1, C_2. By (1.4), the anti-involution corresponding to C_i is represented by the linear map

$$\tilde{\rho}_i : v \longmapsto (J_0 \bar{h}_i)\bar{v}.$$

Their composition $\rho_1 \circ \rho_2$ is a collineation which is represented by the matrix

$$J_0 \bar{h}_1 \ J_0 h_2 \ \in \mathbf{GL}(2, \mathbb{C})$$

where J_0 is defined in (1.3). The collineation $\rho_1 \circ \rho_2$ is represented by an elliptic element of $\mathbf{PGL}(2, \mathbb{C})$ fixing $C_1 \cap C_2$, and is conjugate to a rotation of angle 2θ represented by the matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \sim \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

having trace (2θ) when the determinant is normalized to equal one.

1.3 Elliptic geometry

Now we develop the Riemannian geometry on $\mathbb{P}^1_{\mathbb{C}} \approx S^2$ induced by the embedding of S^2 as the sphere $S^2(r)$ of radius $r > 0$ in Euclidean space $\mathbb{R}^3$. To illustrate the formal properties of these metrics, the radius r is a parameter throughout our discussion. Recall that the curvature of $S^2(r)$ equals r^{-2}.

For a detailed discussion of the geometry of S^2, see Coxeter [34], §VI.

1.3.1 *Isometries*

The isometry group of $S^2(r)$ is the orthogonal group $\mathbf{O}(3)$ consisting of two components. The identity component $\mathbf{SO}(3)$ consists of rotations. The other component consists of orientation-reversion isometries, and contains two kinds of involutions: reflections in hyperplanes, and the antipodal map

$$\alpha_O : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}$$

Exercise 1.3.1 *The general orientation-reversing isometry of S^2 is a composition of a reflection in a plane and a rotation leaving invariant the plane. For example,*

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is a reflection in the xy-plane for $\theta = 0$ and the antipodal map for $\theta = \pm\pi$.

If H is a plane in $\mathbb{R}^3$ containing the origin O, then the Euclidean reflection ι_H in H is the unique isometry whose fixed-point set is H. It follows that the *great circle* $H \cap S^2(r)$ is a totally geodesic subset of $S^2(r)$ and thus a geodesic. Since through any point on $S^2(r)$ in any direction tangent to $S^2(r)$ there exists a great circle, the geodesics on $S^2(r)$ are precisely the great circles.

Suppose that $x, y \in S^2(r)$. Their distance—*as measured on* $S^2(r)$—equals the shortest length of an arc of the great circle on $S^2(r)$ joining x to y. Thus $d(x, y)$ equals the radius r multiplied by the angle subtended by the lines spanned by x and y:

$$\cos\left(\frac{d(x,y)}{r}\right) = |x/\|x\| \cdot y/\|y\||$$

which we rewrite as

$$\cos^2\left(\frac{d(x,y)}{r}\right) = \frac{(x \cdot y)(y \cdot x)}{(x \cdot x)(y \cdot y)}. \tag{1.9}$$

This expression is *homogeneous* in x, y, that is it depends only on the lines $[x]$, $[y]$ spanned by x, y respectively. (Such expressions for non-Euclidean distance were considered by Cayley [26]; compare the discussion in Coxeter [34].)

In particular, if $x, y \in S^2(r)$ then $d(x, y) \leq \pi r$, with equality if and only if x, y are antipodal. Thus the (Riemannian) *diameter* of $S^2(r)$ equals πr.

Every geodesic through a point x also contains its antipode $\alpha_O(x)$. Furthermore, if $x, y \in S^2(r)$ are not antipodal, then the plane they generate meets $S^2(r)$ in the unique geodesic joining x, y. Thus, folowing Felix Klein, define the *elliptic plane* $\mathbf{E}^2_{\mathbb{R}}$ as the quotient of $S^2(r)$ by the antipodal map. $\mathbf{E}^2_{\mathbb{R}}$ identifies with the real projective plane. The restriction ds^2 of the Euclidean metric to $S^2(r)$ is invariant under α_O, and defines a Riemannian metric on $\mathbf{E}^2_{\mathbb{R}}$ of curvature r^{-2}. The diameter of $\mathbf{E}^2_{\mathbb{R}}$ equals $\pi r/2$ and the area is $2\pi r^2$.

1.3.2 *Stereographic projection*

To better visualize $\mathbb{P}^1_{\mathbb{C}}$ we relate it to the sphere $S^2(r)$ by stereographic projection. We give $\mathbb{P}^1_{\mathbb{C}}$ the Riemannian metric induced from the Euclidean metric on this sphere. With this metric, $\mathbb{P}^1_{\mathbb{C}}$ is the *complex elliptic line* and denoted $\mathbf{E}^1_{\mathbb{C}}$. Choose a parameter $r > 0$.

Exercise 1.3.2 *Embed $\mathbb{C}$ as the affine plane in $\mathbb{R}^3$ by*

$$\zeta = x + iy \longmapsto \begin{bmatrix} rx \\ ry \\ 0 \end{bmatrix}. \tag{1.10}$$

The South pole

$$S = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}$$

corresponds to the origin in $\mathbb{C}$. The North pole

$$N = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$$

corresponds to $\infty \in \mathbb{P}^1_{\mathbb{C}}$. Define stereographic projection $\Sigma : \mathbb{C} \cup \{\infty\} \longrightarrow S^2(r)$ *as follows. If $\zeta \in \mathbb{C}$, then $\Sigma(\zeta)$ is the unique point in which the line joining the North pole to the point corresponding to ζ in $\mathbb{R}^3$ intersects $S^2(r)$. Explicitly*

$$\begin{aligned} \Sigma : \mathbb{P}^1_{\mathbb{C}} &\longrightarrow S^2(r) \\ \zeta &\longmapsto \frac{r}{|\zeta|^2 + 1} \begin{bmatrix} \mathrm{Re}(\zeta) \\ \mathrm{Im}(\zeta) \\ |\zeta|^2 - 1 \end{bmatrix} \\ \infty &\longmapsto \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \end{aligned}$$

and its inverse is

$$\begin{aligned} \Sigma^{-1} : S^2 &\longrightarrow \mathbb{P}^1_{\mathbb{C}} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\longmapsto \frac{x + iy}{r - z}. \end{aligned}$$

We chose the embedding (1.10) so that the equator $S^2(r) \cap (\mathbb{R}^2 \times \{0\})$ corresponds to the unit circle $|\zeta| = 1$ in $\mathbb{C}$.

Exercise 1.3.3 *Let ds^2 denote the Euclidean metric tensor (first fundamental form) on $S^2(r)$. Show that the Riemannian metric on $\mathbb{P}^1_{\mathbb{C}}$ induced by stereographic projection Σ equals*

$$\Sigma^*(ds^2) = \frac{4r^2}{(1+\zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}. \tag{1.11}$$

In particular the restriction of Σ to $\mathbb{C}$ is conformal with respect to the Euclidean metric on $\mathbb{C}$. As $r \longrightarrow \infty$, these Riemannian metrics do not converge to a Riemannian metric.

Exercise 1.3.4 *Instead of the embedding defined in (1.10), use*

$$\zeta = x + iy \longmapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

to define an embedding $\Sigma' : \mathbb{C} \cup \{\infty\} \longrightarrow S^2(r)$ which no longer maps the unit circle to the equator. Then

$$(\Sigma')^*(ds^2) = \frac{4r^4}{(r^2+\zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}$$

which converges to (4 times) the Euclidean metric as $r \longrightarrow \infty$.

Here is how the antipodal map on $S^2(r)$ appears in the stereographic picture of $\mathbb{P}^1_{\mathbb{C}}$:

Exercise 1.3.5 *Show that the diagram*

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\Sigma} & S^2(r) \\ {\scriptstyle \alpha_{\mathfrak{E}}}\downarrow & & \downarrow{\scriptstyle \alpha_O} \\ \mathbb{P}^1_{\mathbb{C}} & \xrightarrow[\Sigma]{} & S^2(r) \end{array}$$

commutes, where $\alpha_{\mathfrak{E}}$ denotes the elliptic anti-involution

$$\begin{aligned} \alpha_{\mathfrak{E}} : \mathbb{P}^1_{\mathbb{C}} &\longrightarrow \mathbb{P}^1_{\mathbb{C}} \\ \zeta &\longmapsto -1/\bar{\zeta} = -\frac{\zeta}{|\zeta|^2}. \end{aligned}$$

Given a point $u \in E^2(r)$, the Euclidean reflection ι_u in the plane $u^\perp$ is an isometry and hence its fixed point set—the great circle $u^\perp \cap S^2(r)$—is a totally geodesic submanifold. Since it has dimension 1, it is actually a geodesic. Furthermore the reflection ι_u commutes with the antipodal map. Thus the hyperbolic anti-involution

$$\Sigma^{-1} \circ \alpha_u \circ \Sigma$$

of $\mathbb{P}^1_{\mathbb{C}}$ commutes with the elliptic anti-involution

$$\Sigma^{-1} \circ \alpha_O \circ \Sigma.$$

Here is a collection of characterizations of great circles:

Exercise 1.3.6 *Let C be a circle. Then the following conditions are equivalent:*

1. *$\Sigma(C)$ is a great circle on $S^2(r)$.*
2. *C is invariant under the elliptic anti-involution $\alpha_{\mathfrak{E}}$.*
3. *The hyperbolic involution fixing C commutes with $\alpha_{\mathfrak{E}}$.*
4. *$C = \mathbb{P}(F)$ where $F \subset \mathbb{V}$ is totally real 2-plane, that is a plane such that the restriction of the Hermitian form $\langle\langle , \rangle\rangle$ to F is real.*
5. *The indefinite Hermitian matrix h_C defining the antipolarity corresponding to C has trace 0.*
6. *C is either a straight line containing the origin or a Euclidean circle whose center ζ and radius R satisfy*

$$R^2 = 1 + |\zeta|^2.$$

In the stereographic representation of $\mathbb{P}^1_{\mathbb{C}}$ the spherical distance can be computed using (1.9). For simplicity we consider the distance between the origin O (corresponding to the South pole) and the point $\Sigma(\zeta) \in S^2(r)$.

Exercise 1.3.7 *The spherical distance between $\Sigma(0)$ and $\Sigma(\zeta)$ equals $2r \tan^{-1} |\zeta|$*

The distance may also be expressed in terms of the cross-ratio and the elliptic anti-involution:

Exercise 1.3.8 *If $x \neq y \in \mathbb{P}^1_{\mathbb{C}}$ are distinct, then $x, y, \alpha_{\mathfrak{E}}(x), \alpha_{\mathfrak{E}}(y)$ lie on a circle and the spherical distance $d(x, y)$ between $\Sigma(x), \Sigma(y)$ satisfies*

$$\mathbf{X}\{x, y, \alpha_{\mathfrak{E}}(x), \alpha_{\mathfrak{E}}(y)\} = \cos^2\left(\frac{d}{2r}\right). \tag{1.12}$$

1.3.3 *The Fubini–Study metric*

Yet another formula for the spherical distance involves the Hermitian structure on $\mathbb{C}^2$, similar to Cayley's formula (1.9). This formula generalizes to complex hyperbolic space (see (3.4)) and we discuss it here for complex elliptic space in arbitrary dimension n.

Consider $\mathbb{C}^{n+1}$ with the standard (positive definite) Hermitian inner product:

$$\langle\langle x, y\rangle\rangle = \sum_{j=1}^{n+1} x_j \bar{y}_j.$$

(We use double angled brackets to denote the standard positive definite Hermitian form.) Then, analogous to (1.9), define a map $d : \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{R}$ by

$$\cos^2\left(\frac{d(x,y)}{r}\right) = \frac{\langle\!\langle x,y\rangle\!\rangle\langle\!\langle y,x\rangle\!\rangle}{\langle\!\langle x,x\rangle\!\rangle\langle\!\langle y,y\rangle\!\rangle} \tag{1.13}$$

where $x, y \in \mathbb{C}^{n+1} - \{0\}$ are nonzero vectors representing elements $[x], [y] \in \mathbb{P}^n_{\mathbb{C}}$ respectively. This expression is homogeneous and hence is well defined on $\mathbb{P}^n_{\mathbb{C}}$.

Exercise 1.3.9 *Prove that d satisfies the triangle inequality and defines the structure of a metric space on $\mathbb{P}^n_{\mathbb{C}}$. Furthermore show this distance function is the distance function defined by the Riemannian structure (1.11) (for $r = 1$).*

This distance was introduced by Fubini [54] and Study [157], and discussed in Coolidge [30]. The resulting *Fubini–Study metric* extends to a Kähler structure on complex projective space. *Complex elliptic n-space* $\mathbf{E}^n_{\mathbb{C}}$ is complex projective space with this structure.

For example, if $\zeta \in \mathbb{C}$, then the Fubini–Study distance between ζ and the origin O equals

$$\rho(0,\zeta) = 2\tan^{-1}|\zeta|.$$

The *metric circle* centered at x of radius ρ consists of all points at distance ρ from x. The preceding discussion implies that the metric circle C_ρ centered at O having Fubini–Study radius ρ is the Euclidean circle centered at O having (Euclidean) radius

$$R = \tan\left(\frac{\rho}{2r}\right).$$

By (1.11), the circumference of C_ρ equals

$$\int_{C_\rho} ds = 2\pi r\sin\left(\frac{\rho}{r}\right). \tag{1.14}$$

Integrating (1.14) with respect to ρ, the area of the disc D_ρ enclosed by C_ρ equals

$$\int_{D_\rho} dA = 2\pi r^2\left(1-\cos\left(\frac{\rho}{r}\right)\right) = 4\pi r^2\sin^2\left(\frac{\rho}{2r}\right).$$

We compute the geodesic curvature κ of C_ρ using the Gauss–Bonnet theorem. Since C_ρ is invariant under a transitive group of rotations about O, its geodesic curvature is constant. Since the Gaussian curvature K equals r^{-2}, the Gauss–Bonnet theorem states

$$\begin{aligned} 2\pi = 2\pi\chi(D_\rho) &= \int_{C_\rho}\kappa\,ds + \int_{D_\rho}K\,dA \\ &= 2\pi r\sin\left(\frac{\rho}{r}\right)\kappa + 4\pi\sin^2\left(\frac{\rho}{2r}\right). \end{aligned}$$

so

$$\kappa = r^{-1}\cot\left(\frac{\rho}{r}\right).$$

In particular when $\rho = \pi r/2$, the metric circle C_ρ is itself a geodesic.

To simplify the remaining calculations, we will assume $r = 1$. To obtain the trigonometric formulas for arbitrary r, simply leave the angles unchanged but divide the distances by r.

By Exercise 1.2.4, the circle C_ρ is associated to a 2-dimensional real subspace $S \subset \mathbb{C}^2$. Such subspaces S possess an invariant, the *holomorphy angle,* which relates to the distance ρ and the geodesic curvature κ. (This invariant is described in more detail in §2.2.1.) Suppose that $S \subset \mathbb{C}^2$ is an $\mathbb{R}$-linear subspace of (real) dimension 2. The real part of the Hermitian form $\langle\langle , \rangle\rangle$

$$(\!(u, v)\!) = \mathrm{Re}\langle\langle u, v\rangle\rangle$$

is the positive definite $\mathbb{R}$-valued inner product on $\mathbb{C}^2$. Choose a basis $v_1, v_2 \in S$ which is orthonormal with respect to $(\!(,)\!)$:

$$(\!(v_i, v_j)\!) = \delta_{ij}.$$

Then $\langle\langle v_1, v_2\rangle\rangle$ is purely imaginary and there exists μ with $0 \le \mu \le \pi/2$ such that

$$\langle\langle v_1, v_2\rangle\rangle = \pm i\cos(\mu).$$

(μ is the smallest angle between vectors in S and vectors in the image iS of S under the complex structure on $\mathbb{C}^2$; compare §2.2.1 for a more complete discussion.)

The holomorphy angle μ equals 0 if and only if S is a complex line (in which case $\mathbb{P}(S)$ is a single point). $\mu = \pi/2$ if and only if f is totally real (see Exercise 1.3.6); that is, $\mathbb{P}(S)$ is a great circle.

Theorem 1.3.10 *The radius of the metric circle $\mathbb{P}(S)$ equals the holomorphy angle μ.*

Proof In a suitable set of coordinates we may assume

$$v_1 = \begin{bmatrix} \sin(\mu/2) \\ \cos(\mu/2) \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -i\sin(\mu/2) \\ i\cos(\mu/2) \end{bmatrix}$$

and S consists of all

$$\begin{bmatrix} \sin(\mu/2)\bar{\xi} \\ \cos(\mu/2)\xi \end{bmatrix}$$

where $\xi \in \mathbb{C}$. Its projectivization consists of all

$$z = \tan\left(\frac{\mu}{2}\right)\frac{\bar{\xi}}{\xi}$$

where ξ ranges over the nonzero complex numbers. Evidently $[f]$ consists of all z with $|z| = \tan(\mu/2)$, which is the metric circle of radius μ about O. □

1.3.4 *Bisectors*

In complex dimension 1, the *metric bisectors* are also geodesics. Let (X, d) be a metric space and let $a, b \in X$ be distinct points. The *bisector equidistant from a and b* is defined to be the set

$$\mathfrak{E}\{a, b\} = \{z \in X \mid \rho(a, z) = \rho(b, z)\}$$

of point of equal distance from a and b. It follows from the distance formula (1.13) that when $(X, d) = \mathbf{E}^1_{\mathbb{C}}$

$$\mathfrak{E}\{[A], [B]\} = \{[Z] \mid Z \in \mathbb{C}^2, H_{A,B}(Z, Z) = 0\}$$

where $H_{A,B}$ is the Hermitian form on $\mathbb{C}^2$ defined by

$$H_{A,B}(Z; W) = \langle\langle A, A\rangle\rangle^{-1}\langle\langle Z, A\rangle\rangle\langle\langle A, W\rangle\rangle - \langle\langle B, B\rangle\rangle^{-1}\langle\langle Z, B\rangle\rangle\langle\langle B, W\rangle\rangle.$$

In particular $\mathfrak{E}\{[A], [B]\}$ corresponds to the Hermitian matrix

$$F_{A,B} = \langle\langle A, A\rangle\rangle^{-1}\bar{A}^{\dagger}A - \langle\langle B, B\rangle\rangle^{-1}\bar{B}^{\dagger}B.$$

The Hermitian matrix corresponding to $H_{A,B}$ has trace zero. Observe that this condition is not invariant under $\mathbf{GL}(2, \mathbb{C})$, but only under the $\mathbf{U}(2)$, the stabilizer of the elliptic anti-polarity. The trace of the matrix representing a bilinear (or Hermitian) form is not intrinsic. However, the trace of a *linear transformation* is intrinsic. Thus if F_1, F_2 are Hermitian matrices, with one of them, say F_2, nondegenerate, then the quantity

$$\mathrm{trace}(F_1 F_2^{-1})$$

is invariant under the action of $\mathbf{GL}(2, \mathbb{C})$ on such pairs (F_1, F_2). Moreover, taking $F_2 = \mathbb{I}$ and $F_1 = H$, we see that trace(H) is invariant under the subgroup of $\mathbf{GL}(2, \mathbb{C})$ stabilizing the Hermitian form represented by the identity matrix.

Hermitian matrices corresponding to the great circles, the circle

$$|\zeta - \zeta_0| = \sqrt{1 + \zeta_0\bar{\zeta}_0}$$

and the straight line $e^{i\theta}\mathbb{R}$, respectively, are

$$F_{\zeta_0} = \begin{bmatrix} 1 & \bar{\zeta}_0 \\ \zeta_0 & 1 \end{bmatrix}, \qquad F_\theta = \begin{bmatrix} 0 & ie^{-i\theta} \\ -ie^{i\theta} & 0 \end{bmatrix}.$$

1.3.5 *Trigonometry*

We briefly review trigonometry in the complex elliptic line. Consider a triangle $\triangle$ with vertices $[\tilde{A}], [\tilde{B}], [\tilde{C}]$ represented by vectors $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{V}$. Denote the side lengths by a, b, c and the angles by $0 < \alpha, \beta, \gamma < \pi$. Applying an automorphism we may assume that

$$\tilde{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} \sin(b/2) \\ \cos(b/2) \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} \sin(a/2)e^{i\gamma} \\ \cos(a/2) \end{bmatrix}.$$

The *first cosine law* is obtained from computing $c = d(A, B)$ by the distance formula (1.13):

$$\begin{aligned} \frac{1+\cos(c)}{2} &= \cos^2(c/2) \\ &= |\sin(a/2)\sin(b/2)e^{i\gamma} + \cos(a/2)\cos(b/2)|^2 \\ &= (\sin(a/2)\sin(b/2)\cos(\gamma) + \cos(a/2)\cos(b/2))^2 \\ &\quad + (\sin(a/2)\sin(b/2)\sin(\gamma))^2 \\ &= \sin^2(a/2)\sin^2(b/2) \\ &\quad + 2\sin(a/2)\sin(b/2)\cos(\gamma)\cos(a/2)\cos(b/2) \\ &\qquad + \cos^2(a/2)\cos^2(b/2))^2 \\ &= \frac{1-\cos(a)}{2}\frac{1-\cos(b)}{2} \\ &\quad + \frac{\cos(\gamma)\sin(a)\sin(b)}{2} \\ &\qquad + \frac{1+\cos(a)}{2}\frac{1+\cos(b)}{2} \\ &= \frac{1}{2} + \frac{1}{2}(\cos(a)\cos(b) + \cos(\gamma)\sin(a)\sin(b)) \end{aligned}$$

obtaining

$$\cos(c) = \cos(a)\cos(b) + \cos(\gamma)\sin(a)\sin(b).$$

For variable r, we obtain

$$\cos(c/r) = \cos(a/r)\cos(b/r) + \cos(\gamma)\sin(a/r)\sin(b/r).$$

which approaches the Euclidean cosine law

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

as $r \longrightarrow \infty$. The Pythagorean theorem follows, by taking $\gamma = \pi/2$:

$$\cos(c/r) = \cos(a/r)\cos(b/r)$$

for any right triangle with hypotenuse c and other sides a, b.

The law of sines can be deduced from the cosine law as follows (see Beardon [9], §7.12, upon which this is based):

$$\begin{aligned}\left(\frac{\sin(c)}{\sin(\gamma)}\right)^2 &= \frac{\sin^2(c)}{1-((\cos(a)\cos(b)-\cos(c))/(\sin(a)\sin(b)))^2}\\ &= \frac{\sin^2(a)\sin^2(b)\sin^2(c)}{\sin^2(a)\sin^2(b)-(\cos(a)\cos(b)-\cos(c))^2}\\ &= \frac{\sin^2(a)\sin^2(b)\sin^2(c)}{(1-\cos^2(a))(1-\cos^2(b))-(\cos(a)\cos(b)-\cos(c))^2}\\ &= \frac{\sin^2(a)\sin^2(b)\sin^2(c)}{1-\cos^2(a)-\cos^2(b)-\cos^2(c)+2\cos(a)\cos(b)\cos(c)}\end{aligned}$$

is symmetrical in a, b, c and therefore

$$\frac{\sin(c)}{\sin(\gamma)} = \frac{\sin(a)}{\sin(\alpha)} = \frac{\sin(b)}{\sin(\beta)}$$

or with varying r:

$$\frac{\sin(c/r)}{\sin(\gamma)} = \frac{\sin(a/r)}{\sin(\alpha)} = \frac{\sin(b/r)}{\sin(\beta)}$$

which approaches the Euclidean sine law as $r \longrightarrow \infty$.

1.3.6 *An area formula*

Here is an algebraic formula for the area for a geodesic triangle in $\mathbf{E}^1_{\mathbb{C}}$. Suppose that A, B, C are points in $\mathbf{E}^1_{\mathbb{C}}$, which are distinct and no two of them are antipodal. As no pair of vertices is antipodal, there is a unique minimizing geodesic segment between any pair of vertices.

The area of a geodesic triangle in $\mathbf{E}^1_{\mathbb{C}}$ equals the *angular defect*

$$(\alpha + \beta + \gamma) - \pi$$

(compare the discussion in Coxeter [35], [34], and Rosenfeld [146]). Here is an alternative formula, for which I have been unable to find a reference. The quantity

$$\langle\!\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle\!\rangle := \langle\!\langle \tilde{A}, \tilde{B} \rangle\!\rangle \langle\!\langle \tilde{B}, \tilde{C} \rangle\!\rangle \langle\!\langle \tilde{C}, \tilde{A} \rangle\!\rangle \tag{1.15}$$

is a nonzero complex number which scales by the positive real number $|\alpha\beta\gamma|^2$ when $\tilde{A}, \tilde{B}, \tilde{C}$ are replaced by $\alpha\tilde{A}, \beta\tilde{B}, \gamma\tilde{C}$ respectively. Thus its argument

$$\arg\langle\!\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle\!\rangle$$

depends only on the equivalence classes $A, B, C \in \mathbb{P}^1_{\mathbb{C}}$, and is denoted $\mathbb{A}(A, B, C)$. (See §2.2.5 for a more detailed discussion.)

Exercise 1.3.11 *Let $A, B, C \in \mathbb{P}^1_{\mathbb{C}}$ be three points, no two of which are antipodal. Let $\triangle$ be the triangle with vertices A, B, C and sides the unique minimizing geodesic segments joining them. Then the area of $\triangle$ equals* $2|\mathbb{A}(A, B, C)|$.

1.4 Hyperbolic geometry

1.4.1 *The complex hyperbolic line*

The *complex hyperbolic line* $\mathbf{H}^1_{\mathbb{C}}$ (or *complex hyperbolic 1-space*) is defined analogously to $\mathbf{E}^1_{\mathbb{C}}$, except that the elliptic anti-polarity is replaced by a hyperbolic anti-polarity η. The null locus $N(\eta)$ is a circle, the *absolute* of this model of $\mathbf{H}^1_{\mathbb{C}}$.

A hyperbolic anti-polarity corresponds to an indefinite Hermitian form, which we now fix as the Hermitian form

$$\langle Z, W\rangle = Z_1\bar{W}_1 - Z_2\bar{W}_2 = \bar{W}^\dagger \mathbb{I}_{1,1} Z$$

where

$$\mathbb{I}_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We refer to the vector space $\mathbb{C}^2$ with the structure defined by this Hermitian form as the *Hermitian vector space* $\mathbb{C}^{1,1}$.

A vector $Z \in \mathbb{C}^{1,1}$ is said to be *negative* (respectively *null, positive*) if and only if $\langle Z, Z\rangle < 0$ (respectively $\langle Z, Z\rangle = 0$, $\langle Z, Z\rangle > 0$). A line $l \subset \mathbb{C}^{1,1}$ is *negative* (respectively *null, positive*) if l consists of negative (respectively null, positive) vectors. The subset of $\mathbb{P}(\mathbb{C}^{1,1})$ consisting of negative lines (respectively null lines) is defined to be the *complex hyperbolic 1-space* $\mathbf{H}^1_{\mathbb{C}}$ (respectively the *absolute* $\partial\mathbf{H}^1_{\mathbb{C}}$). In the Gaussian representation of $\mathbb{P}(\mathbb{C}^{1,1}) = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex hyperbolic line is the *unit disc* $|\zeta| < 1$.

Exercise 1.4.1 *If $v, w \in \mathbb{C}^{1,1}$ are negative vectors, then*

$$\frac{\langle v, w\rangle\langle w, v\rangle}{\langle v, v\rangle\langle w, w\rangle} \geq 1 \tag{1.16}$$

and depends only on the lines $[v], [w] \in \mathbf{H}^1_{\mathbb{C}}$.

In analogy with (1.13), define the *hyperbolic distance* between two points $[u], [v] \in \mathbf{H}^1_{\mathbb{C}}$ by

$$\cosh^2\left(\frac{d([u],[v])}{2}\right) = \frac{\langle v, w\rangle\langle w, v\rangle}{\langle v, v\rangle\langle w, w\rangle}.$$

This defines a metric on $\mathbf{H}^1_{\mathbb{C}}$ whose infinitesimal form is *the Poincaré metric*

$$\frac{4d\zeta d\bar{\zeta}}{(1-\zeta\bar{\zeta})^2}$$

which has constant curvature -1.

Through every $U, V \in \mathbf{H}^1_{\mathbb{C}}$ exists a circle $c(U,V)$ which is orthogonal to the absolute $\partial\mathbb{C}^{1,1}$. The unique anti-involution fixing that circle preserves the absolute and thus the intersection $\mathbf{H}^1_{\mathbb{C}} \cap c(U,V)$ is a geodesic in $\mathbf{H}^1_{\mathbb{C}}$. Furthermore

$$c(U,V) \cap \partial\mathbf{H}^1_{\mathbb{C}}$$

consists of two points U_∞, V_∞ and the hyperbolic distance $d(U,V)$ can be expressed in terms of the cross-ratio:

$$\mathbf{X}\{U_\infty, V_\infty, U, V\} = e^{d(U,V)}.$$

Exercise 1.4.2 *Find a formula, similar to (1.12), expressing the distance in terms of the hyperbolic anti-involution and cross-ratio.*

The subgroup $\mathbf{U}(1,1)$ of unitary automorphisms of $\mathbb{C}^{1,1}$ acts by projective transformations preserving the absolute and acts isometrically on $\mathbf{H}^1_{\mathbb{C}}$.

Exercise 1.4.3 *Sometimes an indefinite Hermitian form given by a matrix which is anti-diagonal is more convenient than a diagonal Hermitian matrix. For example, for the indefinite Hermitian form*

$$\langle Z, W \rangle = Z_1 \bar{W}_2 + Z_2 \bar{W}_1$$

the corresponding unitary group identifies with $\mathbf{SL}(2,\mathbb{R})$ *and the absolute corresponds to the real projective line* $\mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1_{\mathbb{C}}$.

Geodesics in $\mathbf{H}^1_{\mathbb{C}}$ are also represented by indefinite Hermitian matrices (corresponding to hyperbolic anti-polarities) in a way similar to the parametrization of great circles by indefinite Hermitian matrices. For example, the Hermitian matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

represents the geodesic γ_I corresponding to the imaginary axis $i\mathbb{R}$ in affine coordinates.

Suppose $\gamma \subset \mathbf{H}^1_{\mathbb{C}}$ is an arbitrary geodesic. The origin $O \in \mathbf{H}^1_{\mathbb{C}}$ is represented by 0 in affine coordinates or, equivalently, the negative vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^{1,1}.$$

Let $p = p(\gamma)$ be the point on γ closest to O. Let δ be the distance from O to p and let θ be the angle that the geodesic from O to p makes with the intersection

$$\mathbf{H}^1_{\mathbb{R}} = \mathbf{H}^1_{\mathbb{C}} \cap \mathbb{P}^1_{\mathbb{R}}$$

(if $p = O$, then $\pi/2 - \theta$ is defined as the angle between $\mathbf{H}^1_{\mathbb{R}}$ and γ). Then a Hermitian matrix F_γ such that $\gamma \subset N(F_\gamma)$ is

$$F_\gamma = \bar{A}^\dagger \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} -\sinh(\delta) & e^{i\theta}\cosh(\delta) \\ e^{-i\theta}\cosh(\delta) & -\sinh(\delta) \end{bmatrix}$$

where the element

$$A = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \begin{bmatrix} \cosh(\delta/2) & -\sinh(\delta/2) \\ -\sinh(\delta/2) & \cosh(\delta/2) \end{bmatrix}$$

of $\mathbf{U}(1,1)$ represents an isometry of $\mathbf{H}^1_{\mathbb{C}}$ taking the vertical geodesic γ_I in affine coordinates) to γ.

This gives a convenient model for the geodesics in $\mathbf{H}^1_{\mathbb{C}}$. If $\delta > 0$, then p completely determines θ. If $\delta = 0$, then θ is arbitrary, with ambiguity determined up to ± 1. It follows that the *2-dimensional real manifold consisting of geodesics in* $\mathbf{H}^1_{\mathbb{C}}$ *identifies with the unit disc in* $\mathbb{C}$ *blown up at the origin.*

1.4.2 *Hypercycles, horocycles and metric circles*

Metric circles in $\mathbf{H}^1_{\mathbb{C}}$ are represented by Euclidean circles. A metric circle of radius ρ has geodesic curvature $\coth(\rho)$.

Exercise 1.4.4 *Let* $u : \mathbf{H}^1_{\mathbb{C}} \longrightarrow \mathbb{R}$ *denote distance to the origin and* θ *the usual angular coordinate on the unit disc. Describe the Poincaré metric tensor*

$$\mathbf{g} = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2} = du^2 + \sinh(u)^2 d\theta^2$$

to obtain hyperbolic polar coordinates

$$z = e^{i\theta}\tanh(u/2)$$

on $\mathbf{H}^1_{\mathbb{C}}$.

Metric circles limit to *horocycles,* which we may regard as "circles centered at infinity." For example, choose a point $p \in \mathbf{H}^1_{\mathbb{C}}$ and a line l in the tangent space at p. Let $z_n \in \mathbf{H}^1_{\mathbb{C}}$ be a path in $\mathbf{H}^1_{\mathbb{C}}$ which converges to a point z_∞ on the absolute. Let C_n be the unique circle centered at z_n which passes through p and is tangent to l at p. Then the sequence C_n converges (for example, in the Hausdorff topology) to $C_\infty \cup \{z_\infty\}$ where C_∞ is a horocycle.

Horocycles have geodesic curvature 1 and can be regarded as limits of *hypercycles* or *equidistant curves.* Let γ be a geodesic. For any $\delta > 0$, the set of points $x \in \mathbf{H}^1_{\mathbb{C}}$ with $d(x,\gamma) < \delta$ has two boundary curves, each of which is a δ-equidistant curve to γ. Such a curve C will be called a *hypercycle* and has geodesic curvature $\tanh(\delta)$. The two endpoints of a hypercycle are the two endpoints of γ and we say that C and γ are *parallel.*

A circle C in $\mathbb{P}^1_{\mathbb{C}}$ intersects $\mathbf{H}^1_{\mathbb{C}}$ in one of four possibilities:

1. The empty set.
2. A metric circle if $C \subset \mathbf{H}^1_{\mathbb{C}}$ is disjoint from the absolute.
3. A horocycle if $C \neq \partial\mathbf{H}^1_{\mathbb{C}}$ and $C \subset \mathbf{H}^1_{\mathbb{C}} \cup \partial\mathbf{H}^1_{\mathbb{C}}$. (In this case C intersects the absolute in one point.)

4. A hypercycle if C meets each of the two components of $\mathbb{P}^1_{\mathbb{C}} - \partial\mathbf{H}^1_{\mathbb{C}}$.

In the last case C intersects the absolute in two points. The angle of intersection θ is related to the distance δ by the following formulas:

$$\delta = \log(\cot(\theta/2)), \qquad \cot(\theta) = \sinh(\delta).$$

Exercise 1.4.5 *Given a geodesic γ, and a point $p \in \mathbf{H}^1_{\mathbb{C}}$, there exists a unique hypercycle $C(p;\gamma)$ through p and parallel to γ. If γ_n is a sequence of geodesics converging to an ideal point $q_\infty \in \partial\mathbf{H}^1_{\mathbb{C}}$, then show that the hypercycles $C(p;\gamma_n)$ converge to the unique horocycle centered at q_∞ passing through p.*

Exercise 1.4.6 *Let S be a 1-parameter subgroup of* $\mathbf{PU}(1,1)$ *and let $p \in \mathbf{H}^1_{\mathbb{C}}$. Show that the orbit $S(p)$ is*

1. *A metric circle (possibly the point $\{p\}$) if S is elliptic.*
2. *A horocycle if S is parabolic.*
3. *A hypercycle (possibly a geodesic) if S is hyperbolic.*

In the first two cases, the center of $S(p)$ is the fixed point of S, and in the last case, $S(p)$ is a hypercycle parallel to the unique geodesic invariant under S.

1.4.3 *Trigonometry*

The formulas of elliptic trigonometry have analogues in hyperbolic trigonometry, and we briefly present these formulas. The proofs are analogous to those in §1.3.5 and left as an exercise. We consider the complex hyperbolic line $\mathbf{H}^1_{\mathbb{C}}(r)$ with curvature $-r^{-2}$. For more details, see Coxeter [35, 34], Fenchel [52] and Thurston [161]. The *first cosine law* in $\mathbf{H}^1_{\mathbb{C}}$ is

$$-\cosh(c/r) + \cosh(a/r)\cosh(b/r) = \cos(\gamma)\sinh(a/r)\sinh(b/r)$$

(which approaches the Euclidean cosine law

$$c^2 = a^2 + b^2 - 2\cos(\gamma)ab$$

as $r \longrightarrow \infty$). Setting $\gamma = \pi/2$ gives the Pythagorean theorem

$$\cosh(c/r) = \cosh(a/r)\cosh(b/r)$$

for any right triangle with hypotenuse c and other sides a, b. The *second cosine law* in $\mathbf{H}^1_{\mathbb{C}}$ is

$$\cos(\gamma) + \cos(\alpha)\cos(\beta) = \cosh(c/r)\sin(\alpha)\sin(\beta) \tag{1.17}$$

whose limit as $r \longrightarrow \infty$ reduces to

$$\cos(\gamma) = \cos(\alpha + \beta)$$

which follows from the Euclidean relation

$$\alpha + \beta + \gamma = \pi.$$

From the cosine law follows the sine law:

$$\frac{\sinh(c/r)}{\sin(\gamma)} = \frac{\sinh(a/r)}{\sin(\alpha)} = \frac{\sinh(b/r)}{\sin(\beta)}.$$

Here is an algebraic formula for the area for a geodesic triangle in $\mathbf{H}^1_{\mathbb{C}}$. Suppose that A, B, C are points in $\mathbf{H}^1_{\mathbb{C}}$, spanning a triangle $\triangle(A, B, C)$. Its area equals the *angular defect:*

$$\pi - (\alpha + \beta + \gamma)$$

(compare the discussion in Coxeter [35], [34], and Rosenfeld [146]). Here is an alternative formula, analogous to 1.3.6.

Exercise 1.4.7 *Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{C}^{n-1}$ be negative vectors representing the vertices A, B, C. As in (1.15),*

$$\langle\!\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle\!\rangle := \langle\!\langle \tilde{A}, \tilde{B} \rangle\!\rangle \langle\!\langle \tilde{B}, \tilde{C} \rangle\!\rangle \langle\!\langle \tilde{C}, \tilde{A} \rangle\!\rangle$$

depends only on the points $A, B, C \in \mathbf{H}^1_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}}$. Let

$$\mathbb{A}(A, B, C) = \arg\langle\!\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle\!\rangle.$$

The area of $\triangle(A, B, C)$ equals $2|\mathbb{A}(A, B, C)|$.

1.4.3.1 *The angle of parallelism* The duality between angle and length is concretely manifest by the relation of *angle of parallelism.* Namely, choose a point $O \in \mathbf{H}^1_{\mathbb{C}}$ and a complete geodesic $l \subset \mathbf{H}^1_{\mathbb{C}}$. Then the angle of parallelism is the angle 2θ at O between the two rays asymptotic to l, which is related to the distance $d = d(O, l)$ from O to l.

Alternatively, choose the point p on l closest to O; then O and p are two vertices of two ideal right triangle whose sides are rays of l. Then the angle at p of this triangle equals θ, and the finite side has length d.

The angle of parallelism θ and the length d are related by the suggestive formulas

$$\begin{aligned}
\sinh(d) &= \cot(2\theta) \\
\cosh(d) &= \csc(2\theta) \\
\tanh(d) &= \cos(2\theta) \\
\coth(d) &= \sec(2\theta) \\
\operatorname{sech}(d) &= \sin(2\theta) \\
\operatorname{csch}(d) &= \tan(2\theta) \\
e^d &= \cot(\theta) \\
e^{-d} &= \tan(\theta).
\end{aligned} \tag{1.18}$$

1.4.4 *The right half-plane model*

A closely related model for hyperbolic geometry is the right half-plane $\mathfrak{H}^1$ described by $\mathrm{Re}(w) > 0$. (Although the upper half-plane is more standard, the right half-plane seems to be more convenient for some of the higher-dimensional generalizations.) The embedding

$$\mathbf{B} : w \mapsto \begin{bmatrix} 1/2 - w \\ 1/2 + w \end{bmatrix}$$

maps $\mathfrak{H}^1 \longrightarrow \mathbb{C}^{1,1}$. Now

$$\langle \mathbf{B}(w_1), \mathbf{B}(w_2) \rangle = -(w_1 + \bar{w}_2),$$

and in particular $\langle \mathbf{B}(w), \mathbf{B}(w) \rangle = -2\mathrm{Re}(w)$. We choose coordinates $t, \delta \in \mathbb{R}$ such that

$$w = e^t(\mathrm{sech}(\delta) + i \tanh(\delta)).$$

The Riemannian metric on $\mathfrak{H}^1$ is

$$\mathbf{g} = \frac{|dw|^2}{\mathrm{Re}(w)^2} = \cosh^2(\delta) dt^2 + d\delta^2.$$

The positive real axis $\mathbb{R}^+$ is the geodesic γ_0 from O to ∞ and $\delta = \delta(w)$ is the distance from w to γ_0.

A closely related set of coordinates on $\mathfrak{H}^1$ uses the same coordinate δ but replaces t by τ where

$$e^\tau = e^t \mathrm{sech}(\delta)$$

so that

$$w = e^\tau (1 + i \sinh(\delta))$$

and

$$\mathbf{g} = \frac{|dw|^2}{\mathrm{Re}(w)^2} = d\tau^2 + (\sinh(\delta) d\tau + \cosh(\delta) d\delta)^2.$$

While the level sets of t are geodesics orthogonal to γ_0, the level sets of τ are horocycles orthogonal to γ_0.

In this model the hypercycles parallel to γ_0 are rays through the origin. In particular the ray γ_m of slope $-\infty < m < \infty$ satisfies

$$m = \sinh(\delta)$$

where δ denotes the (signed) distance from γ_m to γ_0. The distance along the hypercycle γ_m from parameter t for $t_1 < t < t_2$ is given by

$$\cosh(\delta)(t_2 - t_1)$$

whereas the distance between the points w_1, w_2 is given by

$$\sinh\left(\frac{\rho(w_1, w_2)}{2}\right) = \cosh(\delta) \sinh\left(\frac{(t_2 - t_1)}{2}\right).$$

1.4.5 *The real hyperbolic plane*

An alternative model for an isomorphic geometry is *real hyperbolic 2-space.* This model is quite different, arising from the *real* projective *plane* $\mathbb{P}^2_{\mathbb{R}}$, rather than the *complex* projective *line*. Consider the Lorentzian vector space $\mathbb{R}^{2,1}$, that is a 3-dimensional real vector space with inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3.$$

The corresponding affine space $\mathbb{E}$ has a parallel flat Lorentzian metric

$$g = dx_1dy_1 + dx_2dy_2 - dx_3dy_3$$

and with this structure (E, g) is a simply connected complete Lorentzian manifold (*Minkowski (2+1)-space*) of zero curvature. The set of all points satisfying

$$\langle x, x\rangle = -1$$

is a two-sheeted hyperboloid for which the restriction of g is a Riemannian metric of constant curvature -1.

The Siegel domain for $\mathbf{H}^2_{\mathbb{R}}$ consists of $w = (w_1, w_2) \in \mathbb{R}^2$ satisfying $f(w) < 0$ where

$$f(w) = 2w_1 - w_2^2.$$

Then

$$\begin{aligned} d\log f(w) &= (2w_1 - w_2^2)^{-1}(2dw_1 - 2w_2dw_2) \\ Dd\log f(w) &= D(f^{-1}df) = f^{-1}d^2f - f^{-2}(df)^2 \\ &= \frac{4}{(2w_1 - w_2^2)^2}\left\{(dw_2 - w_1dw_1)^2 + (2w_1 - w_2^2)dw_2^2\right\} \end{aligned}$$

and $Dd\log f(w)$ is the metric on $\mathbf{H}^2_{\mathbb{R}}$. Two vector fields whose flows generate the affine automorphism group are

$$w_2\frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}, \qquad 2w_2\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2}.$$

Exercise 1.4.8 *A projectively equivalent model for* $\mathbf{H}^2_{\mathbb{R}}$ *is the unit ball* $(x_1, x_2) \in \mathbb{R}^2$ *where*

$$x_1^2 + x_2^2 < 1.$$

Show that in this model the metric tensor is given by

$$\frac{4}{(1 - x_1^2 - x_2^2)^2}(x_1dx_1 + x_2dx_2)^2 + \frac{4}{1 - x_1^2 - x_2^2}(dx_1^2 + dx_2^2).$$

1.4.6 *Trigonometry and Gram matrices*

The trigonometric formulas—in particular the laws of cosines—in $\mathbf{H}^2_{\mathbb{R}}$ can be expressed succinctly and abstractly in terms of 3×3 symmetric matrices as follows. Associated to a triangle are two triples, the *side lengths* a, b, c and *vertex angles* α, β, γ. These invariants define symmetric bilinear forms on dual vector spaces, which we describe in terms of Gram matrices. The *edge matrix* is the symmetric matrix

$$E = \begin{bmatrix} 1 & \cosh(c) & \cosh(b) \\ \cosh(c) & 1 & \cosh(a) \\ \cosh(b) & \cosh(a) & 1 \end{bmatrix}$$

and the *vertex matrix* equals

$$V = \begin{bmatrix} 1 & -\cos(\gamma) & -\cos(\beta) \\ -\cos(\gamma) & 1 & -\cos(\alpha) \\ -\cos(\beta) & -\cos(\alpha) & 1 \end{bmatrix}.$$

Writing

$$D = \begin{bmatrix} \sinh(a) & 0 & 0 \\ 0 & \sinh(b) & 0 \\ 0 & 0 & \sinh(c) \end{bmatrix}$$

the *matrix form of the law of cosines* is

$$E^{-1} = \det(E)^{-1} D^{\dagger} V D \tag{1.19}$$

where

$$\det(E) = 1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c).$$

Equation (1.19) includes both versions of the laws of cosines, six equations in all. (Compare Coxeter [34], Thurston [161].)

2

ALGEBRAIC AND GEOMETRIC BACKGROUND

This chapter contains prerequisites for the main text. The previous chapter was meant to illustrate these ideas which are developed in greater generality here. After a fairly lengthy discussion of linear algebra in complex vector spaces (in particular real structures, Hermitian and symplectic structures), we review the analytic and differential-geometric background concerning Kähler manifolds. We emphasize symplectic and contact geometry. This chapter concludes with a discussion of "Heisenberg spaces" which model the boundary of complex hyperbolic space.

2.1 Linear algebra

2.1.1 *Complex structures on a real vector space*

Let E be a complex vector space. By considering only scalar multiplications by $\mathbb{R} \subset \mathbb{C}$, there is a real vector space $E_{\mathbb{R}}$ whose points and operations are identical with those of E, which we call the *real vector space underlying* E. Multiplication by $i = \sqrt{-1}$ defines an automorphism $\mathbb{J} : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}$ satisfying

$$\mathbb{J} \circ \mathbb{J} = -\mathbb{I} \tag{2.1}$$

(where $\mathbb{I}$ denotes the identity map). An endomorphism $\mathbb{J}$ of a real vector space satisfying (2.1) is called a *complex structure* and the correspondence

$$E \mapsto (E_{\mathbb{R}}, \mathbb{J})$$

defines an isomorphism of categories between complex vector spaces and real vector spaces with complex structure.

Let $E = (E_{\mathbb{R}}, \mathbb{J})$ be a complex vector space. Then $-\mathbb{J}$ is also a complex structure on $E_{\mathbb{R}}$. We call the complex vector space $\bar{E}$ corresponding to $E = (E_{\mathbb{R}}, -\mathbb{J})$ the *conjugate* of E. Let E, E' be complex vector spaces. A linear map $f : E_{\mathbb{R}} \longrightarrow E'_{\mathbb{R}}$ which is complex linear with respect to the complex structure $-\mathbb{J}$ on $E_{\mathbb{R}}$ and a complex structure $\mathbb{J}'$ on E' satisfies

$$f(\mathbb{J}u) = -\mathbb{J}' f(u)$$

for every $u \in E$ is said to be an *anti-linear* $E \longrightarrow E'$. Clearly anti-linear maps $E \longrightarrow E'$ correspond to linear maps $\bar{E} \longrightarrow E'$.

If $S \subset E$ is a vector subspace, then its underlying real vector subspace $S_{\mathbb{R}}$ is a subspace of $E_{\mathbb{R}}$ that is invariant under $\mathbb{J}$. Conversely, suppose that S is a subspace of $E_{\mathbb{R}}$ that is invariant under $\mathbb{J}$. Then S underlies a complex-linear subspace of $E_{\mathbb{R}}$. We refer to such a subspace of $E_{\mathbb{R}}$ as a *complex subspace.* Then S is *complex;* that is, there exists a $\mathbb{C}$-linear subspace $F \subset E$ such that $S = F_{\mathbb{R}}$ if and only if S is $\mathbb{J}$-invariant.

A *line* in a complex vector space E is a 1-dimensional complex-linear subspace. The space of lines of E is the *projective space* $\mathbb{P}(E)$ associated to E. Since a line is determined by any nonzero vector it contains, projective space $\mathbb{P}(E)$ may also be described as the quotient of the subset $E - \{0\}$ of nonzero vectors in E by the action of $\mathbb{C}^*$ by scalar multiplication. The quotient map is called *projectivization* and is denoted

$$\mathbb{P} : E - \{0\} \longrightarrow \mathbb{P}(E).$$

Projective space has the natural structure of a complex manifold of dimension $\dim_{\mathbb{C}}(E) - 1$. An automorphism $f : E \longrightarrow E$ induces a biholomorphic automorphism of $\mathbb{P}(E)$, called a *collineation* or *projective automorphism.* The *collineation group* or *projective group* of $\mathbb{P}(E)$ is the quotient $\mathbf{PGL}(E)$ of the general linear group $\mathbf{GL}(E)$ by the central subgroup of scalar multiplications. $\mathbf{PGL}(E)$ acts effectively on $\mathbb{P}(E)$ by collineations.

The linear map $\mathbb{J}$ extends to a $\mathbb{C}$-linear endomorphism of the complexification $E_{\mathbb{C}} = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The complexification $E_{\mathbb{C}}$ splits as a direct sum of two eigenspaces:

$$E^{1,0} = \{u \in E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \mid \mathbb{J}u = iu\}$$

—the *vectors of type (1,0)*—and

$$E^{0,1} = \{u \in E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \mid \mathbb{J}u = -iu\}$$

—the *vectors of type (0,1).* Then

$$\begin{aligned} E &\longrightarrow E^{1,0} \\ u &\mapsto u^{1,0} = \frac{1}{2}(u - i\mathbb{J}u) \end{aligned}$$

and

$$\begin{aligned} \bar{E} &\longrightarrow E^{0,1} \\ u &\mapsto u^{0,1} = \frac{1}{2}(u + i\mathbb{J}u) \end{aligned}$$

are isomorphisms of $\mathbb{C}$-vector spaces.

2.1.2 *Real structures on a complex vector space*

A complex vector space E may arise as the complexification of a real vector space S:

$$E = S_{\mathbb{C}} = S \otimes_{\mathbb{R}} \mathbb{C}.$$

In this case we may try to recover S as a *real form* of E.

Let S be a vector space over $\mathbb{R}$. Composition of the isomorphism $S \cong S \otimes_{\mathbb{R}} \mathbb{R}$ with the inclusion $S \otimes_{\mathbb{R}} \mathbb{R} \hookrightarrow S \otimes_{\mathbb{R}} \mathbb{C}$ induced by the field extension $\mathbb{R} \hookrightarrow \mathbb{C}$ induces an inclusion $S \hookrightarrow S_{\mathbb{C}}$. Furthermore complex conjugation $\mathbb{C} \longrightarrow \mathbb{C}$ induces an anti-linear involution ρ of $S_{\mathbb{C}}$ defined by

$$\rho(s \otimes z) = s \otimes \bar{z}$$

whose fixed-point set is S.

Let E be a complex vector space (with complex structure $\mathbb{J} : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}$) and $S \subset E_{\mathbb{R}}$ be a (real) linear subspace. The following conditions are equivalent:

1. $S \cap \mathbb{J}(S) = 0$.
2. The composition

$$S \otimes \mathbb{C} \longrightarrow E_{\mathbb{C}} \longrightarrow E^{1,0} \cong E$$

   is injective (where the second map is projection onto (1,0)).

In this case we say that S is *purely real.* (Sometimes this concept is called "totally real," which for us has a different meaning; see §2.2.1.)

A maximal purely real subspace is called a *real form* of E. In particular if $\dim_{\mathbb{C}}(E) = n$ and $S \subset E_{\mathbb{R}}$ is purely real with $\dim_{\mathbb{R}}(S) = n$, then S is a real form of E. If $S \subset E$ is a real form of E, then $E \cong S \otimes_{\mathbb{R}} \mathbb{C}$.

A *real structure* on a complex vector space E is an anti-linear involution ρ of E. If $S \subset E$ is a real form of E, then

$$\rho(s \otimes z) = s \otimes \bar{z}$$

defines a real structure ρ on E whose fixed-point set is S. Conversely if $\rho : E \longrightarrow E$ is a real structure, then $\rho^2 = \mathbb{I}$ implies that $E_{\mathbb{R}}$ decomposes as a direct sum $E_{\mathbb{R}} = E_\rho^+ \oplus E_\rho^-$ of the eigenspaces

$$E_\rho^+ = \{u \in E_{\mathbb{R}} \mid \rho(u) = u\}$$

(that is, the fixed-point set) and

$$E_\rho^- = \{u \in E_{\mathbb{R}} \mid \rho(u) = -u\},$$

each of which is a real form of E. It follows that for every complex vector space E there is a natural bijection between the set of real forms of E and the set $\mathfrak{R}(E)$ of real structures on E.

2.1.3 *Projective equivalence of real structures*

Let E be a complex vector space. The group of unit complex scalars acts on $\mathfrak{R}(E)$ as follows. Let $S \subset E_{\mathbb{R}}$ be a real form of E. For any nonzero complex number ζ, the image $\zeta S \subset E$ is a real form of E. Furthermore $\zeta S = S$ if and only if $\zeta \in \mathbb{R}$. Thus $\mathbb{C}^*/\mathbb{R}^* \cong \mathbb{T}$ naturally acts on $\mathfrak{R}(E)$.

Here is the corresponding action on real structures. Let ρ be a real structure. Then for any complex scalar $\zeta \in \mathbb{C}^*$

$$\rho^{\zeta} : x \mapsto \rho(\zeta x) = \bar{\zeta}\rho(x)$$

defines an anti-linear map which has order 2 if and only if $|\zeta| = 1$. If S is the fixed-point set of ρ, then $\zeta^{-1/2} \cdot S$ is the fixed-point set of ρ^{ζ}.

We say that real structures $\rho, \rho' \in \mathfrak{R}(E)$ are *projectively equivalent* if there exists $\zeta \in \mathbb{C}$ such that $\rho' = \rho^{\zeta}$. Equivalently, two real forms $S, S' \subset E$ correspond to projectively equivalent real structures if there exists $\zeta \in \mathbb{C}$ such that $S' = \zeta S$. One example of this relationship has already been discussed: the complementary real forms $E_{\rho}^{\pm}$ associated to a real structure $\rho \in \mathfrak{R}(E)$ are projectively equivalent since

$$E_{\rho}^{-} = E_{-\rho}^{+}, \qquad E_{\rho}^{+} = E_{-\rho}^{-}.$$

Since a real structure $\rho \in \mathfrak{R}(E)$ is anti-linear, the map it defines on projective space $\mathbb{P}(E)$ is an anti-holomorphic projective involution, or, following Cartan [24], an *anti-involution.* On $\mathbb{P}^n(\mathbb{C})$ all anti-involutions arise from real structures on $\mathbb{C}^{n+1}$ provided that n is even.

Exercise 2.1.1 *Relate anti-involutions on $\mathbb{P}^n(\mathbb{C})$ to real structures on $\mathbb{C}^{n+1}$ when n is odd.*

Two real structures define the same map on projective space if and only if they are projectively equivalent. In that case the corresponding fixed-point sets of the induced anti-collineation are equal. (Compare Cartan [24] and Jacobowitz [92], §9.1.)

2.1.4 *Matrix representation of real structures*

Suppose that ρ_0 is a fixed real structure on E and ρ is an arbitrary real structure on E. Then the composition $\phi = \rho \circ \rho_0 : E \longrightarrow E$ is a $\mathbb{C}$-linear automorphism of E which satisfies

$$\phi \circ \rho_0 \circ \phi = \rho_0; \tag{2.2}$$

conversely, given a real structure ρ_0 on E any $\mathbb{C}$-linear automorphism ϕ of E satisfying (2.2) determines a new real structure ρ on E by

$$\rho = \phi \circ \rho_0.$$

To interpret these conditions in terms of matrices, let $E = \mathbb{C}^n$ and ρ_0 be the standard real structure given by complex conjugation. Let A be the matrix representing the endomorphism ϕ. Then ρ is given by

$$\rho(z) = A(\bar{z}).$$

and the condition that ρ has order 2 is

$$A \cdot \bar{A} = \mathbb{I}. \tag{2.3}$$

Observe that this condition implies that the set of eigenvalues of A is closed under inversion in the unit circle

$$\begin{aligned} \iota_{\mathbb{T}} : \mathbb{C}^* &\longrightarrow \mathbb{C}^* \\ z &\longmapsto (\bar{z})^{-1}. \end{aligned}$$

Suppose that P is a matrix representing a $\mathbb{C}$-linear automorphism ψ of E; then $\psi \circ \rho \circ \psi^{-1}$ is a real structure on E whose corresponding real form equals $\psi(E_\rho^+)$. The corresponding matrix is $P \cdot A \cdot \bar{P}^{-1}$.

Here is an explicit example in complex dimension $n = 1$.

A real structure ρ on $\mathbb{C}^2$ which leaves invariant the lines $l_\pm$ spanned by vectors

$$v_\pm = \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix}$$

corresponds to a matrix having the form

$$A = \zeta \begin{bmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{bmatrix}$$

where $\zeta \in \mathbb{T}$ is a unit complex number. (A matrix $A \in \mathbf{GL}(2, \mathbb{C})$ leaves invariant $l_\pm$ if and only if its diagonal entries are equal and its off-diagonal entries are equal.) A vector $0 \neq Z \in \mathbb{C}^2$ is real with respect to ρ if and only if $A(\bar{Z}) = Z$; that is,

$$|z - i\cot(\theta)| = |\csc(\theta)|$$

where $z = Z_1/Z_2$ is the inhomogeneous coordinate. This set is the circle centered at $i\cot(\theta)$, of radius $|\csc(\theta)|$, which meets the real axis at angle θ. In the hyperbolic geometry of the unit disc $|z| < 1$ (in homogeneous coordinates described by $\langle Z, Z\rangle < 0$), this circle determines a hypercycle passing through the points $\mathbb{P}(q_\pm) = \pm 1$ at hyperbolic distance $\sinh^{-1}(\tan(\theta))$. This hypercycle has geodesic curvature $\sin(\theta)$.

2.2 Hermitian linear algebra

Consider a nondegenerate symmetric bilinear form $⟪,⟫ : E_\mathbb{R} \times E_\mathbb{R} \to \mathbb{R}$. If $⟪,⟫$ is $\mathbb{J}$-invariant (that is, if $\mathbb{J}$ is orthogonal with respect to $⟪,⟫$):

$$⟪\mathbb{J}x, \mathbb{J}y⟫ = ⟪x, y⟫$$

for all $x, y \in E_\mathbb{R}$), then

$$\begin{aligned} \omega : E_\mathbb{R} \times E_\mathbb{R} &\longrightarrow \mathbb{R} \\ \omega(x, y) &= ⟪x, \mathbb{J}y⟫ \end{aligned}$$

is nondegenerate, skew-symmetric and bilinear that is, a *symplectic form*. Such a symplectic form is *of type (1,1) with respect to* $\mathbb{J}$; that is, it satisfies

$$\omega(\mathbb{J}x, \mathbb{J}y) = \omega(x, y).$$

Given $(E_{\mathbb{R}}, \mathbb{J})$ there is a bijection between $\mathbb{J}$-invariant nondegenerate symmetric bilinear forms $(\!(,)\!)$ and symplectic forms ω of type (1,1). Then ω is *positive (respectively negative)* if and only if the corresponding symmetric bilinear form $(\!(,)\!)$ is positive (respectively negative) definite.

If E is a complex vector space, then a Hermitian structure on E consists of a map $\langle,\rangle : E \times E \longrightarrow \mathbb{C}$ satisfying the following properties:

1. (Linearity) For any $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u_1, u_2, v \in E$,
$$\langle \lambda_1 u_1 + \lambda_2 u_2, v\rangle = \lambda_1 \langle u_1, v\rangle + \lambda_2 \langle u_2, v\rangle.$$
2. (Symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
3. (Nondegeneracy) For each nonzero $u \in E$, the map $v \mapsto \langle v, u\rangle$ is a nonzero linear functional.

The underlying real vector space carries a $\mathbb{J}$-invariant nondegenerate symmetric bilinear form, defined by

$$\begin{aligned}(\!(,)\!) : E_{\mathbb{R}} \times E_{\mathbb{R}} &\longrightarrow \mathbb{R}\\ (\!(x, y)\!) &= \mathrm{Re}\langle x, y\rangle\end{aligned}$$

and a symplectic structure of type (1,1) with respect to $\mathbb{J}$ by

$$\begin{aligned}\omega : E_{\mathbb{R}} \times E_{\mathbb{R}} &\longrightarrow \mathbb{R}\\ \omega(x, y) &= \mathrm{Im}\langle x, y\rangle.\end{aligned} \tag{2.4}$$

A Hermitian structure is *positive* (respectively *negative)* if and only if the corresponding symmetric bilinear form $(\!(,)\!)$ is positive (respectively negative) definite. For example, the standard positive definite Hermitian structure $\langle\!\langle,\rangle\!\rangle$ on $\mathbb{C}^n$ is given by

$$\langle\!\langle z, w\rangle\!\rangle = \sum_{j=1}^{n} z_j \bar{w}_j.$$

Another Hermitian structure which we use frequently is the one on $\mathbb{C}^{n,1}$—this is $\mathbb{C}^{n+1}$ with the Hermitian form defined by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j - z_{n+1}\bar{w}_{n+1}.$$

The Hermitian structure on E and the orthogonal and symplectic structures on $E_{\mathbb{R}}$ are related by

$$\langle u, v\rangle = (\!(u, v)\!) + i\omega(u, v).$$

Under these correspondences, the following three categories are isomorphic:

1. Hermitian vector spaces (complex vector spaces with Hermitian structure).
2. Real vector spaces with complex structure $\mathbb{J}$ and a $\mathbb{J}$-invariant nondegenerate symmetric bilinear form.
3. Symplectic vector spaces (real vector spaces with a nondegenerate skew-symmetric bilinear form ω) with a complex structure $\mathbb{J}$ for which ω is of type (1,1) with respect to $\mathbb{J}$.

Let E be a Hermitian vector space. The group $\mathbf{U}(E)$ of unitary automorphisms of E consists of all $A \in \mathbf{End}(E)$ such that $A^*A = I$ where A^* denotes adjoint with respect to the Hermitian form

$$\langle AX, Y\rangle = \langle X, A^*Y\rangle.$$

Its Lie algebra $\mathfrak{u}(E)$ consists of skew-adjoint endomorphisms: $A + A^* = 0$. (Notice that $A \in \mathbf{End}(E)$ is skew-adjoint if and only if iA is self-adjoint.)

Suppose that (V, ω) is a symplectic vector space. Let $U \subset V$ be a linear subspace. Then the *radical of* U is the subspace

$$\sqrt{U} = \{u \in U \mid \omega(u, v) = 0 \quad \forall v \in U\}$$

and ω defines a nondegenerate symplectic structure ω_F on the *reduced space* $U/\sqrt{U}$ by the obvious formula

$$\omega_U(u + \sqrt{U}, v + \sqrt{U}) = \omega(u, v).$$

A subspace $U \subset V$ is *symplectic* if and only if the restriction of ω to U is nondegenerate, that is if $\sqrt{U} = 0$. In that case the restriction ω to U defines a symplectic vector space $(U, \omega|_U)$.

2.2.1 *Real subspaces of Hermitian vector spaces*

Let $S \subset E_{\mathbb{R}}$ be an $\mathbb{R}$-linear subspace of $E_{\mathbb{R}}$, where E is a complex vector space. If E is a Hermitian vector space with positive Hermitian structure and $S \subset E_{\mathbb{R}}$ is complex, then S is a symplectic subspace of $(E_{\mathbb{R}}, \omega)$.

Now suppose that $\langle, \rangle$ is a Hermitian structure on E. We say that a subspace $S \subset E_{\mathbb{R}}$ is *totally real* if and only if S and its image $\mathbb{J}(S)$ are orthogonal with respect to $(\!(,)\!)$. (Note that a totally real subspace is purely real, but not conversely.) Since $(\!(u^{1,0}, v^{1,0})\!) = \mathrm{Re}\langle u, v\rangle$ and $(\!(u, v)\!) = \mathrm{Re}\langle u, v\rangle$ and $\langle u, \mathbb{J}v\rangle = -i\langle u, v\rangle$ for $u, v \in E$, we see that S is totally real if and only if the Hermitian product $\langle u, v\rangle$ is real for all $u, v \in S$. Since the symplectic structure is the imaginary part of the Hermitian structure (2.4), a real subspace is totally real (with respect to the Hermitian structure) if and only if it is *isotropic* (with respect to the symplectic structure):

$$\omega(x, y) = 0$$

for all $x, y \in S$.

2.2.2 *Angle of holomorphy*

Let $\langle , \rangle$ be a positive Hermitian form on a complex vector space E. The underlying real vector space $E_{\mathbb{R}}$ inherits a positive definite inner product and a complex structure $\mathbb{J}$. Let $S_1, S_2 \subset E_{\mathbb{R}}$ be linear subspaces. Define the *angle* $\angle(S_1, S_2)$ *between* S_1 *and* S_2 as the smallest angle $\angle(s_1, s_2)$ between nonzero vectors $s_1 \in S_1$ and $s_2 \in S_2$. Since each S_i is invariant under multiplication by -1, this angle lies between 0 and $\pi/2$.

Exercise 2.2.1 *Two subspaces* $S_1, S_2 \subset E_{\mathbb{R}}$ *are orthogonal if and only if* $\langle\langle s_1, s_2 \rangle\rangle = 0$ *for all* $s_1 \in S_1, s_2 \in S_2$.

Let $v_1, v_2 \in E$ be nonzero vectors. They span complex lines $\mathbb{C}v_i \subset E_{\mathbb{R}}$ whose angle satisfies $\angle(\mathbb{C}v_1, \mathbb{C}v_2) \leq \angle(v_1, v_2)$. Here is an algebraic expression for this angle in terms of the Hermitian structure:

Lemma 2.2.2

$$\cos(\angle(\mathbb{C}v_1, \mathbb{C}v_2)) = \frac{|\langle v_1, v_2 \rangle|}{\|v_1\|\|v_2\|}.$$

By contrast,

$$\cos(\angle(v_1, v_2)) = \frac{|\mathrm{Re}\langle v_1, v_2 \rangle|}{\|v_1\|\|v_2\|}.$$

Proof Replace v_j by $v_j/\|v_j\|$ to assume that v_1, v_2 are unit vectors. Minimizing the angle between vectors in $\mathbb{C}v_1$ and $\mathbb{C}v_2$ is equivalent to maximizing its cosine. The cosine between unit vectors in these planes equals

$$\cos(\angle(e^{i\psi_1}v_1, e^{i\psi_2}v_2)) = \mathrm{Re}(e^{i(\psi_1 - \psi_2)}\langle v_1, v_2 \rangle)$$

(for $\psi_1, \psi_2 \in \mathbb{R}$). The maximum value of this expression over all $\psi_1, \psi_2 \in \mathbb{R}$ equals $|\langle v_1, v_2 \rangle|$ as desired. □

Let $S \subset E_{\mathbb{R}}$ be an $\mathbb{R}$-linear subspace of dimension 2. Its *angle of holomorphy* $\mu(S)$ is defined as the angle $\angle(S, \mathbb{J}(S))$ between S and $\mathbb{J}(S)$. (Compare §1.3.3.) Clearly $\mu(S) = 0$ (respectively $\pi/2$) if and only if S is complex (respectively totally real).

Theorem 2.2.3 *Let* v_1, v_2 *be unit vectors in* $\mathbb{C}^n$ *with the usual Hermitian structure* $\langle\langle , \rangle\rangle$. *Let* $\mathrm{span}_{\mathbb{R}}(v_1, v_2)$ *denote the real 2-plane spanned by* v_1 *and* v_2 *and let* $\mu = \mu(\mathrm{span}_{\mathbb{R}}(v_1, v_2))$ *denote its angle of holomorphy. If* $\{v_1, v_2\}$ *is orthonormal, then*

$$\cos\mu = |\mathrm{Re}\langle\langle v_1, \mathbb{J}(v_2) \rangle\rangle| \tag{2.5}$$

and, in general,

$$\sin(\angle(\mathbb{C}v_1, \mathbb{C}v_2)) = \sin(\mu)\sin(\angle(v_1, v_2)). \tag{2.6}$$

Proof Let $S = \mathrm{span}_{\mathbb{R}}(\mathrm{v}_1, \mathrm{v}_2)$ and let $\mu = \mu(S)$ be the angle of holomorphy. Then $\cos(\mu)$ equals the maximum value of

$$\begin{aligned}&\langle\!\langle \cos(\theta_1)v_1 + \sin(\theta_1)v_2, \cos(\theta_2)\mathbb{J}v_1 + \sin(\theta_2)\mathbb{J}v_2 \rangle\!\rangle \\ &\qquad = (\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2))\langle\!\langle v_1, \mathbb{J}v_2\rangle\!\rangle = \sin(\theta_2 - \theta_1)\langle\!\langle v_1, \mathbb{J}v_2\rangle\!\rangle\end{aligned}$$

over all $\theta_1, \theta_2 \in \mathbb{R}$. This maximum value equals $|\langle\!\langle v_1, \mathbb{J}v_2\rangle\!\rangle|$, proving (2.5).

To prove (2.6), let $\phi = \angle(\mathbb{C}v_1, \mathbb{C}v_2)$ and $\theta = \angle(v_1, v_2)$. It suffices to consider the case $n = 2$. Two unit vectors in $\mathbb{C}^2$ which span complex lines meeting at angle ϕ may be represented (by applying an element of $\mathbf{U}(2)$) by vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \cos(\phi)e^{i\psi} \\ \sin(\phi) \end{bmatrix}.$$

The phase ψ and the angle θ are related by

$$\begin{aligned}\cos(\theta) &= \frac{\mathrm{Re}\langle\!\langle v_1, v_2\rangle\!\rangle}{\sqrt{\langle\!\langle v_1, v_1\rangle\!\rangle\langle\!\langle v_2, v_2\rangle\!\rangle}} \\ &= \cos(\phi)\cos(\psi).\end{aligned} \tag{2.7}$$

Replace v_2 by

$$\begin{aligned}\tilde{v}_2 &= \frac{v_2 - (\mathrm{Re}\langle\!\langle v_1, v_2\rangle\!\rangle)v_1}{\|v_2 - (\mathrm{Re}\langle\!\langle v_1, v_2\rangle\!\rangle)v_1\|} \\ &= \frac{1}{\sqrt{\sin^2(\phi) + \cos^2(\phi)\sin^2(\psi)}}\begin{bmatrix} i\cos(\phi)\sin(\psi) \\ \sin(\phi)\end{bmatrix}\end{aligned}$$

to make $\{v_1, \tilde{v}_2\}$ orthonormal. Apply (2.5):

$$\begin{aligned}\cos(\mu) &= |\mathrm{Re}\langle\!\langle v_1, \mathbb{J}\tilde{v}_2\rangle\!\rangle| \\ &= \frac{\cos(\phi)\sin(\psi)}{\sqrt{\sin^2(\phi) + \cos^2(\phi)\sin^2(\psi)}} \\ &= (1 + \tan^2(\phi)\csc^2(\psi))^{-1/2}\end{aligned}$$

which implies

$$1 + \tan^2(\phi)\csc^2(\psi) = \sec^2(\mu).$$

Therefore

$$\tan^2(\phi)\csc^2(\psi) = \tan^2(\mu)$$

which, when combined with (2.7), yields

$$\cos^2(\theta) = \cos^2(\phi) - \sin^2(\phi)\cot^2(\mu).$$

Applying the identities $\cos^2(\theta) = 1 - \sin^2(\theta)$, $\cos^2(\phi) = 1 - \sin^2(\phi)$ and $(1 + \cot^2(\mu))\sin^2(\mu) = 1$, one deduces

$$\sin(\phi) = \sin(\mu)\sin(\theta)$$

proving (2.6) as desired. □

Here is another invariant, useful for the calculations discussed later. Let Π_j denote the orthogonal projection of $\mathbb{C}^2$ onto U_j:

$$\Pi_j(v) = \langle\!\langle v, u_j\rangle\!\rangle u_j.$$

Define the angle $\eta = \eta(u_1, u_2)$ to be the angle between u_1 and $\Pi_1(u_2)$ in U_1. The function $\eta(u_1, u_2)$ is symmetric:

$$\begin{aligned}\eta(u_1, u_2) &= \angle(u_1, \langle\!\langle u_2, u_1\rangle\!\rangle u_1)\\ &= \cos^{-1}\left(\mathrm{Re}\langle\!\langle u_2, u_1\rangle\!\rangle / |\langle\!\langle u_2, u_1\rangle\!\rangle|\right)\\ &= \cos^{-1}\left(\mathrm{Re}\langle\!\langle u_1, u_2\rangle\!\rangle / |\langle\!\langle u_1, u_2\rangle\!\rangle|\right)\\ &= \eta(u_2, u_1)\end{aligned}$$

and satisfies

$$\cos(\theta) = \cos(\phi)\cos(\eta) \tag{2.8}$$

$$\begin{aligned}\cos(\mu)\sin(\theta) &= \sqrt{\sin^2(\theta) - \sin^2(\phi)}\\ &= \sqrt{\cos^2(\phi) - \cos^2(\theta)}\\ &= \cos(\phi)\sin(\eta).\end{aligned} \tag{2.9}$$

Here is a normal form for such vectors expressed in the parameters ϕ, η. Every pair of unit vectors $u_1, u_2 \in \mathbb{C}^2$ with $\phi(u_1, u_2) = \phi$ and $\eta(u_1, u_2) = \eta$ is equivalent by an element of $\mathbf{U}(2)$ to the pair

$$u_1 = \begin{bmatrix}1\\0\end{bmatrix}, \qquad u_2 = \begin{bmatrix}e^{i\eta}\cos(\phi)\\ \sin(\phi)\end{bmatrix}. \tag{2.10}$$

We shall use the parameters θ, ϕ, μ for the Riemannian angle, complex angle and holomorphy angle, respectively. These three parameters satisfy (2.6). Other authors use different conventions and I apologize for introducing another naming convention which is consistent with none of the preceding ones. For example, Giraud calls our complex angle ϕ the "premier pseudo-angle" and η satisfying (2.8) or (2.9) the "deuxième pseudo-angle" and develops trigonometric identities using these quantities. Hsiang [89] denotes our Riemannian angle θ by λ and uses θ to denote what we call

$$\begin{aligned}\psi &= \cos^{-1}(\operatorname{Re}\langle\langle u_1, iu_2\rangle\rangle)\\ &= \cos^{-1}(\cos(\mu)\sin(\theta)).\end{aligned}$$

Brehm [17] uses λ for the Riemannian angle (our "θ"), but his use of ϕ and ψ is the same as ours. Aravinda and Leuzinger [4] use Hsiang's notation, and introduce an angle μ which is complementary to our complex angle ϕ.

2.2.3 *Totally real subspaces*

Lemma 2.2.4 *Let E be a vector space with positive definite Hermitian structure $\langle\langle,\rangle\rangle$ and let $\mathbf{U}(E)$ denote its group of unitary automorphisms.*

1. *An orthonormal basis of a totally real subspace S is an orthonormal basis of the Hermitian vector space $S \otimes_{\mathbb{R}} \mathbb{C}$ it spans.*
2. *Every totally real subspace lies in a maximal totally real subspace. A totally real subspace $S \subset E_{\mathbb{R}}$ is maximal if and only if $\dim(S) = \dim_{\mathbb{C}}(E)$.*
3. *For each k, $\mathbf{U}(E)$ acts transitively on totally real k-dimensional subspaces of E.*
4. *For each $0 \leq k \leq n$, $\mathbf{U}(E)$ acts transitively on complex k-dimensional subspaces of E.*

Proof Let $S \subset E_{\mathbb{R}}$ be a totally real subspace of dimension k. Choose an orthonormal basis $x_1, \ldots, x_k$:

$$\operatorname{Re}\langle\langle x_j, x_k\rangle\rangle = (\!(x_j, x_k)\!) = \delta_{jk}.$$

Since S is totally real,

$$\operatorname{Im}\langle\langle x_j, x_k\rangle\rangle = (\!(x_j, \mathbb{J}x_k)\!) = 0$$

from which it follows

$$\langle\langle x_j, x_k\rangle\rangle = \delta_{jk}$$

proving 1.

Since $S \subset E_{\mathbb{R}}$ is totally real, $S \otimes_{\mathbb{R}} \mathbb{C}$ embeds in E. Thus $\dim_{\mathbb{R}}(S) \leq \dim_{\mathbb{C}}(E)$. If

$$\dim_{\mathbb{R}}(S) = k < n = \dim_{\mathbb{C}}(E)$$

then an orthonormal basis $\{x_1, \ldots, x_k\}$ of S extends to an orthonormal basis $\{x_1, \ldots, x_n\}$ of E; the $\mathbb{R}$-linear span P of $\{x_1, \ldots, x_n\}$ is then an n-dimensional (hence maximal) totally real subspace of E, proving 2. Now the unique linear map taking the standard unitary basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ to $\{x_1, \ldots, x_n\}$ is unitary. Furthermore it maps the totally real subspace $\mathbb{R}^k$ to P, from which 3 follows. To prove 4, let $F \subset E$ be a k-dimensional $\mathbb{C}$-linear subspace and choose a unitary basis $(f_1, \ldots, f_k)$ of F over $\mathbb{C}$. Extend $(f_1, \ldots, f_k)$ to a unitary basis f of E. The unique linear map taking the standard basis to f is unitary and sends $\mathbb{C}^k$ to F. □

A real form of a complex vector space (defined in §2.1.2) is a maximal purely real subspace. When this vector space has a Hermitian structure, we require that a real form is compatible with the Hermitian structure in the following sense: a *real form* of a Hermitian vector space E is a maximal *totally real* subspace of $E_{\mathbb{R}}$. (Compare Nicas [133].)

2.2.4 *Real structures on a Hermitian vector space*

Suppose that E is a (nondegenerate) Hermitian vector space. If $S \subset E$ is a real form of E, then $E_{\mathbb{R}}$ decomposes as an *orthogonal* direct sum

$$E_{\mathbb{R}} = S \oplus \mathbb{J}(S) \cong S \otimes_{\mathbb{R}} \mathbb{C}$$

and there is a unique real structure ρ of $E_{\mathbb{R}} \to E_{\mathbb{R}}$ (corresponding to complex conjugation) whose fixed-point set is S. Since S and $\mathbb{J}(S)$ are totally real, ρ is compatible with the Hermitian structure on E:

$$\langle \rho(x), \rho(y) \rangle = \overline{\langle x, y \rangle}, \tag{2.11}$$

that is ρ is *anti-unitary* with respect to $\langle , \rangle$. In this way real forms of a Hermitian vector space correspond to compatible real structures.

Here is an interpretation of these conditions in terms of matrices. Continuing the notation of §2.1.4, consider $E = \mathbb{C}^n$ and ρ_0 the standard real structure. Let $e_1, \ldots, e_n$ be the standard basis of E. In terms of a basis a Hermitian structure on E is described by a nondegenerate Hermitian $n \times n$ matrix h:

$$h_{i\bar{j}} = \langle e_i, e_j \rangle.$$

We shall assume that ρ_0 is compatible with the Hermitian structure determined by h; this is equivalent to the condition that $h_{i\bar{j}} \in \mathbb{R}$ for each $i, j = 1, \ldots, n$. As in §2.1.4, any other real structure ρ is determined by a matrix A satisfying (2.3). In terms of A the anti-unitarity condition (2.11) is equivalent to the condition

$$\bar{A}^{\dagger} h A = h$$

(whence A is *unitary with respect to* h) which can be rewritten

$$A = h^{-1} (\bar{A}^{\dagger})^{-1} h. \tag{2.12}$$

For a matrix A satisfying (2.3), condition (2.12) is equivalent to

$$A = h^{-1} A^{\dagger} h,$$

the condition that ϕ is *symmetric with respect to* the real symmetric matrix h. Thus, compatible real structures on a Hermitian vector space are parametrized by *unitary symmetric matrices.* (Compare the discussion in Jacobowitz [92] and Cartan [24].)

If S is the fixed-point set of ρ as above, then the restriction of $\langle , \rangle$ to S is a nondegenerate $\mathbb{R}$-valued symmetric bilinear form. Unitary automorphisms of a Hermitian vector space preserving a compatible real structure are accordingly orthogonal automorphisms of the corresponding symmetric bilinear form on S. Conversely, if S is a real vector space with nondegenerate inner product then the inner product on S naturally extends to a Hermitian structure on $S \otimes \mathbb{C}$. If g is an orthogonal automorphism of S, then g extends to a unitary automorphism of $S \otimes \mathbb{C}$ which commutes with the canonical real structure on $S \otimes \mathbb{C}$. In this way the space of compatible real structures may be identified with the homogeneous space $\mathbf{U}(E)/\mathbf{O}(S)$.

2.2.5 *Hermitian triple products*

An important invariant of triples of vectors in a Hermitian vector space is the *Hermitian triple product.* In §1.15 the product $\langle\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle\rangle$ was used to describe the area of a triangle in the complex elliptic or hyperbolic line. Cartan [21] uses the triple product to define his *invariant angulaire,* which we discuss extensively in §7.1. Brehm[17] uses a Hermitian triple product to describe his *shape invariant* for triangles (see §3.2).

Let $(E, \langle , \rangle)$ be a Hermitian vector space. Vectors $Z_1, \ldots, Z_n \in E$ in a Hermitian vector space span a totally real subspace if and only if each Hermitian product $\langle Z_i, Z_j \rangle \in \mathbb{R}$. Obviously every 1-dimensional real subspace is totally real. Given two vectors Z_1, Z_2, there exist scalars $\lambda_i \in \mathbb{C}^*$ such that $\tilde{Z}_i = \lambda_i Z_i$ span a totally real subspace. Simply take $\lambda_1 = 1$, and $\lambda_2 = \langle Z_1, Z_2 \rangle$ if $\langle Z_1, Z_2 \rangle \neq 0$.

Suppose $Z_1, Z_2, Z_3 \in E$ are three vectors. The three complex lines they span may or may not contain nonzero vectors spanning (over $\mathbb{R}$) a totally real subspace. A convenient criterion involves their *Hermitian triple product:*

$$\langle Z_1, Z_2, Z_3 \rangle = \langle Z_1, Z_2 \rangle \langle Z_2, Z_3 \rangle \langle Z_3, Z_1 \rangle \in \mathbb{C}. \tag{2.13}$$

Although this complex number depends on the vectors Z_1, Z_2, Z_3, its *argument* depends only on their images $\mathbb{P}(Z_1), \mathbb{P}(Z_2), \mathbb{P}(Z_3) \in \mathbb{P}(E)$ in projective space. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$ be nonzero scalars write $\tilde{Z}_i = \lambda_i Z_i$. Then

$$\langle \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \rangle = |\lambda_1|^2 |\lambda_2|^2 |\lambda_3|^2 \langle Z_1, Z_2, Z_3 \rangle. \tag{2.14}$$

(Compare Jacobowitz [92], §9.1, Theorem 4.) Thus

$$\arg(\langle \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \rangle) = \arg(\langle Z_1, Z_2, Z_3 \rangle)$$

as claimed.

Lemma 2.2.5 *Suppose $\langle Z_1, Z_2, Z_3\rangle \neq 0$. The following conditions are equivalent:*

1. $\langle Z_1, Z_2, Z_3\rangle$ *is real.*
2. *There exist nonzero scalars*

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$$

such that

$$\tilde{Z}_1 = \lambda_1 Z_1, \ \tilde{Z}_2 = \lambda_2 Z_2, \ \tilde{Z}_3 = \lambda_3 Z_3$$

span a totally real subspace.

Proof If $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ span a totally real subspace, then each Hermitian product $\langle \tilde{Z}_i, \tilde{Z}_j\rangle$ is real and by (2.14), $\langle \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3\rangle$ is real. Conversely, suppose that $\langle Z_1, Z_2, Z_3\rangle \in \mathbb{R}$. Setting $\lambda_1 = 1$ and $\lambda_2 = \langle Z_1, Z_2\rangle$, we have $\langle \tilde{Z}_1, \tilde{Z}_2\rangle$ is real. Setting $\lambda_3 = \langle Z_1, Z_3\rangle$, we have $\langle \tilde{Z}_3, \tilde{Z}_1\rangle$ is real. Since $\langle \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3\rangle \in \mathbb{R}$, it follows that $\langle \tilde{Z}_2, \tilde{Z}_3\rangle$ is real. Since all Hermitian products are real, it follows that $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ span a totally real subspace. □

2.2.6 *Outer products on a Hermitian vector space*

Let E^* denote the complex vector space dual to E and let $\bar{E}$ denote its conjugate. The Hermitian form induces an isomorphism of complex vector spaces

$$\begin{aligned} \bar{E} &\longrightarrow E^* \\ v &\longmapsto v^* \end{aligned}$$

where

$$v^* : u \longmapsto \langle u, v\rangle$$

is the linear functional associated to v. The algebra $\mathbf{End}(E)$ consisting of $\mathbb{C}$-linear endomorphisms is canonically isomorphic to $E \otimes_{\mathbb{C}} E^*$. Let $x, y \in E$. Their *Hermitian outer product* is the endomorphism of E defined by

$$xy^* : u \longmapsto \langle u, y\rangle x.$$

Indeed this construction defines an isomorphism of complex vector spaces:

$$\begin{aligned} E \otimes_{\mathbb{C}} \bar{E} &\longrightarrow \mathbf{End}(E) \\ x \otimes y &\longmapsto xy^*. \end{aligned}$$

Clearly

$$\langle xy^*(u), v\rangle = \langle u, yx^*(v)\rangle$$

whence xy^* and yx^* are adjoint endomorphisms (implicitly using the anti-linear isomorphism $E \to \bar{E}$ identifying E with $\bar{E}$). The map

$$\rho(x \otimes y) = -y \otimes x$$

defines a real structure on $E \otimes_{\mathbb{C}} \bar{E}$ whose fixed-point set corresponds to the Lie subalgebra $\mathfrak{u}(E)$ of skew-adjoint endomorphisms of E. In particular $xy^* + yx^*$ is self-adjoint, while $xy^* - yx^*$ and $i(xy^* + yx^*)$ are skew-adjoint.

2.2.7 *Cross-products*

Let E be a 3-dimensional Hermitian vector space. Then the complex line $\wedge^3 E$ inherits a Hermitian structure defined by

$$\langle a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3 \rangle = \begin{vmatrix} \langle a_1, b_1 \rangle & \langle a_2, b_1 \rangle & \langle a_3, b_1 \rangle \\ \langle a_1, b_2 \rangle & \langle a_2, b_2 \rangle & \langle a_3, b_2 \rangle \\ \langle a_1, b_3 \rangle & \langle a_2, b_3 \rangle & \langle a_3, b_3 \rangle \end{vmatrix}.$$

This Hermitian structure is positive if and only if E is positive or has signature (1,2), and is negative if and only if E is negative or has signature (2,1). Choose $\omega_0 \in \wedge^3 E$ compatible with the Hermitian structure in the sense that

$$\langle \omega_0, \omega_0 \rangle = \pm 1.$$

(Such a generator is unique up to multiplication by a unit complex number.) There is an alternating bilinear map $\boxtimes : E \otimes E \longrightarrow \bar{E}$ defined by

$$\langle a, b \boxtimes c \rangle \, \omega_0 = a \wedge b \wedge c$$

called *Hermitian cross-product.* Clearly

$$\langle a, a \boxtimes b \rangle = \langle b, a \boxtimes b \rangle = 0$$

and if a, b are linearly independent vectors, they span the 2-plane

$$(a \boxtimes b)^{\perp} = \{u \in E \mid \langle u, a \boxtimes b \rangle = 0\}.$$

For the remainder of this section we assume that $E = \mathbb{C}^{2,1}$ so that

$$\begin{vmatrix} \langle a, a \rangle & \langle b, a \rangle & \langle c, a \rangle \\ \langle a, b \rangle & \langle b, b \rangle & \langle c, b \rangle \\ \langle a, c \rangle & \langle b, c \rangle & \langle c, c \rangle \end{vmatrix} = -\langle a \boxtimes b, c \rangle \langle c, a \boxtimes b \rangle \leq 0. \tag{2.15}$$

Taking $c = a \boxtimes b$, we obtain

$$\langle a \boxtimes b, a \boxtimes b \rangle^2 = - \begin{vmatrix} \langle a, a \rangle & \langle b, a \rangle & 0 \\ \langle a, b \rangle & \langle b, b \rangle & 0 \\ 0 & 0 & \langle a \boxtimes b, a \boxtimes b \rangle \end{vmatrix}$$

whence

$$\langle a \boxtimes b, a \boxtimes b \rangle = \langle b, a \rangle \langle a, b \rangle - \langle b, b \rangle \langle a, a \rangle. \tag{2.16}$$

Therefore if a, b are linearly independent null vectors then $a \boxtimes b$ is positive.

2.2.8 *Relation with Hermitian triple products*

The nonnegativity formula (2.15) implies conditions on the Hermitian triple product $\langle a, b, c\rangle$ defined in (2.13). Suppose that $a, b, c \in \mathbb{C}^{2,1}$ are null vectors; then

$$\mathrm{Re}\langle a, b, c\rangle = \frac{1}{2}\begin{vmatrix} \langle a, a\rangle & \langle b, a\rangle & \langle c, a\rangle \\ \langle a, b\rangle & \langle b, b\rangle & \langle c, b\rangle \\ \langle a, c\rangle & \langle b, c\rangle & \langle c, c\rangle \end{vmatrix} \leq 0. \tag{2.17}$$

This will be interpreted geometrically in §7.1.1.

For the next application let a, b be null vectors and c a positive vector. Then by (2.15),

$$|\langle a \boxtimes b, c\rangle|^2 = |\langle a, b\rangle|^2\langle c, c\rangle - 2\mathrm{Re}\langle a, c\rangle\langle c, b\rangle\langle b, a\rangle \tag{2.18}$$

and (dividing by $|\langle a, b\rangle|^2\langle c, c\rangle$) we obtain

$$1 - 2\mathrm{Re}(\eta(a, b; c)) = \frac{|\langle a \boxtimes b, c\rangle|^2}{|\langle a, b\rangle|^2\langle c, c\rangle} \geq 0 \tag{2.19}$$

where $\eta(a, b; c)$ is defined as

$$\eta(a, b; c) = \frac{\langle c, b\rangle\langle a, c\rangle}{\langle a, b\rangle\langle c, c\rangle}.$$

It follows from (2.19) that the complex number $\eta(a, b; c)$ satisfies

$$\mathrm{Re}(\eta(a, b; c)) \leq \frac{1}{2}.$$

This result will be interpreted geometrically in 7.3.1.

Closely related is the decomposition formula. Suppose that a, b are linearly independent null vectors. Then $\langle a, b\rangle \neq 0$ and

$$c = \langle a, b\rangle^{-1} a \boxtimes b$$

satisfies

$$\langle c, c\rangle = 1$$

and the Hermitian form on $\mathbb{C}^{2,1}$ decomposes as

$$\mathbb{I}_{2,1} = c \otimes c^* + \frac{1}{2}(a \otimes b^* + b \otimes a^*).$$

For any $x, y \in E$ we have

$$\langle x, y\rangle = \langle x, c\rangle\langle c, y\rangle + \frac{1}{2}(\langle x, b\rangle\langle a, y\rangle + \langle x, a\rangle\langle b, y\rangle). \tag{2.20}$$

In a later application we suppose that $Q_j^{\pm}$ $(j = 1, 2)$ are null vectors normalized so that $\langle Q_j^-, Q_j^+\rangle = 2$. Then $Q_j^0 = \frac{1}{2} Q_j^- \boxtimes Q_j^+$ satisfies $\langle Q_j^0, Q_j^0\rangle = 1$. By (2.18),

$$|\langle Q_1^0, Q_2^0\rangle|^2 = |\langle \tfrac{1}{2} Q_1^- \boxtimes Q_1^+, Q_2^0\rangle|^2 = 1 - \operatorname{Re}\left(\langle Q_1^-, Q_2^0\rangle\langle Q_2^0, Q_1^+\rangle\right)$$

and by taking $x = Q_1^-, y = Q_1^+, a = Q_2^-, b = Q_2^+, c = Q_2^0$ in (2.20), we obtain

$$\langle Q_1^-, Q_2^0\rangle\langle Q_2^0, Q_1^+\rangle = 2 - \frac{\langle Q_1^-, Q_2^+\rangle\langle Q_2^-, Q_1^+\rangle + \langle Q_1^-, Q_2^-\rangle\langle Q_2^+, Q_1^+\rangle}{2}$$

and thus

$$|\langle Q_1^0, Q_2^0\rangle|^2 = -1 + \frac{\operatorname{Re}\left(\langle Q_1^-, Q_2^+\rangle\langle Q_2^-, Q_1^+\rangle + \langle Q_1^-, Q_2^-\rangle\langle Q_2^+, Q_1^+\rangle\right)}{2}. \tag{2.21}$$

Explicitly, in $\mathbb{C}^{2,1}$ we have

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \boxtimes \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{vmatrix} \bar{a}_1 & \bar{b}_1 & \vec{e}_1 \\ \bar{a}_2 & \bar{b}_2 & \vec{e}_2 \\ -\bar{a}_3 & -\bar{b}_3 & \vec{e}_3 \end{vmatrix} = \begin{bmatrix} \bar{a}_3\bar{b}_2 - \bar{a}_2\bar{b}_3 \\ \bar{a}_1\bar{b}_3 - \bar{a}_3\bar{b}_1 \\ \bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1 \end{bmatrix}.$$

2.2.9 *The real structure preserving three lines*

Applying these ideas, we derive a formula for the real structure fixing a triple of null vectors. Let E be a 3-dimensional Hermitian vector space and $\{e_1, e_2, e_3\}$ a basis of a totally real subspace $F \subset E$. We find the compatible real structure fixing the corresponding three points in projective space. Assume e_1, e_2, e_3 are normalized so that all their Hermitian products are real. Let

$$\begin{aligned} C_1 &= e_2 \boxtimes e_3 \\ C_2 &= e_3 \boxtimes e_1 \\ C_3 &= e_1 \boxtimes e_2 \end{aligned}$$

be the corresponding cross-products, so that $\langle C_j, e_k\rangle = 0$ if $j \neq k$. Define

$$\rho(Z) = \sum_{j=1}^{3} \frac{\langle C_j, Z\rangle}{\langle C_j, e_j\rangle} e_j.$$

Then $\rho : E \longrightarrow E$ is evidently anti-linear and fixes each e_j. We compute that ρ has order 2:

$$\begin{aligned} \rho(\rho(Z)) &= \sum_{j=1}^{3} \frac{1}{\langle C_j, e_j\rangle} \Big\langle C_j, \sum_{i=1}^{3} \frac{\langle C_i, Z\rangle}{\langle C_i, e_i\rangle} e_i \Big\rangle e_j \\ &= \sum_{i,j=1}^{3} \frac{\langle Z, C_i\rangle}{\langle e_i, C_i\rangle} \frac{\langle C_j, e_i\rangle}{\langle C_j, e_j\rangle} e_j \\ &= \sum_{j=1}^{3} \frac{\langle Z, e_j\rangle}{\langle e_j, C_j\rangle} e_j = Z \end{aligned}$$

(because

$$\left\langle \sum_{j=1}^{3} \frac{\langle Z, e_j \rangle}{\langle e_j, C_j \rangle} e_j, C_i \right\rangle = \langle Z, C_i \rangle$$

and C_1, C_2, C_3 form a basis for $\mathbb{C}^{2,1}$). To see that ρ is anti-unitary, write

$$Z = \sum_{j=1}^{3} z^j e_j, \qquad W = \sum_{j=1}^{3} w^j e_j$$

so that $\langle C_j, Z \rangle = z^j \langle C_j, e_j \rangle$ and $\langle C_j, W \rangle = w^j \langle C_j, e_j \rangle$. Then

$$\begin{aligned}
\langle \rho(Z), \rho(W) \rangle &= \sum_{i,j=1}^{3} \left\langle \frac{\langle C_i, Z \rangle}{\langle C_i, e_i \rangle} e_i, \frac{\langle C_j, W \rangle}{\langle C_j, e_j \rangle} e_j \right\rangle \\
&= \sum_{i,j=1}^{3} \bar{z}^i \langle e_i, e_j \rangle w^j \\
&= \sum_{i,j=1}^{3} w^j \langle e_j, e_i \rangle \bar{z}^i \quad \text{(since } \langle e_i, e_j \rangle \text{ are real)} \\
&= \langle W, Z \rangle.
\end{aligned}$$

Thus ρ is the desired real structure.

2.3 Symplectic geometry

Let M be a smooth manifold. A *symplectic structure on* M is a closed nondegenerate exterior 2-form ω on M. In particular for each $p \in M$, the alternating tensor $\omega_p : T_p \times T_p \longrightarrow \mathbb{R}$ determines a symplectic structure on the tangent space T_pM. The condition $d\omega = 0$ that ω is closed is a local compatibility of the ω_p between nearby tangent spaces. Nondegeneracy of ω is equivalent to the top exterior power ω^n (where $\dim M = 2n$) being a volume form. The *Darboux theorem* (see, for example, McDuff–Salamon [120], Weinstein [167]) asserts that for any point $p \in M$ there exists an open neighborhood U and a smooth coordinate system $\psi : U \longrightarrow \mathbb{R}^{2n}$ such that

$$\omega = \psi^* \sum_{j=1}^{n} dx_j \wedge dy_j$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are coordinates on $\mathbb{R}^{2n}$.

An exterior 2-form ω is exact if $\omega = d\alpha$ for a 1-form α; abusing terminology a 1-form α is an *exact symplectic structure* if and only if $d\alpha$ is a symplectic structure. The cotangent bundle of any smooth manifold M has a canonical exact symplectic structure invariant under the diffeomorphism group of M.

Given a symplectic structure ω on M, every smooth function $f : M \longrightarrow \mathbb{R}$ determines a vector field Hf, the *Hamiltonian vector field* which is defined by the condition that for every vector field Y on M,

$$\omega(Hf, Y) = df(Y) = Yf.$$

(This is formally analogous to the construction of the gradient vector field associated to a function f on a smooth Riemannian manifold. The Riemannian metric establishes an isomorphism between the tangent bundle and the cotangent bundle. Under the corresponding isomorphism between 1-forms and vector fields, the differential df corresponds to the gradient vector field. Whereas flows of gradient vector fields distort the metric and increase the function f, Hamiltonian flows conserve both the symplectic structure and the potential function. These conservation laws immediately follow from ω being closed and alternating.)

The ring $C^\infty(M)$ of smooth functions on M inherits the structure of a Lie algebra under *Poisson bracket:*

$$[f,g] = \omega(Hf, Hg) = -(Hf)g = (Hg)f.$$

Let $\mathbf{Symp}(M,\omega)$ denote the Lie algebra of vector fields preserving ω (that is, vector fields X such that the Lie derivative $\mathfrak{L}_X\omega = 0$). Let $H^1(M)$ be the first de Rham cohomology group with the structure of an abelian Lie algebra. The locally constant functions form a central subalgebra $H^0(M) \subset C^\infty(M)$ and the diagram of Lie algebra homomorphisms

$$0 \longrightarrow H^0(M) \longrightarrow C^\infty(M) \longrightarrow \mathbf{Symp}(M,\omega) \longrightarrow H^1(M) \longrightarrow 0$$

is exact. If $[f,g] = 0$, then the flow of the Hamiltonian vector field Hf leaves invariant the function g.

The *momentum mapping* and *symplectic quotient* construction for symplectic manifolds (due to Marsden and Weinstein [115]) unifies many examples in geometry and reveals structure. We develop both complex projective space and many of the objects in complex hyperbolic in terms of this construction.

Suppose that $S \subset M$ is a smooth submanifold. Define for each $x \in S$,

$$N_x(S) = \sqrt{T_xS} = \{u \in T_xS \mid \omega(u,v) = 0 \quad \forall v \in T_xS\}.$$

If the dimension of $N_x(S)$ is a constant function of x, then $N(S)$ is a vector subbundle of the tangent bundle TS and is integrable (a consequence of $d\omega = 0$). Let $\mathfrak{N}$ denote the foliation of S tangent to $N(S)$. The symplectic form ω defines a symplectic structure on each normal space $T_xS/N_x(S)$ as in §2.2. In particular if the leaf space $S/\mathfrak{N}$ is a smooth manifold, it inherits the structure of a symplectic manifold, the *symplectic quotient* of $S \subset M$.

2.3.1 *Momentum mappings and symplectic quotients*

Particularly interesting is when G is a Lie group acting symplectically on (M,ω) whose Lie algebra $\mathfrak{g}$ acts by Hamiltonian vector fields. Suppose that there is a

Lie algebra homomorphism $\phi : \mathfrak{g} \longrightarrow C^\infty(M)$ such that for each $\xi \in \mathfrak{g}$, the vector field $H(\phi(\xi))$ infinitesimally generates the flow $\{\exp(t\xi) \mid t \in \mathbb{R}\}$. (Such a homomorphism is called *a Hamiltonian action.*) Dually the *momentum mapping*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

defined by

$$\mu(x)(\xi) = \phi(\xi)(x)$$

(for $x \in M$, $\xi \in \mathfrak{g}$) is equivariant with respect to the action of G on M and the coadjoint action of G on the vector space dual to its Lie algebra. In particular each orbit in M is mapped onto an orbit in $\mathfrak{g}^*$.

Suppose 0 is a *regular value* of μ; that is, for each $p \in \mu^{-1}(0)$, the differential

$$d\mu_p : T_pM \longrightarrow T_0\mathfrak{g}^*$$

is surjective. Applying this procedure to $\mu^{-1}(0)$ yields a symplectic structure on the quotient $\mu^{-1}(0)/G$, provided the orbit space is a smooth manifold. This is the *Marsden–Weinstein reduction* of the Hamiltonian action, introduced in [115].

Here is an example. Let E be a Hermitian vector space with Hermitian structure $\langle , \rangle$. The corresponding symplectic vector space is $E_{\mathbb{R}}$ with symplectic structure ω defined by

$$\omega(v, w) = \mathrm{Im}\langle v, w\rangle.$$

Denote the group of unitary automorphisms of E by $\mathbf{U}(E)$ and its Lie algebra consisting of skew-adjoint endomorphisms of E by $\mathfrak{u}(E)$. Identify $\mathfrak{u}(E)$ with its $\mathbb{R}$-linear dual vector space via the *trace form*

$$\begin{aligned} \mathfrak{u}(E) \times \mathfrak{u}(E) &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto X \cdot Y := \mathrm{trace}(XY). \end{aligned}$$

Since X, Y are skew-adjoint,

$$\mathrm{trace}(XY) = \mathrm{trace}(X^*Y^*) = \mathrm{trace}(Y^*X^*) = \overline{\mathrm{trace}(XY)} \in \mathbb{R}.$$

Theorem 2.3.1 *The quadratic map of real vector spaces*

$$\begin{aligned} \mu : E_{\mathbb{R}} &\longrightarrow \mathfrak{u}(E) \\ v &\mapsto \frac{i}{2} vv^* \end{aligned}$$

(where vv^* is defined as in §2.2.6) is an equivariant momentum mapping for the action of $\mathbf{U}(E)$ on the symplectic vector space $(E_{\mathbb{R}}, \omega)$.

Proof We first prove equivariance. Let $P \in \mathbf{U}(E)$ and $v \in E$. Then $P^* = P^{-1}$ and

$$\mu(Pv) = \frac{i}{2}(Pvv^*P^*) = \frac{i}{2}(Pvv^*P^{-1}) = \mathrm{Ad}(P)\left(\frac{i}{2}vv^*\right) = \mathrm{Ad}(P)\mu(v).$$

We next show that μ is a momentum mapping. This means that for each $\xi \in \mathfrak{u}(E)$, the corresponding vector field Ξ on $E_{\mathbb{R}}$ has Hamiltonian potential function

$$p \mapsto \mu(p) \cdot \xi.$$

This means that for each tangent vector $w \in T_pE_{\mathbb{R}}$,

$$\omega_p(\Xi(p), w) = d\mu_p(w) \cdot \xi.$$

Let $v \in E$ correspond to a point p of $E_{\mathbb{R}}$ and $u \in E$ correspond to the tangent vector w to $E_{\mathbb{R}}$. In terms of the skew-Hermitian matrix ξ, the tangent vector $\Xi(p)$ is the velocity vector to the trajectory

$$v \mapsto e^{t\xi}(v)$$

and equals the vector $\xi \cdot v$ given by matrix multiplication. Hence

$$\begin{aligned}
\omega_p(\Xi(p), w) &= \mathrm{Im}\langle \xi \cdot v, u\rangle \\
&= \frac{i}{2}(-2i\mathrm{Im}\langle v, \xi(u)\rangle) \\
&= \frac{i}{2}(-\langle v, \xi(u)\rangle + \langle \xi(u), v\rangle) \\
&= \frac{i}{2}(\langle \xi(v), u\rangle + \langle \xi(u), v\rangle) \\
&= \frac{1}{2}\mathrm{Re}\ \mathrm{trace}(i(vu^* + uv^*)\xi)
\end{aligned}$$

Now express the differential of the momentum mapping in terms of matrices:

$$\begin{aligned}
d\mu_{(v)}(u) &= \left.\frac{d}{dt}\right|_{t=0} \frac{i}{2}(v + tu)(v + tu)^* \\
&= \frac{i}{2}(vu^* + uv^*).
\end{aligned}$$

Thus

$$d\mu_{(v)}(u) \cdot \xi = \frac{1}{2}\mathrm{Retrace}(i(vu^* + uv^*)\xi) = \omega_p(\Xi(p), w)$$

as desired. $\square$

2.3.2 *The momentum mapping in coordinates*

In terms of a basis $\{e_j\}_{1\leq j\leq n}$ of E over $\mathbb{C}$, consider the Hermitian matrix $H = (h_{i\bar{j}})_{1\leq i,j\leq n}$:

$$h_{i\bar{j}} = \langle e_i, e_j\rangle.$$

If $v = v^i e_i \in E$, then

$$v^* = h_{i\bar{j}}\overline{v^j}e^i$$

where $\{e^i\}_{1\leq i\leq n}$ is the basis of E^* dual to $\{e_j\}_{1\leq j\leq n}$. Then μ is given by

$$\mu(v)_{ij} = ih_{i\bar{k}}\overline{v^k}v_j.$$

Let $E = \mathbb{C}^n$ with the standard positive definite Hermitian structure. Then

$$\mu(z) = \frac{i}{2}\begin{bmatrix} z_1\bar{z}_1 & \dots & z_1\bar{z}_n \\ \vdots & \vdots & \vdots \\ z_n\bar{z}_1 & \dots & z_n\bar{z}_n \end{bmatrix}.$$

For the standard indefinite structure $E = \mathbb{C}^{n,1}$ corresponding to the diagonal Hermitian matrix $\mathbb{I}_n \oplus (-1)$, there is an analogous formula (see 3.1.3):

$$\mu(z) = \frac{i}{2}\begin{bmatrix} z_1\bar{z}_1 & \dots & z_1\bar{z}_n & -z_1\bar{z}_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ z_n\bar{z}_1 & \dots & z_n\bar{z}_n & -z_n\bar{z}_{n+1} \\ z_{n+1}\bar{z}_1 & \dots & z_{n+1}\bar{z}_n & -z_{n+1}\bar{z}_{n+1} \end{bmatrix}.$$

2.3.3 *Projective space as a symplectic quotient*

Complex projective space itself is a symplectic quotient $S^{2n+1}/\mathbb{T}$. Let $M = \mathbb{C}^{n+1} - \{0\}$ and consider the unitary $\mathbb{T}$-action by scalar multiplication. This action embeds in the action of the unitary group $\mathbf{U}(n+1)$ on $\mathbb{C}^{n+1}$ (indeed, it is the *center* of $\mathbf{U}(n+1)$). The embedding of $\mathbb{T}$ in $\mathbf{U}(n+1)$ is given by

$$\zeta \longmapsto \zeta\mathbb{I}_{n+1}$$

and the corresponding Lie algebra projection is given by

$$\begin{aligned} \mathfrak{u}(n+1) &\longrightarrow \mathbb{R} \\ X &\longmapsto -i\mathrm{trace}(X). \end{aligned}$$

Therefore the Hamiltonian potential function $f : M \longrightarrow \mathbb{R}$ for the $\mathbb{T}$-action in $\mathbf{U}(n+1)$ is given by the composition

$$\begin{aligned} M &\xrightarrow{\mu} \mathfrak{u}(n+1) \longrightarrow \mathbb{R} \\ v &\longmapsto -i\mathrm{trace}\left(\frac{i}{2}vv^*\right) = \frac{1}{2}\langle\!\langle v, v\rangle\!\rangle. \end{aligned}$$

The level sets of the momentum mapping are thus the concentric spheres $S^{2n+1}(R)$ and the corresponding symplectic quotients are the projective spaces $S^{2n+1}(R)/\mathbb{T} = \mathbb{P}(\mathbb{C}^{n+1})$. (Compare [81], §23, pp.162–164).

From this construction one derives the Kähler form and the whole Fubini–Study Kähler geometry of $\mathbb{P}^n_{\mathbb{C}}$, since the complex structure on M descends to a complex structure on the quotient. We shall obtain the Bergman Kähler geometry of complex hyperbolic space by a completely analogous construction (§3.1.3).

2.4 Complex analysis

A *complex manifold* (of complex dimension n) is a smooth manifold M with a distinguished atlas of charts into $\mathbb{C}^n$ such that the coordinate changes are local biholomorphic automorphisms of domains in $\mathbb{C}^n$. The tangent bundle TM of M is a *complex vector bundle.* It is a real vector bundle with a *complex structure*, an endomorphism $\mathbb{J} : TM \to TM$ such that $\mathbb{J}^2 = -I$. (This endomorphism is the $\mathbb{R}$-linear map corresponding to multiplication by $i = \sqrt{-1}$.) The complexified tangent bundle $TM \otimes \mathbb{C}$ then decomposes as a direct sum

$$TM \otimes \mathbb{C} = T^{1,0}(M) \oplus T^{0,1}(M)$$

where $T^{1,0}(M)$ is the $+i$-eigenspace of the ($\mathbb{C}$-linear) map

$$\mathbb{J} \otimes \mathbb{C} : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$$

and $T^{0,1}(M)$ is the $-i$-eigenspace. Similarly its dual, the complexified cotangent bundle, splits as

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$$

where $\Lambda^{1,0}(M)$ (respectively $\Lambda^{0,1}(M)$) is the complex vector bundle dual to $T^{1,0}(M)$ (respectively $T^{0,1}(M)$). The kth exterior power

$$\Lambda^k(TM) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} M$$

where

$$\Lambda^{p,q} M = \Lambda^p(\Lambda^{1,0} M) \otimes_{\mathbb{C}} \Lambda^q(\Lambda^{0,1} M).$$

This decomposition is called the *Hodge decomposition.*

The spaces of sections decompose into Hodge types as well. The space $\Omega^k(M, \mathbb{C})$ of $\mathbb{C}$-valued smooth exterior k-forms on M splits into Hodge summands:

$$\Omega^k(M; \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

In terms of the complex structure an exterior 2-form $\omega \in \Omega^2(M; \mathbb{C})$ has Hodge type $(1, 1)$ if and only if

$$\omega(\mathbb{J}X, \mathbb{J}Y) = \omega(X, Y)$$

for all vectors $X, Y \in TM$. Suppose that $\omega \in \Omega^2(M;\mathbb{R})$ has Hodge type (1,1) and is nondegenerate. Then ω is the imaginary part of a (possibly indefinite) nondegenerate Hermitian metric: there exists a Hermitian metric $\langle , \rangle$ such that

$$\omega(X,Y) = \frac{1}{2i}(\langle X,Y\rangle - \langle Y,X\rangle).$$

Such a 2-form ω necessarily has type (1,1)).

The corresponding metric is Kähler if and only if

1. ω is *positive,* that is $\omega(\mathbb{J}X, X) > 0$ for all nonzero real tangent vectors X, and
2. ω is *closed,* that is $d\omega = 0$.

The latter condition implies that (M,ω) is a symplectic manifold.

A *Kähler structure* can thus be defined in any of the following equivalent ways:

1. A complex structure with a closed, positive (1,1)-form.
2. A Riemannian structure with a complex structure such that the corresponding exterior 2-form is closed.
3. A symplectic structure with a compatible integrable almost complex structure which is positive.

(For more discussion of the equivalent definitions of Kähler structures, see §2 of [125].) The complexified exterior derivative $d : \Omega^k(M;\mathbb{C}) \longrightarrow \Omega^{k+1}(M;\mathbb{C})$ splits as $d = \partial + \bar\partial$ where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar\partial : \Omega^{p,q} \to \Omega^{p,q+1}$. The integrability condition $d \circ d = 0$ becomes

$$\partial \circ \partial = \bar\partial \circ \bar\partial = \partial\bar\partial + \bar\partial\partial = 0.$$

2.4.1 *Kähler potentials*

The decomposition $d = \partial + \bar\partial$ suggests the use of the real operator

$$d^c = -i(\partial - \bar\partial) = 2\mathrm{Im}(\partial)$$

which satisfies:

$$2\partial = d + id^c, \qquad 2\bar\partial = d - id^c.$$

(See [78], [166], [168]—but beware of the different convention used in [78].)

Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. Its *complex Hessian* is the (1,1)-form

$$\partial\bar\partial f = -\frac{i}{2} dd^c f.$$

For every closed (1,1)-form ω, *locally* there exists a function f such that $\omega = \partial\bar\partial f$. That is, for every point $p \in M$, a neighborhood U of p in M and a function $f : U \longrightarrow \mathbb{C}$ exist such that $\partial\bar\partial f = -\frac{i}{2} dd^c f = \omega$. In particular every Kähler form is locally of the form $\partial\bar{\partial}f$. We call such a function f a *Kähler potential* for ω. If f, g are functions such that $\partial\bar{\partial}f = \partial\bar{\partial}g$, then $f - g$ is a pluriharmonic function. Locally every pluriharmonic function is the real part of a holomorphic function. (If the first Betti number vanishes, then every pluriharmonic function is the real part of a *globally defined holomorphic function.*) On the other hand, if M is a Kähler manifold, then a function $f : M \longrightarrow \mathbb{R}$ is harmonic if and only if $\partial\bar{\partial}f$ is orthogonal to the Kähler form Φ: for if $\Lambda : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1}$ denotes contraction with Φ, then standard first-order Kähler identities (as in [78]) imply:

$$\Delta f = d^*df = [\Lambda, d^c]df = \Lambda d^c df.$$

The real operator d^c relates to the complex structure ([166],p.34) as follows. To state this relation, it is enough to assume that M is only an *almost complex manifold*: a real manifold M together with an endomorphism $\mathbb{J} : TM \longrightarrow TM$ satisfying $\mathbb{J} \circ \mathbb{J} + \mathbb{I} = 0$. Let $\Omega^*(M)$ be the graded algebra of real-valued exterior differential forms on M. Then $\mathbb{J}$ acts as an automorphism of $\Omega^*(M)$ as follows. Let

$$\Omega^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

be the decomposition into Hodge types and let

$$\Pi^{p,q} : \Omega^*(M) \otimes \mathbb{C} \longrightarrow \Omega^{p,q}(M)$$

be the corresponding projection. Then the operator

$$\mathbb{J} = \bigoplus_{p,q} i^{p-q}\Pi^{p,q} : \Omega^*(M) \otimes \mathbb{C} \longrightarrow \Omega^*(M) \otimes \mathbb{C}$$

restricts to an automorphism of the graded $\mathbb{R}$-algebra $\Omega^*(M)$. We claim that

$$d^c = \mathbb{J}^{-1} \circ d \circ \mathbb{J}. \tag{2.22}$$

To see this, note that since ∂ is a derivation of degree 1 of $\Omega^*(M) \otimes \mathbb{C}$ so is its imaginary part $d^c = 2\mathrm{Im}(\partial)$. Since $\mathbb{J}$ is an automorphism of the graded algebra $\Omega^*(M)$ and d is a derivation of degree 1, the map $\mathbb{J} \circ d \circ (\mathbb{J})^{-1}$ is also a derivation of degree 1. On functions $\mathbb{J}$ acts trivially and on a 1-form η (regarded as a linear map $\eta : TM \longrightarrow \mathbb{R}$), $\mathbb{J}(\eta)$ is the composition $\eta \circ \mathbb{J}$; thus if $f \in \Omega^0(M)$ is a function,

$$\mathbb{J}^{-1} \circ d \circ \mathbb{J} f = -\mathbb{J} \circ (\partial f + \bar{\partial} f) = -i(\partial f - \bar{\partial} f) = d^c f$$

as desired. One checks that $d^c \circ d^c = 0$ and since $d^2 = 0$, $(\mathbb{J}^{-1} \circ d \circ \mathbb{J})^2 = 0$. In particular d^c annihilates $\mathbb{J}(d\Omega^0(M))$. It follows that both d^c and $\mathbb{J}^{-1} \circ d \circ \mathbb{J}$ annihilate the image $\mathbb{J}^{-1} \circ d \circ \mathbb{J}(\Omega^0(M))$ and since $\Omega^*(M)$ is generated as a graded algebra by its subalgebras $\Omega^0(M)$ and $\mathbb{J}^{-1} \circ d \circ \mathbb{J}(\Omega^0(M))$, the two derivations d^c and $\mathbb{J}^{-1} \circ d \circ \mathbb{J}$ are equal, proving (2.22).

2.4.2 *Curvature of Kähler manifolds*

Let M be a Kähler manifold and let $p \in M$. Its curvature tensor

$$R_p : \wedge^2 T_pM \longrightarrow \mathbf{End}(T_pM)$$

satisfies extra conditions ([100], [77], [11], [125]):

1. For any tangent vectors $v_1, v_2 \in T_pM$, the curvature $R(v_1 \wedge v_2)$ lies in the unitary subalgebra $\mathfrak{u}(T_pM)$ of skew-adjoint endomorphisms.
2. As a linear map $\wedge^2 T_pM \longrightarrow \mathbf{End}(T_pM)$, the curvature is of type (1,1):

$$R(\mathbb{J}v_1, \mathbb{J}v_2) = R(v_1, v_2).$$

As proved in Kobayashi–Nomizu [100] (§IX, Theorem 7.5, p.168), the corresponding *sectional curvature* function of real 2-planes in T_pM is completely determined by the sectional curvature function restricted to complex lines in T_pM. If the sectional curvature of every complex line in TM equals κ, then M is said to have *constant holomorphic sectional curvature* κ. In that case the sectional curvature of a 2-dimensional subspace $S \subset TM$ equals

$$\kappa \frac{1 + 3\cos^2(\alpha(S))}{4}$$

where $\alpha(S)$ is the angle of holomorphy defined in §2.2.1. (Compare Kobayashi–Nomizu [100] ,§IX, Proposition 7.4, and Gray [77], §6.10, pp.102–103.)

In particular if $\kappa = -1$ the sectional curvature is negative, pinched between -1 and $-1/4$. The curvature tensor is given by

$$R(X, Y) = \kappa(XY^* - YX^* + 2i\mathrm{Im}\langle X, Y\rangle I)$$

in terms of the outer product operation defined in §2.2.6. Note that the presence of the term

$$2i\mathrm{Im}\langle X, Y\rangle = X^*Y - YX^* = -\mathrm{trace}(XY^* - YX^*)$$

(where X^*Y denotes the scalar *inner product*) implies that the trace of $R(X, Y) \in \mathbf{End}(E)$ equals

$$(n + 1)2i\mathrm{Im}\langle X, Y\rangle.$$

2.5 Contact geometry and CR geometry

2.5.1 *Contact structures*

Let M be a smooth manifold and E a *hyperplane field* on M, that is a vector bundle $E \subset TM$ of real codimension 1. A *calibration* of E consists of a 1-form $\omega : TM \longrightarrow \mathbb{R}$ such that for each $x \in M$, the hyperplane $E_x \subset T_xM$ equals the kernel of the linear functional $\omega_x : T_xM \longrightarrow \mathbb{R}$. If ω_1, ω_2 are two calibrations of a hyperplane field, there exists a function $s : M \longrightarrow \mathbb{R}$ such that $\omega_1 = s\omega_2$. We define a real line bundle

$$A(E) = \{u \in T^*M \mid u(E) = 0\},$$

the *annihilator bundle* of E. Calibrations of E are nonvanishing sections of $A(E)$. Clearly $A(E)$ is a line bundle dually paired to the quotient line bundle TM/E and if $s \in \Gamma(A(E))$ is a calibration, there is a dual section of TM/E, denoted s^{-1}, such that $s \otimes s^{-1}$ pairs to the unit section of the trivial $\mathbb{R}$-bundle over M under the canonical pairing

$$A(E) \otimes TM/E \longrightarrow \mathbb{R}.$$

(In case these line bundles are nontrivial, we pass to a double cover which orients them. Henceforth we shall assume that TM/E and $A(E)$ are trivial line bundles, so that E can be calibrated.) For an intuitive introduction to contact structures on 3-manifolds, see §3.7 of Thurston [161].

If (E, ω) is a calibrated hyperplane field, then its *Levi form* is the skew-symmetric bilinear form $\Lambda^2(E) \to \mathbb{R}$ defined by the restriction of the exterior derivative $(d\omega)_x$ to E_x. For example, E is integrable (tangent to a codimension 1 foliation of M) if and only if its Levi form is identically zero. At the other extreme, ω is said to be a *contact form* if and only if $(d\omega)_x$ is nondegenerate for all $x \in M$. (In that case, E has even rank $2m$ so that M has dimension $2m+1$.) Equivalently, the restriction of the top exterior power $(d\omega)^m$ to $E = \mathrm{Ker}(\omega)$ is everywhere nonzero—that is, $\omega \wedge d\omega^m$ is a volume form on M. We say that the hyperplane field E is a *contact structure* if and only if there exists a contact 1-form ω calibrating it. This condition is independent of the choice of ω:

Theorem 2.5.1 *Let E be a contact structure and let ω be any 1-form calibrating E. Then ω is a contact 1-form; that is, the restriction $d\omega$ to E is nondegenerate.*

Proof By hypothesis, there exists a 1-form ω_0 calibrating E such that the restriction $d\omega_0$ is nondegenerate. Since ω also calibrates E, there exists a smooth function $f : M \longrightarrow \mathbb{R}$ such that $\omega = e^f\omega_0$. But

$$d\omega = e^f(df \wedge \omega_0 + d\omega_0) \equiv e^f d\omega_0 \qquad (\mathrm{mod}\ \omega_0)$$

so the restriction of $d\omega$ to E is also nondegenerate. □

The *Darboux theorem* states that given any contact form ω on M and $p \in M$, there exists an open neighborhood $U \subset M$ of p and a smooth coordinate system $\psi : U \longrightarrow \mathbb{R}^{2m+1}$ such that

$$\omega|_U = \psi^* \left(dv + 2\sum_{j=1}^{m}(x_j dy_j - y_j dx_j) \right)$$

where $x_1, y_1, \ldots, x_m, y_m, v$ are coordinates on $\mathbb{R}^{2m+1}$. We call such a coordinate system *Darboux coordinates*. Later we shall describe this in the more abstract context of "Heisenberg spaces."

2.5.2 *Legendrian submanifolds of contact manifolds*

Let (M^{2n-1}, E) be a contact manifold. A k-dimensional submanifold $S \subset M$ is *horizontal* if and only if it is everywhere tangent to E; that is, for each $s \in S$, its tangent space $T_sS \subset E_s$.

Lemma 2.5.2 *Let $S \subset M$ be a horizontal submanifold of a contact manifold (M^{2n-1}, E). Then* $\dim S < n$.

Proof Consider a calibration ω for E. Then for each $s \in S$, the restriction of $d\omega$ to E_s is a symplectic structure. Furthermore $T_sS \subset E_s$ is isotropic with respect to this symplectic structure: if $X_s, Y_s \in T_sS$ are tangent vectors, extend them to smooth vector fields on S. Then

$$d\omega(X_s, Y_s) = (X\omega(Y) - Y\omega(X) - \omega([X, Y]))(s) = 0$$

since X_s, Y_s and $[X, Y](s) \in T_sS$ all lie in $T_sS \subset E_s = \mathrm{Ker}(\omega)$. Since T_sS is isotropic, it follows that $\dim T_sS \leq \frac{1}{2} \dim E_s = n - 1$. □

If in addition S has dimension $n-1$, then S is said to be *Legendrian*. In particular T_sS is a Lagrangian subspace of E_s.

2.5.3 *Automorphisms of contact structures*

Let E be a contact structure on a smooth $(2m + 1)$-dimensional manifold M. We describe the infinitesimal automorphisms of the contact structure E (our treatment basically follows Kobayashi [99]). In particular vector fields whose local flows preserve E (infinitesimal automorphisms of E) correspond naturally to sections of TM/E—nonsingular vector fields preserving E correspond to calibrations of E. This is analogous to the existence of Hamiltonian potentials for symplectic vector fields and at the end of this section we describe a construction making this analogy precise.

Let ξ be a vector field on a contact manifold (M, E) generating a local flow $\{\Xi_t\}$ on M. Then the following conditions are equivalent:

1. $\{\Xi_t\}$ preserves E; that is, $d\Xi_t(E_p) = E_{\Xi_t(p)}$ where $d\Xi_t$ denotes the differential of Ξ_t (wherever it is defined).
2. For each section $s \in \Gamma(E)$ the Lie derivative $\mathfrak{L}_\xi(s)$ is a section of E.
3. For some (and hence every) calibration ω of E, there exists a function $f : M \longrightarrow \mathbb{R}$ such that $\mathfrak{L}_\xi(\omega) = f\omega$.
4. $\mathfrak{L}_\xi\omega \wedge \omega = 0$.

Such a vector field is called a *contact vector field.* The contact vector fields on the contact manifold (M, E) comprise a Lie subalgebra $\mathrm{Vect}(M, E) \subset \Gamma(TM)$.

Theorem 2.5.3 *Let $\Gamma(TM) \longrightarrow \Gamma(TM/E)$ denote the map arising from the quotient projection $TM \longrightarrow TM/E$. Then its restriction*

$$\Pi_E : \mathrm{Vect}(M, E) \longrightarrow \Gamma(TM/E)$$

is an isomorphism of vector spaces.

(Compare Kobayashi [99], I.7.1, or McDuff–Salamon [120].)

Proof Suppose that ω calibrates E. (If necessary pass to a double cover of M to orient TM/E.) We first show that Π_E is injective. Suppose that $\Pi_E(\xi) = 0$. Then $\xi \in \Gamma(E)$ and $\omega(\xi) = 0$. Cartan's formula $\mathfrak{L}_\xi = d\iota_\xi + \iota_\xi d$ implies that $0 = \mathfrak{L}_\xi \omega = \iota_\xi d\omega$. But since $d\omega|_E$ is nondegenerate, $\xi = 0$. Thus Π_E is injective.

Now we show that Π_E is surjective. We begin by choosing a vector field corresponding to the nonzero section $(\omega_0)^{-1} \in \Gamma(A(E))$. Choose any vector field ξ_1 on M such that $\omega(\xi_1) = 1$. The 1-form

$$\iota_{\xi_1}(d\omega) : TM \longrightarrow \mathbb{R}$$

restricts to a linear functional on E. Since $d\omega|_E$ is nondegenerate, there exists a section $Y \in \Gamma(E)$ such that $d\omega(Y, U) = d\omega(X, U)$ for all $U \in \Gamma(E)$. The vector field $\xi = \xi_1 - Y$ satisfies $\omega(\xi) = 1$ and $\mathfrak{L}_\xi \omega = 0$.

Now an arbitrary section of TM/E is given by $f\omega^{-1}$ where $f : M \longrightarrow \mathbb{R}$ is a smooth function (possibly with zeros). Then we must find a vector field ξ_f such that $\omega(\xi_f) = f$ and $\mathfrak{L}_{\xi_f}\omega \equiv 0 \pmod{\omega}$. By nondegeneracy of $d\omega|_E$, there is a unique vector field $\beta \in \Gamma(E)$ such that the restriction to E of

$$df + \iota_\beta(d\omega)$$

is zero. Then $\xi_f = f\xi + \beta$ is the desired vector field. □

The vector field ξ is called the *contact vector field* or the *characteristic vector field* corresponding to the contact 1-form ω. Clearly an arbitrary vector field ξ_f preserving E also preserves ω if and only if it commutes with ξ. In terms of the "potential function" $f = \omega(\xi_f)$, this is equivalent to $\xi f = 0$; that is, f is constant along the flow of ξ.

An alternative version of this construction of contact vector fields uses the Hamiltonian construction of vector fields on a symplectic manifold as follows. The cotangent bundle T^*M has a canonical exact symplectic structure; with respect to this symplectic structure, the subbundle $A(E) \subset T^*M$ is a symplectic submanifold. Let α_E be the corresponding exact symplectic structure on $A(E)$; for any calibration $\omega \in \Gamma(A(E))$, the pullback $\omega^*(\alpha_E)$ of the 1-form α_E under the map $\omega : M \longrightarrow A(E)$ equals the 1-form $\omega \in \Omega^1(M)$. In particular the contact structure E is induced from the hyperplane field $\mathrm{Ker}(\alpha_E)$ on $A(E)$ by any nonzero section ω of $A(E)$.

A section s of TM/E defines a function on $f_s : A(E) \longrightarrow \mathbb{R}$ which is linear on each fiber. The corresponding Hamiltonian vector field $H(f_s)$ generates a flow on $A(E)$ which preserves the level sets of f_s, each of which maps diffeomorphically onto M under the bundle projection. In the terminology of Eliashberg [46], the exact symplectic manifold $A(E)$ is the *symplecticization* of the contact manifold (M, E). Later we shall prove that the symplecticization of the "model" contact manifold—Heisenberg space—is complex hyperbolic space.

2.5.4 *CR-structures*

Suppose that M is an n-dimensional complex manifold and $W \subset M$ is a real hypersurface. Then although W will not have a complex structure, the complex structure on M nonetheless determines a geometry on W. For each $x \in W$, the tangent space $T_x W$ possesses a unique *maximal complex subspace*

$$E_x = T_x M \cap \mathbb{J} T_x M$$

and the collection of all E_x forms a complex $(n-1)$-dimensional vector bundle embedded as a real hyperplane in the tangent bundle of W. For any odd-dimensional (real) manifold W, a (real) hyperplane field $E \subset TW$ with a complex structure $\mathbb{J} : E \longrightarrow E$ is called a *CR-structure* provided it satisfies the following integrability condition (analogous to integrability of almost complex structures): the sections of the $(1,0)$-subspace $E^{1,0} \subset TW \otimes \mathbb{C}$ consisting of all $e \in E \otimes \mathbb{C}$ with $\mathbb{J}e = ie$ are closed under Lie bracket. If $X \in \Gamma(E)$ is a section of E, then $X' = X - i\mathbb{J}X$ is a section of $E^{1,0}$ with real part X.

If $(E, \mathbb{J})$ is a CR-structure, then a 1-form ω *calibrates* $(E, \mathbb{J})$ if and only if

1. ω calibrates E, that is $E = \mathrm{Ker}(\omega)$.
2. The Levi form is of type (1,1) with respect to $\mathbb{J}$; that is,

$$d\omega(\mathbb{J}X, \mathbb{J}Y) = d\omega(X, Y)$$

for $X, Y \in \Gamma(E)$.

A calibration of a CR-structure $(E, \mathbb{J})$ is traditionally called a *pseudo-Hermitian structure.*

Under the integrability condition, condition 2 above actually follows from condition 1. If $X \in \Gamma(E)$, then its projection into $E^{1,0}$ is given by the complex vector field

$$X' = X - i\mathbb{J}X$$

and since

$$[X', Y'] = ([X, Y] - [\mathbb{J}X, \mathbb{J}Y]) - i([\mathbb{J}X, Y] + [X, \mathbb{J}Y])$$

the integrability condition that $E^{1,0}$ is stable with respect to Lie bracket is actually three conditions on $X, Y \in \Gamma(E)$:

1. $[X, Y] - [\mathbb{J}X, \mathbb{J}Y] \in \Gamma(E)$;
2. $[\mathbb{J}X, Y] + [X, \mathbb{J}Y] \in \Gamma(E)$;
3. $-[X, Y] + [\mathbb{J}X, \mathbb{J}Y] = \mathbb{J}([\mathbb{J}X, Y] + [X, \mathbb{J}Y])$.

If ω is any 1-form with $E = \mathrm{Ker}(\omega)$, then

$$d\omega(X, Y) = -\omega([X, Y])$$

for $X, Y \in \Gamma(E)$. By condition 1 above, it follows that

$$d\omega(\mathbb{J}X, \mathbb{J}Y) = d\omega(X, Y)$$

for $X, Y \in \Gamma(E)$; that is, the Levi form is of type (1,1).

2.5.5 *Calibrations from defining functions*

If $f : M \longrightarrow \mathbb{R}$ is a smooth defining function for W; that is, 0 is a regular value of f and $W = f^{-1}(0)$, then $d^c f$ is a calibration for the CR-structure on W induced from its embedding in the complex manifold M. For

$$\begin{aligned} TW &= \operatorname{Ker}(df : TM \longrightarrow \mathbb{R}) \\ \mathbb{J}TW &= \operatorname{Ker}(df \circ \mathbb{J} : TM \longrightarrow \mathbb{R}) = \operatorname{Ker}(d^c f) \end{aligned}$$

Thus the CR-structure

$$E = TW \cap \mathbb{J}TW \subset TW$$

is calibrated by the 1-form $d^c f$ restricted to W.

A CR-structure is *Levi flat* if and only if the subbundle E is integrable. In terms of a calibration $\omega \in \Omega^1(M)$, this is equivalent to $\omega \wedge d\omega = 0$. A CR-structure is *nondegenerate* if and only if there is a calibration ω such that $\omega \wedge (d\omega)^{n-1}$ is nonzero; equivalently the restriction of the *Levi form* $d\omega$ to $E \times E$ is a nondegenerate skew-symmetric form. A CR-structure $(E, \mathbb{J})$ is *strongly pseudoconvex* if and only if there exists a calibration ω such that the Levi form is a positive definite Hermitian form on E; that is, if for all $X \in \Gamma(E)$, $X \neq 0$,

$$d\omega(X, \mathbb{J}X) > 0.$$

This condition is readily seen to be independent of the choice of calibration: every other such calibration is given by $\omega' = e^f \omega$ for some function f; then

$$\begin{aligned} d\omega'(X, \mathbb{J}X) &= e^f (df \wedge \omega + d\omega)(X, \mathbb{J}X) \\ &= 0 + e^f d\omega(X, \mathbb{J}X) > 0 \end{aligned}$$

(since $\omega(X) = \omega(\mathbb{J}X) = 0$). Thus a real hypersurface W in a complex manifold M is strongly pseudoconvex if there is a defining function $f : M \longrightarrow \mathbb{R}$ such that $dd^c f$ is a positive (1,1)-form when restricted to $E = \operatorname{Ker}(df) \cap \operatorname{Ker}(d^c f)$.

2.6 Heisenberg spaces

The purpose of this section is to develop the abstract notion of a *Heisenberg space,* a space globally modelled on the Heisenberg group (that is, a *principal homogeneous space* for the Heisenberg group). The boundary of complex hyperbolic space has the natural structure of a compactified Heisenberg space with complex structure. Heisenberg spaces form the natural abstract setting to discuss the asymptotics of complex hyperbolic geometry.

2.6.1 *Heisenberg spaces and symplectic vector spaces*

Let (V, ω) be a symplectic vector space. Associated to (V, ω), we define a group structure on $V \times \mathbb{R}$ with operation

$$(\xi_1, v_1) \cdot (\xi_2, v_2) = (\xi_1 + \xi_2, v_1 + v_2 + 2\omega(\xi_1, \xi_2))$$

(the factor of 2 is included so that this group law coincides with the conventions of the literature, for example [104]). The resulting group is called the *Heisenberg* *group* $H = H(V, \omega)$. The center of H is the subset $C \cong \{0\} \times \mathbb{R}$ and projection $V \times \mathbb{R} \longrightarrow V$ defines an isomorphism of groups $H/C \longrightarrow V$ (where V is given the vector space group structure). The commutator map

$$\begin{aligned} [,] : H \times H &\longrightarrow H \\ (A, B) &\longmapsto ABA^{-1}B^{-1} \end{aligned}$$

takes values in C and defines a nondegenerate skew-symmetric bilinear form

$$H/C \times H/C \longrightarrow C$$

which we identify with the symplectic form

$$4\omega : V \times V \longrightarrow \mathbb{R}.$$

One sees that the correspondence

$$(V, \omega) \longmapsto H(V, \omega)$$

defines an isomorphism from the category of symplectic vector spaces to a suitable category of "Heisenberg groups." The objects in this category are groups H which are central extensions

$$0 \longrightarrow C \longrightarrow H \longrightarrow V \longrightarrow 0$$

(where C is the center of G and V is a vector group) together with a fixed isomorphism $C \longrightarrow \mathbb{R}$ such that the map defined by commutator

$$H/C \times H/C \longrightarrow C$$

is a nondegenerate skew-symmetric bilinear form. These constructions are most naturally set in the abstract context of affine spaces. If G is a topological group, then a *principal G-homogeneous space* is a space X equipped with a simply transitive left G-action. An *affine space* is a principal G-homogeneous space X where G is a vector space. G is called the *vector space underlying* X and its operations are *translations*. Thus for every pair of "points" $x, y \in X$ there is a unique "vector" v translating x to y. Evidently a vector space is nothing more than an affine space together with a choice of base point. If V is a vector space, then V acts simply transitively on itself under

$$\begin{aligned} V \times V &\longrightarrow V \\ (v, x) &\longmapsto v + x \end{aligned}$$

and thus V has the structure of an affine space. However, V has a distinguished point, namely 0. A *pointed affine space* is an affine space together with a distinguished point. The categories of vector spaces and pointed affine spaces are canonically isomorphic. If X_1, X_2 are affine spaces, then a map $f : X_1 \longrightarrow X_2$ is *affine* if for every translation $t_1 : X_1 \longrightarrow X_1$ there exists a translation $t_2 : X_2 \longrightarrow X_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle t_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

commutes.

In differential-geometric terms, an affine space is a simply connected smooth manifold with a complete flat torsion-free affine connection.

A *symplectic affine space* is a space compatibly endowed with both the structure of an affine space and a symplectic structure. That is, a symplectic affine space is an affine space with a symplectic structure invariant under translations. Alternatively a symplectic affine space is an affine space whose underlying vector space is a symplectic vector space. (In terms of connections, a symplectic affine space is an affine space with parallel symplectic structure.) Darboux's theorem for smooth symplectic manifolds asserts that every symplectic manifold is locally symplectomorphic to a symplectic affine space.

In a similar way, we define a *Hermitian affine space* to be an affine space where the associated vector space of translations has a Hermitian structure.

A Heisenberg space is the analogue of a symplectic affine space where a Heisenberg group replaces the model symplectic vector space.

Definition 2.6.1 *A* Heisenberg space *is a principal G-homogeneous space $\mathcal{H}$ where G is the Heisenberg group $H(V, \omega)$ corresponding to a symplectic vector space (V, ω).*

A Heisenberg space $\mathcal{H}$ admits a fibration in a natural way: the group of central translations defines an $\mathbb{R}$-action on $\mathcal{H}$, whose orbit space $\mathcal{H}/C$ has the natural structure of a symplectic affine space. We call the vector field generating this $\mathbb{R}$-action the *vertical vector field* and denote it by ν. Clearly ν is invariant under the action of the Heisenberg group.

An *invariant contact form* on a Heisenberg space $\mathcal{H}$ is defined as follows. Choose a point $x \in \mathcal{H}$ and a linear functional $\alpha_x : T_x\mathcal{H} \longrightarrow \mathbb{R}$ such that $\alpha_x(\nu_x) \neq 0$ where ν is the vertical vector field on $\mathcal{H}$. There is a unique extension of $\alpha_x \in T_x^*\mathcal{H}$ to an $H(V, \omega)$-invariant 1-form on $\mathcal{H}$: explicitly, if $g \in H(V, \omega)$, then

$$\alpha_{gx} = \alpha_x \circ dg^{-1} : T_{gx}\mathcal{H} \longrightarrow \mathbb{R}.$$

One checks that α is not closed; indeed $d\alpha = \Pi_V^*(\omega)$ where $\Pi_V : \mathcal{H} \longrightarrow \mathcal{H}/C$ is vertical projection. (Indeed, α defines a connection on the line bundle $H \longrightarrow \mathcal{H}/C$ whose curvature is the exterior 2-form ω.) Nondegeneracy of ω on $\mathcal{H}/C$ implies α is a contact form.

In the sense of §2.5.3, ν is the contact vector field corresponding to the contact 1-form α.

To obtain an expression in coordinates, we start with symplectic coordinates for V: linear coordinates $x_1, y_1, \ldots, x_m, y_m$ such that

$$\omega = 2\sum_{i=1}^{m} dx_i \wedge dy_i.$$

Then we may write $\mathcal{H}$ as $V \times \mathbb{R}$ where $v \in \mathbb{R}$ denotes the vertical coordinate, the vertical vector field is

$$\nu = \frac{\partial}{\partial v}$$

and the invariant contact form is

$$\alpha = dv + 2\sum_{i=1}^{m} x_i dy_i - y_i dx_i.$$

2.6.2 *Horizontal affine subspaces*

Let $P \subset \mathcal{H}$ be a horizontal affine subspace and $p \in P$. Horizontality implies $\Pi_{\mathbb{V}}$ maps P injectively onto an isotropic affine subspace P' of the symplectic affine space $\mathcal{H}/C$. Since $\Pi_{\mathbb{V}} : E_p \longrightarrow \mathcal{H}/C$ is an isomorphism, $\Pi_{\mathbb{V}}^{-1}(P') \subset T_p\mathcal{H}$ is the unique CR-horizontal subspace mapping to P'. Hence P is the unique affine subspace containing p tangent to $\Pi_{\mathbb{V}}^{-1}(P')$.

Thus *CR-horizontal affine subspaces containing* p bijectively correspond to *isotropic subspaces of* E_p. Furthermore Legendrian affine subspaces containing p correspond to Lagrangian subspaces of E_p.

2.6.3 *Complex analysis on Heisenberg spaces*

If (V, ω) underlies a Hermitian structure (that is, $V = S_{\mathbb{R}}$ for a complex vector space S and $\omega(x,y) = \mathrm{Im}\langle\langle x, y\rangle\rangle$ for a Hermitian structure $\langle\langle , \rangle\rangle$ on S), then

$$\alpha_{\mathcal{H}} = dv + 2\mathrm{Im}\langle\langle d\xi, \xi\rangle\rangle.$$

Darboux's theorem for contact structures asserts that if M is a smooth manifold and α is a contact form on M, for each point $x \in M$ there exists a neighborhood U of x in M and an embedding $\psi : U \longrightarrow \mathcal{H}$ such that

$$\alpha = \psi^* \alpha_{\mathcal{H}}.$$

The corresponding coordinates

$$(x_1, y_1, \ldots, x_m, y_m, v)$$

we call *Heisenberg coordinates* on M.

In Heisenberg coordinates the connection can be described explicitly as follows. The coordinate vector fields

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}$$

on $\mathcal{H}/C$ lift to invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i\nu, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i\nu$$

which satisfy the commutation relations

$$[X_i, X_j] = [Y_i, Y_j] = 0, \qquad [X_i, Y_j] = -4\delta_{ij}\nu.$$

The vector fields $X_1, Y_1, \ldots, X_m, Y_m, \nu$ form a basis of $H(V,\omega)$-invariant vector fields on $\mathcal{H}$ and at each point $x \in \mathcal{H}$, the vectors $X_1, Y_1, \ldots, X_m, Y_m$ span the contact structure $E_x = \operatorname{Ker}(\alpha_x) \subset T_x\mathcal{H}$.

We can define complex vector fields

$$Z_1, \bar{Z}_1, \ldots, Z_m, \bar{Z}_m$$

by

$$Z_j = \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial v}$$

$$\bar{Z}_j = \frac{1}{2}(X_j + iY_j) = \frac{\partial}{\partial \bar{z}_j} - iz_j\frac{\partial}{\partial v}$$

so that $Z_1, \ldots, Z_m$ span $E^{1,0} \subset T\mathcal{H} \otimes \mathbb{C}$, $\bar{Z}_1, \ldots, \bar{Z}_m$ span $E^{0,1} \subset T\mathcal{H} \otimes \mathbb{C}$ and

$$[Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = 0, \qquad [Z_j, \bar{Z}_k] = -2i\delta_{jk}\nu.$$

If $f : \mathcal{H} \longrightarrow \mathbb{R}$ is any smooth function, then

$$\begin{aligned} df &= (\nu f)\alpha + \sum_{j=1}^{m}(X_j f)dx_j + (Y_j f)dy_j \\ &= (\nu f)\alpha + \sum_{j=1}^{m}(Z_j f)dz_j + (\bar{Z}_j f)d\bar{z}_j \end{aligned}$$

and the contact vector field corresponding to the calibration $f^{-1}\alpha$ is given by

$$\begin{aligned} \xi_f &= f\nu + \frac{1}{4}\sum_{j=1}^{m}(X_j f)Y_j - (Y_j f)X_j \\ &= f\nu - \frac{i}{2}\sum_{j=1}^{m}(Z_j f)\bar{Z}_j - (\bar{Z}_j f)Z_j. \end{aligned}$$

Exercise 2.6.2 *Compute the following examples of contact vector fields with the given potentials:*

1. *If* $f = 1$, *then* $\xi_f = \frac{\partial}{\partial v}$.
2. *If* $f = 2v$, *then* $\xi_f = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2v\frac{\partial}{\partial v}$.
3. *If* $f = 2y$, *then* $\xi_f = \frac{\partial}{\partial x} + y\frac{\partial}{\partial v}$.
4. *If* $f = r^2 = x^2 + y^2$, *then* $\xi_f = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.
5. *If* $f = r^4 + v^2$, *then*

$$\begin{aligned}\xi_f &= 2(x^2+y^2)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + 4v\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + (4v^2 - (x^2+y^2)^2)\frac{\partial}{\partial v} \\ &= -2r^2\frac{\partial}{\partial \theta} + 2rv\frac{\partial}{\partial r} + (4v^2 - r^4)\frac{\partial}{\partial v}.\end{aligned}$$

6. *If* $f = 2(yr^2 - 2vx)$, *then*

$$\begin{aligned}\xi_f &= 2(yr^2 - 2vx)\iota^*\left(2\frac{\partial}{\partial y}\right) \\ &= (3y^2 - 2x^2)\frac{\partial}{\partial x} + (2v - 4xy)\frac{\partial}{\partial y} - (y^3 + 2xv)\frac{\partial}{\partial v}.\end{aligned}$$

(Hint for 6: Use inversion in the unit circle $|\zeta| = 1$, $v = 0$. *Let*

$$f^\iota(\zeta, v) = (r^2 + 4v^2) f\left(\frac{\zeta}{r^2 + 2iv}, \frac{-v}{r^4 + 4v^2}\right).$$

Then

$$\iota^*\omega = (r^4 + 4v^2)^{-1}\omega$$

and if f *generates a contact vector field* ξ, *then* f^ι *generates* $\iota^*(\xi)$.*)*

Exercise 2.6.3 *The condition that contact vector field* ξ_f *associated to a smooth function* $f : \mathcal{H} \longrightarrow \mathbb{R}$ *generates a flow by holomorphic maps if* f *satisfies a system of differential equations analogous to the Cauchy–Riemann equations. To this end, express* ξ_f *in terms of the complex vector fields* Z *and* $\bar{Z}$*:*

$$\begin{aligned}\xi_f &= \frac{1}{2}(-Xf \cdot Y + Yf \cdot X) + fV \\ &= i(Zf \cdot \bar{Z} - \bar{Z}f \cdot Z)) + fV.\end{aligned}$$

Since $[Z, \bar{Z}] = iV$ *and* $[Z, V] = [\bar{Z}, V] = 0$, *it follows that*

$$[\xi_f, Z] = iZ\bar{Z}f \cdot Z - iZZf \cdot \bar{Z}$$

(necessarily Lie bracket with ξ_f *preserves* $E \otimes \mathbb{C}$*) so Lie bracket with* ξ_f *preserves* $E^{1,0}$ *if and only if* $ZZf = 0$.

Since $Z = \frac{1}{2}(X - iY)$, *we see that* $ZZf = (XX - YY)f - i(XY + YX)f$ *and*

$$(XX - YY)f = f_{xx} - f_{yy} - 2xf_{yv} - 2yf_{xv} + (y^2 - x^2)f_{vv}$$

and

$$(XY + YX)f = 2\left(f_{xy} + xf_{xv} + yf_{yv} - xyf_{vv}\right).$$

2.6.4 *Rumin's "curl" operator*

In his thesis [148] (compare also [149]) Michel Rumin introduces a de Rham complex canonically associated to a contact manifold (M^{2m+1}, ξ). We briefly outline Rumin's theory as exercises in contact, CR and Heisenberg geometry.

Let (M^{2m+1}, ξ) be a contact manifold and let $\Omega^k(M)$ denote the space of exterior k-forms on M and $\mathcal{I} \subset \Omega^*(M)$ the differential ideal annihilating ξ (the ideal generated by θ and $d\theta$, where θ is a calibrating 1-form). Let $\mathcal{J} \subset \Omega^*(M)$ be the differential ideal comprising exterior forms η such that $\eta \wedge \theta = \eta \wedge d\theta = 0$. Then the exterior derivative defines differentials

$$d^j : \mathcal{A}^k = \Omega^k(M)/\mathcal{I}^k \longrightarrow \mathcal{A}^{k+1} = \Omega^{k+1}(M)/\mathcal{I}^{k+1}$$

for $j < k$ and

$$d^j : \mathcal{A}^k = \mathcal{J}^k \longrightarrow \mathcal{A}^{k+1} = \mathcal{J}^{k+1}$$

for $j > k + 1$. The Levi form $d\theta$ defines a symplectic structure on ξ which pairs cochains of complementary degree. This pairing gives Poincaré duality on the cochain level.

The middle differential d^m is the interesting one, since it is second order (all the others are first order). Consider the case $m = 1$; that is, a 3-dimensional contact manifold. If $\psi \in \Omega^1(M)$ is a 1-form, then a unique function $f \in \Omega^0(M)$ exists such that

$$d(\psi + f\theta) \wedge \theta = 0.$$

Define

$$D(\psi \mod \mathcal{I}^1) = d(\psi + f\theta).$$

The resulting complex is locally exact and resolves the constant sheaf $\mathbb{R}$. Thus it can be used to compute the cohomology of M.

On Heisenberg space, the explicit form for the operator D is particularly suggestive. Let

$$Z = \frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial v}, \qquad \bar{Z} = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial v}, \qquad \nu = \frac{\partial}{\partial v}$$

be the basis of left-invariant complex vector fields on the Heisenberg group H. The dual basis of complex 1-forms is

$$dz, d\bar{z}, \theta = dv + 2\text{Im}(z d\bar{z})$$

and the exterior derivative of a function is given by

$$df = (\nu f)\theta + Zf dz + \bar{Z} f d\bar{z}.$$

3

THE BALL MODEL

We introduce complex hyperbolic space as the unit ball in $\mathbb{C}^n$, the contractible region defined by a real hyperquadric in $\mathbb{P}^n_{\mathbb{C}}$. As a symplectic (Kähler) quotient, $\mathbf{H}^n_{\mathbb{C}}$ inherits a Kähler structure. $\mathbf{H}^n_{\mathbb{C}}$ is practically characterized by the property that complex linear subspaces are holomorphic totally geodesic submanifolds. Furthermore every totally geodesic submanifold is either holomorphic or totally real. We compute the Kähler structure: the Riemannian metric tensor as well as the Kähler symplectic form. A formula for the distance function similar to (1.9) is derived. A formula for the geodesic between two points is also given. The next section explores trigonometry in complex hyperbolic (and elliptic) space. A discussion of the various laws of sines and cosines is given, as well as a qualitative discussion of the asymptotic properties of the Pythagorean theorem. Cartan's generalization of the concurrence of the altitudes is discussed. More formulas are given in the third section of this chapter, including a discussion of orthogonal projections onto totally geodesic subspaces.

3.1 The unit ball in $\mathbb{C}^n$ and its projective model

3.1.1 *The Hermitian vector space* $\mathbb{C}^{n,1}$

Let E be a complex vector space of dimension $n+1$. The *projective space associated to* E is the space $\mathbb{P}(E)$ of all lines (1-dimensional complex linear subspaces through the origin) in E, given the quotient topology. Alternatively $\mathbb{P}(E)$ is the quotient $(E - \{0\})/\mathbb{C}^*$ of the space of nonzero vectors in E by the group of nonzero complex scalars acting by scalar multiplication. The quotient map $\mathbb{P} : E - \{0\} \longrightarrow \mathbb{P}(E)$ is then a holomorphic principal $\mathbb{C}^*$-bundle over $\mathbb{P}(E)$. The group $\mathbf{GL}(E)$ of linear transformations of E induces a group $\mathbf{PGL}(E)$ of *collineations* or *projective transformations* of $\mathbb{P}(E)$.

Let $\mathbb{C}^{n,1}$ be the $(n+1)$-dimensional complex vector space consisting of $(n+1)$-tuples

$$Z = \begin{bmatrix} Z' \\ Z_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}$$

with the Hermitian pairing

$$\begin{aligned} \langle Z, W \rangle &= \langle\!\langle Z', W' \rangle\!\rangle - Z_{n+1}\bar{W}_{n+1} \\ &= Z_1\bar{W}_1 + \cdots + Z_n\bar{W}_n - Z_{n+1}\bar{W}_{n+1} \end{aligned}$$

(where, as usual, Z' denotes a vector in $\mathbb{C}^n$ and $Z_{n+1} \in \mathbb{C}$). We denote the group of (unitary) automorphisms of $\mathbb{C}^{n,1}$ by $\mathbf{U}(n,1)$. For any unit complex number

ζ, scalar multiplication by ζ lies in $\mathbf{U}(n,1)$; the corresponding subgroup is the center of $\mathbf{U}(n,1)$ and we denote it by $\mathbb{T}$.

A vector Z is said to be *negative (respectively null, positive)* if and only if the Hermitian inner product $\langle Z, Z\rangle$ is negative (respectively null, positive). *Complex hyperbolic n-space* $\mathbf{H}^n_{\mathbb{C}}$ is defined to be the subset of $\mathbb{P}(\mathbb{C}^{n,1})$ consisting of negative lines in $\mathbb{C}^{n,1}$. The *absolute*, also called the *boundary* of $\mathbf{H}^n_{\mathbb{C}}$, is the subset $\partial\mathbf{H}^n_{\mathbb{C}}$ of $\mathbb{P}(\mathbb{C}^{n,1})$ consisting of null lines in $\mathbb{C}^{n,1}$. The image $\mathbf{PU}(n,1)$ of $\mathbf{U}(n,1)$ in $\mathbf{PGL}(\mathbb{C}^{n,1})$ is the full group of biholomorphisms of $\mathbf{H}^n_{\mathbb{C}}$.

Let $\mathbb{C}^n$ be complex n-space with the standard positive definite Hermitian inner product

$$\langle\!\langle z, w\rangle\!\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

and let $\mathbf{U}(n)$ denote its (compact) group of unitary automorphisms. We shall identify $\mathbf{H}^n_{\mathbb{C}}$ with the unit ball

$$\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \langle\!\langle z, z\rangle\!\rangle < 1\}$$

as follows. Let

$$\begin{aligned} \mathbf{A} : \mathbb{C}^n &\longrightarrow \mathbb{P}(\mathbb{C}^{n,1}) \\ z' &\mapsto \begin{bmatrix} z' \\ 1 \end{bmatrix} \end{aligned}$$

be the biholomorphic embedding of $\mathbb{C}^n$ onto the affine patch of $\mathbb{P}(\mathbb{C}^{n,1})$ defined by $Z_{n+1} \neq 0$ (in homogeneous coordinates). Since any vector in $\mathbb{C}^{n,1}$ with homogeneous coordinate $Z_{n+1} = 0$ is positive, $\mathbf{H}^n_{\mathbb{C}} \subset \mathbf{A}(\mathbb{C}^n)$ and $\mathbf{A}$ identifies $\mathbb{B}^n$ with $\mathbf{H}^n_{\mathbb{C}}$ and $\partial\mathbb{B}^n = S^{2n-1} \subset \mathbb{C}^n$ with $\partial\mathbf{H}^n_{\mathbb{C}}$. The hyperplane "at infinity" $\mathbb{P}(\mathbb{C}^{n,1}) - \mathbf{A}(\mathbb{C}^n)$ is the orthogonal complement $O^\perp$ where the vector

$$O = \begin{bmatrix} 0' \\ 1 \end{bmatrix} \in \mathbb{C}^{n,1}$$

corresponds to the origin in $\mathbb{C}^n$.

Exercise 3.1.1 *Prove that the stabilizer of O in $\mathbf{SU}(n,1)$ is isomorphic to the unitary group $\mathbf{U}(n)$ of $\mathbb{C}^n$.*

3.1.2 *Complex reflections*

Suppose that $F \subset \mathbb{C}^{n,1}$ is a complex linear subspace of $\mathbb{C}^{n,1}$ such that the restriction of the Hermitian structure to F is nondegenerate. Then there is an orthogonal direct-sum decomposition

$$\mathbb{C}^{n,1} = F \oplus F^\perp$$

where

$$F^{\perp} = \{Z \in \mathbb{C}^{n,1} \mid \langle Z, f \rangle = 0 \quad \forall f \in F\}$$

is the orthogonal complement of F. If $\Pi_F : \mathbb{C}^{n,1} \to F$ denotes the orthogonal projection, then

$$\langle Z, W \rangle = \langle \Pi_F(Z), \Pi_F(W) \rangle + \langle \Pi_{F^\perp}(Z), \Pi_{F^\perp}(W) \rangle$$

where $\Pi_{F^\perp}(Z) = Z - \Pi_F(Z)$ is the orthogonal projection onto $F^\perp$. If F is 1-dimensional,

$$\Pi_F(Z) = \frac{\langle Z, V \rangle}{\langle V, V \rangle} V$$

where V generates F. If ζ is a unit complex number, the *complex reflection in F with reflection factor ζ* is the transformation

$$\varrho_F^\zeta : X \mapsto \Pi_F(X) + \zeta \Pi_{F^\perp}(X)$$

and evidently lies in $\mathbf{U}(n,1)$. If ζ is a pth root of unity, then ϱ_F^ζ has order p. We shall refer to a complex reflection of order 2 (that is, when $\zeta = -1$) as an *inversion.*

Exercise 3.1.2 *Show that the orthogonal projection of $\mathbf{H}_{\mathbb{C}}^n$ onto a complex linear submanifold L of (complex) dimension m is the restriction of the projection Π of projective space onto the projective subspace $\hat{L}$ containing L. (Recall that Π is a projective map—that is, incidence preserving—which maps the complement of a projective subspace of dimension $n - m - 1$ onto $\hat{L}$.)*

These formulas can be written conveniently in terms of Hermitian outer products (see §2.2.6). Let F be the hyperplane polar to a vector $\phi \in \mathbb{C}^{n,1}$. Orthogonal projection onto F is given by

$$\Pi_F = I - \phi \otimes \phi^*$$

and complex reflection with factor ζ is given by

$$\varrho_F^\zeta = I + (\zeta - 1)\phi \otimes \phi^*.$$

Lemma 3.1.3 $\mathbf{PU}(n,1)$ *acts transitively on the set $\mathbf{H}_{\mathbb{C}}^n$ of negative lines in $\mathbb{C}^{n,1}$. The unitary group $\mathbf{U}(n,1)$ acts transitively on the set of vectors $Z \in \mathbb{C}^{n,1}$ such that $\langle Z, Z \rangle = -1$.*

Proof Let $\tilde{x}, \tilde{y} \in \mathbb{C}^{n,1}$ be negative vectors. By multiplying $\tilde{x}, \tilde{y}$ by complex numbers we obtain vectors $X, Y \in \mathbb{C}^{n,1}$ with $\mathbb{P}(\tilde{x}) = \mathbb{P}(X)$ and $\mathbb{P}(\tilde{y}) = \mathbb{P}(Y)$ satisfying:

$$\langle X, X \rangle = \langle Y, Y \rangle = -1, \qquad \langle X, Y \rangle < 0. \tag{3.1}$$

Explicitly

$$X = (-\langle \tilde{x}, \tilde{x} \rangle)^{-1/2} \tilde{x}$$
$$Y = -(-\langle \tilde{y}, \tilde{y} \rangle)^{-1/2} |\langle \tilde{x}, \tilde{y} \rangle|^{-1} \langle \tilde{x}, \tilde{y} \rangle \tilde{y}.$$

Let $M = X + Y$; then

$$\langle M, M \rangle = -2 + 2\mathrm{Re}\langle X, Y \rangle < 0$$

so that M spans a negative line. Inversion ϱ in M

$$\varrho : Z \mapsto -Z + 2\frac{\langle Z, M \rangle}{\langle M, M \rangle} M$$

lies in $\mathbf{U}(n,1)$ and maps $\varrho(X) = Y$, $\varrho(Y) = X$. Thus $\mathbf{U}(n,1)$ acts transitively on the set of negative lines in $\mathbb{C}^{n,1}$.

For the second assertion consider an arbitrary pair $(\tilde{x}, \tilde{y})$ satisfying $\langle \tilde{x}, \tilde{x} \rangle = \langle \tilde{y}, \tilde{y} \rangle = -1$. Let $X = \tilde{x}$ and

$$Y = -\frac{\langle \tilde{x}, \tilde{y} \rangle}{|\langle \tilde{x}, \tilde{y} \rangle|} \tilde{y}.$$

Let ϱ be the above inversion which interchanges X and Y'. Composition of ϱ with scalar multiplication by

$$-\frac{\langle \tilde{y}, \tilde{x} \rangle}{|\langle \tilde{x}, \tilde{y} \rangle|} \in \mathbb{T}$$

is an element of $\mathbf{U}(n,1)$ taking X to Y. □

Exercise 3.1.4 *Let $x, y \in \mathbf{H}^n_{\mathbb{C}}$. Choose negative vectors $\tilde{x}, \tilde{y} \in \mathbb{C}^{n,1}$ such that*

$$x = \mathbb{P}(\tilde{x})$$
$$y = \mathbb{P}(\tilde{y}).$$

Show that the midpoint $\mathbf{mid}(x, y)$ *with respect to the Bergman metric discussed in §3.1.3 is represented by the vector*

$$\begin{aligned} M(X,Y) &= \mu(X+Y) \\ &= (-\langle \tilde{x}, \tilde{x} \rangle)^{-1/2} \tilde{x} - (-\langle \tilde{y}, \tilde{y} \rangle)^{-1/2} |\langle \tilde{x}, \tilde{y} \rangle|^{-1} \langle \tilde{x}, \tilde{y} \rangle \tilde{y}. \end{aligned} \tag{3.2}$$

Since $\mathbf{U}(n)$ acts transitively on the unit sphere

$$S^{2n-1} \subset \mathbb{C}^n,$$

Lemma 3.1.3 implies:

Corollary 3.1.5 $\mathbf{PU}(n,1)$ *acts transitively on the unit tangent bundle of* $\mathbf{H}^n_{\mathbb{C}}$.

The method of Lemma 3.1.6 applies to null vectors as well.

Lemma 3.1.6 $\mathbf{PU}(n,1)$ *acts transitively on the set* $\partial\mathbf{H}^n_{\mathbb{C}}$ *of null lines in* $\mathbb{C}^{n,1}$. *The unitary group* $\mathbf{U}(n,1)$ *acts transitively on the set of null vectors* $Z \in \mathbb{C}^{n,1}$.

Proof Suppose that $x, y \in \partial\mathbf{H}^n_{\mathbb{C}}$ are distinct null lines; we find a complex reflection in $\mathbf{U}(n,1)$ taking x to y. Since $x \neq y$ there exist representative null vectors $\tilde{x}, \tilde{y} \in \mathbb{C}^{n,1}$ such that

$$x = \mathbb{P}(\tilde{x})$$
$$y = \mathbb{P}(\tilde{y})$$

and $\langle\tilde{x}, \tilde{y}\rangle \neq 0$. By appropriate scaling (for example, replacing $\tilde{y}$ by $\langle\tilde{x}, \tilde{y}\rangle\tilde{y}$), we may assume $\langle\tilde{x}, \tilde{y}\rangle$ is real. Let $M = \tilde{x} + \tilde{y}$. Then $\langle M, M\rangle = 2\langle\tilde{x}, \tilde{y}\rangle$ and inversion in M interchanges $\tilde{x}$ and $\tilde{y}$. For the second assertion, suppose $\tilde{x}, \tilde{y} \in \mathbb{C}^{n,1}$ are linearly independent null vectors. Let

$$\zeta = \frac{\langle\tilde{x}, \tilde{y}\rangle\tilde{y}}{|\langle\tilde{x}, \tilde{y}\rangle|} \in \mathbb{T}.$$

Then the composition of the complex reflection in $M = \tilde{x} + \zeta\tilde{y}$ with scalar multiplication by ζ is an element of $\mathbf{U}(n,1)$ mapping $\tilde{x}$ to $\tilde{y}$. □

3.1.3 *Complex hyperbolic space as a symplectic quotient*

We develop the geometry of $\mathbf{H}^n_{\mathbb{C}}$ as a Kähler quotient similar to the construction of the Fubini–Study Kähler structure on projective space discussed in §2.3.3. (For a similar discussion for $\mathbb{P}^n_{\mathbb{C}}$, see [1], §2.5, p.40 and also [120], §4.21.) Recall that a Kähler structure on a manifold is equivalent to a complex structure $\mathbb{J}$ and a symplectic structure ω, compatible in the sense that ω is a positive (1,1)-form with respect to $\mathbb{J}$.

$\mathbf{H}^n_{\mathbb{C}}$ inherits a complex structure as an open subset of $\mathbb{P}(\mathbb{C}^{n,1})$. In the affine coordinates on $\mathbf{H}^n_{\mathbb{C}}$ given by the biholomorphic embedding

$$\mathbf{A} : \mathbf{H}^n_{\mathbb{C}} \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{P}(\mathbb{C}^{n,1})$$

(discussed in 3.1.1) the operators $d = \partial + \bar{\partial}$ expressing the complex structure on $\mathbb{B}^n \subset \mathbb{C}^n$ derive from the corresponding operators on $\mathbb{P}(\mathbb{C}^{n,1})$.

The symplectic structure on $\mathbf{H}^n_{\mathbb{C}}$ is induced from that of $\mathbb{C}^{n,1}$ by a symplectic quotient construction. Recall that the symplectic structure on $\mathbb{C}^{n,1}$ is given by

$$\omega(X, Y) = \mathrm{Im}\langle X, Y\rangle.$$

Consider the Hamiltonian function $f : \mathbb{C}^{n,1} \longrightarrow \mathbb{R}$:

$$f(X) = -\frac{1}{2}\langle X, X\rangle$$

with corresponding flow

$$Z \mapsto e^{-it}Z$$

which is periodic with period 4π. This flow defines an action of the unit circle $\mathbb{T} \subset \mathbb{C}^*$. Its infinitesimal generator is the vector field

$$\tau(Z) = -\mathrm{Im}\left(Z\frac{\partial}{\partial Z}\right) = \frac{1}{2}\left(X\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial X}\right)$$

which is the real part of the holomorphic (1,0)-vector field

$$\tau(Z)^{1,0} = -i\sum_{j=1}^{n} Z_n\frac{\partial}{\partial Z_n}.$$

This follows readily from the calculation (a special case of Theorem 2.3.1)

$$\begin{aligned} df_{(Z)}(u) &= \left.\frac{d}{dt}\right|_{t=0} f(Z+tu) \\ &= -\mathrm{Re}\langle Z, u\rangle \\ &= -\mathrm{Im}\langle iZ, u\rangle \\ &= \omega(\tau(Z), u) \end{aligned}$$

where $Z \in \mathbb{C}^{n,1}$ and $u \in T_Z\mathbb{C}^{n,1} \cong \mathbb{C}^{n,1}$. (§2.3.3 gives the analogous construction for projective space.)

Let $\kappa \in \mathbb{R}$. The corresponding symplectic quotient $f^{-1}(\kappa)/\mathbb{T}$ inherits a symplectic structure Φ_κ. This exterior 2-form is of type (1,1) with respect to $\mathbb{J}$:

$$\Phi_\kappa(\mathbb{J}v_1, \mathbb{J}v_2) = \Phi_\kappa(v_1, v_2)$$

for vectors v_1, v_2 tangent to $f^{-1}(\kappa)/\mathbb{T}$. The various level sets $f^{-1}(\kappa)/\mathbb{T}$ for $\kappa > 0$ all identify via maps

$$\mathbf{H}^n_{\mathbb{C}} \longrightarrow f^{-1}(\kappa)/\mathbb{T}.$$

Let $Z \in f^{-1}(\kappa)$. The tangent space to $f^{-1}(\kappa)/\mathbb{T}$ at the equivalence class $[Z]$ naturally identifies with the orthogonal complement $Z^\perp$ which is a positive definite subspace of the indefinite Hermitian vector space $T_Z\mathbb{C}^{n,1} \cong \mathbb{C}^{n,1}$. Hence Φ_κ is a *positive* (1,1)-form. Since it is closed, Φ_κ (together with the complex structure $\mathbb{J}$) defines a *Kähler structure* on $\mathbf{H}^n_{\mathbb{C}}$.

However, explicit determination of this Kähler structure requires a slight modification of this construction. Although the embedding $\mathbf{A} : \mathbf{H}^n_{\mathbb{C}} \cong \mathbb{B}^n \hookrightarrow \mathbb{C}^{n,1}$ is holomorphic and defines a slice of the $\mathbb{C}^*$-action on negative vectors, it does not map into a single level set of $f : \mathbb{C}^{n,1} \longrightarrow \mathbb{R}$. Since the level sets $f^{-1}(\kappa)$ and $f^{-1}(\kappa')$ identify under scalar multiplication by $\sqrt{\kappa/\kappa'}$, it is convenient to replace Φ by a closed (1,1)-form

$$\Phi' = 2i\partial\bar{\partial}\log f$$

which is

1. invariant under scalar multiplications (because $f(\lambda Z) = |\lambda|^2 f(Z)$);
2. a constant scalar multiple of Φ on $f^{-1}(\kappa)$.

Furthermore the restriction of Φ' to any line through the origin is identically zero. Now

$$\Phi = dd^c f = 2i\partial\bar{\partial} f$$

and

$$\partial\bar{\partial}\log f = f^{-1}\partial\bar{\partial} f - (f^{-1}\partial f) \wedge (f^{-1}\bar{\partial} f).$$

Since $df = \partial f + \bar{\partial} f$, the restrictions of ∂f and $\bar{\partial} f$ to a level set $f^{-1}(\kappa)$ are linearly dependent. Therefore the restriction of $(f^{-1}\partial f) \wedge (f^{-1}\bar{\partial} f)$ to $f^{-1}(\kappa)$ is zero. Hence $\kappa\Phi'$ and Φ have identical restrictions to $f^{-1}(\kappa)$. Thus the symplectic structure induced on $\mathbb{B}^n$ from $f^{-1}(\kappa)/\mathbb{T}$ by $\mathbf{A} : \mathbf{H}^n_{\mathbb{C}} \cong \mathbb{B}^n \longrightarrow \mathbb{C}^{n,1}$ equals

$$\begin{aligned}
\Phi_\kappa &= 2\kappa i\partial\bar{\partial}\log(f \circ \mathbf{A}) \\
&= 2\kappa i\partial\bar{\partial}\log(1 - \langle\langle z, z\rangle\rangle) \\
&= 2\kappa i\partial\{(1 - \langle\langle z, z\rangle\rangle)^{-1}(-\langle\langle z, dz\rangle\rangle)\} \\
&= \frac{-2\kappa i}{(1 - \langle\langle z, z\rangle\rangle)^2}\left\{\langle\langle z, dz\rangle\rangle \wedge \langle\langle dz, z\rangle\rangle - (1 - \langle\langle z, z\rangle\rangle)\left(\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j\right)\right\} \\
&= \frac{2\kappa i}{(1 - \langle\langle z, z\rangle\rangle)^2}\left\{\left(\sum_{j=1}^{n} \bar{z}_j dz_j\right) \wedge \left(\sum_{k=1}^{n} z_k d\bar{z}_k\right) + (1 - \langle\langle z, z\rangle\rangle)\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j\right\}
\end{aligned}$$

The corresponding Riemannian metric tensor may be written as

$$2\kappa(1 - \langle\langle z, z\rangle\rangle)^{-2}\left\{\langle\langle z, dz\rangle\rangle\langle\langle dz, z\rangle\rangle + (1 - \langle\langle z, z\rangle\rangle)\langle\langle dz, dz\rangle\rangle\right\} \tag{3.3}$$

(correcting a misprint in [100], pp.162,169; see also Toledo [162], p.20 (4.1)). Summarizing:

Theorem 3.1.7 *The expression*

$$\Phi_\kappa = \frac{2\kappa i}{(1 - \langle\langle z, z\rangle\rangle)^2} \times \left\{\left(\sum_{j=1}^{n} \bar{z}_j dz_j\right) \wedge \left(\sum_{k=1}^{n} z_k d\bar{z}_k\right) + (1 - \langle\langle z, z\rangle\rangle)\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j\right\}$$

defines a $\mathbf{PU}(n,1)$*-invariant positive closed (1,1)-form on* $\mathbb{B}^n$*. Hence it defines an invariant Kähler structure on* $\mathbb{B}^n$*. Furthermore the map*

$$\mathbb{B}^n \longrightarrow \mathbb{C}^{n,1}$$
$$z' \longmapsto \sqrt{\frac{2\kappa}{1-\|z\|^2}}\begin{bmatrix} z' \\ 1 \end{bmatrix}$$

is a symplectic embedding into $f^{-1}(\kappa) \subset \mathbb{C}^{n,1}$.

Exercise 3.1.8 *Show that the Bergman metric can be described by the following formula. Let* $\mathbb{P}$ *denote projectivization and let* $x \in \mathbf{H}^n_{\mathbb{C}}$ *be a point defined by the negative vector* $\tilde{v} \in \mathbb{C}^{n,1}$, $\mathbb{P}(\tilde{x}) = x$. *Let* $V_1, V_2 \in T_x\mathbf{H}^n_{\mathbb{C}}$ *be tangent vectors which are the image of* $\tilde{v}_1, \tilde{v}_2 \in T_{\tilde{x}}\mathbb{C}^{n,1} \approx \mathbb{C}^{n,1}$:

$$V_i = d\mathbb{P}(\tilde{v}_i)$$

for $i = 1, 2$. *Then*

$$g_x(V_1, V_2) = \frac{\langle \tilde{v}_1, \tilde{v}_2\rangle\langle \tilde{x}, \tilde{x}\rangle - \langle \tilde{v}_1, \tilde{x}\rangle\langle \tilde{x}, \tilde{v}_2\rangle}{\langle \tilde{x}, \tilde{x}\rangle}.$$

(For the analogous formula in $\mathbf{E}^n_{\mathbb{C}}$, see [1], §2.5, p.40.)

The Bergman metric on $\mathbf{H}^n_{\mathbb{C}}$ satisfies the following basic property:

Theorem 3.1.9 *Let* $L \subset \mathbb{P}(\mathbb{C}^{n,1})$ *be a complex line which meets* $\mathbf{H}^n_{\mathbb{C}}$. *The restriction of the Hermitian metric* g_κ *to* $L \cap \mathbf{H}^n_{\mathbb{C}}$ *has constant curvature* $-2/\kappa$.

Proof Let L_0 denote the line consisting of points $[Z] \in \mathbb{P}(\mathbb{C}^{n,1})$ with homogeneous coordinates $Z_2 = \cdots = Z_n = 0$ and consider the inhomogeneous coordinate $z = Z_1/Z_{n+1}$. Then the restriction

$$(g_\kappa)|_{L_0} = 2\kappa(1-|z|^2)^{-2}dzd\bar{z}$$

is $\kappa/2$ times the Poincaré metric of constant curvature -1. Thus $(g_\kappa)|_{L_0}$ has constant curvature $-2/\kappa$.

For a general line L, consider a point $x \in L$ and a unit vector $v \in T_xL$. By Corollary 3.1.5 there exists a collineation $A \in \mathbf{PU}(n,1)$ taking the origin to x and a unit vector in T_0L_0 to v. Hence A takes L_0 to L; since A is an isometry of g_κ the restriction $g_\kappa|_L$ has constant curvature $-2/\kappa$ as claimed. □

3.1.4 *Complex-linear subspaces*

Theorem 3.1.10 *Let* $F \subset \mathbb{P}(\mathbb{C}^{n,1})$ *be a complex m-dimensional projective subspace which intersects* $\mathbf{H}^n_{\mathbb{C}}$. *Then* $F \cap \mathbf{H}^n_{\mathbb{C}}$ *is a totally geodesic holomorphic submanifold biholomorphically isometric to* $\mathbf{H}^m_{\mathbb{C}}$.

Proof Clearly $F \cap \mathbf{H}^n_{\mathbb{C}}$ is a holomorphic submanifold. Let $\tilde{F} = \pi^{-1}(F) \subset \mathbb{C}^{n,1}$ be the $(m+1)$-dimensional linear subspace corresponding to $F \subset \mathbb{P}(\mathbb{C}^{n,1})$. The 1-parameter subgroup

$$R(\tilde{F}) = \{\varrho^{\zeta}_{\tilde{F}} \mid \zeta \in \mathbb{T}\}$$

of complex reflections in $\tilde{F}$ (see §3.1.2) determines a subgroup of $\mathbf{PU}(n,1)$ whose fixed-point set in $\mathbf{H}^n_{\mathbb{C}}$ equals F. Therefore F is totally geodesic. Since $\mathbf{PU}(n,1)$

acts transitively on $\mathbf{H}_{\mathbb{C}}^n$, by applying an automorphism we may assume that F contains the origin. Since $\mathbf{U}(n) \subset \mathbf{PU}(n,1)$ (the stabilizer of the origin) acts transitively on m-dimensional complex linear subspaces of $\mathbb{C}^n$, there exists $A \in \mathbf{PU}(n,1)$ such that $A(F) = \mathbb{C}^m \times \{0'\}$ and A takes $F \cap \mathbf{H}_{\mathbb{C}}^n$ by a biholomorphic isometry to $\mathbf{H}_{\mathbb{C}}^m$. □

Thus given any $x \in \mathbb{B}^n$ and a complex linear subspace $F \subset T_x\mathbb{B}^n$ of dimension k, a unique holomorphic totally geodesic submanifold contains x and is tangent to F. We shall call such a holomorphic submanifold a *$\mathbb{C}^k$-plane.* A $\mathbb{C}^1$-plane is called a *complex geodesic.* The intersection of a $\mathbb{C}^k$-plane with $\partial\mathbf{H}_{\mathbb{C}}^n$ is a smoothly embedded sphere S^{2k-1} in $\partial\mathbf{H}_{\mathbb{C}}^n \approx S^{2n-1}$, which we call a *$\mathbb{C}^k$-chain.* A $\mathbb{C}^1$-chain we call a *chain,* following Cartan [21], [22].

Similarly, the intersection of $\mathbf{H}_{\mathbb{C}}^n$ with a complex hyperplane is a totally geodesic holomorphic complex hypersurface, which we call a *complex hyperplane in* $\mathbf{H}_{\mathbb{C}}^n$; its boundary is a smoothly embedded $(2n-1)$-sphere in $\partial\mathbf{H}_{\mathbb{C}}^n$, which we call a *hyperchain.* A complex hyperplane $H \subset \mathbb{P}(\mathbb{C}^{n,1})$ which intersects $\mathbf{H}_{\mathbb{C}}^n$ corresponds to a positive line, namely $H = \mathbb{P}(v^\perp)$ where $v \in \mathbb{C}^{n,1}$ is positive and $v^\perp$ is the complex linear hyperplane consisting of vectors $u \in \mathbb{C}^{n,1}$ such that $\langle u, v\rangle = 0$. We call v a *polar vector* and the positive complex line it spans the *polar line.*

Theorem 3.1.11 1. *Any pair of distinct points in* $\mathbf{H}_{\mathbb{C}}^n \cup \partial\mathbf{H}_{\mathbb{C}}^n$ *lies on a unique complex geodesic.*

2. *Given a nonzero tangent vector* $v \in T_x\mathbf{H}_{\mathbb{C}}^n$ *at* $x \in \mathbf{H}_{\mathbb{C}}^n$ *there is a unique complex geodesic containing* x *and tangent to* v.
3. *Any pair of distinct points of* $\partial\mathbf{H}_{\mathbb{C}}^n$ *lies on a unique chain.*

Proof Since any pair of distinct points in $\mathbb{P}(\mathbb{C}^{n,1})$ lies on a unique projective line, two points in $\mathbf{H}_{\mathbb{C}}^n \cup \partial\mathbf{H}_{\mathbb{C}}^n$ span a unique complex geodesic. Similarly, any tangent vector v at a point $x \in \mathbb{P}(\mathbb{C}^{n,1})$ is tangent to a unique complex line, whence the second assertion. The third assertion follows immediately from the first. □

3.1.5 *Orthogonal projections onto totally geodesic subspaces*

Associated to a complex linear subspace $F \subset \mathbf{H}_{\mathbb{C}}^n$ is an *orthogonal projection* $\Pi_F : \mathbf{H}_{\mathbb{C}}^n \longrightarrow F$.

Let M be a simply connected complete Riemannian manifold of nonpositive curvature and $W \subset M$ a closed totally geodesic submanifold. The *orthogonal projection* $\Pi : M \longrightarrow W$ is defined as follows. For any $x \in M$, the distance function

$$\begin{aligned} M &\longrightarrow \mathbb{R} \\ u &\longmapsto \rho(u, x) \end{aligned}$$

is a strictly convex function along geodesics in M. Since W is totally geodesic, its restriction to W is also strictly convex and hence has a unique minimum. Let

$\Pi(x) \in W$ be the point of W closest to x. Clearly the geodesic from x to $\Pi(x)$ is orthogonal to W at $\Pi(x)$.

When there is an *inversion* in W, that is an order 2 isometry ι whose fixed points comprise W, then $\Pi(x)$ is the midpoint of the geodesic segment joining x to $\iota(x)$.

Consider a complex totally geodesic submanifold $h \subset \mathbb{B}^n$. Then there exists a complex linear subspace $H \subset \mathbb{C}^{n,1}$ such that $h = \mathbb{P}(H) \cap \mathbb{B}^n$. Let

$$H^\perp = \{u \in \mathbb{C}^{n,1} \mid \langle u, H \rangle = 0\}$$

be the Hermitian complement to H; then the Hermitian projection

$$\Pi_H : \mathbb{C}^{n,1} \longrightarrow H \subset \mathbb{C}^{n,1}$$

onto H with kernel $H^\perp$ determines a projective map Π_h defined on the complement $\mathbb{P}(\mathbb{C}^{n,1}) - \mathbb{P}(H^\perp)$. Since $h \neq \emptyset$, $H^\perp$ is positive and thus there is a well-defined map $\Pi_h : \mathbf{H}^n_{\mathbb{C}} \to h$ (indeed Π_h extends to the absolute $\partial\mathbf{H}^n_{\mathbb{C}}$). A simple calculation shows that Π_h is the orthogonal projection onto the totally geodesic holomorphic submanifold h. The fibers are the totally geodesic holomorphic submanifolds of complementary dimension which meet h orthogonally in single points.

From now on, we normalize the curvature by taking $\kappa = 2$. Thus a complex geodesic has constant curvature -1. The *holomorphic sectional curvature* of a tangent vector v is defined to be the sectional curvature of the complex line (considered as a real tangent 2-plane) spanned by v. Since every tangent vector to $\mathbf{H}^n_{\mathbb{C}}$ is tangent to a complex geodesic, Theorems 3.1.9 and 3.1.10 imply that $\mathbf{H}^n_{\mathbb{C}}$ *has constant holomorphic sectional curvature* -1. See §2.4.2 and the references given there for a discussion of the sectional curvature and the curvature tensor.

3.1.6 *Synthetic geometry of the Bergman metric*

The Bergman metric can be described in terms of synthetic projective geometry as follows. (Compare Fig. 3.1.) Let $x, y \in \mathbb{B}^n$ be a pair of distinct points; let $\overleftrightarrow{xy}$ be the unique complex line they span. The Bergman metric restricts on $\overleftrightarrow{xy} \cap \mathbb{B}^n$ to the Poincaré metric of constant curvature -1 given by:

$$\frac{4R^2\, dz d\bar{z}}{(R^2 - r^2)^2}$$

where R denotes the radius of the disc and $r = r(z)$ is the distance to the center. Since $\overleftrightarrow{xy}$ is totally geodesic, the distance between x and y equals the distance between x and y in $\overleftrightarrow{xy}$ with respect to this Poincaré metric. Furthermore the geodesic from x to y is the Poincaré geodesic in $\overleftrightarrow{xy}$ joining x and y, that is the circular arc in $\overleftrightarrow{xy}$ orthogonal to $\partial(\overleftrightarrow{xy})$.

3.1.7 *The distance formula*

The distance function in $\mathbf{H}^n_{\mathbb{C}}$ has the following useful algebraic description in terms of the Hermitian structure on $\mathbb{C}^{n,1}$. Compare Theorem 2.1 of Epstein [48],

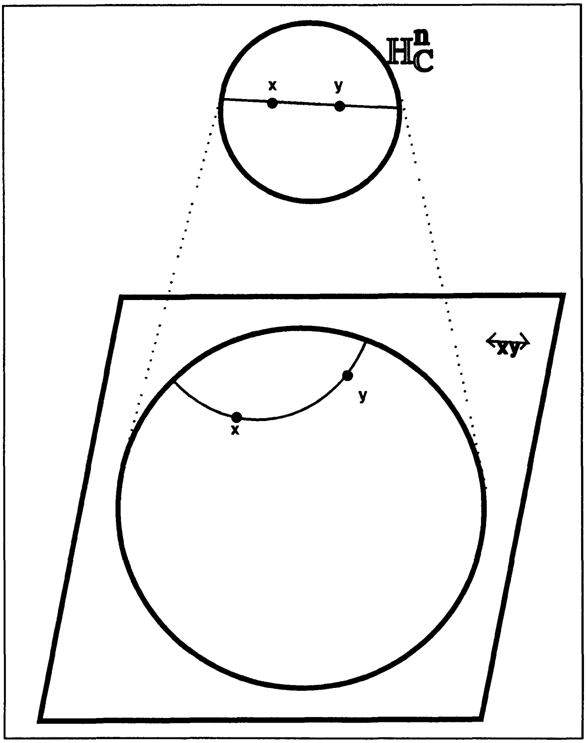

FIG. 3.1. The Bergman metric

Mostow [127], [128]. (Beware of their different conventions concerning the curvature: Epstein normalizes the holomorphic sectional curvature to be -4 and Mostow chooses it to be -2.) This formula is analogous to the familiar fact that the angle $\angle(a, b)$ between nonzero vectors a, b in Euclidean space is given by

$$\cos(\angle(a,b)) = \frac{|a \cdot b|}{\|a\|\|b\|} = \sqrt{\frac{(a \cdot b)(b \cdot a)}{(a \cdot a)(b \cdot b)}}.$$

Since the distance $\rho(A, B)$ between the corresponding lines $A = a\mathbb{R}, B = b\mathbb{R}$ in the elliptic geometry on real projective space equals the angle $\angle(a, b)$,

$$\cos^2 \rho(A,B) = \frac{(a \cdot b)(b \cdot a)}{(a \cdot a)(b \cdot b)}.$$

Let $x, y \in \mathbf{H}^n_{\mathbb{C}}$ be points corresponding to vectors $X, Y \in \mathbb{C}^{n,1}$; then the distance between them is given by

$$\cosh^2\left(\frac{\rho(x,y)}{2}\right) = \frac{\langle X,Y\rangle\langle Y,X\rangle}{\langle X,X\rangle\langle Y,Y\rangle}. \tag{3.4}$$

Clearly if $\lambda, \mu \in \mathbb{C}^*$ are scalars, then the right-hand side is unchanged if X and Y are replaced by λX and μY and thus depends only on x and y; furthermore it is invariant under the action of $\mathbf{U}(n,1)$ on pairs (X,Y). By applying homotheties to X and Y and elements of $\mathbf{U}(n,1)$ to the pair, we may assume that

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Y = \begin{bmatrix} z \\ 1 \end{bmatrix}$$

where $\|z\| < 1$ and

$$\frac{\langle X,Y\rangle\langle Y,X\rangle}{\langle X,X\rangle\langle Y,Y\rangle} = \frac{1}{1-\|z\|^2}.$$

But the distance $\rho(x,y)$ may be computed as follows. Since the Poincaré metric restricts to the square of

$$ds = 2\frac{dr}{1-r^2} = d\log\frac{1+r}{1-r} = 2d\tanh^{-1}(r)$$

on the affine $\mathbb{R}$-line

$$\left\{X + r\frac{1}{\|(Y-X)\|}(Y-X) \;\middle|\; -1 < r < 1\right\}$$

containing X, Y, the Euclidean coordinate r is related to the hyperbolic distance $\rho = \rho(x,y)$ by

$$r = \tanh(\rho/2)$$

from which it follows

$$\cosh^2\left(\frac{\rho}{2}\right) = \frac{1}{1-r^2} = \frac{\langle X,Y\rangle\langle Y,X\rangle}{\langle X,X\rangle\langle Y,Y\rangle}$$

as claimed

Some of the relations between these hyperbolic trigonometric functions are illustrated in Fig. 3.2. Here s denotes a real parameter, say representing the hyperbolic distance from the origin to a point p of Euclidean distance $\tanh(s/2)$. Then the point represented by $\tanh(s)$ is the midpoint of the the two points

$$\tanh(s) \pm i\operatorname{sech}(s)$$

where the unit circle meets its orthogonal circle passing through p. This figure illustrates the following formulas:

$$\begin{aligned} \tanh(s/2) &= \coth(s) - \operatorname{csch}(s) \\ \coth(s/2) &= \coth(s) + \operatorname{csch}(s) \\ \tanh^2(s) + \operatorname{sech}^2(s) &= 1. \end{aligned}$$

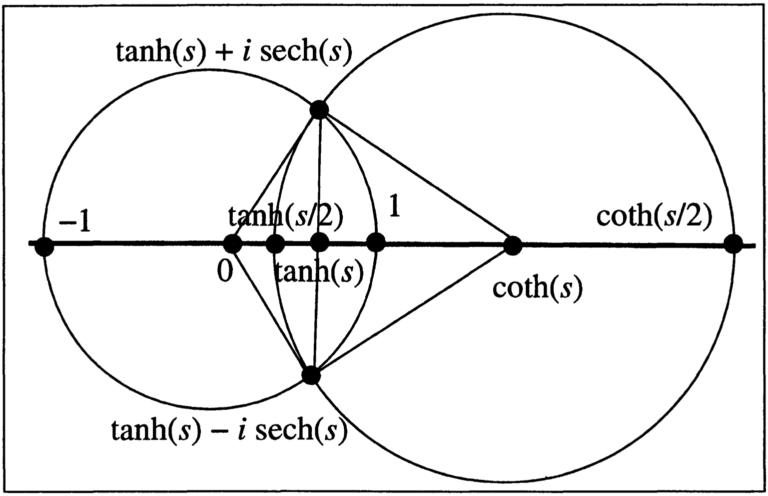

FIG. 3.2. Metric relations for the unit circle and an orthogonal circle

3.1.8 *The Kähler potential associated to a point*

The construction of the Bergman metric in §3.1.3 implies that the function

$$f(z) = \log(1 - \langle\langle z, z\rangle\rangle) \tag{3.5}$$

is a Kähler potential function for $\mathbb{B}^n$ invariant under the stabilizer $\mathbf{U}(n)$ of the origin O. We henceforth call it the *Kähler potential associated to* O, denoting it ψ_O. (Compare §2.4.1.) In terms of the distance function,

$$\psi_O(z) = 2 \log \operatorname{sech}\left(\frac{\rho(z, O)}{2}\right). \tag{3.6}$$

(See §6.1.1 for a Kähler potential associated to more general complex linear subspaces.)

Another interpretation of the Kähler potential (3.5) involves the *Bergman kernel* $K(z, \zeta)$, the kernel function defining the projection of $L^2(\mathbb{B}^n)$ onto the subspace of holomorphic functions (see, for example, Rudin [147], Krantz [107], Theorem 1.4.21). Since

$$K(z, \zeta) = \frac{n!}{\pi^n}(1 - \langle\langle z, \zeta\rangle\rangle)^{-n-1}$$

the Bergman metric equals

$$\partial\bar{\partial} \log K(z, z).$$

An important relationship between the analysis and the geometry of $\mathbb{B}^n$ is that biholomorphic mappings of $\mathbb{B}^n$ are isometries of the Bergman metric.

3.1.9 *Totally real totally geodesic subspaces*

Corresponding to the compatible real structures on $\mathbb{C}^{n,1}$ are the *real forms* of $\mathbf{H}^n_{\mathbb{C}}$; that is, the maximal totally real totally geodesic subspaces of $\mathbf{H}^n_{\mathbb{C}}$. (A totally real totally geodesic subspace is maximal if and only if it has dimension n.) For example, the standard real structure on $\mathbb{C}^{n,1}$ corresponds to the standard real form $\mathbb{R}^{n,1} \subset \mathbb{C}^{n,1}$ whose corresponding real structure is given by complex conjugation:

$$\rho_0 : \mathbb{C}^{n,1} \longrightarrow \mathbb{C}^{n,1}$$
$$\begin{bmatrix} Z' \\ Z_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} \bar{Z}' \\ \bar{Z}_{n+1} \end{bmatrix}.$$

Since ρ is anti-linear, it maps complex lines in $\mathbb{C}^{n,1}$ to complex lines and hence induces an (anti-holomorphic) map ι of $\mathbb{P}(\mathbb{C}^{n,1})$ whose fixed-point set $\mathbb{P}(W_\rho)$ is a real projective space of dimension n. (For the usual real structure ρ_0 above, this real projective space consists of all points which have real coordinates.) Real structures ρ, ρ_0 are *projectively equivalent* (§2.1.3) if the maps they induce on projective space coincide. Two real structures ρ and ρ_0 are projectively equivalent if and only if $\phi = \rho \circ \rho_0$ is a homothety if and only if the fixed-point sets $\mathbb{P}(W_\rho) = \mathbb{P}(W_{\rho_0})$ are identical.

A compatible real structure ρ on $\mathbb{C}^{n,1}$ induces an anti-holomorphic isometry of the Bergman metric which has order 2. In particular its fixed-point set is a totally real totally geodesic submanifold S of $\mathbb{B}^n$. The fixed-point set S of ι determines ι uniquely so we write $\iota = \iota_S$ and call ι_S the *inversion* in S.

For the usual real structure, this submanifold is *real hyperbolic n-space* $\mathbf{H}^n_{\mathbb{R}}$ in its Beltrami–Klein projective model with constant curvature $-1/4$. The corresponding metric tensor on $\mathbf{H}^n_{\mathbb{R}}$ is given by

$$\frac{4}{\left(1 - \sum_{j=1}^n x_j^2\right)^2} \left\{ \sum_{i=1}^n \Big(1 - \sum_{j \neq i} x_j^2\Big) dx_i^2 + \sum_{1 \leq i \neq j \leq n} x_i x_j \, dx_i dx_j \right\}. \tag{3.7}$$

$\mathbf{PU}(n,1)$ acts transitively on $\mathbb{B}^n$ and $\mathbf{U}(n)$ acts transitively on the set of totally real k-dimensional subspaces of $\mathbb{C}^n$ (for any $0 < k < n$). Consequently, for any point $x \in \mathbb{B}^n$ and every totally real k-dimensional subspace $S \subset T_x\mathbb{B}^n$, $\mathrm{Exp}_x(S)$ is the unique totally real totally geodesic subspace through x tangent to S.

Every $\mathbb{R}^k$-plane $P^k \subset \mathbf{H}^n_{\mathbb{C}}$ lies in infinitely many $\mathbb{R}^m$-planes P' if $k < m \leq n$ and lies in a unique $\mathbb{C}^k$-plane $P_{\mathbb{C}}$. Such totally geodesic subspaces enjoy the following orthogonality property:

Lemma 3.1.12 *Let $k < m \leq n$ and P be an $\mathbb{R}^k$-plane contained in an $\mathbb{R}^m$-plane $P' \subset \mathbf{H}^n_{\mathbb{C}}$. Let $P_{\mathbb{C}}$ be the unique $\mathbb{C}^k$-plane containing P. Then $P = P' \cap P_{\mathbb{C}}$ and for each $x \in P$ the subspaces P' and $P_{\mathbb{C}}$ meet orthogonally at x. If ι' denotes inversion in an $\mathbb{R}^n$-plane containing P', then ι' preserves $P_{\mathbb{C}}$ and acts on $P_{\mathbb{C}}$ by inversion in P.*

Proof We first show that $P = P' \cap P_{\mathbb{C}}$. If $x \in P$, then the tangent space of $P' \cap P_{\mathbb{C}}$ at x is a totally real subspace of the complex subspace $T_x P_{\mathbb{C}} = T_x P \otimes \mathbb{C}$ containing $T_x P$ and hence must equal $T_x P$. Since two totally geodesic subspaces having the same tangent space are identical, it follows that $P = P' \cap P_{\mathbb{C}}$.

Next we show that ι' preserves $P_{\mathbb{C}}$ and restricts to the conjugation in the totally real totally geodesic subspace $P \subset P_{\mathbb{C}}$. For any $x \in P$ we claim ι' preserves the tangent space $T_x P_{\mathbb{C}}$: since P lies in the fixed-point set of ι', $\iota'(x) = x$ and $d\iota'(T_x P) = T_x P$. As ι' is anti-holomorphic,

$$d\iota' \circ \mathbb{J}_x = -\mathbb{J}_x \circ d\iota'$$

(where $\mathbb{J}_x : T_x\mathbf{H}^n_{\mathbb{C}} \longrightarrow T_x\mathbf{H}^n_{\mathbb{C}}$ denotes the almost complex structure), the tangent space

$$T_x P_{\mathbb{C}} = T_x P \oplus \mathbb{J}_x T_x P$$

is also invariant under $d\iota'$. By uniqueness of totally geodesic subspaces with given tangent space, $\iota'(P_{\mathbb{C}}) = P_{\mathbb{C}}$ as claimed. The restriction $\iota'|_{P_{\mathbb{C}}}$ is then an anti-holomorphic involution fixing P.

Two totally geodesic subspaces A, B meet orthogonally at $x \in A \cap B$ if and only if each vector in the orthogonal complement of $T_x(A \cap B) \subset T_x A$ is orthogonal to each vector in the orthogonal complement of $T_x(A \cap B) \subset T_x B$. A vector in $T_x P_{\mathbb{C}}$ orthogonal to $T_x P$ must be of the form $\mathbb{J}_x v$ where $v \in T_x P$. If $w \in T_x P'$, then $\langle v, w\rangle \in \mathbb{R}$ (since $T_x P'$ is totally real) and

$$\langle\!\langle \mathbb{J}_x v, w \rangle\!\rangle = \mathrm{Re}\big(i\langle v, w\rangle\big) = 0;$$

that is, $\mathbb{J}_x v$ and w are orthogonal, as desired. This completes the proof of Lemma 3.1.12. □

If $x, y \in \mathbf{H}^n_{\mathbb{R}}$ are points in the ball with real coordinates, then the complex geodesic C they span is invariant with respect to conjugation ρ_0. The restriction of ρ_0 to C is an anti-holomorphic involution of C and its fixed-point set γ is therefore a geodesic in C, indeed the geodesic containing x and y. On the other hand, $\gamma = C \cap \mathbf{H}^n_{\mathbb{R}}$ is an $\mathbb{R}$-linear subspace. It follows that the geodesic containing x and y is a Euclidean straight line segment (indeed a Euclidean diameter of C).

3.1.10 *Fermi coordinates on totally geodesic surfaces*

Figures 3.3 and 3.4 depict the two models of 2-dimensional hyperbolic geometry which are contained in complex hyperbolic space as totally geodesic surfaces. A complex geodesic ($\mathbf{H}^1_{\mathbb{C}}$) is drawn in Fig. 3.3 and a totally real geodesic 2-plane ($\mathbf{H}^2_{\mathbb{R}}$) is drawn in Fig. 3.4. The two surfaces intersect in a (real) geodesic ($\mathbf{H}^1_{\mathbb{R}}$), drawn as the horizontal interval in both pictures. In each picture are drawn the family of geodesics orthogonal to $\mathbf{H}^1_{\mathbb{R}}$ at intervals of length $1/2$ as well as the family of hypercycles at distance multiples of $1/2$ from $\mathbf{H}^1_{\mathbb{R}}$.

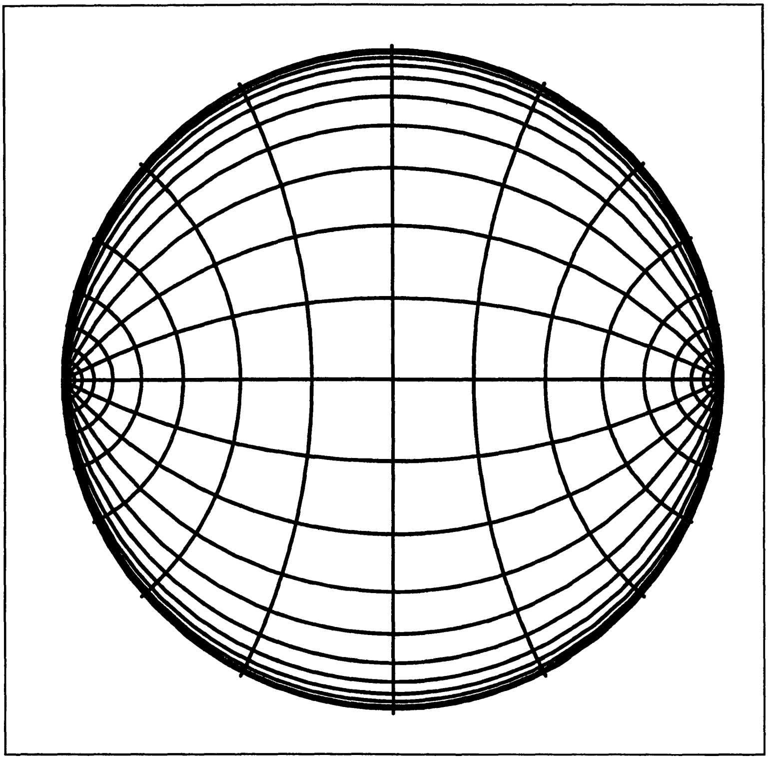

FIG. 3.3. Fermi coordinates in a complex geodesic

3.1.11 *Classification of totally geodesic submanifolds*

Using the general theory of symmetric spaces (see Helgason [84], Kobayashi-Nomizu [100], Wolf [169] or Loos [113]), one can readily prove that the only totally geodesic subspaces of $\mathbf{H}^n_{\mathbb{C}}$ are either complex linear subspaces or totally real totally geodesic submanifolds. Here is a sketch of the proof.

$\mathbf{H}^n_{\mathbb{C}}$ is a *Riemannian symmetric space;* that is, a Riemannian manifold M such that through every point $x \in M$ there is an isometry σ fixing x and whose differential equals $-I$ on T_xM. (Compare Helgason [84], Kobayashi-Nomizu [100] or Loos [113].) Let G denote its group of isometries and $\mathfrak{g}$ its Lie algebra. Then σ induces an automorphism (the *Cartan involution*) on $\mathfrak{g}$, whose subalgebra of fixed points is the Lie algebra of the isotropy group of G at $x \in M$ and whose -1-eigenspace is a subspace $\mathfrak{p} \subset \mathfrak{g}$ naturally identified with the tangent space T_xM under the differential

$$\mathfrak{g} \leftrightarrow T_eG \longrightarrow T_xM$$

of the evaluation map

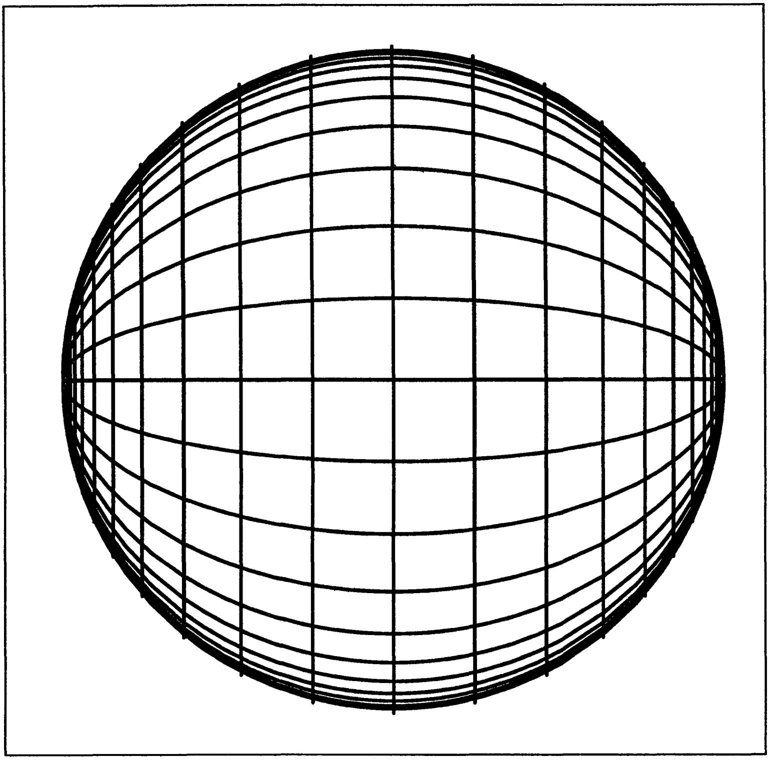

FIG. 3.4. Fermi coordinates in a totally real 2-plane

$$\begin{aligned} G &\longrightarrow M \\ g &\longmapsto g(x). \end{aligned}$$

A totally geodesic submanifold W through x is uniquely determined by its tangent space $T_xW \subset T_xM$ by

$$W = \mathrm{Exp}_x(T_xW)$$

Then a subspace $\mathfrak{m} \subset \mathfrak{p} \leftrightarrow T_xM$ corresponds to a totally geodesic submanifold if and only if

$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}.$$

($\mathfrak{m}$ is called a *Lie triple system.*)

We apply this to $\mathbf{H}^n_{\mathbb{C}}$ as follows, using the description of the curvature tensor in §2.4.2. If $A, B \in \mathfrak{p}$ correspond to tangent vectors $a, b \in T_xM$ respectively, then the curvature transformation $R(a, b) : T_xM \longrightarrow T_xM$ identifies with the linear mapping

$$\mathrm{ad}[B, A] : \mathfrak{p} \longrightarrow \mathfrak{p}$$

which by §2.4.2 equals

$$R(a, b) = -(ab^* - ba^* + 2i\mathrm{Im}\langle a, b\rangle I)$$

where $a, b \in T_x\mathbf{H}^n_{\mathbb{C}} \cong \mathbb{C}^n$.

Exercise 3.1.13 *Suppose the* $\mathbb{R}$*-linear subspace* $\mathfrak{m} \subset (\mathbb{C}^n)_{\mathbb{R}}$ *corresponds to a totally geodesic submanifold. Then either* $\mathfrak{m}$ *is orthogonal to* $\mathbb{J}\mathfrak{m}$ *or* $\mathfrak{m}$ *is invariant under* $\mathbb{J}$.

It follows that every totally geodesic submanifold of $\mathbf{H}^n_{\mathbb{C}}$ is either totally real or complex.

Exercise 3.1.14 *Prove that* $\mathbf{PU}(n, 1)$ *is a* simple *Lie group: a normal Lie subgroup is either trivial or* $\mathbf{PU}(n, 1)$ *itself.*

3.2 Trigonometry

Trigonometry in $\mathbf{H}^n_{\mathbb{C}}$ may be regarded as an extension of trigonometry in the hyperbolic plane. A triangle in $\mathbf{H}^n_{\mathbb{C}}$ will generally not lie in a totally geodesic submanifold. To understand the relations between the geometric invariants of a triangle in $\mathbf{H}^n_{\mathbb{C}}$, we compare the relations with the corresponding relations in constant curvature -1 and $-1/4$. The material in this section is adapted from the papers of Giraud [65], Hsiang[89], Brehm [17], and Leuzinger [111], [112].

Non-Euclidean trigonometry is reviewed in §1.3.5, §1.4.3. In classical trigonometry, each side (respectively vertex) of a triangle contributes one invariant, namely its length (respectively its angle). The sine and cosine laws relate these six quantities. Other geometric quantities such as the area, inradius, the three exradii, the circumradius, and the altitudes also relate to these quantities. The "first cosine law" relates the length of one side to the other two side lengths and an opposite angle. When the opposite angle is a right angle, the first cosine law reduces to the Pythagorean theorem. The "second cosine law" is dual to the first cosine law, relating an angle to the length of the opposite side and the other two angles. The second cosine law degenerates in Euclidean geometry to the fact that the angle sum of a Euclidean triangle equals π. The sine law is somewhat more conceptual: the ratio between the sines of the angles and the hyperbolic sines of the sides is independent of the opposite vertex side pair. The hyperbolic sine of a side is proportional to the area of a metric disc having that side as diameter, giving a more conceptual interpretation of the sine law.

Trigonometry in complex hyperbolic space and complex projective space is richer, as the notion of triangle is augmented. Given any triple of points A, B, C there are unique geodesic segments joining any pair of them. However, there are also unique "complex geodesics" joining any pair of them as well. The "ordinary" triangle $\triangle(A, B, C)$ formed by the geodesic segments from A to B, B to C, and C to A embeds in a "complex" triangle formed by the complex geodesics $\overleftrightarrow{AB}$,

$\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$. We call these three complex geodesics the *complex sides* of $\triangle(A, B, C)$. We find that the usual law of sines in the real hyperbolic plane extends directly when the angles are replaced by angles between the complex sides; this is the "first sine law," discussed in the papers by Giraud [65], Hsiang [89], Brehm [17] and Leuzinger [111, 112].

In complex hyperbolic space a single invariant attaches to a side, namely its length. However, two invariants attach to a vertex: the ordinary Riemannian angle θ, and the "complex angle" ϕ defined as the angle between the two complex sides. Other invariants—most notably the angle of holomorphy μ of the real 2-plane tangent to the two sides at the given vertex and the angle η formed by the projections of the sides onto a complex side—can be expressed simply in terms of these angles using formulas from §2.2.2.

A more symmetrical invariant is Brehm's *shape invariant* which is defined by a Hermitian triple product as defined in §2.2.5. We refer the reader to Brehm [17] for these relationships.

Leuzinger [111, 112] develops trigonometry in the considerably more general context of symmetric spaces of noncompact type. He gives a conceptual description of the sine and cosine laws in terms of the symplectic geometry of actions of groups of isometries on various homogeneous spaces. See Leuzinger [111, 112].

3.2.1 *The first law of cosines*

The distance formula (3.4) implies trigonometric formulas. We consider a geodesic triangle $\triangle$ with vertices A, B, C. The corresponding sides are

$$a = \rho(B, C), \qquad b = \rho(C, A), \qquad c = \rho(A, B)$$

and the angles at the vertices are

$$\alpha = \angle(CAB), \qquad \beta = \angle(ABC), \qquad \gamma = \angle(BCA).$$

Then the *law of cosines* in a space of constant curvature $-k^2 < 0$ is

$$\cosh(kc) = \cosh(ka)\cosh(kb) - \cos(\gamma)\sinh(ka)\sinh(kb). \tag{3.8}$$

The special case when $\gamma = \pi/2$ is the *Pythagorean theorem:*

$$\cosh(kc) = \cosh(ka)\cosh(kb). \tag{3.9}$$

From the law of cosines follows the corresponding *law of sines:*

$$\frac{\sinh(ka)}{\sin(\alpha)} = \frac{\sinh(kb)}{\sin(\beta)} = \frac{\sinh(kc)}{\sin(\gamma)}.$$

(To deduce the law of sines from the law of cosines, apply the law of cosines and write $(\sinh(ka)/\sin(\alpha))^2$ as an expression symmetric in A, B, C. The law of cosines also implies *the second law of cosines* (see Beardon [9],§7.12, p.149):

$$\cosh(kc) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

The asymptotics of these formulas as $k \longrightarrow 0$ give corresponding formulas in Euclidean trigonometry.

3.2.2 *The laws of sines*

Two distinct complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$ intersect in either the empty set or a point. At a point of intersection, the complex geodesics intersect at an angle ϕ where $0 \leq \phi \leq \pi/2$. In the following let a, b, c denote the side lengths and ϕ_A, ϕ_B, ϕ_C denote the angles between the complex sides at the vertices A, B, C respectively. We assume that the holomorphic sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ equals -1 (so that each complex geodesic has Gaussian curvature -1).

Theorem 3.2.1 *(The first sine law)*

$$\frac{\sinh(a/2)}{\sin(\phi_A)} = \frac{\sinh(b/2)}{\sin(\phi_B)} = \frac{\sinh(c/2)}{\sin(\phi_C)}. \tag{3.10}$$

When $\triangle$ lies in a totally real geodesic 2-plane, then the complex angles ϕ_A, ϕ_B, ϕ_C equal the usual Riemannian angles and (3.10) reduces to the usual law of sines in a hyperbolic plane of curvature $-1/4$. When $\triangle$ lies in a complex geodesic, then

$$\phi_A = \phi_B = \phi_C = 0$$

and this theorem is vacuous. The first sine law can be found in [65] (18), p.71, [89], Theorem 2(ii) and [17], Theorem 2 (8).

Denote by $\theta_A, \theta_B, \theta_B$ the three usual (Riemannian) angles at the respective vertices. Let μ_A, μ_B, μ_C denote the holomorphy angles of the two sides at the respective vertices.

Theorem 3.2.2 *(The second sine law)*

$$\frac{\sinh(a)}{\cos(\mu_A)\sin(\theta_A)} = \frac{\sinh(b)}{\cos(\mu_B)\sin(\theta_B)} = \frac{\sinh(c)}{\cos(\mu_C)\sin(\theta_C)}. \tag{3.11}$$

When $\triangle$ lies in a complex geodesic, then

$$\mu_A = \mu_B = \mu_C = 0$$

and (3.11) reduces to the usual law of sines in a hyperbolic plane of curvature -1. When $\triangle$ lies in a totally real geodesic 2-plane,

$$\mu_A = \mu_B = \mu_B = \pi/2$$

and (3.11) is vacuous. The second sine law can be found in [65] (20),p.73, [89], Theorem 2(i) and [17], Theorem 2 (9).

We present the first law of cosines in two equivalent forms. This result appears in the above-mentioned papers as well, although our formulation is somewhat different. The first version involves the lengths $a/2, b/2, c/2$. Consider it a perturbation of the law of cosines in the hyperbolic plane of curvature $-1/4$. Its left-hand side is an expression which vanishes for real hyperbolic space; its right-hand side is a multiple of $\cos(\mu_C)$ which vanishes exactly when the triangle lies in a totally real totally geodesic subspace.

Theorem 3.2.3 *(First cosine law: version 1)*

$$\begin{aligned}\cosh^2(c/2) - \big(\cosh(a/2)\cosh(b/2) - \cos(\theta_C)\sinh(a/2)\sinh(b/2)\big)^2 \\ = \cos^2(\mu_C)\sin^2(\theta_C)\sinh^2(a/2)\sinh^2(b/2).\end{aligned}$$

The second version involves the lengths a, b, c and may be regarded as a perturbation of the law of cosines in a hyperbolic plane of curvature -1. Both involve the holomorphy angle μ_C which is $\pi/2$ if $\triangle$ lies in a totally real geodesic subspace and is 0 if $\triangle$ lies in a complex geodesic. We have written the equation so that the "correction term" on the right-hand side is positive. We write the second version using the same convention:

Theorem 3.2.4 *(First cosine law: version 2)*

$$\begin{aligned}\big(\cosh(a)\cosh(b) - \cos(\theta_C)\sinh(a)\sinh(b)\big) - \cosh(c) \\ = 2\sin^2(\mu_C)\sin^2(\theta_C)\sinh^2(a/2)\sinh^2(b/2).\end{aligned}$$

(Observe that $\sinh(a/2) = (\cosh(a) - 1)/2$ can be expressed in terms of a, etc. Also the expression $\sin(\mu_C)\sin(\theta_C)$ equals $\sin(\phi_C)$.)

Specializing to the case of a right triangle, we obtain the correponding *Pythagorean theorems.* We suppose that the Riemannian angle $\theta_C = \pi/2$ at the vertex C. Then the tangent vectors at C to the sides a and b span a real 2-plane of holomorphy angle μ_C. The two versions of the Pythagorean theorem express the length of the hypotenuse c in terms of the side lengths a, b and the holomorphy angle μ_C:

Corollary 3.2.5 *(The Pythagorean theorem: 2 versions)*

$$\cosh^2\left(\frac{c}{2}\right) = \cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right) + \cos^2(\mu_C)\sinh^2\left(\frac{a}{2}\right)\sinh^2\left(\frac{b}{2}\right)$$

$$\cosh(c) = \cosh(a)\cosh(b) - 2\sin^2(\mu_C)\sinh^2\left(\frac{a}{2}\right)\sinh^2\left(\frac{b}{2}\right).$$

Let O be a point in $X = \mathbf{H}^2_{\mathbb{C}}$ and v_1, v_2 be two geodesic rays emanating from O. Denote their (real) tangent vectors by $u_1, u_2 \in T_O X \cong \mathbb{C}^2$. Denote the complex lines they span by $U_j = u_j\mathbb{C} \subset T_O X$. Let $\Pi_j : T_O X \longrightarrow U_j$ denote orthogonal projection onto U_j. The usual (positive definite) Hermitian form on $\mathbb{C}^2$ will be denoted by $\langle\langle , \rangle\rangle$ and the corresponding norm by $\|u\| = \sqrt{\langle\langle v, v\rangle\rangle}$.

The usual Riemannian angle

$$\theta = \angle(v_1, v_2) = \angle(u_1, u_2)$$

between v_1, v_2 is given by the Riemannian inner product

$$(v_1, v_2) = \mathrm{Re}\langle\langle v_1, v_2\rangle\rangle$$

as follows:

$$\cos(\theta) = \frac{(u_1, u_2)}{\|u_1\|\,\|u_2\|} = \frac{\mathrm{Re}\langle\langle u_1, u_2\rangle\rangle}{\|u_1\|\,\|u_2\|}.$$

This angle (the *real angle* between u_1, u_2) assumes values between 0 and π. The *complex angle* ϕ equals $\angle(U_1, U_2)$, that is the *smallest angle* between a vector $\xi_1 u_1 \in U_1$ and a vector $\xi_2 u_2 \in U_2$ where $\xi_1, \xi_2 \in \mathbb{C}^*$, which is determined by

$$\cos(\phi) = \frac{|\langle\langle u_1, u_2\rangle\rangle|}{\|u_1\|\,\|u_2\|}.$$

Unlike the real angle, the complex angle takes values between 0 and $\pi/2$. Furthermore, by definition, $\phi \le \theta$ and $\phi \le \pi - \theta$.

The angle pair

$$(\theta, \phi) \in [0, \pi] \times [0, \pi/2]$$

completely determines the pair of rays in $\mathbb{C}^2$ up to the action of the unitary group $\mathbf{U}(2)$. However, for various purposes, we introduce two other invariants related to θ and ϕ. The *holomorphy angle* $\mu = \mu(P)$ of a real 2-plane $P \subset \mathbb{C}^2$ is the angle between P and its image $\mathbb{J}(P)$ under the complex structure $\mathbb{J}$. If u_1, u_2 are an orthonormal basis of P, then $\langle\langle u_1, u_2\rangle\rangle$ is purely imaginary and

$$\cos(\mu) = \mathrm{Im}\langle\langle u_1, u_2\rangle\rangle = (u_1, \mathbb{J}(u_2)).$$

In general, μ relates to θ and ϕ by

$$\sin(\phi) = \sin(\mu)\sin(\theta). \tag{3.12}$$

Since μ is the angle between linear subspaces, $0 \le \mu \le \pi/2$. When $\mu = 0$, the 2-plane P is a complex line; when $\mu = \pi/2$, the 2-plane P is totally real.

A pair of points in $\mathbf{H}^2_{\mathbb{C}}$ is determined up to isometry by the distance ρ between them. Here is a normal form for pairs: applying an element of $\mathbf{U}(2,1)$, we may assume that a pair of negative vectors representing $[v], [w] \in \mathbf{H}^2_{\mathbb{C}}$ separated by distance ρ is given by

$$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} \sinh(\rho/2) \\ 0 \\ \cosh(\rho/2) \end{bmatrix}. \tag{3.13}$$

The first law of sines is discussed in [65] (18), p.71, [89], Theorem 2(ii) and [17], Theorem 2 (8). The proof presented here is more conceptual, starting from a basic result analogous to a Pythagorean theorem in $\mathbf{H}^2_{\mathbb{R}}$ (curvature $-1/4$). If $\triangle(A,B,C)$ is a right triangle in $\mathbf{H}^2_{\mathbb{R}}$ with a right angle at C, then the sine of angle A satisfies

$$\sin(A) = \frac{\sinh(a/2)}{\sinh(c/2)}. \tag{3.14}$$

3.2.3 *Complex triangles*

Consider a general triangle $\triangle = \triangle(A,B,C)$ in $\mathbf{H}^2_{\mathbb{C}}$. Let $\overleftrightarrow{AB}$ denote the complex line spanned by vertices A, B and denote by

$$\Pi : \mathbf{H}^2_{\mathbb{C}} \longrightarrow \overleftrightarrow{AB} \subset \mathbf{H}^2_{\mathbb{C}}$$

orthogonal projection. The projection $\Pi(C)$ is the point of the complex side $\overleftrightarrow{AB}$ closest to the vertex C and its inverse image

$$h_C = \Pi^{-1}\Pi(C)$$

is a complex geodesic which we call the *complex altitude*. h_C meets the complex side $\overleftrightarrow{AB}$ at $\Pi(C)$. The two complex geodesics h_C and $\overleftrightarrow{AB}$ are orthogonal. (However, as discussed by Coolidge [30], [31], and Cartan [24], h_C generally will not intersect the real side c.)

In general, right triangles in $\mathbf{H}^2_{\mathbb{C}}$ tend to be complicated. However, a particularly tractable case occurs when two complex sides are orthogonal: $\phi_C = \pi/2$. (Since

$$\phi_C \le \theta_C \le \pi/2$$

this condition implies the real angle θ_C is a right angle as well.)

Suppose that u_1, u_2 are vectors in $\mathbb{C}^2$ spanning orthogonal complex lines $U_1, U_2 \subset \mathbb{C}^2$. Then together u_1 and u_2 span over $\mathbb{R}$ a totally real linear subspace of $\mathbb{C}^2$. If u_1, u_2 are unit vectors in $T_{\Pi(C)}X$ tangent to the real geodesics from $\Pi(C)$ to C and A respectively, then u_1 and u_2 span orthogonal complex lines. The exponential image $\mathrm{Exp}(P)$ of the real 2-plane $P \subset T_{\Pi(C)}X$ spanned by u_1, u_2 is a totally real totally geodesic submanifold of $\mathbf{H}^2_{\mathbb{C}}$. Then $\triangle(A, \Pi(C), C)$ is a right triangle inside $\mathrm{Exp}(P) \cong \mathbf{H}^2_{\mathbb{R}}$ with angle ϕ_A at A and $\rho(A,C) = b$.

Applying (3.14) to $\triangle(A, \Pi(C), C)$ yields

$$\sinh(\rho(C, \Pi(C))) = \sin(\phi_A)\sinh(b/2). \tag{3.15}$$

A similar argument applied to vertex B yields

$$\sinh(\rho(C, \Pi(C))) = \sin(\phi_B)\sinh(a/2). \tag{3.16}$$

Combining (3.15) and (3.16) yields the first sine law:

$$\frac{\sinh(a/2)}{\sin(\phi_A)} = \frac{\sinh(b/2)}{\sin(\phi_B)}$$

The second sine law is more complicated, but the general outline of the proof is similar. Rather than expressing the "complex altitude" in two different ways, the second sine law involves the altitude of the projection of the triangle onto the complex side $\overleftrightarrow{CA}$. We present the proof here, based on the following formula for the altitude, whose proof is deferred until later:

Lemma 3.2.6 *Let* $\Pi : \mathbf{H}^2_{\mathbb{C}} \longrightarrow \overleftrightarrow{CA}$ *be orthogonal projection onto* $\overleftrightarrow{CA}$ *and let* $b \subset \overleftrightarrow{CA}$ *be the real side (the real geodesic) joining* C *to* A. *The length* h' *of the altitude of* $\Pi(\triangle)$ *is the distance* $\rho(\Pi(B), b)$ *in* $\overleftrightarrow{CA}$ *and satisfies*

$$\sinh(h') = \frac{\sin(\theta_C)\cos(\mu_C)\sinh(a)}{\cosh^2(a/2) - \cos^2(\phi_C)\sinh^2(a/2)}.$$

Rewrite Lemma 3.2.6 in the form

$$\sin(\theta_C)\cos(\mu_C)\sinh(a) = \sinh(h')\left(\cosh^2(a/2) - \cos^2(\phi_C)\sinh^2(a/2)\right).$$

We show that the right-hand side is symmetrical in A, C. To this end,

$$\begin{aligned}\cosh^2(a/2) - \cos^2(\phi_C)\sinh^2(a/2) &= (1 + \sinh^2(a/2)) \\ &\quad - (1 - \sin^2(\phi_C))\sinh^2(a/2) \\ &= 1 + \sinh^2(a/2)\sin^2(\phi_C) \\ \text{(by 3.10)} &= 1 + \sinh^2(c/2)\sin^2(\phi_A) \\ &= \cosh^2(c/2) - \cos^2(\phi_A)\sinh^2(c/2).\end{aligned}$$

(Compare [65], p.72.) Thus

$$\sin(\theta_C)\cos(\mu_C)\sinh(a) = \sin(\theta_A)\cos(\mu_A)\sinh(c)$$

from which the second sine law follows.

To derive the remaining formulas, we resort to an explicit choice of coordinates, finding a normal form for triangles. Working in the Hermitian vector space $\mathbb{C}^{2,1}$, let vertex A correspond to the origin. The normal forms derived in the first two sections provide coordinates for the other two vertices, sides, complex sides, etc. Alternatively, choose A, B, C to correspond to the standard basis and calculate instead with the Gram matrix—the Hermitian matrix corresponding to the Hermitian form with respect to this basis. This approach is used in [34] and [161].

The vertex $C = [\tilde{C}]$ corresponds to the vector $\tilde{C} \in \mathbb{C}^{2,1}$

$$\tilde{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Use the normal forms (3.13), (2.10) to represent the vertices $A = [\tilde{A}]$ and $B = [\tilde{B}]$ by vectors

$$\tilde{A} = \begin{bmatrix} \sinh(b/2) \\ 0 \\ \cosh(b/2) \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} e^{i\eta}\cos(\phi)\sinh(a/2) \\ \sin(\phi)\sinh(a/2) \\ \cosh(a/2) \end{bmatrix}. \tag{3.17}$$

For brevity write $\phi = \phi_C$, $\eta = \eta_C$, $\theta = \theta_C$ and $\mu = \mu_C$.

The first cosine law follows immediately by applying the distance formula to this normal form for the triangle. We compute $c = \rho(A, B)$. Namely,

$$\begin{aligned}
\cosh^2(c/2) &= \frac{\langle \tilde{A}, \tilde{B}\rangle\langle \tilde{B}, \tilde{A}\rangle}{\langle \tilde{A}, \tilde{A}\rangle\langle \tilde{B}\tilde{B}\rangle} \\
&= |\cos(\phi)e^{i\eta}\sinh(a/2)\sinh(b/2) - \cosh(a/2)\cosh(b/2)|^2 \\
&= (\cos(\phi)\cos(\eta)\sinh(a/2)\sinh(b/2) - \cosh(a/2)\cosh(b/2))^2 \\
&\quad + \cos^2(\phi)\sin^2(\eta)\sinh^2(a/2)\sinh^2(b/2) \\
&= (\cosh(a/2)\cosh(b/2) - \cos(\theta)\sinh(a/2)\sinh(b/2))^2 \\
&\quad + \cos^2(\mu)\sin^2(\theta)\sinh^2(a/2)\sinh^2(b/2)
\end{aligned}$$

(by (2.8) and (2.9)). This completes the proof of the first version of the first cosine formula (Theorem 3.2.3).

To obtain the second version of the first cosine law, use the substitutions

$$\cosh^2(t/2) = \frac{\cosh(t)+1}{2}, \qquad \sinh^2(t/2) = \frac{\cosh(t)-1}{2}$$

and $2\cosh(t/2)\sinh(t/2) = \sinh(t)$ to replace the above expression by one involving the lengths a, b, c rather than $a/2, b/2, c/2$. Here are the details:

$$\begin{aligned}
\cosh^2(c/2) &= \big(\cosh(a/2)\cosh(b/2) - \cos(\theta)\sinh(a/2)\sinh(b/2)\big)^2 \\
&\quad + \cos^2(\mu)\sin^2(\theta)\sinh^2(a/2)\sinh^2(b/2) \\
&= \cosh^2(a/2)\cosh^2(b/2) \\
&\quad - 2\cos(\theta)\sinh(a/2)\sinh(b/2)\cosh(a/2)\cosh(b/2) \\
&\quad + \big(\cos^2(\theta) + \sin^2(\theta)\cos^2(\mu)\big)\sinh^2(a/2)\sinh^2(b/2) \\
&= \cosh^2(a/2)\cosh^2(b/2) - \frac{1}{2}\cos(\theta)\sinh(a)\sinh(b) \\
&\quad + \big(1 - \sin^2(\theta)\sin^2(\mu)\big)\sinh^2(a/2)\sinh^2(b/2)
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
\frac{\cosh(c)+1}{2} &= \frac{\cosh(a)+1}{2}\cdot\frac{\cosh(b)+1}{2} + \frac{\cosh(a)-1}{2}\cdot\frac{\cosh(b)-1}{2} \\
&\quad - \frac{1}{2}\cos(\theta)\sinh(a)\sinh(b) \\
&\quad - \sin^2(\theta)\sin^2(\mu)\sinh^2(a/2)\sinh^2(b/2)
\end{aligned}$$

which by (3.12) yields

$$\cosh(c) = \cosh(a)\cosh(b) - \cos(\theta)\sinh(a)\sinh(b) \\ - 2\sin^2(\phi)\sinh^2(a/2)\sinh^2(b/2).$$

Proof of Lemma 3.2.6 Using the normal form for the triangle, we compute the altitude of the projection of a triangle into one of its complex sides. The complex side $\overleftrightarrow{CA}$ consists of all $[Z]$ of the form

$$Z = \begin{bmatrix} Z_1 \\ 0 \\ Z_3 \end{bmatrix}$$

and orthogonal projection is

$$\Pi : \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \mapsto \begin{bmatrix} Z_1 \\ 0 \\ Z_3 \end{bmatrix}.$$

Thus

$$\Pi(\tilde{B}) = \begin{bmatrix} e^{i\eta}\cos(\phi)\sinh(a/2) \\ 0 \\ \cosh(a/2) \end{bmatrix}.$$

In these coordinates $\overleftrightarrow{CA}$ is represented by the complex hyperbolic line $\mathbf{H}^1_{\mathbb{C}}$. In the inhomogeneous coordinate z corresponding to the point

$$\begin{bmatrix} z \\ 0 \\ 1 \end{bmatrix} \in \mathbf{H}^1_{\mathbb{C}}$$

b lies on the real axis $\mathbf{H}^1_{\mathbb{R}} \subset \mathbf{H}^1_{\mathbb{C}}$. Since the distance $\rho(z, b)$ from a point z in the unit disc $\mathbf{H}^1_{\mathbb{C}}$ to the geodesic b contained in $\mathbf{H}^1_{\mathbb{R}}$ satisfies

$$\sinh(\rho(z,b)) = \frac{2\mathrm{Im}(z)}{1 - |z|^2}$$

and $\Pi(B)$ corresponds to the point with

$$z = e^{i\eta}\cos(\phi)\tanh(a/2)$$

we see that

$$\begin{aligned} \sinh(h'_b) &= \frac{2\mathrm{Im}(z)}{1 - |z|^2} \\ &= \frac{2\sin(\eta)\cos(\phi)\tanh(a/2)}{1 - \cos^2(\phi)\tanh^2(a/2)} \\ &= \frac{2\sin(\eta)\cos(\phi)\sinh(a/2)\cosh(a/2)}{\cosh^2(a/2) - \cos^2(\phi)\sinh^2(a/2)} \\ &= \frac{\sin(\theta)\cos(\mu)\sinh(a)}{\cosh^2(a/2) - \cos^2(\phi)\sinh^2(a/2)} \end{aligned}$$

(using (2.9)) as desired. The proof of Lemma 3.2.6 is complete. □

Exercise 3.2.7 *Using the normal form for triangles, compute the Gram matrix of Hermitian products*

$$\begin{bmatrix} \langle\tilde{A},\tilde{A}\rangle & \langle\tilde{A},\tilde{B}\rangle & \langle\tilde{A},\tilde{C}\rangle \\ \langle\tilde{B},\tilde{A}\rangle & \langle\tilde{B},\tilde{B}\rangle & \langle\tilde{B},\tilde{C}\rangle \\ \langle\tilde{C},\tilde{A}\rangle & \langle\tilde{C},\tilde{B}\rangle & \langle\tilde{C},\tilde{C}\rangle \end{bmatrix} = \begin{bmatrix} -1 & \bar{\xi} & \cosh(b/2) \\ \xi & -1 & \cosh(a/2) \\ \cosh(b/2) & \cosh(a/2) & -1 \end{bmatrix}$$

where $\xi = \sinh(a/2)\sinh(b/2)\cos(\phi)e^{i\eta} - \cosh(a/2)\cosh(b/2)$. *Eliminate* $\cos(\theta) = \cos(\phi)\cos(\eta)$ *to obtain:*

$$\begin{aligned}\cos(\theta_C) = \mathrm{Re}\Big\{&\langle\tilde{A},\tilde{B}\rangle\langle\tilde{B},\tilde{A}\rangle\langle\tilde{C},\tilde{C}\rangle + \langle\tilde{A},\tilde{C}\rangle\langle\tilde{C},\tilde{B}\rangle\langle\tilde{B},\tilde{A}\rangle\Big\} \\ &\Big/ \Big\{\sqrt{(\langle\tilde{C},\tilde{A}\rangle\langle\tilde{A},\tilde{C}\rangle - \langle\tilde{C},\tilde{C}\rangle\langle\tilde{A},\tilde{A}\rangle)\langle\tilde{A},\tilde{A}\rangle} \\ &\quad \cdot \sqrt{(\langle\tilde{C},\tilde{B}\rangle\langle\tilde{B},\tilde{C}\rangle - \langle\tilde{C},\tilde{C}\rangle\langle\tilde{B},\tilde{B}\rangle)\langle\tilde{B},\tilde{B}\rangle}\Big\}.\end{aligned}$$

Compare this formula with the matrix form of the law of cosines in §1.4.6.

Exercise 3.2.8 *Formulate and prove a version of the second law of cosines (1.17) in complex hyperbolic space. In addition to its length, associate to a side the angle of rotation relating the parallel translate (along the side) of the vector at one endpoint tangent to the adjacent side to the corresponding vector at the other side.*

3.2.4 *Angle of parallelism*

Consider a right triangle with one finite side of length d and an infinite hypotenuse as in §1.4.3.1. The vertex with right angle will be denoted p and the other finite vertex will be denoted O. The tangent vectors at p to the two sides span a real 2-plane $P \subset T_p\mathbf{H}^n_{\mathbb{C}}$ with holomorphy angle $\nu = \mu(P)$.

Exercise 3.2.9 *Let* θ *and* μ *denote the Riemannian angle and holomorphy angle, respectively, associated with the vertex* O. *The complex angle* ϕ *at* O *is determined from* μ, θ *by (2.6) Then*

$$\begin{aligned}\cos(\theta) + i\sin(\theta)\cos(\mu) &= \tanh(d/2)\left(1 + \cos(\nu)\frac{\cos(\nu)\mathrm{sech}^2(d/2) - 2i\mathrm{csch}(d)}{1+\cos^2(\nu)\tanh^2(d/2)}\right) \\ &= \tanh(d) + i\ \mathrm{sech}(d) \\ &\quad + \big(\cos(\nu) - 1\big)\cdot\left(\frac{\tanh(d) + i\ \mathrm{sech}(d) - 1}{\tanh(d/2) - i\ \cos(\nu)}\right).\end{aligned}$$

When the triangle lies in a totally real totally geodesic submanifold, $\nu = \pi/2$, and this formula yields ues

$$\cos(\theta) = \cos(\phi) = \tanh(d/2)$$
$$\mu = 0.$$

When the triangle lies in a complex totally geodesic submanifold, $\nu = \pi/2$, and this formula yields

$$\cos(\theta) = \tanh(d)$$

$$\mu = \phi = 0.$$

These two formulas are consistent with (1.18) in $\mathbf{H}^1_{\mathbb{C}}$ and the analogous formula in $\mathbf{H}^2_{\mathbb{R}}$ respectively.

This discussion is a slight modification of Proposition A.2 of Aravinda-Leuzinger [4], which uses some different conventions. In particular, [4] normalizes the holomorphic sectional curvature to be -4 rather than -1 and uses a different set of vertex invariants (see the end of §2.2.2). Thus Proposition A.2 of [4] in our normalization and our notation would be

$$\sin^2(\phi) = \frac{\sin^2(\nu)}{1+\sinh^2(d/2)(1+\cos^2(\nu))}$$
$$\cos^2(\mu)\sin^2(\theta) = \frac{\cos^2(\nu)}{(1+\sinh^2(d/2)(1+\cos^2(\nu)))^2}.$$

3.2.5 *Comparing real and complex hyperbolic space*

Using the general law of cosines (Theorems 3.2.3 and 3.2.4) we can compare complex hyperbolic space (whose curvature is pinched between -1 and $-1/4$) and real hyperbolic spaces of curvature -1 and $-1/4$. (This material was worked out with Bernhard Leeb.) We fix the lengths a, b and the angle γ and regard the length c as a function of the angle of holomorphy μ. Then Theorems 3.2.3 and 3.2.4 imply that $c(\mu)$ is a monotone decreasing function of μ where $0 \le \mu \le \pi/2$. Now Theorem 3.2.3 implies that $c(\mu) \ge c(\pi/2)$ with equality holding if and only if $\mu = \pi/2$; that is, if P is totally real. On the other hand, Theorem 3.2.4 implies that $c(\mu) \le c(0)$ with equality holding if and only if $\mu = 0$; that is, if P is complex.

We restate these results in terms of a specific pair of *comparison maps:* Let $\mathbf{H}^{2n}_{\mathbb{R}}(K)$ denote a $2n$-dimensional real hyperbolic space of constant curvature $K < 0$. Choose base points O in $\mathbf{H}^n_{\mathbb{C}}$ and $\mathbf{H}^{2n}_{\mathbb{R}}(K)$ for $K = -1, -1/4$ and isomorphisms of the tangent spaces

$$T_O\mathbf{H}^{2n}_{\mathbb{R}}(-1) \longrightarrow T_O\mathbf{H}^n_{\mathbb{C}} \longrightarrow T_O\mathbf{H}^{2n}_{\mathbb{R}}(-1/4). \tag{3.18}$$

Letting X denote any of these three Hadamard manifolds, the exponential map defines a diffeomorphism

$$\mathrm{Exp} : T_O X \longrightarrow X.$$

Composing with (3.18), we obtain a sequence of comparison maps:

$$\mathbf{H}^{2n}_{\mathbb{R}}(-1) \longrightarrow \mathbf{H}^n_{\mathbb{C}} \longrightarrow \mathbf{H}^{2n}_{\mathbb{R}}(-1/4). \tag{3.19}$$

Exercise 3.2.10 *Use the two versions of the first cosine law to prove that the comparison maps (3.19) are distance decreasing.*

3.2.6 *Asymptotics of the Pythagorean theorem*

Asymptotic properties of the Pythagorean theorem give geometric interpretations of the angle of holomorphy. For example, if the right triangle is small $(a, b, c \sim 0)$, the approximations

$$\cosh(x) \sim 1 + x^2/2 + x^4/24$$

yield

$$c^2 \sim a^2 + b^2 + \frac{1}{3}\left(\frac{1 + 3\cos^2(\mu)}{4}\right) a^2 b^2 \tag{3.20}$$

where

$$\kappa(P) = -\frac{1 + 3\cos^2(\mu)}{4}$$

is the sectional curvature (compare §2.4.2) of $P \subset T_O \mathbf{H}^n_{\mathbb{C}}$.

For large triangles, the approximation

$$\cosh(x) \sim \frac{e^x}{2}$$

for $a \sim \infty$ yields

$$c - (a + b) \sim \log\left(\frac{1 + \cos^2(\mu)}{4}\right) \tag{3.21}$$

as $a, b, c \sim \infty$.

3.2.7 *Complex altitudes and the orthocenter*

Let $a, b, c \in \mathbf{H}^n_{\mathbb{C}}$ respectively. Then the complex altitudes (see §3.2.3) are the complex geodesics h_a, h_b, h_c defined by

$$\begin{aligned} h_a &= \overleftrightarrow{a\iota_{bc}(a)} \\ h_b &= \overleftrightarrow{b\iota_{ca}(b)} \\ h_c &= \overleftrightarrow{c\iota_{ca}(b)} \end{aligned}$$

where ι_{bc} is the inversion in the *complex side* $\overleftrightarrow{bc}$, etc. of the triangle $\Delta(a, b, c)$.

The following theorem is taken from Cartan's book [24], Ch.III, §166, pp.228–230. The proof given below is Cartan's original proof. It is an analogue of the theorem in plane (Euclidean, hyperbolic or elliptic) geometry that in any triangle, the altitudes concur. If $x, y \in \mathbf{H}^n_{\mathbb{C}}$, then we denote the complex geodesic they span by $\overleftrightarrow{xy}$.

Theorem 3.2.11 *Let $a, b, c \in \mathbf{H}^n_{\mathbb{C}}$ be the vertices of a triangle $\Delta(a,b,c)$. Then the complex altitudes h_a, h_b, h_c of $\Delta(a,b,c)$ are concurrent if and only if a, b, c lie in a totally geodesic 2-plane.*

Proof We may assume that $n = 2$. If a, b, c are linearly dependent their projective images lie in a complex geodesic. Represent a, b, c by coordinate vectors in a Hermitian vector space of signature (2,1):

$$\tilde{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

be the matrix of Hermitian products (Gram matrix). Then the complex geodesics spanned by the three vertices are dual to the vectors

$$\begin{aligned} \overleftrightarrow{bc} &\longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ \overleftrightarrow{ca} &\longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ \overleftrightarrow{ab} &\longleftrightarrow \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and the complex altitudes have dual vectors

$$\begin{aligned} h_a &\longleftrightarrow \begin{bmatrix} 0 & -h_{31} & h_{21} \end{bmatrix} \\ h_b &\longleftrightarrow \begin{bmatrix} h_{32} & 0 & -h_{12} \end{bmatrix} \\ h_c &\longleftrightarrow \begin{bmatrix} -h_{23} & h_{13} & 0 \end{bmatrix}. \end{aligned}$$

The complex altitudes h_a, h_b, h_c are concurrent if and only if the three dual vectors are linearly dependent, that is if and only if

$$0 = \begin{vmatrix} 0 & -h_{31} & h_{21} \\ h_{32} & 0 & -h_{12} \\ -h_{23} & h_{13} & 0 \end{vmatrix} = h_{12}h_{23}h_{31} - h_{21}h_{13}h_{32}.$$

Since H is a Hermitian matrix, $h_{12}h_{23}h_{31}$ and $h_{21}h_{13}h_{32}$ are complex conjugate. Hence the complex altitudes concur if and only if the Hermitian triple product $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ is real, that is if and only if a, b, c lie in a totally real totally geodesic subspace. □

Cartan proves also a converse statement, which may be interpreted as the equivalence of the two decompositions of *spinal spheres* (see §5.1). The "foot" of the complex altitude from a equals the orthogonal projection of a onto the complex geodesic containing b and c.

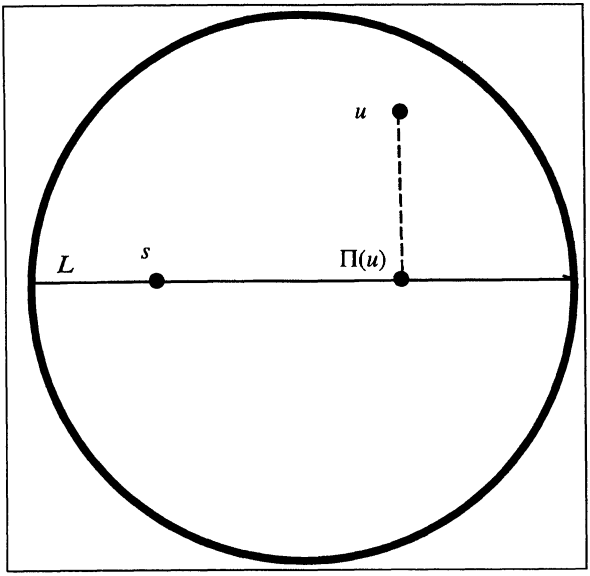

FIG. 3.5. Proof of Lemma 3.2.13

Theorem 3.2.12 *Let α be the real geodesic from b to c and let $\Pi_{\overleftrightarrow{bc}}$ denote the orthogonal projection onto $\overleftrightarrow{bc}$. Then*

$$\Pi_{\overleftrightarrow{bc}}(a) \in \alpha$$

if and only if $(a, b, c) \in$ **Real**.

The proof is an elementary consequence of coexistence of the slice and meridian decompositions and will be given in §5.1.6. Compare also Corollary 7.1.5.

For the next two results, recall that a *Lambert quadrilateral* (or *isosceles birectangle*) is a quadrilateral having three right angles.

Lemma 3.2.13 *Let $L \subset \mathbb{B}^n$ be a complex linear subspace with orthogonal projection Π. Then for all $u \in \mathbb{B}^n - L$ and $s \in L$, the geodesics from $\Pi(u)$ to u and to s are orthogonal and span a totally real totally geodesic 2-plane. Furthermore*

$$\cosh\left(\frac{\rho(u,s)}{2}\right) = \cosh\left(\frac{\rho(u,\Pi(u))}{2}\right)\cosh\left(\frac{\rho(\Pi(u),s)}{2}\right).$$

Conversely suppose that $n = 2$. Then if u, v, s span a right triangle on a totally real 2-plane then the orthogonal projection onto the complex geodesic spanned by v, s maps u to v and the orthogonal projection onto the complex geodesic spanned by u, v maps s to v.

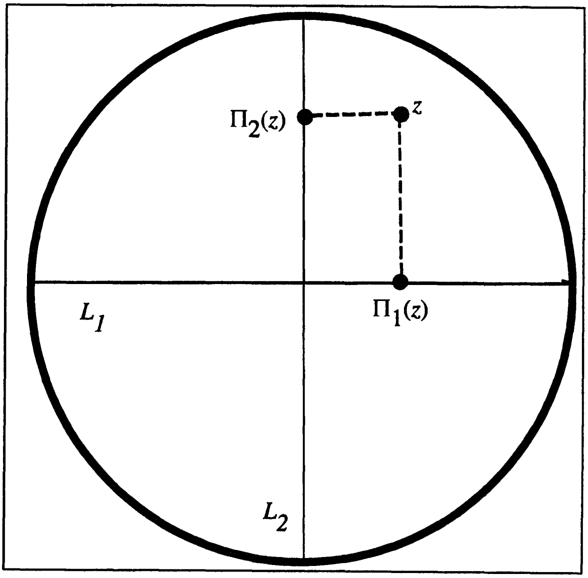

FIG. 3.6. Proof of Lemma 3.2.14

Lemma 3.2.14 *Let $L_1, L_2 \subset \mathbb{B}^2$ be orthogonal complex geodesics and let $z \in \mathbb{B}^n$. Let $\Pi_i : \mathbb{B}^n \longrightarrow L_i$ denote orthogonal projection. Then $z, \Pi_1(z), \Pi_2(z), L_1 \cap L_2$ lie in a totally real geodesic 2-plane and form a Lambert quadrilateral. Furthermore*

$$\sinh\left(\frac{\rho(z,\Pi_1(z))}{2}\right) = \cosh\left(\frac{\rho(z,\Pi_2(z))}{2}\right)\sinh\left(\frac{\rho(\Pi_2(z), L_1 \cap L_2)}{2}\right).$$

Proof of Lemma 3.2.13 (Compare Fig. 3.5.) Find a 2-dimensional complex linear subspace containing z and meeting the hyperplanes orthogonally; working in this subspace we may assume $n = 2$.

By applying an automorphism we may assume $L = \{0\} \times \mathbb{B}^1 \subset \mathbb{B}^2$ so that $\Pi(z_1, z_2) = (0, z_2)$. Furthermore by applying an automorphism preserving L we may assume that $\Pi(u) = 0$, whence $u = (0, u_2)$. Since geodesics through 0 are Euclidean straight lines we see that the two geodesics joining $\Pi(u)$ to u and to s have tangent vectors

$$\vec{\mathbf{v}}_s = (v_1, 0), \qquad \vec{\mathbf{v}}_u = (0, v_2)$$

and $\langle \vec{\mathbf{v}}_s, \vec{\mathbf{v}}_u \rangle = 0$. The vanishing of the real part of $\langle \vec{\mathbf{v}}_s, \vec{\mathbf{v}}_u \rangle$ is equivalent to orthogonality of the vectors $\vec{\mathbf{v}}_u$ and $\vec{\mathbf{v}}_s$ and vanishing of the imaginary part of $\langle \vec{\mathbf{v}}_s, \vec{\mathbf{v}}_u \rangle$ implies that the two vectors span a totally real subspace of $T_0\mathbb{B}^n$. It follows that the $\mathbb{R}$-linear 2-plane $\mathbb{R}s \oplus \mathbb{R}u$ meets $\mathbb{B}^2$ in a totally geodesic, totally real submanifold. The second assertion is just the Pythagorean theorem.

Conversely, let P be the $\mathbb{R}$-plane containing s, v, u and let S (respectively U) be the complex geodesic containing s, v (respectively v, u). It suffices to prove that S and U are orthogonal at $v = S \cap U$. The tangent space to S at v is spanned by the tangent vector $\vec{\mathbf{v}}_s$ to the geodesic ray from v to s and $\mathbb{J}\vec{\mathbf{v}}_s$. The tangent space to U at v is spanned by the tangent vector $\vec{\mathbf{v}}_u$ to the geodesic ray from v to u and $\mathbb{J}\vec{\mathbf{v}}_u$. Therefore it suffices to check the following:

$$\vec{\mathbf{v}}_u \perp \vec{\mathbf{v}}_s, \qquad \mathbb{J}\vec{\mathbf{v}}_u \perp \vec{\mathbf{v}}_s, \qquad \vec{\mathbf{v}}_u \perp \mathbb{J}\vec{\mathbf{v}}_s, \qquad \mathbb{J}\vec{\mathbf{v}}_u \perp \mathbb{J}\vec{\mathbf{v}}_s.$$

The first orthogonality follows from u, v, s being a right triangle and the second two orthogonalities follow from $(u, v, s) \in$ **Real**; the fourth one follows from the first. This completes the proof of Lemma 3.2.13. □

Proof of Lemma 3.2.14 (Compare Fig. 3.6.) Choose coordinates so that $L_1 \cap L_2$ is the origin, $L_1 = \mathbb{B}^1 \times \{0\}$, $L_1 = \{0\} \times \mathbb{B}^1$ and $z = (z_1, z_2)$. Then $\Pi_1(z) = (z_1, 0)$, $\Pi_2(z) = (0, z_2)$. Then

$$P = \{(\zeta_1, \zeta_2) \in \mathbb{B}^2 \mid \operatorname{Im}(\bar{z}_1 \zeta_1) = 0\}$$

is a totally real geodesic 2-plane containing $z, \Pi_1(z), \Pi_2(z), L_1 \cap L_2$.

The last assertion is trigonometry of Lambert quadrilaterals. In a Lambert quadrilateral whose sides are a_1, b_1, a_2, b_2 (here b_1 and b_2 are adjacent sides which are not orthogonal and a_i is the side opposite b_i), the sides satisfy:

$$\sinh\left(\frac{b_1}{2}\right) = \cosh\left(\frac{b_2}{2}\right) \sinh\left(\frac{a_1}{2}\right). \tag{3.22}$$

(This formula can easily be derived from the two trigonometric identities in [9], Theorem 7.17.1, p.157, where the lengths are halved because the hyperbolic plane has curvature $-1/4$.) Taking

$$b_1 = \rho(z, \Pi_1(z)), \qquad b_2 = \rho(z, \Pi_2(z)), \qquad a_1 = \rho(\Pi_2(z), L_1 \cap L_2)$$

completes the proof of Lemma 3.2.14. □

3.3 Computations in the ball model

3.3.1 *Polar vectors for complex hyperplanes*

In this section we discuss the geometry of complex geodesics and complex linear subspaces in general. As usual, we concentrate on the case $n = 2$ and many of our results reduce to this case. Of special importance are complex geodesics and complex hyperplanes (which coincide when $n = 2$). If $H \subset \mathbf{H}^n_{\mathbb{C}}$ is a totally geodesic complex hyperplane, then $H = \mathbb{P}(\tilde{H})$ where $\tilde{H} \subset \mathbb{C}^{n,1}$ is a complex linear hyperplane. $\tilde{H}^\perp$ is then a positive line and $\mathbb{P}(\tilde{H}^\perp)$ is a point in the complement $\mathbb{P}(\mathbb{C}^{n,1}) - \overline{\mathbf{H}^n_{\mathbb{C}}}$. Thus the totally geodesic complex hyperplanes in $\mathbf{H}^n_{\mathbb{C}}$ bijectively correspond to the points of $\mathbb{P}(\mathbb{C}^{n,1}) - \overline{\mathbf{H}^n_{\mathbb{C}}}$. The space of complex hyperplanes in $\mathbf{H}^n_{\mathbb{C}}$ identifies with the "outside" of the ball in projective space. A geometric set of "center/radius" coordinates parametrizes this space when $n = 2$ (§4.3.4).

3.3.2 *Orthogonal projections of complex geodesics*

Let H_1, H_2 be complex hyperplanes determined by polar vectors u_1, u_2 respectively: $\mathbb{P}(u_i^\perp) = H_i$. Let U be the complex linear subspace spanned by u_1 and u_2. Then $\dim(U) = 2$ unless $H_1 = H_2$ and there are three cases, depending on whether U is positive, degenerate or indefinite. The following conditions are equivalent:

1. U is positive;
2. $U^\perp$ is indefinite;
3. there exists a negative vector in
$$U^\perp = \{v \in \mathbb{C}^{n,1} \mid \langle v, u_i \rangle = 0, i = 1, 2\} = u_1^\perp \cap u_2^\perp.$$
(Such a negative vector corresponds to a point in $H_1 \cap H_2$.)

Similarly U is degenerate if and only if H_1 and H_2 are disjoint in $\mathbf{H}_{\mathbb{C}}^n$ but $\overline{H_1} \cap \overline{H_2}$ is nonempty. (In this case the intersection is a single ideal point in $\partial\mathbf{H}_{\mathbb{C}}^n$.) Finally, U is indefinite if and only if $\overline{H_1}$ and $\overline{H_2}$ are disjoint in $\mathbf{H}_{\mathbb{C}}^n \cup \partial\mathbf{H}_{\mathbb{C}}^n$. In each of these cases there is an $\mathbb{C}^2$-plane $L \subset \mathbf{H}_{\mathbb{C}}^n$ which is orthogonal to H_1 and H_2 and if $\Pi_L : \mathbf{H}_{\mathbb{C}}^n \cup \partial\mathbf{H}_{\mathbb{C}}^n \longrightarrow L$ denotes orthogonal projection, then $H_i = \Pi_L^{-1}(H_i \cap L)$. Thus we reduce to the case when $n = 2$.

If A and B are distinct complex geodesics corresponding to polar vectors $a, b \in \mathbb{C}^{2,1}$, consider the cross-product $c = a \boxtimes b$ as defined in §2.2.7. Three possibilities arise:

1. c is negative—in this case A and B intersect in the point $\mathbb{P}(c)$ corresponding to the negative vector c—and we write $A \chi B$;
2. c is null—in this case A and B are *asymptotic (or parallel)* at the point $\mathbb{P}(c) \in \partial\mathbf{H}_{\mathbb{C}}^2$—and we write $A \parallel B$;
3. c is positive—then A and B are *ultraparallel,* that is they are disjoint and have a (necessarily unique) common orthogonal complex geodesic, which is polar to c—and we write $A)(B$.

Normalizing $\langle a, a \rangle = \langle b, b \rangle = 1$ and using (2.16) these three cases respectively correspond to:

1. $|\langle a, b \rangle| < 1$, in which case $|\langle a, b \rangle| = \cos(\theta)$ where θ is the angle of intersection between A and B;
2. $|\langle a, b \rangle| = 1$;
3. $|\langle a, b \rangle| > 1$, in which case $|\langle a, b \rangle| = \cosh(\rho/2)$, where ρ is the distance between A and B.

Let H_1, H_2 be two complex hyperplanes in $\mathbf{H}_{\mathbb{C}}^n$; we determine the image of H_2 under the orthogonal projection $\Pi_1 : \mathbf{H}_{\mathbb{C}}^n \longrightarrow H_1$ onto H_1. There exists a common orthogonal $\mathbb{C}^2$-plane L and $H_i \cap L$ is a complex geodesic in L such that

$$H_i = \Pi_L^{-1}(H_i \cap L).$$

Let $\Pi' : L \longrightarrow L \cap H_1$ be orthogonal projection; then

$$\Pi_1(H_2) = (\Pi_L)^{-1}(\Pi'(L \cap H_2))$$

and we reduce to the case $n = 2$. In that case H_1, H_2 are complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$ and we distinguish three cases:

1. $H_1 \between H_2$: Let $x = H_1 \cap H_2$; then Π_1 maps H_2 diffeomorphically onto the geometric ball about x of radius
$$2 \tanh^{-1} \cos(\angle(H_1, H_2)).$$
2. $H_1 \parallel H_2$: Let $x = \overline{H_1} \cap \overline{H_2}$; then Π_1 maps H_2 diffeomorphically onto a geometric horoball centered at x.
3. $H_1)(H_2$: Let x be the point on H_1 closest to H_2; alternatively let H_0 be the common orthogonal complex geodesic to H_1 and H_2—then $x = H_0 \cap H_1$. Then the distance between H_1 and H_2 equals
$$\rho(H_1, H_2) = \rho(H_0 \cap H_1, H_0 \cap H_2)$$
and Π_1 maps H_2 diffeomorphically onto the geometric ball about x of radius
$$2 \tanh^{-1} \operatorname{sech}\left(\frac{\rho(H_1, H_2)}{2}\right).$$

Theorem 3.3.1 *Let $n = 2$. Suppose that γ is the geodesic segment between $x, y \in \overline{\mathbf{H}^n_{\mathbb{C}}}$ and let $\Sigma \subset \mathbf{H}^n_{\mathbb{C}}$ be a complex geodesic. Let $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ be orthogonal projection. If $\Pi_\Sigma(\gamma)$ is a geodesic segment in Σ, then there exists a totally real totally geodesic 2-plane P containing $x, y, \Pi_\Sigma(x), \Pi_\Sigma(y)$. Necessarily P meets Σ orthogonally in the geodesic $\Pi_\Sigma(\gamma)$.*

Proof Let Γ be the complex geodesic containing x, y. We distinguish three cases, depending on whether Γ meets Σ, is asymptotic to Σ, or is ultraparallel to Σ. In all three cases, the following lemma (whose proof is straightforward and omitted) will be useful:

Lemma 3.3.2 *Suppose that $D \subset \mathbf{H}^1_{\mathbb{C}}$ is a disc of radius R centered at x_0 and let γ be a geodesic in $\mathbf{H}^1_{\mathbb{C}}$ which intersects D. Then γ meets ∂D orthogonally if and only if γ contains x_0. More precisely,*

$$\cos(\angle(\gamma, \partial D)) = \frac{\sinh(\rho(x_0, \gamma))}{\sinh(R)}.$$

Similarly, if D is a horodisc centered at x_0 which meets a geodesic γ, then γ is orthogonal to ∂D if and only if $x_0 \in \gamma$.

Suppose first that Γ meets Σ and let $x_0 = \Gamma \cap \Sigma$. In that case the restriction $\Pi_\Sigma|_\Gamma$ maps Γ biholomorphically onto a disc D of radius

$$R = 2 \tanh^{-1} \cos(\angle(\Sigma, \Gamma))$$

centered at x_0.

Now $\Pi_\Sigma(\gamma)$ is a geodesic in Σ if and only if $\Pi_\Sigma(\gamma)$ meets ∂D orthogonally, which, by Lemma 3.3.2 above, means that γ passes through x_0. Since $x_0 \in \Sigma$, it follows that $\Pi_\Sigma(\gamma)$ also passes through x_0. By Lemma 3.2.13, unique $\mathbb{R}$-planes P_x, P_y contain the geodesic right triangles

$$P_x \supset \Delta(x, \Pi_\Sigma(x), x_0), \qquad P_y \supset \Delta(y, \Pi_\Sigma(y), x_0), \tag{3.23}$$

respectively. It will suffice to prove $P_x = P_y$; this follows from the claim that P_x and P_y have the same tangent space at x_0 and are hence equal. The tangent space to P_x is spanned by a vector tangent to the segment of γ from x_0 to x and a vector tangent to the segment of $\Pi_\Sigma(\gamma)$ from x_0 to $\Pi_\Sigma(x)$. Similarly, the tangent space to P_y is spanned by a vector tangent to the segment of γ from x_0 to y and a vector tangent to the segment of $\Pi_\Sigma(\gamma)$ from x_0 to $\Pi_\Sigma(y)$; hence $T_{x_0}P_x = T_{x_0}P_y$ and $P_x = P_y$, proving the claim in the first case.

The next case, when $\Sigma \parallel \Gamma$, is proved analogously. Let $x_0 = \bar{\Sigma} \cap \bar{\Gamma} \in \partial\mathbf{H}^n_{\mathbb{C}}$. By Lemma 3.2.13, $\mathbb{R}$-planes P_x, P_y contain the geodesic right triangles (with ideal vertex x_0) as in (3.23). The rest of the proof in this case is completely analogous.

Finally we consider the case when $\Sigma)(\Gamma$. Let σ_0 be the point of Σ closest to Γ and γ_0 the point of Γ closest to Σ; that is, $\rho(\Sigma, \Gamma) = \rho(\sigma_0, \gamma_0)$. Then $\Pi_\Sigma|_\Gamma$ maps Γ biholomorphically onto a disc D of radius

$$R = 2\tanh^{-1}\operatorname{sech}\left(\rho(\Sigma, \Gamma)\right)$$

centered at σ_0 and $\Pi_\Sigma(\gamma_0) = \sigma_0$. Now $\Pi_\Sigma(\gamma)$ is a geodesic in Σ if and only if $\Pi_\Sigma(\gamma)$ meets ∂D orthogonally, which, by Lemma 3.3.2 above, means that γ passes through γ_0.

Let C be the unique complex geodesic orthogonal to Σ and Γ (so that $C \cap \Sigma = \sigma_0$ and $C \cap \Gamma = \gamma_0$) and let $c \subset C$ be the geodesic containing γ_0 and σ_0. We claim that x (respectively y) maps to γ_0 under orthogonal projection Π_C onto C. To this end it suffices to show that x, γ_0, σ_0 form a right triangle in an $\mathbb{R}$-plane, which follows from Lemma 3.2.13, since O is the orthogonal projection of σ_0 onto Γ. Applying Lemma 3.2.14 with $L_1 = \Sigma$ and $L_2 = C$ we see that $x, \gamma_0, \Pi(x), \sigma_0$ are the vertices of a Lambert quadrilateral on an $\mathbb{R}$-plane which we denote by P_x. Similarly $y, \gamma_0, \Pi(y), \sigma_0$ form a Lambert quadrilateral on an $\mathbb{R}$-plane P_y. The $\mathbb{R}$-planes P_x and P_y coincide because they have the same tangent space $T_{\gamma_0}\gamma + T_{\gamma_0}c$ at γ_0 (they contain a common pair of distinct geodesics, γ and c). $\square$

3.3.3 *The exponential map and the Cartan decomposition*

Let $O \in \mathbb{B}^n$ be the origin and let $e_O \in \mathbb{C}^{n,1}$ be the corresponding vector

$$\begin{bmatrix} 0' \\ 1 \end{bmatrix}.$$

The stabilizer of O is easily seen to be the unitary group whose embedding

$$\mathbf{U}(n) \hookrightarrow \mathbf{PU}(n, 1)$$

is the composition of the inclusion

$$\begin{aligned}\mathbf{U}(n) &\longrightarrow \mathbf{U}(n,1) \\ A &\mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}\end{aligned} \tag{3.24}$$

with projection $\mathbf{U}(n,1) \longrightarrow \mathbf{PU}(n,1)$. The Lie algebra of $\mathbf{PU}(n,1)$ equals $\mathfrak{g} = \mathfrak{su}(n,1)$ consisting of matrices

$$M(X,\zeta) = \begin{bmatrix} X & \zeta \\ \zeta^* & -\mathrm{trace}(X) \end{bmatrix}$$

where $X \in \mathfrak{u}(n)$ satisfies $X^* = -X$ and $\zeta \in \mathbb{C}^n$. (We write $(n+1)$-square matrices in $(n+1) \times (n+1)$ block form. For any matrix Y we denote its conjugate transpose by Y^*.) The corresponding embedding of Lie algebras

$$\mathfrak{u}(n) \xrightarrow{\cong} \mathfrak{k} \subset \mathfrak{g} = \mathfrak{su}(n,1)$$

is given by

$$X \mapsto \begin{bmatrix} X - \frac{1}{n+1}\mathrm{trace}(X)\mathbb{I}_n & 0 \\ 0 & -\frac{1}{n+1}\mathrm{trace}(X) \end{bmatrix}.$$

Then $\iota_O : M(X,\zeta) \mapsto M(X,-\zeta)$ is a *Cartan involution*; the corresponding maximal compact subgroup K of $G = \mathbf{PU}(n,1)$ is $\mathbf{U}(n)$, embedded as in (3.24). Geometrically, K is the stabilizer of the origin $O \in G/K = \mathbb{B}^n$ and $\iota_O : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ is reflection in O.

In the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the compact subalgebra $\mathfrak{k}$ is as above and $\mathfrak{p}$ consists of matrices

$$p_\zeta = M(\zeta, 0) = \begin{bmatrix} 0_n & \zeta \\ \zeta^* & 0 \end{bmatrix}$$

where $\zeta \in \mathbb{C}^n$; as representations of $\mathbf{U}(n)$, $\mathfrak{k}$ is the adjoint representation and $\mathfrak{p}$ is the standard n-dimensional unitary representation of $\mathbf{U}(n)$.

We compute the restriction of the exponential map $\mathrm{Exp} : \mathfrak{g} \longrightarrow G$ to $\mathfrak{p} \subset \mathfrak{g}$. Let $\zeta \in \mathbb{C}^n$; then $\zeta\zeta^*$ is an $n \times n$ matrix whose trace is the scalar $\zeta^*\zeta = \langle\langle \zeta, \zeta \rangle\rangle = \|\zeta\|^2$. Thus

$$\begin{aligned}
p_\zeta^2 &= \begin{bmatrix} \zeta\zeta^* & 0 \\ 0 & \zeta^*\zeta \end{bmatrix} \\
p_\zeta^3 &= \begin{bmatrix} 0 & \zeta\zeta^*\zeta \\ \zeta^*\zeta\zeta^* & 0 \end{bmatrix} = \|\zeta\|^2 p_\zeta \\
p_\zeta^4 &= \begin{bmatrix} \zeta\zeta^*\zeta\zeta^* & 0 \\ 0 & \zeta^*\zeta\zeta^*\zeta \end{bmatrix} = \|\zeta\|^2 p_\zeta^2,
\end{aligned}$$

and for $n > 0$,

$$(p_\zeta)^{2n+1} = \|\zeta\|^{2n} p_\zeta, \quad (p_\zeta)^{2n+2} = \|\zeta\|^{2n} p_\zeta^2.$$

Thus

$$\begin{aligned}
P_\zeta = \exp(p_\zeta) &= \mathbb{I} + \sum_{n=0}^{\infty} \frac{p_\zeta^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{p_\zeta^{2n+2}}{(2n+2)!} \\
&= \mathbb{I} + \sum_{n=0}^{\infty} \frac{\|\zeta\|^{2n+1}}{(2n+1)!} \|\zeta\|^{-1} p_\zeta + \sum_{n=0}^{\infty} \frac{\|\zeta\|^{2n+2}}{(2n+2)!} \|\zeta\|^{-2} p_\zeta^2 \\
&= \mathbb{I} + \frac{\sinh \|\zeta\|}{\|\zeta\|} p_\zeta + \frac{\cosh \|\zeta\| - 1}{\|\zeta\|^2} p_\zeta^2 \\
&= \begin{bmatrix} \mathbb{I}_{n-1} + (\cosh \|\zeta\| - 1)/\|\zeta\|^2 \, \zeta\zeta^* & (\sinh \|\zeta\|)/\|\zeta\| \, \zeta \\ (\sinh \|\zeta\|)/\|\zeta\| \, \zeta^* & \cosh \|\zeta\| \end{bmatrix}.
\end{aligned}$$

P_ζ takes the origin 0 to

$$z = \frac{\tanh \|\zeta\|}{\|\zeta\|} \zeta$$

in $\mathbb{B}^n$. Since

$$\frac{z}{\|z\|} = \frac{\zeta}{\|\zeta\|}, \quad \cosh \|\zeta\| = \frac{1}{\sqrt{1 - \|z\|^2}}, \quad \sinh \|\zeta\| = \frac{\|z\|}{\sqrt{1 - \|z\|^2}},$$

$$P_z = \begin{bmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{1 - \|z\|^2}} \begin{bmatrix} \|z\|^{-2}(1 - \sqrt{1 - \|z\|^2}) zz^* & z \\ z^* & 1 \end{bmatrix}$$

takes 0 to z. (P_z is the unique *strictly hyperbolic* element of $\mathbf{PU}(n, 1)$ taking 0 to z along the unique geodesic joining 0 to z.) The Cartan decomposition of $\mathbf{PU}(n, 1)$ can now be easily obtained. Let $g \in \mathbf{PU}(n, 1)$ and $z = g^{-1}(0)$. Then $(P_z)^{-1} = P_{-z}$ and

$$gP_z(0) = g(g^{-1}(0)) = 0$$

so $U = gP_z$ fixes 0 and thus lies in $\mathbf{U}(n)$. Thus $g = UP_{-z}$ and we obtain the *Cartan decomposition:*

$$\mathbf{PU}(n, 1) = \mathbf{U}(n) \cdot \exp(\mathfrak{p}). \tag{3.25}$$

3.3.4 *Growth of volume of geodesic balls*

We shall compute how the volume of a geodesic ball of radius ρ grows as a function of the distance ρ (compare [77], §6.4]). For example, in the Poincaré disc (complex hyperbolic 1-space), a circle of radius ρ has circumference $4\pi \sinh(\rho)$ and encloses an area $4\pi(\cosh(\rho) - 1)$, both of which grow like e^ρ.

At the point $(0, \ldots, 0, r) \in \mathbb{B}^n$ the Kähler form has the value

$$\Phi = 4i \left(\sum_{j=1}^{n-1} \frac{1}{1-r^2} dz_j \wedge d\bar{z}_j + \frac{1}{(1-r^2)^2} dz_n \wedge d\bar{z}_n \right)$$

and the volume form has the value

$$\begin{aligned} \frac{1}{n!}\Phi^n &= (4i)^n \frac{1}{(1-r^2)^{n+1}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= 8^n \frac{1}{(1-r^2)^{n+1}} d\mathrm{vol} \end{aligned}$$

where

$$\begin{aligned} d\mathrm{vol} &= \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= r^{2n-1} dr d\sigma \end{aligned}$$

is the Euclidean volume form on $\mathbb{C}^n$ and $d\sigma$ is the volume form on the unit sphere $r = 1$. Since the Euclidean distance r from 0 is related to the hyperbolic distance ρ by

$$r = \tanh\left(\frac{\rho}{2}\right)$$

the volume of a ball of radius ρ is given by

$$\begin{aligned} \mathrm{vol}(\mathrm{B}(\rho)) &= \int_{2\tanh^{-1}(r) \le \rho} 8^n \frac{r^{2n-1}}{(1-r^2)^{(n+1)}} dr d\sigma \\ &= \frac{1}{2} 8^n \sigma_{2n-1} \int_0^\rho \sinh^{2n-1}(R/2)\cosh(R/2)\, dR \\ &= \frac{8^n \sigma_{2n-1}}{2n} \sinh^{2n}(\rho/2) \sim \frac{8^n \sigma_{2n-1}}{2n} e^{n\rho} \end{aligned}$$

(where $\sigma_{2n-1} = 2\pi^n/n!$ is the Euclidean volume of the unit sphere $S^{2n-1} \subset \mathbb{C}^n$). Thus the volume of a geodesic ball has exponential growth rate n. In contrast, in a real hyperbolic space $\mathbf{H}^n_{\mathbb{R}}(K)$ of dimension n and curvature $K < 0$, the volume of a geodesic ball of radius ρ grows like

$$\mathrm{vol}(B(\rho)) \sim e^{\sqrt{-K}(n-1)\rho}.$$

In particular the exponential growth rate of volume in the totally real geodesic subspace $\mathbf{H}^n_{\mathbb{R}} \subset \mathbf{H}^n_{\mathbb{C}}$ is $(n-1)/2$ as compared with the exponential growth rate n of the ambient $\mathbf{H}^n_{\mathbb{C}}$.

3.3.5 *Geodesics in the ball model*

Here is a simple formula for the geodesic spanned by two points $\xi_1, \xi_2 \in \partial \mathbf{H}^n_{\mathbb{C}}$. Represent ξ_1 and ξ_2 by null vectors $\tilde{\xi}_1, \tilde{\xi}_2 \in \mathbb{C}^{n,1}$. Rescaling, normalize the pair as follows:

$$\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle = -\frac{1}{2}.$$

Transitivity of the action of $\mathbf{U}(n,1)$ on pairs of null vectors with fixed nonzero Hermitian product implies existence of $g \in \mathbf{U}(n,1)$ such that

$$g(\tilde{\xi}_1) = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}, \qquad g(\tilde{\xi}_2) = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

A point on the geodesic spanned by $g(\tilde{\xi}_1)$ and $g(\tilde{\xi}_2)$ is given by

$$\begin{bmatrix} 0 \\ \sinh(t) \\ \cosh(t) \end{bmatrix} = e^{-t} g(\tilde{\xi}_1) + e^t g(\tilde{\xi}_2).$$

Hence the geodesic joining ξ_1 and ξ_2 is given by the path of vectors in $\mathbb{C}^{n,1}$:

$$\tilde{\xi}(t) = e^{-t}\tilde{\xi}_1 + e^t \tilde{\xi}_2.$$

Theorem 3.3.3 *Let $\xi_1, \xi_2 \in \partial \mathbb{B}^n$. Define*

$$w = \frac{\operatorname{Im}\langle\langle \xi_1, \xi_2 \rangle\rangle}{1 - \operatorname{Re}\langle\langle \xi_1, \xi_2 \rangle\rangle} \in \mathbb{R}.$$

Then the geodesic with endpoints ξ_1 and ξ_2 is given by

$$\xi(t) = \frac{1 + iw}{1 + e^t + iw}\xi_1 + \frac{e^t}{1 + e^t + iw}\xi_2$$

for $t \in \mathbb{R}$.

This geodesic is a Euclidean straight line segment if and only if $\langle\langle \xi_1, \xi_2 \rangle\rangle$ is real; that is, if ξ_1, ξ_2 lie in a totally real geodesic subspace containing the origin.

An alternative proof uses the direct construction of geodesics. Let $L \subset \mathbb{C}^2$ denote the (affine) complex line joining ξ_1, ξ_2. Parametrize L as

$$\alpha(\lambda) = \frac{1 - \lambda}{2}\xi_1 + \frac{1 + \lambda}{2}\xi_2$$

where $\lambda \in \mathbb{C}$ and $L \cap \mathbb{B}^n$ is the complex geodesic containing ξ_1 and ξ_2. (Note that $\alpha(-1) = \xi_1$ and $\alpha(1) = \xi_2$.) Parametrize this complex geodesic isometrically by the Poincaré disc

$$\Delta = \{s \in \mathbb{C} \mid |s| < 1\}.$$

The following lemma (whose proof is standard and hence omitted) is useful:

Lemma 3.3.4 *Let E be a complex vector space with positive definite Hermitian form $\langle\langle , \rangle\rangle$ and let $\xi_1, \xi_2 \in E$ be distinct points. Let L denote the affine line in E containing ξ_1, ξ_2. Then the point on L closest to the origin equals*

$$\frac{\langle\langle \xi_2, \xi_1 - \xi_2 \rangle\rangle}{\langle\langle \xi_1 - \xi_2, \xi_1 - \xi_2 \rangle\rangle}\xi_1 + \frac{\langle\langle \xi_1, \xi_2 - \xi_1 \rangle\rangle}{\langle\langle \xi_2 - \xi_1, \xi_2 - \xi_1 \rangle\rangle}\xi_2.$$

Since $\langle\langle \xi_1, \xi_1 \rangle\rangle = \langle\langle \xi_2, \xi_2 \rangle\rangle = 1$ and $\xi_1 \neq \xi_2$,

$$|\langle\langle \xi_1, \xi_2 \rangle\rangle| \leq 1, \qquad \mathrm{Re}\langle\langle \xi_1, \xi_2 \rangle\rangle < 1$$

and by the lemma above the point on L closest to the origin $0 \in \mathbb{B}^n$ is

$$\begin{aligned} \xi_0 &= \frac{1 - \langle\langle \xi_2, \xi_1 \rangle\rangle}{2(1 - \mathrm{Re}\langle\langle \xi_1, \xi_2 \rangle\rangle)}\xi_1 + \frac{1 - \langle\langle \xi_1, \xi_2 \rangle\rangle}{2(1 - \mathrm{Re}\langle\langle \xi_1, \xi_2 \rangle\rangle)}\xi_2 \\ &= \alpha(-iw) \end{aligned}$$

where w is as above.

The map

$$\begin{aligned} \Delta &\xrightarrow{\phi} \mathbb{B}^n \cap L \\ s &\mapsto \xi_0 + s(\xi_0 - \xi_2) \end{aligned}$$

is affine, conformal and maps Δ isometrically onto $\mathbb{B}^n \cap L$. Since ϕ maps

$$\begin{aligned} -1 &\mapsto \xi_2 \\ (1 - iw)/(1 + iw) &\mapsto \xi_1, \end{aligned}$$

it maps the Poincaré geodesic in Δ joining -1 and $(1 - iw)/(1 + iw)$ to the geodesic in $\mathbb{B}^n \cap L$ joining ξ_1 and ξ_2. Now the linear fractional transformation

$$\lambda \mapsto \frac{1 + \lambda}{1 - \lambda}$$

takes the left half-plane $\mathrm{Re}(\lambda) < 0$ to Δ mapping ∞ to -1; the image of the geodesic

$$\{-e^t - iw\}_{t \in \mathbb{R}}$$

from $-iw$ to ∞ in the left half-plane is the geodesic from $(1 - iw)/(1 + iw)$ to -1 in Δ. Thus

$$\phi\left(\frac{1 - e^t - iw}{1 + e^t + iw}\right) = \xi(t) = \frac{1 + iw}{1 + e^t + iw}\xi_1 + \frac{e^t}{1 + e^t + iw}\xi_2$$

is the desired geodesic.

3.3.6 *Orthogonal projection onto real hyperbolic space*

This section derives the formula for the orthogonal projection onto a maximal totally real geodesic subspace. Let $\mathbf{H}^n_{\mathbb{R}} \cong \mathbb{B}^n_{\mathbb{R}} \subset \mathbb{B}^n$ be the maximal totally real subspace consisting of points in $\mathbb{B}^n$ with real coordinates; then the orthogonal projection $\Pi_{\mathbb{R}}$ associates to a point of $\mathbb{B}^n$ the point of $\mathbb{B}^n_{\mathbb{R}}$ to which it is closest. Let $z \mapsto \bar{z}$ be complex conjugation; its fixed-point set equals $\mathbb{B}^n_{\mathbb{R}}$.

For any $z \in \mathbb{B}^n$, let

$$\begin{aligned}\alpha(z) &= \frac{1 - \operatorname{Re}\langle\langle z, \bar{z}\rangle\rangle}{\operatorname{Im}\langle\langle z, \bar{z}\rangle\rangle} \\ &= \frac{1 - (\!(\operatorname{Re}(z), \operatorname{Re}(z))\!) + (\!(\operatorname{Im}(z), \operatorname{Im}(z))\!)}{2(\!(\operatorname{Re}(z), \operatorname{Im}(z))\!)} \\ &\in \mathbb{R} \cup \{\infty\}\end{aligned} \tag{3.26}$$

and (compare Fig. 3.7)

$$\begin{aligned}\beta(z) &= -\frac{\alpha}{|\alpha|}\exp(\sinh^{-1}|\alpha|) \\ &= \begin{cases} \alpha(z) - \sqrt{\alpha(z)^2 + 1} & \text{if } \operatorname{Im}\langle\langle z, \bar{z}\rangle\rangle \neq 0 \text{ and } \alpha(z) > 0, \\ 0 & \text{if } \operatorname{Im}\langle\langle z, \bar{z}\rangle\rangle = 0, \text{ that is if } \alpha(z) = \infty \\ -\alpha(z) + \sqrt{\alpha(z)^2 + 1} & \text{if } \operatorname{Im}\langle\langle z, \bar{z}\rangle\rangle \neq 0 \text{ and } \alpha(z) < 0. \end{cases}\end{aligned} \tag{3.27}$$

Then orthogonal projection $\Pi_{\mathbb{R}} : \mathbb{B}^n \longrightarrow \mathbb{B}^n_{\mathbb{R}}$ is given by

$$\Pi_{\mathbb{R}}(z) = \operatorname{Re}(z) + \beta(z)\operatorname{Im}(z). \tag{3.28}$$

In particular, $\Pi_{\mathbb{R}}(z) = \operatorname{Re}(z)$ if and only if

$$(\!(\operatorname{Re}(z), \operatorname{Im}(z))\!) = 0.$$

For any $z \in \mathbb{B}^n$, the orthogonal projection $\Pi_{\mathbb{R}}(z)$ equals the midpoint of the geodesic segment joining z and $\bar{z}$. Let $Z = \mathbf{A}(z) \in \mathbb{C}^{n,1}$ be a vector representing $z \in \mathbb{B}^n$. Choose $\eta(z) \in \mathbb{C}$ such that

$$\eta(z)^2 = -\langle Z, \bar{Z}\rangle = 1 - \langle\langle z, \bar{z}\rangle\rangle.$$

By (3.2) the midpoint of z and $\bar{z}$ is represented by the vector in $\mathbb{C}^{n,1}$

$$\begin{aligned}M(z, \bar{z}) &= \overline{\eta(z)}Z + \eta(z)\bar{Z} \\ &= 2\operatorname{Re}(\eta(z))\operatorname{Re}(Z) + 2\operatorname{Im}(\eta(z))\operatorname{Im}(Z) \\ &= 2\operatorname{Re}(\eta(z)) \cdot \{\operatorname{Re}(Z) + \beta(z)\operatorname{Im}(Z)\}\end{aligned}$$

where

$$\beta(z) = \frac{\operatorname{Im}(\eta(z))}{\operatorname{Re}(\eta(z))} \in \mathbb{R} \cup \{\infty\}$$

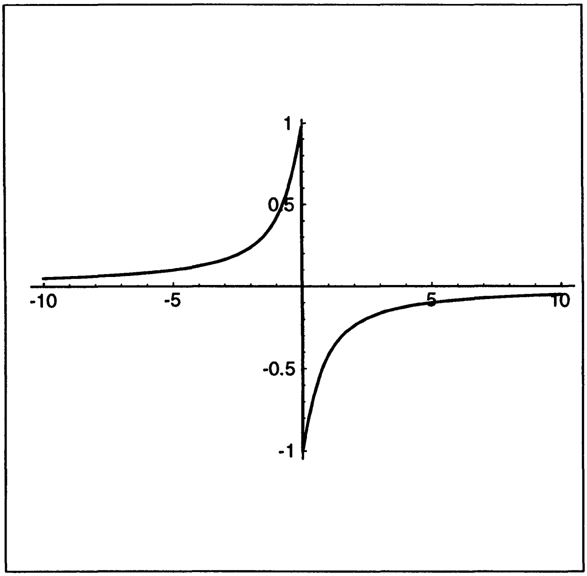

FIG. 3.7. Graph of $\beta = \pm(|\alpha| - \sqrt{\alpha^2+1})$

is independent of the choice of $\eta(z)$. Since the last coordinate of $\mathrm{Re}(Z)$ equals 1 and the last coordinate of $\mathrm{Im}(Z)$ equals 0, the vector $M(z,\bar{z})$ corresponds to the point

$$\mathrm{Re}(z) + \beta(z)\mathrm{Im}(z) \in \mathbb{B}^n_{\mathbb{R}}.$$

Since for all complex numbers η,

$$\frac{\mathrm{Re}(\eta^2)}{\mathrm{Im}(\eta^2)} = \frac{1}{2}\left\{\frac{\mathrm{Re}(\eta)}{\mathrm{Im}(\eta)} - \frac{\mathrm{Im}(\eta)}{\mathrm{Re}(\eta)}\right\},$$

$\beta(z)$ satisfies

$$\frac{\beta(z)^{-1} - \beta(z)}{2} = \frac{\mathrm{Re}\langle\langle z,\bar{z}\rangle\rangle - 1}{\mathrm{Im}\langle\langle z,\bar{z}\rangle\rangle}.$$

Therefore

$$\beta(z) = \alpha(z) \pm \sqrt{\alpha(z)^2+1} \tag{3.29}$$

where $\alpha(z)$ is given by (3.26). Since $\mathrm{Re}(\eta(z)^2) > 0$,

$$|\beta(z)|^2 = \left|\frac{\mathrm{Im}(\eta(z))}{\mathrm{Re}(\eta(z))}\right|^2 < 1,$$

the sign in (3.29) is the opposite of that of $\alpha(z)$, proving that $\beta(z)$ is indeed defined as in (3.27). The proof of (3.28) is now complete.

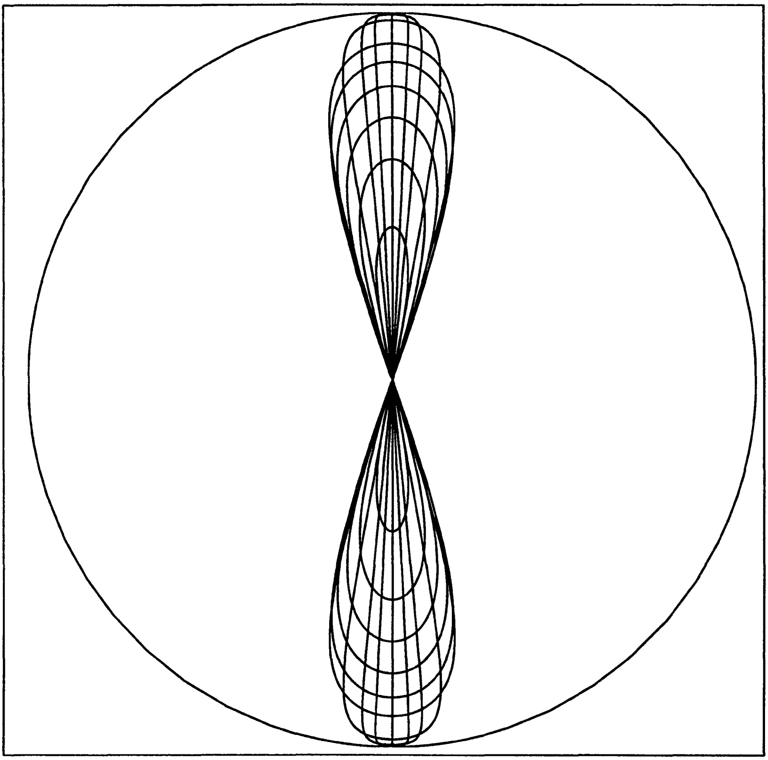

FIG. 3.8. Orthogonal projection of complex geodesics onto totally real subspace

Figure 3.8 depicts the orthogonal projection of the chains bounding the complex geodesics $z_1 = ir$ in $\mathbb{B}^2$, for $-1 < r < 1$, to the totally real geodesic subspace $\mathbf{H}^2_{\mathbb{R}}$.

4

THE PARABOLOID MODEL AND HEISENBERG GEOMETRY

4.1 The Siegel domain

The unit ball model is the view of $\mathbf{H}^n_{\mathbb{C}}$ from a point inside—the origin. Another model arises for $\mathbf{H}^n_{\mathbb{C}}$ when viewed from a point p_∞ on the absolute. This model generalizes the upper half-plane model of $\mathbf{H}^1_{\mathbb{C}}$. Complex hyperbolic space is now represented as an unbounded domain, a *Siegel domain* in $\mathbb{C}^n$. Just as the unit ball model is symmetric under the stabilizer $\mathbf{U}(n)$ of the origin, the Siegel domain model is invariant under the stabilizer of p_∞. The complement in $\mathbb{P}(\mathbb{C}^{n,1})$ of the hyperplane polar to the origin O is an affine space in which the stabilizer of O acts affinely. Similarly the complement of the hyperplane H_∞ polar to $\partial\mathbf{H}^n_{\mathbb{C}}$ at p_∞ is an affine space in which the stabilizer of p_∞ acts affinely. The polar hyperplane H_∞ is tangent to $\partial\mathbf{H}^n_{\mathbb{C}}$ at p_∞. In corresponding affine coordinates on the complement $\mathbb{P}(\mathbb{C}^{n,1}) - H(p_\infty)$, the stabilizer of p_∞ is represented by *affine transformations*. The unipotent transformations in this stabilizer form a subgroup isomorphic to the Heisenberg group $\mathcal{H}$. Since $\mathcal{H}$ acts simply transitively on $\partial\mathbf{H}^n_{\mathbb{C}} - \{p_\infty\}$, we may represent $\partial\mathbf{H}^n_{\mathbb{C}}$ as the 1-point compactification of $\mathcal{H}$. We may interpret the stabilizer of p_∞ as "Heisenberg similarity transformations"—analogous to the group of Euclidean similarity transformations acting on the boundary of real hyperbolic space. In this chapter we describe how complex hyperbolic geometry degenerates into Heisenberg geometry on the absolute.

4.1.1 *The Cayley transform*

Choose a point $q \in \partial\mathbf{H}^n_{\mathbb{C}}$; such a point corresponds to a null line spanned by a null vector $Q \subset \mathbb{C}^{n,1}$. A unique $\mathbb{C}$-hyperplane $H(q)$ is tangent to $\partial\mathbf{H}^n_{\mathbb{C}}$ at q; it corresponds to the linear hyperplane $Q^\perp \subset \mathbb{C}^{n,1}$. The affine patch on $\mathbb{P}(\mathbb{C}^{n,1})$ complementary to $H(q)$ contains $\mathbf{H}^n_{\mathbb{C}}$ as an *unbounded domain* and we denote this embedding $\mathbf{B} : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \mathbb{P}(\mathbb{C}^{n,1}) - H(q)$. Specifically let Q be the vector

$$\tilde{p}_\infty = \begin{bmatrix} 0' \\ -1 \\ 1 \end{bmatrix} \in \mathbb{C}^{n,1}$$

whence $H(p_\infty)$ consists of all points having homogeneous coordinates

$$\begin{bmatrix} z' \\ z_n \\ -z_n \end{bmatrix}. \tag{4.1}$$

The map

$$\mathbf{B} : \mathbb{C}^n \longrightarrow \mathbb{P}(\mathbb{C}^{n,1}) - H$$
$$\begin{pmatrix} w' \\ w_n \end{pmatrix} \longmapsto \begin{bmatrix} w' \\ 1/2 - w_n \\ 1/2 + w_n \end{bmatrix}$$

is the desired affine embedding; $\mathbf{H}^n_{\mathbb{C}}$ corresponds to the *Siegel domain* $\mathfrak{H}^n$ consisting of points $w \in \mathbb{C}^n$ satisfying

$$2\mathrm{Re}(w_n) - \langle\langle w', w' \rangle\rangle > 0.$$

The two sets of inhomogeneous (affine) coordinates in the Siegel domain and the ball, respectively, are related by the *Cayley transform*:

$$z \in \mathbb{B}^n \longleftrightarrow w \in \mathfrak{H}^n$$
$$z_j = \frac{2w_j}{1+2w_n} \qquad w_j = \frac{z_j}{1+z_n} \quad (1 \leq j < n)$$
$$z_n = \frac{1-2w_n}{1+2w_n} \qquad w_n = \left(\frac{1}{2}\frac{1-z_n}{1+z_n}\right).$$

Under the Cayley transform the boundary $\partial\mathbb{B}^n$ corresponds to the real hypersurface

$$\partial\mathfrak{H}^n = \{w \in \mathbb{C}^n \mid 2\mathrm{Re}(w_n) - \langle\langle w', w' \rangle\rangle = 0\}$$

together with the ideal point p_∞.

This ideal point corresponds to the point

$$\begin{pmatrix} 0' \\ -1 \end{pmatrix} \in \partial\mathbb{B}^n$$

and the origin $0 \in \partial\mathfrak{H}^n \cap \mathbb{C}^n$ corresponds to the point

$$\begin{pmatrix} 0' \\ 1 \end{pmatrix} \in \partial\mathbb{B}^n.$$

Observe that with our convention $\mathfrak{H}^1$ is the right half-plane, not the more customary upper half-plane in $\mathbb{C}$. The subspace $\{0\} \times \mathfrak{H}^1 \subset \mathfrak{H}^n$ is the complex geodesic corresponding to the complex geodesic $\{0\} \times \mathbb{B}^1 \subset \mathbb{B}^n$. Orthogonal projection in both cases is the linear projection on the last Cartesian factor.

4.1.2 *Busemann functions and horospheres*

We next compute the Bergman metric in the Siegel domain model by deriving a defining function (a Kähler potential) adapted to the ideal point q. To this end, consider a unit speed geodesic $x_t \in \mathbb{B}^n$ such that $\lim_{t\to\infty} x_t = q$. The corresponding *Busemann function* (see [6]) is defined as

$$h_Q(z) = \lim_{t\to\infty} (\rho(z, x_t) - t).$$

As in §3.1.8, the Kähler potential associated to a point $x \in \mathbb{B}^n$ is given by

$$\begin{aligned}\psi_{x_t}(z) &= 2\log\operatorname{sech}\left(\frac{\rho(z,x_t)}{2}\right)\\ &= \log\frac{\langle\tilde{x}_t,\tilde{x}_t\rangle\langle Z,Z\rangle}{\langle\tilde{x}_t,Z\rangle\langle Z,\tilde{x}_t\rangle}\end{aligned}$$

where $\tilde{x}_t, Z \in \mathbb{C}^{n,1}$ are negative vectors corresponding to x_t, z respectively.

Explicitly, the points in $\mathbb{B}^n$ corresponding to the vectors

$$\tilde{x}_t = \begin{bmatrix} 0' \\ -\tanh(t/2) \\ 1 \end{bmatrix}$$

move (as $t \longrightarrow +\infty$) at unit speed along a geodesic towards q corresponding to

$$Q = \begin{bmatrix} 0' \\ -1 \\ 1 \end{bmatrix}$$

and let $Z = \mathbf{A}(z)$ be as above, $\langle\!\langle z, z\rangle\!\rangle < 1$. Now, as $t \longrightarrow +\infty$,

$$\begin{aligned}\rho(z,x_t) &= 2\cosh^{-1}\frac{|z_n\sinh(t/2)+\cosh(t/2)|}{\sqrt{1-\langle\!\langle z,z\rangle\!\rangle}}\\ &\sim 2\cosh^{-1}\left(\frac{|z_n+1|}{2\sqrt{1-\langle\!\langle z,z\rangle\!\rangle}}e^{t/2}\right)\\ &\sim 2\log\left(\frac{|z_n+1|}{\sqrt{1-\langle\!\langle z,z\rangle\!\rangle}}e^{t/2}\right)\\ &= t+\log\frac{|z_n+1|^2}{1-\langle\!\langle z,z\rangle\!\rangle}\end{aligned}$$

whence the Busemann function equals

$$\begin{aligned}h_Q(z) &= \log\frac{|z_n+1|^2}{1-\langle\!\langle z,z\rangle\!\rangle}\\ &= -\log\Psi_Q(z) \qquad (4.2)\end{aligned}$$

where

$$\Psi_Q(z) = \exp(-h_Q(z)) = \frac{-\langle Z, Z\rangle}{\langle Z, Q\rangle\langle Q, Z\rangle}.$$

Since

$$h_Q(z) + \log(1 - \langle\!\langle z, z\rangle\!\rangle) = \log|1 + z_n|^2$$

is pluriharmonic, it follows that $-h_Q(z) = \log \Psi_Q(z)$ is a Kähler potential function. By (4.2), this Kähler potential depends only on the null vector $Q \in \mathbb{C}^{n,1}$ representing the ideal point q.

For any null vector $Q \in \mathbb{C}^{n,1}$, define the *Kähler potential associated to* Q :

$$\psi_Q(z) = \log \Psi_Q(z) = 2\mathrm{Re}(w_n) - \langle\!\langle w', w'\rangle\!\rangle.$$

Note that if $\lambda \in \mathbb{C}^*$,

$$\psi_{\lambda Q} = \psi_Q - \log|\lambda|^2$$

and thus an ideal point determines a Kähler potential *up to an additive constant.*

We write $w' = (w_1, \ldots, w_{n-1}) \in \mathbb{C}^{n-1}$ and denote the standard (positive) Hermitian form in $\mathbb{C}^{n-1}$ by $\langle\!\langle , \rangle\!\rangle$. In terms of the coordinates $(w', w_n) \in \mathbb{C}^n$ on $\mathfrak{H}^n$ the Kähler potential associated to O equals

$$\begin{aligned}\log(1 - \langle\!\langle z, z\rangle\!\rangle) &= \log|1 + z_n|^2 - h_Q(z) \\ &= \log\frac{4}{|1 + 2w_n|^2} + \log \Psi_Q(z)\end{aligned}$$

where

$$\begin{aligned}\Psi_Q(w) &= \exp(-h_Q(z)) \\ &= 2\mathrm{Re}(w_n) - \langle\!\langle w', w'\rangle\!\rangle \end{aligned} \tag{4.3}$$

is the defining function for the Siegel domain $\mathfrak{H}^n \subset \mathbb{C}^n$. The Kähler form on $\mathfrak{H}$ is

$$\begin{aligned}-4i\partial\bar\partial\log(1 - \langle\!\langle z, z\rangle\!\rangle) &= -4i\partial\bar\partial\log\Psi_Q(z) \\ &= -4i\partial\bar\partial\log(w_n + \bar{w}_n - \langle\!\langle w', w'\rangle\!\rangle) \\ &= \frac{4i}{f(w)^2}\{\,(dw_n - \langle\!\langle dw', w'\rangle\!\rangle) \wedge (d\bar{w}_n - \langle\!\langle w', dw'\rangle\!\rangle) \\ &\quad + f(w)\langle\!\langle dw', dw'\rangle\!\rangle\,\}\end{aligned}$$

(as

$$\partial\bar\partial\log\frac{4}{|1 + 2w_n|^2} = 0$$

since $2/(1 + 2w_n)$ is holomorphic). The metric tensor is

$$\mathrm{g}(w) = \frac{4}{f(w)^2}\{(dw_n - \langle\!\langle dw', w'\rangle\!\rangle)(d\bar{w}_n - \langle\!\langle w', dw'\rangle\!\rangle) + f(w)\langle\!\langle dw', dw'\rangle\!\rangle\}. \tag{4.4}$$

4.1.3 *Root-space decomposition of* $\mathfrak{su}(n,1)$

(Compare §1A of [104].) To understand the structure of the group $\mathbf{PU}(n,1)$ of automorphisms of $\mathbf{H}^n_{\mathbb{C}}$, we consider a hyperbolic 1-parameter subgroup and its generator, an element of $\mathfrak{su}(n,1)$. We shall write $(n+1)$-square matrices in $((n-1)+1+1) \times ((n-1)+1+1)$ block form; that is,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & a_{22} & a_{23} \\ A_{31} & a_{32} & a_{33} \end{bmatrix}$$

where A_{11} is an $(n-1)$-square matrix, $A_{1j} \in \mathbb{C}^{n-1}$ are column vectors, $A_{i1} \in (\mathbb{C}^{n-1})^*$ are row vectors and $a_{ij} \in \mathbb{C}$ are scalars for $i, j > 1$. If A is a matrix, we denote by A^* its conjugate transpose. Thus

$$\mathbf{U}(n,1) = \{A \in \mathbf{GL}(n+1,\mathbb{C}) \mid A^*\mathbb{I}_{n,1}A = \mathbb{I}_{n,1}\}$$

and

$$\mathfrak{su}(n,1) = \{A \in \mathfrak{sl}(n+1,\mathbb{C}) \mid A^*\mathbb{I}_{n,1} + \mathbb{I}_{n,1}A = 0\}$$

where $\mathbb{I}_{n,1}$ denotes the $(n+1) \times (n+1)$ matrix $\mathbb{I}_n \oplus (-1)$.

Now the 1-parameter subgroup consisting of

$$h_t = e^{t\eta} = \begin{bmatrix} \mathbb{I}_{n-1} & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{bmatrix}$$

$(t \in \mathbb{R})$ with infinitesimal generator

$$\eta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathfrak{su}(n,1)$$

acts on $\mathbb{B}^n$, fixing the pair of points $(0, \pm 1) \in \partial\mathbb{B}^n$ and translating along the geodesic (at constant speed 2, see §1.4)

$$\{(0, \tanh(s)) \mid s \in \mathbb{R}\} = \{0\} \times (-1,1) \subset \mathbb{B}^n$$

by the linear fractional transformation

$$\begin{bmatrix} z' \\ z_n \end{bmatrix} \mapsto \begin{bmatrix} (\sinh(t)z_n + \cosh(t))^{-1} z' \\ (\sinh(t)z_n + \cosh(t))^{-1}(\cosh(t)z_n + \sinh(t)) \end{bmatrix}$$

where

$$z' = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \in \mathbb{C}^{n-1}.$$

(Compare [125], pp.64–65.)

An isometry of $\mathbf{H}_{\mathbb{C}}^n$ leaving invariant a geodesic γ, acts by translation along γ and whose differential acts on normal vectors by parallel translation will be called *(strictly) hyperbolic.* In the paraboloid model h_t fixes both the origin 0 and p_∞, and translates along the line

$$\{0'\} \times (0, \infty) \subset \mathfrak{H}^n$$

by the affine transformation (an inhomogeneous dilation)

$$\begin{bmatrix} w' \\ w_n \end{bmatrix} \mapsto \begin{bmatrix} e^{-t}w' \\ e^{-2t}w_n \end{bmatrix}.$$

The centralizer of this flow is a direct product

$$G_0 = \{h_t\}_{t\in\mathbb{R}} \times \mathbf{U}(n-1)$$

of the flow itself with the unitary group $\mathbf{U}(n-1)$. An element $A \in \mathbf{U}(n-1)$ acts on $\mathfrak{H}^n$ by

$$\begin{bmatrix} w' \\ w_n \end{bmatrix} \mapsto \begin{bmatrix} Aw' \\ w_n \end{bmatrix}.$$

Decompose the Lie algebra $\mathfrak{g} = \mathfrak{su}(n,1)$ into the eigenspaces

$$\mathfrak{g}_k = \operatorname{Ker}(\operatorname{ad}\eta - k\mathbb{I})$$

of $\operatorname{ad}\eta$. (The eigenspace $\mathfrak{g}_k$ is nonzero only for $k = 0, \pm1, \pm2$.)

$$\mathfrak{g}_0 = \left\{ \left. \begin{bmatrix} A & 0 & 0 \\ 0 & -\frac{1}{2}\operatorname{trace}(A) & t \\ 0 & t & -\frac{1}{2}\operatorname{trace}(A) \end{bmatrix} \right| A \in \mathfrak{u}(n-1), t \in \mathbb{R} \right\}$$

$$\mathfrak{g}_{-1} = \left\{ \left. \begin{bmatrix} 0_{n-1} & \zeta & \zeta \\ -\zeta^* & 0 & 0 \\ \zeta^* & 0 & 0 \end{bmatrix} \right| \zeta \in \mathbb{C}^{n-1} \right\}$$

$$\mathfrak{g}_{-2} = \left\{ \left. \begin{bmatrix} 0_{n-1} & 0 & 0 \\ 0 & iv/2 & iv/2 \\ 0 & -iv/2 & -iv/2 \end{bmatrix} \right| v \in \mathbb{R} \right\}$$

$$\mathfrak{g}_{1} = \left\{ \left. \begin{bmatrix} 0_{n-1} & \zeta & -\zeta \\ -\zeta^* & 0 & 0 \\ -\zeta^* & 0 & 0 \end{bmatrix} \right| \zeta \in \mathbb{C}^{n-1} \right\}$$

$$\mathfrak{g}_2 = \left\{ \begin{bmatrix} 0_{n-1} & 0 & 0 \\ 0 & iv/2 & -iv/2 \\ 0 & iv/2 & -iv/2 \end{bmatrix} \middle| v \in \mathbb{R} \right\}.$$

As usual $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$. The Lie algebra of the stabilizer G_- of

$$\tilde{p_\infty} = \begin{bmatrix} 0' \\ -1 \\ 1 \end{bmatrix} \in \partial \mathbf{H}^n_{\mathbb{C}}$$

equals

$$\mathfrak{g}_- = \bigoplus_{j=0}^{2} \mathfrak{g}_{-j},$$

the semidirect product of $\mathfrak{g}_0$ with the subalgebra

$$\mathfrak{n}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}.$$

$\mathfrak{n}_-$ is the Lie algebra of the Heisenberg group associated to the symplectic vector space $(\mathbb{C}^{n-1}, \omega)$ where the symplectic product of $\zeta_1, \zeta_2 \in \mathbb{C}^{n-1}$ is given by

$$\omega(\zeta_1, \zeta_2) = 4 \operatorname{Im}\langle\langle \zeta_1, \zeta_2 \rangle\rangle.$$

(Compare §2.6.1.) This follows readily from the identity

$$\left[\begin{bmatrix} 0 & \zeta_1 & \zeta_1 \\ -\zeta_1^* & 0 & 0 \\ -\zeta_1^* & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \zeta_2 & \zeta_2 \\ -\zeta_2^* & 0 & 0 \\ -\zeta_2^* & 0 & 0 \end{bmatrix} \right] = 4 \operatorname{Im}\langle\langle \zeta_1, \zeta_2 \rangle\rangle \begin{bmatrix} 0 & 0 & 0 \\ 0 & i/2 & i/2 \\ 0 & -i/2 & -i/2 \end{bmatrix}.$$

This is the Heisenberg group with which we shall be concerned.

Similarly the Lie algebra of the stabilizer G_+ of the origin

$$\tilde{p}_0 = \begin{bmatrix} 0' \\ 1 \\ 1 \end{bmatrix} \in \partial \mathbf{H}^n_{\mathbb{C}}$$

equals

$$\mathfrak{g}_+ = \bigoplus_{j=0}^{2} \mathfrak{g}_j,$$

the semidirect product of $\mathfrak{g}_0$ with the Heisenberg Lie algebra

$$\mathfrak{n}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

4.1.4 *Interpretation by outer products*

The outer products of §2.2.1 give a convenient way of directly describing these transformations. Let $P_0, P_\infty \in \mathbb{C}^{n,1}$ be null vectors and $P^\perp \subset \mathbb{C}^{n,1}$ the positive hyperplane consisting of all $\zeta \in \mathbb{C}^{n,1}$ such that $\langle \zeta, P_0 \rangle = \langle \zeta, P_\infty \rangle = 0$.

Exercise 4.1.1 *Show that*

$$\eta = -\frac{1}{2}(P_0 P_\infty^* - P_\infty P_0^*)$$

is the infinitesimal generator of the hyperbolic 1-parameter subgroup $\{h_t\}_{t\in\mathbb{R}}$ *above taking*

$$\begin{aligned} \eta : P_0 &\mapsto P_0, \\ P_\infty &\mapsto -P_\infty, \\ P^\perp &\mapsto 0. \end{aligned}$$

Furthermore the root spaces can be expressed in terms of outer products by

$$\begin{aligned} \mathfrak{g}_{-1} &= \{\zeta P_\infty^* - P_\infty \zeta^* \mid \zeta \in P^\perp\} \\ \mathfrak{g}_1 &= \{\zeta P_0^* - P_0 \zeta^* \mid \zeta \in P^\perp\} \\ \mathfrak{g}_{-2} &= i P_\infty P_\infty^* \mathbb{R} \\ \mathfrak{g}_2 &= i P_0 P_0^* \mathbb{R}. \end{aligned}$$

4.2 Heisenberg geometry

Using the root-space decomposition of the isometry group, we find a group of unipotent isometries which acts simply transitively on the complement of a fixed point. This gives coordinates on the absolute, making it a *Heisenberg space* in the sense of §2.6.

4.2.1 *The Heisenberg group fixing a point on the absolute*

Let $\mathfrak{N} = \mathfrak{N}_-$ denote the group corresponding to the Heisenberg Lie algebra fixing p_∞. Its exponential map is

$$\begin{aligned} \exp : \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} &\longrightarrow \mathfrak{N} \\ \begin{bmatrix} 0_{n-1} & \zeta & \zeta \\ -\zeta^* & iv/2 & iv/2 \\ \zeta^* & -iv/2 & -iv/2 \end{bmatrix} &\mapsto H(\zeta, v) \\ &= \begin{bmatrix} \mathbb{I}_{n-1} & \zeta & \zeta \\ -\zeta^* & 1 - \frac{1}{2}(\|\zeta\|^2 - iv) & -\frac{1}{2}(\|\zeta\|^2 - iv) \\ \zeta^* & \frac{1}{2}(\|\zeta\|^2 - iv) & 1 + \frac{1}{2}(\|\zeta\|^2 - iv) \end{bmatrix} \end{aligned}$$

and $\mathfrak{N}$ acts simply transitively on $\partial \mathbf{H}^n_{\mathbb{C}} - \{p_\infty\} \subset \mathbb{P}(\mathbb{C}^{n,1})$ by the evaluation map:

$$\begin{aligned} \mathfrak{N} &\longrightarrow \partial \mathbf{H}^n_{\mathbb{C}} - \{p_\infty\} \\ H(\zeta, v) &\longmapsto H(\zeta, v) \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\zeta \\ 1 - \|\zeta\|^2 + iv \\ 1 + \|\zeta\|^2 - iv \end{bmatrix}. \end{aligned} \tag{4.5}$$

Thus *the complement of a point in the boundary of complex hyperbolic space has the natural structure of a Heisenberg space,* as defined in §2.6.

In the Siegel domain model, the stabilizer

$$G_- = \mathfrak{N} \rtimes \mathbf{U}(n-1) \cdot \{h_t\}_{t \in \mathbb{R}}$$

of p_∞ acts by affine transformations $\mathbb{C}^n \longrightarrow \mathbb{C}^n$:

$$H(\zeta, v) A h_t : \begin{pmatrix} w' \\ w_n \end{pmatrix} \mapsto \begin{pmatrix} A e^{-t} w' + \zeta \\ e^{-2t} w_n + \langle\!\langle w', \zeta \rangle\!\rangle + \frac{1}{2}(\|\zeta\|^2 - iv) \end{pmatrix}$$

where $w' \in \mathbb{C}^{n-1}$. Each horosphere also has the structure of a Heisenberg space. Following Goldman–Parker [73], we introduce *horospherical coordinates*

$$(z, u, v) \in \mathbb{C}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}$$

on the Siegel domain $\mathfrak{H}^n$ where

$$\begin{aligned} z &= w' \in \mathbb{C}^{n-1} \\ u + iv &= 2\bar{w}_n - \langle\!\langle w', w' \rangle\!\rangle. \end{aligned}$$

Then the horospherical height function satisfies $f(w) = u$. The transformation $H(\zeta, \nu)$ maps the point with horospherical coordinates (z, u, v) to the point with horospherical coordinates

$$(z + \zeta, u, v + 2\mathrm{Im}\langle\!\langle \zeta, z \rangle\!\rangle + \nu).$$

For consistency we have considered the Hermitian form defined by the diagonal matrix $\mathbb{I}_{n+1} \oplus -1$. However, for many purposes it is more convenient to work with the Hermitian form defined by the *nondiagonal* matrix

$$F = \begin{bmatrix} 0 & 0 & i/2 \\ 0 & \mathbb{I}_n & 0 \\ -i/2 & 0 & 0 \end{bmatrix}.$$

(Exercise 1.4.3 describes the 1-dimensional version.) This Hermitian structure is used by Burns–Shnider [19], p.226 and [20] which is related to Hermitian structures used by Epstein [48] (the "second Hermitian form") and Parker [137], among others, since this provides several notational advantages:

Exercise 4.2.1 *Show that the unitary group of the Hermitian form F satisfies the following:*

1. *The first and last coordinate vectors are null.*
2. *The Cartan subgroup $\{h_t\}_{t\in\mathbb{R}}$ is represented by diagonal matrices.*
3. *The root spaces $\mathfrak{g}_1, \mathfrak{g}_2$ and $\mathfrak{g}_{-1}, \mathfrak{g}_{-2}$ consist of upper-triangular and lower-triangular matrices respectively.*

Relate this matrix algebra to the matrix algebras described above by the matrix representing the Cayley transform.

Exercise 4.2.2 *Show that the Hermitian form $\langle , \rangle$ on $\mathbb{C}^{n,1}$ induces the following operation on $\mathfrak{N}$:*

$$\begin{aligned}\langle H(\zeta_1, v_1), H(\zeta, v_2)\rangle &= 4\langle\zeta_1, \zeta_2\rangle - 2(\|\zeta_1\|^2 + \|\zeta_2\|^2) \\ &\quad + 2i(v_1 - v_2) \\ &= -2\|\zeta_1 - \zeta_2\|^2 \\ &\quad + 2i(2\mathrm{Im}\langle\zeta_1, \zeta_2\rangle + (v_1 - v_2)).\end{aligned}$$

4.2.2 *Heisenberg geometry*

We have seen that the boundary of $\mathbf{H}^n_{\mathbb{C}}$ is the 1-point compactification of a Heisenberg space associated to the (positive definite) Hermitian vector space $\mathbb{C}^{n-1}$ in the sense of §2.6.1. This gives convenient coordinates for $\partial\mathbf{H}^n_{\mathbb{C}}$. Now we shall discuss the geometry of $\partial\mathbf{H}^n_{\mathbb{C}}$ from this point of view. As the Siegel model is an affine patch whose ideal hyperplane is tangent to $\partial\mathbf{H}^n_{\mathbb{C}}$ at p_∞, the stabilizer of p_∞ will be represented by *affine* transformations in this model. This section discusses in detail the representation of these transformations in the affine geometry of Heisenberg space.

The Heisenberg group is a nilpotent real Lie group of dimension $2n-1$, which is a nontrivial central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{N} \longrightarrow \mathbb{C}^{n-1} \longrightarrow 0.$$

The explicit central extension is determined by (twice) the symplectic pairing arising as the imaginary part of the Hermitian structure on $\mathbb{C}^{n-1}$:

$$\begin{aligned}\mathbb{C}^{n-1} \times \mathbb{C}^{n-1} &\longrightarrow \mathbb{R} \\ (\zeta_1, \zeta_2) &\longmapsto 2\mathrm{Im}\langle\!\langle\zeta_1, \zeta_2\rangle\!\rangle.\end{aligned}$$

This means that we may represent the elements of $\mathfrak{N}$ (and hence the points of $\mathcal{H}$) by pairs (ζ, v) where $\zeta \in \mathbb{C}^{n-1}$ and $v \in \mathbb{R}$ with multiplication

$$(\zeta_1, v_1) \cdot (\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\mathrm{Im}\langle\!\langle\zeta_1, \zeta_2\rangle\!\rangle).$$

Therefore left-translation by $(z, u) \in \mathfrak{N}$ is given by

$$(\zeta, v) \longmapsto (\zeta + z, v + u + 2\mathrm{Im}\langle\!\langle z, \zeta\rangle\!\rangle);$$

we refer to such transformations $\mathcal{H} \longrightarrow \mathcal{H}$ as *Heisenberg translations*.

The center of $\mathfrak{N}$ consists of the *vertical translations*

$$(\zeta, v) \mapsto (\zeta, v + u)$$

where $u \in \mathbb{R}$. We henceforth identify $\mathfrak{N}$ concretely with the group of Heisenberg translations of $\mathcal{H}$. The vertical translations form a 1-parameter group of automorphisms of $\mathcal{H}$ (corresponding to $\mathfrak{g}_{-2}$ in the root-space decomposition) and the vector field generating this flow will be called the *vertical vector field*

$$\nu = \frac{\partial}{\partial v}.$$

The subset $\mathbb{V}$ of $\mathcal{H}$ defined by $\zeta = 0$ will be called the *vertical axis* and corresponds to the center of $\mathfrak{N}$. The unitary group $\mathbf{U}(n-1)$ acts on $\mathcal{H}$ by

$$A : (\zeta, v) \mapsto (A\zeta, v)$$

for $A \in \mathbf{U}(n-1), \zeta \in \mathbb{C}^{n-1}, v \in \mathbb{R}$. We refer to these transformations as *Heisenberg rotations about the vertical axis.* In general a *Heisenberg rotation* is any transformation conjugate by a Heisenberg translation to a Heisenberg rotation about the vertical axis.

Exercise 4.2.3 *Show that any transformation conjugate in* $\mathbf{PU}(n,1)$ *to a Heisenberg rotation about the vertical axis is a Heisenberg rotation.*

The group generated by $\mathfrak{N}$ and $\mathbf{U}(n-1)$ acts isometrically with respect to several left-invariant metrics. We refer to this group as the group of *Heisenberg isometries* and denote it by $\mathbf{Isom}(\mathcal{H})$; it has the structure of a semidirect product $\mathfrak{N} \rtimes \mathbf{U}(n-1)$. It may be characterized as the stabilizer of some (and in fact any) horosphere centered at p_∞; equivalently $\mathbf{Isom}(\mathcal{H})$ is the stabilizer of the lift of p_∞ to a null vector in $\mathbb{C}^{n,1}$. For $n = 2$, the geometry of $\mathcal{H}$ invariant under $\mathbf{Isom}(\mathcal{H})$) is "nil-geometry" in the sense of Thurston [161], [159] (see also Scott [155]), one of the eight geometries uniformizing 3-manifolds (§3.8, [161]).

The group of nonzero complex numbers $\mathbb{C}^*$ acts by *Heisenberg (complex) dilations about the origin*:

$$\lambda : (\zeta, v) \mapsto (\lambda\zeta, |\lambda|^2 v).$$

A general *Heisenberg (complex) dilation* will be any transformation conjugate by a Heisenberg translation to a Heisenberg dilation about the origin. Clearly the Heisenberg dilations about the origin and the Heisenberg rotations about the vertical axis intersect in the group of scalar multiplications by unit complex numbers $\mathbb{T} = \mathbb{C}^* \cap \mathbf{U}(n-1)$. They generate the group

$$\mathbf{Sim}_0(\mathcal{H}) \cong \mathbb{R}_+ \times \mathbf{U}(n-1)$$

where $\mathbb{R}_+$ is the group of *real Heisenberg dilations about the origin* where $\lambda \in \mathbb{R}_+$. It corresponds to the hyperbolic 1-parameter subgroup $\{h_t\}_{t \in \mathbb{R}}$ discussed in §4.1.3.

The group $\mathbf{Sim}_0(\mathcal{H})$ is analogous to the group $\mathbf{Sim}_0(\mathbb{R}^n) = \mathbb{R}_+ \times \mathbf{O}(n)$ of linear similarity transformations of $\mathbb{R}^n$ (the stabilizer of the origin in $\mathbf{Sim}(\mathbb{R}^n)$). The group

$$\mathbf{Sim}(\mathcal{H}) = (\mathbb{R}_+ \times \mathbf{U}(n-1)) \ltimes \mathfrak{N}$$

generated by Heisenberg dilations about the origin, Heisenberg rotations about the vertical axis, and Heisenberg translations is analogous to the full group of similarity transformations of $\mathbb{R}^n$ and hence we call it the *Heisenberg similarity group*. In the root-space decomposition it corresponds to $\exp(\mathfrak{g}_-)$.

The *scale factor homomorphism*

$$\lambda : \mathbf{Sim}(\mathcal{H}) \longrightarrow \mathbb{R}_+$$

has kernel the group $\mathbf{Isom}(\mathcal{H})$ of Heisenberg isomëtries. The (usual Euclidean) parallel volume form $d\mathrm{vol}$ on $\mathcal{H}$ is $\mathbf{Isom}(\mathcal{H})$-invariant and if $g \in \mathbf{Sim}(\mathcal{H})$, then

$$g^* d\mathrm{vol} = \det(g)^{-1} d\mathrm{vol}$$

where

$$\det(g) = \lambda(g)^n.$$

If $g \in \mathbf{Sim}(\mathcal{H})$, then either $\lambda(g) = 1$ (and $g \in \mathbf{Isom}(\mathcal{H})$) or g is conjugate to the product of the Heisenberg dilation $(\zeta, v) \mapsto (\lambda\zeta, \lambda^2 v)$ with a Heisenberg rotation about the vertical axis.

Heisenberg geometry "fibers" over the "unitary-similarity" geometry of complex Euclidean space $\mathbb{C}^{n-1}$ in the following sense. Let $\mathbf{Sim}(\mathbb{C}^{n-1})$ denote the group of complex affine transformations of $\mathbb{C}^{n-1}$ generated by homotheties $\mathbb{C}^*$, unitary transformations $\mathbf{U}(n-1)$ and translations $\mathbb{C}^{n-1}$; that is, the semidirect product

$$\mathbb{C}^{n-1} \rtimes (\mathbb{C}^* \times \mathbf{U}(n-1)).$$

The kernel of the natural homomorphism $\Pi : \mathbf{Sim}(\mathcal{H}) \longrightarrow \mathbf{Sim}(\mathbb{C}^{n-1})$ consists of vertical translations. Furthermore vertical projection $\Pi_{\mathrm{V}} : \mathcal{H} \longrightarrow \mathbb{C}^{n-1}$ is equivariant respecting Π, in the sense that for every $g \in \mathbf{Sim}(\mathcal{H})$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\Pi_{\mathrm{V}}} & \mathbb{C}^{n-1} \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \Pi(g)} \\
\mathcal{H} & \xrightarrow[\Pi_{\mathrm{V}}]{} & \mathbb{C}^{n-1}
\end{array}$$

4.2.3 *Vertical projection and complex perspective*

In terms of projective geometry, vertical projection corresponds to *complex perspective* (compare [73]). Identify points of $\mathcal{H}$ with real geodesics in $\mathbf{H}^n_{\mathbb{C}}$ emanating from q_∞; vertical chains correspond to complex geodesics from q_∞. The (parabolic) pencil of complex geodesics incident to q_∞ has the natural structure of a complex affine space modeled on $\mathbb{C}^{n-1}$ and the inclusion of real geodesics in complex geodesics corresponds to vertical projection. The perspective mapping is given in the Siegel domain model by

$$\mathfrak{H}^n \longrightarrow \mathbb{C}^{n-1}$$
$$\begin{pmatrix} w' \\ w_n \end{pmatrix} \mapsto w'$$

which evidently maps $\mathbb{C}$-affine subspaces in $\mathfrak{H}^n$ to $\mathbb{C}$-affine subspaces in $\mathbb{C}^{n-1}$.

4.2.4 *The CR-structure of Heisenberg space*

The natural geometric structure of $\partial\mathbf{H}^n_{\mathbb{C}}$ is its CR-structure, which admits a simple expression in Heisenberg coordinates on $\mathcal{H} = \partial\mathbf{H}^n_{\mathbb{C}} - \{p_\infty\}$. In particular it is calibrated by a 1-form which is *affine;* that is, its coefficients are polynomials of degree 1.

The derivation of this contact 1-form involves the Kähler potential associated to p_∞ (related to the Busemann function). By §4.1.2, the Busemann function h_Q and the horospherical height function f are related by (4.3) and either of these functions defines the foliation of $\mathfrak{H}^n$ by horospheres. f is evidently invariant under $\mathbf{U}(n-1) \subset G_-$ and transforms under the dilations $\{h_t\}_{t\in\mathbb{R}}$ by

$$f \circ h_t = e^{-2t} f.$$

As in §2.4.1, the 1-form (a primitive of the Kähler form)

$$d^c \log f = \frac{2}{f}\mathrm{Im}\Big(dw_n - \sum_{j=1}^{n-1} \bar{w}_j dw_j\Big)$$

defines a family of calibrated CR-structures on the horospheres which is invariant under the stabilizer G_- of $p_\infty \in \partial\mathbf{H}^n_{\mathbb{C}}$.

Let ϕ be a positive real number; since $\mathfrak{N}$ acts simply transitively on each horosphere $f^{-1}(\phi)$, the Bergman metric on $\mathbf{H}^n_{\mathbb{C}}$ pulls back under the (left-)action

$$\mathfrak{N} \longrightarrow f^{-1}(\phi) \subset \mathbf{H}^n_{\mathbb{C}}$$

to a left-invariant Riemannian metric g_ϕ on $\mathfrak{N}$. Explicitly

$$\mathrm{g}_\phi = \frac{1}{\phi}(dv - 2\mathrm{Im}\langle\langle \zeta, d\zeta\rangle\rangle)^2 + 4d\zeta^* d\zeta. \tag{4.6}$$

Now

$$\langle\!\langle d\zeta, d\zeta \rangle\!\rangle = \sum_{i=1}^{n-1} d\zeta_i d\bar{\zeta}_i = \sum_{i=1}^{n-1} dx_i^2 + dy_i^2$$

(where $\zeta_i = x_i + y_i$) and

$$\omega = dv - 2\mathrm{Im}(\langle\!\langle \zeta, d\zeta \rangle\!\rangle) = dv + 2\sum_{i=1}^{n-1} (x_i dy_i - y_i dx_i) \tag{4.7}$$

is the contact 1-form calibrating the CR-structure on $\mathfrak{N}$. Now

$$\omega, dx_1, dy_2, \ldots, dx_{n-1}, dy_{n-1}$$

form a basis for the left-invariant 1-forms on $\mathfrak{N}$ and the left-invariant Riemannian metric is given by

$$g_\phi = \left\{ \frac{\omega^2}{\phi} + 4\sum_{i=1}^{n-1}(dx_i^2 + dy_i^2) \right\}.$$

Observe that (4.6) and (4.7) give expressions for the contact 1-form and the restriction of the Bergman metric to horospheres which are polynomials in Heisenberg coordinates.

As $\phi \longrightarrow 0$, the pullbacks of g_ϕ to $\mathfrak{N}$ do not converge. However, as observed in Korányi [102], §4, the dual tensor fields (the symmetric contravariant 2-tensor fields)

$$g_\phi^* = \phi\left(\frac{\partial}{\partial v}\right)^2 + \frac{1}{4}\sum_{j=1}^{n-1}\left(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial v}\right)^2 + \left(\frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial v}\right)^2$$

do converge to a contravariant symmetric 2-tensor field g_0^* as $\phi \to 0$. This tensor field is degenerate (as a bilinear form on $T^*\mathcal{H}$) although its restriction to $\mathrm{Ann}(\nu) \cong E^*$ is nondegenerate.

Using this infinitesimal expression, length can be defined for a smooth curve—a curve which is not Legendrian is defined to have infinite length. The resulting *Carnot–Carathéodory metric*

$$m : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$$

(compare [80], [121], [134]) is a natural metric on $\mathcal{H}$ which uniformly scales under dilation: if $\delta : \mathcal{H} \longrightarrow \mathcal{H}$ is dilation with scale factor $\lambda > 0$ and $x, y \in \mathcal{H}$, then

$$m(\delta x, \delta y) = \lambda m(x, y).$$

To calibrate the CR-structure on $\partial\mathbf{H}^n_{\mathbb{C}}$, replace the function $\log f$ (which diverges as $\phi \longrightarrow 0$) with f. Since

$$\begin{aligned} d^c f &= f d^c \log f \\ &= 2\mathrm{Im}\big(dw_n - \sum_{j=1}^{n-1} \bar{w}_j dw_j\big) \\ &= -dv - 2\mathrm{Im} \sum_{j=1}^{n-1} \bar{\zeta}_j d\zeta_j \end{aligned}$$

the CR-structures on the horospheres $f^{-1}(\phi)$ converge to the CR-structure on $\partial\mathbf{H}^n_{\mathbb{C}}$ which is calibrated by $d^c f$. In Heisenberg coordinates this calibration is given by (4.3) above and the contact vector field (see §2.5.3) corresponding to this calibration is the vertical vector field ν on $\mathcal{H}$.

4.3 Chains

Chains.
My baby's got me wrapped up in chains
And they ain't the kind you can see.
These chains of love got a hold on me.

Chains.
I can't break away from these chains.
Can't run around 'cause I'm not free.
These chains of love won't let me be.

Gerry Goffin and Carole King, 1962

4.3.1 *Chains in Heisenberg geometry*

Now we discuss the representation of chains (boundaries of complex geodesics) in Heisenberg space and more generally the boundaries of complex linear subspaces. Chains passing through ∞ are represented as the fibers of the vertical projection $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}^{n-1}$, vertical straight lines defined by $\zeta = \zeta_0$. We call such chains *vertical.* A chain not containing ∞ is called *finite.* A finite chain is represented by an ellipse whose vertical projection is a (Euclidean) circle in $\mathbb{C}^{n-1}$. (See §8 of Jacobowitz [92] for an alternative treatment of this material.)

If C is a complex-linear subspace then there is a unique inversion whose fixed-point set is C, which we denote by ι_C. Conversely the fixed-point set of an inversion is a complex linear subspace. For example, inversion in the vertical axis $\zeta = 0$ is the usual Euclidean reflection $(\zeta, v) \mapsto (-\zeta, v)$ in $\mathbb{V}$ on Heisenberg coordinates, represented by the diagonal matrix

$$\tilde{\iota}_{\mathbb{V}} = \begin{bmatrix} -\mathbb{I}_{n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{SU}(n,1). \tag{4.8}$$

Suppose now that a, b bound complex-linear subspaces A, B such that

$$\dim_{\mathbb{C}}(A) + \dim_{\mathbb{C}}(B) = n.$$

Two such subspaces (or their bounding chains) are said to be *orthogonal* if they satisfy the conclusions of the following elementary theorem:

Lemma 4.3.1 *The following conditions are equivalent:*

1. ι_a *and* ι_b *commute;*
2. $\iota_a(b) = b$*;*
3. $\iota_b(a) = a$*;*
4. $(\iota_a \iota_b)^2 = I$*;*
5. *the complex subspaces* A, B *intersect orthogonally in* $\mathbf{H}^n_{\mathbb{C}}$*;*
6. A *is a fiber of the orthogonal projection onto* B*;*
7. B *is a fiber of the orthogonal projection onto* A*.*

Proof Since two involutions commute if and only if their product is an involution, 1 is equivalent to 4. Since the centralizer of a tranformation preserves its fixed-point set, and the fixed-point set ι_a precisely equals a, 2 implies 1. Similarly 3 implies 1. Conversely, assuming 2,

$$\iota_a(\iota_b)\iota_a^{-1} = \iota_{\iota_a b} = \iota_b$$

from which 1 follows. Similarly 1 implies 3. Finally, by the trichotomy of §3.3.2, A and B are ultraparallel, asymptotic or intersect at an angle θ. If $A)(B$ or $A \parallel B$, then $\iota_A \iota_B$ has infinite order; if $A \between B$, then $\iota_A \iota_B$ is equivalent to a rotation of angle 2θ. In particular $\iota_A \iota_B$ is an involution if and only if $2\theta = \pi$. □

4.3.2 *Inversion in a finite hyperchain*

Consider the hyperchain $c = \partial L$ where $L = \mathbf{H}^{n-1}_{\mathbb{C}} \times \{0\} \subset \mathbf{H}^n_{\mathbb{C}}$. Inversion in c is represented by the diagonal matrix

$$\tilde{\iota}_c = \begin{bmatrix} \mathbb{I}_{n-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{SU}(n,1) \tag{4.9}$$

which in the ball and Siegel domain models for $\mathbf{H}^n_{\mathbb{C}}$ is given by

$$\begin{pmatrix} z' \\ z_n \end{pmatrix} \mapsto \begin{pmatrix} z' \\ -z_n \end{pmatrix} \quad \text{on } \mathbb{B}^n, \qquad \begin{pmatrix} w' \\ w_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(w_n)^{-1} w' \\ \frac{1}{4}(w_n)^{-1} \end{pmatrix} \quad \text{on } \mathfrak{H}^n.$$

As a map of Heisenberg space this inversion $\iota_c : \mathcal{H} \longrightarrow \mathcal{H}$ is given by

$$\iota : \begin{pmatrix} \zeta \\ v \end{pmatrix} \mapsto \begin{pmatrix} \zeta/(|\zeta|^2 - iv) \\ -v/(|\zeta|^4 + v^2) \end{pmatrix}. \tag{4.10}$$

The inversion ι_c interchanges the origin $(0,0) \in \mathcal{H}$ and ∞; moreover, if δ is a Heisenberg dilation about the origin, then $\delta\iota$ is the inversion in the chain $\delta^{1/2}(\partial H)$ and therefore

$$\delta\iota = \delta^{1/2}\iota\delta^{-1/2}.$$

For example, the hyperchains ∂L orthogonal to $\mathbb{V}$ are Euclidean hyperspheres in $\mathbb{C}^{n-1} \times \{v_0\} \subset \mathcal{H}$. Specifically they are determined by a *height* $v_0 \in \mathbb{R}$ and a *radius* $r_0 > 0$:

$$\partial L = \{(\zeta, v) \in \mathcal{H} \mid \|\zeta\| = r_0, \quad v = v_0\}. \tag{4.11}$$

To see this, consider a positive vector

$$Z = \begin{bmatrix} Z' \\ Z_n \\ Z_{n+1} \end{bmatrix} \in \mathbb{C}^{n,1}$$

polar to L; it follows from (4.8) that $Z' = 0' \in \mathbb{C}^{n-1}$. By applying vertical Heisenberg translation of magnitude

$$-\mathrm{Im}\left(\frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}}\right)$$

we may assume that O_L is the origin $(0', 0) \in \mathcal{H}$ and (by scalar multiplication by a complex number of unit length)

$$Z = \begin{bmatrix} 0' \\ 1 + r_0^2 \\ 1 - r_0^2 \end{bmatrix}$$

from which (4.11) follows.

4.3.3 *Boundaries of complex-linear subspaces*

Let $L^k \subset \mathbf{H}_{\mathbb{C}}^n$ be a $\mathbb{C}^k$-plane and $\partial L = \bar{L} \cap \partial\mathbf{H}_{\mathbb{C}}^n$ its bounding $\mathbb{C}^k$-chain. There are two cases, depending on whether ∞ lies in ∂L. If $\infty \in \partial L$, then ∂L is *vertical*, and if $\infty \notin \partial L$, then ∂L is *finite*. We denote the inversion (complex reflection of order 2) in L by ι_L. If ∂L is finite, then $\iota_L(\infty) \in \mathcal{H}$ is called the *center* of ∂L and will be denoted by O_L.

Theorem 4.3.2 *Let $\partial L \subset \partial \mathbf{H}^n_{\mathbb{C}} = \mathcal{H} \cup \{\infty\}$ be a $\mathbb{C}^k$-chain and let $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}^{n-1}$ be vertical projection.*

1. *If ∂L is vertical, then $\Pi_{\mathbb{V}}(\partial L - \{p_\infty\})$ is a $\mathbb{C}$-affine subspace of $\mathbb{C}^{n-1}$ of dimension $k-1$ and*
$$\partial L - \{p_\infty\} = \Pi_{\mathbb{V}}^{-1}\left(\Pi_{\mathbb{V}}(\partial L - \{p_\infty\})\right).$$
2. *If ∂L is finite, then $\Pi_{\mathbb{V}}$ maps ∂L bijectively onto a (Euclidean) sphere in $\mathbb{C}^{n-1}$ of real dimension $2k-1$.*
3. *Two $\mathbb{C}^k$-chains having the same vertical projection differ by a vertical translation.*

Proof For 1, since $p_\infty \in \partial L$, inversion ι_L in L is a projective transformation fixing p_∞ and hence the unique complex hyperplane $H(p_\infty)$ tangent to $\partial \mathbf{H}^n_{\mathbb{C}}$ at p_∞ as in §4.1. Thus in the corresponding affine patch on $\mathfrak{H}^n$, the inversion ι_L acts by affine transformations. In particular its fixed-point set $L - \{p_\infty\}$ is a complex affine subspace. As the complex lines passing through p_∞ are bounded by the vertical chains, ∂L is a union of vertical chains and by §4.2.3, $\Pi_{\mathbb{V}}(\partial L)$ is a complex affine subspace of dimension $k-1$.

Suppose that ∂L is a finite $\mathbb{C}^k$-chain with center O_L. A unique vertical $\mathbb{C}^{k+1}$-chain L^+ contains L. (Take the boundary of the $(k+1)$-dimensional subspace corresponding to the span

$$\mathbb{P}(\tilde{p_\infty} + \mathbb{P}^{-1}(L))$$

of p_∞ and L in $\mathbb{P}(\mathbb{C}^{n,1})$.) By 1, $\Pi_{\mathbb{V}}(\partial L^+)$ is a k-dimensional $\mathbb{C}$-affine subspace of $\mathbb{C}^{n-1}$. Replacing $\mathbb{C}^{n-1}$ by $\Pi_{\mathbb{V}}(\partial L^+)$, we may assume that $k = n-1$.

Let τ be a Heisenberg translation taking O_L to the vertical axis $\mathbb{V}$; then $\tau(L)$ is a $\mathbb{C}^k$-chain centered on $\mathbb{V}$. Since $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}^{n-1}$ is equivariant with respect to Heisenberg translations of $\mathcal{H}$ and Euclidean translations of $\mathbb{C}^{n-1}$, by replacing L by $\tau(L)$, we may assume that $O_L \in \mathbb{V}$.

Now $\mathbb{V}$ is the unique chain containing O_L and ∞, and ι_L interchanges O_L and ∞. Therefore $\iota_L(\mathbb{V}) = \mathbb{V}$; that is, L and $\mathbb{V}$ are orthogonal. Applying the previous discussion of hyperchains centered on $\mathbb{V}$, 2 follows. 3 follows similarly. □

4.3.4 *Center and radius of finite chains*

We shall mainly concentrate on the case in complex dimension $n = 2$, where chains are uniquely parametrized by a *center* (a point in $\mathcal{H}$) and a *radius* (a positive real number). In this case, Theorem 4.3.2 implies two important corollaries.

Corollary 4.3.3 *The vertical projection of a chain is either a point or a Euclidean circle in $\mathbb{C}$.*

Corollary 4.3.4 *Two chains having the same vertical projection differ by a vertical translation.*

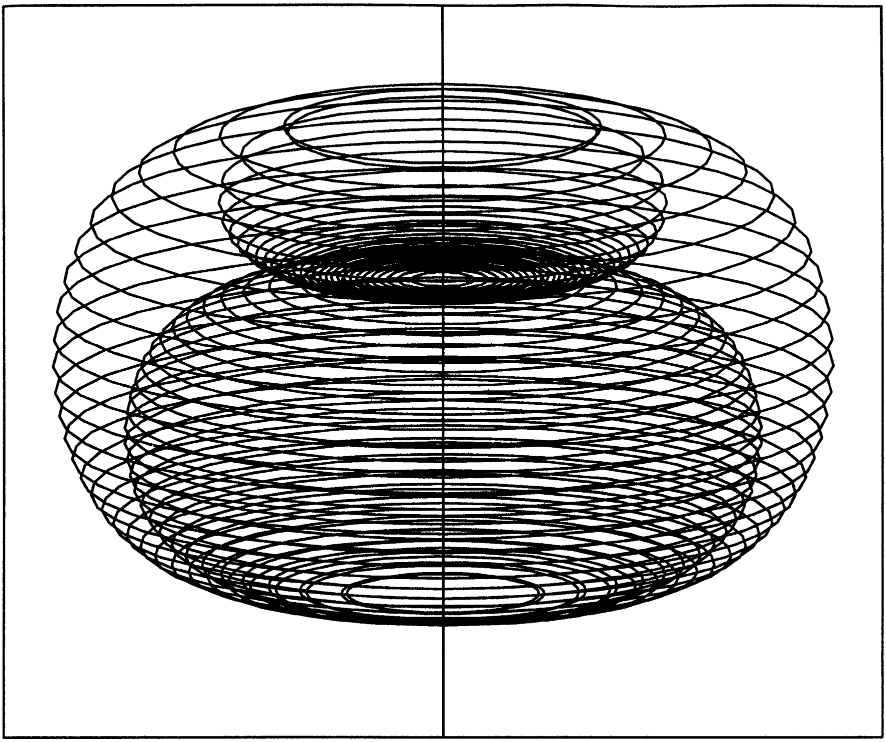

FIG. 4.1. Chains orthogonal to vertical axis

If $c \subset \mathcal{H}$ is a finite chain, then the radius of the Euclidean circle $\Pi_{\mathrm{V}}(c) \subset \mathbb{C}$ is called the *radius* of c. Some chains orthogonal to the vertical axis are illustrated in Fig. 4.1.

The vertical chain defined by $\zeta = \zeta_0$ has polar vector

$$\begin{bmatrix} 1 \\ -\bar{\zeta}_0 \\ \bar{\zeta}_0 \end{bmatrix}.$$

The finite chain with center $(\zeta_0, v_0) \in \mathcal{H}$ and radius $r_0 > 0$ has polar vector

$$\begin{bmatrix} 2\bar{\zeta}_0 \\ 1 + r_0^2 - \zeta_0\bar{\zeta}_0 + iv_0 \\ 1 - r_0^2 + \zeta_0\bar{\zeta}_0 - iv_0 \end{bmatrix}$$

and consists of all $(\zeta, v) \in \mathcal{H}$ satisfying the equations

$$|\zeta - \zeta_0| = r_0, \qquad v = v_0 - 2\,\mathrm{Im}(\bar{\zeta}_0\zeta) \tag{4.12}$$

(an ellipse of eccentricity $\sqrt{1 + |\zeta_0|^2}$).

Theorem 4.3.5 1. *Let $p, q \in \mathcal{H}$ be distinct points. Then there exists a unique chain passing through p and q.*

2. *Let $p \in \mathcal{H}$ and $\xi \in T_p\mathcal{H} - E_p$ be a tangent vector at p transverse to the CR-structure at p. Then there exists a unique chain passing through p tangent to ξ.*

Proof For 1, a unique projective line in $\mathbb{P}(\mathbb{C}^{n,1})$ passes through $p, q \in \mathbb{P}(\mathbb{C}^{n,1})$. Its boundary is the desired chain. For 2, since ξ is a nonzero tangent vector, a unique projective line passes through p tangent to ξ. Since $p \in \mathcal{H}$ and $\xi \notin E_p$, this line is not tangent to $\partial\mathbf{H}^n_{\mathbb{C}} \subset \mathbb{P}(\mathbb{C}^{n,1})$ and meets $\partial\mathbf{H}^n_{\mathbb{C}}$ in a chain. □

The following elementary result immediately follows from the transitivity of $\mathbf{PU}(n,1)$ on chains and (4.11):

Theorem 4.3.6 *Let $c \subset \partial\mathbf{H}^n_{\mathbb{C}}$ be a hyperchain and let $p \notin c$. The unique chain passing through p orthogonal to c equals the orbit of p under the group of complex reflections in c.*

This chain is evidently the inverse image $\Pi_c^{-1}(\Pi_c(p))$ where Π_c is orthogonal projection onto the complex hyperplane bounded by c.

A significant difference exists between the differentiable structure of $\partial\mathbf{H}^n_{\mathbb{C}}$ and its analogue in real hyperbolic geometry. The inversion ι_c gives a smooth chart at ∞ in $\partial\mathbf{H}^n_{\mathbb{C}} = \mathcal{H} \cup \{\infty\}$. For example, if U is a neighborhood of ∞ in Heisenberg space then a function $f : U \longrightarrow \mathbb{R}$ is smooth if and only if $f \circ \iota$ is smooth. This differentiable structure is different from the "usual" differentiable structure on the 1-point compactification of $\mathbb{R}^{2n-1}$ carrying conformal geometry on S^{2n-1}. Distinct vertical lines in $\mathcal{H}$ are vertical chains which meet at ∞ and are thus transverse, while in the conformal compactification vertical lines are circles which are tangent at ∞.

Exercise 4.3.7 *Show that the* vertical plane *in $\mathcal{H}$ defined by* $\mathrm{Re}(\zeta) = 0$ *is a smooth surface $\mathcal{H}$ although its closure in $\mathcal{H} \cup \{\infty\}$ is not smooth. (This surface is an example of a* fan, *discussed extensively in Goldman–Parker [73]. Compare §8.1.3 and Fig. 4.2.)*

Exercise 4.3.8 *Let $P_1, P_2 \subset \mathcal{H}$ be contact hyperplanes such that $P_1 \cap P_2$ is a straight line. Then $\bar{P}_i = P_i \cup \{\infty\}$ for $i = 1, 2$ are smooth surfaces in $\mathcal{H} \cup \{\infty\}$ which intersect tangentially at ∞.*

Exercise 4.3.9 *A Heisenberg similarity $\phi \in$ $\mathbf{Sim}(\mathcal{H})$ extends to a diffeomorphism of $\mathcal{H} \cup \{\infty\}$. Show that its differential at ∞ equals the identity map of the tangent space $T_\infty(\mathcal{H} \cup \{\infty\})$ if and only if ϕ is a Heisenberg translation. In general relate the differential at ∞ to the linear part of ϕ as an affine transformation of $\mathcal{H}$.*

4.3.5 *The centroid of a chain in Heisenberg space*

The center and radius parameters of a chain can be characterized in terms of the contact 1-form on Heisenberg space. Let C be a finite chain. Let ω be the

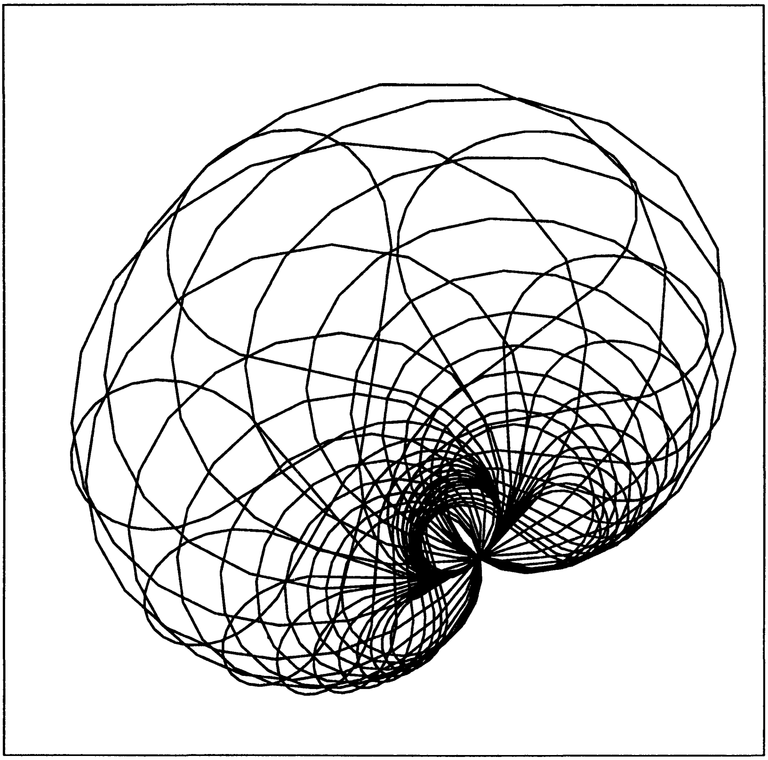

FIG. 4.2. Fan with vertex at the origin

calibration of the CR-structure invariant under Heisenberg translation. Then $\omega|_C$ is a 1-form on C (which we may normalize to have integral 1 over C). The measure μ on C dual to $\omega|_C$ extends to a probability measure on $\mathcal{H}$ supported on C. The center of C is the centroid of μ.

Given a chain $C \subset \mathcal{H}$, there exists a unique parametrization $\tau : \mathbb{R} \longrightarrow C$ such that $\tau^*\omega = dt$ (where t is the coordinate on $\mathbb{R}$).

Furthermore this parametrization lies in the preferred projective class; that is, there exists a 1-parameter subgroup $\{g_s\}_{s\in\mathbb{R}}$ of automorphisms leaving C invariant such that

$$g_s(\tau(t)) = \tau(t+s).$$

($\{g_s\}_{s\in\mathbb{R}}$ is elliptic if C is finite; otherwise it is unipotent.) Moreover, the period of τ (the minimum t such that $\tau|_{[s,s+t]}$ is surjective) equals $4\pi R^2$ where R is the radius of the chain. If C is the unit circle, then this parametrization is

$$\tau(s) = (e^{is/2}, 0).$$

If C is the vertical axis, then this parametrization is

$$\tau(s) = (0, s).$$

4.3.6 *Foliating Heisenberg space by orthogonal chains*

Let $\Sigma \subset \mathbf{H}^2_{\mathbb{C}}$ be the complex geodesic $\{0\} \times \mathbf{H}^1_{\mathbb{C}}$ bounded by the vertical axis $\mathbb{V} \subset \mathcal{H}$. The ubiquitous expression

$$\Upsilon(\zeta, v) = \|\zeta\|^2 - iv$$

represents the composition of the orthogonal projection $\Pi_\Sigma : \mathbf{H}^2_{\mathbb{C}} \longrightarrow \Sigma$ with the embedding $\Sigma \hookrightarrow \mathbb{C}$ as the right half-plane $\mathfrak{H}^1$.

Furthermore the restriction of the projection Π_Σ to the boundary $\mathcal{H} - \mathbb{V}$ "embeds" hyperbolic geometry in Heisenberg geometry—whereby points of the hyperbolic plane correspond to the chains in $\mathcal{H} - \mathbb{V}$ orthogonal to $\mathbb{V}$. These chains foliate $\mathcal{H} - \mathbb{V}$ and the symmetric 2-form

$$\Upsilon^* \left(\frac{|dw|^2}{\|\mathrm{Re}(w)\|^2} \right) = (d \log \|\zeta\|^2)^2 + |\zeta|^{-4} dv^2$$

defines a hyperbolic metric transverse to the foliation of $\mathcal{H} - \mathbb{V}$ by chains orthogonal to $\mathbb{V}$.

The stabilizer of $\mathbb{V}$ is the image of the composition

$$\mathbf{U}(1,1) \longrightarrow \mathbf{U}(2,1) \rightarrow \mathbf{PU}(2,1)$$

which embeds $\mathbf{U}(1,1) \hookrightarrow \mathbf{PU}(2,1)$ by

$$A \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \in \mathbf{U}(2,1)$$

for $A \in \mathbf{U}(1,1)$. The center of $\mathbf{U}(1,1)$ is the group of complex reflections pointwise fixing $\mathbb{V}$. Orthogonal projection

$$\Pi_\Sigma : \mathbf{H}^2_{\mathbb{C}} \longrightarrow \Sigma$$

is represented in the ball model by

$$\Pi_\Sigma(z_1, z_2) = (0, z_2)$$

and is equivariant with respect to the homomorphism $\mathbf{U}(1,1) \longrightarrow \mathbf{PU}(1,1)$ (where $\mathbf{PU}(1,1)$ is the group of holomorphic automorphisms of $\Sigma \cong \mathbf{H}^1_{\mathbb{C}}$).

Lemma 4.3.10 *The subgroup* $\mathbf{SU}(1,1) \subset \mathbf{U}(1,1)$ *acts simply transitively on the complement* $\mathcal{H} - \mathbb{V}$.

Proof Clearly $\mathcal{H} - \mathbb{V}$ identifies with $\partial\mathbf{H}^2_{\mathbb{C}} - \partial\mathbf{H}^1_{\mathbb{C}}$. Let $x \in \mathbf{H}^1_{\mathbb{C}}$ and consider the stabilizer G of $x \in \mathbf{SU}(1,1)$. Since Π_Σ is $\mathbf{SU}(1,1)$-equivariant and $\mathbf{PU}(1,1)$ acts transitively on $\mathbf{H}^1_{\mathbb{C}}$, transitivity of $\mathbf{SU}(1,1)$ on $\partial\mathbf{H}^2_{\mathbb{C}} - \partial\mathbf{H}^1_{\mathbb{C}}$ follows from transitivity of G on $\partial\Pi_\Sigma^{-1}(x)$. Furthermore $\mathbf{SU}(1,1)$ acts freely on $\partial\mathbf{H}^2_{\mathbb{C}} - \partial\mathbf{H}^1_{\mathbb{C}}$ if

G acts freely on $\partial\Pi_\Sigma^{-1}(x)$. Thus it suffices to show that G acts simply transitively on $\partial\Pi_\Sigma^{-1}(x)$.

Take x to be the origin $(0,0)$ in the ball model. Its stabilizer G in $\mathbf{SU}(1,1)$ is the image of diagonal matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{bmatrix}$$

in $\mathbf{PU}(2,1)$ where $|\zeta| = 1$. In the ball model, G acts by

$$(z_1, z_2) \longmapsto (\zeta z_1, \zeta^2 z_2). \tag{4.13}$$

As $\partial\Pi_\Sigma^{-1}(x)$ consists of all $(z_1, 0) \in \partial\mathbb{B}^2$ with $|z_1| = 1$, (4.13) implies that G acts simply transitively on $\partial\Pi_\Sigma^{-1}(x)$ as claimed. □

The action of $\mathbf{SU}(1,1)$ on Σ is not effective, since the inversion corresponding to $\zeta = -1$ acts trivially on $\mathbf{H}^1_{\mathbb{C}}$. (However, $\mathbf{U}(1,1)$ does act effectively on $\mathcal{H} - \mathbb{V}$ by the preceding lemma.) Identify

$$\mathcal{H} - \mathbb{V} \cong \partial\mathbf{H}^2_{\mathbb{C}} - \partial\mathbf{H}^1_{\mathbb{C}}$$

as a $\mathbf{U}(1,1)$-homogeneous fiber bundle over the hyperbolic plane $\Sigma \approx \mathbf{H}^1_{\mathbb{C}}$ as follows. Using notation from the previous proof, the stabilizer of x in $\mathbf{U}(1,1)$ is the subgroup G' generated by G and the group of complex reflections $\mathbb{T}_x$ about x. (G' is the direct product $G \times \mathbb{T}_x$.) The complex reflection component is the determinant of this embedding

$$\xi : G' \subset \mathbf{U}(1,1) \xrightarrow{\det} \mathbb{T}.$$

In the ball model G' acts by

$$(z_1, z_2) \longmapsto (\xi\zeta z_1, \xi\zeta^2 z_2)$$

where $|\zeta||\xi| = 1$. Taking $z_2 = 0$ as above, we identify the isotropy representation of G' on the fiber $\Pi_\Sigma^{-1}(x)$.

The evaluation map $\mathbf{SU}(1,1) \longrightarrow \mathcal{H} - \mathbb{V}$ fits into a commutative diagram:

$$\begin{array}{ccc} \mathbf{SU}(1,1) & \longrightarrow & \mathcal{H} - \mathbb{V} \\ \downarrow & & \downarrow \\ \mathbf{SU}(1,1)/G & \longrightarrow & \Sigma \end{array}$$

The quotient map $\mathbf{SU}(1,1) \longrightarrow \mathbf{SU}(1,1)/G$ is equivariantly isomorphic to a (homogeneous) circle bundle over the hyperbolic plane. This circle bundle corresponds to a square root of the unit tangent circle bundle of the hyperbolic plane; that is, the circle bundle corresponding to the inverse of the complex line bundles of spinors. Passing to discrete subgroups of $\mathbf{SU}(1,1)$ produces interesting examples of Seifert manifolds with spherical CR-structures (compare [19], [43], [94]). The orbits of the group of complex reflections in $\mathbb{V}$ are chains and define a Seifert fibration of the quotient. In particular if S is a closed orientable surface of genus $g > 1$, then corresponding to an embedding $\pi_1(S) \hookrightarrow \mathbf{PU}(1,1)$ there is a subgroup $\Gamma \subset \mathbf{U}(1,1) \subset \mathbf{PU}(2,1)$ such that $(\mathcal{H} - \mathbb{V})/\Gamma$ is a spherical CR-manifold diffeomorphic to the oriented circle bundle over S with Euler class $1-g$. The quotient of this manifold by the involution determined by inversion in $\mathbb{V}$ is homeomorphic to the unit tangent bundle of S.

4.3.7 *Extending Fuchsian groups to* $\mathbf{H}^n_{\mathbb{C}}$

The groups considered here are a special case of a general construction which leads to interesting complex hyperbolic manifolds. Namely, let $\Gamma \subset \mathbf{SU}(n,1)$ be a discrete subgroup. The composition

$$\Gamma \hookrightarrow \mathbf{SU}(n,1) \longrightarrow \mathbf{PU}(n+k,1)$$

defines an action of Γ on $\mathbf{H}^{n+k}_{\mathbb{C}}$. The quotient $M = \mathbf{H}^{n+k}_{\mathbb{C}}/\Gamma$ is a complex hyperbolic $(n+k)$-manifold containing the holomorphic totally geodesic submanifold $S = \mathbf{H}^n_{\mathbb{C}}/\Gamma$. Orthogonal projection

$$\Pi : \mathbf{H}^{n+k}_{\mathbb{C}} \longrightarrow \mathbf{H}^n_{\mathbb{C}}$$

defines an $\mathbf{SU}(n,1)$-equivariant $\mathbf{H}^k_{\mathbb{C}}$-fibration which descends to an $\mathbf{H}^k_{\mathbb{C}}$-fibration

$$M = \mathbf{H}^{n+k}_{\mathbb{C}}/\Gamma \longrightarrow \mathbf{H}^n_{\mathbb{C}}/\Gamma = S$$

for which S is a section.

Suppose that S is a closed manifold. Then M cannot be a Stein manifold, as it contains a compact holomorphic submanifold S. Furthermore its complex hyperbolic structure is *rigid* in a certain sense. Any representation $\phi : \Gamma \longrightarrow \mathbf{PU}(n+1,1)$ sufficiently near the inclusion $\Gamma \hookrightarrow \mathbf{PU}(n+1,1)$ stabilizes a complex totally geodesic submanifold; that is, ϕ is conjugate to a representation stabilizing $\mathbf{H}^n_{\mathbb{C}}$. For $n = 1$, this "local rigidity" theorem is due to Goldman [68], and for $n > 1$ to Goldman–Millson [71]. A much stronger "global rigidity" theorem is proved by Toledo [163] for $n = 1$ and Corlette [32] for $n = 2$. Another extension of the local rigidity, whose hypotheses involve the Hausdorff dimension (suitably adapted to Heisenberg geometry) of the limit set, is due to Yue [174].

Examples of non-cocompact Fuchsian lattices in $\mathbf{SU}(1,1)$ which are not locally rigid (in an appropriate sense) have been constructed by Gusevskii and Parker [82]. They construct a whole family of deformations of the inclusion of a non-cocompact lattice $\Gamma \subset \mathbf{SU}(1,1)$ where the holonomy around each boundary component is mapped to parabolic (but not unipotent) elements. Compare also §7.1.5.

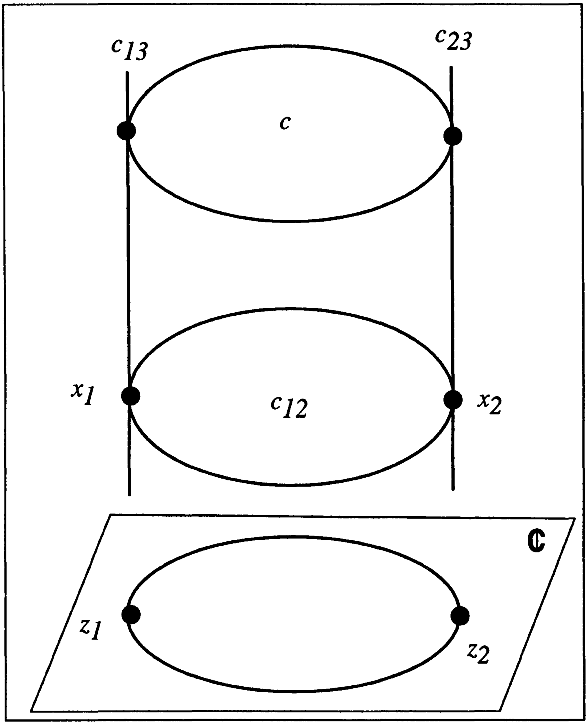

FIG. 4.3. Sketch of proof of Theorem 4.3.11

4.3.8 *Simple synthetic geometry of chains*

In [21], [22] Cartan notes that in some ways chains resemble geodesics on a Riemannian manifold and in some ways they don't. Through any two distinct points of $\partial\mathbf{H}^n_{\mathbb{C}}$ passes a unique chain. Through any points $x_1, \ldots, x_m \in \partial\mathbf{H}^n_{\mathbb{C}}$ *in general position* passes an $(m-1)$-dimensional linear subspace. In particular every triple of points in $\partial\mathbf{H}^n_{\mathbb{C}}$ lies in the boundary of a complex hyperbolic 2-space. (For this reason, and for simplicity of exposition, we only consider the case of complex dimension $n = 2$.)

Despite their similarities with geodesics, chains cannot be fit together to form analogues of totally geodesic surfaces as follows:

Theorem 4.3.11 *Let x_1, x_2, x_3 be a triple of distinct points not lying on a common chain and let c_{12}, c_{23} and c_{31} be the unique chains containing the pairs x_1 and x_2, x_2 and x_3, x_3 and x_1 respectively. Suppose that c is a chain which intersects c_{12}, c_{23} and c_{31}. Then c contains x_1, x_2 or x_3.*

Proof Apply an automorphism to assume x_3 is the point at infinity in Heisenberg space. Let $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}$ denote vertical projection and let $z_1 = \Pi_{\mathbb{V}}(x_1)$ and

$z_2 = \Pi_V(x_2)$. Then c_{13} and c_{23} are the vertical chains $\Pi_V^{-1}(z_1)$ and $\Pi_V^{-1}(z_2)$ respectively. Since x_1, x_2, x_3 do not lie on a common chain, $z_1, z_2 \in \mathbb{C}$ are distinct. (Compare Fig. 4.3.)

Assume that c does not contain x_1, x_2 or x_3 to obtain a contradiction. Since $x_3 \notin c$, the projection $\Pi_V(c)$ is a Euclidean circle in $\mathbb{C}$ containing z_1 and z_2. By assumption c meets c_{12} in a point x. If $x \in c_{31}$, then

$$x = x_1 = c_{31} \cap c_{12},$$

contradicting $x_1 \notin c$. Thus $x \notin c_{31}$. Similarly $x \notin c_{23}$. Therefore $z = \Pi_V(x)$ is distinct from z_1 and z_2. The projections $\Pi_V(c)$ and $\Pi_V(c_{12})$ are Euclidean circles in $\mathbb{C}$ (by Corollary 4.3.3). As they each contain the three distinct points $z, z_1, z_2 \in \mathbb{C}$, they coincide. That is, the chains c and c_{12} have the same vertical projection. By Corollary 4.3.4 the chains c and c_{12} differ by a vertical translation. Since they intersect, they must actually be identical. Therefore $c = c_{12}$, contradicting $x_1 \notin c$. □

4.3.9 *Chain-preserving transformations*

One of the main results of [21] is the following, analogous to the well-known result of von Staudt that a transformation of projective space preserving the relation of collinearity must be a projective automorphism— *a transformation preserving collinearity is a collineation.*

Theorem 4.3.12 *(Cartan) Let $f : \partial\mathbf{H}^n_{\mathbb{C}} \longrightarrow \partial\mathbf{H}^n_{\mathbb{C}}$ be a (not necessarily continuous) injection such that for $x = (x_1, x_2, x_3) \in C_3(\partial\mathbf{H}^n_{\mathbb{C}})$,*

$$x \in \mathbf{Chain} \Longleftrightarrow f(x) \in \mathbf{Chain}.$$

Then f is an automorphism (possibly anti-holomorphic) of $\partial\mathbf{H}^n_{\mathbb{C}}$.

Proof By applying an automorphism, we may assume that f fixes ∞. Since a chain through ∞ is a vertical line the space of chains through ∞ may be identified with the complex line $\mathbb{C}$ and the map which assigns to a point $x \neq \infty$ in Heisenberg space the vertical chain containing x is the vertical projection $\Pi_V : \mathcal{H} \longrightarrow \mathbb{C}$. Since f takes chains to chains and fixes ∞ it follows that f induces a map $\bar{f}$ of $\mathbb{C} = \Pi_V(\mathcal{H})$ for which the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "f"] \arrow[d, "\Pi_V"'] & \mathcal{H} \arrow[d, "\Pi_V"] \\ \mathbb{C} \arrow[r, "\bar{f}"'] & \mathbb{C} \end{tikzcd}$$

commutes. Since vertical projections of chains are circles in $\mathbb{C}$, the transformation $\bar{f} : \mathbb{C} \longrightarrow \mathbb{C}$ is a circle-preserving transformation of the plane. Such a transformation $\mathbb{C} \longrightarrow \mathbb{C}$ must be a complex affine transformation. (Here is a brief proof. By §6.71 of [35], a circle-preserving transformation of the inversive plane must be a Möbius transformation—the result needed here follows from this once given the fact that a circle-preserving transformation of $\mathbb{C}$ preserves lines (circles through ∞). But the line through two points a, b consists of the complement of the union of all circles through a, b. Therefore a circle-preserving transformation of the plane must also preserve lines and hence is a Euclidean similarity.) By composing with a Heisenberg similarity transformation we may assume that $\bar{f}$ is the identity and that $f(0) = 0$. Suppose that $x \in \mathcal{H}$ is not fixed under f and that $z = \Pi_{\mathbf{V}}(x) \neq 0 \in \mathbb{C}$. Let c be the chain passing through 0 and x. Then $f(c)$ is another chain through the origin distinct from c (since $f(x) \neq x$ have the same vertical projection and c is not vertical). Hence the projections $\Pi_{\mathbf{V}}(c), \Pi_{\mathbf{V}}(f(c))$ are distinct circles in $\mathbb{C}$ containing 0 and z, contradicting the fact that $\bar{f}$ is the identity map on $\mathbb{C}$. Thus f fixes all points off the vertical axis, and by repeating the above argument with 0 replaced by any point not on the vertical axis, it follows that f is the identity. □

4.4 $\mathbb{R}$-circles

4.4.1 *$\mathbb{R}$-spheres in Heisenberg geometry*

Chains (and more generally, boundaries of complex k-dimensional totally geodesic subspaces) provide a rich class of geometric objects in Heisenberg space. In addition, boundaries of totally real totally geodesic subspaces interact curiously with chains. We call the boundary of an $\mathbb{R}^k$-plane an *$\mathbb{R}^{k-1}$-sphere* (following Mostow [127]). An $\mathbb{R}^1$-sphere is called an *$\mathbb{R}$-circle*. We shall call an $\mathbb{R}^{n-1}$-sphere in $\mathbf{H}^n_{\mathbb{C}}$ an *$\mathbb{R}$-form*. Note that an $\mathbb{R}^{k-1}$-sphere is a smooth manifold of (real) dimension $k-1$. As usual, we concentrate on the case $n = 2$, where $\mathbb{R}$-circles and $\mathbb{R}$-forms coincide and are 1-dimensional.

We first determine $\mathbb{R}$-spheres in Heisenberg coordinates on $\partial\mathbf{H}^n_{\mathbb{C}}$. (See §9 of Jacobowitz [92] for an alternative treatment of this material.) The space $\mathfrak{R}(\mathbf{H}^n_{\mathbb{C}})$ of $\mathbb{R}$-forms of $\mathbf{H}^n_{\mathbb{C}}$ naturally identifies with the homogeneous space $\mathbf{PU}(n,1)/\mathbf{PO}(n,1)$

4.4.2 *$\mathbb{R}$-spheres are horizontal*

Recall (2.5.2) that a submanifold $S \subset \partial\mathbf{H}^n_{\mathbb{C}}$ is *CR-horizontal* if and only if S is everywhere tangent to the CR-structure $E \subset T\partial\mathbf{H}^n_{\mathbb{C}}$.

Theorem 4.4.1 *If $P \subset \mathbf{H}^n_{\mathbb{C}}$ is an $\mathbb{R}^k$-plane, then the $\mathbb{R}$-sphere $\partial P \subset \partial\mathbf{H}^n_{\mathbb{C}}$ is CR-horizontal with respect to the canonical CR-structure on $\partial\mathbf{H}^n_{\mathbb{C}}$.*

Proof Let $p \in \partial P$. Since the canonical CR-structure on $\partial\mathbf{H}^n_{\mathbb{C}}$ is defined as

$$E_p = T_p(\partial\mathbf{H}^n_{\mathbb{C}}) \cap \mathbb{J}T_p(\partial\mathbf{H}^n_{\mathbb{C}}),$$

it suffices to prove that

$$\mathbb{J}(T_p\partial P) \subset T_p(\partial\mathbf{H}^n_{\mathbb{C}}).$$

By a change of coordinates we may assume that $P = \mathbb{B}^k_{\mathbb{R}} \times \{0_{n-k}\}$ consists of points

$$\begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with $x_j \in \mathbb{R}$ and that

$$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $T_p\partial\mathbf{H}^n_{\mathbb{C}} = i\mathbb{R} \times \mathbb{C}^{n-1}$ and $T_p(\partial P) = \{0\} \times \mathbb{R}^{k-1} \times \{0\}$. Now

$$\mathbb{J}T_p(\partial P) = \{0\} \times i\mathbb{R}^{k-1} \times \{0\} \subset i\mathbb{R} \times \mathbb{C}^{n-1} = T_p\partial\mathbf{H}^n_{\mathbb{C}}$$

as desired. □

4.4.3 *Infinite $\mathbb{R}$-spheres*

Just as for a chain, an $\mathbb{R}$-sphere in Heisenberg geometry is one of two types, depending on whether or not it passes through ∞. An $\mathbb{R}$-sphere S is *infinite* if and only if $\infty \in S$ and othewise it is *finite*. In Heisenberg coordinates infinite $\mathbb{R}$-spheres have a particularly pleasant form:

Theorem 4.4.2 *Let $S \subset \partial\mathbf{H}^n_{\mathbb{C}} = \mathcal{H} \cup \{\infty\}$ be an $\mathbb{R}^k$-sphere containing ∞. Then $S - \{\infty\}$ is an affine subspace of $\mathcal{H}$ such that vertical projection $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}^{n-1}$ maps $S - \{\infty\}$ injectively onto an isotropic affine subspace of $\mathbb{C}^{n-1}$.*

Remark 4.4.3 *The projection $\Pi_{\mathbb{V}}(S - \{\infty\})$ actually determines the affine subspace*

$$S - \{\infty\} \subset \mathcal{H}$$

uniquely, once a base point $p \in S - \{\infty\}$ is chosen. If $L \subset \mathbb{C}^{n-1}$ is an isotropic $\mathbb{R}$-affine subspace of real dimension k containing $\Pi_{\mathbb{V}}(p)$, then there is a unique k-dimensional $\mathbb{R}$-affine subspace $\tilde{L} \subset \mathcal{H}$ such that $\Pi_{\mathbb{V}} : \tilde{L} \rightarrow L$ is injective (see §2.6.2).

Proof Inversion ι in S is an anti-involution of $\mathbb{P}(\mathbb{C}^{n,1})$. Since $p_\infty \in S$ and ι preserves $\partial\mathbf{H}^n_{\mathbb{C}}$, the $\mathbb{C}$-hyperplane $H(p_\infty) \subset \mathbb{P}(\mathbb{C}^{n,1})$ tangent to $\partial\mathbf{H}^n_{\mathbb{C}}$ at p_∞ is ι-invariant (see §4.1). In the corresponding affine patch (the Siegel domain $\mathfrak{H}^n$), ι is an anti-holomorphic affine transformation, whose fixed-point set is a real affine subspace. By Theorem 4.4.1, such a subspace is CR-horizontal. $\Pi_{\mathbb{V}}$ maps it bijectively onto a Lagrangian subspace of $\mathbb{C}^{n-1}$ (by §2.5.2). □

For $n = 2$, any $\mathbb{R}$-line in $\mathbb{C}$ is Lagrangian, so this result implies:

Corollary 4.4.4 *An infinite $\mathbb{R}$-circle in $\partial\mathbf{H}^n_{\mathbb{C}}$ is represented by an $\mathbb{R}$-affine line in $\mathcal{H}$ which is CR-horizontal.*

For example, the infinite $\mathbb{R}$-circle corresponding to the standard totally real subspace $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$ is the *real axis*

$$\mathbb{R} \times \{0\} \subset \mathcal{H} = \{(\zeta, v) \in \mathcal{H} \mid \operatorname{Im}(\zeta) = v = 0\}.$$

Every infinite $\mathbb{R}$-circle in $\mathcal{H}$ is obtained from $\mathbb{R} \times \{0\}$ by a Heisenberg isometry.

Exercise 4.4.5 *Let R_1, R_2 be infinite $\mathbb{R}$-circles and let $\bar{R}_i = R_i \cup \{\infty\}$ be their closures in $\mathcal{H} \cup \{\infty\}$. Then $\bar{R}_1$ and $\bar{R}_2$ are tangent at ∞ if and only if the vertical projections $\Pi_{\mathbb{V}}(R_1)$ and $\Pi_{\mathbb{V}}(R_2)$ are parallel lines in $\mathbb{C}$. Otherwise, the angle in E_∞ betwen $\bar{R}_1$ and $\bar{R}_2$ equals the angle between the straight lines $\Pi_{\mathbb{V}}(R_1)$ and $\Pi_{\mathbb{V}}(R_2)$.*

4.4.4 *Finite $\mathbb{R}$-circles and lemniscates*

In this section we determine the representation of finite $\mathbb{R}$-spheres in Heisenberg geometry. Then we consider parametrizations of all $\mathbb{R}$-circles in $\mathbf{H}^2_{\mathbb{C}}$. For simplicity, we assume $n = 2$ throughout the remainder of this section. In particular we only consider $\mathbb{R}$-circles in a Heisenberg space of real dimension 3 (complex dimension 2). The key definition is the following:

Definition 4.4.6 *A* lemniscate *is any curve in $\mathbb{C}$ obtained from the locus (depicted in Fig. 4.4)*

$$|z|^4 + \operatorname{Re}(z^2) = 0 \tag{4.14}$$

by complex affine transformations (similarity transformations).

A lemniscate $l \subset \mathbb{C}$ has a unique singular point, which we call its *double point* and denote by $\mathbf{double}(l)$. For example, if l is defined by (4.14), then $\mathbf{double}(l) = (0, 0)$.

Theorem 4.4.7 *A finite $\mathbb{R}$-circle is the CR-horizontal lift of a lemniscate in $\mathbb{C}$.*

Theorem 4.4.7 determines the finite $\mathbb{R}$-circle uniquely. A smooth Legendrian curve $\{\gamma(t)\}_{0 \le t \le 1}$ is completely determined by the following data:

1. A single point on it (for example $\gamma(0)$).
2. Its vertical projection $\Pi_{\mathbb{V}} \circ \gamma$ (a curve in $\mathbb{C}^{n-1}$).

Let

$$\eta = \operatorname{Im}\langle\langle \zeta, d\zeta \rangle\rangle$$

be the 1-form on $\mathbb{C}^{n-1}$ such that

$$\omega = dv - \Pi^*_{\mathbb{V}}(\eta)$$

calibrates the CR-structure. (Recall this means that for any point $p \in \mathcal{H}$, the CR-structure at p is the kernel of the linear map $\omega_p : T_p\mathcal{H} \longrightarrow \mathbb{R}$.) Then $\int_\gamma \omega = 0$ if and only if the vertical coordinate $v(\gamma(t))$ satisfies

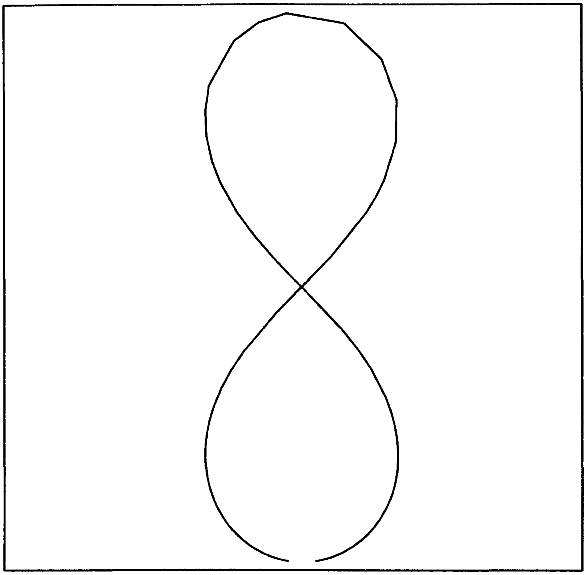

FIG. 4.4. Vertical projection of the purely imaginary $\mathbb{R}$-circle

$$v(\gamma(t)) = \int_0^t \eta((\Pi_{\mathbb{V}} \circ \gamma)'(\tau))\, d\tau.$$

In particular γ is a closed curve if and only if its vertical projection encloses zero area (algebraically). Since the two halves of a lemniscate enclose the same area but with opposite orientation, the Legendrian lift of a lemniscate is a closed curve in $\mathcal{H}$.

4.4.5 *The purely imaginary finite* $\mathbb{R}$*-circle*

Finite $\mathbb{R}$-circles have a geometric description similar to that of chains, in terms of a "center" and "radius," except the radius of an $\mathbb{R}$-circle is a nonzero *complex* number. The *center* of a finite $\mathbb{R}$-circle $R \in \mathfrak{R}$ is defined as the image of ∞ under inversion ι_R in R. (Infinite $\mathbb{R}$-circles are degenerate cases, centered at ∞.)

Consider the finite $\mathbb{R}$-circle centered at the origin and with radius 1. This $\mathbb{R}$-circle bounds the purely imaginary real form $\mathbb{J}_0\mathbb{B}^2_{\mathbb{R}} \subset \mathbb{B}^2$ corresponding to the *purely imaginary real structure* on $\mathbb{C}^{n,1}$ given by the real structure corresponding to the matrix $\mathbb{I}_{2,1}$:

$$\begin{aligned} \mathbb{C}^{2,1} &\longrightarrow \mathbb{C}^{2,1} \\ Z &\longmapsto \mathbb{I}_{2,1}\bar{Z}. \end{aligned}$$

We denote this $\mathbb{R}$-circle by $R_{\mathbb{J}}$ and call it the *purely imaginary* $\mathbb{R}$*-circle.* Every finite $\mathbb{R}$-circle is obtained from $R_{\mathbb{J}}$ by a Heisenberg similarity (see §4.4.7).

4.4.6 *Equations for finite* ℝ*-circles*

Explicitly $R_{\mathbb{J}}$ is described by the equations

$$\operatorname{Re}\left(\frac{1-|\zeta|^2+iv}{1+|\zeta|^2-iv}\right)=\operatorname{Re}\left(\frac{2\zeta}{1+|\zeta|^2-iv}\right)=0 \tag{4.15}$$

which in cylindrical coordinates ($\zeta = re^{i\theta}, v$) on $\mathcal{H}$ reduce to

$$\begin{aligned} r^2 &= -\cos(2\theta) \\ v &= \cot(\theta)(1+r^2)=\sin(2\theta) \end{aligned}$$

or, equivalently,

$$r^2+iv=-e^{2i\theta}.$$

Thus the vertical projection of $R_{\mathbb{J}}$ is the lemniscate of Bernoulli (the first of the above equations)

$$(x^2+y^2)^2+x^2-y^2=0$$

in the $\zeta = x+iy$-plane. (See Fig. 4.4.) Inversion in $R_{\mathbb{J}}$ is then given by

$$\iota_{\mathbb{J}}:(\zeta,v)\mapsto\left(\frac{-\bar{\zeta}}{|\zeta|^2+iv},\frac{v}{|\zeta|^4+v^2}\right). \tag{4.16}$$

In Cartesian coordinates, this ℝ-circle is described by

$$(x^2+y^2)^2+x^2-y^2=v-\frac{x}{y}\left(1+x^2+y^2\right)=0 \tag{4.17}$$

and in cylindrical coordinates by

$$r=\sqrt{-\cos(2\theta)}, \quad v=\sin(2\theta) \tag{4.18}$$

where θ ranges over the set where $\cos(2\theta)\geq 0$, for example

$$\pi/4\leq\theta\leq 3\pi/4, \qquad 5\pi/4\leq\theta\leq 7\pi/4. \tag{4.19}$$

Using the change of coordinates

$$\begin{aligned} \tan(\theta) &= \frac{t+t^{-1}}{2} \\ t &= \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right) \end{aligned}$$

this curve admits the rational parametrization

$$x=\frac{2t(1-t^2)}{4t^2+(1+t^2)^2}, \qquad y=\frac{(1+t^2)(1-t^2)}{4t^2+(1+t^2)^2}, \qquad v=\frac{4t(1+t^2)}{4t^2+(1+t^2)^2} \tag{4.20}$$

for $-\infty\leq t\leq\infty$.

4.4.7 *The center and complex radius of a finite* $\mathbb{R}$*-circle*

The moduli space $\mathfrak{R}$ of all $\mathbb{R}$-circles is more complicated than the moduli space of chains. (A convenient model for the moduli space of chains is the complement of $\mathbf{H}^2_{\mathbb{C}}$ in $\mathbb{P}(\mathbb{C}^{2,1})$, obtained by using polar vectors.) The 5-dimensional real homogenous space

$$\mathfrak{R} \cong \mathbf{PU}(2,1)/\mathbf{PO}(2,1)$$

cannot be a domain in complex projective space. However, one may still stratify $\mathfrak{R}$ in a way similar to the stratification of the space of chains by vertical chains (the stratum is a complex line) and finite chains (with center/radius parameters in $\mathcal{H} \times \mathbb{R}_+$). Here $\mathfrak{R}$ is stratified by a space of infinite $\mathbb{R}$-circles and a stratum of finite $\mathbb{R}$-circles (with center/radius parameters in $\mathcal{H} \times \mathbb{C}^*$).

Theorem 4.4.8 *Every finite* $\mathbb{R}$*-circle is the image of* $R_{\mathbb{J}}$ *under a Heisenberg similarity. Conversely the only nontrivial Heisenberg similarity transformation which leaves* $R_{\mathbb{J}}$ *invariant is inversion*

$$\iota_{\mathbb{V}} : (\zeta, v) \longmapsto (-\zeta, v)$$

in the vertical axis.

Proof Consider a finite $\mathbb{R}$-circle $R \subset \mathcal{H}$ with inversion $\iota_R : \mathcal{H} \longrightarrow \mathcal{H}$. Apply Heisenberg translation taking the center $\iota_R(\infty)$ of R to the origin $0 \in \mathcal{H}$ in order to assume R is centered at 0. Inversion ι_R leaves invariant the chain containing 0 and $\infty = \iota_R(0)$, which is the vertical axis $\mathbb{V}$. The restriction of ι_R to $\mathbb{V}$ is an anti-involution, which necessarily has two fixed points. Since ι_R interchanges 0 and ∞, the fixed points of $\iota_R|\mathbb{V}$ are $(0, \pm r_0^2)$ where $r_0 > 0$. Applying Heisenberg dilation δ by r_0, we may assume that ι_R fixes $(\pm 1, 0) \in \mathbb{V}$.

Let $R_{\mathbb{J}}$ denote the purely imaginary $\mathbb{R}$-circle as above and denote its inversion by $\iota_{\mathbb{J}}$. Since $(\pm 1, 0) \in R_{\mathbb{J}}$ and $\iota_{\mathbb{J}}(\mathbb{V}) = \mathbb{V}$, the composition $\iota_R \circ \iota_{\mathbb{J}}$ leaves $\mathbb{V}$ invariant and fixes $(0,0), (\pm 1, 0), \infty$. Thus the holomorphic automorphism $\iota_R \circ \iota_{\mathbb{J}}$ fixes $\mathbb{V}$, and is a complex reflection in $\mathbb{V}$:

$$\varrho_{\mathbb{V}}^{\theta} : (\zeta, v) \longmapsto (e^{i\theta}\zeta, v)$$

for some θ. Therefore

$$\begin{aligned} \iota_R &= \varrho_{\mathbb{V}}^{\theta} \circ \iota_{\mathbb{J}} \\ &= \varrho_{\mathbb{V}}^{\theta} \circ \iota_{\mathbb{J}} \\ &= \varrho_{\mathbb{V}}^{\theta/2} \circ \iota_{\mathbb{J}} \circ \varrho_{\mathbb{V}}^{-\theta/2} \end{aligned}$$

whence $R = \varrho_{\mathbb{V}}^{\theta/2}(R_{\mathbb{J}})$. In this way R is obtained from $R_{\mathbb{J}}$ by applying a composition of Heisenberg similarities.

Conversely suppose that $g \in \mathbf{Sim}(\mathcal{H})$ is a Heisenberg similarity leaving $R_{\mathbb{J}}$ invariant. (We assume that g is holomorphic.)

As g is a Heisenberg similarity, $g(\infty) = \infty$. Since $R_{\mathbb{J}}$ uniquely determines the inversion $\iota_{\mathbb{J}}$,

$$g \circ \iota_{\mathbb{J}} = \iota_{\mathbb{J}} \circ g$$

and

$$\begin{aligned} g(0) &= g \circ \iota_{\mathbb{J}}(\infty) \\ &= \iota_{\mathbb{J}} \circ g(\infty) \\ &= \iota_{\mathbb{J}}(\infty) \\ &= 0. \end{aligned}$$

Since $\mathbb{V}$ is the unique chain containing 0 and ∞, g leaves $\mathbb{V}$ invariant, acting by dilation on $\mathbb{V}$. Since g is holomorphic and leaves invariant $\mathbb{V} \cap R_{\mathbb{J}} = \{(\pm 1, 0)\}$, its restriction to $\mathbb{V}$ must be the identity. Thus g is a complex reflection $\varrho_{\mathbb{V}}^{\theta}$ about $\mathbb{V}$. Its vertical projection is rotation about the origin $0 \in \mathbb{C}$ through angle θ preserving the lemniscate $\Pi_{\mathbb{V}}(R_{\mathbb{J}}) \subset \mathbb{C}$. Thus $e^{i\theta} = \pm 1$ as claimed. □

Definition 4.4.9 *If R is a finite $\mathbb{R}$-circle, then a unique Heisenberg translation T exists such that the center of $T(R)$ is the origin. There is a Heisenberg dilation (unique up to $\iota_{\mathbb{V}}$)*

$$\delta : (\zeta, v) \longmapsto (re^{i\theta}\zeta, r^2 v)$$

such that $T(R) = \delta(R_{\mathbb{J}})$. The complex radius *of R is defined to be the complex number $r^2 e^{2i\theta}$ (which by Theorem 4.4.8 is uniquely defined by R).*

(A family of chains and $\mathbb{R}$-circles related by Heisenberg translations is depicted in Figs. 4.5, 4.6, 4.7, 4.8.)

4.4.8 *Unitary-symmetric matrices for finite $\mathbb{R}$-circles*

Next we derive formulas for matrices representing some of the examples of $\mathbb{R}$-circles discussed above. Let R be the finite $\mathbb{R}$-circle centered at the origin having complex radius z^2, where $z = re^{i\theta} \in \mathbb{C}^*$; by the preceding definition, $R = \delta(R_{\mathbb{J}})$ where δ is dilation by z on $\mathcal{H}$ about the origin. Since δ is represented by the matrix

$$g(r,\theta) = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & (r + r^{-1})/2 & (r^{-1} - r)/2 \\ 0 & (r^{-1} - r)/2 & (r + r^{-1})/2 \end{bmatrix} \in \mathbf{U}(2,1), \tag{4.21}$$

§2.1.4 implies that the unitary-symmetric matrix representing $R = \delta(R_{\mathbb{J}})$ is

$$g\mathbb{I}_{n,1}\bar{g}^{-1} = \begin{bmatrix} e^{2i\theta} & 0 & 0 \\ 0 & (r^2 + r^{-2})/2 & -(r^{-2} - r^2)/2 \\ 0 & (r^{-2} - r^2)/2 & -(r^2 + r^{-2})/2 \end{bmatrix}.$$

The space of finite $\mathbb{R}$-circles forms a 5-dimensional manifold, the top stratum of the space $\mathfrak{R}$ of all $\mathbb{R}$-circles.

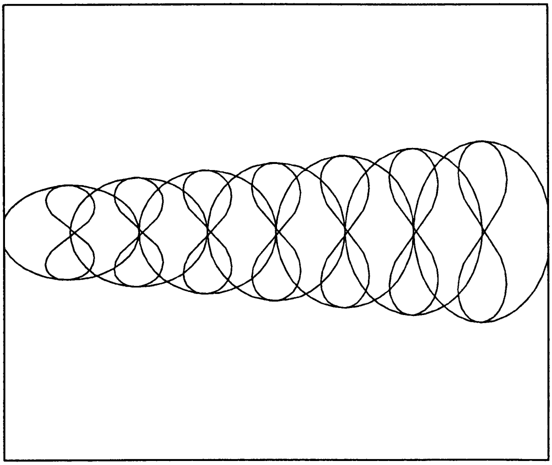

FIG. 4.5. Chains and $\mathbb{R}$-circles along real axis

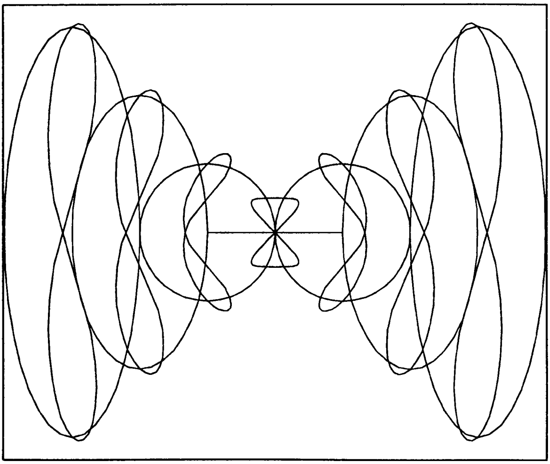

FIG. 4.6. Chains and $\mathbb{R}$-circles along real axis: another view

4.4.9 *Parameters for infinite $\mathbb{R}$-circles*

Finally consider an infinite $\mathbb{R}$-circle R. Then $R \cap \mathbb{V}$ is a point $p = (0, v_0)$ with coordinate $v_0 \in \mathbb{R}$. Furthermore R is a straight line in the horizontal plane E_p defined by $v = v_0$. Such a line is determined by the following data:

1. The height v_0.
2. The angle $\theta = \angle(R, i\mathbb{R} \times \{p\})$ with the imaginary axis $(0 \leq \theta \leq \pi)$.

Composition of dilation by $e^{i\theta}$ (rotation by θ) and vertical translation by v_0 corresponds to the matrix

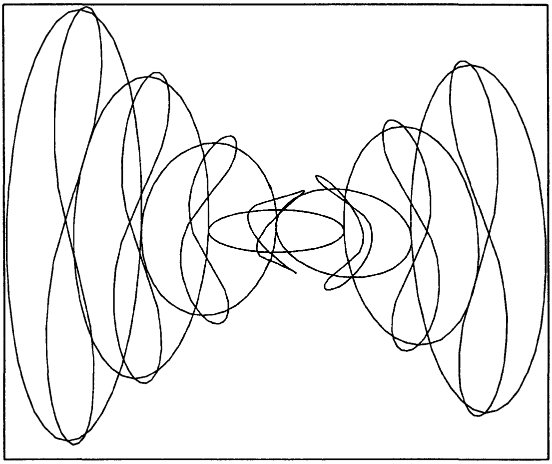

FIG. 4.7. Chains and ℝ-circles along real axis: yet another view

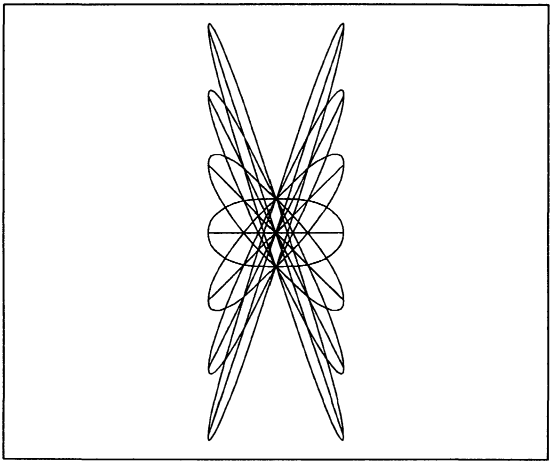

FIG. 4.8. Chains and ℝ-circles: still another view

$$T = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 - iv_0/2 & -iv_0/2 \\ 0 & iv_0/2 & 1 + iv_0/2 \end{bmatrix}.$$

Thus

$$R_0^\theta = \delta(i\mathbb{R} \times \{0\}) = \{(\zeta, v) \in \mathcal{H} \mid \mathrm{Re}(e^{i\theta}\zeta) = 0, v = v_0\}.$$

A unitary-symmetric matrix corresponding to the ℝ-circle through the origin with angle θ with the real axis is

$$T \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T = \begin{bmatrix} -e^{2i\theta} & 0 & 0 \\ 0 & 1+iv_0 & iv_0 \\ 0 & -iv_0 & 1-iv_0 \end{bmatrix}.$$

Thus infinite $\mathbb{R}$-circles meeting $\mathbb{V}$ form a 2-dimensional stratum of $\mathfrak{R}$, parametrized by $(v_0, \theta) \in \mathbb{R} \times [0, \pi)$.

Let R be an infinite $\mathbb{R}$-circle not passing through the origin. Then R is completely determined by the point

$$p_R = (\zeta, v) \in \mathcal{H} - \mathbb{V}$$

closest to the vertical axis (in the Euclidean metric). This parameter completely determines the straight line R.

For example, Formula (4.5), for Heisenberg translation $H(\zeta, 0)$ by 0, implies that the unitary-symmetric matrix corresponding to the $\mathbb{R}$-circle with closest-point parameter $p = (re^{i\theta}, 0) \in \mathcal{H} - \mathbb{V}$ is

$$g(r,\theta) \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & -2 \\ -2 & 2 & 3 \end{bmatrix} g(r,\theta).$$

Thus infinite $\mathbb{R}$-circles not meeting $\mathbb{V}$ form a 3-dimensional stratum of $\mathfrak{R}$, parametrized by the *closest-point* $p_R \in \mathcal{H} - \mathbb{V}$.

Assembling these two strata of infinite $\mathbb{R}$-circles, the space of all infinite $\mathbb{R}$-circles is the Cartesian product of $\mathbb{R}$ with $\mathbb{C} \approx \mathbb{R}^2$ blown up at the origin: an infinite $\mathbb{R}$-circle R is parametrized by "cylindrical coordinates"

$$(\rho(R), \theta(R), v(R)) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$$

(where $(0, \theta_0, v_0)$ is identified with $(0, -\theta_0, v_0)$) as follows. Let $R \subset \mathcal{H}$ be an infinite $\mathbb{R}$-circle and let $x_R \in \mathcal{H}$ be the point on R at minimum Euclidean distance from the vertical axis $\mathbb{V}$. Let $T \in \mathfrak{N}$ be the Heisenberg translation taking 0 to x_R; then $T^{-1}(R)$ is an infinite $\mathbb{R}$-circle through 0 and hence equals $R_{\theta+\frac{\pi}{2}}$ for some $\theta \in [0, \pi)$. If R doesn't meet $\mathbb{V}$ (that is, $\Pi_{\mathbb{V}}(x_R) \neq 0$) then θ is the usual cylindrical coordinate of the vertical projection $\Pi_{\mathbb{V}}(x_R) = \zeta_0 \in \mathbb{C}$. Taking $v(R)$ to be the vertical component of x_R and ρ to be the cylindrical coordinate defined by

$$\Pi_{\mathbb{V}}(x_R) = \zeta_0 = \rho(R) e^{i\theta(R)}$$

yields the desired parametrization by $(\rho, \theta, v) \in [0, \infty) \times [0, \pi) \times \mathbb{R}$. In particular the infinite $\mathbb{R}$-circle with parameters ρ, θ, v_0 is

$$\begin{aligned} R &= T(R_{\theta+\frac{\pi}{2}}) \\ &= \{(\zeta, v) \in \mathcal{H} \mid \bar{\zeta}_0 \zeta = |\zeta_0|^2 + i(v - v_0)\} \\ &= \{(x, y, v) \in \mathcal{H} \mid x\cos(\theta) + y\sin(\theta) = \rho, \\ &\qquad x\rho\sin(\theta) - y\rho\cos(\theta) + v = v_0\}. \end{aligned}$$

4.4.10 *Interpretation of ℝ-circle parameters in affine geometry*

The affine geometry of Heisenberg space provides a characterization of the closest-point x_R of an infinite ℝ-circle:

Theorem 4.4.10 *Let R be an infinite ℝ-circle with inversion ι_R and $\mathbb{V}$ the vertical axis. Then for any $v \in \mathbb{V}$, the point x_R on R closest to $\mathbb{V}$ (with respect to Euclidean distance) equals the midpoint of v and $\iota_R(v)$.*

Proof For any $u \in \mathcal{H}$ let $M(u) = \frac{1}{2}(u + \iota_R(u))$ denote the midpoint of u and its image under inversion ι_R in R. If $u_0 \in \mathbb{V}$ is a vertical vector, then

$$\iota_R(u + u_0) = \iota_R(u) - u_0$$

and

$$M(u + u_0) = \frac{1}{2}\left(u + u_0 + \iota_R(u) - u_0\right) = M(u)$$

and by applying a vertical translation we may assume that the closest point to R on $\mathbb{V}$ is the origin 0. Then the point x_R on R closest to $\mathbb{V}$ lies on the horizontal plane $v = 0$. Suppose first that $R \cap \mathbb{V} = \emptyset$; apply an element of $\mathbf{Sim}_0(\mathcal{H})$ to assume that $x_R = (1, 0) \in \mathcal{H}$. Then $\Pi_{\mathbb{V}}(R)$ is the line $\mathrm{Re}(\zeta) = 1$ and R is the line

$$\{(1 + iy, y) \mid y \in \mathbb{R}\}$$

with conjugation given by

$$(\zeta, v) \mapsto (2 - \bar{\zeta}, 2\mathrm{Im}(\zeta) - v)$$

and $M((0, v)) = (1, 0)$ as desired. If R crosses $\mathbb{V}$, then by applying a Heisenberg rotation we may assume that R equals the real axis $\mathrm{Im}(\zeta) = 0$ and $\iota_R(\zeta, v) = (\bar{\zeta}, -v)$ and $M((0, v)) = (0, 0)$ as desired. □

Similarly, the center of a finite ℝ-circle admits a characterization in affine geometry:

Theorem 4.4.11 *Let $R \subset \mathcal{H}$ be a finite ℝ-circle and let $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}$ be vertical projection. Then there exist two points p_1, p_2 such that $\Pi_{\mathbb{V}}(p_1) = \Pi_{\mathbb{V}}(p_2)$ and the restriction of $\Pi_{\mathbb{V}}$ to $R - \{p_1, p_2\}$ is injective. The center of R is the midpoint of p_1, p_2 in Heisenberg space.*

Proof By Theorem 4.4.8, it suffices to consider the purely imaginary ℝ-circle, where the above claims can be checked directly. □

4.4.11 *ℝ-circles with given direction*

Theorem 4.4.12 *Let $x_0, x_1 \in \partial \mathbf{H}^n_{\mathbb{C}}$ be distinct points and $l \subset E_{x_0}$ be a line in the CR-structure at x_0. Then a unique ℝ-circle passes through x_0, x_1 which is tangent to l at x_0.*

Proof Choose Heisenberg coordinates so that x_0 is the origin and x_1 is the point at infinity in Heisenberg space. Then $\mathbb{R}$-circles through x_0, x_1 are the $\mathbb{R}$-lines in E_{x_0}. □

The following corollary to Theorem 4.4.12 will be used in Theorem 9.2.1 to analyze tangencies of spinal spheres:

Lemma 4.4.13 *Let C_1, C_2 be two distinct chains which intersect at p. Suppose that $q \in C_1 - \{p\}$. Then there exists a unique $\mathbb{R}$-circle R such that*

1. $p, q \in R$;
2. $T_pR \subset T_pC_1 + T_pC_2$.

Proof Two chains which are tangent at a point coincide. Because chains are nowhere Legendrian, the subspace

$$T_pC_1 + T_pC_2 \subset T_p\dot{\mathcal{H}}$$

of $T_p\mathcal{H}$ is a 2-plane transverse to E_p. Thus

$$l = (T_pC_1 + T_pC_2) \cap E_p$$

is 1-dimensional. By Theorem 4.4.12, a unique $\mathbb{R}$-circle R passes through p, q which is tangent to l at p. □

4.4.12 *$\mathbb{R}$-circles orthogonal to a fixed chain*

Another application of Theorem 4.4.12 involves the parametrization of the collection of $\mathbb{R}$-circles orthogonal to a fixed chain C by an $\mathbb{RP}^{2n-3}$-bundle over the set of unordered pairs of distinct points of C. An $\mathbb{R}$-circle R is orthogonal to C if and only if R meets C in exactly two points x, y. Of course, x and y uniquely determine C. The $\mathbb{R}$-circle R is completeley determined by x, y and its tangent space at x. Thus given a line $l \subset E_x$, a unique $\mathbb{R}$-circle $R(x, y; l)$ passes through x and y and is tangent to l at x. An arbitrary pair of distinct points $x, y \in \partial\mathbf{H}^n_{\mathbb{C}}$ determines an isomorphism of (real) projective spaces defined by

$$\begin{aligned} \mathbb{P}(E_x) &\longrightarrow \mathbb{P}(E_y) \\ l &\longmapsto T_yR(x, y; l). \end{aligned}$$

This family of isomorphisms $\mathbb{P}(E_x) \to \mathbb{P}(E_y)$ is natural with respect to the action of the automorphism group on pairs (x, y). (Compare §7.1.5.)

Exercise 4.4.14 *(Schwartz [153]) Let C be a chain, $p \in C$ and $q \notin C$. Then there a unique $\mathbb{R}$-circle R exists such that $p, q \in R$ and $R \cap C$ contains p and exactly one other point.*

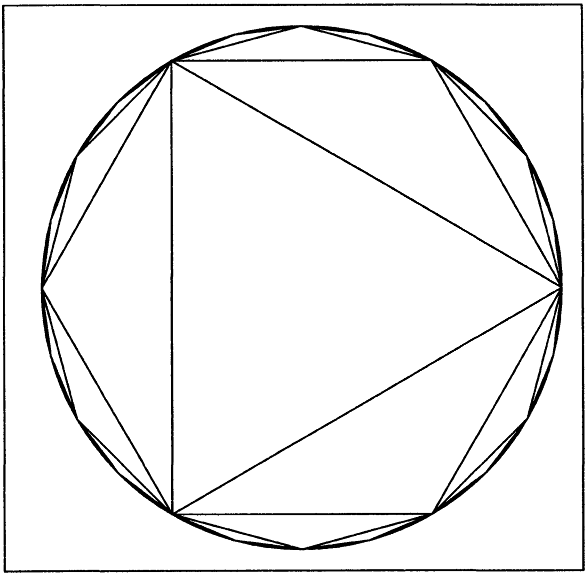

FIG. 4.9. Tiling $\mathbf{H}^2_{\mathbb{R}}$ by ideal triangles

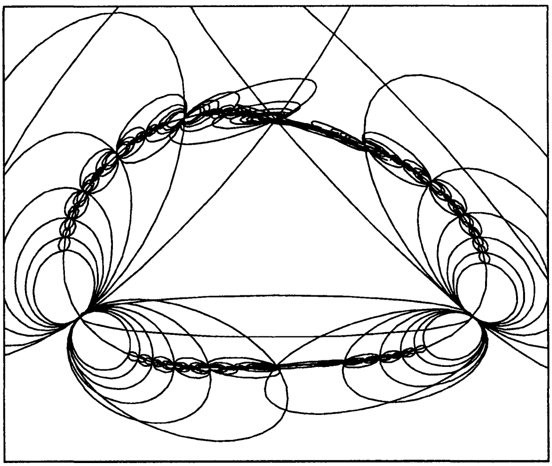

FIG. 4.10. An ideal triangle group in $\mathbf{PO}(2,1) \subset \mathbf{PU}(2,1)$

4.4.13 *Example: triangle groups from* $\mathbf{PO}(2,1)$

Here is a general construction of discrete groups whose limit set is an $\mathbb{R}$-sphere. Suppose that Γ is a discrete subgroup of $\mathbf{PO}(n,1)$, which we regard as a discrete subgroup of $\mathbf{PU}(n,1)$ under the embedding $\mathbf{PO}(n,1) \hookrightarrow \mathbf{PU}(n,1)$. Then Γ acts properly on $\mathbf{H}^n_{\mathbb{C}}$; indeed it acts properly on the complement $\partial\mathbf{H}^n_{\mathbb{C}} - \partial\mathbf{H}^n_{\mathbb{R}}$.

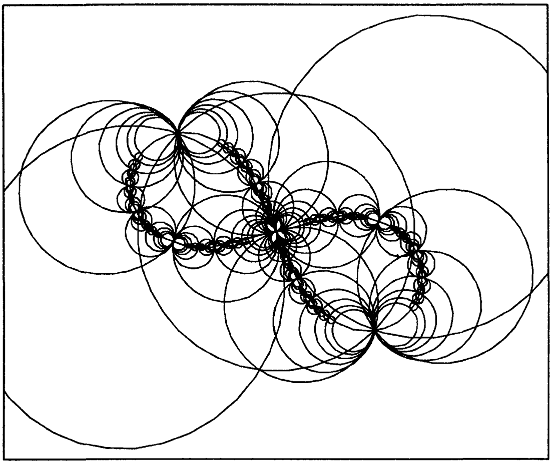

FIG. 4.11. An ideal triangle group: top view

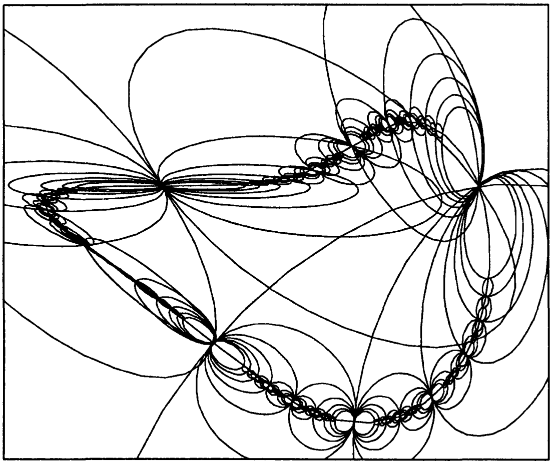

FIG. 4.12. An ideal triangle group: generic view

Orthogonal projection $\Pi_{\mathbb{R}} : \overline{\mathbf{H}^n_{\mathbb{C}}} \longrightarrow \overline{\mathbf{H}^n_{\mathbb{R}}}$ is Γ-equivariant. $\Pi_{\mathbb{R}}^{-1}(\partial\mathbf{H}^n_{\mathbb{C}}) = \partial\mathbf{H}^n_{\mathbb{R}}$ so

$$\Pi_{\mathbb{R}}(\partial\mathbf{H}^n_{\mathbb{C}} - \partial\mathbf{H}^n_{\mathbb{R}}) \subset \mathbf{H}^n_{\mathbb{R}}$$

and since Γ acts properly on $\mathbf{H}^n_{\mathbb{C}} \supset \mathbf{H}^n_{\mathbb{R}}$, Γ acts properly on $\partial\mathbf{H}^n_{\mathbb{C}} - \partial\mathbf{H}^n_{\mathbb{R}}$.

Here is a specific example of such a group. Consider an $\mathbb{R}$-circle R and three distinct points $p_1, p_2, p_3 \subset R$. Let C_{ij} denote the unique chain containing p_i and p_j and let ι_{ij} denote inversion in C_{ij}. Then the group Γ generated by $\iota_{12}, \iota_{23}, \iota_{31}$ is discrete (it is an ideal triangle group acting on the real hyperbolic plane, as drawn in Fig. 4.9) and its limit set is R. Such a group is pictured in Figs. 4.10–4.12.

These pictures show the configuration of chains $C_{12}, C_{2\dot{3}}, C_{31}$, their images under words of small length, and the limit set $\mathbb{R}$-circle R. These groups are flexible as subgroups of $\mathbf{PU}(2,1)$; their nearby deformations remain discrete with limit sets fractal Jordan curves. For more information, see [69, 70, 72, 75, 153].

5

BISECTORS AND SPINAL SPHERES

In $\mathbf{H}_{\mathbb{C}}^n$ totally geodesic (real) hypersurfaces do not exist, unlike in (real) hyperbolic space. A reasonable substitute are the metric *bisectors*. These real analytic hypersurfaces come about as close as possible to being totally geodesic. They are minimal hypersurfaces of cohomogeneity 1, all equivalent under the automorphisms of $\mathbf{H}_{\mathbb{C}}^n$, and enjoy two natural decompositions into totally geodesic submanifolds. In a precise sense (Theorem 6.1.2), bisectors are dual to (real) geodesics, and are also parametrized by an unoriented pair of points on $\partial\mathbf{H}_{\mathbb{C}}^n$.

The intersection of a bisector with the absolute is a smooth hypersurface in $\partial\mathbf{H}_{\mathbb{C}}^n$ diffeomorphic to a sphere. We call such hypersurfaces *spinal spheres.* (Mostow calls a bisector a *spinal surface.*)

5.1 Two decompositions of bisectors

Giraud [65] and Mostow [128] describe the structure of bisectors in terms of a foliation by complex hyperplanes orthogonal to the spine σ. This structure theorem parametrizes bisectors in terms of the unordered pair $\partial\sigma \subset \partial\mathbf{H}_{\mathbb{C}}^n$ and shows that there is a bijective correspondence between bisectors and geodesics (Theorem 5.1.6). On the absolute, the slice decomposition of a bisector $\mathfrak{E}$ defines a decomposition of the spinal sphere $\partial\mathfrak{E}$ into hyperchains, which we also call *slices.*

Bisectors enjoy another decomposition into totally real, totally geodesic submanifolds, which we call the *meridian decomposition.* (Spinal spheres have a corresponding decomposition into $\mathbb{R}$-spheres.) In this and subsequent chapters, we shall explore and exploit the interaction between these two decompositions.

5.1.1 *Bisectors equidistant from a pair of points*

Let $z_1, z_2 \in \mathbf{H}_{\mathbb{C}}^n$ be two distinct points. The *bisector equidistant from z_1 and z_2* (or the *bisector of* $\{z_1, z_2\}$) is defined as

$$\mathfrak{E}\{z_1, z_2\} = \{z \in \mathbf{H}_{\mathbb{C}}^n \mid \rho(z_1, z) = \rho(z_2, z)\}.$$

An *equidistant hypersurface* or *bisector* is a subset $\mathfrak{E} = \mathfrak{E}\{z_1, z_2\}$ for some pair z_1, z_2. (Mostow also calls such subsets "spinal surfaces.") In this case, we shall say that $\mathfrak{E}$ is *equidistant from z_1* (or *equidistant from z_2*).

The boundary of a bisector in $\partial\mathbf{H}_{\mathbb{C}}^n$ is by definition *a spinal sphere* in $\partial\mathbf{H}_{\mathbb{C}}^n$.

Let z_1, z_2 be as above. Let $\Sigma \subset \mathbf{H}_{\mathbb{C}}^n$ be the complex geodesic spanned by z_1 and z_2; we call Σ the *complex spine (or $\mathbb{C}$-spine) of $\mathfrak{E}$ (with respect to $\{z_1, z_2\}$).* The *spine of $\mathfrak{E}$ (with respect to $\{z_1, z_2\}$)* equals

$$\sigma\{z_1, z_2\} = \mathfrak{E}\{z_1, z_2\} \cap \Sigma = \{z \in \Sigma \mid \rho(z_1, z) = \rho(z_2, z)\};$$

that is, the orthogonal bisector of the geodesic segment joining z_1 and z_2 in Σ. (Compare Fig. 5.1.) We shall soon prove that the spine and complex spine of $\mathfrak{E}$ depend intrinsically on the hypersurface $\mathfrak{E}$—and not on the pair $\{z_1, z_2\}$ used to define $\mathfrak{E}$.

5.1.2 *Slice decomposition*

Theorem 5.1.1 *(Giraud, Mostow) Let $\mathfrak{E}$, Σ and σ be as above. Let $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ be orthogonal projection onto Σ. Then*

$$\mathfrak{E} = \Pi_\Sigma^{-1}(\sigma) = \bigcup_{s \in \sigma} \Pi_\Sigma^{-1}(s).$$

Proof Let $z \in \mathbf{H}^n_{\mathbb{C}}$. Then Lemma 3.2.13 implies

$$\cosh\left(\frac{\rho(z, z_j)}{2}\right) = \cosh\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right) \cosh\left(\frac{\rho(\Pi_\Sigma(z), z_j)}{2}\right)$$

for $j = 1, 2$ so

$$\begin{aligned} z \in \mathfrak{E}\{z_1, z_2\} &\Longleftrightarrow \rho(z, z_1) = \rho(z, z_2) \\ &\Longleftrightarrow \rho(\Pi_\Sigma(z), z_1) = \rho(\Pi_\Sigma(z), z_2) \\ &\Longleftrightarrow \Pi_\Sigma(z) \in \sigma\{z_1, z_2\} \end{aligned}$$

as desired. □

Definition 5.1.2 *(Mostow) The complex hyperplanes $\Pi_\Sigma^{-1}(s)$, for $s \in \sigma$, are called the* slices *of $\mathfrak{E}$ (with respect to $\{z_1, z_2\}$).*

Since orthogonal projection $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ is a real analytic fibration, we obtain:

Corollary 5.1.3 *A bisector is a real analytic real hypersurface in $\mathbf{H}^n_{\mathbb{C}}$ diffeomorphic to $\mathbb{R}^{2n-1}$. A spinal sphere is a real analytic real hypersurface in $\partial\mathbf{H}^n_{\mathbb{C}}$ diffeomorphic to S^{2n-2}.*

(More general results, valid in simply connected complete manifolds of negative sectional curvature, are discussed in Ehrlich–ImHof [45] and ImHof [91].)

5.1.3 *Slices, spine and complex spine are intrinsic*

The slices of a bisector $\mathfrak{E}$ are independent of the defining pair $\{z_1, z_2\}$ because of the following:

Lemma 5.1.4 *A bisector is Levi-flat and its maximal holomorphic submanifolds are its slices.*

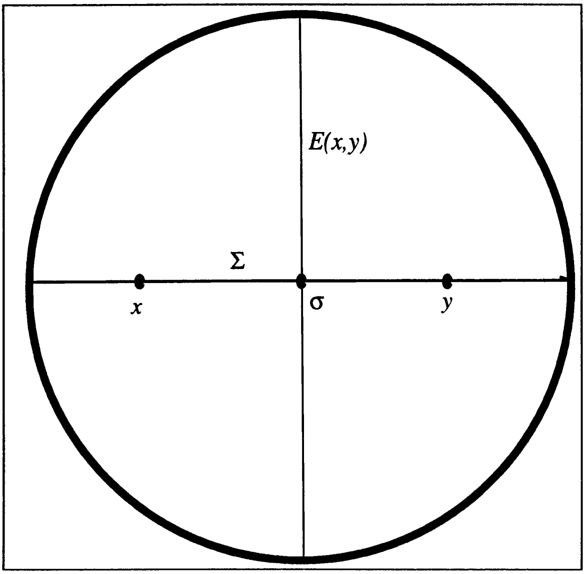

FIG. 5.1. The slice decomposition of a bisector

Proof Clearly each slice is a $\mathbb{C}$-hyperplane in $\mathbf{H}^n_{\mathbb{C}}$ and hence is a holomorphic submanifold. Furthermore each $x \in \mathfrak{E}$ lies in the unique slice $S_x = \Pi_\Sigma^{-1}(\Pi_\Sigma(x))$ and $T_x S_x \subset T_x\mathfrak{E}$ is a complex subspace of real codimension 1. Thus $T_x S_x$ is the maximal complex subspace of $T_x\mathfrak{E}$ and the integral submanifolds of the CR-structure on $\mathfrak{E}$ are the slices $\mathfrak{E}$. □

Corollary 5.1.5 *Suppose that $\{z_1, z_2\}$ and $\{z_1', z_2'\}$ are two pairs of distinct points in $\mathbf{H}^n_{\mathbb{C}}$ such that $\mathfrak{E} = \mathfrak{E}\{z_1, z_2\}$ and $\mathfrak{E}' = \mathfrak{E}\{z_1', z_2'\}$ are equal. Then the slices (respectively spine, complex spine) of $\mathfrak{E}$ with respect to $\{z_1, z_2\}$ equal the slices (respectively spine, complex spine) of $\mathfrak{E}'$ with respect to $\{z_1', z_2'\}$.*

Proof By Lemma 5.1.4, the decomposition of a bisector into slices is the foliation tangent to its induced CR-structure; hence the slices of $\mathfrak{E}$ depend intrinsically on the geometry of $\mathfrak{E}$. Since each slice is orthogonal to Σ, any pair of distinct slices of a bisector are ultraparallel and their unique common orthogonal complex geodesic equals Σ. Thus $\mathfrak{E}$ completely determines the complex spine Σ. Since $\sigma = \Sigma \cap \mathfrak{E}$, the bisector $\mathfrak{E}$ uniquely determines its spine. □

The slices, spine and complex spine of a bisector can be characterized by Riemannian properties (§5.5).

We call the endpoints of the spine of $\mathfrak{E}$ the *vertices* of the bisector $\mathfrak{E}$ (or the spinal sphere $\mathfrak{S}$). Since a geodesic $\sigma \subset \mathbf{H}^n_{\mathbb{C}}$ is completely determined by the unordered pair $\partial\sigma \subset \partial\mathbf{H}^n_{\mathbb{C}}$ consisting of its endpoints, bisectors (or spinal spheres) are completely parametrized by unordered pairs of distinct points in $\partial\mathbf{H}^n_{\mathbb{C}}$.

5.1.4 *Duality between bisectors and geodesics*

We have associated to every bisector a geodesic—its spine. Conversely if $\sigma \subset \mathbf{H}^n_{\mathbb{C}}$ is a geodesic a unique bisector $\mathfrak{E} = \mathfrak{E}_\sigma$ has spine σ. Given σ, there exists a unique complex geodesic $\Sigma \supset \sigma$. Let $R_\sigma : \Sigma \longrightarrow \Sigma$ be the unique reflection whose fixed-point set is σ. Choose an arbitrary point $z_1 \in \Sigma - \sigma$ and let $z_2 = R_\sigma(z_1)$. Then

$$\mathfrak{E}\{z_1, z_2\} = \Pi_\Sigma^{-1}(\sigma)$$

is a bisector having σ as spine.

Theorem 5.1.6 *There is a natural bijective correspondence between bisectors in $\mathbf{H}^n_{\mathbb{C}}$ and geodesics in $\mathbf{H}^n_{\mathbb{C}}$.*

§6.1.3 interprets this symplectically.

The preceding argument also gives the following characterization of the complex spine in terms of the equidistance relation:

Theorem 5.1.7 *Let $\mathfrak{E}$ be a bisector with spine σ and complex spine Σ. Then $\mathfrak{E}$ is equidistant from $z \in \mathbf{H}^n_{\mathbb{C}}$ if and only if $z \in \Sigma - \sigma$.*

Proof The definitions of Σ and σ imply that if $\mathfrak{E}$ is equidistant from $z \in \mathbf{H}^n_{\mathbb{C}}$ then $z \in \Sigma - \sigma$. Conversely, suppose that $z \in \Sigma - \sigma$ and let $z' \in \Sigma - \sigma$ be the image of z under reflection in σ. Then z, z' span Σ and σ is the geodesic in Σ equidistant from z, z' as desired. □

Definition 5.1.8 *Two bisectors $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathbf{H}^n_{\mathbb{C}}$ are* (strictly) coequidistant *if*

$$(\Sigma_1 - \sigma_1) \cap (\Sigma_2 - \sigma_2) \neq \emptyset$$

where Σ_i (respectively σ_i) denotes the complex spine (respectively spine) of $\mathfrak{E}_i$. By Theorem 5.1.7, this condition is equivalent to the existence of distinct points $z_0, z_1, z_2 \in \mathbf{H}^n_{\mathbb{C}}$ such that $\mathfrak{E}_i = \mathfrak{E}\{z_0, z_1\}$. We say that $\mathfrak{E}_1, \mathfrak{E}_2$ are (weakly) coequidistant *if $\Sigma_1 \cap \Sigma_2 \neq \emptyset$.*

5.1.5 *Vectors polar to slices of bisectors*

The family of slices of a bisector is analogous to the family of points comprising a geodesic. Here is an explicit formula for vectors polar to the slices of a bisector with given vertices:

Exercise 5.1.9 *Let Q_+ and Q_- be linearly independent null vectors in $\mathbb{C}^{n,1}$. Then necessarily $\langle Q_+, Q_- \rangle \neq 0$. Let $q_\pm = \mathbb{P}(Q_\pm)$ be the corresponding points on $\partial\mathbf{H}^n_{\mathbb{C}}$. Replace $Q_\pm$ by scalar multiples to assume that $\langle Q_+, Q_- \rangle = -2$. Define, for $t > 0$,*

$$\begin{aligned} S(t) &= S_{(q_-, q_+)}(t) \\ &= \frac{1}{2}\left(t\, Q_- + t^{-1}\, Q_+\right). \end{aligned}$$

Then $S(t)$ satisfies $\langle S(t), S(t) \rangle = 1$ and is polar to a slice of the bisector with vertices $q_\pm$.

For the pair

$$Q_- = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad Q_+ = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

the vectors

$$S(e^{-d/2}) = \begin{bmatrix} 0 \\ \sinh(d/2) \\ \cosh(d/2) \end{bmatrix}$$

are polar to the pencil of slices.

5.1.6 *The meridianal decomposition*

Bisectors decompose into totally real totally geodesic subspaces as well as into complex hyperplanes:

Theorem 5.1.10 *Let $\sigma \subset \mathbf{H}^n_{\mathbb{C}}$ be a geodesic. For each $k \geq 2$, the bisector $\mathfrak{E}$ having spine σ is the union of all $\mathbb{R}^k$-planes containing σ.*

Proof Let Σ be the complex geodesic containing σ and choose points $z_1, z_2 \in \Sigma$ such that $\mathfrak{E} = \mathfrak{E}\{z_1, z_2\}$. Let P be an $\mathbb{R}^n$-plane containing σ; we first show that $P \subset \mathfrak{E}$. It suffices to show that for $y \in P$ that $\rho(z_1, y) = \rho(z_2, y)$. Let ι_P denote anti-involution in P; by Lemma 3.1.12, ι_P preserves Σ and acts on Σ by reflection in $\sigma \subset \Sigma$. Thus ι_P interchanges z_1 and z_2 and, as desired,

$$\rho(z_1, y) = \rho(\iota_P(z_1), \iota_P(y)) = \rho(z_2, y).$$

Conversely, each $y \in \mathfrak{E}$ lies in a totally real totally geodesic subspace containing σ. We first derive the formula for the vertices of the bisector equidistant from a pair of points $z_1, z_2 \in \mathbf{H}^n_{\mathbb{C}}$: Represent z_1, z_2 by vectors $Z_1, Z_2 \in \mathbb{C}^{n,1}$. Normalize this pair so that $\langle Z_1, Z_1 \rangle = \langle Z_2, Z_2 \rangle = -1$ and $\langle Z_1, Z_2 \rangle > 0$. Then

$$\langle Z_1, Z_2 \rangle = \cosh\left(\frac{\rho(z_1, z_2)}{2}\right).$$

The complex geodesic Σ containing z_1, z_2 consists of the image in $\mathbf{H}^n_{\mathbb{C}}$ of the complex 2-plane spanned by Z_1 and Z_2. A $\mathbb{C}$-linear combination $aZ_1 + bZ_2$ determines a point equidistant from z_1 and z_2 only if $|a| = |b|$. In particular the endpoints of the bisector in Σ equidistant from z_1 and z_2 are represented by vectors

$$V_1 = Z_1 + \xi Z_2, \qquad V_2 = Z_2 + \xi Z_1$$

where

$$\xi = \operatorname{sech}\left(\frac{\rho(z_1, z_2)}{2}\right) + i \tanh\left(\frac{\rho(z_1, z_2)}{2}\right).$$

If $y \in \mathfrak{E}$, there exists $Y \in \mathbb{C}^{n,1}$ representing y; the distance formula (3.4) implies

$$\langle Y, Z_1\rangle\langle Z_1, Y\rangle = \langle Y, Z_2\rangle\langle Z_2, Y\rangle$$

and

$$\langle Y, V_1\rangle\langle V_2, Y\rangle = \langle Y, Z_1\rangle\langle Z_2, Y\rangle + \langle Y, Z_2\rangle\langle Z_1, Y\rangle \\ + \xi\langle Y, Z_1\rangle\langle Z_1, Y\rangle + \bar{\xi}\langle Y, Z_2\rangle\langle Z_2, Y\rangle$$

is real. By replacing Y by a complex multiple, we may assume that both $\langle Y, V_1\rangle$ and $\langle Y, V_2\rangle$ are real. Since $\langle V_1, V_2\rangle$ is real, Y, V_1, V_2 span a totally real linear subspace of $\mathbb{C}^{n,1}$ which projects to an $\mathbb{R}^2$-plane in $\mathbf{H}^n_{\mathbb{C}}$ containing σ and y. (Compare the discussion in §2.2.) □

Intersecting these totally real totally geodesic subspaces with $\partial\mathbf{H}^n_{\mathbb{C}}$, we obtain:

Corollary 5.1.11 *Let $u_1, u_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ be distinct points. Then for each $k \geq 1$ the spinal sphere with vertices u_1, u_2 is the union of all $\mathbb{R}^k$-spheres containing u_1 and u_2.*

We refer to the $\mathbb{R}^n$-planes (respectively $\mathbb{R}$-spheres) containing the spine (respectively the vertices) as the *meridians* of the bisector $\mathfrak{E}$ (respectively spinal sphere $\mathfrak{S}$). Figure 5.2 depicts one meridian of a spinal sphere drawn by its slices.

With the pair of decompositions of a bisector, we can now prove Theorem 3.2.12 as promised in §3.2.7.

Proof of Theorem 3.2.12 Let $\mathfrak{S}$ be the bisector with spine α. By the slice decomposition (§5.1.2), the first condition is equivalent to $a \in \mathfrak{S}$. By the meridian decomposition (§5.1.6), the second condition is equivalent to $a \in \mathfrak{S}$. □

5.1.7 *Example: the horizontal plane*

To illustrate these ideas, we discuss one example in detail: the bisector $\mathfrak{E}$ with vertices $p_0, p_\infty \in \partial\mathbf{H}^n_{\mathbb{C}}$ (see §4.1).

$$\tilde{p}_\infty = \begin{bmatrix} 0' \\ -1 \\ 1 \end{bmatrix}, \qquad \tilde{p}_0 = \begin{bmatrix} 0' \\ 1 \\ 1 \end{bmatrix}.$$

The complex spine of $\mathfrak{E}$ is the complex geodesic $\Sigma = \{0\} \times \mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^n_{\mathbb{C}}$ bounded by the vertical axis $\mathbb{V} \subset \mathcal{H}$. Orthogonal projection $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ is

$$\begin{pmatrix} z' \\ z_n \end{pmatrix} \longmapsto \begin{pmatrix} 0' \\ z_n \end{pmatrix}.$$

Since the geodesic σ spanned by p_0 and p_∞ equals $\sigma = \{0\} \times \mathbf{H}^1_{\mathbb{R}}$, in both the ball and paraboloid models the bisector

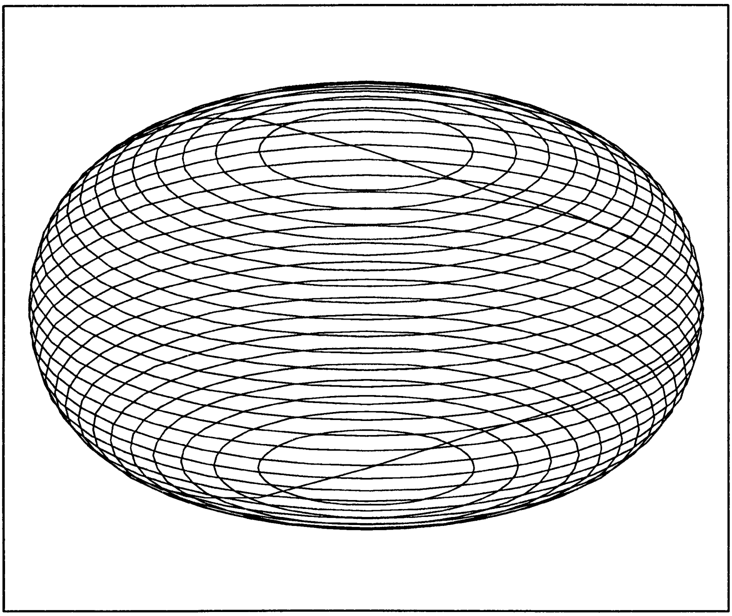

FIG. 5.2. A meridian and several slices of the unit spinal sphere

$$\mathfrak{C} = \{z \in \mathbb{B}^n \mid z_n \in \mathbb{R}\} = \{w \in \mathfrak{H}^n \mid w_n \in \mathbb{R}\}$$

consists of all vectors with last coordinate real. Its slices are

$$S_t = \{(z', t) \in \mathbb{B}^n \mid \|z'\|^2 < 1 - t^2\}$$

for $|t| < 1$ (in the ball model) and

$$S_u = \{(w', u) \in \mathfrak{H}^n \mid \|w'\|^2 < 2u\}$$

for $u > 0$ (in the paraboloid model). The corresponding spinal sphere $\partial\mathfrak{C} \subset \partial\mathbf{H}^n_{\mathbb{C}}$ is given in Heisenberg coordinates by the *horizontal hyperplane*

$$\{(\zeta, v) \in \mathcal{H} \mid v = 0\} \approx \mathbb{C}^{n-1}$$

(which corresponds to the contact hyperplane at the origin). The slices are the concentric round spheres centered at the origin and the meridians correspond to the totally real linear subspaces of $\mathbb{C}^{n-1}$. (If $n = 2$, then the meridians are just the Euclidean straight lines in the horizontal plane (infinite $\mathbb{R}$-circles) which pass through the origin.) This bisector is clearly invariant under $\mathbf{Sim}_0(\mathcal{H})$.

All bisectors and spinal spheres having ∞ as a vertex are obtained from these by Heisenberg translation. In particular the spinal sphere with vertices ∞ and $(\zeta_0, v_0) \in \mathcal{H}$ is

$$E_{(\zeta_0,v)} = \{(\zeta, v) \in \mathcal{H} \mid v = v_0 - 2\,\mathrm{Im}\langle\langle \zeta, \zeta_0 \rangle\rangle\}.$$

We distinguish this subset of $\mathcal{H}$ from its tangent space $T_{(\zeta_0,v)}E_{(\zeta_0,v)}$, which is the contact hyperplane $E_(\zeta_0, v)$, although it is tempting to identify these two spaces. Following Schwartz [153], we call $E_{(\zeta_0,v)} \subset \mathcal{H}$ the *prolongation* of

$$E_{(\zeta_0,v)} \subset T_{(\zeta_0,v)}\mathcal{H}.$$

The identification of the contact hyperplane with its prolongation arises from the usual identification

$$T_{(\zeta_0,v)}\mathcal{H} \longrightarrow \mathcal{H}$$

of an affine space with its tangent space arising from *Euclidean* translations of $\mathcal{H}$.

Exercise 5.1.12 *(Compare Gusevskii–Parker [82] and Schwartz [153].) Let $W \subset \mathcal{H}$ be a Euclidean plane which does not contain a vertical line. Then there exists a unique $w_0 \in W$ such that $W \cup \{\infty\}$ is the spinal sphere $\mathcal{S}\{w_0, \infty,\}$. In other words, $W \subset \mathcal{H}$ identifies with the prolongation of the contact hyperplane $E_{w_0} \subset T_{w_0}\mathcal{H}$.*

In contrast, spinal spheres which contain ∞ but not as a vertex can be quite complicated; Figures 5.5– 5.8 depict such surfaces.

5.1.8 *Example: the unit spinal sphere*

Here is an example of a spinal sphere which is a compact subset of $\mathcal{H}$, and is somewhat analogous to a "unit sphere" in Heisenberg space. Consider the bisector $\mathfrak{E}_0$ with vertices $(0, \pm 1) \in \mathcal{H}$. (Its complex spine is again bounded by the vertical axis.) In the ball model, the corresponding spine is the geodesic

$$\left\{ \begin{bmatrix} 0' \\ it \end{bmatrix} \in \mathbb{B}^n \,\middle|\, -1 < t < 1 \right\}$$

and

$$\mathfrak{E} = \{(z', it) \in \mathbb{B}^n \mid \|z'\|^2 < 1 - t^2\}$$

in the ball model. In the paraboloid model,

$$\mathfrak{E} = \left\{ (w', e^{i\theta}) \in \mathfrak{H}^n \,\middle|\, \|w'\|^2 < 2\cos(\theta),\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right\}$$

It is bounded by the spinal sphere $\partial\mathfrak{E}$ described by

$$\|\zeta\|^4 + v^2 = 1$$

in Heisenberg coordinates. This spinal sphere is depicted in Fig. 5.3.

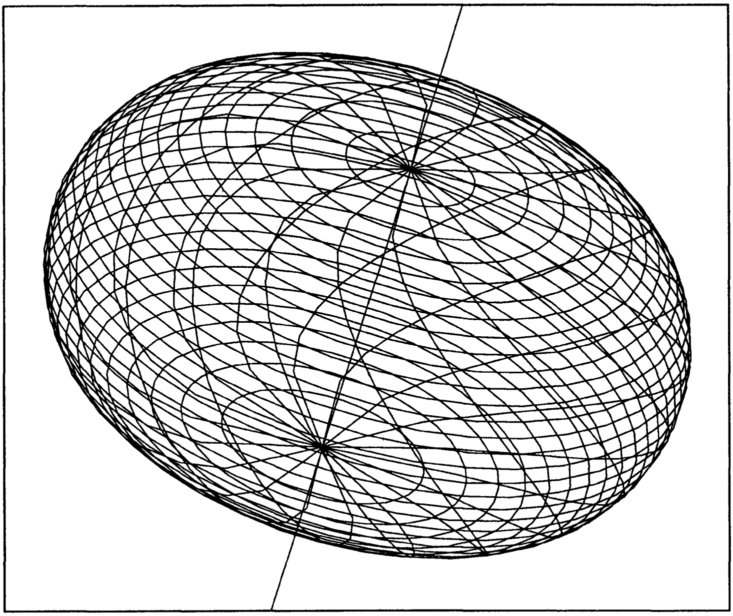

FIG. 5.3. Slices, meridian and complex spine of a vertical spinal sphere

5.1.9 *Vertical spinal spheres*

A bisector or spinal sphere whose complex spine Σ is bounded by a vertical chain is called *vertical.* Equivalently Σ contains p_∞. Every vertical spinal sphere is either a contact hyperplane $E_{(\zeta_0,v_0)}$ (if one of the vertices is ∞) or obtained from the bisector $\mathfrak{C}_0$ above by Heisenberg translations and dilations.

One can define (following [102]) a geometry in which vertical spinal spheres bound metric balls. A convenient "center" of a vertical spinal sphere is given by the midpoint of the two vertices (intrinsically defined as the harmonic conjugate of ∞ with respect to the two vertices, taken along the complex spine). Use the *gauge*

$$\begin{aligned} |\ | : \mathcal{H} &\longrightarrow \mathbb{R} \\ (\zeta, v) &\longmapsto (\|\zeta\|^4 + v^2)^{1/4} \end{aligned}$$

to define a distance

$$\begin{aligned} d_G : \mathcal{H} \times \mathcal{H} &\longrightarrow \mathbb{R} \\ ((\zeta_1, v_1), (\zeta_2, v_2)) &\longmapsto |(\zeta_1, v_1)^{-1}(\zeta_2, v_2)| \end{aligned}$$

making $\mathcal{H}$ into a metric space in which the metric spheres are vertical spinal spheres. For example, the "unit" spinal sphere (discussed in §5.1.8) bounds the unit ball centered at the origin in Heisenberg space. Furthermore this metric is invariant under Heisenberg isometries and scales uniformly under Heisenberg similarity transformations

$$d_G(\phi x_1, \phi x_2) = \lambda(\phi) d_G(x_1, x_2)$$

for $x_1, x_2 \in \mathcal{H}$ and $\phi \in \mathbf{Sim}(\mathcal{H})$. It follows that $(\mathcal{H}, d_G)$ is quasi-isometric to Heisenberg space with its Carnot–Carathéodory metric. Furthermore the Carnot–Carathéodory metric is the inner metric corresponding to d_G. (Compare Cygan [37], Gromov [79], Miner [122], Mitchell [121], Pansu [134] and Parker [135].)

5.1.10 *Great spinal spheres*

Just as "horizontal" spinal spheres have a vertex at ∞, *great spinal spheres* are characterized by the property that their spine contains the origin (in the ball model). These spinal spheres are analogous to the great circles in elliptic geometry. A bisector $\mathfrak{E}$ whose spine σ contains the origin in $\mathbb{B}^n$ is represented by the intersection of an $\mathbb{R}$-linear hyperplane in $\mathbb{C}^n$ with $\mathbb{B}^n$. To see this, let $v \in T_0\mathbb{B}^n \approx \mathbb{C}^n$ be a vector tangent to σ; then a normal vector to the corresponding bisector equals the image $\mathbb{J}v$ and consists of all vectors $u \in \mathbb{B}^n$ such that

$$\mathrm{Im}\langle\!\langle u, v\rangle\!\rangle = 0.$$

In particular its slices are the $\mathbb{C}$-hyperplanes

$$\langle\!\langle u, v\rangle\!\rangle = t$$

for $-1 < t < 1$ and its meridians are the totally real linear subspaces of $\mathbb{C}^n$ containing v.

5.1.11 *Slices and meridians of spinal spheres*

Let $\mathfrak{S} \subset \mathcal{H}$ be a spinal sphere. The CR-structure on $\mathcal{H}$ restricts to a hyperplane field on $\mathfrak{S}$ which has singularities precisely at its vertices. In the case $n = 2$, this hyperplane field is a *line field* on $\mathfrak{S}$ whose integral curves are precisely the meridians. The following characterization of slices of a spinal sphere is implicit in Cartan [21]:

Theorem 5.1.13 *The only hyperchains contained in a spinal sphere are its slices.*

Proof Since $\mathbf{PU}(n,1)$ acts transitively on the set of spinal spheres, it suffices to consider the spinal sphere $\mathcal{S}$ with vertices ∞ and $(0', 0)$ which is the horizontal hyperplane $v = 0$. Its slices are the chains which are given by the Euclidean spheres centered at the origin. On the other hand, the v-coordinate of a hyperchain c is constant (by (4.12)) if and only if its center lies on the vertical axis. Hence a hyperchain which lies on $\mathcal{S}$ must be one of the slices of $\mathcal{S}$. $\square$

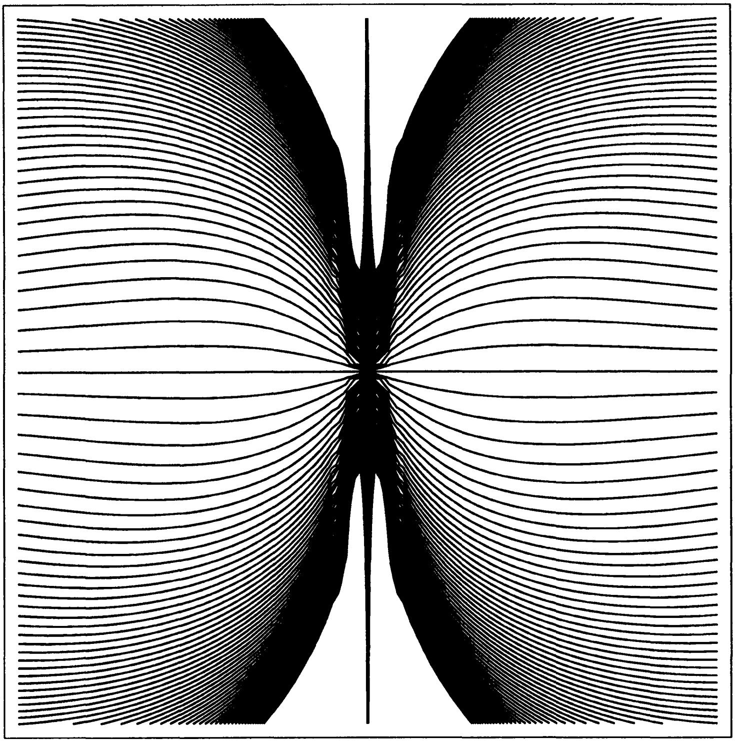

FIG. 5.4. Level sets of function defining spinal with vertices $(\pm 1, 0)$

5.1.12 *Infinite spinal spheres*

Spinal spheres passing through ∞ (but not having ∞ as a vertex) have a particularly interesting form. (Examples are depicted in Figs. 5.5–5.8.)

Exercise 5.1.14 *Show that the spinal sphere* $\mathfrak{S}$ *with vertices* $(1,0)$ *and* $(-1,0)$ *in Heisenberg coordinates is given by*

$$v = -\frac{y}{x}\left(1 + x^2 + y^2\right)$$

The vertical slice—the slice containing ∞*—is defined by* $x = y = 0$*. The horizontal meridian—the meridian containing* ∞*—is defined by* $v = y = 0$*. Compute the other slices and meridians of* $\mathfrak{S}$ *in Heisenberg coordinates.*

Perhaps the easiest way to do the above calculation is to use Cartan's angular invariant, discussed in §7.1. In particular, see Corollary 7.1.5. The level sets of the function $v = v(x, y)$ are depicted in Fig. 5.4.

5.2 Automorphisms of bisectors

5.2.1 *Bisectors are characterized by their vertices*

If $\mathfrak{E}$ is a bisector, the endpoints of its spine in $\partial\mathbf{H}_{\mathbb{C}}^n$ are called the *vertices* of $\mathfrak{E}$. Since a geodesic in $\partial\mathbf{H}_{\mathbb{C}}^n$ is determined by the (unordered pair of distinct) endpoints, and $\mathbf{PU}(n,1)$ acts transitively on the set of such pairs, $\mathbf{PU}(n,1)$ acts transitively on geodesics in $\mathbf{H}_{\mathbb{C}}^n$. Thus Theorem 5.1.6 implies that $\mathbf{PU}(n,1)$ acts transitively on bisectors. Furthermore the stabilizer in $\mathbf{PU}(n,1)$ of a bisector equals the stabilizer of its spine, a group which is isomorphic to

$$(\mathbb{Z}/2 \ltimes \mathbb{R}_+) \times \mathbf{U}(n-1)$$

(which identifies with $\mathbb{Z}/2 \ltimes \mathbb{C}^*$ for $n = 2$). The identity component $\mathbb{R}_+ \times \mathbf{U}(n-1)$ is generated by the hyperbolic 1-parameter group $\mathbb{R}_+$ stabilizing σ and a group $\mathbf{U}(n-1)$ of unitary "rotations" leaving invariant each slice and pointwise fixing σ. (For $n = 2$, this group $\mathbf{U}(1)$ consists of complex reflections.) The cyclic group $\mathbb{Z}/2$ is generated by inversion in a slice of $\mathfrak{E}$.

5.2.2 *Inversions in slices*

In a constant curvature geometry (Euclidean space, real hyperbolic space or the sphere), a bisector H is equidistant from *any* point $x \notin H$. For there is a unique reflection ι_H whose fixed-point set is H, which is totally geodesic. Moreover, H equals the hypersurface equidistant from x and its reflected image $\iota_H(x)$. However, in $\mathbf{H}_{\mathbb{C}}^n$, no isometry exists whose fixed-point set is a real hypersurface, since otherwise such a hypersurface would be totally geodesic. Instead for each slice S of a bisector $\mathfrak{E}$, inversion ι_S in S leaves $\mathfrak{E}$ invariant, but pointwise fixes only the slice S. Inversion ι_S acts by reflection on the spine σ, interchanging the two endpoints and fixing the point $S \cap \sigma$.

Theorem 5.2.1 *Let $S \subset \mathbf{H}_{\mathbb{C}}^n$ be a complex hyperplane and let ι_S denote inversion in S.*

1. *Suppose that $u_1, u_2 \in \partial\mathbf{H}_{\mathbb{C}}^n$. Then S is a slice of the bisector $\mathfrak{E}$ having vertices u_1, u_2 if and only if ι_S interchanges u_1 and u_2.*
2. *Suppose that $z_1, z_2 \in \mathbf{H}_{\mathbb{C}}^n$. Then ι_S interchanges z_1 and z_2 if and only if S is the slice of the bisector $\mathfrak{E}\{z_1, z_2\}$ equidistant from z_1, z_2 which contains the midpoint $\mathbf{mid}(z_1, z_2)$.*

Proof To prove 1, let σ (respectively Σ) denote the geodesic (respectively complex geodesic) through u_1, u_2 and let $\Pi : \mathbf{H}_{\mathbb{C}}^n \longrightarrow \Sigma$ denote orthogonal projection. Suppose that S is a slice of the bisector $\mathfrak{E}$ having σ as spine. Then S intersects Σ orthogonally in the single point $\Pi(S) = x \in \sigma \subset \mathbf{H}_{\mathbb{C}}^n$. Since $S \perp \Sigma$, the inversion ι_S leaves Σ invariant and its restriction of ι_S to Σ is inversion in x. Since $x \in \sigma$, it follows that ι_S leaves σ invariant and interchanges the two endpoints u_1, u_2 of σ.

Conversely, suppose that ι_S interchanges u_1 and u_2. Since σ is the unique geodesic having endpoints u_1 and u_2, inversion ι_S preserves σ and its restriction to σ is reflection in some point $x \in \sigma$. Since x is fixed under ι_S, x must lie in S and since the differential $d\iota_S$ restricted to $T_x\sigma \subset T_x\mathbf{H}^n_{\mathbb{C}}$ is $-\mathbb{I}$, the geodesic σ is orthogonal to the fixed-point set S of ι_S at $x \in \sigma$. Therefore S is a slice of $\mathfrak{E}$. The proof of 1 is complete.

The proof of 2 uses the following lemma:

Lemma 5.2.2 *Let f be an isometry of a Riemannian manifold M and let W be an open subset of the fixed-point set of f. Suppose that $S \subset M$ is an f-invariant totally geodesic submanifold and $x \in S \cap W$ has the property that the differential of $df_x : T_xS \longrightarrow T_xS$ does not have 1 as an eigenvalue. Then W and S are orthogonal at x.*

Proof By the hypothesis on W, the differential df_x restricts to the identity on $T_xW \subset T_xM$ and by the hypothesis on S, the tangent space T_xS lies in the sum of the other eigenspaces of df_x acting on T_xM. Since df_x is a linear isometry of T_xM, eigenspaces for different eigenvalues of df_x are orthogonal. Thus $T_xS \perp T_xW$, and W and S are orthogonal at x. □

Now let $\mu \in \mathbf{H}^n_{\mathbb{C}}$ be the midpoint $\mathbf{mid}(z_1, z_2)$. Suppose first that ι_S interchanges z_1 and z_2. Let Π_S denotes orthogonal projection onto S; then we claim

$$\mu = \Pi_S(z_1) = \Pi_S(z_2). \tag{5.1}$$

Let γ be the geodesic segment from z_1 to z_2. Then ι_S preserves γ and fixes μ:

$$\begin{aligned}\iota_S(\mu) &= \mathbf{mid}(\iota_S(z_1), \iota_S(z_2)) \\ &= \mathbf{mid}(\iota_S(z_2), \iota_S(z_1)) \\ &= \mu,\end{aligned}$$

whence $\mu \in S$. The differential $d\iota_S$ at μ has derivative -1 on $T_\mu\gamma$ and applying Lemma 5.2.2, γ meets S orthogonally at μ. It follows that $\gamma \subset \Pi_S^{-1}(\mu)$, proving (5.1).

We now show that S is a slice of $\mathfrak{E}$. For any $s \in S$,

$$\begin{aligned}\cosh\left(\frac{\rho(z_1, s)}{2}\right) &= \cosh\left(\frac{\rho(z_1, \mu)}{2}\right)\cosh\left(\frac{\rho(\mu, s)}{2}\right) \\ &= \cosh\left(\frac{\rho(z_2, \mu)}{2}\right)\cosh\left(\frac{\rho(\mu, s)}{2}\right) \\ &= \cosh\left(\frac{\rho(z_2, s)}{2}\right)\end{aligned}$$

(by (3.8)) and $s \in \mathfrak{E}\{z_1, z_2\}$. It follows that $S \subset \mathfrak{E}\{z_1, z_2\}$. By Lemma 5.1.4, the maximal holomorphic submanifolds of a bisector are its slices, and a $\mathbb{C}$-hyperplane contained in a bisector must be a slice. Thus S is a slice of $\mathfrak{E}\{z_1, z_2\}$.

Conversely, suppose that S is the slice of $\mathfrak{E}\{z_1, z_2\}$ containing μ and let σ (respectively Σ) be the spine (respectively complex spine) of $\mathfrak{E}\{z_1, z_2\}$. Since (by definition) S is orthogonal to σ at μ, the inversion ι_S preserves σ and restricts to inversion in μ on σ. Thus any geodesic in Σ through μ is invariant under ι_S so σ is ι_S-invariant. Finally, ι_S interchanges z_1 and z_2 (because μ is equidistant from z_1 and z_2) as claimed. The proof of 2 is complete. □

5.2.3 *Inversions in meridians*

Corollary 5.2.3 *Suppose that $P^n \subset \mathbf{H}^n_{\mathbb{C}}$ is an $\mathbb{R}^n$-plane and let ι_P denote inversion in P.*

1. *If $z_1, z_2 \in \mathbf{H}^n_{\mathbb{C}}$, then P is a meridian of $\mathfrak{E}\{z_1, z_2\}$ if and only if ι_P interchanges z_1 and z_2.*
2. *Suppose $u_1, u_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$. Then P is a meridian of the bisector $\mathfrak{E}$ with vertices u_1, u_2 if and only if ι_P fixes u_1 and u_2.*

Proof Let Σ be the complex geodesic containing z_1 and z_2 and let $\sigma \subset \Sigma$ be the spine of $\mathfrak{E}\{z_1, z_2\}$. If ι_P interchanges z_1 and z_2, then clearly $\iota_P(\Sigma) = \Sigma$ and ι_P acts by reflection in $\sigma \subset \Sigma$. In particular σ is fixed by ι_P and thus $\sigma \subset P$. Therefore P is a meridian of $\mathfrak{E}\{z_1, z_2\}$.

Conversely, suppose that P is a meridian of $\mathfrak{E}\{z_1, z_2\}$; then $P \cap \Sigma$ contains σ and since P is totally real, the totally real geodesic submanifold $P \cap \Sigma$ of Σ equals σ. By Lemma 3.1.12, P intersects Σ orthogonally and ι_P restricts on Σ to reflection in σ. It follows that ι_P interchanges z_1 and z_2. The proof of 1 is now complete.

To prove 2, suppose that $\mathfrak{E}$ is a bisector with vertices u_1, u_2 and let σ be (as above) the geodesic containing u_1 and u_2. Then ι_P fixes u_1 and u_2 if and only if $u_1, u_2 \in P$, that is if and only if $\sigma \subset P$, that is P is a meridian of $\mathfrak{E}$. The proof of Corollary 5.2.3 is complete. □

5.2.4 *Bisectors with a given slice*

Theorem 5.2.4 *Given two ultraparallel $\mathbb{C}$-hyperplanes H_1, H_2 in $\mathbf{H}^n_{\mathbb{C}}$, there is a unique bisector $\mathfrak{E}$ having H_1, H_2 as slices.*

Proof Here is an explicit formula. Choose vectors $u_1, u_2 \in \mathbb{C}^{n,1}$ polar to H_1 and H_2, normalized as follows:

$$\langle u_i, u_i \rangle = 1, \qquad \langle u_1, u_2 \rangle > 1.$$

Let

$$u_\pm = u_1 - \left(\langle u_1, u_2 \rangle \pm \sqrt{\langle u_1, u_2 \rangle^2 - 1}\right) u_2.$$

Then u_+ and u_- are null vectors. Let $\mathfrak{E}$ be the bisector having $\mathbb{P}(u_+)$ and $\mathbb{P}(u_-)$ as vertices; we claim that H_1 and H_2 are slices of $\mathfrak{E}$. To this end, suppose that $v \in \mathbb{C}^{n,1}$ is a nonpositive vector corresponding to a point in $\bar{H}_2$ (the case that v corresponds to a point in $\bar{H}_1$ is completely analogous); then $\langle v, u_2 \rangle = 0$ and the Hermitian triple product (compare §2.2.5)

$$\langle v, u_+, u_- \rangle = \langle v, u_1 \rangle \langle u_1, v \rangle \langle u_-, u_+ \rangle$$

is real (since $\langle u_-, u_+ \rangle \in \mathbb{R}$). It follows that $\mathbb{P}(v) \in \mathfrak{E}$ as desired. □

Remark 5.2.5 *As a limiting case of the preceding theorem, observe that given a $\mathbb{C}$-hyperplane $H \subset \mathbf{H}_{\mathbb{C}}^n$ and $q \in \partial\mathbf{H}_{\mathbb{C}}^n - \partial H$, then the bisector with vertices q and $\iota_H(q)$ has H as a slice.*

Remark 5.2.6 *Theorem 5.2.4 is "dual" to the existence of a geodesic joining a pair of points in $\mathbf{H}_{\mathbb{C}}^n$. A bisector is a real pencil of complex hyperplanes, as a geodesic is a real pencil of points.*

5.2.5 *Inverting vertices of spinal spheres*

Here is another characterization of spinal spheres, in terms of their vertices:

Theorem 5.2.7 *Let $q_1, q_2 \in \partial\mathbf{H}_{\mathbb{C}}^n$ be distinct points. Then the spinal sphere $\mathfrak{S}(q_1, q_2)$ having vertices q_1, q_2 consists of all images $\iota_c(q_1)$ of q_1 under inversions ι_c in chains c containing q_2.*

Proof Choose Heisenberg coordinates so that q_1 is the origin $(0,0)$ and q_2 is the point at infinity. Then a chain passing through infinity is a vertical chain and

$$\mathfrak{S}(q_1, q_2) = \{(\zeta, 0) \in \mathcal{H} \mid \zeta \in \mathbb{C}\}$$

is the horizontal plane in Heisenberg coordinates. Suppose that $p \in \mathfrak{S}(q_1, q_2)$; then there exists $\zeta_0 \in \mathbb{C}$ such that $p = (\zeta_0, 0)$. If c is the vertical chain $\zeta = \zeta_0/2$ with inversion

$$\iota_c : (\zeta, v) \mapsto (\zeta_0 - \zeta, v - 2\mathrm{Im}(\overline{\zeta_0}\zeta))$$

then $\iota_c(q_1) = p$. Conversely, any chain through $q_2 = \infty$ has the above form for $\zeta_0 \in \mathbb{C}$ and inverts $(0,0)$ to $(\zeta_0, 0) \in \mathfrak{S}(q_1, q_2)$. □

5.2.6 *Tangent $\mathbb{R}$-circles and unipotent automorphisms*

We prove now some general lemmas in synthetic Heisenberg geometry which will be used later in §9.2.1 to investigate tangential pairs of spinal spheres. Unlike chains, $\mathbb{R}$-circles are not uniquely determined by a tangent direction. The main result (Lemma 5.2.9) characterizes pairs of two $\mathbb{R}$-circles which intersect tangentially at a point $p \in \partial\mathbf{H}_{\mathbb{C}}^n$ in terms of unipotent automorphisms fixing p. The following elementary fact about inverting meridians of spinal spheres in the slices will be used:

Lemma 5.2.8 *Let $\mathfrak{S}$ be a spinal sphere with a slice C. Let ι be inversion in C. Then ι leaves each meridian of $\mathfrak{S}$ invariant.*

Proof (Compare §3.1.12.) Let R be a meridian of $\mathfrak{S}$ and let p, q be the vertices of $\mathfrak{S}$. Then $R \cap C$ consists of two points and R is the unique $\mathbb{R}$-circle containing the four distinct points $p, q, R \cap C$. Since ι interchanges p and q (by (5.1)), it follows that $\iota(R)$ is an $\mathbb{R}$-circle containing $q, p, R \cap C$. By uniqueness $\iota(R) = R$ as desired. □

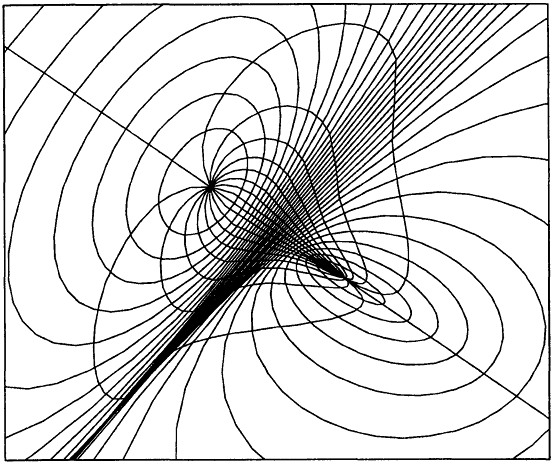

FIG. 5.5. A spinal sphere passing through infinity: another view

Lemma 5.2.9 *Let R_1 and R_2 be two $\mathbb{R}$-circles which intersect at $p \in \partial\mathbf{H}^n_{\mathbb{C}}$. Then R_1 and R_2 are tangent at p if and only if there exists a unipotent automorphism u fixing p such that $u(R_1) = R_2$.*

Proof If u is a unipotent automorphism fixing p with $u(R_1) = R_2$, then the differential du restricts to the identity map on the contact hyperplane $E_p \subset T_p\mathcal{H}$. Thus

$$T_pR_1 = T_pR_2$$

as desired. Conversely, choose Heisenberg coordinates so that $p = \infty$. Then (by Corollary 4.4.4), R_1 and R_2 are represented by Euclidean straight lines, and the stabilizer of p is the Heisenberg similarity group whose unipotent elements form the subgroup U of Heisenberg translations. Let $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}$ denote vertical projection. Then R_1 is the image of R_2 under a Heisenberg translation if and only if $\Pi_{\mathbb{V}}(R_1)$ and $\Pi_{\mathbb{V}}(R_2)$ are parallel straight lines in $\mathbb{C}$. Thus we suppose that $\Pi_{\mathbb{V}}(R_1)$ and $\Pi_{\mathbb{V}}(R_2)$ intersect transversely at $z \in \mathbb{C}$ to obtain a contradiction as follows. There exists a Heisenberg translation $u \in U$ such that R_1 and $R_2' = u(R_2)$ intersect at $\tilde{z} \in \Pi_{\mathbb{V}}^{-1}(z)$. Since u acts trivially on E_p, the $\mathbb{R}$-circles R_1 and R_2' are tangent at p. Let C be a slice of the spinal sphere $\mathfrak{S}$ with vertices p and $\tilde{z}$ and ι inversion in C. Then R_1 and R_2' are meridians of $\mathfrak{S}$. By Theorem 5.2.1 ι interchanges p and $\tilde{z}$ but preserves each meridian R_1, R_2' (by Lemma 5.2.8 above). Therefore ι maps the transverse intersection of R_1 and R_2' at $\tilde{z}$ to the tangential intersection of R_1 and R_2' at p. This contradiction completes the proof of Lemma 5.2.9. □

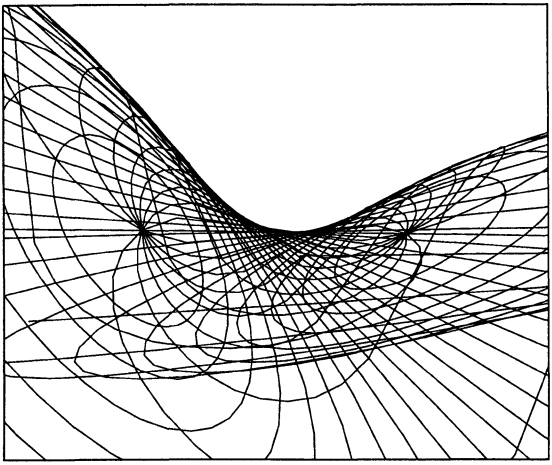

FIG. 5.6. A spinal sphere passing through infinity

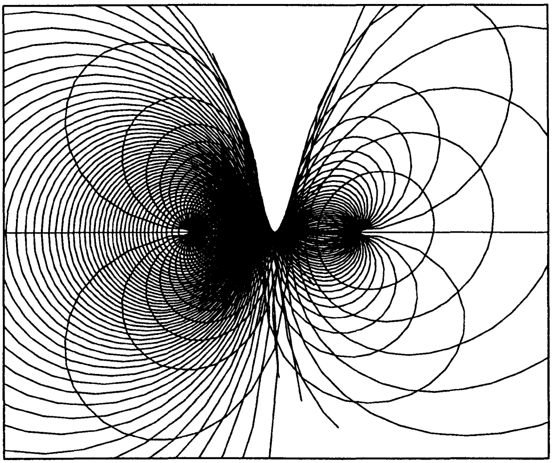

FIG. 5.7. Top view of a spinal sphere with real axis meridian

Exercise 5.2.10 *(Schwartz) Let $S \subset \mathcal{H}$ be a spinal sphere not containing ∞ with meridians M_θ, where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Let $\Pi_{\mathbb{V}} : \mathcal{H} \longrightarrow \mathbb{C}$ be vertical projection. Then the set*

$$\{\mathbf{double}(\Pi_{\mathbb{V}}(M_\theta)) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

of double points of the lemniscates $\Pi_{\mathbb{V}}(M_\theta) \subset \mathbb{C}$ is a circle in $\mathbb{C}$.

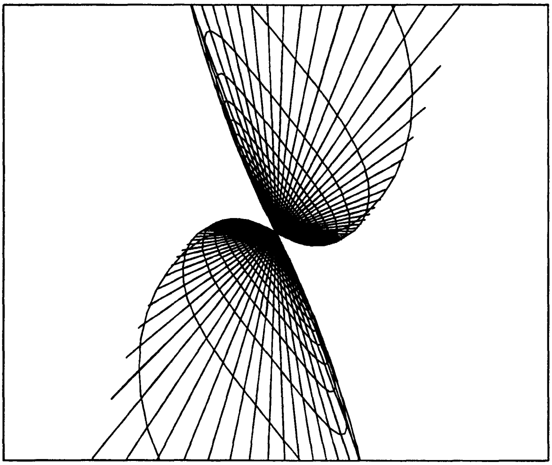

FIG. 5.8. Side view of a spinal sphere with real axis meridian

5.3 Elementary bisector intersections

5.3.1 *Cospinal pairs*

A fundamental problem in complex hyperbolic geometry is the classification of bisector intersections. The simplest case of bisector intersections occurs for pairs of bisectors which have the same complex spine. In that case we say that the bisectors are *cospinal.*

Suppose that $\mathfrak{E}_1, \mathfrak{E}_2$ are distinct bisectors whose respective spines σ_1, σ_2 lie in a common complex geodesic Σ. Let $\Pi_\Sigma : \mathbf{H}_{\mathbb{C}}^n \longrightarrow \Sigma$ be orthogonal projection; The slice decomposition §5.1.2 implies

$$\mathfrak{E}_i = \Pi^{-1}(\sigma_i).$$

We distinguish three cases:

1. σ_1 and σ_2 intersect; in this case $\mathfrak{E}_1 \cap \mathfrak{E}_2$ equals the $\mathbb{C}^{n-2}$-plane $\Pi_\Sigma^{-1}(\sigma_1 \cap \sigma_2)$. As observed in Mostow [128], Theorem 3.2.4, the angle of intersection of these two hypersurfaces is constant (see Theorem 5.3.1 below).
2. σ_1 and σ_2 are parallel; in that case the slices of $\mathfrak{E}_1$ are ultraparallel to the slices of $\mathfrak{E}_2$ although the spinal spheres $\partial\mathfrak{E}_1$ and $\partial\mathfrak{S}_2$ share a common vertex.
3. σ_1 and σ_2 are ultraparallel.

Intersections of cospinal bisectors are particularly simple, as first observed by Mostow [128]:

Theorem 5.3.1 *(Mostow [128], Theorem 3.2.4) Suppose that σ_1 and σ_2 are geodesics in a complex geodesic Σ which intersect at a point p with angle θ. Then at each point $u \in \mathfrak{E}_1 \cap \mathfrak{E}_2 = \Pi_\Sigma^{-1}(p)$, the hypersurfaces $\mathfrak{E}_1$ and $\mathfrak{E}_2$ intersect at angle θ.*

Proof Let ν be the normal bundle of

$$\mathfrak{C}_1 \cap \mathfrak{C}_2 = \Pi_\Sigma^{-1}(p)$$

in $\mathbf{H}^n_\mathbb{C}$. Since the restriction of the differential

$$d\Pi_\Sigma : \nu_u \longrightarrow T_p\Sigma$$

is an isometry, the angle between normal vectors to $\mathfrak{C}_1$ and $\mathfrak{C}_2$ equals that of their images. □

In terms of the 2-dimensional hyperbolic geometry transverse to the foliation of $\mathcal{H} - \mathbb{V}$ by orthogonal hyperchains (see §4.3.6), geodesics correspond to spinal spheres whose vertices lie on $\mathbb{V}$, or equivalently bisectors whose spines lie on the complex geodesic Σ bounded by $\mathbb{V}$. (Figure 4.1 depicts three vertical spinal spheres obtained as the inverse image of an ideal triangle in $\mathfrak{H}^1$ under Π_Σ.) If $\sigma \subset \Sigma$ is a geodesic, and $P \supset \sigma$ is a meridian of $\Pi^{-1}(\sigma)$, then inversion ι_P in P preserves Σ (compare Lemma 3.1.12) and its restriction to Σ is reflection in σ. If P and P' are two meridians, then ι_P and $\iota_{P'}$ differ by an automorphism which is the identity on Σ and acts by a unitary transformation on the normal bundle to Σ (if $n = 2$, such a transformation must necessarily be a complex reflection in Σ). Any two inversions act identically on the leaf space of the orthogonal foliation.

Polyhedra bounded by cospinal bisectors arise naturally when polygons in $\mathbf{H}^1_\mathbb{C}$ are extended to polyhedra in $\mathbf{H}^n_\mathbb{C}$ under orthogonal projection. For example, suppose that Γ is a discrete group acting on $\mathbf{H}^n_\mathbb{C}$ preserving the complex geodesic Σ. Let $u \in \Sigma$ be a point and let $\Delta \subset \Sigma$ be the Dirichlet fundamental region based at u for the restriction of Γ to Σ. Then the Dirichlet fundamental region in $\mathbf{H}^n_\mathbb{C}$ based at u equals the polyhedron $\Pi_\Sigma^{-1}(\Delta)$ whose faces are cospinal bisectors.

5.3.2 *Comeridianal pairs: Cartan's configuration*

Another interesting case of bisector intersections occurs for bisectors sharing a common meridian. The material in this section is motivated by Cartan's paper [21] which characterizes automorphisms of Heisenberg space as chain-preserving and $\mathbb{R}$-circle-preserving transformations. If X is a set and $k > 1$ is an integer, denote by $\mathcal{C}_k(X)$ the set of all ordered k-tuples of distinct points of X. If $x = (x_1, x_2) \in \mathcal{C}_2(\partial\mathbf{H}^n_\mathbb{C})$, we denote by $\mathbf{chain}\{x_1, x_2,\}$ the unique chain containing x_1 and x_2. We define a ternary relation as follows. If $x = (x_1, x_2, x_3) \in \mathcal{C}_3(\partial\mathbf{H}^n_\mathbb{C})$, then we say that $x \in \mathbf{Chain}$ if and only if x_1, x_2, x_3 all lie on the same chain; that is, if

$$\mathbf{chain}\{x_1, x_2\} = \mathbf{chain}\{x_2, x_3\} = \mathbf{chain}\{x_3, x_1\}.$$

Similarly, if $x = (x_1, \ldots, x_k) \in \mathcal{C}_k(\mathbf{H}^n_\mathbb{C} \cup \partial\mathbf{H}^n_\mathbb{C})$, we say that $x \in \mathbf{Real}$ if and only if $x_1, \ldots, x_k$ all lie in a totally real geodesic subspace. For $p_1 \neq p_2 \in \partial\mathbf{H}^n_\mathbb{C}$, denote the spinal sphere with vertices p_1, p_2 by $\mathcal{S}\{v_1, v_2\}$.

5.3.3 $\mathbb{R}$-*circle-preserving transformations*

To relate these two relations, Cartan considers a remarkable configuration of one chain and seven $\mathbb{R}$-circles. By Theorem 4.3.12, a transformation of $\mathcal{H}$ which preserves the chain relation must be an automorphism. In [21], Cartan goes on to ask whether an $\mathbb{R}$-circle-preserving transformation is necessarily an automorphism as well. In particular he proves:

Theorem 5.3.2 *Let $f : \partial\mathbf{H}^n_{\mathbb{C}} \longrightarrow \partial\mathbf{H}^n_{\mathbb{C}}$ be a (not necessarily continuous) injection such that for $x = (x_1, x_2, x_3) \in \mathcal{C}_3(\partial\mathbf{H}^n_{\mathbb{C}})$,*

$$x \in \mathbf{Real} \Longleftrightarrow f(x) \in \mathbf{Real}.$$

Then f is an automorphism (possibly anti-holomorphic) of $\partial\mathbf{H}^n_{\mathbb{C}}$.

This theorem follows readily from Theorem 4.3.12 and the following curious characterizations of the chain relation in terms of $\mathbb{R}$-circles:

Theorem 5.3.3 *(Cartan) Let $(x_1, x_2, x_3) \in \mathcal{C}_3(\partial\mathbf{H}^n_{\mathbb{C}})$ be a triple of distinct points. Then $(x_1, x_2, x_3) \notin$ **Chain** if and only if there exists $x_1' \neq x_1, x_2, x_3$ such that the triples x_1, x_1', x_2 and x_1, x_1', x_3 lie on $\mathbb{R}$-circles.*

Theorem 5.3.4 *(Cartan) Suppose that x_1, x_2, x_3 are three distinct points in $\partial\mathbf{H}^n_{\mathbb{C}}$. Then*

$$(x_1, x_2, x_3) \in \mathbf{Chain}$$

if and only if there exist points y_1, y_2, y_3, y_4 lying on an $\mathbb{R}$-circle containing none of x_1, x_2, x_3 such that each of the six triples

$$\begin{array}{ll} \{y_1, y_2, x_1\}, & \{y_3, y_4, x_1\}, \\ \{y_1, y_2, x_2\}, & \{y_3, y_4, x_2\}, \\ \{y_1, y_2, x_3\}, & \{y_3, y_4, x_3\} \end{array}$$

lies on an $\mathbb{R}$-circle.

Figure 5.9 schematically depicts this configuration of seven $\mathbb{R}$-circles. (Segments of the $\mathbb{R}$-circles are drawn more lightly than segments of chains.) Figure 5.10 depicts the vertical projection of this configuration. Figures 5.11 and 5.12 show other views of the same picture.

5.3.4 *Triples not lying on a chain*

We first prove Theorem 5.3.3. The proof of Theorem 5.3.4 occupies the rest of this section and involves the classification of pairs of bisectors which possess a common meridian. In light of the meridianal characterization (Corollary 5.1.11) of spinal spheres, Theorem 5.3.3 is equivalent to the following:

Theorem 5.3.5 *Let $(x_1, x_2, x_3) \in \mathcal{C}_3(\partial\mathbf{H}^n_{\mathbb{C}})$ be a triple of distinct points. Then the following conditions are equivalent:*

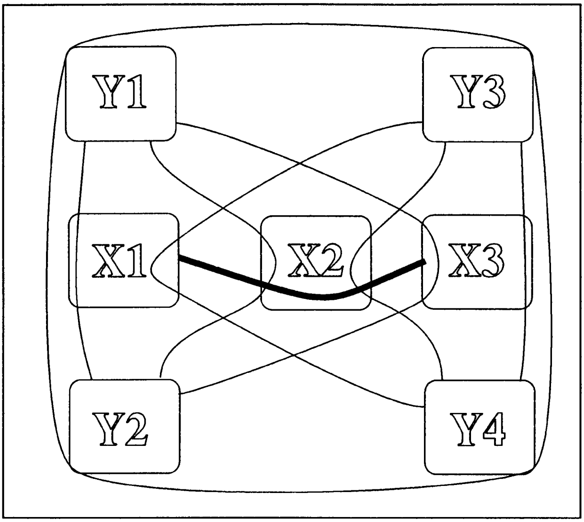

FIG. 5.9. Cartan's configuration: a schematic diagram

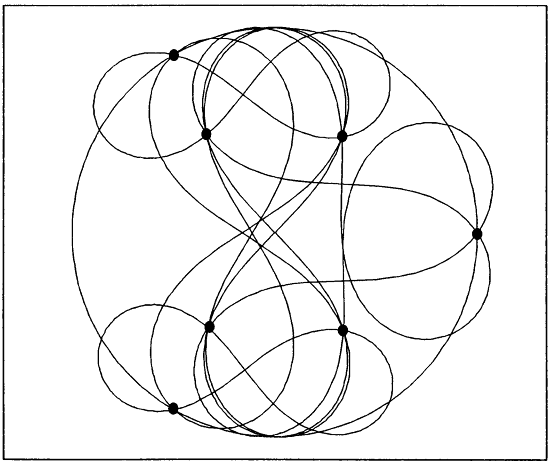

FIG. 5.10. Vertical view of Cartan's configuration

1. $(x_1, x_2, x_3) \notin$ **Chain**.
2. *There exists a spinal sphere with vertex* x_1 *containing* x_2 *and* x_3.
3. *There exists a spinal sphere with vertex* x_2 *containing* x_3 *and* x_1.
4. *There exists a spinal sphere with vertex* x_3 *containing* x_1 *and* x_2.

Proof It suffices to prove $1 \iff 2$, as the other cases follow by symmetry. Choose coordinates so that $x_1 = \infty$; then a spinal sphere with vertex at x_1

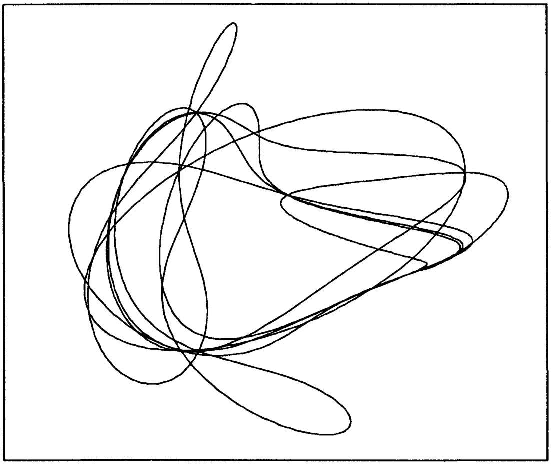

FIG. 5.11. Generic view of Cartan's configuration

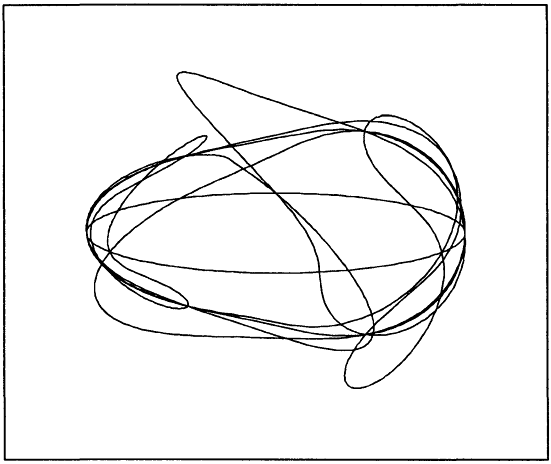

FIG. 5.12. Side view of Cartan's configuration

is a contact hyperplane E_u (where u is the other vertex). Such a hyperplane is transverse to vertical chains and meets a vertical chain in exactly one finite point. In particular $(x_1, x_2, x_3) \in$ **Chain** implies that x_2 and x_3 lie in the same vertical chain, contradicting $x_2 \neq x_3$. Thus 2 implies 1.

Conversely, suppose 1. We must find a point x_1' distinct from x_1, x_2, x_3 such that $x_2, x_3 \in \mathcal{S}\{x_1, x_1'\}$. Since

$$p \in \mathcal{S}\{x_1, x_1'\}$$
$$\Updownarrow$$
$$(p, x_1, x_1') \in \mathbf{Real}$$
$$\Updownarrow$$
$$(x_1', p, x_1) \in \mathbf{Real}$$
$$\Updownarrow$$
$$x_1' \in \mathcal{S}\{x_1, p\}$$

we must show that the two spinal spheres $\mathcal{S}\{x_1, x_2\}$, $\mathcal{S}\{x_1, x_3\}$ have nonempty intersection. Choosing coordinates so that $x_1 = \infty$, $x_2 = (0,0) \in \mathcal{H}$ and $x_3 = (\zeta_0, v_0) \in \mathcal{H}$, condition 1 is just the condition $\zeta_0 \neq 0$. The two spinal spheres

$$\mathcal{S}\{x_1, x_2\} = \{(\zeta, v) \in \mathcal{H} \mid v = 0\},$$
$$\mathcal{S}\{x_1, x_3\} = \{(\zeta, v) \in \mathcal{H} \mid v = v_0 - 2\mathrm{Im}(\bar{\zeta}_0\zeta)\}$$

are parallel if and only if $\zeta_0 = 0$. Since $\zeta_0 \neq 0$, these two spinal spheres have nonempty intersection:

$$\mathcal{S}\{x_1, x_2\} \cap \mathcal{S}\{x_1, x_3\} = \{(\zeta, 0) \in \mathcal{H} \mid v_0 = 2\mathrm{Im}(\bar{\zeta}_0\zeta)\}$$

which is a Euclidean line. Choose $x_1' \in \mathcal{S}\{x_1, x_2\} \cap \mathcal{S}\{x_1, x_3\}$; then $\mathcal{S}\{x_1, x_1'\}$ is the desired spinal sphere. □

5.3.5 *Classification of comeridianal pairs*

Two bisectors $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are *comeridianal* if and only if they have a common meridian; that is, if their spines lie in a common totally real totally geodesic 2-plane. In view of the meridianal decomposition (Corollary 5.1.11), $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are co- meridianal if and only if $\mathfrak{S}_1 \cap \mathfrak{S}_2$ contains an $\mathbb{R}^2$-plane.

Figure 6.12 depicts three comeridianal spinal spheres, whose spines intersect. Figures 5.14, 5.15, and 5.16 depict views of a pair of spinal spheres whose intersection consists of a chain and a meridian.

We analyze the intersection of comeridianal bisectors. Denote the spine (respectively complex spine) of $\mathfrak{E}_i$ by σ_i (respectively Σ_i) and let P denote the totally real totally geodesic 2-plane containing σ_1 and σ_2. As in the cospinal case, we distinguish three cases, here depending on whether the spines intersect, are asymptotic, or are ultraparallel.

Theorem 5.3.6 *Suppose that $\mathfrak{E}_1 \neq \mathfrak{E}_2$ are bisectors which have a common meridian P. Let $\sigma_i \subset P$ be the spine of $\mathfrak{E}_i$. Then exactly one of the following occurs:*

1. *σ_1 and σ_2 intersect; in this case $\Sigma_1 \cap \Sigma_2 = \sigma_1 \cap \sigma_2$ and the intersection $\mathfrak{E}_1 \cap \mathfrak{E}_2 = P$.*
2. *σ_1 and σ_2 are asymptotic; in this case Σ_1 and Σ_2 are asymptotic complex geodesics and again $\mathfrak{E}_1 \cap \mathfrak{E}_2 = P$.*

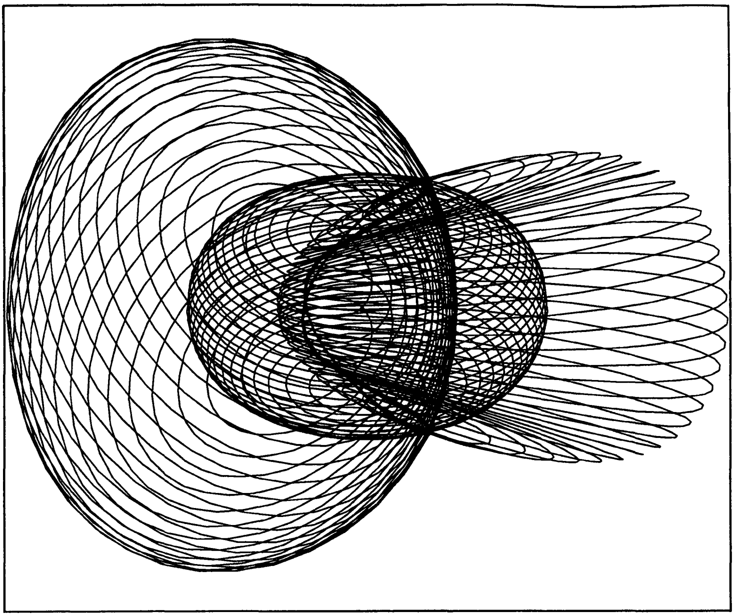

FIG. 5.13. Spinal spheres with a common meridian

3. *σ_1 and σ_2 are ultraparallel geodesics in P.*

In the last case, let σ denote the geodesic in P orthogonal to both σ_1 and σ_2 and Σ the complex geodesic containing it. Then $\Sigma \perp \Sigma_i$ at $\sigma_i \cap \sigma$ and by the slice decomposition (§5.1.2), $\Sigma \subset \mathfrak{E}_i$. Indeed, $\mathfrak{E}_1 \cap \mathfrak{E}_2 = P \cup \Sigma$, where $P \cap \Sigma = \sigma$.

To prove these claims, we choose coordinates so that $P = \mathbb{B}^2_{\mathbb{R}}$ and $\sigma_1 = \{0\} \times \mathbb{B}^1_{\mathbb{R}}$ has endpoints

$$\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$$

so that $\mathfrak{E}_1$ consists of all $(z_1, z_2) \in \mathbb{B}^2$ such that

$$\mathrm{Im}(z_2) = 0. \tag{5.2}$$

For case 1, suppose that σ_2 has vertices

$$\begin{pmatrix} \sin(\theta) \\ \pm\cos(\theta) \end{pmatrix}$$

so that $\mathfrak{E}_2$ consists of all $(z_1, z_2) \in \mathbb{B}^2$ such that

$$\mathrm{Im}(\sin(\theta) z_1 + \cos(\theta) z_2) = 0. \tag{5.3}$$

If $\sin(\theta) \neq 0$, the set of simultaneous solutions of (5.2) and (5.3) consists of all points with real coordinates; that is, $\mathfrak{E}_1 \cap \mathfrak{E}_2 = \mathbb{B}^2_{\mathbb{R}}$ as claimed.

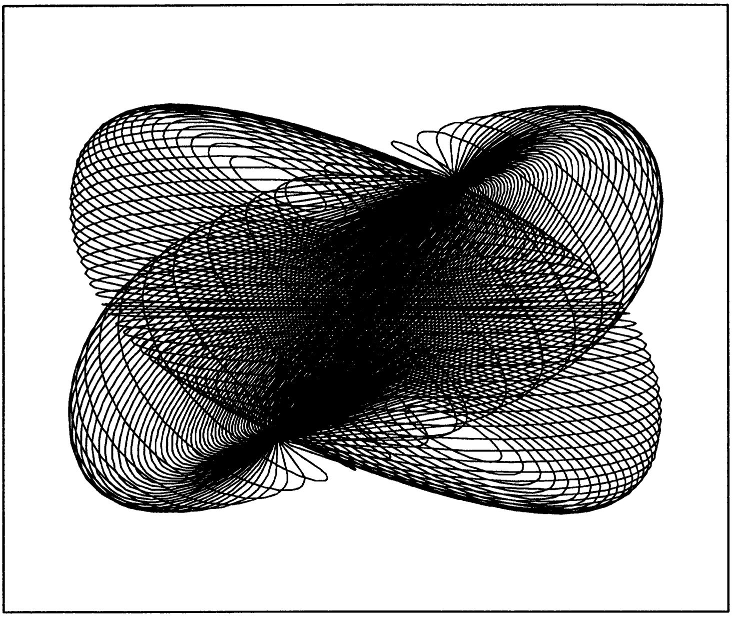

FIG. 5.14. A pair of spinal spheres with both a common slice and meridian

For case 2, suppose that σ_2 has vertices

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so that $\mathfrak{C}_2$ consists of all $(z_1, z_2) \in \mathbb{B}^2$ such that

$$\operatorname{Im}(1 + \bar{z}_1)(1 - z_2) = 0. \tag{5.4}$$

Once again the set of simultaneous solutions of (5.2) and (5.4) equals $\mathbb{B}^2_{\mathbb{R}}$.

For case 3, suppose that σ_2 has vertices

$$\begin{pmatrix} \tanh(\tau/2) \\ \pm\operatorname{sech}(\tau/2) \end{pmatrix}$$

for $\tau = \rho(\sigma_1, \sigma_2) > 0$. Then $\mathfrak{C}_2$ consists of all $(z_1, z_2) \in \mathbb{B}^2$ satisfying

$$\operatorname{Im}\big\{(1 - z_1 \tanh(\tau/2))\bar{z}_2\big\} = 0.$$

Then $z \in \mathfrak{C}_1 \cap \mathfrak{C}_2$ if and only if $z_2 = 0$ or $\operatorname{Im}(z_1) = \operatorname{Im}(z_2) = 0$. In other words,

$$\mathfrak{C}_1 \cap \mathfrak{C}_2 \quad = \quad \mathbb{B}^2_{\mathbb{R}} \bigcup_{(\mathbb{B}^1_{\mathbb{R}} \times \{0\})} (\mathbb{B}^1 \times \{0\})$$

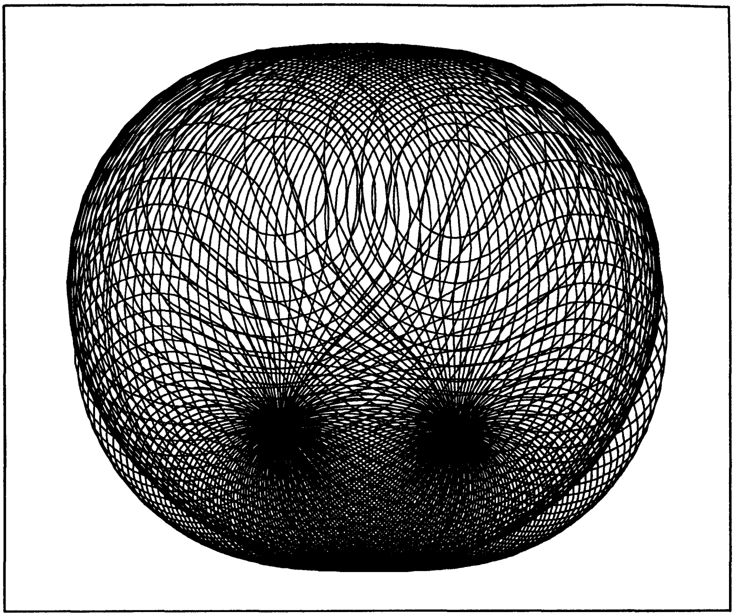

FIG. 5.15. Vertical projection of pair of spinal spheres with common slice and meridian

as claimed. (These hypersurfaces do not intersect transversely, of course, and indeed the intersection is not a smooth manifold.) The intersection of the corresponding spinal spheres at infinity is a union of a chain and an $\mathbb{R}$-circle which meet tangentially in two points (having saddle-type contact).

Exercise 5.3.7 *The angle of intersection of the two bisectors varies between $-\pi/2$ and $\pi/2$.*

For more information on the angle of intersection between two bisectors, see Hsieh [90].

Corollary 5.3.8 *Let $\partial\mathfrak{C}_1, \partial\mathfrak{C}_2$ be two spinal spheres whose intersection properly contains an $\mathbb{R}$-circle ∂P. Then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are comeridianal and their spines are ultraparallel. There exists a common slice Σ such that $\mathfrak{C}_1 \cap \mathfrak{C}_2 = P \cup \Sigma$.*

Proof Since ∂P is a Legendrian curve, it must lie on an integral curve of the CR-structure on $\mathfrak{S}_i$. Thus ∂P must be a meridian of $\partial\mathfrak{C}_i$. Since ∂P uniquely determines P, P must be a common meridian of both $\mathfrak{C}_1$ and $\mathfrak{C}_2$. By Theorem 5.3.6, either $\partial\mathfrak{C}_1 \cap \partial\mathfrak{C}_2 = P$ or σ_1 and σ_2 are ultraparallel, in which case $\mathfrak{C}_1 \cap \mathfrak{C}_2$ is the union of P with a common slice. $\square$

We now prove Theorem 5.3.4. Suppose that C is a chain containing the three distinct points x_1, x_2, x_3 and let ι_C be inversion in C. Choose $y_1 \notin C$ and let

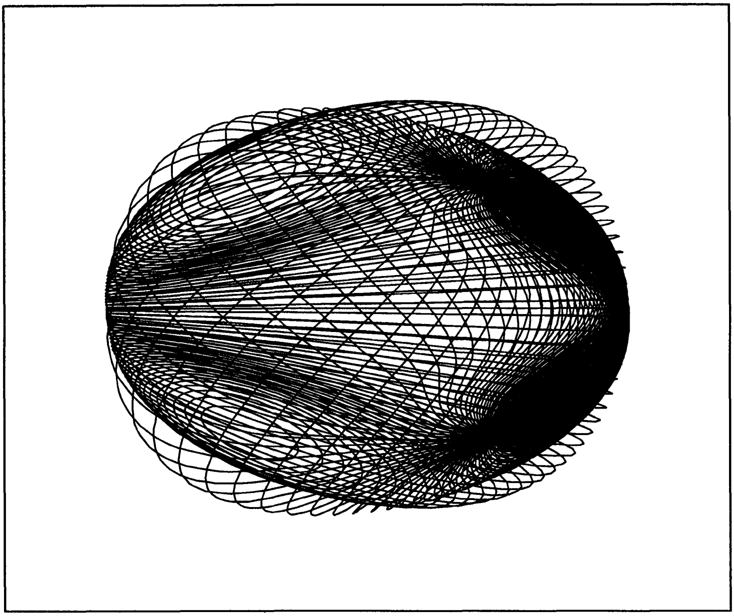

FIG. 5.16. Another view of pair of spinal spheres with common slice and meridian

$y_2 = \iota_C(y_1)$. Then C is a slice of the bisector $\mathfrak{E}_1$ with vertices y_1, y_2 (by (5.1)). Let y_3 be a point of the complement $\mathfrak{E}_1 - (C \cup \{y_1, y_2\})$ and let $y_4 = \iota_C(y_3)$. Let $\mathfrak{E}_2$ be the bisector with vertices y_3, y_4. By (5.1), C is a slice of $\mathfrak{E}_2$. Let P be the meridian of $\mathfrak{E}_1$ containing y_3. Then P is orthogonal to C and invariant under ι_C. Therefore $y_4 \in P$. Thus y_1, y_2, y_3, y_4 all lie on the $\mathbb{R}$-circle ∂P. Furthermore, since $x_i \in C \subset \mathfrak{E}_j$, the meridians of $\mathfrak{E}_j$ are $\mathbb{R}$-circles which contain x_i and the vertices of $\mathfrak{E}_j$, as desired.

Conversely, suppose that a configuration of seven $\mathbb{R}$-circles containing the seven points

$$x_1, x_2, x_3, y_1, y_2, y_3, y_4$$

satisfies the incidence relations of Theorem 5.3.4. Let $\mathfrak{E}_1$ (respectively $\mathfrak{E}_2$) be the bisector with vertices y_1, y_2 (respectively y_3, y_4). Let ∂P be the $\mathbb{R}$-circle containing y_1, y_2, y_3, y_4; by Corollary 5.2.3 $\partial P \subset \mathfrak{E}_1 \cap \mathfrak{E}_2$. Similarly, since each x_i lies on an $\mathbb{R}$-circle with y_1, y_2 (respectively y_3, y_4), $x_i \in \mathfrak{E}_1$ (respectively $x_i \in \mathfrak{E}_2$). Therefore x_1, x_2, x_3 lie on an intersection of two spinal spheres which properly contains the $\mathbb{R}$-circle ∂P. Since $x_i \notin \partial P$, Corollary 5.3.8 implies that the existence of a chain C such that $(\mathfrak{E}_1 \cap \mathfrak{E}_2) - \partial P \subset C$. Thus $x_1, x_2, x_3 \in C$ as desired. The proof of Theorem 5.3.4 is now complete.

5.4 Calibrating the CR-structure at infinity

5.4.1 *Distortion functions*

Now we relate bisectors to CR calibrations associated to geometric entities in $\mathbf{H}_{\mathbb{C}}^n$. We shall see that spinal spheres are level sets of the Poisson kernel function. Recall from §2.5.5 that if $\mathcal{U}$ is a neighborhood of $\partial\mathbf{H}_{\mathbb{C}}^n \subset \mathbb{P}(\mathbb{C}^{n,1})$ and $\Psi : \mathcal{U} \longrightarrow \mathbb{R}$ is a smooth map such that 0 is a regular value of Ψ and $\Psi^{-1}(0) = \partial\mathbf{H}_{\mathbb{C}}^n$, then the restriction of $d^c\Psi$ to $\partial\mathbf{H}_{\mathbb{C}}^n$ is a contact 1-form ω calibrating the CR-structure on $\partial\mathbf{H}_{\mathbb{C}}^n$. We denote the space of calibrations of the CR-structure of $\partial\mathbf{H}_{\mathbb{C}}^n$ by $\mathbf{Cal}(\partial\mathbf{H}_{\mathbb{C}}^n)$. If $\omega \in \mathbf{Cal}(\partial\mathbf{H}_{\mathbb{C}}^n)$, then

$$\nu = \omega \wedge (d\omega)^{n-1}$$

is a volume form on $\partial\mathbf{H}_{\mathbb{C}}^n$. If g is an automorphism, then $g^*\omega \in \mathbf{Cal}(\partial\mathbf{H}_{\mathbb{C}}^n)$ and a function $\mathbf{J}_{g,\omega}$ exists such that

$$g^*\omega = \mathbf{J}_{g,\omega} \cdot \omega.$$

The *distortion function* $\mathbf{J}_{g,\omega}$ *of* g *with respect to* ω satisfies the following *chain rule*

$$\mathbf{J}_{gh,\omega} = \mathbf{J}_{g,\omega}\mathbf{J}_{h,g^*\omega} = \mathbf{J}_{g,\omega}g^*(\mathbf{J}_{h,\omega}).$$

Define the *isometric sphere of* g *with respect to* ω to be the subset comprising $z \in \partial\mathbf{H}_{\mathbb{C}}^n$ such that $\mathbf{J}_{g,\omega}(z) = 1$; that is, the locus of points where the calibration's value is unchanged by g.

The Kähler potential $\psi_x : \mathbf{H}_{\mathbb{C}}^n \longrightarrow \mathbb{R}$ associated to a point x determines a function $\Psi_x : \mathbf{H}_{\mathbb{C}}^n \longrightarrow \mathbb{R}$ which smoothly extends to a defining function for $\partial\mathbf{H}_{\mathbb{C}}^n$ on a neighborhood of $\mathbf{H}_{\mathbb{C}}^n \subset \mathbb{P}(\mathbb{C}^{n,1})$. Namely,

$$\Psi_x(z) = e^{\psi_x(z)} = \operatorname{sech}^2\left(\frac{\rho(z,x)}{2}\right)$$

extends to a smooth defining function on a neighborhood of $\mathbf{H}_{\mathbb{C}}^n \subset \mathbb{P}(\mathbb{C}^{n,1})$. The defining function associated to the origin equals

$$\Psi_0(z) = 1 - \|z\|^2.$$

Denote the corresponding calibration by

$$\omega_x = d^c\Psi_x|_{\partial\mathbf{H}_{\mathbb{C}}^n} \in \mathbf{Cal}(\partial\mathbf{H}_{\mathbb{C}}^n)$$

and the corresponding volume form by

$$\nu_x = \omega_x \wedge (d\omega_x)^{n-1}.$$

When x is the origin in $\mathbf{H}_{\mathbb{C}}^n$,

$$\omega_x = \frac{2(dv - 2\mathrm{Im}\langle\!\langle\zeta, d\zeta\rangle\!\rangle)}{(1 + \|\zeta\|^2)^2 + v^2}$$

in Heisenberg coordinates on $\partial\mathbf{H}_{\mathbb{C}}^n$.

We shall see that an isometric sphere defined with respect to one of the above calibrations ω_x is a spinal sphere equidistant from x.

Lemma 5.4.1 *If $g \in \mathbf{PU}(n,1)$, then $g^*\Psi_x = \Psi_{g^{-1}x}$ and $g^*\omega_x = \omega_{g^{-1}x}$.*

Proof

$$\begin{aligned} g^*\Psi_x(z) &= \Psi_x(gz) \\ &= \operatorname{sech}^2\left(\frac{\rho(gz,x)}{2}\right) \\ &= \operatorname{sech}^2\left(\frac{\rho(z,g^{-1}x)}{2}\right) \\ &= \Psi_{g^{-1}x}(z). \end{aligned}$$

For the second assertion apply d^c and restrict to $\partial\mathbf{H}^n_{\mathbb{C}}$. □

5.4.2 *Dirichlet isometric spheres*

For fixed $x, y \in \mathbf{H}^n_{\mathbb{C}}$, the function $\phi_{x,y} : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \mathbb{C}$ defined by

$$\phi_{x,y}(z) = \sqrt{\frac{1-\|x\|^2}{1-\|y\|^2}\,\frac{1-\langle\!\langle z,y\rangle\!\rangle}{1-\langle\!\langle z,x\rangle\!\rangle}}$$

is a holomorphic function of $z \in \mathbf{H}^n_{\mathbb{C}}$. The definition of Ψ_x implies that

$$\frac{\Psi_x(z)}{\Psi_y(z)} = |\phi_{x,y}(z)|^2.$$

Now

$$d^c\Psi_x = d^c(\phi_{x,y}\bar{\phi}_{x,y}\Psi_y) = \Psi_y(d^c|\phi_{x,y}|^2) + |\phi_{x,y}|^2 d^c\Psi_y$$

so on $\partial\mathbf{H}^n_{\mathbb{C}}$,

$$\omega_x = |\phi_{x,y}|^2\omega_y$$

and

$$\nu_x = |\phi_{x,y}|^{2n}\nu_y.$$

In particular the set of points $z \in \mathbf{H}^n_{\mathbb{C}}$ such that $\omega_x(z) = \omega_y(z)$ equals the equidistant hypersurface $\mathfrak{E}\{x,y\}$. As a special case, for any automorphism $g \in \mathbf{PU}(n,1)$, the isometric sphere of g with respect to ω_x bounds the equidistant hypersurface

$$\{z \in \mathbf{H}^n_{\mathbb{C}} \mid g^*\nu_x(z) = \nu_x(z)\} = \mathfrak{E}\{x, g^{-1}(x)\}.$$

In analogy with the faces of Dirichlet fundamental regions in real hyperbolic space, we refer to these bisectors (as well as their boundaries) as *Dirichlet isometric spheres.*

5.4.3 *Relation with the Poisson kernel*

The family of calibrations ω_x and volume forms ν_x intimately relates to the *Poisson integral formula* on $\mathbf{H}^n_{\mathbb{C}}$ (compare §3.3 of Rudin [147]). For $\zeta \in \partial\mathbf{H}^n_{\mathbb{C}}$, the Radon–Nikodym derivative

$$\frac{\nu_x}{\nu_O} = \frac{(1-\|x\|^2)^n}{|1-\langle\!\langle \zeta, x\rangle\!\rangle|^{2n}}$$

(in the ball model) is a harmonic function of x (with respect to the Laplace–Beltrami operator for the Bergman metric). Furthermore, every continuous function f on $\partial\mathbf{H}^n_{\mathbb{C}}$ extends to a harmonic function on $\mathbf{H}^n_{\mathbb{C}}$ by the Poisson integral formula:

$$f(z) = \int_{\partial\mathbf{H}^n_{\mathbb{C}}} f(\zeta)\, d\nu_x(\zeta).$$

For example, we can represent bisectors as level sets of bounded harmonic functions as follows. Let $\mathfrak{S}$ denote a bisector with spine σ and complex spine Σ. Let $\rho_\sigma : \Sigma \longrightarrow \mathbb{R}$ be the function representing signed distance in Σ to σ and let $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ denote orthogonal projection (evidently a holomorphic map). Then

$$\phi(z) = \frac{1}{\pi}\tan^{-1}\sinh(\rho_\sigma\circ\Pi_\Sigma(z))$$

is a harmonic function bounded between -1 and 1 such that $\phi^{-1}(0) = \mathfrak{S}$. Indeed, its values on $\partial\mathbf{H}^n_{\mathbb{C}}$ are ± 1 on the two components of the complement of $\partial\mathfrak{S}$. In the Siegel domain $\mathfrak{H}^n$ with

$$\Sigma = \mathfrak{H}^1 = \{(0', w_n) \in \mathbb{C}^n \mid \mathrm{Re}(w_n) > 0\}$$

and $\sigma = \mathfrak{H}^1_{\mathbb{R}} = \{0'\} \times \mathbb{R}_+$, the bisector is represented by

$$\mathfrak{S} = (\mathbb{C}^{n-1} \times \mathbb{R}) \cap \mathfrak{H}^n$$

and orthogonal projection by

$$\Pi_\Sigma(w) = (0', w_n).$$

Signed distance of $\Pi_\Sigma(w)$ to σ is given by

$$\sinh(\rho_\sigma\circ\Pi_\Sigma(w)) = \frac{\mathrm{Im}(w)}{\mathrm{Re}(w)}.$$

Thus

$$\tan^{-1}\sinh(\rho_\sigma\circ\Pi_\Sigma(w)) = \mathrm{Im}(\log(w_n))$$

(where log is the branch of the logarithm function defined on the right half-plane taking 1 to 0) is the imaginary part of the holomorphic function $\log(w_n)$ on $\mathfrak{H}^n$ and is evidently (pluri-)harmonic.

5.4.4 *Calibrations associated with boundary points*

An analogous construction exists for calibrations associated to points on the boundary. Let $q \in \partial \mathbf{H}^n_{\mathbb{C}}$ be an ideal point and let $Q \in \mathbb{P}^{-1}(q) \subset \mathbb{C}^{n,1}$ be a null vector projecting to q. Let Ψ_Q be the corresponding defining function:

$$\Psi_Q(Z) = \frac{-\langle Z, Z\rangle}{\langle Z, Q\rangle\langle Q, Z\rangle}$$

(Ψ_Q is undefined at q). By 4.1.2, Ψ_Q is related to the Busemann function at q by

$$\Psi_Q(z) = e^{-h_q(z)}.$$

We obtain CR calibrations and volume forms by

$$\omega_q = d^c\Psi_Q \mid_{\partial \mathbf{H}^n_{\mathbb{C}}}, \qquad \nu_q = \omega_q \wedge (d\omega_q)^{n-1}$$

and, as for the calibrations associated to points in $\mathbf{H}^n_{\mathbb{C}}$,

$$g^*\omega_q = \omega_{g^{-1}(q)}, \qquad g^*\nu_q = \nu_{g^{-1}(q)}.$$

However, unlike the volume forms associated to interior points, the total volume of ν_q is *infinite.*

Given null vectors $P, Q \in \mathbb{C}^{n,1}$, we obtain a holomorphic function

$$\phi_{P,Q}(z) = \frac{\langle Z, P\rangle}{\langle Z, Q\rangle}$$

such that

$$\frac{\Psi_Q(z)}{\Psi_P(z)} = |\phi_{P,Q}(z)|^2$$

and it follows that

$$\omega_P = |\phi_{P,Q}|^2\omega_Q$$

and

$$\nu_P = |\phi_{P,Q}|^{2n}\nu_Q.$$

5.4.5 *Ford isometric spheres and the Bruhat decomposition*

These calibrations can be used to define the analogue of *Ford isometric spheres* (Ford [53]) for automorphisms of Heisenberg geometry.

Namely, fix a point $q \in \partial \mathbf{H}^n_{\mathbb{C}}$ and a null vector $Q \in \mathbb{C}^{n,1}$ projecting to q. Let $\Psi_Q : \mathbb{P}(\mathbb{C}^{n,1} - Q^\perp) \longrightarrow \mathbb{R}$ be the corresponding defining function and $\omega = \omega_Q$ the corresponding calibration. For convenience take q to be the point at infinity in Heisenberg space. Recall that a spinal sphere is *vertical* if its complex spine contains infinity; that is, it is a vertical chain. We shall see that the isometric spheres corresponding to the calibrations ω_Q are precisely the vertical spinal spheres. Indeed, we have the following trichotomy of automorphisms, closely related to the Bruhat decomposition of $\mathbf{PU}(n, 1)$:

Theorem 5.4.2 *Let $g \in \mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}})$. Then either:*

1. *g preserves ω and g is a Heisenberg isometry; or*
2. *$g^*\omega = \lambda^2\omega$ where $\lambda^2 \neq 1$ is constant and g is a Heisenberg dilation of scale factor λ; or*
3. *the isometric sphere of g with respect to ω is a vertical spinal sphere.*

For example, consider inversion ι_C in the hyperchain $C = S^{2n-3} \times \{0\}$ discussed in §4.10; we claim that the isometric sphere for ι_C equals the spinal sphere $\mathcal{S}$ with vertices $\{(0, \pm 1)\}$, given by

$$\{(\zeta, v) \in \mathcal{H} \mid \|\zeta\|^4 + v^2 = 1\}.$$

Since

$$d(\|\zeta\|^2 - iv) = 2\mathrm{Re}\langle\!\langle d\zeta, \zeta\rangle\!\rangle - idv$$

the calibration ω pulls back to

$$\iota_C^*\omega(\zeta, v) = \frac{1}{\Upsilon(\zeta, v)}\omega(\zeta, v) \tag{5.5}$$

where

$$\Upsilon(\zeta, v) = \|\zeta\|^4 + v^2.$$

Thus

$$\mathbf{J}_{\iota_C,\omega}(\zeta, v) = (\|\zeta\|^4 + v^2)^{-1}$$

and the isometric sphere of ι_C equals the unit spinal sphere (of §5.1.8) as claimed.

Proof of Theorem 5.4.2 Let $g \in \mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}})$. If $g(p_\infty) = p_\infty$, then $g \in \mathbf{Sim}(\mathcal{H})$ and scales ω by a constant factor λ^2. Either g preserves ω (and g is a product of a Heisenberg translation and a Heisenberg rotation) or $\lambda \neq 1$ and g is a Heisenberg dilation.

We first describe the *Bruhat decomposition* of $\mathbf{PU}(n, 1)$. Let ι_C denote inversion in the unit sphere (interchanging ∞ and $0 \in \mathcal{H}$), described in §2.9. Then

$$\mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}}) = \mathbf{Sim}(\mathcal{H}) \amalg (\mathbf{Sim}(\mathcal{H})\iota_C\mathbf{Isom}(\mathcal{H})).$$

Suppose that g is an automorphism which is not a Heisenberg similarity; that is, g does not fix ∞. Let τ be the unique Heisenberg translation taking $g^{-1}(\infty)$ to 0. A unique Heisenberg similarity transformation s exists such that

$$g = s\iota_C\tau.$$

To see this, let $s = g\tau^{-1}\iota_C$. Then $s(\infty) = \infty$ (since $\iota_C(\infty) = 0$ and $\tau g^{-1}(\infty) = 0$) so $s \in \mathbf{Sim}(\mathcal{H})$. Let $\lambda = \lambda(s)$ be the scale factor of s; that is, s is the composition of a Heisenberg isometry with Heisenberg dilation by λ. By (5.5),

$$g^*\omega = (s\iota_C\tau)^*\omega = \lambda^2\tau^*\left(\frac{1}{\Upsilon}\omega\right) = \frac{\lambda^2}{\Upsilon\circ\tau}\cdot\omega$$

so the isometric sphere of g with respect to ω equals the set of points with Heisenberg coordinates (ζ, v) satisfying

$$\Upsilon(\tau(\zeta, v)) = \lambda^2;$$

that is, the spinal sphere having vertices

$$\tau^{-1}(0, \pm\lambda).$$

The proof of Theorem 5.4.2 is now complete. □

For example, let R be a finite $\mathbb{R}$-circle with inversion ι_R. Let C be the vertical chain containing the center of R, that is $C = \mathbf{chain}\{\infty, \iota_R\infty,\}$. Then C is invariant under ι_R and the restriction of ι_R to C is an orientation-reversing involution of C. Such a transformation necessarily has two fixed points $R \cap C$, which are the vertices of the isometric sphere of ι_R with respect to ω.

As another example, consider a nonvertical hyperchain C with inversion ι_C and let $(\iota_C)^{\pm\frac{1}{2}}$ be the complex reflections of order 4 about C. Then the isometric sphere of ι_C is the spinal sphere with vertices

$$(\iota_C)^{\pm\frac{1}{2}}(\infty).$$

For more information on Ford isometric spheres in complex hyperbolic space, see Parker [135].

5.4.6 *Calibrations associated to totally geodesic subspaces*

Calibrations of the CR-structure are naturally associated to totally geodesic submanifolds S. Let G_S denote the stabilizer of S.

Suppose first that S is a holomorphic totally geodesic subspace of complex dimension k. We may assume that S equals $(\{0\} \times \mathbf{H}^k_{\mathbb{C}}) \subset \mathbf{H}^n_{\mathbb{C}}$ and so

$$G_S = \mathbb{P}(\mathbf{U}(k,1) \times \mathbf{U}(n-k))$$

is its stabilizer. By (6.1) the function

$$\Psi_S(z) = \operatorname{csch}^2\left(\frac{\rho(z,S)}{2}\right) = \frac{1 - \|z\|^2}{\sum_{j=1}^{n-k} z_j\bar{z}_j}$$

is a G_S-invariant defining function for $\partial\mathbf{H}^n_{\mathbb{C}}$ with a pole along S. Then

$$\omega_S = d^c\Psi_S|_{\partial\mathbf{H}^n_{\mathbb{C}}}$$

is a G_S-calibration singular along ∂S. In Heisenberg coordinates:

$$\omega_S = \frac{dv - 2\,\mathrm{Im}\langle\!\langle\zeta, d\zeta\rangle\!\rangle}{\sum_{j=1}^{n-k}|\zeta_j|^2}.$$

5.4.7 *Calibrations associated to real forms*

We next consider the calibration associated to a real form P of $\mathbf{H}^n_{\mathbb{C}}$. It suffices to consider the case $P = \mathbf{H}^n_{\mathbb{R}}$ with inversion ι_P. Then

$$\psi_P(z) = \log \operatorname{sech} \rho(z, P)$$

is a Kähler potential function associated with P (and invariant under the stabilizer G_P of P), where $\rho(z, P)$ denotes distance from z to P. The distance formula

$$\begin{aligned}\cosh \rho(z, P) &= \cosh\left(\frac{\rho(z, \iota_P(z))}{2}\right)\\ &= \sqrt{\frac{\langle Z, \iota_P(Z)\rangle\langle \iota_P(Z), Z\rangle}{\langle Z, Z\rangle\langle \iota_P(Z), \iota_P(Z)\rangle}}\\ &= \frac{|1 - \langle\!\langle z, \iota_P(z)\rangle\!\rangle|}{1 - \langle\!\langle z, z\rangle\!\rangle}\end{aligned}$$

implies

$$\psi_P(z) = \psi_O(z) + h(z) + \overline{h(z)}$$

where ψ_O is the Kähler potential defined in §3.1.8 and

$$\begin{aligned}h(z) &= \frac{1}{2}\log(1 - \langle z, \iota_P(Z)\rangle)\\ &= \frac{1}{2}\log\frac{2w_n - \sum_{j=1}^{n-1} w_j^2}{(1 + 2w_n)^2}\end{aligned}$$

is holomorphic on $\mathbf{H}^n_{\mathbb{C}}$. Thus $\psi_P - \psi_O$ is pluriharmonic and

$$\Phi = -4i\partial\bar{\partial}\psi_O = -4i\partial\bar{\partial}\psi_P = 2dd^c\phi.$$

Hence $2d^c\psi_P$ is a $G_P \cong \mathbf{PO}(n, 1)$-invariant primitive for Φ.

Thus if $\Gamma \subset \mathbf{PO}(n, 1) \subset \mathbf{PU}(n, 1)$ is a discrete group, the quotient manifold $M = \mathbf{H}^n_{\mathbb{C}}/\Gamma$ has the property that the Kähler form is *canonically* exact. Since the integral of the Kähler form over a holomorphic submanifold is nonzero, M has no compact holomorphic submanifolds. Thus (as first observed by Burns and Shnider in [19], Proposition 6.4), M is a Stein manifold.

Here is a G_P-invariant calibration. Let

$$\Psi_P(z) = e^{\psi_P(z)} = \frac{\langle Z, Z\rangle\langle \iota_P(Z), \iota_P(Z)\rangle}{\langle Z, \iota_P(Z)\rangle\langle \iota_P(Z), Z\rangle}.$$

In Heisenberg coordinates, the corresponding calibration is

$$d^c\Psi_P|_{\partial\mathbf{H}^n_{\mathbb{C}}} = \frac{dv - 2\mathrm{Im}\langle\!\langle \zeta, d\zeta\rangle\!\rangle}{|\sum_{j=1}^{n-1}\zeta_j^2 - \|\zeta\|^2 + iv|}.$$

The denominator has the alternative form:

$$2\sqrt{\left(\sum_{j=1}^{n-1} y_j^2\right)^2 + \left(\frac{v}{2} + \sum_{j=1}^{n-1} x_j y_j\right)^2}\ .$$

5.5 Differential geometry of bisectors

In this section we compute the curvature and second fundamental form of a bisector. The main technique involves "cylindrical coordinates" adapted to a geodesic in $\mathbf{H}_{\mathbb{C}}^n$ in which these tensor fields are particularly tractable. Some of the results of this section were obtained jointly with Kevin Corlette. For more information see Hsieh's thesis [90].

5.5.1 *Cylindrical coordinates*

For simplicity consider only the case of hypersurfaces in complex dimension $n = 2$ and work in the Siegel domain

$$\mathfrak{H}^2 = \{(w_1, w_2) \in \mathbb{C}^2 \mid f(w) > 0\}$$

where

$$f(w) = w_2 + \bar{w}_2 - w_1\bar{w}_1$$

is a defining function and

$$\mathfrak{E} = \{w \in \mathfrak{H}^2 \mid w_2 \in \mathbb{R}\}$$

is the bisector. The complex spine of $\mathfrak{E}$ is the complex geodesic

$$\Sigma = \{0\} \times \mathfrak{H}^1$$

and the spine is

$$\sigma = \{0\} \times \mathbb{R}^+.$$

Orthogonal projection $\mathfrak{H}^2 \longrightarrow \Sigma$ is given by the complex coordinate function w_2. For each $\zeta \in \mathfrak{H}^1$ let C_ζ denote the complex geodesic

$$w_2^{-1}(\zeta) = \mathfrak{H}^2 \cap (\mathbb{C} \times \{\zeta\}).$$

If $\zeta \in \sigma$ (that is, ζ is real), then C_ζ is a slice of the bisector $\mathfrak{E}$. On each complex geodesic C_ζ choose "hyperbolic polar coordinates" (u, θ) (see §1.4.2) where $u : \mathfrak{E} \longrightarrow \mathbb{R}$ denotes distance to the spine and θ is the usual angular variable:

$$w_1 = \sqrt{2}e^{t/2}(e^{i\theta}\tanh(u/2))$$

where $u \geq 0$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$. Let $t : \mathfrak{H}^2 \longrightarrow \mathbb{R}$ be the "spinal coordinate" corresponding to distance along σ and let $s : \mathfrak{H}^1 \longrightarrow \mathbb{R}$ denote (signed) distance to σ:

$$w_2 = e^t(1 + i\sinh(s))$$

so that $\mathfrak{E}$ is defined by the equation $s = 0$. (Compare §1.4.4 for the 1-dimensional case, where (t, s) are denoted (τ, δ).) Then (u, θ, t), where

$$\begin{aligned} &u \geq 0, \\ &\theta \in \mathbb{R}/(2\pi\mathbb{Z}), \\ &t \in \mathbb{R}, \end{aligned}$$

define global coordinates on $\mathfrak{E}$, with the usual ambiguity of polar coordinates that, for $u = 0$, the coordinate θ is arbitrary.

In summary, $\mathfrak{H}^2$ has coordinates (s, t, u, θ), where

$$s, t \in \mathbb{R}, \quad u > 0, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

The bisector $\mathfrak{E}$ is defined by $s = 0$, the complex spine Σ is defined by $u = 0$ (θ indeterminate), and the spine σ by $s = u = 0$.

The fibers P_θ of θ are the meridians of $\mathfrak{E}$. In these coordinates we have

$$f(w) = 2e^t \operatorname{sech}^2(u/2)$$

(where f is the function—the Kähler potential associated to a point—defined in (3.5)) and

$$w_1^{-1} dw_1 = d\log(w_1) = \frac{1}{2}dt + \frac{1}{2}\frac{\operatorname{sech}^2(u/2)}{\tanh(u/2)}du + id\theta,$$

implying

$$\frac{2}{\sqrt{f(w)}} dw_1 = e^{i\theta}\left\{(\sinh(u/2)dt + \operatorname{sech}(u/2)du) + 2i\sinh(u/2)d\theta\right\}.$$

Now

$$w_2^{-1} dw_2 = d\log w_2 = dt + (\tanh(s) + i\operatorname{sech}(s))\, ds$$

$$dw_2 = e^t\left\{dt + i(\sinh(s)dt + \cosh(s)ds)\right\}$$

$$\bar{w}_1 dw_1 = e^t\left\{\tanh^2(u/2)dt + \operatorname{sech}^2(u/2)\tanh(u/2)du + 2i\tanh^2(u/2)d\theta\right\}$$

$$\frac{2}{f(w)}(dw_2 - \bar{w}_1 dw_1) = dt - \tanh(u/2)du \\ + i\left\{\cosh^2(u/2)(\cosh(s)ds + \sinh(s)dt) - 2\sinh^2(u/2)d\theta\right\}.$$

From this and (4.4), the metric tensor is

$$\begin{aligned}
\mathrm{g}(w) &= \left|\frac{2}{\sqrt{f}}dw_1\right|^2 + \left|\frac{2}{f}(dw_2 - \bar{w}_1 dw_1)\right|^2 \\
&= (\sinh(u/2)dt + \operatorname{sech}(u/2)du)^2 \\
&\quad + (\sinh(u/2)d\theta)^2 \\
&\qquad + (dt - \tanh(u/2)du)^2 \\
&\qquad + \left\{\cosh^2(u/2)(\cosh(s)ds + \sinh(s)dt) - 2\sinh^2(u/2)d\theta\right\}^2.
\end{aligned}$$

An orthonormal frame for $\mathfrak{H}^2$ is

$$\begin{aligned}
\xi_1 &= \operatorname{sech}(u/2)\left(\frac{\partial}{\partial s} - \tanh(s)\frac{\partial}{\partial t} + \frac{1}{2}\operatorname{sech}^2(s)\frac{\partial}{\partial\theta}\right) \\
\xi_2 &= \operatorname{sech}(u/2)\operatorname{sech}(s)\left(\frac{\partial}{\partial t} + \frac{1}{2}\sinh(s)\frac{\partial}{\partial\theta}\right) \\
\xi_3 &= \frac{\partial}{\partial u} \\
\xi_4 &= \frac{1}{2}\operatorname{sech}(u/2)\operatorname{csch}(u/2)\frac{\partial}{\partial\theta}
\end{aligned}$$

which is adapted to $\mathfrak{C}$ in the sense that ξ_1 is the unit normal vector to $\mathfrak{C}$ and ξ_1, ξ_2, ξ_3 are all tangent to $\mathfrak{C}$. The dual orthonormal coframe is

$$\begin{aligned}
\phi^1 &= \cosh(u/2)ds \\
\phi^2 &= \cosh(u/2)(\cosh(s)dt + \sinh(s)ds) \\
\phi^3 &= du \\
\phi^4 &= \sinh(u/2)\cosh(u/2)\left\{2d\theta - (\sinh(s)dt + \cosh(s)ds)\right\}.
\end{aligned}$$

Now

$$\begin{aligned}
d\phi^1 &= \frac{1}{2}\tanh(u/2)\phi^3 \wedge \phi^1 \\
d\phi^2 &= \frac{1}{2}\tanh(u/2)\phi^3 \wedge \phi^2 + \operatorname{sech}(u/2)\tanh(s)\phi^1 \wedge \phi^2 \\
d\phi^3 &= 0 \\
d\phi^4 &= \coth(u)\phi^3 \wedge \phi^4 - \tanh(u/2)\phi^1 \wedge \phi^2.
\end{aligned}$$

In matrix form,

$$d\phi = \omega \wedge \phi$$

where

$$\phi = \begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{bmatrix}$$

and the connection form equals

$$\begin{aligned} \omega = \frac{1}{2}\tanh(u/2) &\begin{bmatrix} 0 & -\phi^4 & -\phi^1 & -\phi^2 \\ \phi^4 & 0 & -\phi^2 & \phi^1 \\ \phi^1 & \phi^2 & 0 & 0 \\ \phi^2 & -\phi^1 & 0 & 0 \end{bmatrix} \\ + \coth(u) &\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi^4 \\ 0 & 0 & -\phi^4 & 0 \end{bmatrix} \\ + \tanh(s)\operatorname{sech}(u/2) &\begin{bmatrix} 0 & \phi^2 & 0 & 0 \\ -\phi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

5.5.2 *Curvature of bisectors*

Now we specialize to the bisector $\mathfrak{E}$. The unit normal vector is ξ_1 and its induced connection $\tilde{\omega}$ is given by setting $s = 0$, $\phi^1 = 0$ and restricting ω to the coframe

$$\begin{aligned} \tilde{\phi}^2 &= \cosh(u/2)dt \\ \tilde{\phi}^3 &= du \\ \tilde{\phi}^4 &= \sinh(u)d\theta \end{aligned}$$

and

$$\tilde{\omega} = \frac{1}{2}\tanh(u/2)\begin{bmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \coth(u)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \phi^4 \\ 0 & -\phi^4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2}\sinh(u/2)dt & 0 \\ \frac{1}{2}\sinh(u/2)dt & 0 & \cosh(u)d\theta \\ 0 & -\cosh(u)d\theta & 0 \end{bmatrix}.$$

The curvature 2-form $\Omega = d\omega - \omega \wedge \omega$ is given by

$\Omega =$

$$\begin{bmatrix} 0 & -\frac{1}{4}\cosh(u/2)du\wedge dt & \frac{1}{2}\sinh(u/2)\cosh(u)dt\wedge d\theta \\ \frac{1}{4}\cosh(u/2)du\wedge dt & 0 & -\sinh(u)d\theta\wedge du \\ \frac{1}{2}\sinh(u/2)\cosh(u)dt\wedge d\theta & \sinh(u)d\theta\wedge du & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & \phi^3\wedge\phi^4 & 0 \\ \phi^4\wedge\phi^3 & 0 & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & \phi^3\wedge\phi^2 & 0 \\ \phi^2\wedge\phi^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\tanh(u/2)}{2\tanh(u)}\begin{bmatrix} 0 & 0 & \phi^4\wedge\phi^2 \\ 0 & 0 & 0 \\ \phi^2\wedge\phi^4 & 0 & 0 \end{bmatrix}.$$

In particular the sectional curvatures of the coordinate 2-planes are given by

$$\kappa_{34}^{\mathfrak{C}} = -1,$$
$$\kappa_{23}^{\mathfrak{C}} = -\frac{1}{4},$$
$$\kappa_{24}^{\mathfrak{C}} = -\frac{\tanh(u/2)}{2\tanh(u)} = -\frac{\left(1+\tanh^2(u/2)\right)}{4}$$

(the first two of these sectional curvatures one knows already, since ξ_2, ξ_3 span a totally geodesic 2-plane which is totally real and ξ_3, ξ_4 span a totally geodesic 2-plane which is complex). The graph of the function $\kappa_{24}^{\mathfrak{C}}(u)$ is given in Fig. 5.17.

5.5.3 *The second fundamental form of a bisector*

Finally we compute the second fundamental form of the bisector with respect to the frame ξ_2, ξ_3, ξ_4 above and the unit normal vector ξ_1. By the Weingarten equations

$$\mathbf{II}(\xi_j, \xi_k) = \mathbf{g}(\xi_1, \nabla_{\xi_j}\xi_k) = \phi^1(\nabla_{\xi_j}\xi_k) = \omega_k^1(\xi_j)$$

(for $j, k = 2, 3, 4$) and the second fundamental form of the level sets of the coordinate function

$$s : \mathfrak{H}^2 \longrightarrow \mathbb{R}$$

equals

$$\mathbf{II} = \begin{bmatrix} \tanh(s)\operatorname{sech}(u/2) & 0 & -\frac{1}{2}\tanh(u/2) \\ 0 & 0 & 0 \\ -\frac{1}{2}\tanh(u/2) & 0 & 0 \end{bmatrix}$$

which on $\mathfrak{C}$ equals

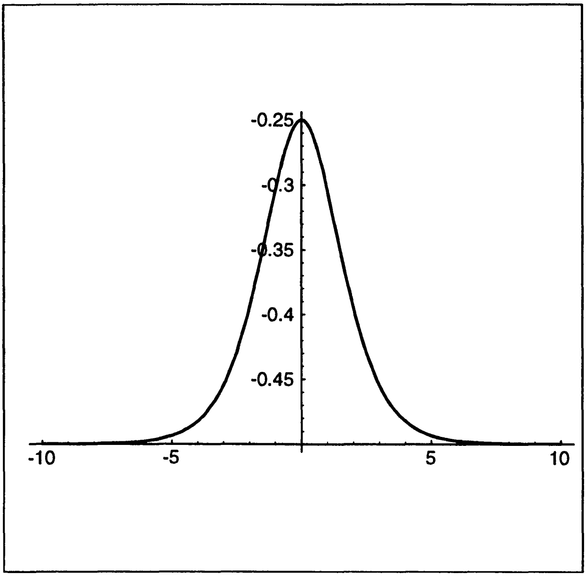

FIG. 5.17. Graph of the function $\tanh(u/2)/(2\tanh(u))$

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2}\tanh(u/2) \\ 0 & 0 & 0 \\ -\frac{1}{2}\tanh(u/2) & 0 & 0 \end{bmatrix}.$$

(Of course the only nonzero entry $\mathbf{II}(\xi_j, \xi_k)$ must be for $\{j,k\} = \{2,4\}$ since the other two coordinate 2-planes are tangent to totally geodesic surfaces.)

The complex structure and the Kähler form can be computed easily in these coordinates. Using formulas above, it follows from $w_1^{-1}dw_1$ and $w_2^{-1}dw_2$ being of type (1,0) that

$$\mathbb{J}(\phi^1) = -\phi^2, \qquad \mathbb{J}(\phi^2) = \phi^1, \qquad \mathbb{J}(\phi^3) = \phi^4, \qquad \mathbb{J}(\phi^4) = -\phi^3$$

and the Kähler form equals

$$\Phi = \phi^2 \wedge \phi^1 + \phi^3 \wedge \phi^4.$$

5.5.4 *Bisectors in complex elliptic space*

Certain bisectors in $\mathbf{H}^n_{\mathbb{C}}$ extend to "bisectors" in the ambient projective space $\mathbb{P}(\mathbb{C}^{n,1})$. Let

$$O = \mathbf{A}(0) \in \mathbb{C}^{n,1}, \quad \langle O, O\rangle = -1$$

define the negative line corresponding to the origin $0 \in \mathbb{B}^n$ and let $\mathbf{U}(n-1) \subset \mathbf{PU}(n,1)$ denote its stabilizer. Let

$$O^\perp = \{X \in \mathbb{C}^{n,1} \mid \langle X, O\rangle = 0\}$$

be the corresponding positive hyperplane. There exists a unique positive definite Hermitian structure $\langle\!\langle , \rangle\!\rangle$ on $\mathbb{C}^{n,1}$ such that

1. $\langle\!\langle O, O^{\perp} \rangle\!\rangle = 0$;
2. $\langle\!\langle X, Y \rangle\!\rangle = \langle X, Y \rangle$ if $X, Y \in O^{\perp}$;
3. $\langle\!\langle O, O \rangle\!\rangle = 1$;

and this positive Hermitian structure is invariant under $\mathbf{U}(n-1)$. Such a Hermitian form induces (by the construction in §3.1.3) *a Fubini–Study Kähler structure* on $\mathbb{P}(\mathbb{C}^{n,1})$ having holomorphic sectional curvature $+1$. If $z_1, z_2 \in \mathbf{H}^n_{\mathbb{C}}$ are points symmetrical about the origin, then the hypersurface in $\mathbb{P}(\mathbb{C}^{n,1})$ equidistant from z_1, z_2 (with respect to the Fubini–Study metric) contains the hypersurface in $\mathbf{H}^n_{\mathbb{C}}$ equidistant from z_1, z_2 (with respect to the Bergman metric). These bisectors in complex projective space have been studied by Lawson [109]. In particular they are minimal hypersurfaces of cohomogeneity 1.

Let $r > 0$; then the real hypersurface $\mathfrak{E}$ consisting of points in $\mathbb{P}(\mathbb{C}^{n,1})$ equidistant from the points z_1, z_2

$$\begin{bmatrix} 0 \\ \pm ri \\ 1 \end{bmatrix}$$

consists of all $\begin{bmatrix} Z' \\ Z_n \\ Z_{n+1} \end{bmatrix}$ for which

$$\bar{Z}_n Z_{n+1} - Z_n \bar{Z}_{n+1} = 0.$$

There is a "spine" and a "slice decomposition" for these equidistant hypersurfaces in projective space as follows. Let Σ be the complex line containing z_1, z_2; in the above coordinates $\Sigma = 0_{n-1} \times \mathbb{CP}^1$. Let $\sigma \subset \Sigma$ be the geodesic in Σ equidistant from z_1 and z_2. Let Σ^* be the $(n-2)$-dimensional projective subspace polar to Σ under the polarity of $\mathbb{P}(\mathbb{C}^{n,1})$ induced by $\langle\!\langle , \rangle\!\rangle$ (that is, Σ^* is the projectivization of the linear subspace of $\mathbb{C}^{n,1}$ which is $\langle\!\langle , \rangle\!\rangle$-orthogonal to the linear subspace corresponding to Σ); in coordinates $\Sigma^* = \mathbb{P}(\mathbb{C}^{n-1} \times \{0\})$. Clearly $\Sigma^* \subset \mathfrak{E}$. Projection away from Σ^* onto Σ defines a projective map

$$\Pi : \mathbb{P}(\mathbb{C}^{n,1}) - \Sigma^* \longrightarrow \Sigma$$
$$\begin{bmatrix} Z' \\ Z_n \\ Z_{n+1} \end{bmatrix} \longmapsto \begin{bmatrix} 0_{n-1} \\ Z_n \\ Z_{n+1} \end{bmatrix}$$

and

$$\mathfrak{E} = \Sigma^* \cup \Pi^{-1}(\sigma).$$

Unlike the case of $\mathbf{H}^n_{\mathbb{C}}$, bisectors in $\mathbb{P}^n_{\mathbb{C}}$ are not smooth manifolds. The singular locus of $\mathfrak{E}$ equals Σ^* and the link of each point of Σ^* is a (real) 2-torus.

Define "cylindrical coordinates" on $\mathfrak{E} - (l \cup l^*)$ by

$$t = \tan\left(\frac{Z_1}{2Z_0}\right), \qquad u = \tan\left|\frac{Z_2}{2Z_0}\right|, \qquad \theta = \arg\left(\frac{Z_2}{2Z_0}\right).$$

The coordinate t defines complex geodesics orthogonal to l passing through l^* with sectional curvature $+1$. The coordinate θ defines totally real geodesic 2-planes passing through l^* with sectional curvature $+1/4$. The induced metric is

$$\mathrm{g} = \cos(u/2)^2 dt^2 + du^2 + \sin(u)^2 d\theta^2$$

in terms of the corresponding orthonormal frame

$$\xi_2 = \sec(u/2)\frac{\partial}{\partial t}, \qquad \xi_3 = \frac{\partial}{\partial u}, \qquad \xi_4 = \csc(\mathrm{u})\frac{\partial}{\partial \theta}$$

the sectional curvatures are

$$\kappa_{23}^{\mathfrak{E}} = \frac{1}{4}, \qquad \kappa_{24}^{\mathfrak{E}} = \frac{\tan(\frac{u}{2})}{2\tan(u)}, \qquad \kappa_{34}^{\mathfrak{E}} = 1$$

and the second fundamental form is determined by

$$\mathbf{II}(e_1, e_3) = \sqrt{\frac{1}{4} - \frac{\tan(u/2)}{2\tan(u)}} = \frac{1}{2}\tan(u/2).$$

5.5.5 *Convexity of bisectors*

Consider a bisector $\mathfrak{E} = \mathfrak{E}_{z_1,z_2} \subset \mathbf{H}_{\mathbb{C}}^n$. That its second fundamental form has only one pair of nonzero entries suggests that a geodesic segment joining two points on $\mathfrak{E}$ must lie entirely in one of the components of $\mathbf{H}_{\mathbb{C}}^n - \mathfrak{E}$, or in a totally geodesic subspace contained in $\mathfrak{E}$.

Theorem 5.5.1 *Let $x, y \in \overline{\mathfrak{E}} \subset \overline{\mathbf{H}_{\mathbb{C}}^n}$ be distinct points and let $\gamma : [a,b] \longrightarrow \overline{\mathbf{H}_{\mathbb{C}}^n}$ be the geodesic with $x = \gamma(a)$ and $y = \gamma(b)$, where $-\infty \leq a < b \leq \infty$. Then the following conditions are equivalent:*

1. *$\gamma(t) \in \mathfrak{E}$ for some $a < t < b$;*
2. *$\gamma(t) \in \mathfrak{E}$ for all $a \leq t \leq b$;*
3. *the orthogonal projection of γ to the complex spine of $\mathfrak{E}$ is a geodesic segment;*
4. *x, y lie in a common slice or meridian of $\mathfrak{E}$.*

Proof Let σ be the spine of $\mathfrak{E}$ and Σ the complex spine and let Π_Σ be orthogonal projection onto Σ so that $\mathfrak{E} = \Pi_\Sigma^{-1}(\sigma)$. Let Γ denote the complex geodesic spanned by x and y. Then by §3.3.2, Π_Σ maps Γ biholomorphically onto either a disc or horodisc in Σ. In particular Π maps γ onto a circular arc in Σ which intersects $\partial\Pi(\Gamma)$ orthogonally. Then the segment s of $\sigma \subset \Sigma$ joining $\Pi(x)$ to $\Pi(y)$ is a circular arc intersecting $\partial\Pi(\Gamma)$ in $\Pi(x)$ and $\Pi(y)$. In the Poincaré metric on $\Pi(\Gamma)$, $\Pi(\gamma)$ is a geodesic and s is a hypercycle equidistant to that geodesic. In particular an interior point of s meets $\Pi(\gamma)$ if and only if $s = \Pi(\gamma)$, proving the equivalence of 1 and 2 with 3. To prove the equivalence with 4, there are two cases: if $\Pi(\gamma)$ is a point $s_0 \in \sigma$, then x, y lie in the slice $\Pi^{-1}(s_0)$. Otherwise by the previous argument we assume $\Pi(\gamma)$ is a geodesic. By Theorem 3.3.1, a totally real 2-plane P contains $x, y, \Pi(x), \Pi(y)$. Since σ is the geodesic containing $\Pi(x)$ and $\Pi(y)$, P is a meridian of $\mathfrak{E}$. $\square$

6

AUTOMORPHISMS

This chapter divides into two separate sections. The first section discusses the symplectic geometry of $\mathbf{H}_{\mathbb{C}}^n$ and constructs Hamiltonian potential functions for various 1-parameter groups of automorphisms. In particular we obtain various geometric formulas relating distance between various objects. By general results of McDuff [119], $\mathbf{H}_{\mathbb{C}}^n$ is symplectomorphic to Euclidean space as well as the cotangent bundle of $\mathbf{H}_{\mathbb{R}}^n$. The second section discusses the classification of conjugacy classes of automorphisms of $\mathbf{H}_{\mathbb{C}}^n$. After some general discussion of the basic elliptic–parabolic–hyperbolic trichotomy, more specific results for $n = 2$ are discussed. In this case (like $\mathbf{SL}(2,\mathbb{C})$) generic conjugacy classes are determined by a single complex number, the *trace* of a lift to $\mathbf{SU}(2,1)$. This gives an appealing picture of the space of conjugacy classes of automorphisms of $\mathbf{H}_{\mathbb{C}}^2$. This theory has been used in [72] (see also Sandler [151]) to obtain partial results concerning discreteness of ideal triangle groups acting on $\mathbf{H}_{\mathbb{C}}^n$. The classification of automorphisms may also be found in Giraud [65].

6.1 Symplectic geometry of $\mathbf{H}_{\mathbb{C}}^n$

One-parameter subgroups of the group $\mathbf{PU}(n,1)$ determine flows on $\mathbf{H}_{\mathbb{C}}^n$ by automorphisms. In particular these flows are Hamiltonian flows for functions $\mathbf{H}_{\mathbb{C}}^n \longrightarrow \mathbb{R}$ related to the geometry of $\mathbf{H}_{\mathbb{C}}^n$. Theorem 2.3.1 may be used to determine the potential function for a given automorphic flow. The level sets of the momentum mapping are then natural submanifolds invariant under the flow and the corresponding symplectic quotient is in many cases complex hyperbolic space. We consider three cases:

1. The flow by complex reflections in a $\mathbb{C}^k$-plane S, in which case S appears as the 0-level set as well as the symplectic quotient (§6.1.1).
2. The flow by the center of the Heisenberg group $\mathfrak{N}$ stabilizing a point $q \in \partial\mathbf{H}_{\mathbb{C}}^n$—here the level sets are the horospheres centered at q and the corresponding symplectic quotients are all $\mathfrak{N}/\mathfrak{Z}(\mathfrak{N}) \approx \mathbb{C}^{n-1}$ (§6.1.2).
3. The flow by a hyperbolic 1-parameter group stabilizing a geodesic σ—here the 0-level set is the bisector $\mathfrak{E}$ having σ as spine and the symplectic quotient is $\mathbf{H}_{\mathbb{C}}^{n-1}$, explicitly identified with (any) slice of $\mathfrak{E}$ (§6.1.3).

6.1.1 *Complex-linear subspaces*

This section discusses various functions related to a $\mathbb{C}^k$-plane $S \subset \mathbf{H}_{\mathbb{C}}^n$. We begin with a formula for the distance from a point to a complex hyperplane (analogous to the distance formula (3.4)). We find a Kähler potential function adapted to S (generalizing the Kähler potential associated to a point; see §3.1.8).

Suppose that S is a complex k-dimensional totally geodesic subspace of $\mathbf{H}^n_{\mathbb{C}}$. Let ι_S denote inversion in S; then for any $z \in \mathbf{H}^n_{\mathbb{C}}$ the geodesic joining z to $\iota_S(z)$ is orthogonal to S and intersects S in the point $\Pi_S(z)$ of S closest to z (here $\Pi_S : \mathbf{H}^n_{\mathbb{C}} \longrightarrow S$ denotes orthogonal projection). Applying an automorphism, assume that $S = \{0\} \times \mathbb{B}^k \subset \mathbf{H}^n_{\mathbb{C}}$, in which case ι_S is represented by the matrix $-\mathbb{I}_{n-k} \oplus \mathbb{I}_{k+1} \in \mathbf{U}(n,1)$. Then (applying (3.4))

$$\begin{aligned}\cosh(\rho(z,S)) &= \cosh\left(\frac{\rho(z,\iota_S(z))}{2}\right)\\ &= 1 + 2\frac{\sum_{j=1}^{n-k} z_j\bar{z}_j}{1-\|z\|^2}.\end{aligned} \tag{6.1}$$

Now let $\varrho(S) = \{\varrho_\zeta\}_{\zeta\in\mathbb{T}}$ denote the (circle) group of complex reflections in S (where ϱ_ζ is represented by $\zeta\mathbb{I}_{n-k} \oplus \mathbb{I}_{k+1}$). The infinitesimal generator is then $\xi = i\mathbb{I}_{n-k} \oplus 0_{k+1}$ and a Hamiltonian potential function for the corresponding flow $e^{-t\xi} = \varrho_{e^{-it}}$ (for $t \in \mathbb{R}$) on $\mathbb{C}^{n,1}$ is

$$\tilde{f} : Z \longmapsto \operatorname{trace}(\mu(Z)\cdot\xi) = \frac{1}{2}\sum_{j=1}^{n-k} Z_j\bar{Z}_j$$

where μ is the equivariant momentum mapping for the linear action of $\mathbf{U}(n,1)$ on $\mathbb{C}^{n,1}$ discussed in Theorem 2.3.1.

Under the symplectic embedding $\mathbf{A} : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \mathbb{C}^{n,1}$ (defined in §3.1.1) and the change of coordinates

$$\begin{aligned}Z_j &= \frac{2}{\sqrt{1-\|z\|^2}}z_j \quad (1 \le j \le n)\\ Z_{n+1} &= \frac{2}{\sqrt{1-\|z\|^2}}\end{aligned}$$

we obtain a $\varrho(S)$-equivariant Hamiltonian function f for the corresponding (projective) action on $\mathbf{H}^n_{\mathbb{C}}$:

$$\begin{aligned}f(z) &= \frac{\sum_{j=1}^{n-k} z_j\bar{z}_j}{1-\|z\|^2}\\ &= \frac{\cosh(\rho(z,S))-1}{2}\\ &= \sinh^2\left(\frac{\rho(z,S)}{2}\right).\end{aligned}$$

In particular the level sets of f bound metric tubular neighborhoods of S. In particular $f^{-1}(0) = S$ and the symplectic quotient

$$f^{-1}(0)/\varrho(S) = S \approx \mathbf{H}_{\mathbb{C}}^{k}.$$

The other symplectic quotients $f^{-1}(\epsilon)/\varrho(S)$ are the projective normal bundles over S.

For example, in the case S is a point, that is $k = 0$, then the Hamiltonian potential for the flow of complex reflections about S is

$$f_S : z \mapsto \sinh^2\left(\frac{\rho(z,S)}{2}\right)$$

and is related to the Kähler potential ψ_S associated to S (see §3.1.8) by

$$f_S(z) = e^{-\psi_S(z)} - 1.$$

In the case of a hyperplane, that is $k = n - 1$, this function determines a Kähler potential (in a sense "dual" to the Kähler potential associated to a point in $\mathbf{H}_{\mathbb{C}}^{n}$). As above, choose coordinates so that $S = \{0\} \times \mathbb{B}^{n-1}$, so that the stabilizer of S is $\mathbf{U}(n-1,1) \subset \mathbf{PU}(n,1)$, embedded in the natural way—projection $\mathbf{U}(n,1) \longrightarrow \mathbf{PU}(n,1)$ restricts injectively to the linear group

$$1 \oplus \mathbf{U}(n-1,1) \subset \mathbf{U}(n,1).$$

The function $\psi_S : \mathbf{H}_{\mathbb{C}}^{n} \longrightarrow \mathbb{R}$

$$\psi_S(z) = 2\log\operatorname{csch}\left(\frac{\rho(z,S)}{2}\right)$$

is a (singular) Kähler potential, evidently invariant under $\mathbf{U}(n-1,1)$. By (3.6),

$$\psi_O(z) = \log(1 - \|z\|^2)$$

is a Kähler potential for $\mathbf{H}_{\mathbb{C}}^{n}$. Since the linear coordinate

$$z_1 : \mathbf{H}_{\mathbb{C}}^{n} \longrightarrow \mathbb{C}$$

is holomorphic, $\log(z_1\bar{z}_1)$ is the real part of a holomorphic function on $\mathbf{H}_{\mathbb{C}}^{n} - S$ and is hence pluriharmonic. (6.1) implies that

$$\psi_S(z) = \log(1 - \|z\|^2) - \log(z_1\bar{z}_1)$$

is a Kähler potential on the complement $\mathbf{H}_{\mathbb{C}}^{n} - S$, the *Kähler potential associated to S*.

We can express these functions in terms of a polar vector in a formula analogous to (3.4) as follows. Let $Z_s \in \mathbb{C}^{n,1}$ be a positive vector polar to S in the sense that $S = \mathbb{P}((Z_s)^{\perp})$.

Theorem 6.1.1 *Let $z \in \mathbf{H}_{\mathbb{C}}^{n}$ be represented by a negative vector $Z \in \mathbb{C}^{n,1}$. Then distance from z to the hyperplane S is determined by*

$$\rho(z,S) = 2\ \sinh^{-1}\sqrt{-\frac{\langle Z, Z_s\rangle\langle Z_s, Z\rangle}{\langle Z, Z\rangle\langle Z_s, Z_s\rangle}}.$$

The proof is a direct consequence of the preceding formulas and is therefore omitted.

6.1.2 *Horospheres and ideal points*

Next consider an ideal point $q \in \partial \mathbf{H}^n_{\mathbb{C}}$ and the 1-parameter subgroup which forms the center of the Heisenberg group stabilizing q (that is, the unipotent radical of the stabilizer of q). As usual choose coordinates so that q is represented by the point p_∞ as in §4.1.1; then the above-mentioned center is $\exp(\mathfrak{g}_{-2})$ in the root-space decomposition of §4.1.3. (In Heisenberg coordinates, this 1-parameter corresponds to vertical translation.) In terms of a null vector $Q \in \mathbb{C}^{n,1}$ representing q, this group has infinitesimal generator (in terms of outer products as in §4.1.4)

$$\xi = iQQ^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & -i \\ 0 & i & i \end{bmatrix}$$

and by a calculation similar to (6.1) implies the function

$$f_Q(z) = \frac{P|Z_n + Z_{n+1}|^2}{2} = \frac{4\ |1 + z_n|^2}{1 - \|z\|^2}$$

defines a Hamiltonian potential function for the corresponding flow. This function is related to the Busemann function for q (see (4.2)) and is related to the Hermitian structure by

$$f_Q(z) = e^{h_c(z)} = \frac{\langle Z, Q\rangle\langle Q, Z\rangle}{\langle Z, Z\rangle}.$$

The level sets are horospheres centered at q and the symplectic quotients are the quotients of these horospheres by $\exp(\mathfrak{g}_{-2})$, which can be identified via the action of the Heisenberg group $\mathfrak{N} = \exp(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1})$ with $\mathfrak{N}/\exp(\mathfrak{g}_{-2}) \cong \mathbb{C}^{n-1}$.

6.1.3 *Geodesics and bisectors*

Bisectors can be realized as level sets of momentum mappings as follows. Let $\sigma \subset \mathbf{H}^n_{\mathbb{C}}$ be a geodesic and let

$$\mathcal{A} = \{e^{t\xi}\}_{t\in\mathbb{R}}$$

be the hyperbolic 1-parameter subgroup stabilizing σ and let $\Sigma \subset \mathbf{H}^n_{\mathbb{C}}$ be the unique complex geodesic containing σ. We normalize ξ so that for any $s_0 \in \sigma$, the geodesic path $\exp(t\xi)(s_0)$ has unit speed.

Theorem 6.1.2 *Let $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ denote orthogonal projection onto Σ. Then the function $f : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \mathbb{R}$ defined by*

$$f(z) = \sinh(\rho(\Pi_\Sigma(z), \sigma))\ \cosh^2\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right)$$

is a Hamiltonian function for the flow corresponding to the $\mathcal{A}$-action on $\mathbf{H}^n_{\mathbb{C}}$, whose 0-level set $f^{-1}(0)$ equals the bisector $\mathfrak{E}$ having σ as spine. The corresponding symplectic quotient $f^{-1}(0)/\mathcal{A}$ equals complex hyperbolic $(n-1)$-space $\mathbf{H}^{n-1}_{\mathbb{C}}$ represented as a slice of $\mathfrak{E}$.

Proof Since $\mathbf{PU}(n,1)$ acts transitively on bisectors, it suffices to consider one particular example. If σ is the geodesic $\{0\} \times \mathbf{H}^1_{\mathbb{R}}$ with endpoints $(0,\ldots,0,\pm 1)$, then ξ is represented by the matrix

$$\xi = \begin{bmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}.$$

Theorem 2.3.1 implies that the flow corresponding to the action of $\mathcal{A}$ on $\mathbb{C}^{n,1}$ has Hamiltonian potential function

$$\tilde{f} : \mathbb{C}^{n,1} \longrightarrow \mathbb{R}$$

given by

$$\tilde{f} : Z \mapsto \operatorname{trace}(\mu(Z) \cdot \xi) = \frac{1}{2}\operatorname{Im}(Z_{n+1}\bar{Z}_n)$$

where

$$\mu : \mathbb{C}^{n,1} \longrightarrow \mathfrak{u}(n,1)$$

is the equivariant momentum mapping defined in Theorem 2.3.1.

We use the ball model of $\mathbf{H}^n_{\mathbb{C}}$. Complex hyperbolic n-space is obtained as the quotient of

$$(\operatorname{trace} \circ \mu)^{-1}(-4i)$$

by $\mathbb{T}$ where the ball $\mathbb{B}^n$ by is mapped to $\mathbb{C}^{n,1}$ by the symplectic embedding

$$\mathbf{H}^n_{\mathbb{C}} \approx \mathbb{B}^n \xrightarrow{\phi} (\operatorname{trace} \circ \mu)^{-1}(-4i) \subset \mathbb{C}^{n,1}$$

$$z \mapsto \frac{2}{\sqrt{1 - \|z\|^2}} \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ 1 \end{bmatrix}.$$

The Hamiltonian function $\tilde{f}$ restricts to the Hamiltonian function $f = \tilde{f} \circ \phi$ for the induced flow of $\mathcal{A}$ on $\mathbf{H}^n_{\mathbb{C}}$:

$$f(z) = \frac{2\operatorname{Im}(z_n)}{1 - \|z\|^2}.$$

Let $\Sigma \subset \mathbf{H}^n_{\mathbb{C}}$ be the complex geodesic containing σ. Then the distance (in Σ) from $\Pi_\Sigma(z)$ to the real geodesic σ is given by

$$\sinh(\rho(\Pi_\Sigma(z), \sigma)) = \frac{2\ \mathrm{Im}(z_n)}{1 - z_n \bar{\zeta}_n}$$

and distance to Σ equals the function $\rho_\Sigma : \mathbf{H}^n_{\mathbb{C}} \to \mathbb{R}$ defined by

$$\begin{aligned}\cosh^2\left(\frac{\rho_\Sigma(z)}{2}\right) &= \cosh^2\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right) \\ &= \frac{1 - |z_n|^2}{1 - \|z\|^2}.\end{aligned}$$

and thus

$$f(z) = \sinh(\rho(\Pi_\Sigma(z), \sigma))\ \cosh^2\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right)$$

as desired. □

6.1.4 *Distance to bisectors*

Theorem 6.1.2 expresses a duality between geodesics (real dimension 1) and bisectors (real codimension 1). We can relate these functions to the distance functions between bisectors and geodesics as follows.

Theorem 6.1.3 *Let σ be a geodesic, $\Sigma \supset \sigma$ the complex geodesic containing it and $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ orthogonal projection. Let $\mathfrak{E} = \Pi_\Sigma^{-1}(\sigma)$ be the bisector having spine σ. Then for every $z \in \mathbf{H}^n_{\mathbb{C}}$, its distance to $\mathfrak{E}$ is given by*

$$\sinh\left(\frac{\rho(z, \mathfrak{E})}{2}\right) = \sinh\left(\frac{\rho(\Pi_\Sigma(z), \sigma)}{2}\right) \cosh\left(\frac{\rho(z, \Sigma)}{2}\right). \tag{6.2}$$

Proof By working in the $\mathbb{C}^2$-plane spanned by σ and z, we may assume that $n = 2$. Let S_t $(-1 < t < 1)$ be the slices of $\mathfrak{E}$, with S_{t_0} the slice closest to z. Then we claim that $\Pi_\Sigma(S_{t_0}) \in \sigma$ is the point of σ closest to $\Pi_\Sigma(z)$. For we may choose coordinates so that

$$\begin{aligned}\sigma &= \{0\} \times \mathbf{H}^1_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}} \\ \Sigma &= \{0\} \times \mathbf{H}^1_{\mathbb{C}}\end{aligned}$$

and S_t is the complex geodesic defined by $z_2 = t$, polar to

$$\tilde{S}_t = \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix}.$$

Thus the value of $t = t_0$ minimizes the function

$$\sinh^2\left(\frac{\rho(z,S_t)}{2}\right) = \frac{\langle \mathbf{A}(z), \tilde{S}_t\rangle\langle \tilde{S}_t, \mathbf{A}(z)\rangle}{\langle \mathbf{A}(z), \mathbf{A}(z)\rangle\langle \tilde{S}_t, \tilde{S}_t\rangle}$$
$$= \frac{|z_2 - t|^2}{(1-\|z\|^2)(1-t^2)}$$

whose critical points are the solutions of the quadratic equation

$$t^2 - 2\frac{1+z_2\bar{z}_2}{z_2+\bar{z}_2}t + 1 = 0. \tag{6.3}$$

The point $\sigma_s = (0,s)$ of σ closest to $\Pi_\Sigma(z)$ minimizes the function

$$\cosh^2\left(\frac{\rho(\Pi_\Sigma(z),\sigma_s)}{2}\right) = \frac{|z_2 s - 1|^2}{(1-\|z\|^2)(1-s^2)}$$

whose critical points also solve (6.3). Since (6.3) admits only one solution with $|t|<1$, the claim follows.

Let $s_t \in S_t$ be the point closest to z. By Lemma 3.2.14, the four points

$$z, \Pi_\Sigma(z), s_t, \Pi_\Sigma(s_t)$$

lie in a totally geodesic $\mathbb{R}^2$-plane, forming a Lambert quadrilateral. Theorem 6.1.3 now follows from Lemma 3.2.14. □

6.1.5 *Distance to a geodesic*

These quantities relate to the distance function to a geodesic as follows:

Theorem 6.1.4 *Let σ be a geodesic and $\Sigma \supset \sigma$ be the unique complex geodesic containing σ. Then for any $z \in \mathbf{H}^n_{\mathbb{C}}$, the distance functions from σ and Σ are related by*

$$\cosh\left(\frac{\rho(z,\sigma)}{2}\right) = \cosh\left(\frac{\rho(z,\Sigma)}{2}\right)\cosh\left(\frac{\rho(\Pi_\Sigma(z),\sigma)}{2}\right).$$

Proof Let $\Pi_\Sigma : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma$ denote orthogonal projection. If $s \in \Sigma$, then $z, \Pi_\Sigma(z), s$ lie on a totally real subspace; this can be seen by choosing coordinates so that

$$\Sigma = \{z \in \mathbf{H}^n_{\mathbb{C}} \mid z_1 = \cdots = z_{n-1} = 0\}$$

so that

$$\Pi_\Sigma(z) = (0,\ldots,0,z_n).$$

By applying an automorphism we may assume that z satisfies $z_n = 0$ and since $s \in \Sigma$, we have $s = (0,\ldots,0,s_n)$. Then the $\mathbb{R}$-linear subspace

$$\{(tz_1,\ldots,tz_{n-1},us_n) \mid t,u \in \mathbb{R}\}$$

meets $\mathbf{H}^n_{\mathbb{C}}$ in a totally real geodesic subspace. Now the triangle

$$\triangle(z, \Pi_\Sigma(z), s)$$

in P has a right angle at the vertex $\Pi_\Sigma(z)$ and by the Pythagorean theorem (for a space of constant curvature $-1/4$, see (3.9) with $k = 1/2$)

$$\cosh\left(\frac{\rho(z,s)}{2}\right) = \cosh\left(\frac{\rho(z,\Pi_\Sigma(z))}{2}\right)\cosh\left(\frac{\rho(\Pi_\Sigma(z),s)}{2}\right). \tag{6.4}$$

Now consider the geodesic $\sigma \subset \Sigma$. (6.4) implies that the point s_0 on σ closest to z equals the point on σ closest to $\Pi_\Sigma(z)$. Thus

$$\rho(z,\sigma) = \rho(\Pi_\Sigma(z),\sigma).$$

Furthermore the closest point on $\mathfrak{C} = \Pi_\Sigma^{-1}(\sigma)$ to z lies on the slice $\Pi_\Sigma^{-1}(s_0) \subset \Pi_\Sigma^{-1}(\sigma)$ closest to $\mathfrak{C} = \Pi_\Sigma^{-1}(\sigma)$. Thus $\rho(z,\mathfrak{C}) = \rho(\Pi_\Sigma(z), s_0) = \rho(\Pi_\Sigma(z),\sigma)$. Moreover, $\rho(z,\Sigma) = \rho(z,\Pi_\Sigma(z))$ and the result follows by the Pythagorean theorem in a totally real 2-plane, (see (3.9) in §3.2.1 with $k = 1/2$). □

6.1.6 *Distance to bisectors and geodesics*

In summary:

$$\cosh\left(\frac{\rho(z,\sigma)}{2}\right) = \cosh\left(\frac{\rho(z,\Sigma)}{2}\right)\cosh\left(\frac{\rho(\Pi_\Sigma(z),\sigma)}{2}\right)$$
$$\sinh\left(\frac{\rho(z,\mathfrak{C})}{2}\right) = \cosh\left(\frac{\rho(z,\Sigma)}{2}\right)\sinh\left(\frac{\rho(\Pi_\Sigma(z),\sigma)}{2}\right).$$

Eliminating $\rho(\Pi_\Sigma(z),\sigma)$ from these two equations, the above distance functions to the submanifolds $\sigma, \Sigma, \mathfrak{C}$ respectively relate as follows:

$$\sinh^2\left(\frac{\rho(z,\sigma)}{2}\right) = \sinh^2\left(\frac{\rho(z,\Sigma)}{2}\right) + \sinh^2\left(\frac{\rho(z,\mathfrak{C})}{2}\right).$$

Thus the Hamiltonian potential for the flow $\mathcal{A}$ is

$$\begin{aligned} f_\sigma(z) &= \sinh\left(\rho(\Pi_\Sigma(z),\sigma)\right)\cosh^2\left(\frac{\rho(z,\Sigma)}{2}\right) \\ &= 2\cosh\left(\frac{\rho(z,\sigma)}{2}\right)\sinh\left(\frac{\rho(z,\mathfrak{C})}{2}\right). \end{aligned}$$

6.2 Classification of automorphisms in dimension 2

This section classifies automorphisms of $\mathbf{H}^2_{\mathbb{C}}$. In particular we find an analogue of the familiar trichotomy of elements of $\mathbf{PGL}(2,\mathbb{C})$ into elliptic, parabolic and hyperbolic types. Also, generic conjugacy classes in $\mathbf{PU}(2,1)$ are determined by a trace invariant in the quotient of $\mathbb{C}$ by the group of cube roots of unity—analogous to the parametrization of generic conjugacy classes in $\mathbf{PGL}(2,\mathbb{C})$ by a trace invariant in the quotient of $\mathbb{C}$ by $\{\pm 1\}$.

6.2.1 *Hyperbolic, parabolic and elliptic automorphisms*

An automorphism g of $\mathbf{H}^n_{\mathbb{C}}$ lifts to a unitary transformation $\tilde{g}$ of $\mathbb{C}^{n,1}$ and the fixed points of g on $\mathbb{P}(\mathbb{C}^{n,1})$ correspond to eigenvectors of $\tilde{g}$. By the Brouwer fixed-point theorem, every automorphism of $\mathbf{H}^n_{\mathbb{C}}$ has a fixed point in $\mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}}$. An automorphism g is *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{C}}$, *parabolic* if it has a unique fixed point on $\partial\mathbf{H}^n_{\mathbb{C}}$, and *(weakly) hyperbolic* or *loxodromic* if it fixes a unique pair of points on $\partial\mathbf{H}^n_{\mathbb{C}}$. If g is loxodromic, then $\tilde{g}$ has one eigenvalue outside the unit circle and one eigenvalue inside the unit circle. An element is *strictly hyperbolic* if it is loxodromic and a lift exists all of whose eigenvalues are real. An element g is elliptic if and only if g generates a cyclic group whose closure is compact. The eigenvalues of the matrix corresponding to g all have norm 1. We say that an elliptic element g is *regular elliptic* if and only if its eigenvalues are distinct. A parabolic element g is *unipotent* if it can be represented by a unipotent element of $\mathbf{U}(n,1)$; that is, a linear transformation having 1 as its only eigenvalue. g is *ellipto-parabolic* if it is not unipotent; in that case a unique invariant complex geodesic Σ exists upon which g acts as a parabolic automorphism (of $\mathbf{H}^1_{\mathbb{C}}$) and g acts by a nontrivial unitary automorphism on the normal bundle of Σ. (Compare the discussion in Giraud [65] and Chen–Greenberg [27].)

Theorem 6.2.1 *The regular elliptic elements of* $\mathbf{PU}(n,1)$ *form an open set.*

Proof Suppose that $g \in \mathbf{PU}(n,1)$ is regular elliptic. Represent g by a linear transformation $\tilde{g} \subset \mathbf{U}(n,1)$ such that the eigenvalues of $\tilde{g}$ are distinct. Since the set of linear transformations with distinct eigenvalues is open, it intersects $\mathbf{U}(n,1)$ in an open set which projects to an open neighborhood of g in $\mathbf{PU}(n,1)$. □

Theorem 6.2.1 and Corollary 6.2.2 hold in more generality, for example if G is the group of automorphisms of a homogeneous Kähler manifold, such as a Hermitian symmetric space or a Griffiths period domain. For G a real semisimple Lie group, the condition that elliptic elements (defined as elements generating precompact cyclic subgroups) have nonempty interior is equivalent to the existence of a compact Cartan subgroup.

6.2.2 *Purely hyperbolic groups are discrete*

The following result is due to Chen–Greenberg [27]; see also Kulkarni [108]. (I am grateful to Ravi Kulkarni for a conversation in 1979 introducing me to some of the ideas.)

Corollary 6.2.2 *Let* $\Gamma \subset \mathbf{PU}(n,1)$ *be Zariski dense. If no element of* Γ *is elliptic, then* Γ *is discrete.*

Proof Let $G = (\bar{\Gamma})^0$ be the identity component of the closure of Γ. We claim that since Γ is Zariski dense, G either equals all of $\mathbf{PU}(n,1)$ or is trivial. For G is a closed connected subgroup of $\mathbf{PU}(n,1)$ and is hence a Lie subgroup. Let $\mathfrak{g}$ be its Lie algebra. Clearly Γ normalizes the Lie algebra $\mathfrak{g}$ of G. Thus $\mathrm{Ad}(\Gamma)$ preserves $\mathfrak{g}$. The condition on $\gamma \in \mathbf{PU}(n,1)$ that

$$\mathrm{Ad}(\gamma)(\mathfrak{g}) = \mathfrak{g}$$

is a polynomial condition on γ. Therefore the normalizer

$$\{\gamma \in G \mid \mathrm{Ad}(\gamma)(\mathfrak{g}) = \mathfrak{g}\}$$

of $\mathfrak{g}$ is an algebraic subgroup of $\mathbf{PU}(n,1)$ containing Γ. Since Γ is Zariski dense, G is a normal closed connected Lie subgroup of $\mathbf{PU}(n,1)$. Such a subgroup of $\mathbf{PU}(n,1)$ is either trivial or $\mathbf{PU}(n,1)$ itself (Exercise ex:punosimple).

$G = \mathbf{PU}(n,1)$ if and only if Γ is dense (with respect to the classical topology). In this case Γ intersects the open set consisting of regular elliptic elements, contradicting the hypothesis that Γ contains no elliptic elements. Thus G is trivial; that is, Γ is discrete. The proof of Corollary 6.2.2 is complete. □

Remark 6.2.3 *This corollary is false in general Lie groups; for example, there exist subgroups of* $\mathrm{SL}(2,\mathbb{C})$ *or* $\mathrm{SL}(3,\mathbb{R})$ *which are dense in the classical topology but contain no elliptic elements.*

6.2.3 *Traces and conjugacy classes in* $\mathbf{SU}(2,1)$

Let $C_3 \subset \mathbb{C}$ denote the set of cube roots of unity. There is a short exact sequence

$$1 \longrightarrow C_3 \longrightarrow \mathbf{SU}(2,1) \longrightarrow \mathbf{PU}(2,1) \longrightarrow 1.$$

Let $\tau : \mathbf{SU}(2,1) \longrightarrow \mathbb{C}$ be the function which assigns to an element of $\mathbf{SU}(2,1)$ its trace. Since an element of $\mathbf{PU}(2,1)$ has three lifts to $\mathbf{SU}(2,1)$ which differ by a cube root of unity, a holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ has a complex number as trace, well-defined only up to multiplication by a cube root of unity.

Our main result involves the real polynomial $f : \mathbb{C} \longrightarrow \mathbb{R}$ defined by

$$f(z) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27. \tag{6.5}$$

Theorem 6.2.4 *The map* $\tau : \mathbf{SU}(2,1) \longrightarrow \mathbb{C}$ *defined by trace is surjective. If* $A_1, A_2 \in \mathbf{SU}(2,1)$ *satisfy* $\tau(A_1) = \tau(A_2) \in \mathbb{C} - f^{-1}(0)$*, then they are conjugate. Suppose* $A \in \mathbf{SU}(2,1)$.

1. *A is regular elliptic if and only if* $f(\tau(A)) < 0$;
2. *A is loxodromic if and only if* $f(\tau(A)) > 0$;
3. *A is ellipto-parabolic if and only if A is not elliptic and* $\tau(A) \in f^{-1}(0) - 3C_3$;
4. *A is a complex reflection (about either a point or a complex geodesic) if and only if A is elliptic and* $\tau(A) \in f^{-1}(0) - 3C_3$;
5. $\tau(A) \in 3C_3$ *if and only if A represents a unipotent automorphism of* $\mathbf{H}^2_{\mathbb{C}}$.

Figure 6.1 depicts the level set $f^{-1}(0)$ of f. Its interior corresponds to conjugacy classes of elliptic elements and its exterior corresponds to conjugacy classes of loxodromic elements. Its boundary corresponds to various parabolic conjugacy classes and its three cusps correspond to the three central elements (and various unipotent conjugacy classes).

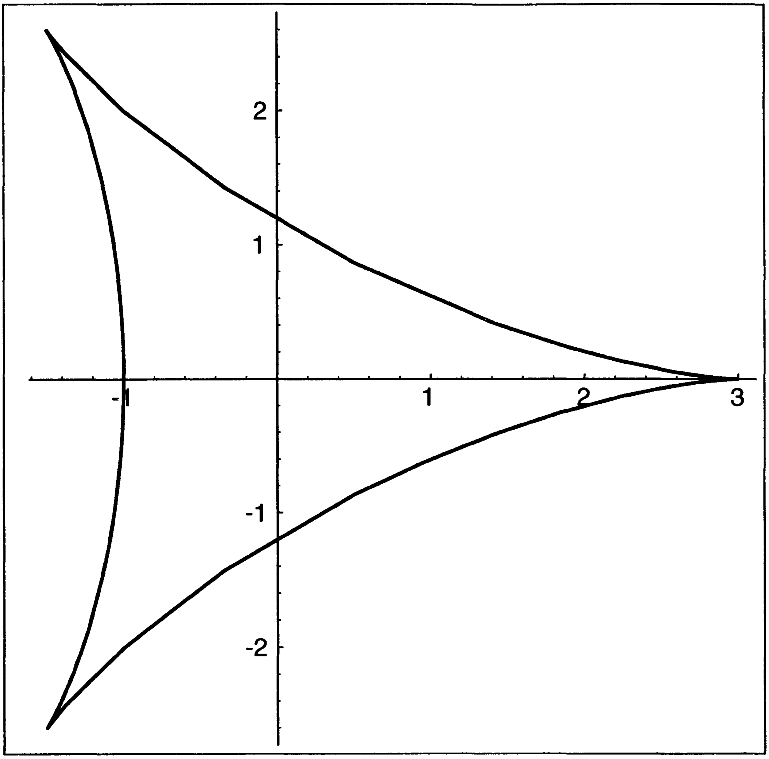

FIG. 6.1. Traces of elliptic elements

Proof The proof of Theorem 6.2.4 uses the following lemma:

Lemma 6.2.5 *Let E be a Hermitian vector space and let A be a unitary automorphism of E. Then the set of eigenvalues of A is invariant under inversion $\iota_{\mathbb{T}}$ in the unit circle in $\mathbb{C}$:*

$$\begin{aligned} \iota_{\mathbb{T}} : \mathbb{C} &\longrightarrow \mathbb{C} \\ (z) &\longmapsto 1/\bar{z}. \end{aligned}$$

(Compare the discussion in §2.1.4.)

Proof of Lemma 6.2.5 A is unitary with respect to a Hermitian form on E defined by a Hermitian matrix M if and only if

$$\bar{A}^{\dagger} M A = M.$$

Equivalently,

$$A = M^{-1}(\bar{A}^{\dagger})^{-1} M.$$

Thus A has the same eigenvalues as $\bar{A}^{-1}$. If λ is an eigenvalue of A, then so is $\iota_{\mathbb{T}}(\lambda) = \bar{\lambda}^{-1}$. $\square$

We resume the proof of Theorem 6.2.4. Lemma 6.2.5 implies that if E has odd dimension each unitary automorphism has at least one eigenvalue of norm 1. Furthermore the eigenvalues not on the unit circle occur in $\iota_{\mathbb{T}}$-invariant pairs.

Let $\chi_A(t)$ be the characteristic polynomial of A:

$$\chi_A(t) = t^3 - xt^2 + yt - 1.$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A are the roots of $\chi_A(t)$. The trace of A is

$$x = \tau(A) = \lambda_1 + \lambda_2 + \lambda_3$$

and the determinant $\det(A) = 1$ equals

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{6.6}$$

The coefficient y in the characteristic polynomial of A equals

$$y = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = \bar{\tau}(A)$$

(by (6.6)). Thus the characteristic polynomial of A equals

$$\chi_A(t) = t^3 - \tau(A)t^2 + \overline{\tau(A)}t - 1.$$

If $A \in \mathbf{SU}(2,1)$, then its three eigenvalues form a set

$$\tilde{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$$

satisfying (6.6) and

$$\lambda \in \tilde{\lambda} \Longrightarrow \iota_{\mathbb{T}}(\lambda) = \bar{\lambda}^{-1} \in \tilde{\lambda}. \tag{6.7}$$

Let $\tilde{\Lambda}$ denote the set of all such unordered triples satisfying (6.6) and Λ denote the set of all such unordered triples $\tilde{\lambda}$ of complex numbers satisfying (6.6) and (6.7). Then $\iota_{\mathbb{T}}$ induces an involution on $\tilde{\Lambda}$ (also denoted $\iota_{\mathbb{T}}$) whose fixed-point set equals Λ. The map

$$\begin{aligned} \chi : \tilde{\Lambda} &\longrightarrow \mathbb{C}^2 \\ \tilde{\lambda} &\longmapsto (\lambda_1 + \lambda_2 + \lambda_3, \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \end{aligned}$$

is bijective, with inverse map

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \tilde{\Lambda} \\ (x, y) &\longmapsto \{t \in \mathbb{C} \mid t^3 - xt^2 + yt - 1 = 0\}. \end{aligned}$$

The involution j on $\mathbb{C}^2$ defined by

$$j(x,y) = (\bar{y}, \bar{x})$$

satisfies

$$\chi \circ \iota_{\mathbb{T}} = j \circ \chi$$

and χ maps the fixed-point-set Λ of $\iota_{\mathbb{T}}$ bijectively onto the fixed-point-set of j on $\mathbb{C}^2$, which is the image of

$$\begin{aligned} e : \mathbb{C} &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto (z, \bar{z}). \end{aligned}$$

Clearly $e \circ \chi|_\Lambda = \tau$.

Let $\tilde{\Lambda}_{\text{sing}} \subset \tilde{\Lambda}$ be the subset comprising unordered triples $\{\lambda_1, \lambda_2, \lambda_3\}$ which are not distinct. By the theory of resultants, χ maps $\tilde{\Lambda}_{\text{sing}}$ bijectively onto

$$\{(x,y) \in \mathbb{C}^2 \mid \tilde{f}(x,y) = 0\}$$

where $\tilde{f}(x,y)$ is the discriminant of $\chi(t) = t^3 - xt^2 + yt - 1$:

$$\begin{aligned} \tilde{f}(x,y) &= \text{Resultant}(\chi, \chi') \\ &= \begin{vmatrix} 1 & -x & y & -1 & 0 \\ 0 & 1 & -x & y & -1 \\ 3 & -2x & y & 0 & 0 \\ 0 & 3 & -2x & y & 0 \\ 0 & 0 & 3 & -2x & y \end{vmatrix} \\ &= -x^2y^2 + 4(x^3 + y^3) - 18xy + 27. \end{aligned}$$

Let $\Lambda_0 = \Lambda - \tilde{\Lambda}_{\text{sing}}$. Then τ maps $\Lambda \cap \tilde{\Lambda}_{\text{sing}}$ bijectively onto $f^{-1}(0)$ and $\Lambda_0 = \Lambda - \tilde{\Lambda}_{\text{sing}}$ bijectively onto $\mathbb{C} - f^{-1}(0)$ where

$$f(z) = (\tilde{f} \circ e)(z) = \tilde{f}(z, \bar{z})$$

as above. Thus $\tau : \mathbf{SU}(2,1) \longrightarrow \mathbb{C}$ is surjective.

Let Λ_0^l be the subset of Λ such that only one $\lambda_i \in \mathbb{T}$. Rearranging $\tilde{\lambda}$ if necessary we may assume that

$$|\lambda_1| > 1, \qquad |\lambda_2| < 1, \qquad |\lambda_3| = 1. \tag{6.8}$$

In particular $\Lambda_0^l \subset \Lambda_0$. Furthermore given λ_1 satisfying (6.8), we uniquely obtain λ_2, λ_3 by

$$\lambda_2 = (\bar{\lambda}_1)^{-1}, \qquad \lambda_3 = \frac{\bar{\lambda}_1}{\lambda_1}$$

proving that Λ_0^l is homeomorphic to the outside of the unit disc in $\mathbb{C}$ and hence connected. Let $\Lambda_0^e = \Lambda_0 - \Lambda_0^l$. If $\lambda \in \mathbb{R}^* - \{\pm 1\}$, then

$$\tau(\lambda, \lambda^{-1}, 1) = \lambda + \lambda^{-1} + 1$$

proving that τ maps the set of all real points of Λ_0^l bijectively onto

$$\mathbb{R} - [-1, 3]$$

upon which f is positive. Since Λ_0^l is connected, τ maps Λ_0^h bijectively onto $f^{-1}(\mathbb{R}^+)$ and τ maps Λ_0^e bijectively onto $f^{-1}(\mathbb{R}^-)$.

Thus $f(\tau(A)) < 0$ if and only if the eigenvalues of A are distinct unit complex numbers, that is if and only if A is regular elliptic. $f(\tau(A)) > 0$ if and only if A has exactly one eigenvalue outside $\mathbb{T}$, one eigenvalue inside $\mathbb{T}$ and one unit eigenvalue, that is if and only if A is loxodromic.

Next consider the case when $f(\tau(A)) = 0$. Evidently $A \in \mathbf{PU}(2,1)$ is unipotent if and only if it has a lift to $\mathbf{SU}(2,1)$ all of whose eigenvalues are equal. Since such an eigenvalue must be a cube root of unity, it follows that the set of eigenvalues of A lies in Λ_2 and $\frac{1}{3}\tau(A) \in C_3$. Conversely if $\frac{1}{3}\tau(A) \in C_3$, then the characteristic polynomial of A is

$$\chi_A(\lambda) = \left(\lambda - \frac{\tau(A)}{3}\right)^3$$

and A has three equal eigenvalues and is projectively equivalent to a unipotent matrix.

Finally consider the case when $\tau(A) \in f^{-1}(0) - 3C_3$. Then A has an eigenvalue $\zeta \in \mathbb{T}$ of multiplicity 2 and another eigenvalue equal to ζ^{-2}. Since $\tau(A) \notin 3C_3$, $\zeta^{-2} \neq \zeta$. There are two cases depending on the Jordan canonical form for A: if A is diagonalizable then A is elliptic and represents a complex reflection. These split into two cases as well, depending on whether the ζ-eigenspace V_ζ is positive or indefinite: if V_ζ is indefinite, then A represents a complex reflection about the complex geodesic corresponding to V_ζ and if V_ζ is positive, then A represents a complex reflection about the point corresponding to the ζ^{-2}-eigenspace (the Hermitian orthogonal complement of V_ζ). If A is not diagonalizable, then A has a repeated eigenvalue λ of norm 1 and has Jordan canonical form

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}.$$

In this case the λ-eigenvector is necessarily null and A is ellipto-parabolic. (In Heisenberg coordinates these are represented by vertical translations composed with nontrivial rotations about the vertical axis.) This concludes the proof of Theorem 6.2.4. □

Thus the set $f^{-1}(0)$ admits the parametric representation

$$x(\zeta) = \zeta^{-2} + 2\zeta$$

where $\zeta \in \mathbb{T}$ ranges over the unit complex numbers. Since

$$\frac{dx(e^{i\theta})}{d\theta} = 2ie^{i\theta}(1 - e^{-3i\theta}),$$

the only singular points of the boundary occur for $\theta \in (2\pi i/3)\mathbb{Z}$ corresponding to the three singular points $3C_3 \subset f^{-1}(0)$. For $\theta \sim 0$,

$$x(\theta) = 3 - 3\theta^2 + i\theta^3 + O(\theta^4)$$

and the singularities of $f^{-1}(0)$ at $3C_3$ are cubic cusps.

7

NUMERICAL INVARIANTS

This chapter summarizes several invariants in complex hyperbolic geometry. First, the "invariante angulaire" discussed by Cartan [21] is presented. This invariant is associated to triples of points on the absolute and has been used by Toledo in his investigation [41, 163, 162] of surface group actions on complex hyperbolic space. Following that is the "complex cross-ratio" introduced by Korányi and Reimann in [105]. Finally, inspired by these invariants and the distance formulas, we introduce an invariant associated to a real geodesic and a complex hyperplane, which is used in the following chapter to investigate intersections of bisectors.

7.1 Cartan's angular invariant

7.1.1 *Angular invariants of point triples*

Associated to a triple x of points in $\partial \mathbf{H}^n_{\mathbb{C}}$ are simple invariants detecting whether x lies on a chain or an $\mathbb{R}$-circle. Following Cartan [21], consider such a triple of distinct points

$$x = (x_1, x_2, x_3) \in \mathcal{C}_3(\partial \mathbf{H}^n_{\mathbb{C}}).$$

Choose lifts $\tilde{x}_i \in \mathbb{C}^{n,1}$ with $\mathbb{P}(\tilde{x}_i) = x_i$. Then the Hermitian triple product

$$\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = \langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle \in \mathbb{C}$$

has negative real part by (2.17). (Since each $\tilde{x}_i$ is a null vector, none of the Hermitian products vanish.) Furthermore replacing $\tilde{x}_i$ by $\xi_i \tilde{x}_i$ (where $\xi_i \in \mathbb{C}^*$ and $i = 1, 2, 3$) multiplies this complex number by the positive real number $|\xi_1 \xi_2 \xi_3|^2$. Thus

$$\mathbb{A}(x) = \arg\left(-\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \right)$$

is independent of the chosen lifts and satisfies

$$-\frac{\pi}{2} \le \mathbb{A}(x) \le \frac{\pi}{2}.$$

We call $\mathbb{A}(x)$ the *Cartan angular invariant* of x. It is the only invariant of a triple of points, in the following sense:

Theorem 7.1.1 *Let x_1, x_2, x_3 and y_1, y_2, y_3 be pairs of distinct triples such that*

$$\mathbb{A}(x_1, x_2, x_3) = \mathbb{A}(y_1, y_2, y_3).$$

Then there exists a holomorphic automorphism $g \in \mathbf{PU}(n, 1)$ such that $g(x_i) = y_i$ for $i = 1, 2, 3$. When $n = 2$, this automorphism is unique unless x_1, x_2, x_3 lie in a chain. For general n, this automorphism is uniquely determined up to an automorphism leaving invariant the 2-plane spanned by x_1, x_2, x_3.

Proof Consider first the case that $x \notin$ **Chain**. Choose lifts $\tilde{x}_i, \tilde{y}_i \in \mathbb{C}^{n,1}$ of the points in $\partial\mathbf{H}^n_{\mathbb{C}}$ to null vectors. Since x_1, x_2, x_3 (respectively y_1, y_2, y_3) are distinct null vectors not contained in chains, both sets $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ and $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ are linearly independent and the Hermitian products $\langle \tilde{x}_i, \tilde{x}_j \rangle$ (respectively $\langle \tilde{y}_i, \tilde{y}_j \rangle$) are nonzero for $i \neq j$. By appropriate rescaling we assume that

$$\langle \tilde{x}_1, \tilde{x}_2 \rangle = \langle \tilde{x}_1, \tilde{x}_3 \rangle = \langle \tilde{y}_1, \tilde{y}_2 \rangle = \langle \tilde{y}_1, \tilde{y}_3 \rangle = 1$$

and that

$$\langle \tilde{x}_2, \tilde{x}_3 \rangle = \langle \tilde{y}_2, \tilde{y}_3 \rangle = e^{i\mathbb{A}} \tag{7.1}$$

where

$$\mathbb{A} = \mathbb{A}(x_1, x_2, x_3) = \mathbb{A}(y_1, y_2, y_3).$$

Now (7.1) implies that the linear map defined by $\tilde{x}_i \mapsto \tilde{y}_i$ is unitary and extends to a unitary automorphism of $\mathbb{C}^{n,1}$. The corresponding projective map is the desired automorphism of $\mathbf{H}^n_{\mathbb{C}}$. When $x, y \in$ **Chain**, one obtains a unique Hermitian isometry between the indefinite 2-planes spanned by x, y respectively, although the extension of this isometry to all of $\mathbb{C}^{2,1}$ will not be unique. □

7.1.2 *Geometric interpretation of Cartan's invariant*

The Cartan invariant enjoys the following geometric interpretation:

Theorem 7.1.2 *Let $(x_1, x_2, x_3) \in C_3(\partial\mathbf{H}^n_{\mathbb{C}})$ and let Σ_{12} (respectively σ_{12}) denote the complex (respectively real) geodesic spanned by x_1 and x_2. Let $\Pi_{12} : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma_{12}$ denote orthogonal projection. Then*

$$|\tan(\mathbb{A}(x_1, x_2, x_3))| = \sinh(\rho(\Pi_{12}(x_3), \sigma_{12})).$$

This identity quantitatively expresses the equality of the meridianal decomposition (corresponding to the left-hand side vanishing) with the slice decomposition (corresponding to the right-hand side vanishing). The function $y = \tan^{-1}(\sinh(x))$ is graphed in Fig.7.1.

Before proving Theorem 7.1.2, we list several elementary properties, whose proofs are immediate from the definition.

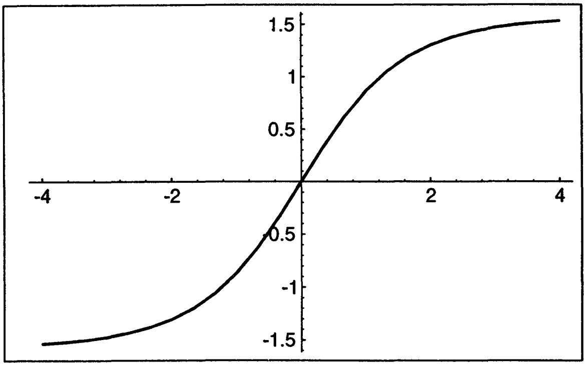

FIG. 7.1. Graph of $\tan^{-1}(\sinh(\mathbb{A}))$

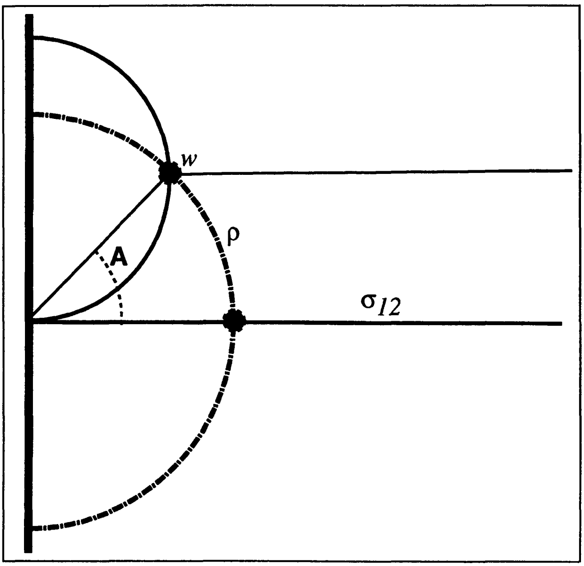

FIG. 7.2. Orthogonal projection of ideal triangle to complex geodesic

1. If $\tau \in \mathfrak{S}_3$ is a permutation, then

$$\mathbb{A}(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = \mathrm{sign}(\tau)\mathbb{A}(x_1, x_2, x_3) \tag{7.2}$$

2. If $g \in \mathbf{PU}(n, 1)$ is a holomorphic automorphism, then clearly $\mathbb{A}(g(x)) = \mathbb{A}(x)$ and if g is an anti-holomorphic automorphism, then $\mathbb{A}(g(x)) = -\mathbb{A}(x)$.

Proof Choose coordinates so that x_1, x_2, x_3 are represented by null vectors

$$\tilde{p}_\infty = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \tilde{p}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} z' \\ z_n \\ 1 \end{bmatrix}$$

respectively, and the definition implies that

$$\tan(\mathbb{A}(x)) = \frac{\mathrm{Im}(w)}{\mathrm{Re}(w)} = \frac{2\ \mathrm{Im}(z_n)}{1 - |z_n|^2} \tag{7.3}$$

where

$$w = (1 + z_n)(1 - \bar{z}_n) = 1 - |z_n|^2 + 2i\mathrm{Im}(z_n).$$

The complex geodesic Σ_{12} containing x_1 and x_2 equals $\{0\} \times \mathfrak{H}^1$, its orthogonal projection is given by

$$\Pi_{12}(z) = (0, z_n),$$

and the real geodesic σ_{12} containing x_1 and x_2 equals

$$\{0\} \times \mathfrak{H}^1_{\mathbb{R}}.$$

(Compare Fig. 7.2.) Since distance from a point $w \in \mathfrak{H}^1$ to the geodesic $\mathfrak{H}^1_{\mathbb{R}}$ is given by

$$\sinh(\rho(w, \mathfrak{H}^1_{\mathbb{R}})) = \frac{\mathrm{Im}(w)}{\mathrm{Re}(w)},$$

Theorem 7.1.2 follows. □

Corollary 7.1.3 *Suppose that $x \in C_3(\partial \mathbf{H}^n_{\mathbb{C}})$. Then*

$$x \in \mathbf{Chain} \iff \mathbb{A}(x) = \pm\frac{\pi}{2}$$

(the sign depending on the orientation of x_1, x_2, x_3 along the chain).

(Compare Cartan [21], [22].)

Proof Suppose that x_1, x_2, x_3 form a positively oriented triple of distinct points on a chain. Then $\Pi_{12}(x_3) = x_3$ and $\rho(x_3, \sigma_{12}) = \infty$. Hence $\mathbb{A}(x) = \tan^{-1}\infty = \pm\pi/2$. To see that the sign is positive, check for a specific value for x_3, that is

$$\tilde{x}_3 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix},$$

so

$$-\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = 4i$$

and $\mathbb{A}(x) = \pi/2$ as desired.

If x_1, x_2, x_3 form a negatively oriented triple along a chain, then (7.2) implies

$$\mathbb{A}(x_1, x_2, x_3) = -\mathbb{A}(x_2, x_1, x_3) = -\frac{\pi}{2}.$$

Conversely, suppose that $\mathbb{A}(x) = \pi/2$. By Theorem 7.1.2, this means that $\Pi_{12}(x_3)$ has infinite distance to σ_{12}, that is $\Pi_{12}(x_3)$ lies on $\partial\mathbf{H}^n_{\mathbb{C}}$. However, $\Pi_{12}^{-1}(\partial\mathbf{H}^n_{\mathbb{C}}) \subset \partial\Sigma_{12}$ and thus x_3 lies on $\partial\Sigma_{12}$. The proof of Corollary 7.1.3 is complete. □

In particular the complex structure induces a natural orientation on each chain, although $\mathbb{R}$-circles do not have such a natural orientation.

Applying (2.2.5) one obtains (compare [21], [22]):

Theorem 7.1.4 *(Cartan) Suppose that $x \in C_3(\partial\mathbf{H}^n_{\mathbb{C}})$. Then*

$$x \in \mathbf{Real} \Longleftrightarrow \mathbb{A}(x) = 0.$$

Corollary 7.1.5 *Let $x, y \in \partial\mathbf{H}^n_{\mathbb{C}}$. Then the spinal sphere with vertices x, y equals the set*

$$\{u \in \partial\mathbf{H}^n_{\mathbb{C}} \mid \mathbb{A}(x, y, u) = 0\}.$$

Proof Let σ be the geodesic in $\mathbf{H}^n_{\mathbb{C}}$ with endpoints x, y. By Corollary 5.1.11, the spinal sphere $\partial\mathfrak{S}$ with spine σ is the union of all $\mathbb{R}$-spheres containing x, y. Since u lies on an $\mathbb{R}$-circle with x and y if and only if $\mathbb{A}(x, y, u) = 0$, the result follows. □

We have the following converse to Theorem 7.1.1:

Lemma 7.1.6 *Let $(x_1, x_2, x_3) \in C_3(\partial\mathbf{H}^2_{\mathbb{C}})$. Suppose that*

$$g \in \mathbf{Aut}(\mathbf{H}^2_{\mathbb{C}})$$

satisfies $g(x_i) = x_i$ for $i = 1, 2, 3$. Then g equals the identity unless:

1. *$x \in$ **Real** and g is conjugation in the $\mathbb{R}^2$-plane containing x_1, x_2, x_3;*
2. *$x \in$ **Chain** and g is a complex reflection in the complex geodesic containing x_1, x_2, x_3.*

(For an alternative discussion of 1 see Jacobowitz [92], §9.1, Theorem 4.)

Proof First we suppose that g is holomorphic, that is $g \in \mathbf{PU}(2,1)$. Choose lifts $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in \mathbb{C}^{2,1}$ and a lift $\tilde{g} \in \mathbf{U}(2,1)$. If $x \notin$ **Chain**, then choose $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ to be linearly independent over $\mathbb{C}$ and form a basis of $\mathbb{C}^{2,1}$. Since the x_i are fixed under g, the $\tilde{x}_i$ are eigenvectors for $\tilde{g}$; that is, there exists $\lambda_i \in \mathbb{C}^*$ such that $\tilde{g}\tilde{x}_i = \lambda_i \tilde{x}_i$. We claim that $\lambda_1 = \lambda_2 = \lambda_3$; that is, $\tilde{g}$ is a scalar matrix.

The basic fact we shall use is the following. Since $\tilde{x}_i$ and $\tilde{x}_j$ are linearly independent null vectors, (2.16) implies that the Hermitian cross-product $\tilde{y}_{ij} = \tilde{x}_i \boxtimes \tilde{x}_j$ is a positive vector in $\mathbb{C}^{2,1}$ and is an eigenvector for $\tilde{g}$ (with eigenvalue $\overline{\lambda_i \lambda_j}$).

If all the eigenvalues of $\tilde{g}$ are distinct, then $\tilde{g}$ is diagonalizable and any eigenvector must be a scalar multiple of an $\tilde{x}_i$. Thus all eigenvectors are null, contradicting the existence of a positive eigenvector above. If two eigenvalues are coincident (say $\lambda_1 = \lambda_2$) then the span E of $\tilde{x}_1$ and $\tilde{x}_2$ is an indefinite complex 2-plane and contains every positive eigenvector for $\tilde{g}$ (since the other eigenspace is a null line). But $\tilde{y}_{13}$ is a positive eigenvector with $\langle \tilde{y}_{13}, E\rangle = 0$, a contradiction.

Next consider the case that $x \in$ **Chain**; that is, when $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are linearly dependent over $\mathbb{C}$. Let E be the complex 2-plane they span; then by an argument similar to the one above, the restriction of $\tilde{g}$ to E must be a scalar. The orthogonal complement $E^\perp$ is a positive line invariant under $\tilde{g}$ and the orthogonal direct sum decomposition

$$\mathbb{C}^{2,1} = E \oplus E^\perp$$

is $\tilde{g}$-invariant. With respect to this decomposition

$$\tilde{g} = \lambda_1 \mathbb{I}_E \oplus \lambda_3 \mathbb{I}_{E^\perp},$$

that is $\tilde{g}$ is a complex reflection in E.

Now suppose that $g \in \mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}})$ is anti-holomorphic. Then g^2 is holomorphic and the results above imply that either $g^2 = \mathbb{I}$ or there exists a complex geodesic Σ with $x_i \in \partial\Sigma$ and g^2 is a complex reflection in Σ. But

$$\begin{aligned} \mathbb{A}(x) &= \mathbb{A}(g(x)) \quad && (\text{since } x = g(x)) \\ &= -\mathbb{A}(x) && (\text{by 2, since } g \text{ is anti-holomorphic}) \end{aligned}$$

so that $x \in$ **Real** by Theorem 7.1.4, completing the proof of Lemma 7.1.6. □

Lemma 7.1.7 *Let $x = (x_1, x_2, x_3) \in C_3(\partial\mathbf{H}^n_{\mathbb{C}})$ be a distinct triple of points. Then an $\mathbb{R}^2$-plane P exists such that inversion ι_P in P satisfies*

$$\iota_P(x_1) = x_2, \qquad \iota_P(x_2) = x_1, \qquad \iota_P(x_3) = x_3.$$

Furthermore P is unique if and only if $x \notin$ **Chain**.

Proof Let $\mathbb{A} = \mathbb{A}(x) \in [-\pi/2, \pi/2]$. Then

$$\tilde{y}_1 = \begin{bmatrix} ie^{i\mathbb{A}} \\ 0 \\ 1 \end{bmatrix}, \qquad \tilde{y}_2 = \begin{bmatrix} -ie^{-i\mathbb{A}} \\ 0 \\ 1 \end{bmatrix}, \qquad \tilde{y}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

are null vectors representing points $y_1, y_2, y_3 \in \partial\mathbf{H}^n_{\mathbb{C}}$ respectively. Furthermore $\mathbb{A}(y) = \mathbb{A}(x)$ and by Lemma 7.1.6, there exists $h \in \mathbf{PU}(2,1)$ with $h(x) = y$. Let P' be the standard real form $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$; inversion ι' in P' is given by complex conjugation $Z \mapsto \bar{Z}$ which interchanges $\tilde{y}_1$ and $\tilde{y}_2$ and fixes $\tilde{y}_3$. The inversion $\iota_P = h^{-1} \circ \iota' \circ h$ in $P = h^{-1}P'$ interchanges x_1 and x_2 while fixing x_3.

For the uniqueness suppose that $g_1, g_2 \in \mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}})$ each take x to y. By 2, each g_i must be anti-holomorphic. Then $g_1(g_2)^{-1}$ is holomorphic, fixes x, and by Lemma 7.1.6 is the identity unless $x \in$ **Chain**. □

7.1.3 *Symmetric triples of points and the barycenter*

Corollary 7.1.8 *Let $\mathfrak{S}_3$ be the symmetric group on $\{1,2,3\}$. Let*

$$x = (x_1, x_2, x_3) \in C_3(\partial \mathbf{H}^2_{\mathbb{C}})$$

be a distinct triple of points. There exists a representation

$$\phi_x : \mathfrak{S}_3 \longrightarrow \mathbf{Aut}(\mathbf{H}^2_{\mathbb{C}})$$

such that for each permutation $\tau \in \mathfrak{S}_3$,

$$\phi_x(\tau) : (x_1, x_2, x_3) \mapsto (x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}). \tag{7.4}$$

*Furthermore, unless $x \in$ **Chain**, $\phi_x(\mathfrak{S}_3)$ has a unique fixed point in $\mathbf{H}^2_{\mathbb{C}}$.*

Proof The only part of this result whose proof is not immediate is the assertion concerning the fixed point. Choose representative null vectors $\tilde{x}_i \in \mathbb{C}^{2,1}$ and consider the following endomorphism of $\mathbb{C}^{2,1}$:

$$\tilde{g} = -\mathbb{I} + \frac{\tilde{x}_1 \tilde{x}_2^*}{\langle \tilde{x}_1, \tilde{x}_2 \rangle} + \frac{\tilde{x}_2 \tilde{x}_3^*}{\langle \tilde{x}_2, \tilde{x}_3 \rangle} + \frac{\tilde{x}_3 \tilde{x}_1^*}{\langle \tilde{x}_3, \tilde{x}_1 \rangle}. \tag{7.5}$$

Since

$$\tilde{g}(\tilde{x}_i) = \frac{\langle \tilde{x}_i, \tilde{x}_{i-1} \rangle}{\langle \tilde{x}_{i+1}, \tilde{x}_{i-1} \rangle} \tilde{x}_{i+1} \tag{7.6}$$

the corresponding projective transformation $g = \mathbb{P}(\tilde{g})$ satisfies (7.4). Furthermore the adjoint endomorphism is

$$\tilde{g}^* = -\mathbb{I} + \frac{\tilde{x}_2 \tilde{x}_1^*}{\langle \tilde{x}_2, \tilde{x}_1 \rangle} + \frac{\tilde{x}_3 \tilde{x}_2^*}{\langle \tilde{x}_3, \tilde{x}_2 \rangle} + \frac{\tilde{x}_1 \tilde{x}_3^*}{\langle \tilde{x}_1, \tilde{x}_3 \rangle}$$

and applying (7.6) twice,

$$\tilde{g}^* \tilde{g}(\tilde{x}_i) = \tilde{x}_i$$

whence $\tilde{g} \in \mathbf{U}(2,1)$. Since $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ constitute a basis of $\mathbb{C}^{2,1}$, it follows from (7.6) that $\tilde{g}$ has trace 0 and determinant $\delta = \langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$; indeed its characteristic polynomial is $t^3 - \delta$. Since $\tilde{g}$ is not a scalar, it is diagonalizable with eigenvalues the three cube roots of unity. Thus $\tilde{g}$ and hence g has order 3. Since g has finite order it must have at least one fixed point in $\mathbb{B}^2$, and such a fixed point is necessarily isolated, since otherwise $\tilde{g}$ would have a repeated eigenvalue. The proof of Corollary 7.1.8 is complete. □

The unique fixed point of g is the *barycenter* of the triple $\{x_1, x_2, x_3\} \subset \partial \mathbf{H}^n_{\mathbb{C}}$.

7.1.4 *Characteristic classes and the angular invariant*

Closely related to this is an invariant related to the Kähler form on $\mathbf{H}^n_{\mathbb{C}}$ used by Toledo in his proof [163] of rigidity of actions of surface groups on $\mathbf{H}^n_{\mathbb{C}}$. For this proof it is useful to extend the Cartan invariant to triples of points in $\mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}}$; no difficulty arises in the consideration of negative vectors in the preceding discussion. If $x = (x_1, x_2, x_3) \in \mathcal{C}_3(\mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}})$ as above, form the 1-cycle $\Delta(x)$ in $\mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}}$ with edges the oriented geodesics with vertices x_i. For any singular 2-chain $\sigma \subset \mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}}$ with $\partial\sigma = \Delta(x)$, the integral of the Kähler form Φ over σ converges and is independent of σ (indeed its absolute value is bounded by π) and we obtain a function

$$\mathcal{C}_3(\partial\mathbf{H}^n_{\mathbb{C}}) \longrightarrow \mathbb{R}$$
$$x \mapsto c(x) = \int_\sigma \Phi = \int_{\Delta(x)} \alpha$$

where α is a 1-form on $\mathbf{H}^n_{\mathbb{C}}$ with $d\alpha = \Phi$.

Explicitly let

$$\alpha = d^c \log \Psi_x$$

where Ψ_x is the Kähler potential function associated to a point $x \in \mathbf{H}^n_{\mathbb{C}}$ (see (3.6) in §3.1.8). This primitive for Φ has the property that its integral over any complete geodesic is finite:

Lemma 7.1.9 *Let $\gamma \subset \mathbf{H}^n_{\mathbb{C}}$ be a complete geodesic and let $\alpha = d^c\psi_x$ be the primitive for the Kähler form associated to a point $x \in \mathbf{H}^n_{\mathbb{C}}$. Then the integral*

$$\int_\gamma \alpha$$

converges.

Proof If $x \in \gamma$, then α restricts identically to zero on γ as can be checked directly by taking x to be the origin. Otherwise choose $y \in \gamma$ and since

$$\begin{aligned}\psi_x(z) - \psi_y(z) = \log(1 - \|x\|^2) &- \log(1 - \|y\|^2)\\ &+ \log(1 - \langle\langle z, y\rangle\rangle) + \log(1 - \langle\langle y, z\rangle\rangle)\\ &- \log(1 - \langle\langle z, x\rangle\rangle) - \log(1 - \langle\langle x, z\rangle\rangle)\end{aligned}$$

we have

$$d^c\psi_x - d^c\psi_y = d\ 2\arg\left(\frac{1 - \langle\langle z, y\rangle\rangle}{1 - \langle\langle z, x\rangle\rangle}\right)$$

and (using the fact that $\int_\gamma d^c\psi_y = 0$)

$$\frac{1}{2}\int_\gamma \alpha = \arg\left(\frac{1 - \langle\langle z_1, y\rangle\rangle}{1 - \langle\langle z_1, x\rangle\rangle}\right) - \arg\left(\frac{1 - \langle\langle z_2, y\rangle\rangle}{1 - \langle\langle z_2, x\rangle\rangle}\right)$$

is finite, where z_1 and z_2 are the two endpoints of γ. The proof of Lemma 7.1.9 is complete. □

This lemma provides a concrete expression for Toledo's integral. (Compare Domic–Toledo [41].) Namely, let x_0 be the barycenter of the triple $\{x_1, x_2, x_3\}$ as defined in Corollary 7.1.8 and let σ_{ij} be the geodesic with endpoints x_i, x_j. For each point $y \in \sigma_{ij}$ let r_y be the geodesic segment from x_0 to y and let

$$\triangle_0(x_1, x_2, x_3) = \bigcup_{y \in \sigma_{12} \cup \sigma_{23} \cup \sigma_{31}} r_y$$

be the cone on $\triangle(x)$ at x_0. Define an explicit form of Toledo's invariant by

$$c(x) = \int_{\triangle_0(x_1,x_2,x_3)} d^c \Psi_{x_0}.$$

An invariant obtained by integrating a closed form over a simplex always satisfies a cocycle property of the following type (see Toledo [163]):

Lemma 7.1.10 *(Toledo) For any* $(x_1, x_2, x_3, x_4) \in \mathcal{C}_4(\partial \mathbf{H}^n_{\mathbb{C}})$,

$$c(x_1, x_2, x_3) - c(x_1, x_2, x_4) + c(x_1, x_3, x_4) - c(x_2, x_3, x_4) = 0. \qquad (7.7)$$

Proof Choose a 3-simplex X with faces $\partial_i X = \tilde{\triangle}(x_1, \ldots, \widehat{x_i}, \ldots, x_4)$. Then the left-hand side of (7.7) equals

$$\int_{\partial_1 X} \Phi - \int_{\partial_2 X} \Phi + \int_{\partial_3 X} \Phi - \int_{\partial_4 X} \Phi = \int_{\partial X} \Phi = 0$$

since Φ is exact. □

Toledo's invariant relates to Cartan's invariant by:

Theorem 7.1.11 $c(x) = 2\mathbb{A}(x)$.

Proof

$$\int_{\triangle(x)} \Phi = \int_{\triangle(x)} (\Pi_{12})^* \Phi_{12} = \int_{\Pi_{12}(\triangle(x))} \Phi_{12}$$

(where Φ_{12} is the area form on Σ_{12}) equals the area of the triangle in Σ_{12} with vertices $x_1, x_2, \Pi_{12}(x_3)$. For taking $x_4 = \Pi_{12}(x_3)$, Lemma 7.1.10 implies that

$$c(x_1, x_2, x_3) - c(x_1, x_2, x_4) = c(x_1, x_3, x_4) - c(x_2, x_3, x_4).$$

But Lemma 3.2.13 implies that x_1, x_3, x_4 (respectively x_2, x_3, x_4) are the vertices of a (right) triangle in a totally real totally geodesic subspace. Since the restriction of the Kähler form Φ to a totally real geodesic subspace is identically zero,

$$c(x_1, x_3, x_4) = c(x_2, x_3, x_4) = 0$$

and $c(x)$ equals the integral of Φ over the triangle in Σ_{12} with vertices x_1, x_2 and $\Pi_{12}(x_3)$. Since the area of the triangle in the Poincaré right half-plane having vertices $0, \infty, z$ is given by

$$\text{area}(\triangle(0, p_\infty, z)) = -2\arg(z)$$

(compare Exercise 1.4.7). Theorem 7.1.11 now follows from Lemma 7.1.6. □

Applying Lemma 7.1.10 to Lemma 7.1.9 we obtain the following cocycle property for Cartan's angular invariant:

Corollary 7.1.12 *Suppose that* $x \in \mathcal{C}_4(\partial \mathbf{H}^n_{\mathbb{C}})$. *Then*

$$\mathbb{A}(x_1,x_2,x_3)+\mathbb{A}(x_1,x_3,x_4)=\mathbb{A}(x_1,x_2,x_4)+\mathbb{A}(x_2,x_3,x_4).$$

A more direct proof follows from the definition. Since

$$\langle \tilde{x}_1,\tilde{x}_2,\tilde{x}_3\rangle\langle \tilde{x}_1,\tilde{x}_3,\tilde{x}_4\rangle = \langle \tilde{x}_1,\tilde{x}_2,\tilde{x}_4\rangle\langle \tilde{x}_2,\tilde{x}_3,\tilde{x}_4\rangle,$$

$$\mathbb{A}(x_1,x_2,x_3)+\mathbb{A}(x_1,x_3,x_4)\equiv\mathbb{A}(x_1,x_2,x_4)+\mathbb{A}(x_2,x_3,x_4) \pmod{2\pi}.$$

But

$$\begin{aligned}\mathcal{C}_4(\partial \mathbf{H}^n_{\mathbb{C}}) &\longrightarrow 2\pi\mathbb{Z}\subset\mathbb{R}\\ x &\longmapsto (\mathbb{A}(x_1,x_2,x_3)+\mathbb{A}(x_1,x_3,x_4))\\ &\qquad -(\mathbb{A}(x_1,x_2,x_4)+\mathbb{A}(x_2,x_3,x_4))\end{aligned}$$

is a continuous function which is zero if x_1,x_2,x_3,x_4 are real vectors. Since $\mathcal{C}_4(\partial \mathbf{H}^n_{\mathbb{C}})$ is connected, this function is identically zero.

7.1.5 *Parallel transport on Heisenberg space*

Given two points $q_1,q_2\in\partial\mathbf{H}^n_{\mathbb{C}}$, no natural identification of the tangent spaces $T_{q_1}\partial\mathbf{H}^n_{\mathbb{C}}$ and $T_{q_2}\partial\mathbf{H}^n_{\mathbb{C}}$ exists. However, the *real projective spaces associated to the CR-structures* do identify as follows. For any $q\in\partial\mathbf{H}^n_{\mathbb{C}}$ let S_q denote the sphere of directions associated to the vector space $(E_q)_{\mathbb{R}}$ (that is, the space of oriented Legendrian lines in the tangent space to $\partial\mathbf{H}^n_{\mathbb{C}}$ at q). (More generally for any $1\le k\le n-1$, consider the Grassmannian $S_q^{(k)}$ of oriented totally real k-planes in E_q; although there is no naturally selected Hermitian metric in E_q, using the preferred *conformal class* there is a well-defined sense of "totally real.") Then for any oriented $\mathbb{R}^k$-plane $P\subset E_{q_1}$ we define a corresponding oriented $\mathbb{R}^k$-plane $\tau_{q_1,q_2}(P)\subset E_{q_2}$ as follows. The CR-structure E_q equals the tangent space $T_q(\mathfrak{S})$ of any spinal sphere having q as a vertex. Furthermore if P is an oriented totally real k-plane in E_{q_1}, then a unique geodesic $\mathbb{R}^{k+1}$-plane $\mathcal{P}$ contains the geodesic σ from q_1 to q_2, which is bounded by an $\mathbb{R}$-sphere whose tangent space at q_1 equals P. (See §4.4.11.) Define

$$\tau_{q_1,q_2}(P)=T_{q_2}(\mathcal{P}).$$

The collection

$$\{\tau_{q_1,q_2}\mid (q_1,q_2)\in\mathcal{C}_2(\partial\mathbf{H}^n_{\mathbb{C}})\}$$

is invariant under $\mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}})$ in the sense that for any automorphism g,

$$\tau_{g(q_1),g(q_2)}\circ dg = dg\circ\tau_{q_1,q_2}$$

as maps $S^k_{q_1}\longrightarrow S^k_{g(q_2)}$. Moreover,

$$\tau_{q_1,q_2}\circ\tau_{q_2,q_1}=\mathbb{I}.$$

Theorem 7.1.13 *If $(q_1, q_2, q_3) \in C_3(\partial \mathbf{H}^n_{\mathbb{C}})$ are distinct points, then the composition*

$$\tau_{q_3,q_1} \circ \tau_{q_2,q_3} \circ \tau_{q_1,q_2} : S^k_{q_1} \longrightarrow S^k_{q_1}$$

equals the map induced by scalar multiplication on E_{q_1} by

$$e^{-3i\mathbb{A}(q_1,q_2,q_3)}.$$

The proof uses another description of τ_{q_1,q_2} in terms of the slices of $\mathfrak{S}(q_1, q_2)$. Suppose that Σ is a slice of $\mathfrak{S} = \mathfrak{S}_{q_1,q_2}$ with inversion ι. Then ι interchanges q_1 and q_2 and τ_{q_1,q_2} is the map induced on the Grassmannian by the differential

$$d\iota : E_{q_1} = T_{q_1}\mathfrak{S} \longrightarrow E_{q_2} = T_{q_2}\mathfrak{S}$$

and reversing orientation (although the linear map $d\iota$ depends on the chosen slice Σ, the resulting map of real projective space is independent of Σ). To see that $-d\iota$ induces τ_{q_1,q_2} directly, it is perhaps easiest to directly compute in Heisenberg coordinates: let q_1 be the origin in Heisenberg coordinates and q_2 the point with coordinates $(r, 0)$ where $r > 0$. Then E_{q_1} and E_{q_2} have respective bases

$$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}, \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - 2r\frac{\partial}{\partial v}\right\}$$

in which the complex structure $\mathbb{J}$ is represented as usual by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since the real axis $\mathbb{R} \times \{0\} \subset \mathcal{H}$ is a meridian of $\mathfrak{S}$,

$$\begin{aligned} \tau_{q_1,q_2} : E_{(0,0)} &\longrightarrow E_{(r,0)} \\ \frac{\partial}{\partial x} &\longmapsto \frac{\partial}{\partial x}. \end{aligned} \tag{7.8}$$

Since $\tau_{q_1,q_2} : E_{(0,0)} \longrightarrow E_{(r,0)}$ is $\mathbb{C}$-linear and commutes with complex reflections about the complex geodesic containing q_1 and q_2, (7.8) implies

$$\begin{aligned} \tau_{q_1,q_2} : E_{(0,0)} &\longrightarrow E_{(r,0)} \\ \frac{\partial}{\partial y} &\longmapsto \frac{\partial}{\partial y} + r\frac{\partial}{\partial v}. \end{aligned}$$

In particular τ_{q_1,q_2} agrees with the differential of the unique Heisenberg translation taking q_1 to q_2. On the other hand the vertical chain $\zeta = r/2$ is a slice of $\mathfrak{S}$ with corresponding inversion

$$\iota(\zeta, v) = (r - \zeta, v - r\mathrm{Im}(\zeta))$$

whose differential takes the above basis of $E_{(0,0)}$ to the negative of the above basis of $E_{(r,0)}$.

Proof of Theorem 7.1.13 Let $t = \tan(\mathbb{A}(q_1, q_2, q_3))$; choose Heisenberg coordinates so that q_1 is the origin $(0,0) \in \mathcal{H}$, q_2 is the point $(1,0)$ as above and q_3 is the point $(1, 1/(2t))$. Then $t = \tan(\mathbb{A})$ where $\mathbb{A} = \mathbb{A}(q_1, q_2, q_3)$. Then E_{q_1}, E_{q_2} and E_{q_3} have bases

$$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}, \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - 2r\frac{\partial}{\partial v}\right\}, \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\},$$

respectively. By the above remarks, τ_{q_1,q_2} maps the above basis of E_{q_1} to the above basis of E_{q_2}. We next calculate τ_{q_2,q_3}. The chain centered at $(1 - it, t)$ having radius $\sqrt{1+t^2} = \sec(\mathbb{A})$ is polar to

$$\begin{bmatrix} 2 \\ (i-t)/(i+t) \\ 1 \end{bmatrix} \in \mathbb{C}^{2,1}$$

and its inversion ι_{23} maps the point with Heisenberg coordinates $(r, 0)$ (where $r \in \mathbb{R}$) to the point whose Heisenberg ζ-coordinate equals

$$\frac{x(t+i)\left\{(t+i) + ix\right\}}{t(i-t) + x(2it - x)}.$$

Therefore the differential $(d\iota)_{q_2} : E_{q_2} \longrightarrow E_{q_3}$ corresponds to multiplication by $\csc(\mathbb{A})(ie^{-i\mathbb{A}})$ and

$$\tau_{q_1,q_2} : \frac{\partial}{\partial x} \mapsto \sin(3\mathbb{A})\frac{\partial}{\partial x} + \cos(3\mathbb{A})\frac{\partial}{\partial y}.$$

Finally we compute τ_{q_3,q_1}. The chain centered at $(0, -\sin(2\mathbb{A})/2)$ with radius $\cos(\mathbb{A})$ is a slice of $\mathfrak{S}(q_3, q_1)$ and its inversion maps the point with Heisenberg coordinates $(x, -1/t)$ (where $x \in \mathbb{R}$) to the point with Heisenberg coordinates

$$\left(\frac{-itx}{1 - itx^2(1+t^2)}, \frac{t^3(1+t^2)x^4}{2\left\{1 + t^2x^4(1+t^2)^2\right\}}\right).$$

Therefore τ_{q_3,q_1} corresponds to multiplication by $-i$ with respect to the usual complex basis of (1,0)-vectors:

$$\tau_{q_3,q_1} : \frac{\partial}{\partial x} \mapsto -\frac{\partial}{\partial y}, \qquad \tau_{q_3,q_1} : \frac{\partial}{\partial y} \mapsto \frac{\partial}{\partial x}.$$

Thus the composition $\tau_{q_1,q_2} \circ \tau_{q_2,q_3} \circ \tau_{q_3,q_1}$ maps

$$\begin{aligned} E_{(0,0)} &\longrightarrow E_{(0,0)} \\ \frac{\partial}{\partial x} &\longmapsto \cos(3\mathbb{A})\frac{\partial}{\partial x} - \sin(3\mathbb{A})\frac{\partial}{\partial y}; \end{aligned}$$

that is, it represents multiplication by $e^{-3i\mathbb{A}}$ on E_{q_1} as desired. The proof of Theorem 7.1.13 is complete. $\square$

Gusevskii and Parker [82] discuss groups generated by reflections in three intersecting $\mathbb{R}$-circles, obtaining a family of discrete groups varying with a real parameter similar to Cartan's invariant. They prove a result very similar to Theorem 7.1.13.

7.1.6 *Geodesic projections*

Let $x \in \mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}}$. *Geodesic projection*

$$\Pi_x : (\mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}} - \{x\}) \longrightarrow \partial\mathbf{H}^n_{\mathbb{C}}$$

is the map which takes $y \in \mathbf{H}^n_{\mathbb{C}} \cup \partial\mathbf{H}^n_{\mathbb{C}} - \{x\}$ to the endpoint of the geodesic ray starting at x and containing y. If x is the point at infinity in Heisenberg space, then we show that the image of a (real or complex) geodesic under Π_x has a particularly nice form in Heisenberg coordinates:

Theorem 7.1.14 *Let $x, u, v \in \partial\mathbf{H}^n_{\mathbb{C}}$ be distinct points, C the complex geodesic spanned by u, v. Let $\mathfrak{S} \subset \partial\mathbf{H}^n_{\mathbb{C}}$ be the spinal sphere with vertex x and slice C. Let γ be the real geodesic with endpoints u, v.*

1. *Geodesic projection Π_x maps C diffeomorphically onto the component C_x of $\mathfrak{S} - \partial C$ not containing x.*
2. *In Heisenberg coordinates with x the point at infinity and $\mathfrak{S}$ the horizontal plane, C_x is a geometric disc in $\mathfrak{S}$ and $\Pi_x(\gamma)$ is a circular arc in C_x orthogonal to its boundary.*

Proof It suffices to consider the case $n = 2$. Take

$$x \leftrightarrow \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad u \leftrightarrow \begin{bmatrix} ie^{i\theta} \\ 0 \\ 1 \end{bmatrix}, \qquad v \leftrightarrow \begin{bmatrix} -ie^{-i\theta} \\ 0 \\ 1 \end{bmatrix}$$

(where $\theta = \mathbb{A}(x, u, v)$ is the Cartan invariant of the triple of points) whence the geodesic spanned by u and v is

$$\begin{aligned}\gamma(t) &= \frac{-t}{\langle u, v\rangle} u + \frac{1}{2t} v \\ &= \begin{bmatrix} (it/2)\sec(\theta) - (i/2t)e^{-i\theta} \\ 0 \\ (t/2)\sec(\theta)e^{i\theta} + 1/(2t) \end{bmatrix}\end{aligned}$$

for $t > 0$, which has Siegel domain coordinates

$$\begin{aligned} w_1 &= ie^{i\theta}\left(\frac{t^2\sec(\theta) - e^{-i\theta}}{t^2\sec(\theta) + e^{i\theta}}\right) \\ w_2 &= \frac{1}{2}. \end{aligned}$$

Geodesic projection from x to $\partial\mathfrak{H}^2$ is

$$\Pi_x : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \longmapsto \begin{pmatrix} w_1 \\ \frac{1}{2}w_1\bar{w}_1 + i\mathrm{Im}(w_2) \end{pmatrix}$$

and so

$$\Pi_x(\gamma(t)) = \begin{pmatrix} ie^{i\theta}(t^2\sec(\theta) - e^{-i\theta})/(t^2\sec(\theta) + e^{i\theta}) \\ \frac{1}{2}\left|(t^2\sec(\theta) - e^{-i\theta})/(t^2\sec(\theta) + e^{i\theta})\right|^2 \end{pmatrix}.$$

These points on $\partial\mathfrak{H}^2$ have Heisenberg coordinates $(\zeta(t), 0)$ where

$$\zeta(t) = ie^{i\theta}\frac{t^2\sec(\theta) - e^{-i\theta}}{t^2\sec(\theta) + e^{i\theta}}$$

describes the circular arc Γ from $ie^{i\theta}$ to $-ie^{-i\theta}$ orthogonal to the unit circle. (This follows because the linear fractional transformation

$$z \mapsto \frac{\sec(\theta)z - e^{-i\theta}}{\sec(\theta)z + e^{i\theta}}$$

maps the positive real axis $\mathbb{R}_+$ onto Γ and the imaginary axis $i\mathbb{R}$ onto the unit circle.) □

Exercise 7.1.15 *Interpret the Cartan invariant θ in terms of the geodesic projection.*

Exercise 7.1.16 *Geodesic projection from finite points admits not nearly as concise a description as geodesic projection from ideal points. Take x to be the origin in the ball $\mathbb{B}^2$ and let C_ρ be a complex geodesic at distance ρ from x. Let $\Delta \subset \mathbb{C}$ be the unit disc. Composition of the parametrization of C_ρ*

$$\begin{aligned} f_{C_\rho} : \Delta &\longrightarrow C_\rho \\ \xi &\longmapsto \begin{pmatrix} \mathrm{sech}(\rho/2)\xi \\ \tanh(\rho/2) \end{pmatrix} \end{aligned}$$

with Π_x is

$$\Pi_x \circ f_{C_\rho}(\xi) = \frac{1}{\sqrt{\mathrm{sech}^2(\rho/2)|\xi|^2 + \tanh^2(\rho/2)}} \begin{pmatrix} \mathrm{sech}(\rho/2)\xi \\ \tanh(\rho/2) \end{pmatrix}$$

on $\partial\mathbf{H}^n_{\mathbb{C}}$. In Heisenberg coordinates, $\Pi_x \circ f_{C_\rho}$ is a diffeomorphism of Δ to the disc

$$\{(\zeta, 0) \in \mathbb{C} \times \{0\} \mid |\zeta| < e^{\rho/2}\}.$$

7.1.7 *Parametrizing ideal triangle groups*

Cartan's angular invariant parametrizes deformations of *ideal triangle groups* in $\mathbf{H}^2_{\mathbb{C}}$, as described in §4.4.13. Three distinct points $p_1, p_2, p_3 \subset \partial\mathbf{H}^n_{\mathbb{C}}$ determine chains C_{12}, C_{23}, C_{31} fixed by inversions $\iota_{12}, \iota_{23}, \iota_{31}$ respectively. A triple of inversions determines a homomorphism

$$\begin{aligned} \phi_{(p_1,p_2,p_3)} : \mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2 &\longrightarrow \mathbf{PU}(2,1) \\ (\gamma_1, \gamma_2, \gamma_3) &\longmapsto (\iota_{12}, \iota_{23}, \iota_{31}). \end{aligned}$$

Two such homomorphisms are conjugate if and only if the corresponding triples (p_1, p_2, p_3) are equivalent. The equivalence class of such a triple is determined by the angular invariant

$$\mathbb{A}(p_1, p_2, p_3) \in [-\pi/2, \pi/2]$$

and thus a well-defined map

$$[-\pi/2, \pi/2] \longrightarrow \mathrm{Hom}(\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2, \mathbf{PU}(2,1))/\mathbf{PU}(2,1)$$

assigns to $u \in [-\pi/2, \pi/2]$ the equivalence class $[\phi_u]$ of $\phi_u = \phi_{(p_1,p_2,p_3)}$ where $\mathbb{A}(p_1, p_2, p_3) = u$. For $u = 0$, such a homomorphism is a discrete embedding (see §4.4.13). The main result of [72] is that for

$$|u| < \tan^{-1}(\sqrt{35}) \approx 0.0893\, \pi/2$$

ϕ_u is a discrete embedding and for

$$|u| > \tan^{-1}(\sqrt{125/3}) \approx 0.0902\, \pi/2 \tag{7.9}$$

it is not a discrete embedding. Condition (7.9) is equivalent to ellipticity of the element $\iota_{12}\iota_{23}\iota_{31}$. In [72] the latter necessary condition is conjectured to be sufficient. Recently, using a geometric construction called "dented tori," Schwartz [153 proved this conjecture:

Theorem 7.1.17 *(Schwartz) Suppose that $\iota_{12}, \iota_{23}, \iota_{31}$ are not all three equal. Then $\iota_{12}, \iota_{23}, \iota_{31}$ generate a discrete group if and only if $\iota_{12}\iota_{23}\iota_{31}$ is not elliptic. Furthermore, in this case, $\iota_{12}, \iota_{23}, \iota_{31}$ freely generate the free product*

$$\langle \iota_{12} \rangle \star \langle \iota_{23} \rangle \star \langle \iota_{31} \rangle$$

of the three cyclic groups of order 2.

7.2 The complex cross-ratio

In [105], Korányi and Reimann introduce a complex-valued invariant associated to an ordered quadruple of distinct points of $\partial \mathbf{H}^n_{\mathbb{C}}$. This invariant generalizes the usual cross-ratio of a quadruple of complex numbers and closely relates to Cartan's angular invariant. Let $(x_1, x_2, x_3, x_4) \in \mathcal{C}_4(\partial \mathbf{H}^n_{\mathbb{C}})$ and let $X_i \in \mathbb{C}^{n,1}$ be a null vector corresponding to x_i. Then the quantity

$$\frac{\langle X_3, X_1 \rangle \langle X_4, X_2 \rangle}{\langle X_4, X_1 \rangle \langle X_3, X_2 \rangle} \tag{7.10}$$

is independent of the chosen lifts X_i depending only on (x_1, x_2, x_3, x_4) ([105], p.294, Remark 1, formula (12)). This *Korányi–Reimann complex cross-ratio* will be denoted $\mathbf{X}\{x_1, x_2, x_3, x_4\} \in \mathbb{C}^*$ and enjoys the following properties:

1. If $g \in \mathbf{PU}(n,1)$, then

$$\mathbf{X}\{g(x_1), g(x_2), g(x_3), g(x_4)\} = \mathbf{X}\{x_1, x_2, x_3, x_4\}.$$

2. If g is an anti-holomorphic automorphism then

$$\mathbf{X}\{g(x_1), g(x_2), g(x_3), g(x_4)\} = \overline{\mathbf{X}\{x_1, x_2, x_3, x_4\}}.$$

3. If $x \in$ **Chain**, then $\mathbf{X}\{x_1, x_2, x_3, x_4\}$ is real and equals the ordinary cross-ratio of the corresponding quadruple of points on the boundary of the unit disc (or half-plane)—for example, the complex cross-ratio of four points on the vertical chain is

$$\mathbf{X}\{\infty, (0,0), (0,1), (0,v)\} = v.$$

4. If $x \in$ **Real**, then $\mathbf{X}\{x_1, x_2, x_3, x_4\}$ is positive and equals the square of the ordinary cross-ratio of the corresponding quadruple of points on the boundary of the unit disc (or half-plane) — for example, the complex cross-ratio of four points on the real axis is

$$\mathbf{X}\{\infty, (0,0), (1,0), (r,0)\} = r^2.$$

5. (Symmetry properties) The complex cross-ratio enjoys the following transformation laws under the group of permutations of $\{1,2,3,4\}$ preserving the partition $\{\{1,2\},\{3,4\}\}$:

$$\begin{aligned} & \mathbf{X}\{xx_1, x_2, x_3, x_4\} &=& \ \mathbf{X}\{x_2, x_1, x_4, x_3\} \\ =\ & \overline{\mathbf{X}\{x_3, x_4, x_1, x_2\}} &=& \ \overline{\mathbf{X}\{x_4, x_3, x_2, x_1\}} \\ =\ & \mathbf{X}\{x_2, x_1, x_3, x_4\}^{-1} &=& \ \mathbf{X}\{x_1, x_2, x_4, x_3\}^{-1} \\ =\ & \overline{\mathbf{X}\{x_4, x_3, x_1, x_2\}}^{-1} &=& \ \overline{\mathbf{X}\{x_3, x_4, x_2, x_1\}}^{-1}. \end{aligned} \tag{7.11}$$

6. (Cyclic products)

$$\begin{aligned} & |\mathbf{X}\{q_1, q_2, q_3, q_4\}\mathbf{X}\{q_1, q_4, q_2, q_3\}\mathbf{X}\{q_1, q_3, q_4, q_2\}| \\ =\ & |\mathbf{X}\{q_1, q_2, q_3, q_4\}\mathbf{X}\{q_4, q_2, q_1, q_3\}\mathbf{X}\{q_3, q_2, q_4, q_1\}| \\ =\ & |\mathbf{X}\{q_1, q_2, q_3, q_4\}\mathbf{X}\{q_2, q_4, q_3, q_1\}\mathbf{X}\{q_4, q_1, q_3, q_2\}| \\ =\ & |\mathbf{X}\{q_1, q_2, q_3, q_4\}\mathbf{X}\{q_2, q_3, q_1, q_4\}\mathbf{X}\{q_3, q_1, q_2, q_4\}| \\ =\ & 1. \end{aligned}$$

(The corresponding cyclic product for the classical cross-ratio—without taking absolute values—equals -1.) These cyclic products relate to the angular invariant by

$$\mathbf{X}\{q_1, q_2, q_3, q_4\}\mathbf{X}\{q_1, q_4, q_2, q_3\}\mathbf{X}\{q_1, q_3, q_4, q_2\} = e^{2i\mathbb{A}(q_2, q_3, q_4)}.$$

7. ([105], Proposition 2) $\mathbf{X}\{x_1, x_2, x_3, x_4\} < 0$ if and only if $x \in$ **Chain** and x_1, x_2 separate x_3, x_4 on this chain;

$$\mathbf{X}\{x_1, x_2, x_3, x_4\} > 0$$

if and only if x_3, x_4 lie on the same orbit of the stabilizer of x_1, x_2.

8. ([105], Proposition 3) Fix $(x_1, x_2, x_3) \in \mathcal{C}_3(\partial \mathbf{H}^n_{\mathbb{C}})$. Then the set

$$\{\mathbf{X}\{x_1, x_2, x_3, x_4\} \mid x_4 \in \partial \mathbf{H}^n_{\mathbb{C}}, x_4 \neq x_1, x_2, x_3\}$$

equals the half-plane

$$\{z \in \mathbb{C} \mid \operatorname{Re}(e^{i\mathbb{A}(x_1,x_2,x_3)} z) \geq 0\}.$$

9. The map

$$(x_1, x_2, x_3, x_4) \mapsto \mathbf{X}\{x_1, x_2, x_3, x_4\}$$

is anti-holomorphic in x_1, x_2 and is holomorphic in x_3, x_4; that is, its differential restricted to

$$\begin{aligned} &E_{x_1} \times E_{x_2} \times E_{x_3} \times E_{x_4} \\ &\qquad \subset T_{x_1}\partial \mathbf{H}^n_{\mathbb{C}} \times T_{x_2}\partial \mathbf{H}^n_{\mathbb{C}} \times T_{x_3}\partial \mathbf{H}^n_{\mathbb{C}} \times T_{x_4}\partial \mathbf{H}^n_{\mathbb{C}} \end{aligned}$$

is anti-linear in the first two arguments and complex linear in the last two arguments.

10. If $q_1, q_2, q_3, q_4 \in \partial \mathbf{H}^n_{\mathbb{C}}$ are distinct points in the boundary of a complex 2-plane (for example, if $n = 2$), then

$$\begin{aligned} 1 = {} & \mathbf{X}\{q_1, q_2, q_3, q_4\} |1 - \mathbf{X}\{q_1, q_4, q_2, q_3\}|^2 \\ & + \overline{\mathbf{X}\{q_1, q_2, q_3, q_4\}}\,(1 - 2\operatorname{Re}\mathbf{X}\{q_1, q_3, q_2, q_4\})\,. \end{aligned} \tag{7.12}$$

The proof of (7.12) involves the Gram determinant

$$\begin{aligned} &\frac{1}{\langle Q_1, Q_3\rangle \langle Q_3, Q_2\rangle \langle Q_2, Q_4\rangle \langle Q_4, Q_1\rangle} \\ &\qquad \times \det \begin{bmatrix} 0 & \langle Q_1, Q_3\rangle & \langle Q_1, Q_2\rangle & \langle Q_1, Q_4\rangle \\ \langle Q_3, Q_1\rangle & 0 & \langle Q_3, Q_2\rangle & \langle Q_3, Q_4\rangle \\ \langle Q_2, Q_1\rangle & \langle Q_2, Q_3\rangle & 0 & \langle Q_2, Q_4\rangle \\ \langle Q_4, Q_1\rangle & \langle Q_4, Q_3\rangle & \langle Q_4, Q_2\rangle & 0 \end{bmatrix} = \\ & -1 + \mathbf{X}\{q_3, q_4, q_2, q_1\}\Big(1 - \mathbf{X}\{q_2, q_4, q_1, q_3\} - \mathbf{X}\{q_1, q_3, q_2, q_4\}\Big) \\ &\quad + \mathbf{X}\{q_1, q_2, q_3, q_4\} \cdot \Big(1 - \mathbf{X}\{q_3, q_2, q_4, q_1\} - \mathbf{X}\{q_1, q_4, q_2, q_3\} \\ &\qquad + \mathbf{X}\{q_1, q_4, q_2, q_3\}\mathbf{X}\{q_3, q_2, q_4, q_1\}\Big) \end{aligned} \tag{7.13}$$

which vanishes because Q_1, Q_2, Q_3, Q_4 are linearly dependent. The symmetry properties (7.11) are then used to relate the right-hand side of (7.13) to the expression in (7.12).

7.2.1 *Geometric interpretation*

The complex cross-ratio

$$\mathbf{X}\{x_1, x_2, x_3, x_4\}$$

admits the following geometric interpretation. Let Σ be the complex geodesic spanned by x_1 and x_2 and σ the real geodesic with endpoints x_1, x_2. Let $\Pi : \overline{\mathbf{H}^n_{\mathbb{C}}} \longrightarrow \Sigma$ be orthogonal projection onto Σ and choose a "linear coordinate" $z_{12} : \Sigma \longrightarrow \mathbb{C}$ such that $z_{12}(\Sigma)$ is the right half-plane

$$\{\zeta \in \mathbb{C} \mid \mathrm{Re}(\zeta) > 0\},$$

$z_{12}(x_1) = \infty$ and $z_{12}(x_2) = 0$. Such a function is uniquely determined up to multiplication by a positive real number. The complex cross-ratio $\mathbf{X}\{x_1, x_2, x_3, x_4\}$ then equals the complex number

$$\frac{z_{12}(\Pi(x_4))}{z_{12}(\Pi(x_3))}$$

which is clearly independent of the choice of z_{12}.

In the case x_1 corresponds to ∞ in Heisenberg space, x_2 is the origin $((0,0)$ in Heisenberg coordinates), then $z_{12} \circ \Pi$ takes the point with Heisenberg coordinates (ζ, v) to

$$\frac{c_0}{2}(\|\zeta\|^2 - iv)$$

(for a constant $c_0 > 0$). (Compare §4.3.6.)

7.2.2 *A reality condition*

Theorem 7.2.1 *Let $x_1, x_2, x_3, x_4 \in \partial\mathbf{H}^n_{\mathbb{C}}$. Then $\mathbf{X}\{x_1, x_2, x_3, x_4\}$ is real if and only if there is an anti-involution ϕ such that $\phi : x_1 \longleftrightarrow x_2$ and $\phi : x_3 \longleftrightarrow x_4$.*

Proof Suppose such an anti-involution ϕ exists. Then

$$\begin{aligned}\mathbf{X}\{x_1, x_2, x_3, x_4\} &= \mathbf{X}\{\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4)\} \\ &= \mathbf{X}\{x_2, x_1, x_4, x_3\} \\ &= \mathbf{X}\{x_1, x_2, x_3, x_4\}\end{aligned}$$

is real.

Conversely, suppose that $\mathbf{X}\{x_1, x_2, x_3, x_4\}$ is real. Let Σ denote the complex geodesic spanned by x_1 and x_2 and $\sigma \subset \Sigma$ the real geodesic spanned by x_1 and x_2. Let

$$\Pi : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \mathbf{H}^1_{\mathbb{R}} \subset \mathbb{C}$$

be the composition of orthogonal projection onto Σ with the map identifying Σ with the right half-plane with $\Pi(x_1) = 0$ and $\Pi(x_2) = \infty$. ($\Pi(\sigma)$ is the positive real axis.) Then $\Pi(x_3)$ and $\Pi(x_4)$ are real multiples of each other. In particular $\Pi(x_3)$ and $\Pi(x_4)$ lie on a hypercycle in $\mathbf{H}^1_{\mathbb{R}}$ with endpoints 0 and ∞. A geodesic $\gamma \subset \Sigma$ bisects $\Pi(x_3)$ and $\Pi(x_4)$ and is orthogonal to this hypercycle. The inverse image $\mathfrak{B} = \Pi^{-1}(\gamma)$ is a bisector and we seek a meridian $\mathcal{M} \subset \mathfrak{B}$ such that the anti-involution $\iota_{\mathcal{M}}$ fixing $\mathcal{M}$ maps x_3 to x_4. To this end, choose any meridian $\mathcal{M}$ and the corresponding anti-involution ι. Then $\Pi(\iota(x_3))$ equals the reflection of $\Pi(x_3)$ in γ which is $\Pi(x_4)$. Thus $\iota(x_3)$ lies in the chain

$$C = \Pi^{-1}(\Pi(x_4)) \cap \partial\mathbf{H}^n_{\mathbb{C}}.$$

For any complex geodesic Σ, the complex reflections in Σ act simply transitively on any chain orthogonal to Σ. Since C is orthogonal to Σ, a complex reflection ϱ takes $\iota(x_3)$ to x_4. The anti-involution $\varrho \circ \iota$ leaves invariant Σ, interchanges x_3 and x_4 and interchanges x_1 and x_2 as desired. □

7.2.3 *A general setting for the cross-ratio*

There is a general approach to the cross-ratio which includes both the classical cross-ratio and the Korányi–Reimann complex cross-ratio. I am grateful to Hillel Furstenberg, John Millson and Chris Stark for conversations clarifying this formalism.

Let X be a smooth manifold and G a group acting on X. We suppose that L_i $(i = 1, 2)$ are G-homogeneous $\mathbb{C}$-line bundles over X (that is, for each $g \in G$ a bundle map $l_i(g) : L_i \longrightarrow L_i$ covers $g : X \longrightarrow X$). Let $L_1 \otimes L_2$ denote the *outer tensor product* of L_1 with L_2, the line bundle over $X \times X$ whose fiber over $(x, y) \in X \times X$ is the tensor product $(L_1)_x \otimes (L_2)_y$. The G-actions on L_i determine a G-action on $L_1 \otimes L_2$ covering the diagonal G-action on $X \times X$.

Suppose that there is a G-equivariant mapping $B : L_1 \otimes L_2 \longrightarrow \mathbb{C}$ of line bundles over $X \times X$, where $\mathbb{C}$ denotes the trivial complex line bundle over $X \times X$.

We consider two cases. In the classical case, $X = \mathbb{CP}^1$, $G = \mathbf{SL}(2, \mathbb{C})$ and L is the holomorphic line bundle of spinors (that is, the (unique) square root of the canonical line bundle). A meromorphic section generating L is $\sqrt{dz}$ which has a simple pole at ∞. A typical local section of L is given by an expression

$$\phi = \phi(z)\sqrt{dz}$$

where ϕ is holomorphic. The action of

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2, \mathbb{C})$$

on X is

$$z \longmapsto g(z) = \frac{az + b}{cz + d}$$

and on L is

$$g^*(\phi(z)\sqrt{dz}) = (cz+d)^{-1}\phi(g(z))\sqrt{dz}.$$

On the product $\mathbb{CP}^1 \times \mathbb{CP}^1$,

$$\theta(x,y) = (x-y)^2 \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}$$

defines a G-invariant holomorphic tensor field (the "infinitesimal cross-ratio"). $\theta^{-1/2}$ defines an G-invariant pairing

$$\begin{aligned} B : L \otimes L &\longrightarrow \mathbb{C} \\ \sqrt{dx} \otimes \sqrt{dy} &\longmapsto x - y. \end{aligned} \tag{7.14}$$

Indeed, G-invariance is just the well-known mean-value formula

$$\frac{g(x)-g(y)}{x-y} = g'(x)^{1/2} g'(y)^{1/2} \tag{7.15}$$

valid for Möbius transformations.

For the Korányi–Reimann cross-ratio, consider $X = \partial \mathbf{H}^n_{\mathbb{C}}$, $G = \mathbf{U}(n,1)$. Recall that X is defined as the projectivization $\mathbb{P}(\mathcal{Q})$ of the set $\mathcal{Q} \subset \mathbb{C}^{n,1}$ of null vectors in $\mathbb{C}^{n,1}$; projectivization $\mathbb{P} : \mathcal{Q} \longrightarrow X$ is a principal $\mathbb{C}^*$-bundle. Take L to be the associated complex line bundle. A point in the total space of L is a pair (x,u) where $x \in \mathbf{H}^n_{\mathbb{C}}$ and $u \in \mathbb{C}^{n,1}$ lies in the line $\mathbb{P}^{-1}(x)$ corresponding to x. Evidently the $\mathbf{U}(n,1)$-action on $\mathbb{C}^{n,1}$ induces a G-action on L covering the action of G on X. The pairing

$$\begin{aligned} B : L \otimes \bar{L} &\longrightarrow \mathbb{C} \\ (x,u) \otimes (y,v) &\longmapsto \langle u, v\rangle \end{aligned} \tag{7.16}$$

is a G-invariant pairing from the outer tensor product $L \otimes \bar{L}$ to $\mathbb{C}$.

Let X^n denote the n-fold Cartesian power of X and let $F_n(X)$ denote the space of smooth functions $f : X^n \longrightarrow \mathbb{C} \cup \{\infty\}$ such that $f^{-1}(\{0,\infty\})$ has measure zero in X^n.

Definition 7.2.2 *We say that two functions $h_1, h_2 \in F_2(X)$ are* equivalent modulo tensor products *if there exist functions $f_1, f_2 \in F_1(X)$ such that*

$$h_1(x,y) = f_1(x) f_2(y) h_2(x,y). \tag{7.17}$$

(The function

$$f_1 \otimes f_2 : (x,y) \mapsto f_1(x) f_2(y)$$

is the tensor product *of f_1 and f_2.) For any function $h \in F_2(X)$ its* quadruplization *is the function $\mathcal{Q}_h \in F_4(X)$ defined by*

$$\mathcal{Q}_h(x,y,z,w) = h(z,x)h(y,w)/(h(z,w)h(y,x)).$$

Lemma 7.2.3 *Two functions $h_1, h_2 \in F_2(X)$ are equivalent modulo tensor products if and only if*

$$\mathcal{Q}_{h_1} = \mathcal{Q}_{h_2}.$$

Proof Clearly (7.17) implies $\mathcal{Q}_{h_1} = \mathcal{Q}_{h_2}$. Conversely suppose h_1 and h_2 have identical quadruplizations. Then $h = h_1/h_2$ has quadruplization $\mathcal{Q}_h = 1$ and

$$h(x,y) = h(x,y_0)h(x_0,y)/h(x_0,y_0)$$

so that h is equivalent modulo tensor products to the constant function 1 and h_1 and h_2 are equivalent modulo tensor products. □

We may succinctly express this as an exact sequence:

$$F_1(X) \otimes F_1(X) \longrightarrow F_2(X) \xrightarrow{\mathcal{Q}} F_4(X).$$

Suppose that σ_i is any section of L_i; then the outer tensor product $\sigma_1 \otimes \sigma_2$ is a section of $L_1 \otimes L_2$ and the function

$$f = B \circ (\sigma \otimes \sigma) : X \times X \longrightarrow \mathbb{C} \cup \{\infty\}$$

has the property that for each $g \in G$, the functions $g^* f$ and f are equivalent modulo tensor products. For there exists a function $j_g : X \longrightarrow \mathbb{C}^*$ such that

$$g^*\sigma(x) = j_g(x)\sigma(gx)$$

and thus

$$g^* f(x,y) = B(g^*\sigma(x), g^*\sigma(y)) = j_g \otimes j_g(x,y) f(x,y).$$

Therefore the quadruplization

$$Q_f : X \times X \times X \times X \longrightarrow \mathbb{C}$$

is G-invariant.

The classical cross-ratio admits an interpretation of this sort, where one quadruplizes the function defined in (7.14). Similarly the complex cross-ratio is obtained by quadruplizing the invariant pairing B defined by the Hermitian pairing in (7.16).

7.3 Real geodesics and complex hyperplanes

This section introduces a complex invariant $\eta(q_1, q_2; c)$ associated to a pair q_1, q_2 of ideal points in $\partial \mathbf{H}^n_{\mathbb{C}}$ and a complex hyperplane c. The pair (q_1, q_2) form the endpoints of an oriented real geodesic, or, alternatively, a bisector. We developed this invariant to help understand Mostow's observation ([128], 3.2.7(1)) that a bisector intersects a complex geodesic in a *hypercycle*. The theory of hypercycles (or *equidistant curves*) in the hyperbolic plane is reviewed in §1.4.2.)

Recall that a hypercycle in hyperbolic space is a curve of constant geodesic curvature $k < 1$. Alternatively, a hypercycle is a boundary component of a collar of constant width δ about a geodesic, where $k = \tanh(\delta)$. See §1.4.2 for more details about hypercycles in $\mathbf{H}^1_{\mathbb{C}}$. We use this invariant later to analyze the intersections of bisectors.

7.3.1 *Invariants of two ideal points and a complex hyperplane*

Let $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ be distinct ideal points in $\mathbf{H}^n_{\mathbb{C}}$ and let c be a complex hyperplane. Let Q_1, Q_2 be null vectors representing q_1, q_2 and C a positive vector polar to c. Then the complex number

$$\eta(Q_1, Q_2; C) = \frac{\langle Q_1, C\rangle\langle C, Q_2\rangle}{\langle Q_1, Q_2\rangle\langle C, C\rangle}$$

is independent of the choices of representative vectors and we denote it $\eta(q_1, q_2; c)$. (We extend the values continuously with the convention that $\eta(q_1, q_2; c) = \infty$ for $q_1 = q_2$.)

Theorem 7.3.1 *Suppose $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ are distinct ideal points, c is a complex hyperplane, and $\eta(q_1, q_2; c) \in \mathbb{C}$ is as above.*

1. *If $g \in \mathbf{PU}(n,1)$ is holomorphic,*
$$\eta(g(q_1), g(q_2); g(c)) = \eta(q_1, q_2; c).$$
2. *If $g \in \mathbf{Aut}(\mathbf{H}^n_{\mathbb{C}})$ is anti-holomorphic, then*
$$\eta(g(q_1), g(q_2); g(c)) = \overline{\eta(q_1, q_2; c)}.$$
3. $\eta(q_2, q_1; c) = \overline{\eta(q_1, q_2; c)}$.
4. $\eta(q_1, q_2; c) = 0$ *if and only if* $q_1 \in \partial c$ *or* $q_2 \in \partial c$.
5. $\mathrm{Re}(\eta(q_1, q_2; c)) \leq \frac{1}{2}$.
6. $\mathrm{Re}(\eta(q_1, q_2; c)) = \frac{1}{2}$ *if and only if q_1 and q_2 span a complex hyperplane orthogonal to c.*
7. $\eta(q_1, q_2; c) = \frac{1}{2}$ *if and only if the inversion in c interchanges q_1 and q_2.*

Only 5–7 are not immediate from the definition. These facts can be deduced from the following geometric interpretation of $\eta(q_1, q_2; c)$:

Theorem 7.3.2 *Let $\Pi_c : \mathbf{H}^n_{\mathbb{C}} \longrightarrow c$ be the orthogonal projection onto c and let γ_i denote the geodesic ray from $\Pi_c(q_i)$ to q_i and γ the geodesic from $\Pi_c(q_1)$ to $\Pi_c(q_2)$. Let θ denote the angle between γ_2 and the image of γ_1 under parallel translation along γ. Then*

$$\eta(q_1, q_2; c) = \frac{1}{1-z} \quad \textit{where}$$
$$z = e^{i\theta} \cosh\left(\frac{\rho(\Pi_c(q_1), \Pi_c(q_2))}{2}\right).$$

Proof of Theorem 7.3.2 Since q_1, q_2, c determine three points in the projective space $\mathbb{P}(\mathbb{C}^{n,1})$, we may assume without loss of generality that $n = 2$. Furthermore the case that $q_1 \in c$ is easily handled, so we assume that c and q_i are not incident. Let $\rho = \rho(\Pi_c(q_1), \Pi_c(q_2))$. Choose coordinates so that

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} e^{i\theta} \\ \sinh(\rho/2) \\ \cosh(\rho/2) \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & \cosh(\rho/2) & \sinh(\rho/2) \\ 0 & \sinh(\rho/2) & \cosh(\rho/2) \end{bmatrix} Q_1$$

from which Theorem 7.3.2 follows. □

Proof of Theorem 7.3.1 The linear fractional transformation

$$\eta = \eta(z) = 1/(1 - z)$$

maps the subset of $\mathbb{C}$ defined by $|z| \geq 1$ to the half-plane $\mathrm{Re}(\eta) \leq \frac{1}{2}$. Theorem 7.3.1(5) now follows since $\cosh(\rho/2) \geq 1$. Since $\eta(z)$ maps the unit circle $|z| = 1$ to the line $\mathrm{Re}(\eta) = \frac{1}{2}$, Theorem 7.3.1(6) follows. Finally Theorem 7.3.1(7) follows since $\eta(z)$ maps $z = -1$ to $\eta = \frac{1}{2}$. □

7.3.2 *Relation to Cartan's invariant*

Theorem 7.3.3 *Let $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ and c a complex geodesic. Let Π_c denote orthogonal projection onto c and let $q_\pm$ denote the endpoints of the geodesic (in c) joining $\Pi_c(q_1)$ to $\Pi_c(q_2)$. Let ι_c denote inversion in c and $\eta = \eta(q_1, q_2; c)$.*

1. $\mathbb{A}(q_1, q_2, q_+) = \arg(1/(1 - \eta))$;
2. $\mathbb{A}(q_1, q_2, q_-) = -\arg(1/(1 - \eta))$;
3. $\mathbb{A}(q_1, q_2, \iota_c(q_2)) = \arg(1 - 2\eta)$;
4. $\mathbb{A}(q_1, q_2, \iota_c(q_1)) = -\arg(1 - 2\eta)$.

Proof Continuing with the preceding coordinates, $\Pi(q_1)$ and $\Pi(q_2)$ are represented by

$$\Pi(Q_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \Pi(Q_2) = \begin{bmatrix} 0 \\ \sinh(\rho/2) \\ \cosh(\rho/2) \end{bmatrix}$$

respectively, and (taking $\rho > 0$) q_+ is represented by

$$Q_+ = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Now

$$\begin{aligned}\mathbb{A}(q_1, q_2, q_+) &= \arg\left(1 - e^{-i\theta}\mathrm{sech}(\rho/2)\right)\\ &= \arg\left(\frac{1}{1-\eta}\right)\end{aligned}$$

as claimed. Thus 2 follows from 1 and Theorem 7.3.1(3) . A representative vector for $\iota_c(q_1)$ is

$$\iota_c(Q_1) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathbb{A}(q_1, q_2, \iota_c(q_1)) = \arg\left(\left(\cosh(\rho/2) + e^{i\theta}\right)\left(\cosh(\rho/2) - e^{-i\theta}\right)\right)$$

but

$$\begin{aligned}1 - 2\bar{\eta} &= \frac{\bar{z}+1}{\bar{z}-1}\\ &= \frac{\left(\cosh(\rho/2) + e^{i\theta}\right)\left(\cosh(\rho/2) - e^{-i\theta}\right)}{|\cosh(\rho/2) - e^{-i\theta}|^2}\end{aligned}$$

from which 4 follows. Thus 3 follows from 4 and Theorem 7.3.1(3) . □

7.3.3 *Distance from a real geodesic to a complex geodesic*

Next we relate the invariant $\eta = \eta(q_1, q_2; c)$ to geometric properties of the geodesic γ having endpoints q_1, q_2 and the complex geodesic C.

Theorem 7.3.4 *Let $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ be distinct points spanning a geodesic γ and let c be a complex geodesic. Then the distance from γ to c is related to the invariant $\eta = \eta(q_1, q_2; c)$ by*

$$\sinh\left(\frac{\rho(\gamma, c)}{2}\right) = |\mathrm{Im}(2\eta)^{1/2}| = \sqrt{2}|\mathrm{Im}(\eta^{1/2})|.$$

In particular, γ meets c if and only if $\eta > 0$, and $\gamma \parallel c$ if and only if $\eta = 0$.

Proof We work in the ball model. Applying an automorphism, assume that the point on γ closest to c is the origin $(0, 0)$ and that the point on c closest to γ is $(\tanh(\rho/2), 0)$ where $\rho = \rho(\gamma, c)$. We may assume that

$$q_1 = \begin{pmatrix} ia \\ \sqrt{1-a^2} \end{pmatrix}, \qquad q_2 = \begin{pmatrix} -ia \\ -\sqrt{1-a^2} \end{pmatrix}$$

for $-1 < a < 1$ and that c is defined by

$$z_1 = \tanh(\rho/2)$$

and has polar vector

$$C = \begin{bmatrix} 1 \\ 0 \\ \tanh(\rho/2) \end{bmatrix}.$$

Taking

$$Q_1 = \begin{bmatrix} ia \\ \sqrt{1-a^2} \\ 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -ia \\ -\sqrt{1-a^2} \\ 1 \end{bmatrix},$$

we compute

$$-2\eta = \frac{(ia - \tanh(\rho/2))^2}{\operatorname{sech}^2(\rho/2)}$$

from which

$$|\mathrm{Re}((-2\eta)^{1/2})| = \sinh(\rho/2),$$

as desired. □

7.3.4 *The complex geodesic spanned by the pair of points*

The expression

$$\sqrt{\mathrm{Re}(1 - 2\eta(q_1, q_c; c))}$$

depends only on the complex geodesic c and the complex geodesic spanned by q_1 and q_2.

Theorem 7.3.5 *Let $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ be distinct points spanning a complex geodesic $\tilde{q}$ and let c be a complex geodesic. Let $\eta = \eta(q_1, q_c; c) \in \mathbb{C}$ be as above. Then*

$$\sqrt{\mathrm{Re}(1-2\eta)} = \begin{cases} \cosh\left(\dfrac{\rho(\tilde{q}, c)}{2}\right) & \text{if } \tilde{q})(c \\ 1 & \text{if } \tilde{q} \parallel c \\ \cos(\angle(\tilde{q}, c)) & \text{if } \tilde{q} \between c. \end{cases}$$

Proof Retaining the same notations as the proof of Theorem 7.3.2, the normalized vector polar to $\tilde{q}$ equals

$$\frac{\tilde{Q}}{\sqrt{\langle \tilde{Q}, \tilde{Q} \rangle}}$$

where

$$\tilde{Q} = Q_1 \boxtimes Q_2 = \begin{bmatrix} \sinh(\rho/2) \\ \cosh(\rho/2) - e^{-i\theta} \\ \sinh(\rho/2) \end{bmatrix}$$

and

$$\frac{|\langle \tilde{Q}, C\rangle|^2}{\langle \tilde{Q}, \tilde{Q}\rangle} = \frac{\sinh^2(\rho/2)}{|\cosh(\rho/2) - e^{-i\theta}|^2}.$$

Now

$$\begin{aligned} 1 - 2\eta &= \frac{z+1}{z-1} \\ &= \frac{z\bar{z} - 1}{|z-1|^2} - 2i\frac{\operatorname{Im}(z)}{|z-1|^2} \\ &= \frac{\sinh^2(\rho/2)}{|e^{i\theta}\cosh(\rho/2) - 1|^2} - 2i\frac{\sin(\theta)\cosh(\rho/2)}{|e^{i\theta}\cosh(\rho/2) - 1|^2} \end{aligned}$$

and Theorem 7.3.5 follows by §3.3.2. □

Corollary 7.3.6 *Suppose that $q_1, q_2 \in \partial \mathbf{H}^n_{\mathbb{C}}$ and c is a complex hyperplane. If*

$$0 < \eta(q_1, q_2; c) \leq \frac{1}{2},$$

then the geodesic γ spanned by q_1, q_2 meets c and

$$\cos(\angle(\gamma, c)) = \sqrt{1 - 2\eta(q_1, q_2; c)}.$$

Proof By §7.3.3, γ meets c. Choose coordinates so that

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

whence $c = \{0\} \times \mathbb{B}^1$. We suppose that $c \cap \gamma$ equals the origin $O \in \mathbb{B}^2$ and we write

$$Q_1 = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 1 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} -\sin(\theta) \\ -\cos(\theta) \\ 1 \end{bmatrix}$$

so that the points of γ correspond to vectors

$$\gamma(r) = \begin{bmatrix} r\sin(\theta) \\ r\cos(\theta) \\ 1 \end{bmatrix}$$

where $-1 < r < 1$. Now unit tangent vectors to c at O are

$$u = \begin{pmatrix} 0 \\ e^{i\psi} \end{pmatrix},$$

where $\psi \in \mathbb{R}$, and a unit tangent vector to γ at O is

$$v = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$

Thus

$$\cos(\angle(u, v)) = \mathrm{Re}\langle\langle u, v\rangle\rangle = \cos(\psi)\cos(\theta)$$

and

$$\cos(\angle(c, v)) = \max_{v \in T_O c} \cos(\angle(u, v)) = \cos(\theta)$$

whence $\angle(c, \gamma) = \theta$. Now

$$\begin{aligned}\eta(q_1, q_2; c) &= \frac{\langle Q_1, C\rangle\langle C, Q_2\rangle}{\langle Q_1, Q_2\rangle\langle C, C\rangle} \\ &= \frac{-\sin^2(\theta)}{-2 \cdot 1} = \frac{1}{2}\sin^2(\theta) = \frac{1}{2}(1 - \cos^2(\theta))\end{aligned}$$

as desired. □

7.3.5 *Projections of geodesics*

In this section we relate the quantity $\eta(q_1, q_2; c)$ to the image of the orthogonal projection of the geodesic Q with endpoints q_1, q_2 to the complex hyperplane c. As usual it will suffice to consider the case $n = 2$.

Theorem 7.3.7 *Let $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ be distinct points spanning a geodesic Q and let c be a complex hyperplane with orthogonal projection $\Pi : \overline{\mathbf{H}^n_{\mathbb{C}}} \longrightarrow c$. Let $\eta = \eta(q_1, q_2; c)$. Then $\Pi(Q)$ is an arc of a hypercycle in c of constant geodesic curvature*

$$\frac{\mathrm{Im}(\eta)}{\sqrt{1 - 2\,\mathrm{Re}(\eta)}}.$$

Proof The proof is based on the following lemma, the proof of which is left as an exercise in trigonometry:

Lemma 7.3.8 *Let $x_0 \in \mathbf{H}^1_{\mathbb{C}}$, $\rho_0 > 0$ and let D be the disc of radius ρ_0 centered at x_0. (In the ball model with x_0 as the origin D has Euclidean radius $R =$ $\tanh(\rho_0/2)$.) Let $p_\pm \in \partial D$ be two points which subtend an angle 2ϕ when viewed from x_0. Let $\gamma \subset D$ be the geodesic from p_+ to p_- with respect to the Poincaré metric on D. Then γ is a curve of constant geodesic curvature*

$$\mathrm{csch}(\rho_0)\tan(\phi) = \frac{R^{-1} - R}{2}\,\tan(\phi).$$

The distance from x_0 to γ relates to the above quantities by

$$\tanh\left(\frac{\rho(x_0, \gamma)}{2}\right) = \tanh\left(\frac{\rho_0}{2}\right)\tan\left(\frac{\phi}{2}\right).$$

We distinguish three cases, depending on whether the complex geodesic $\mathcal{Q}_{\mathbb{C}}$ containing $\mathcal{Q}$ is ultraparallel to c, asymptotic to c, or intersects c. In each case choose coordinates in which c is represented as $\{0\} \times \mathbb{B}^1$ with polar vector

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

If $\mathcal{Q}_{\mathbb{C}})(c$, then take

$$\mathcal{Q}_{\mathbb{C}} = \left\{(\sqrt{1-R^2}, R\xi) \in \mathbb{B}^2 \mid \xi \in \mathbb{C}, |\xi| \leq 1\right\}$$

where

$$R = \operatorname{sech}\left(\frac{\rho(c, \mathcal{Q}_{\mathbb{C}})}{2}\right)$$

and

$$q_1 = \begin{pmatrix} \sqrt{1-R^2} \\ R(\sin(\phi) + i\cos(\phi)) \end{pmatrix}, \qquad q_2 = \begin{pmatrix} \sqrt{1-R^2} \\ R(\sin(\phi) - i\cos(\phi)) \end{pmatrix}$$

whence

$$\eta = -\frac{1-R^2}{2R^2\cos(\phi)}\,(\cos(\phi) + i\sin(\phi)).$$

Now $\Pi(\mathcal{Q}_{\mathbb{C}})$ is represented (in the ball model) as the disc centered at the origin of Euclidean radius R and $\Pi(\mathcal{Q})$ is the circular arc orthogonal to $\partial\Pi(\mathcal{Q}_{\mathbb{C}})$ which joins the two points:

$$\Pi(q_1) = R(\sin(\phi) + i\cos(\phi)), \qquad \Pi(q_2) = R(\sin(\phi) - i\cos(\phi))$$

Applying Lemma 7.3.8,

$$-\frac{\operatorname{Im}(\eta)}{\sqrt{1 - 2\operatorname{Re}(\eta)}} = \frac{R^{-1} - R}{2}\tan(\phi)$$

as desired.

Next consider the case that $\mathcal{Q}_{\mathbb{C}} \chi c$. Take

$$\mathcal{Q}_{\mathbb{C}} = \left\{(\sqrt{1-R^2}\xi, R\xi) \in \mathbb{B}^2 \mid \xi \in \mathbb{C}, |\xi| \leq 1\right\}$$

where

$$R = \cos(\angle(c, \mathcal{Q}_{\mathbb{C}}))$$

and

$$q_1 = \begin{pmatrix} \sqrt{1-R^2}(\sin(\phi) + i\cos(\phi)) \\ R(\sin(\phi) + i\cos(\phi)) \end{pmatrix}, \quad q_2 = \begin{pmatrix} \sqrt{1-R^2}(\sin(\phi) - i\cos(\phi)) \\ R(\sin(\phi) - i\cos(\phi)) \end{pmatrix}$$

whence

$$\eta = -\frac{1-R^2}{2\cos(\phi)}(\cos(\phi) - i\sin(\phi)).$$

The rest of the proof is similar to the ultraparallel case.

The case that $\mathcal{Q}_{\mathbb{C}} \parallel c$ is handled similarly. □

7.3.6 *Intersecting bisectors with complex hyperplanes*

Fundamental to our investigation of intersections is the observation of Mostow [128]: *A bisector meets a complex geodesic in a hypercycle.* The invariant $\eta(q_1, q_2; c) \in \mathbb{C}$ quantitatively refines Mostow's result:

Theorem 7.3.9 *Let $\mathfrak{E}$ be the bisector with vertices q_1 and q_2. Suppose that c is a complex hyperplane which is not a slice of $\mathfrak{E}$ and neither q_1 nor q_2 lie in ∂c. Then $\mathfrak{E}$ meets c if and only if $\eta = \eta(q_1, q_2; c)$ satisfies*

$$\operatorname{Im}(\eta)^2 + 2\operatorname{Re}(\eta) < 1. \tag{7.18}$$

In that case the intersection $\mathfrak{E} \cap c$ can be described as follows. Let S be the complex 2-plane containing q_1 and q_2 and orthogonal to c (S is the intersection with the ball of the affine plane containing q_1, q_2 and the polar to c) and

$$\Pi_S : \mathbf{H}^n_{\mathbb{C}} \longrightarrow S$$

orthogonal projection onto S. Then there exists a hypercycle $h \subset S$ of geodesic curvature

$$\tanh(\rho) = \frac{|\operatorname{Im}(\eta)|}{\sqrt{1 - 2\operatorname{Re}(\eta)}}$$

such that

$$\mathfrak{E} \cap c = (\Pi_S)^{-1}(h).$$

We henceforth denote the region of $\mathbb{C}$ defined by (7.18) by $\mathfrak{P}$. It is depicted in Fig. 7.3.

Proof The 2-plane S is the intersection of the projective plane in $\mathbb{P}(\mathbb{C}^{n,1})$ spanned by q_1, q_2 and the polar point $\mathbb{P}(C)$. Since

$$c = (\Pi_S)^{-1}(S \cap c)$$

and

$$\mathfrak{E} = (\Pi_S)^{-1}(S \cap \mathfrak{E}),$$

it suffices to work entirely within S, that is to consider the case $n = 2$.

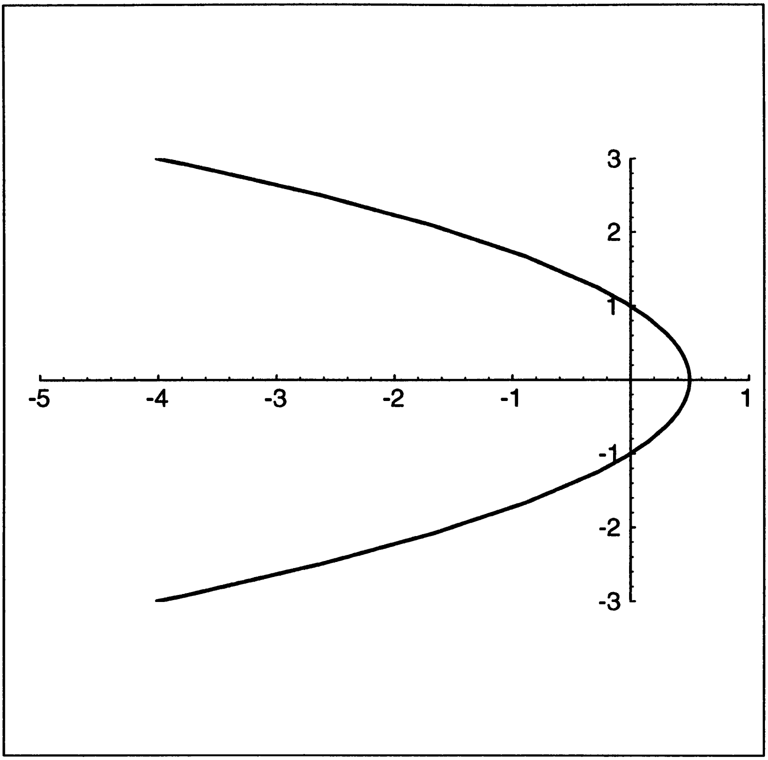

FIG. 7.3. Parabolic region $\mathfrak{P}$ of allowable $\eta(s)$

To this end choose Heisenberg coordinates so that q_1 corresponds to ∞ and q_2 corresponds to the origin in $\mathcal{H}$. Let ι_c denote inversion in c. Since c is neither a slice nor the complex spine of $\mathfrak{C}$,

$$\iota_c(q_1) \neq q_2$$

and ∂c is a chain whose center is neither ∞ nor the origin $0 \in \mathcal{H}$. Furthermore if the center of ∂c lies on the vertical axis (the chain bounding the $\mathbb{C}$-spine Σ of $\mathfrak{C}$), then c is orthogonal to Σ. Either c is a slice of $\mathfrak{C}$ or is disjoint from $\mathfrak{C}$. Thus we assume the center of c does not lie on the vertical axis. Applying a Heisenberg complex dilation, we assume that

$$\iota(q_1) = \text{center}(\partial c) = (1, v_0) \in \mathcal{H}$$

where $v_0 \in \mathbb{R}$ and that ∂c has radius $r_0 > 0$. Under these assumptions, $\partial\mathfrak{C}$ is the horizontal plane $v = 0$ and $\mathfrak{C}$ is the bisector in $\mathbb{B}^2$ defined by

$$\operatorname{Im}(z_2) = 0. \tag{7.19}$$

Furthermore the complex geodesic c is given by

$$c = \{(z_1, z_2) \in \mathbb{B}^2 \mid 2z_1 + (r_0^2 - iv_0)z_2 = 2 - r_0^2 + iv_0\} \tag{7.20}$$

with polar vector

$$C = \begin{bmatrix} 2 \\ r_0^2 + iv_0 \\ 2 - r_0^2 - iv_0 \end{bmatrix}.$$

Thus

$$\eta(q_1, q_2; c) = \frac{1}{2}\Big(1 - r_0^{-2} + i\frac{v_0}{r_0^2}\Big). \tag{7.21}$$

In the ball model, $\mathfrak{C}$ is represented by an $\mathbb{R}$-affine hyperplane (7.19) and c by a $\mathbb{C}$-affine hyperplane (7.20), so the intersection equals the intersection of $\mathbb{B}^2$ with the $\mathbb{R}$-affine line defined by

$$z_1(t) = 1 - \frac{1}{2}(r_0^2 - iv_0)(t+1)$$
$$z_2(t) = t.$$

c intersects $\mathfrak{C}$ if and only if $|v_0| \leq 2r_0$, in which case $c \cap \mathfrak{C}$ is a chord of the circle $\partial c \subset \partial\mathbb{B}^2$ and hence a hypercycle in c. We now compute its geodesic curvature. After solving a quadratic equation, one determines the endpoints $\partial c \cap \partial\mathfrak{C}$ are the following points in $\mathbb{B}^2$:

$$\xi_1 = \begin{pmatrix} 1 - (r_0^2 - iv_0)(2 + r_0^2 + \sqrt{4r_0^2 - v_0^2})/(r_0^4 + v_0^2 + 4) \\ -1 + (4 + 2r_0^2 + 2\sqrt{4r_0^2 - v_0^2})/(r_0^4 + v_0^2 + 4) \end{pmatrix}$$
$$\xi_2 = \begin{pmatrix} 1 - (r_0^2 - iv_0)(2 + r_0^2 - \sqrt{4r_0^2 - v_0^2})/(r_0^4 + v_0^2 + 4) \\ -1 + (4 + 2r_0^2 - 2\sqrt{4r_0^2 - v_0^2})/(r_0^4 + v_0^2 + 4) \end{pmatrix}.$$

Now

$$\langle \xi_1, \xi_2 \rangle = 1 + \frac{-8r_0^2 + 2v_0^2 + 2iv_0\sqrt{4r_0^2 - v_0^2}}{r_0^4 + v_0^2 + 4}. \tag{7.22}$$

By Lemma 3.3.4, the point on c (the complex geodesic containing ξ_1 and ξ_2) closest to the origin $0 \in \mathbb{B}^2$ equals

$$\xi_0 = \frac{1 + iw}{2}\xi_1 + \frac{1 - iw}{2}\xi_2$$

where

$$w = \frac{\operatorname{Im}\langle \xi_1, \xi_2 \rangle}{1 - \operatorname{Re}\langle \xi_1, \xi_2 \rangle} = \frac{v_0}{\sqrt{4r_0^2 - v_0^2}}.$$

Then the affine conformal map $\beta : \Delta \longrightarrow c$

$$\beta(s) = \xi_0 + s(\xi_0 - \xi_2)$$

is an isometry and takes -1 to ξ_1 and $e^{i\psi} = (1 - iw)/(1 + iw)$ to ξ_2; it maps the Euclidean line segment σ from -1 to $e^{i\psi}$ to the Euclidean line segment from ξ_1

to ξ_2. The line segment $\sigma \subset \Delta$ is a ρ-hypercycle (see §7.20 of Beardon [9]) where ρ is given by

$$\sinh(\rho) = \left|\tan\frac{\psi}{2}\right| = |w| = \frac{|v_0|}{\sqrt{4r_0^2 - v_0^2}}$$

with geodesic curvature

$$\tanh(\rho) = \frac{v_0}{2r_0} = \frac{|\mathrm{Im}(\eta)|}{\sqrt{1 - 2\mathrm{Re}(\eta)}}$$

as desired. □

7.3.7 *When the invariant is real*

Consider what it means for $\eta = \eta(q_1, q_2; c)$ to be real. Let $\mathfrak{S}$ be the spinal sphere with vertices q_1, q_2. Since $\mathrm{Re}(\eta) \leq \frac{1}{2}$ in general, if $\eta \in \mathbb{R}$, then either $\eta = \frac{1}{2}$ or $\eta < \frac{1}{2}$. Now $\eta = \frac{1}{2}$ if and only if the inversion ι_c in c interchanges q_1 and q_2. By Theorem 5.2.1, this occurs if and only if c is a slice of $\mathfrak{S}$. The following corollary concerns the other cases when η is real.

Corollary 7.3.10 *The following conditions are equivalent:*

1. $\eta < \frac{1}{2}$*;*
2. $\mathfrak{S} \cap c$ *is a geodesic in* c*;*
3. $\mathbb{A}(q_1, q_2, \iota(q_1)) = 0$*;*
4. $\mathbb{A}(q_1, q_2, \iota(q_2)) = 0$*;*
5. *the orthogonal projection* $\Pi(c)$ *to* c *of the geodesic joining* q_2 *to* q_1 *is a geodesic segment in* c*.*

If these hypotheses are satisfied the intersection $\mathfrak{S} \cap c$ *(of 1) and the projection* $\Pi(c)$ *(of 5) are the same geodesic.*

Proof $1 \iff 2$ follows from Theorem 7.3.9 and $1 \iff 3$ (respectively $1 \iff 4$) follows from Theorem 7.3.3(3) (respectively Theorem 7.3.3(4)) since $z \mapsto 1 - 2z$ maps $(-\infty, \frac{1}{2})$ to $\mathbb{R}_+$. Finally $1 \iff 5$ by Theorem 7.3.7. □

7.3.8 *The distance from a complex geodesic to a bisector*

Alternatively, condition (7.18) follows from the next formula for the distance from a complex hyperplane to a bisector:

Theorem 7.3.11 *Let* c *be a complex hyperplane and let* $q_1, q_2 \in \partial\mathbf{H}^n_{\mathbb{C}}$ *be distinct points. Let* $\mathfrak{S}$ *be the bisector with vertices* q_1, q_2*. Let*

$$\eta = \eta(q_1, q_2; c) \in \mathbb{C}$$

be as above. Then

$$\sqrt{|\eta| + \mathrm{Re}(\eta)} = \begin{cases} \cosh\left(\dfrac{\rho(\mathfrak{S}, c)}{2}\right) & \text{if } \mathfrak{S} \cap c = \emptyset \\ \cos(\angle(\mathfrak{S}, c)) & \text{if } \mathfrak{S} \cap c \neq \emptyset. \end{cases}$$

Proof We may choose coordinates so that

$$C = \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

are representative vectors in $\mathbb{C}^{2,1}$. Then

$$Q(t) = \frac{1}{2}(t^{-1}Q_1 + tQ_2)$$

are vectors polar to the slices $q(t)$ of $\mathfrak{S}$ for $t > 0$ which satisfy

$$\langle Q(t), Q(t)\rangle = 1.$$

Now

$$\frac{\langle Q(t), C\rangle\langle C, Q(t)\rangle}{\langle C, C\rangle\langle Q(t), Q(t)\rangle} = \begin{cases} \cosh^2\left(\dfrac{\rho(q(t), c)}{2}\right) \text{ if } q(t) \cap c = \emptyset \\ \cos^2(\angle(q(t), c)) \text{ if } q(t) \cap c \neq \emptyset \end{cases}$$

and

$$\frac{\langle Q(t), C\rangle\langle C, Q(t)\rangle}{\langle C, C\rangle\langle Q(t), Q(t)\rangle} = \frac{|(1 + u_2)t^{-1} + (1 - u_2)t|^2}{4(1 - u_1\bar{u}_1 - u_2\bar{u}_2)}.$$

Now

$$\eta(q_1, q_2; c) = \frac{1}{2}\frac{(1 - u_2)(1 + \bar{u}_2)}{1 - u_1\bar{u}_1 - u_2\bar{u}_2}.$$

For any complex numbers A, B, the minimum value of

$$|At + Bt^{-1}|^2$$

for $t > 0$ equals

$$2(\mathrm{Re}(A\bar{B}) + |A\bar{B}|)$$

and taking $A = 1 - u_2$, $B = 1 + u_2$, Theorem 7.3.11 follows.

Theorem 7.3.11 gives an alternative proof of condition (7.18) since

$$\mathfrak{P} = \{z \in \mathbb{C} \mid |z| + \mathrm{Re}(z) < 1\}.$$

(Note that $|z| + \mathrm{Re}(z) \geq 0$ for all z.)

7.3.9 *Relation with the complex cross-ratio*

The invariant $\eta(q_1, q_2; c)$ is related to the Korányi–Reimann complex cross-ratio by

$$|\eta(q_1, q_2; c)|^{-2} = \mathbf{X}\{q_1, q_2, \iota_c(q_2), \iota_c(q_1)\} \tag{7.23}$$

and

$$1 - 2\eta(q_1, q_2; c) = \frac{\mathbf{X}\{q_1, q_2, \iota_c(q_2), \iota_c(q_1)\} + \mathbf{X}\{q_1, \iota_c(q_2), q_2, \iota_c(q_1)\} - 1}{\mathbf{X}\{q_1, q_2, \iota_c(q_2), \iota_c(q_1)\}} \tag{7.24}$$

where ι_c denotes inversion in c. The proof will be given in Theorem 9.2.7, in the discussion of spinal spheres with a common slice.

7.3.10 *Orthogonal projections of bisectors*

The invariant $\eta(q_1, q_2; C)$ parametrizes equivalence classes of pairs $(\mathfrak{E}, C)$ where $\mathfrak{E}$ is a bisector (with vertices q_1, q_2) and C is a complex geodesic. Associated to this pair is a natural domain in C, which arises in the study of intersections of bisectors. Let Π_C denote orthogonal projection onto C; then for each slice S of $\mathfrak{E}$, the image $\Pi_C(S) \subset C$ is a geometric disc (or horodisc if $S \parallel C$). Compare the discussion in §3.3.2. Thus the projection $\Pi_C(\mathfrak{E})$ is a union of discs. However, it is generally not geodesically convex (in the Poincaré metric, of course).

Figures 7.4, 7.5, 7.6, 7.7, 7.8, 7.9 illustrate examples of such projections. The circles are the projections of the slices. The circular arc is the projection of the geodesic with endpoints q_1, q_2. It is a segment of a hypercycle joining $\Pi_C(q_1)$ and $\Pi_C(q_2)$.

These pictures of projections of bisectors were drawn using the following formula for the slices of a bisector with given vertices.

Exercise 7.3.12 *Let $u_c \in \mathbb{C}^2 - \mathbb{B}^2$ be a vector polar to a complex geodesic c. Define*

$$u_c^{\perp} = \|u\|^{-1} \begin{bmatrix} -\bar{u}_2 \\ \bar{u}_1 \end{bmatrix} = \|u\|^{-1} J_0 u.$$

Then $\|u^{\perp}\| = 1$ and $\langle\langle u_c^{\perp}, u_c \rangle\rangle = 0$. The image of the affine map

$$\begin{aligned} A_c : \mathbb{C} &\longrightarrow \mathbb{C}^2 \\ \zeta &\longmapsto \|u_c\|^{-2} u_c + \zeta\sqrt{1 - \|u\|^{-2}}\, u_c^{\perp} \end{aligned}$$

does not contain $0 \in \mathbb{C}^2$. The composition

$$\mathbb{P} \circ A_c : \mathbb{C} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

maps $\mathbb{C}$ to the complex line containing the complex geodesic c since

$$\begin{aligned} \langle\langle A_c(\zeta), u_c \rangle\rangle &= \|u\|^{-2} \langle\langle u_c, u_c \rangle\rangle \\ &\quad + \zeta\sqrt{1 - \|u\|^{-2}} \langle\langle u_c^{\perp}, u_c \rangle\rangle = 1 \end{aligned}$$

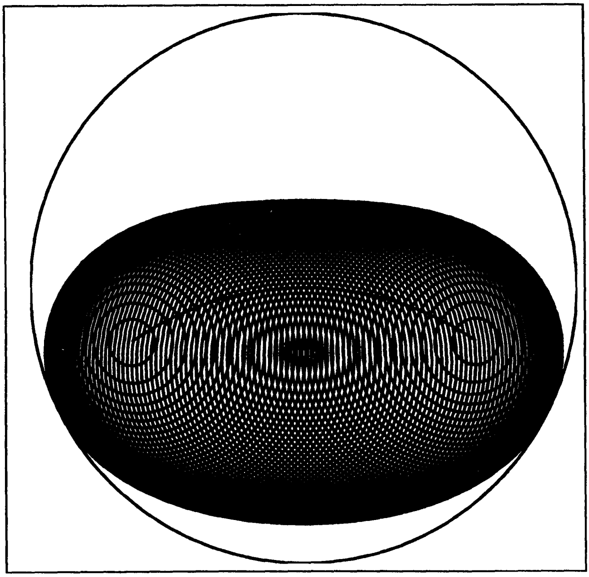

FIG. 7.4. Projection of bisector with $\eta = -0.44i$

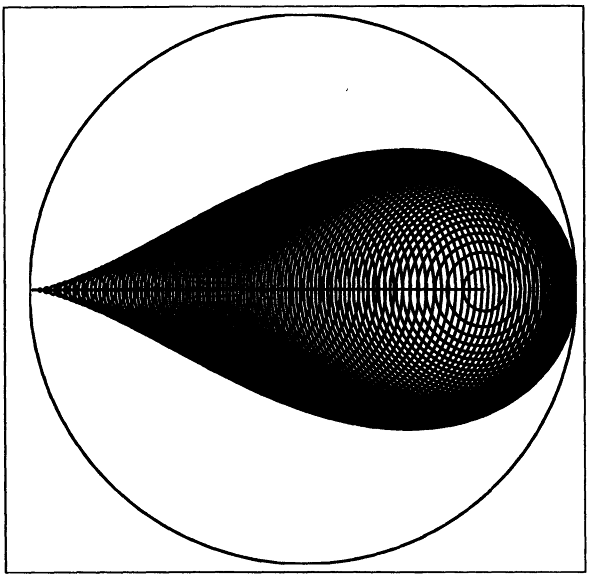

FIG. 7.5. Projection of bisector with $\eta = 0$

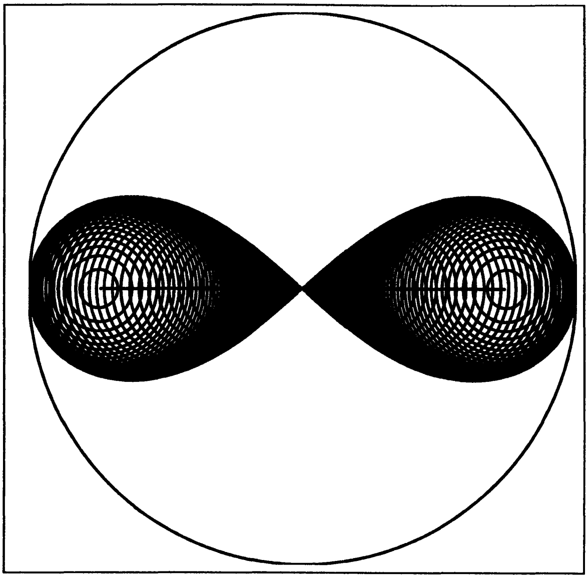

FIG. 7.6. Projection of bisector with $\eta = -0.4$

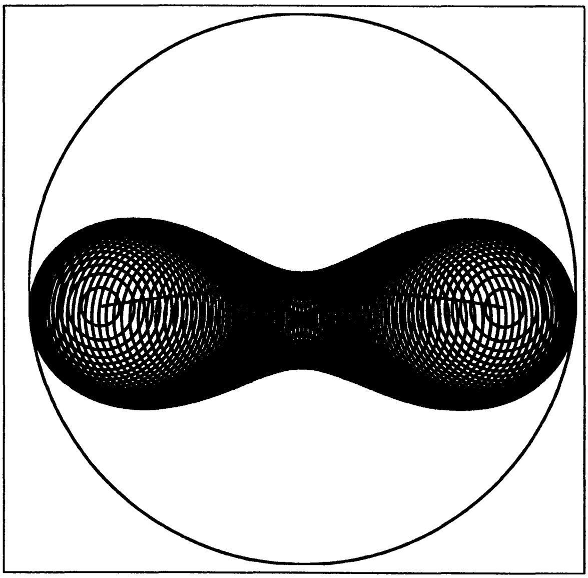

FIG. 7.7. Projection of bisector with $\eta = -0.37 - 0.16i$

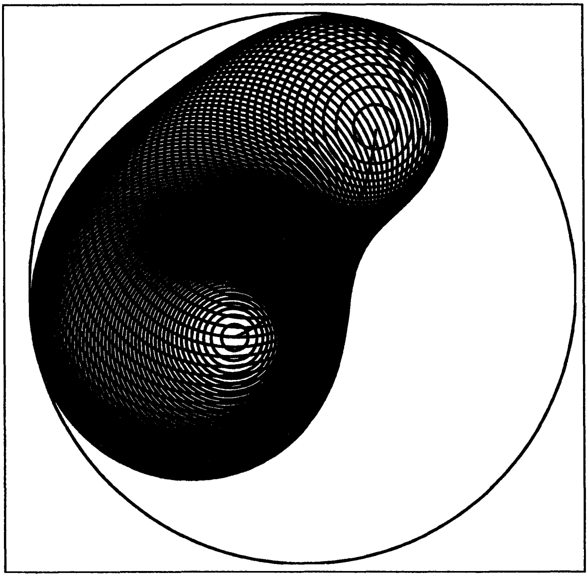

FIG. 7.8. Projection of bisector with $\eta = -1.03 + 0.94i$

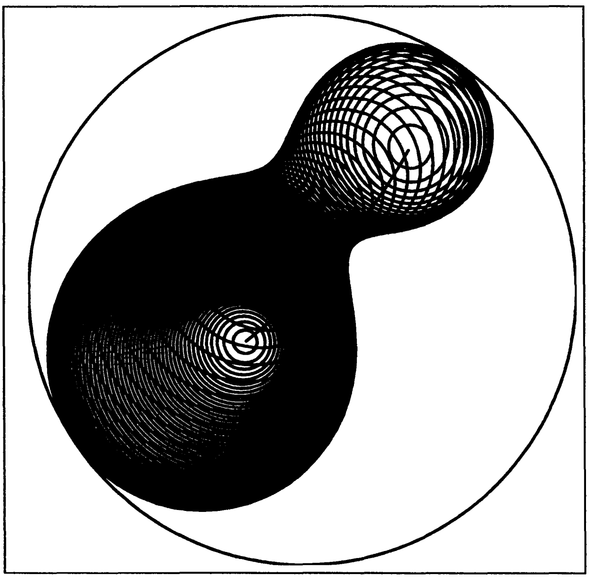

FIG. 7.9. Projection of bisector with $\eta = -1.54 + 0.42i$

Furthermore A_c takes the unit disc $|\zeta| \leq 1$ to C: if $|\zeta| = 1$, then

$$\langle\langle A_c(\zeta), A_c(\zeta)\rangle\rangle = \|u\|^{-4}\langle\langle u_c, u_c\rangle\rangle + (1 - \|u_c\|^{-2}) = 1.$$

The projections of the slices are then the images of the unit circle under A_c. We do not know explicit analytic expressions for these subsets of $\mathbf{H}^1_{\mathbb{C}}$.

8

EXTORS IN PROJECTIVE SPACE

The last two chapters deal with the general theory of bisector intersections. This chapter discusses a class of real hypersurfaces (called "extors") in complex projective space which extend and generalize bisectors in complex hyperbolic and elliptic geometry. The complex projective theory of extors is simpler than that of bisectors in the metric geometries. Thus we classify intersections of extors in $\mathbb{P}^n_{\mathbb{C}}$ as a first step towards understanding the intersection of bisectors. This was all motivated by Giraud [65], and our main goal is to generalize a uniqueness theorem of Giraud, whereby the intersection of an unbalanced pair $\mathfrak{E}_1, \mathfrak{E}_2$ of bisectors can lie in at most one bisector other than $\mathfrak{E}_1$ or $\mathfrak{E}_2$. Giraud's theorem is most appropriately cast in the general setting of complex projective geometry, and thus we return to this context to develop the foundations of the theory of bisectors in complex hyperbolic geometry.

8.1 Extending bisectors to extors

Complex projective geometry contains *complex elliptic geometry* by imposing on $\mathbb{P}^n_{\mathbb{C}}$ the Fubini–Study metric, or equivalently an elliptic anti-polarity. Extors are obtained by abstracting the metric bisectors of complex elliptic geometry.

Let $x, y \in X$ be a pair of distinct points in a metric space (X, ρ). The *metric bisector* defined by (x, y) is the set

$$\mathfrak{E}(x, y) = \{z \in \mathbb{P}^n_{\mathbb{C}} \mid \rho(x, z) = \rho(y, z)\}. \tag{8.1}$$

8.1.1 *Characterization of extors*

Two subsets $A, B \subset \mathbb{P}^n_{\mathbb{C}}$ are *projectively equivalent* if there is a collineation $g \in \mathbf{PGL}(n+1, \mathbb{C})$ such that $g(A) = B$. Then an *extor* is by definition a submanifold projectively equivalent to a metric bisector in $\mathbf{E}^n_{\mathbb{C}}$. Indeed, we may characterize extors in four different ways.

Theorem 8.1.1 *Let $\mathfrak{B} \subset \mathbb{P}^n_{\mathbb{C}}$ be a real analytic hypersurface. The following conditions are equivalent:*

1. *$\mathfrak{B}$ is projectively equivalent to a metric bisector in $\mathbb{P}^n_{\mathbb{C}}$ with respect to the Fubini–Study metric; that is, there exists a collineation $g \in \mathbf{PGL}(n+1, \mathbb{C})$ and a pair of distinct points $x, y \in \mathbb{P}^n_{\mathbb{C}}$ such that $g(\mathfrak{B}) = \mathfrak{E}(x, y)$.*
2. *$\mathfrak{B}$ contains an open subset $\mathfrak{B}'$ projectively equivalent to a metric bisector in $\mathbf{H}^n_{\mathbb{C}}$ with respect to the Bergman metric, that is, there exists a collineation $g \in \mathbf{PGL}(n+1, \mathbb{C})$ and a pair of distinct points $x, y \in \mathbf{H}^n_{\mathbb{C}}$ such that $g(\mathfrak{B}') = \mathfrak{E}(x, y)$.*

3. *There exists a rank 2 indefinite Hermitian form*

$$\Phi : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$$

such that $\mathfrak{B} = \{[Z] \in \mathbb{P}^n_{\mathbb{C}} \mid h(Z,Z) = 0\}$.

4. *There exists a codimension 2 projective subspace* $F \subset \mathfrak{B}$ *and circle* $c \subset l_F$ *(where* l_F *denotes the projective line of hyperplanes containing* F*) such that* $\mathfrak{B} = \Pi_F^{-1}(c) \cup F$ *where* $\Pi_F : \mathbb{P}^n_{\mathbb{C}} - F \to l_F$ *denotes projection.*

If $F \subset \mathbb{P}^n_{\mathbb{C}}$ is a codimension 2 subspace, then the *conormal line* l_F may be indentified with the projective space associated to the 2-dimensional quotient vector space $\mathbb{C}^{n+1}/\tilde{F}$, where

$$\tilde{F} = \mathbb{P}^{-1}(F) \cup \{0\} \subset \mathbb{C}^{n+1}$$

is the codimension 2 *linear* subspace of $\mathbb{C}^{n+1}$ associated to F. When F corresponds to a bisector in $\mathbf{H}^n_{\mathbb{C}}$, l_F is concretely realized in $\mathbb{P}^n_{\mathbb{C}}$ (via the anti-polarity defining $\mathbf{H}^n_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$) as the projective line $F^{\perp} \subset \mathbb{P}^n_{\mathbb{C}}$ containing the complex spine, as defined in §5.1.

Recall that a *circle* $R \subset \mathbb{P}^1_{\mathbb{C}}$ is either a Euclidean circle or a Euclidean straight line (with ∞ adjoined) in $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; see §1.2. Such a "circle" is called a "chaine" in Cartan [24] although that terminology conflicts with his earlier terminology (Cartan [21]) which we have adopted here.

For $n = 1$, an extor is just a circle in the sphere $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, as above. Geodesics in the Poincaré disc $\mathbf{H}^1_{\mathbb{C}}$ are circular arcs orthogonal to the absolute. In this way circles are the analytic continuations of geodesics in $\mathbf{H}^1_{\mathbb{C}}$. Similary, higher-dimensional extors are the analytic continuations of metric bisectors in $\mathbf{H}^n_{\mathbb{C}}$.

8.1.2 *Indefinite Hermitian forms*

Before proving the equivalence of the four characterizations of extors, we describe the third condition in more detail. Let

$$\Phi : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$$

be a rank 2 indefinite Hermitian form. Then the *extor defined by* Φ is the subset of $\mathbb{P}^n_{\mathbb{C}}$ consisting of $[Z]$ satisfying

$$\Phi(Z, Z) = 0.$$

The connection with indefinite Hermitian forms arises from the distance formulas in $\mathbf{E}^n_{\mathbb{C}}$ and $\mathbf{H}^n_{\mathbb{C}}$. Write $V = \mathbb{C}^{n+1}$. Let $x, y \in \mathbf{E}^n_{\mathbb{C}}$ be distinct points. There exists points $X, Y \in V$ such that $x = \mathbb{P}(X)$ and $y = \mathbb{P}(Y)$ respectively. The Fubini–Study distance between x and y is defined by

$$\cos^2(\rho(x,y)/2) = \frac{\langle\!\langle X, Y \rangle\!\rangle \langle\!\langle Y, X \rangle\!\rangle}{\langle\!\langle X, X \rangle\!\rangle \langle\!\langle Y, Y \rangle\!\rangle}$$

where $\langle\!\langle , \rangle\!\rangle$ is the standard positive definite Hermitian form on V. Then (8.1) is equivalent to

$$\Phi_{X,Y}(Z,Z) = 0$$

where

$$\Phi_{X,Y}(Z_1,Z_2) := \langle\!\langle X,Z_2\rangle\!\rangle\langle\!\langle Z_1,X\rangle\!\rangle\langle\!\langle Y,Y\rangle\!\rangle - \langle\!\langle Y,Z_2\rangle\!\rangle\langle\!\langle Z_1,Y\rangle\!\rangle\langle\!\langle X,X\rangle\!\rangle$$

is a rank 2 indefinite Hermitian form, corresponding to the Hermitian matrix

$$\Phi_{X,Y} = \langle Y,Y\rangle XX^* - \langle X,X\rangle YY^*$$

in terms of the outer product construction of §2.2.6.

Similarly, by the distance formula (3.4), the metric bisector in $\mathbf{H}^n_{\mathbb{C}}$ equidistant from points $x,y \in \mathbf{H}^n_{\mathbb{C}}$ represented by negative vectors $X,Y \in \mathbb{C}^{n,1}$ is defined by

$$\Phi_{X,Y}(Z,Z) = 0, \quad \langle Z,Z\rangle < 0$$

where

$$\Phi_{X,Y}(Z_1,Z_2) := \langle X,Z_2\rangle\langle Z_1,X\rangle\langle Y,Y\rangle - \langle Y,Z_2\rangle\langle Z_1,Y\rangle\langle X,X\rangle \tag{8.2}$$

is a rank 2 Hermitian form. In this way certain rank 2 indefinite Hermitian forms define metric bisectors in $\mathbf{E}^n_{\mathbb{C}}$ and $\mathbf{H}^n_{\mathbb{C}}$.

In general, let

$$\Phi : V \times V \longrightarrow \mathbb{C}$$

be an indefinite Hermitian form of rank 2. Since Φ has rank 2, its *radical*

$$\sqrt{\Phi} = \{v \in V \mid \Phi(v,w) = 0\, \forall w \in V\}$$

is a $\mathbb{C}$-linear subspace of codimension 2. Since Φ is indefinite, there exists a basis

$$v_1, v_2, \dots, v_{n+1} \in V$$

such that

$$v_1,\dots,v_{n-1} \in \sqrt{\Phi}, \Phi(v_{n-1},v_{n-1}) = +1, \Phi(v_{n+1},v_{n+1}) = -1, \Phi(v_i,v_j) = 0,$$

for $i \neq j$. If Φ is represented by a Hermitian matrix $\hat{\Phi}$

$$\Phi(v,w) = \bar{v}^\dagger \hat{\Phi} w$$

then the matrix T transforming the standard basis to $(v_1,\dots,v_{n+1})$ satisfies

$$\bar{T}^\dagger \hat{\Phi} T = \hat{\Phi}_0$$

where

$$\hat{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0_{n-1} \end{bmatrix}.$$

That is, all indefinite rank 2 Hermitian forms on V are equivalent under $\mathbf{GL}(n+1, \mathbb{C})$ and all extors defined by indefinite rank 2 Hermitian forms are projectively equivalent.

Exercise 8.1.2 *Let V be a complex vector space with a nondegenerate Hermitian inner product $\langle , \rangle$. Show that linear maps $f : V \longrightarrow \bar{V}$ bijectively correspond to sesquilinear pairings $\Phi : V \times V \times \mathbb{C}$ under*

$$\Phi(u, v) = \langle f(u), v \rangle.$$

Show that Φ is Hermitian if and only if f is self-adjoint (with respect to $\langle , \rangle$).

Exercise 8.1.3 *Suppose that $V = \mathbb{C}^{n,1}$ with Hermitian structure $\langle , \rangle$. Then a rank 2 indefinite Hermitian form Φ arises from a metric bisector in $\mathbf{H}^n_{\mathbb{C}}$; that is, $\Phi = \Phi_{X,Y}$ for $X, Y \in \mathbb{C}^{n,1}$ if and only if the corresponding linear map*

$$f = f_\Phi : \mathbb{C}^{n,1} \longrightarrow \mathbb{C}^{n,1}$$

is semisimple, has trace 0, rank 2 and is self-adjoint. In particular f has exactly two nonzero eigenvalues, each with multiplicity 1. In this way metric bisectors $\mathfrak{E}$ in $\mathbf{H}^n_{\mathbb{C}}$ correspond to pairs of distinct null lines in $\mathbb{C}^{n,1}$ (the vertices *of $\mathfrak{E}$); compare §5.1.3, §5.1.4.*

If f is not semisimple, then $\mathfrak{E}$ corresponds to a fan; see Exercise 4.3.7 or Goldman–Parker [73]. A fan is a limit of spinal spheres where the vertices coalesce. The Hermitian matrix

$$\Phi = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

defines the fan given by $\mathrm{Re}(\zeta) = 0$ *in Heisenberg coordinates. The corresponding self-adjoint endomorphism of $\mathbb{C}^{2,1}$ is*

$$f = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

which is indefinite, has rank 2, but is nilpotent.

Exercise 8.1.4 *Consider $V = \mathbb{C}^{n+1}$ with the standard Hermitian structure $\langle\langle , \rangle\rangle$. Metric bisectors in $\mathbf{E}^n_{\mathbb{C}}$ bijectively correspond to geodesics in $\mathbf{E}^n_{\mathbb{C}}$.*

Suppose that Φ is a rank 2 indefinite Hermitian form on a complex vector space V defining an extor $\mathfrak{E}$. Then the projectivization $\mathbb{P}(\sqrt{\Phi})$ is a codimension 2 projective subspace F, which we call the *focus* of $\mathfrak{E}$. The Hermitian form Φ defines a nondegenerate indefinite Hermitian form Φ' on the quotient space $V' = V/\sqrt{\Phi}$ which is 2-dimensional. The projective line $\mathbb{P}'$ associated to V' identifies with the collection of all linear hyperplanes in V which contain $\sqrt{\Phi}$. The extor defined by Φ' is a circle $\mathfrak{E}' \subset \mathbb{P}'$. The linear projection $V \longrightarrow V'$ defines a *perspective mapping* (or projection)

$$\Pi_F : \mathbb{P}(V) - F \longrightarrow \mathbb{P}'$$

and the extor defined by Φ equals

$$\mathfrak{E} = \Pi_F^{-1}(\mathbb{P}') \cup F. \tag{8.3}$$

Conversely, suppose that $F \subset \mathbb{P}^n_{\mathbb{C}}$ is a codimension 2 subspace. Then the collection l_F of all hyperplanes in $\mathbb{P}^n_{\mathbb{C}}$ containing F has the structure of a projective line (in fact the projective line associated to the 2-dimensional vector space $V/\tilde{F}$, where

$$\tilde{F} = \mathbb{P}^{-1}(F) \cup \{0\} \subset V$$

is the linear subspace corresponding to F). Let $R \subset l_F$ be a circle; R may be defined either as the vanishing locus of an indefinite Hermitian form Φ' on $V/\tilde{F}$ (an anti-polarity on l_F) or as the fixed-point set of a hyperbolic anti-involution. (Recall that these two structures are related by the unique *null polarity* (§1.1.4) on l_F. Compare §1.2.) Let $\Pi_F : \mathbb{P}^n_{\mathbb{C}} - F \longrightarrow l_F$ denote projection. Then

$$\mathfrak{E}(F, R) := \Pi_F^{-1}(\mathbb{P}') \cup F$$

is the extor defined by the rank 2 indefinite Hermitian form Φ on V defined by

$$\Phi(v, w) = \Phi'(\tilde{\Pi}_F(v), \tilde{\Pi}_F(w))$$

where $\tilde{\Pi}_F : V \longrightarrow V/\tilde{F}$ is linear projection.

Exercise 8.1.5 *Suppose that V is a complex vector space with Hermitian structure $\langle , \rangle$. Let $X, Y \in V$ and define $\Phi_{X,Y}$ by (8.2).*

1. *X, Y are linearly dependent if and only $\Phi_{X,Y}$ is zero.*
2. *If X, Y are linearly independent, then $\sqrt{\Phi_{X,Y}}$ is the orthogonal complement of the 2-plane spanned by X and Y.*

Suppose that $\mathfrak{E}$ is an extor corresponding to a bisector in either complex elliptic or hyperbolic geometry. Let $F^{\perp}$ be the complex line dual to the focus F of $\mathfrak{E}$. Let $x \in \mathbf{E}^n_{\mathbb{C}}$ or $x \in \mathbf{H}^n_{\mathbb{C}}$ respectively. Show that there exists a point $y \in \mathbf{E}^n_{\mathbb{C}}$ (or $y \in \mathbf{H}^n_{\mathbb{C}}$ respectively) such that $\mathfrak{E}$ is the bisector equidistant from x and y if and only if $x \in F^{\perp} - \mathfrak{E}$ (respectively $x \in \mathbf{H}^n_{\mathbb{C}} \cap F^{\perp} - \mathfrak{E}$). (Compare Theorem 5.1.7.)

Other real analytic hypersurfaces in $\mathbf{H}^n_{\mathbb{C}}$ are defined by extors, namely the *Clifford cones.* For example, consider the extor $\mathfrak{C}'$ which extends the subset

$$\mathfrak{C} = \left\{ z \in \mathbb{B}^2 \;\middle|\; |z_1| = |z_2| \right\}.$$

$\mathfrak{C}'$ is defined by the Hermitian matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\mathfrak{C}$ is a cone over the Clifford torus

$$\mathfrak{C}' \cap \partial \mathbf{H}^n_{\mathbb{C}} = \left\{ z \;\middle|\; |z_1| = |z_2| = \sqrt{2}/2 \right\}.$$

Like bisectors and fans, Clifford cones enjoy foliations by complex geodesics (defined by $z_1 = \zeta z_2$ for $|\zeta| = 1$ in the above example) and totally real subspaces ($z_1 = \zeta \bar{z}_2$ in the above example). Furthermore, just as bisectors are preimages of Poincaré geodesics under projection $\mathbf{H}^2_{\mathbb{C}} \longrightarrow \mathbf{H}^1_{\mathbb{C}}$ and fans are preimages of Euclidean geodesics under projection $\mathbf{H}^2_{\mathbb{C}} \longrightarrow \mathbb{C}$, Clifford cones are preimages of circles under the projection $\mathbf{H}^2_{\mathbb{C}} - \{O\} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ taking a point x in $\mathbf{H}^2_{\mathbb{C}} - \{O\}$ to the complex line through O and x. (The set of such complex lines forms a $\mathbb{P}^1_{\mathbb{C}}$.)

Analogous to the elliptic–parabolic–hyperbolic trichotomy for circles in $\mathbf{H}^1_{\mathbb{C}}$ (metric circles–horocycles–hypercycles (including geodesics), there are three distinguished classes of extors in $\mathbf{H}^n_{\mathbb{C}}$: Clifford cones, fans, and a generalization of metric bisectors (preimages of hypercycles under projection $\mathbf{H}^2_{\mathbb{C}} \longrightarrow \mathbf{H}^1_{\mathbb{C}}$). Two views of a Clifford torus in Heisenberg space (the intersection of a Clifford cone with the absolute—foliated by chains and $\mathbb{R}$-circles—are depicted in Fig. 8.1 and 8.2.

8.2 Topology and symmetry of an extor

In this section we compute the automorphism group of an extor, decompose an extor in two ways—slices and meridians—and determine the topological type of an extor.

8.2.1 *Automorphisms of an extor*

We determine the group of collineations preserving an extor $\mathfrak{E}$. Since $\mathbf{PGL}(n+1, \mathbb{C})$ acts transitively on the collection of extors in $\mathbb{P}^n_{\mathbb{C}}$, suppose that $\mathfrak{E}$ is defined by the Hermitian form

$$\Phi(Z, W) = Z_n \bar{W}_{n+1} - Z_{n+1} \bar{W}_n.$$

The stabilizer of the focus $F = \mathbb{P}^{n-2} \subset \mathbb{P}^n$ is represented by matrices of the form

$$\begin{bmatrix} A & D \\ 0 & B \end{bmatrix} \tag{8.4}$$

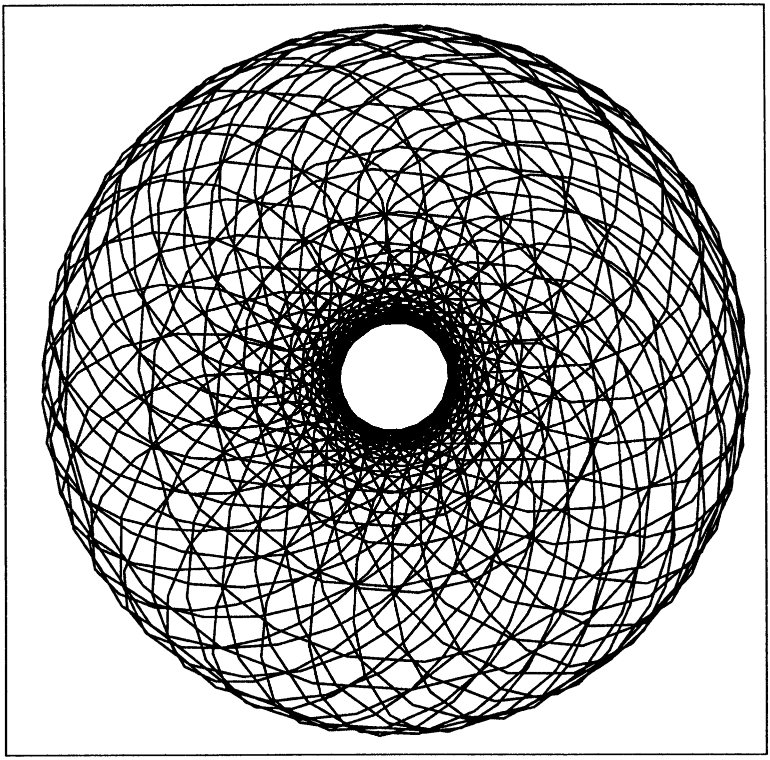

FIG. 8.1. Vertical projection of a Clifford torus

where $A \in \mathbf{GL}(n-1)$, $B \in \mathbf{GL}(2)$ and D is an $(n-1) \times 2$ matrix. The projection (perspective mapping) from F is defined by

$$\Pi : \mathbb{P}^n - \mathbb{P}^{n-2} \longrightarrow \widehat{\mathbb{C}}$$
$$\begin{bmatrix} Z' \\ Z_n \\ Z_{n+1} \end{bmatrix} \longmapsto \frac{Z_n}{Z_{n+1}}$$

where $Z' \in \mathbb{C}^{n-1}$ and $Z_n, Z_{n+1} \in \mathbb{C}$ and the corresponding extor is defined by

$$\mathfrak{E} = \left\{ \begin{bmatrix} Z' \\ Z_n \\ Z_{n+1} \end{bmatrix} \middle| \, Z_n \bar{Z}_{n+1} \in \mathbb{R} \right\}.$$

The stabilizer G of $\mathfrak{E} \subset \mathbb{P}^n$ is the projectivization of the group of matrices of the form (8.4) where $B \in \mathbb{C}^* \cdot \mathrm{SL}(2, \mathbb{R})$. It is a Lie group of (real) dimension $2(n-1)(n+1) + 5 - 2 = 2n^2 + 1$.

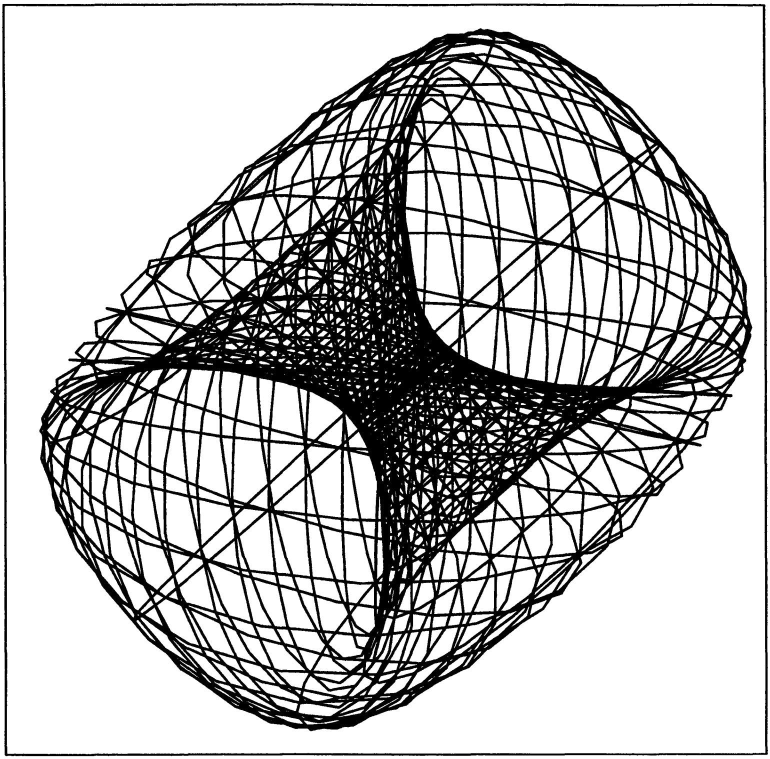

FIG. 8.2. Side view of a Clifford torus

8.2.2 *Topology of an extor*

Next we shall describe the topology of an extor. Several different affine patches provide different points of view.

Consider the affine patch A defined by $z_{n+1} = 1$ and let $A_\infty \approx \mathbb{P}^{n-1}$ be the hyperplane at infinity, comprising points in $\mathbb{P}(\mathbb{C}^{n+1})$ with homogeneous coordinates $z_{n+1} = 0$. In the resulting affine coordinates, $A \cap \mathfrak{E}$ consists of all $z \in \mathbb{C}^n$ such that $z_n \in \mathbb{R}$ and $\mathfrak{E} \cap A_\infty = F$ forms a codimension 2 projective subspace defined by $z_n = 0$ in homogeneous coordinates. Choose a point $p_0 \in F$; we determine the fiber at p_0 of the tubular neighborhood of F in $\mathfrak{E}$. Indeed, p_0 corresponds to a nonzero vector $w'_0 \in \mathbb{C}^{n-1}$, uniquely defined up to multiplication by a nonzero complex scalar. Choose such a representative w'_0 and unique nearby representatives w' for $p \in F$ near w. (That is, choose a local section $p \mapsto w' = w'(p)$ for the projectivization map $\mathbb{C}^n - \{0\} \longrightarrow \mathbb{P}^{n-1}$.) Then points of $A \cap \mathfrak{E}$ are given in affine coordinates by vectors in $\mathbb{C}^n$ of the form

$$\begin{bmatrix} re^{i\theta}\cos(\psi)w' \\ r\sin(\psi) \end{bmatrix}$$

where $\theta, \psi \in \mathbb{R}$ and $r \gg 0$. In particular for fixed $r \gg 0$ and $w' = w'(p)$, the corresponding subset of $A \cap \mathfrak{E}$ is a 2-torus.

Theorem 8.2.1 *Let $\mathfrak{E} = \mathfrak{E}(F, R)$ be an extor in $\mathbb{P}^n_{\mathbb{C}}$. Then the intersection of a tubular neighborhood F in $\mathbb{P}^n_{\mathbb{C}}$ with $\mathfrak{E}$ is diffeomorphic to a K-bundle over F, where K is a cone over a 2-torus.*

In particular, F equals the set of (topologically) singular points of $\mathfrak{E}$. We call F the *cospine* of $\mathfrak{E}$. (This terminology derives from the case $\mathfrak{E}$ corresponds to a spinal hypersurface in complex hyperbolic space, in which case F is the subspace polar to the complex spine. Giraud [65] calls F the "point double" of $\mathfrak{E}$.)

Similarly the *slices* of $\mathfrak{E}$—the fibers of Π—can be characterized in terms of the complex structure. The complex structure on $\mathbb{P}^n_{\mathbb{C}}$ restricts to a Levi-flat CR-structure (a foliation by holomorphic submanifolds) on $\mathfrak{E} - F$ whose leaves are the slices of $\mathfrak{E}$. In particular R can be reconstructed from $\mathfrak{E}$ as the leaf space of the induced CR-structure.

The automorphism group of $\mathfrak{E}$ has two orbits: the principal orbit $\mathfrak{E} - F$ and the singular stratum F. Indeed, the subgroup of matrices of type (8.4) where $B = \mathbb{I}_2$ is the identity matrix and $D = \mathbf{0}_{n-1,2}$ is the zero matrix acts by the full projective group on F and is transitive there. Furthermore the subgroup where $A = \mathbb{I}_{n-1}$ is the identity,

$$D = \begin{bmatrix} 0 & z_1 \\ \vdots & \vdots \\ 0 & z_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & z_{n+1} \end{bmatrix},$$

acts simply transitively (by Euclidean translations in the affine model) on $\mathfrak{E} - F$. Here is an explicit description in coordinates. Take $\tilde{F}$ to be

$$\mathbb{C}^{n-1} \times \{0\} \subset \mathbb{C}^{n+1} = V.$$

Then l_F is a projective line $\mathbb{P}^1_{\mathbb{C}}$ with affine coordinate $\zeta = Z_n/Z_{n+1}$. Take R to be the chain described by the unit circle $|\zeta| = 1$. Then $\mathfrak{E}$ is described in homogeneous coordinates as

$$\mathfrak{E} = \{[Z] \in \mathbb{P} \mid_n \bar{Z}_n - Z_{n+1}\bar{Z}_{n+1} = 0\} \tag{8.5}$$

corresponding to the Hermitian matrix

$$H = \begin{bmatrix} 0 \dots 0 & 0 \\ \vdots \quad\; \vdots & \vdots \\ 0 \dots 1 & 0 \\ 0 \dots 0 & -1 \end{bmatrix}.$$

Analysis of the topology of $\mathfrak{E}$ involves the two affine charts α_1 and α_{n+1}. First consider the chart

$$\alpha_{n+1} : \mathbb{C}^n \longrightarrow \mathbf{A}_{n+1} = \mathbb{P} - \mathbb{P}_{n+1}.$$

The ideal hyperplane $\mathbb{P}_{n+1}$ for α_{n+1} is defined by $Z_{n+1} = 0$, intersecting $\mathfrak{E}$ in the focus $F = \mathbb{P}(\mathbb{C}^{n-1} \times \{0\})$. The affine patch $\mathfrak{E} - \mathfrak{E} \cap \mathbb{P}_{n+1}$ corresponds to

$$\alpha_{n+1}^{-1}(\mathfrak{E}) = \{\zeta \in \mathbb{C}^n \mid |\zeta_n| = 1\} = \mathbb{C}^{n-1} \times S^1$$

decomposing into slices $\mathbb{C}^{n-1} \times \{r\}$ for $r \in S^1$. The space $\mathfrak{E}$ is homeomorphic to an adjunction space

$$(\mathbb{C}^{n-1} \times S^1) \cup \mathbb{P}^{n-1}(\mathbb{C})$$

as follows. Compactify $\mathbb{C}^{n-1}$ as the interior of a $(2n-2)$-cell e^{2n-2}, with boundary $\partial e^{2n-2} \approx S^{2n-3}$. Then $\mathfrak{E}$ is homeomorphic to the adjunction space

$$(e^{2n-2} \times S^1) \cup_f \mathbb{P}^{n-1}(\mathbb{C})$$

with identification map

$$f : \partial(e^{2n-2} \times S^1) \approx S^{2n-3} \times S^1 \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})$$

obtained by composing the Hopf fibration

$$S^{2n-3} \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})$$

with projection $S^{2n-3} \times S^1 \longrightarrow S^{2n-3}$.

Next consider the affine chart $\alpha_1 : \mathbb{C}^n \longrightarrow \mathbf{A}_1 = \mathbb{P} - \mathbb{P}_1$. The extor $\mathfrak{E}$ intersects the ideal hyperplane $\mathbb{P}_1$ in an extor in $\mathbb{P}_1$. Using these affine patches, one can inductively build extors from extors in lower-dimensional spaces, beginning with the case $n = 1$, in which an extor is just a circle (a chain in $\mathbb{P}^1_{\mathbb{C}}$). The corresponding patch of $\mathfrak{E}$ is

$$\alpha_1^{-1}(\mathfrak{E}) = \{\zeta \in \mathbb{C}^n \mid \zeta_{n-1}\bar{\zeta}_{n-1} - \zeta_n\bar{\zeta}_n = 0\} = \mathbb{C}^{n-2} \times C$$

where

$$C = \left\{ \begin{bmatrix} \zeta_{n-1} \\ \zeta_n \end{bmatrix} \in \mathbb{C}^2 \,\middle|\, \zeta_{n-1}\bar{\zeta}_{n-1} - \zeta_n\bar{\zeta}_n = 0 \right\}$$

is a cone over the real 2-torus $\mathbb{T} \times \mathbb{T}$ using coordinates

$$\zeta_{n-1} = re^{i\theta_1}, \qquad \zeta_n = re^{i\theta_2}$$

where $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T} \times \mathbb{T}$ and $r = |\zeta_{n-1}| = |\zeta_n| \geq 0$.

Finally we consider one last affine patch, in which $\mathfrak{C}$ is represented as an adjunction space with one $(2n-1)$-cell. The ideal hyperplane is

$$H = \{[Z] \mid Z_n + Z_{n+1} = 0\}$$

which is a slice of $\mathfrak{C}$. In the affine chart

$$\alpha : \mathbb{C}^n \longrightarrow \mathbb{P} - H$$

$$\xi \longmapsto \begin{bmatrix} i\xi_1 \\ \vdots \\ i\xi_{n-1} \\ \frac{1}{2}(\xi_n + i) \\ \frac{1}{2}(-\xi_n + i) \end{bmatrix}$$

the Hermitian form defining $\mathfrak{C}$ is

$$Z_n \bar{Z}_n - Z_{n+1} \bar{Z}_{n+1} = \operatorname{Im}(\xi_n)$$

and

$$\alpha^{-1}(\mathfrak{C}) = \mathbb{C}^{n-1} \times \mathbb{R}$$

is a $(2n-1)$-cell. The slices of $\mathfrak{C}$ are the complex hyperplanes $\mathbb{C}^{n-1} \times \{\xi_n\}$ for $\xi_n \in \mathbb{R}$, in addition to the hyperplane H.

$\mathfrak{C}$ may be reconstructed as an adjunction space as follows. Compactify

$$\alpha^{-1}(\mathfrak{C}) = \mathbb{C}^{n-1} \times \mathbb{R}$$

as the interior of a $(2n-1)$-cell e^{2n-1}. Its boundary ∂e^{2n-1} is a $(2n-2)$-sphere attached to the ideal slice H by a map

$$f : S^{2n-2} \longrightarrow H \approx \mathbb{P}^{n-1}(\mathbb{C}).$$

Since e^{2n-1} is represented as the $\mathbb{R}$-linear hyperplane $\mathbb{C}^{n-1} \times \mathbb{R} \subset \mathbb{C}^n$, the attaching map f is the restriction of the Hopf fibration (the attaching map for $\mathbb{P}^n_{\mathbb{C}}(\mathbb{C})$ itself) to the intersection of S^{2n-1} with this $\mathbb{R}$-linear hyperplane, which is equivalent to the equatorially embedded $S^{2n-2} \subset S^{2n-1}$. The Hopf fibration is the quotient map for the S^1-action on S^{2n-1} described by

$$\xi \mapsto e^{i\theta} \xi$$

where $\xi \in S^{2n-1} \subset \mathbb{C}^n$ and $e^{i\theta} \in S^1$. The orbit of $\xi \in S^{2n-2}$ either lies completely in S^{2n-2} or meets S^{2n-2} in exactly two antipodal points, depending on whether $\xi_n = 0$ or $\xi_n \neq 0$. The restriction of f to the equator $S^{2n-3} \subset S^{2n-2}$ defined by $\xi_n = 0$ is just the Hopf fibration

$$S^{2n-3} \longrightarrow \mathbb{P}^{n-2}(\mathbb{C}).$$

The restriction of f to the upper hemisphere $\xi_n > 0$ is a diffeomorphism onto an affine patch $\mathbb{P}^{n-1}(\mathbb{C}) - \mathbb{P}^{n-2}(\mathbb{C})$. The restriction of f to the lower hemisphere is the composition of the restriction of f to the upper hemisphere with the antipodal map of S^{2n-2}.

Theorem 8.2.2 *Let $\mathfrak{E}$ be an extor with focus F and spine $R \subset l_F$.*

1. *There exists a diffeomorphism $\phi : \mathfrak{E} - F \longrightarrow \mathbb{C}^{n-1} \times S^1$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{E} - F \arrow[r, "\phi"] \arrow[d, "\Pi_F"'] & \mathbb{C}^{n-1} \times S^1 \arrow[d] \\ R \arrow[r, "\cong"'] & S^1 \end{tikzcd}$$

   where the last vertical arrow is Cartesian projection.

2. *There exists a regular neighborhood N of F in $\mathfrak{E}$ which is homeomorphic to a product $F \times \mathrm{cone}(T^2)$.*
3. *$\mathfrak{E}$ is homeomorphic to the identification space of $[0,\infty] \times T^2 \times \mathbb{P}^{n-2}(\mathbb{C})$ where $\{0\} \times T^2 \times \mathbb{P}^{n-2}(\mathbb{C})$ is identified by the projection*

$$\{0\} \times T^2 \times \mathbb{P}^{n-2}(\mathbb{C}) \longrightarrow \mathbb{P}^{n-2}$$

   and $\{\infty\} \times T^2 \times \mathbb{P}^{n-2}(\mathbb{C})$ is identified by the projection

$$\{\infty\} \times T^2 \times \mathbb{P}^{n-2}(\mathbb{C}) \longrightarrow S^1$$

   induced by a projection $T^2 \longrightarrow S^1$.

8.2.3 *Meridians*

The slices of an extor define a singular foliation by complex hyperplanes. Similarly there exists a singular foliation of $\mathfrak{E}$ by purely real projective subspaces ("chaines spatiales normales" in the terminology of Cartan [24]). We call these submanifolds the *meridians* of $\mathfrak{E}$, since they extend the meridians of bisectors (see §5.1.6).

Let $\mathfrak{E}$ be a bisector with focus F and defined by a circle $R \subset \nu_F$. A *meridian* of $\mathfrak{E}$ is the fixed-point set of an anti-involution $\rho : \mathbb{P} \longrightarrow \mathbb{P}$ which is compatible with $\mathfrak{E}$ in the following sense. Let ρ_ν be the anti-involution of the projective line ν_F defining R. Then ρ *extends* ρ_ν in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P} - F \arrow[r, "\Pi_F"] \arrow[d, "\rho"'] & \nu_F \arrow[d, "\rho_\nu"] \\ \mathbb{P} - F \arrow[r, "\Pi_F"'] & \nu_F \end{tikzcd}$$

In particular every point fixed by ρ will lie in $\mathfrak{E} = \Pi_F^{-1}(R)$. Choose $A \in \mathbf{GL}(n,\mathbb{C})$ such that $v \mapsto A(\bar{v})$ is a linear anti-involution of $\mathbb{C}^n$. Its fixed-point set is a purely real linear subspace of $\mathbb{C}^n$. The $\mathbb{R}$-linear map

$$\rho_A : \mathbb{V}_{\mathbb{R}} \longrightarrow \mathbb{V}_{\mathbb{R}}$$
$$\begin{bmatrix} Z' \\ Z_n \\ Z_{n+1} \end{bmatrix} \longmapsto \begin{bmatrix} A\bar{Z}' \\ \bar{Z}_{n+1} \\ \bar{Z}_n \end{bmatrix}$$

is a linear anti-involution. The fixed points of the corresponding anti-involution of $\mathbb{P}$ lie in $\mathfrak{E}$: if ρ_A fixes $[Z]$, then there exists $\lambda \in \mathbb{C}^*$ such that $\rho_A(Z) = \lambda Z$, which implies that

$$Z_n \bar{Z}_n - Z_{n+1}\bar{Z}_{n+1} = 0$$

(and necessarily $|\lambda| = 1$). If $R_A \subset (\mathbb{C}^n)_{\mathbb{R}}$ is the fixed point set of A, then the projective fixed-point set of ρ_A on $\mathbb{V}$ equals

$$\mathbf{PFix}(\rho_A) = (\mathbb{C}^* \dot{R}_A) \times \left\{ \begin{bmatrix} Z_n \\ Z_{n+1} \end{bmatrix} \middle| Z_n\bar{Z}_n - Z_{n+1}\bar{Z}_{n+1} = 0 \right\}$$

and the meridian M_A is its image in $\mathbb{P}$.

8.3 Pairs of extors

Now we discuss in detail properties of the intersection of extors.

An extor $\mathfrak{E}$ is determined by a pair (F, R) where $F \subset \mathbb{P}^n_{\mathbb{C}}$ is the focus and $R \subset l_F$ is a circle. The simplest case, and the most degenerate, occurs when the two extors share a focus. In this case we say that the extors are *confocal,* and the intersections of confocal extors reduce to intersections of circles in $\mathbb{P}^1_{\mathbb{C}}$. In particular equivalence classes of confocal pairs of extors form a moduli space of real dimension 1. Confocal extors arise in Dirichlet/Ford polyhedra determined from points in a complex-linear submanifold, what we have called *cospinal* bisectors in §5.3.1.

In the nonconfocal case, there are four equivalence classes, *when the complex dimension* $n = 2$, depending on the incidence relations between the foci F_i and the opposite circles $R_j \subset l_{F_i}$. Bisectors arising from Dirichlet/Ford polyhedra are coequidistant/covertical, and for $n = 2$, the conditions of coequidistance/coverticality are generic. For $n > 2$, the additional case that the foci are transverse arises.

8.3.1 *Confocal pairs of extors*

A pair $(\mathfrak{E}_1, \mathfrak{E}_2)$ of extors $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathbb{P}$ is said to be *confocal* if $\mathfrak{E}_1$ and $\mathfrak{E}_2$ have the same focus. Denote their common focus by F, its normal line by l_F and the corresponding projection by $\Pi_F : \mathbb{P} - F \longrightarrow l_F$. Let $R_i \subset F$ denote the spine of $\mathfrak{E}_i$. Corresponding to the four equivalence classes of pairs of chains in a projective line, there are four families of equivalence classes of confocal pairs of extors:

1. $R_1 = R_2$;
2. $R_1 \cap R_2 = \emptyset$;

3. $R_1 \cap R_2$ consists of one point—in this case R_1 and R_2 are tangent;
4. $R_1 \cap R_2$ consists of two points—in this case R_1 and R_2 meet transversely.

Note that pairs (R_1, R_2) which are not tangent are classified up to $\mathbf{PGL}(2, \mathbb{C})$-equivalence by one modulus, in the cases that $R_1 \cap R_2$ consists of zero or two points. If $R_1 \cap R_2 = \emptyset$, then the product of the inversions in R_1 and R_2 is a hyperbolic element of $\mathbf{PGL}(2, \mathbb{C})$ whose dilation factor parametrizes equivalence classes of such pairs. If $R_1 \cap R_2$ consists of two points, then the product of the inversions fixing R_1 and R_2 is an elliptic element of $\mathbf{PGL}(2, \mathbb{C})$ whose rotation angle parametrizes equivalence classes of such pairs.

Suppose that $(\mathfrak{E}_1, \mathfrak{E}_2)$ is a confocal pair as above. The intersection equals

$$\mathfrak{E}_1 \cap \mathfrak{E}_2 = F \cup \Pi_F^{-1}(R_1 \cap R_2).$$

Except when $\mathfrak{E}_1 = \mathfrak{E}_2$, the intersection consists of the union of zero, one or two slices. If $R_1 \cap R_2 = \emptyset$, then $\mathfrak{E}_1 \cap \mathfrak{E}_2 = F \approx \mathbb{P}^{n-2}(\mathbb{C})$. If $R_1 \cap R_2$ consists of one point, then $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is a single slice (a projective hyperplane). If $R_1 \cap R_2$ consists of two points, then $\mathfrak{E}_1 \cap \mathfrak{E}_2$ consists of two slices; that is, two projective hyperplanes which intersect along F.

8.3.2 *Balanced pairs of extors*

A particularly interesting case of intersections of extors $\mathfrak{E}_i = \mathfrak{E}(F_i, R_i)$ occurs when $\Pi_{F_1}(F_2) \in R_1$ and $\Pi_{F_2}(F_1) \in R_2$. In this case we say that $(\mathfrak{E}_1, \mathfrak{E}_2)$ is a *balanced pair* of extors. (If only one of these incidence relations holds, we say the pair is *semi-balanced.*)

Balanced pairs of extors arose in §5.3.5 for bisectors possessing both a common slice and a common meridian.

Theorem 8.3.1 *Let $(\mathfrak{E}_1, \mathfrak{E}_2)$ be a balanced pair of extors in $\mathbb{P}^2_{\mathbb{C}}$. Then there exists a purely real projective plane P and a (complex) projective line l which intersect along a chain in l such that $\mathfrak{E}_1 \cap \mathfrak{E}_2 = P \cup l$. In other words, there exists a collineation $g \in \mathbf{PGL}(3, \mathbb{C})$ such that*

$$\mathfrak{E}_1 \cap \mathfrak{E}_2 = g(\mathbb{P}^2_{\mathbb{R}} \cup \mathbb{P}^1_{\mathbb{C}}).$$

Unlike generic pairs of extors, there will be infinitely many extors $\mathfrak{E}$ containing $\mathfrak{E}_1 \cap \mathfrak{E}_2$ if $(\mathfrak{E}_1, \mathfrak{E}_2)$ is balanced.

Suppose that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are two extors whose foci F_1 and F_2 are distinct. We consider only the case $n = 2$. (By applying projection to $F_1 \cap F_2$, we may reduce to this case.) There are four possibilities (essentially only three, since 2 and 3 are equivalent by interchanging $\mathfrak{E}_1$ and $\mathfrak{E}_2$). If $R \subset W(p)$, we say that q is *incident to* R if and only if either $q = p$ or $\Pi_p(q) \subset R$.

1. F_1 is incident to R_2 and F_2 is incident to R_1;
2. F_1 is not incident to R_2 and F_2 is incident to R_1;
3. F_1 is incident to R_2 and F_2 is not incident to R_1;
4. F_1 is not incident to R_2 and F_2 is not incident to R_1.

The first case corresponds to balanced pairs, the second two cases to semi-balanced pairs, and the last case to unbalanced pairs.

We analyze these cases separately. We choose coordinates so that

$$F_1 = \begin{bmatrix}1 \, 0 \, 0\end{bmatrix}, \qquad F_2 = \begin{bmatrix}0 \, 1 \, 0\end{bmatrix}.$$

In case 1, we may take R_1 and R_2 to be given by the real structures

$$\rho_1 : \begin{bmatrix} * \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} * \\ \bar{z}_2 \\ \bar{z}_3 \end{bmatrix}, \qquad \rho_2 : \begin{bmatrix} z_1 \\ * \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{z}_1 \\ * \\ \bar{z}_3 \end{bmatrix},$$

respectively, or alternatively by Hermitian matrices

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

respectively. The circles R_1 and R_2 are defined by $z_1/z_3 \in \mathbb{R} \cup \{\infty\}$ and $z_2/z_3 \in \mathbb{R} \cup \{\infty\}$ respectively. It follows that $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is the union $\mathbb{P}^1(\mathbb{C}) \cup \mathbb{P}^2(\mathbb{R})$ of a complex line (defined by $z_3 = 0$) and a real plane in $\mathbb{P}^n_{\mathbb{C}}$; that is, $(\mathfrak{E}_1, \mathfrak{E}_2)$ is a balanced pair of extors. This is the only case (for distinct vertices) when $\mathfrak{E}_1 \cap \mathfrak{E}_2$ contains a holomorphic curve.

8.3.3 *Semi-balanced pairs of extors*

In the second case, we may take ρ_1 as above, but replace ρ_2 by the real structure

$$\rho_2 : \begin{bmatrix} z_1 \\ * \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{z}_3 \\ * \\ \bar{z}_1 \end{bmatrix}$$

which defines the circle

$$|z_1/z_3| = 1$$

in $\mathbb{P}(W_{F_2})$. The corresponding Hermitian matrix is

$$\Phi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then $\mathfrak{E}_1 \cap \mathfrak{E}_2 - F_2$ is defined by

$$z_2/z_3 \in \mathbb{R} \cup \{\infty\}, \qquad |z_1/z_3| = 1$$

and (since $z_3 \neq 0$), we can identify $\mathfrak{E}_1 \cap \mathfrak{E}_2$ in homogeneous coordinates with

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \in \mathbb{R}, \quad |z_2| = 1\}$$

(which is a Euclidean cylinder). The full intersection is the 1-point compactification (the ideal point being F_2) of this cylinder. The link of F_2 is a disjoint union of two circles. The smooth part is purely real (as a real submanifold of $\mathbb{P}^n_{\mathbb{C}}$). The third case is completely analogous to the second case.

8.3.4 *Clifford tori*

The last case is generic in complex dimension $n = 2$. Choose coordinates so that

$$\rho_2 : \begin{bmatrix} z_1 \\ * \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{z}_3 \\ * \\ \bar{z}_1 \end{bmatrix}, \qquad \rho_1 : \begin{bmatrix} * \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} * \\ \bar{z}_3 \\ \bar{z}_2 \end{bmatrix}.$$

The corresponding Hermitian matrices are

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In this case

$$\mathfrak{E}_1 \cap \mathfrak{E}_2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; |z_1| = |z_2| = 1 \right\}$$

is a purely real torus in $\mathbb{P}^n_{\mathbb{C}}$, which we call a *Clifford torus.* Such a torus T can be characterized as a principal orbit of a maximal (compact) torus in $\mathbf{PGL}(3, \mathbb{C})$.

Following Giraud [65], we call an intersection of two bisectors of this sort a *rough edge* ("arête gauche").

Exercise 8.3.2 *Suppose $n = 2$ and let Φ_1, Φ_2 be two rank 2 indefinite Hermitian matrices determining bisectors $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathbf{H}^2_{\mathbb{C}}$. Then $\mathfrak{E}_1$ and $\mathfrak{E}_2$ form:*

1. *A confocal pair if and only if* $\mathrm{Ker}(t_1\Phi_1 + t_2\Phi_2)$ *is constant.*
2. *A balanced pair if and only if* $\det(t_1\Phi_1 + t_2\Phi_2)$ *is constant (necessarily zero) but* $\mathrm{Ker}(t_1\Phi_1 + t_2\Phi_2)$ *is nonconstant.*
3. *A semi-balanced pair if and only if* $\det(t_1\Phi_1 + t_2\Phi_2)$ *equals $ct_1^2t_2$ or $ct_1t_2^2$ for some nonzero c.*
4. *An unbalanced pair if and only if*

$$\det(t_1\Phi_1 + t_2\Phi_2) = t_1t_2(c_1t_1 + c_2t_2)$$

for nonzero c_1, c_2.

As an example, consider three points $x, x_1, x_2 \in \mathbf{H}^2_{\mathbb{R}}$ and the bisectors in $\mathbf{H}^2_{\mathbb{C}}$ equidistant from x and x_1, x_2 respectively. Let $\rho_i = \rho(x, x_i)$ for $i = 1, 2$ and $\theta = \angle(x_1, x, x_2)$. We may choose coordinates so that the points are represented by vectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \sinh(\rho_1) \\ 0 \\ \cosh(\rho_1) \end{bmatrix}, \begin{bmatrix} \sinh(\rho_2)\cos(\theta) \\ \sinh(\rho_2)\cos(\theta) \\ \cosh(\rho_2) \end{bmatrix}$$

respectively. Hermitian matrices corresponding to the bisectors $\mathfrak{E}(x, x_1)$ and $\mathfrak{E}(x, x_2)$ are

$$\Phi_1 = \begin{bmatrix} \sinh^2(\rho_1) & 0 & -\cosh(\rho_1)\sinh(\rho_1) \\ 0 & 1 & 0 \\ -\cosh(\rho_1)\sinh(\rho_1) & 0 & -\sinh^2(\rho_1) \end{bmatrix}$$

and

$$\Phi_2 =$$

$$\begin{bmatrix} \sinh^2(\rho_2)\cos^2(\theta) & \sinh^2(\rho_2)\cos(\theta)\sin(\theta) & -\sinh(\rho_2)\cosh(\rho_2)\cos(\theta) \\ \sinh^2(\rho_2)\cos(\theta)\sin(\theta) & \sinh^2(\rho_2)\sin^2(\theta) & -\sinh(\rho_2)\cosh(\rho_2)\sin(\theta) \\ \sinh(\rho_2)\cosh(\rho_2)\cos(\theta) & \sinh(\rho_2)\cosh(\rho_2)\sin(\theta) & -\sinh^2(\rho_2) \end{bmatrix}$$

and

$$\det(t_1\Phi_1 + t_2\Phi_2) = t_1t_2(t_1 + t_2)\sinh^2(\rho_2)\sinh^2(\rho_1)\sin^2(\theta).$$

8.3.5 *Giraud's theorem*

Theorem 8.3.3 *(Giraud [65]) Suppose that $\mathfrak{E}_1, \mathfrak{E}_2$ are two bisectors with respective complex spines Σ_1, Σ_2 such that*

1. *Σ_1 and Σ_2 are distinct;*
2. *$\Sigma_1 \cap \mathfrak{E}_2 = \Sigma_1 \cap \mathfrak{E}_2 = \emptyset$.*

Then there is at most one bisector containing $\mathfrak{E}_1 \cap \mathfrak{E}_2$ besides $\mathfrak{E}_1$ and $\mathfrak{E}_2$.

In the case $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are coequidistant, that is there exists $x_0, x_1, x_2 \in \mathbf{H}^n_{\mathbb{C}}$ such that

$$\mathfrak{E}_i = \mathfrak{E}(x_0, x_i)$$

for $i = 1, 2$, then the third bisector containing $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is $\mathfrak{E}(x_1, x_2)$. Coequidistant bisectors which are not cospinal are always unbalanced in this sense:

Lemma 8.3.4 *Suppose that $x, x_1, x_2 \in \mathbf{H}^n_{\mathbb{C}}$ do not lie on a complex geodesic. Then the extors $\mathfrak{E}_i$ containing the bisectors $\mathfrak{E}(x, x_i)$ intersect in a Clifford torus.*

Proof By considering the 2-dimensional subspace spanned by x, x_1, x_2, it suffices to assume that $n = 2$. Represent the points x, x_1, x_2 by vectors $X, X_1, X_2 \in \mathbb{C}^{n,1}$ satisfying

$$\langle X, X\rangle = \langle X_1, X_1\rangle = \langle X_2, X_2\rangle = -1.$$

Then $\mathfrak{E}_i$ consists of all $[Z] \in \mathbb{P}(\mathbb{C}^{n,1})$ such that

$$H_i(Z) := \langle Z, X\rangle\langle X, Z\rangle - \langle Z, X_1\rangle\langle X_1, Z\rangle = 0.$$

The polar of $\mathfrak{E}_i$ is $p_i = [X \boxtimes X_i]$. By Giraud's theorem 8.3.3 it suffices to show that $p_1 \notin \mathfrak{E}_2$ and $p_2 \notin \mathfrak{E}_1$. Since x, x_1, x_2 do not lie on a complex geodesic, the corresponding vectors X, X_1, X_2 form a basis of $\mathbb{C}^{2,1}$ and $\langle X \boxtimes X_2, X_1\rangle \neq 0$. Now

$$\begin{aligned} H_1(X \boxtimes X_2] &= \langle X \boxtimes X_2, X\rangle\langle X, X \boxtimes X_2\rangle - \langle X \boxtimes X_2, X_1\rangle\langle X_1, X \boxtimes X_2\rangle \\ &= -|\langle X \boxtimes X_2, X_1\rangle|^2 \neq 0. \end{aligned}$$

Thus $p_2 \notin \mathfrak{E}_1$. Similarly $p_1 \notin \mathfrak{E}_2$. □

(This may also be deduced by the algebraic criteria in Exercise 8.3.2.)

Corollary 8.3.5 *The faces of a Dirichlet or Ford polyhedron in $\mathbf{H}_{\mathbb{C}}^n$ lie on extors which are either cospinal or unbalanced.*

8.3.6 *Proof of Giraud's theorem*

We present a slight variation of Giraud's proof, based on the following result ([65], Lemme 3, p.59):

Lemma 8.3.6 *Suppose that E is a 3-dimensional complex vector space and*

$$\xi_1, \xi_2, \xi_3, \eta_1, \eta_2 \in E^*$$

are linear functionals such that ξ_1, ξ_2, ξ_3 are linearly independent. Suppose that for every $v \in E$,

$$|\xi_1(v)| = |\xi_2(v)| = |\xi_3(v)| \tag{8.6}$$

implies that

$$|\eta_1(v)| = |\eta_2(v)|. \tag{8.7}$$

Then $\eta_1\eta_1^* - \eta_2\eta_2^* \in \Lambda^{1,1}E^*$ is a scalar multiple of one of

1. $\xi_1\xi_1^* - \xi_2\xi_2^*$,
2. $\xi_2\xi_2^* - \xi_3\xi_3^*$, *or*
3. $\xi_3\xi_3^* - \xi_1\xi_1^*$.

The proof of Lemma 8.3.6 uses the following elementary fact:

Lemma 8.3.7 *Let $T \subset E$ denote the torus defined by*

$$|z_1| = |z_2| = |z_3| = 1 \tag{8.8}$$

and suppose that H is a Hermitian matrix such that the function $h : E \longrightarrow \mathbb{R}$ defined by

$$h(v) = \bar{v}^\dagger H v$$

vanishes on T. Then H is a diagonal matrix.

Proof Let

$$v(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} e^{i\theta_1} \\ e^{i\theta_2} \\ e^{i\theta_3} \end{bmatrix} \in T$$

and

$$\begin{aligned} f(\theta_1, \theta_2, \theta_3) &:= h(v(\theta_1, \theta_2, \theta_3)) \\ &= H_{11} + H_{22} + H_{33} \\ &\quad + 2\mathrm{Re}\left(H_{12}e^{i(\theta_2-\theta_1)} + H_{13}e^{i(\theta_3-\theta_1)} + H_{23}e^{i(\theta_3-\theta_2)}\right) \end{aligned}$$

is identically zero. In particular, since the functions $e^{i(\theta_2-\theta_1)}, e^{i(\theta_3-\theta_1)}, e^{i(\theta_3-\theta_2)}$ are linearly independent, the trace

$$H_{11} + H_{22} + H_{33} = 0.$$

Since

$$\frac{\partial f}{\partial \theta_1} = 2 \text{ Im} \left(H_{12} e^{i(\theta_2-\theta_1)} - H_{31} e^{i(\theta_3-\theta_1)} \right)$$

is zero,

$$\frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} = 2 \text{ Re} \left(H_{21} e^{i(\theta_1-\theta_2)} \right)$$

is also zero. If $H_{21} \neq 0$, taking

$$\theta_1 - \theta_2 = \arg(H_{21})$$

leads to a contradiction. Thus $H_{21} = 0$ and similarly $H_{ij} = 0$ for $i \neq j$. □

Proof of Lemma 8.3.6 Let H be the Hermitian matrix representing the Hermitian form $\eta_1\eta_1^* - \eta_2\eta_2^*$. By Lemma 8.3.7, H is a real diagonal matrix. Since H has rank 2, it must have one eigenvalue zero, and hence determinant zero, which implies that it is a multiple of one of the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as desired. □

Proof of Theorem 8.3.3 Extending $\mathfrak{E}_1$ and $\mathfrak{E}_2$ to a pair of extors, the first hypothesis implies that the pair is not confocal and the second hypothesis implies it is unbalanced. Since an indefinite rank 2 Hermitian matrix of trace 0 must be of the form $\eta_1\eta_1^* - \eta_2\eta_2^*$, we may assume that $\mathfrak{E}$ is defined by $\eta_1\eta_1^* - \eta_2\eta_2^*$, and $\mathfrak{E}_i$ is defined by $\xi_i\xi_i^* - \xi_3\xi_3^*$ for $i = 1, 2$. Then (8.6) is the condition that $[v] \in \mathbb{E}_1 \cap \mathfrak{E}_2$ and (8.7) expresses $[v] \in \mathfrak{E}$. Changing coordinates so that ξ_1, ξ_2, ξ_3 is the coordinate basis, apply Lemma 8.3.7 to deduce Theorem 8.3.3. □

Here is a more informative and geometric (but longer) proof of Giraud's theorem. Lemma 8.3.7 implies that the only extors containing the Clifford torus

$$T = \{[Z] \in \mathbb{P}^2_{\mathbb{C}} \mid |Z_1| = |Z_2| = |Z_3|\}$$

are the three extors

$$\mathfrak{E}_1 = \{[Z] \in \mathbb{P}^2_{\mathbb{C}} \mid Z_2\bar{Z}_2 - Z_3\bar{Z}_3 = 0\}$$
$$\mathfrak{E}_2 = \{[Z] \in \mathbb{P}^2_{\mathbb{C}} \mid Z_3\bar{Z}_3 - Z_1\bar{Z}_1 = 0\}$$
$$\mathfrak{E}_3 = \{[Z] \in \mathbb{P}^2_{\mathbb{C}} \mid Z_1\bar{Z}_1 - Z_2\bar{Z}_2 = 0\}$$

such that

$$T = \mathfrak{E}_1 \cap \mathfrak{E}_2 \cap \mathfrak{E}_3 = \mathfrak{E}_1 \cap \mathfrak{E}_2 = \mathfrak{E}_1 \cap \mathfrak{E}_3 = \mathfrak{E}_2 \cap \mathfrak{E}_3.$$

In the chart $\alpha_3 : \mathbb{C}^2 \longrightarrow \mathbf{A}_3$ defined by

$$\alpha_3(z_1, z_2) = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}$$

these extors are

$$\begin{aligned}
\alpha_3^{-1}(\mathfrak{E}_1) &= \{\zeta \in \mathbb{C}^2 \mid |\zeta_2| = 1\} \\
\alpha_3^{-1}(\mathfrak{E}_2) &= \{\zeta \in \mathbb{C}^2 \mid |\zeta_1| = 1\} \\
\alpha_3^{-1}(\mathfrak{E}_3) &= \{\zeta \in \mathbb{C}^2 \mid |\zeta_1| = |\zeta_2|\}.
\end{aligned}$$

Observe that $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$ have distinct slices, describing three disjoint chains in the dual projective space.

Let $\mathfrak{E}$ be an arbitrary extor which contains T. Giraud's theorem can be deduced by showing that the slices of $\mathfrak{E}$ must be the slices of $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$. Parametrize the lines in $\mathbb{P}^2_{\mathbb{C}}$ by the dual projective space $\mathbb{P}^*$ consisting of projective equivalence classes of $m = [m_1 m_2 m_3]$ where $[m]$ corresponds to the line

$$l_m = \{[Z] \in \mathbb{P}^2_{\mathbb{C}} \mid m_1 Z_1 + m_2 Z_2 + m_3 Z_3 = 0\}.$$

We may classify the possible ways a line l_m intersects T:

1. $l \cap T = \emptyset$;
2. l intersects T transversely in two points;
3. $l \cap T$ is a single point;
4. $l \cap T$ is diffeomorphic to a circle and l is a slice of $\mathfrak{E}_1$, $\mathfrak{E}_2$ or $\mathfrak{E}_3$.

The last two cases occur when l_m is not transverse to T. The collection of such $[m] \in \mathbb{P}^*$ equals

$$T^* = \{[m] \in \mathbb{P}^* \mid \pm|m_1| \pm |m_2| \pm |m_3| = 0\}.$$

Let $\mu_i = |m_i|$. Case 4 corresponds to when one $\mu_i = 0$. The collection Σ_i of all $[m]$ with $\mu_i = 0$ forms a circle, which for $i = 3$ consists of

$$m = \begin{bmatrix} \zeta & 1 & 0 \end{bmatrix}$$

for $|\zeta| = 1$. In this case l_m is a slice of $\mathfrak{E}_i$.

Case 3 corresponds to when all $\mu_i > 0$; for fixed μ, the collection of all $[m]$ forms a 2-torus parametrized by

$$m = \begin{bmatrix} \zeta_1\mu_1 & \zeta_2\mu_2 & \mu_3 \end{bmatrix}.$$

Case 1 (l_m disjoint from T) occurs when μ_1, μ_2, μ_3 satisfy the strict triangle inequalities. Case 2 occurs when one strict triangle inequality and two strict "reverse-triangle inequalities" are satisfied, for example

$$\begin{aligned} \mu_1 &> \mu_2 + \mu_3 \\ \mu_2 &< \mu_3 + \mu_1 \\ \mu_3 &< \mu_1 + \mu_2. \end{aligned}$$

We parametrize the slices $S(t)$ of $\mathfrak{E}$ by $S(t) = l_{[m(t)]}$ for $t \in \mathbb{R}$, where $m(t)$ is a smooth path in $(\mathbb{C}^3)^*$ and $[m(t)]$ is the corresponding smooth path in $\mathbb{P}^*$. Since $\mathfrak{E} \supset T$, the intersections of the slices $l_m \subset \mathfrak{E}$ with T must fill out all of T. We show that each slice corresponds to one of Σ_1, Σ_2 or Σ_3. Suppose that l_m is a slice of $\mathfrak{E}$ which is transverse to T; then a whole neighborhood U of l_m in the slices of $\mathfrak{E}$ exists such that if $S \in U$, then S is also transverse to T. The union

$$\bigcup_{S \in U} S \cap T$$

would then be a 1-dimensional open subset of T, a contradiction. Hence no slice of $\mathfrak{E}$ which meets T is transverse to T.

Thus the slices of $\mathfrak{E}$ meeting T all lie in T^*, and determine a smooth path lying in T^* for $t \in \mathbb{R}$. In particular there is a nonempty open interval $I \subset \mathbb{R}$ such that

$$[m(t)] \in T^* - (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$$

for $t \in I$. This latter set has three components corresponding to the three conditions

$$\begin{aligned} \mu_1 &= \mu_2 + \mu_3 \\ \mu_2 &= \mu_3 + \mu_1 \\ \mu_3 &= \mu_1 + \mu_2 \end{aligned}$$

and we may assume $m(t)$ satisfies one of them, for example

$$|m_3| = |m_1| + |m_2|$$

(and the other two are handled similarly). The intersection of T with such an l_m is then

$$T \cap l_m = \left\{ \begin{bmatrix} \bar{m}_1/|m_1| \\ \bar{m}_2/|m_2| \\ -1 \end{bmatrix} \right\}$$

which depends smoothly on $m = m(t)$. As before, since

$$T \subset \mathfrak{E} = \bigcup_{t \in \mathbb{R}} S(t)$$

the union

$$\mathfrak{E}_I = \bigcup_{t \in I} S(t)$$

is an open subset of $\mathfrak{E}$ which intersects T in the nonempty open subset

$$\bigcup_{t \in I} (S(t) \cap T)$$

which is the image of a smooth path $I \longrightarrow T$, contradicting $\dim(T) = 2$. Therefore the only paths $m(t)$ in $\mathbb{P}^*$ which meet T lie in Σ_1, Σ_2 or Σ_3 as claimed.

Exercise 8.3.8 *Suppose that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are a semi-balanced pair of extors. Show that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are the only extors containing $\mathfrak{E}_1 \cap \mathfrak{E}_2$.*

8.3.7 *Transverse foci*

Suppose that $n \geq 3$. Then the foci correspond to codimension 2 subspaces $\tilde{F}_1, \tilde{F}_2 \subset \mathbb{C}^{n+1}$. We distinguish three cases, depending on the codimension of $\tilde{F}_1 + \tilde{F}_2$:

1. $\operatorname{codim}(\tilde{F}_1 + \tilde{F}_2) = 2$. In this case $\tilde{F}_1 = \tilde{F}_2$ and the two extors are *confocal.*
2. $\operatorname{codim}(\tilde{F}_1 + \tilde{F}_2) = 1$.
3. $\tilde{F}_1 + \tilde{F}_2 = \mathbb{C}^{n+1}$. In this case $\tilde{F} = \tilde{F}_1 \cap \tilde{F}_2$ has codimension 4.

In the second case $\tilde{F} = \tilde{F}_1 \cap \tilde{F}_2$ has dimension $n-2$. Let P be the projective plane associated to the quotient $\mathbb{C}^{n+1}/\tilde{F}$ and let $\Pi : \mathbb{P}^n_{\mathbb{C}} - F \longrightarrow P$ denote projection. Then there exist extors $\mathfrak{E}'_1, \mathfrak{E}'_2 \subset P$ such that

$$\begin{aligned}
\mathfrak{E}_1 &= \Pi^{-1}(\mathfrak{E}'_1) \cup F, \\
\mathfrak{E}_2 &= \Pi^{-1}(\mathfrak{E}'_2) \cup F, \\
\mathfrak{E}_1 \cap \mathfrak{E}_2 &= \Pi^{-1}(\mathfrak{E}'_1 \cap \mathfrak{E}'_2) \cup F,
\end{aligned}$$

and we can reduce to the case $n = 2$.

In the last case, we may assume that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ correspond to the Hermitian matrices

$$\Phi_1 = \begin{bmatrix} 0_{n-3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0_{n-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

whence $\mathfrak{C}_1 \cap \mathfrak{C}_2$ consists of points $[Z] \in \mathbb{P}^n_{\mathbb{C}}$ whose homogeneous coordinates satisfy

$$Z_{n-2}\bar{Z}_{n-2} - Z_{n-1}\bar{Z}_{n-1} = Z_n\bar{Z}_n - Z_{n+1}\bar{Z}_{n+1} = 0.$$

The mapping

$$\begin{aligned} p : \mathfrak{C}_1 \cap \mathfrak{C}_2 &\longrightarrow \mathbb{P}^1_{\mathbb{C}} \\ [Z] &\longmapsto \frac{Z_{n-1}}{Z_{n+1}} \end{aligned}$$

is surjective and its restriction to $p^{-1}(\mathbb{C}^*)$ is a trivial T^2-bundle over $\mathbb{C}^*$. The singular fibers $p^{-1}(0)$ and $p^{-1}(\infty)$ are each circles S^1 which are attached to $p^{-1}(\mathbb{C}^*)$ along $(1,0)$-curves and $(0,1)$-curves on the fibers, respectively. In particular, $\mathfrak{C}_1 \cap \mathfrak{C}_2$ may be identified with an identification space of $T^2 \times \mathbb{P}^1_{\mathbb{C}}$ where $T^2 \times \{0, \infty\}$ is collapsed to $S^1 \times \{0, \infty\}$ under the map

$$\begin{aligned} (e^{i\theta}, e^{i\psi}, \infty) &\mapsto (e^{i\theta}, \infty) \\ (e^{i\theta}, e^{i\psi}, 0) &\mapsto (e^{i\psi}, 0). \end{aligned}$$

Note that the invariant of the Hermitian forms takes the form (when $n = 3$):

$$\det(t_1\Phi_1 + t_2\Phi_2) = t_1^2 t_2^2.$$

For $n > 3$, the intersection of two extors whose foci F_1, F_2 are transverse can be reduced to that of extors $\mathfrak{C}_1', \mathfrak{C}_2' \subset \nu_F$ where $\nu_F = \mathbb{P}(V/(\tilde{F}_1 \cap \tilde{F}_2))$ is the conormal space to $F = F_1 \cap F_2$. Let

$$\Pi_F : \mathbb{P}^n_{\mathbb{C}} - F \longrightarrow \nu_F$$

denote projection. Then

$$\mathfrak{C}_1 \cap \mathfrak{C}_2 = F \cup \Pi_F^{-1}(\mathfrak{C}_1' \cap \mathfrak{C}_2').$$

9

INTERSECTIONS OF BISECTORS

Using the classification of intersections of extors, we specialize to bisectors $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathbf{H}^n_{\mathbb{C}}$, and determine how they can intersect. In real hyperbolic space, bisectors are totally geodesic and so are their intersections. In particular bisector intersections are necessarily connected. However, in complex hyperbolic geometry, this is no longer true. We have already discussed two cases of bisector intersections: cospinal pairs (where the spines lie in a common complex geodesic, see §5.3.1) and comeridianal pairs (where the spines lie in a common meridian, see §5.3.5). Cospinal families of bisectors arose from the orthogonal projections onto complex geodesics. Comeridianal pairs arose in the discussion of Cartan's configuration of seven $\mathbb{R}$-circles characterizing triples of points lying on a chain.

We begin with the general properties of intersections of a pair $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathbf{H}^n_{\mathbb{C}}$ of bisectors in $\mathbf{H}^n_{\mathbb{C}}$. Unless $\mathfrak{E}_1$ and $\mathfrak{E}_2$ possess a common slice (the "cotranchal" case), they intersect transversely. In particular the intersection is transverse if $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are *coequidistant* (the complex spines intersect) or *covertical* (the complex spines are asymptotic). Lemma 9.1.5 describes the structure of the components. Using Mostow's observation (Theorem 7.3.9) that the slices of $\mathfrak{E}_1$ intersect $\mathfrak{E}_2$ in hypercycles which foliate $\mathfrak{E}_1 \cap \mathfrak{E}_2$, we see that components of $\mathfrak{E}_1 \cap \mathfrak{E}_2$ bijectively correspond to components of $\partial\mathfrak{E}_1 \cap \partial\mathfrak{E}_2$. Thus connectedness questions about intersections of bisectors are equivalent to the corresponding questions about the intersections of the spinal spheres bounding them.

§9.1–§9.1.4, develop a general theory of bisector intersections using the invariant (see §7.3.1)

$$\eta = \eta(q_-, q_+; c) \in \mathbb{C}$$

associated to two points $q_-, q_+ \in \partial\mathbf{H}^n_{\mathbb{C}}$ (the vertices of $\mathfrak{E}_1$) and a complex geodesic (a slice of $\mathfrak{E}_2$). Applying η to Mostow's slice decomposition, a pair of bisectors determines a component of a hyperbola in $\mathbb{C}$. The intersection of this hyperbola with the parabolic region $\mathfrak{P}$ defined by (7.18) corresponds to the intersection $\mathfrak{E}_1 \cap \mathfrak{E}_2$. The main result of this section is that the intersection of two bisectors (respectively spinal spheres) has at most two connected components. (Indeed this intersection may be disconnected, as discussed in §9.1.2.)

§9.2.1 deals with tangencies of spinal spheres. The main result is that if two spinal spheres are tangent at a point for which the respective slices are distinct, then the given point of tangency is isolated in the intersection. The remaining component of the intersection is empty, or another isolated point or a circle. A general result (Theorem 9.1.6) is that the intersection of two bisectors has at most two components. (In general such intersections can be disconnected.) The main result (Theorem 9.2.6) is that two bisectors which are respectively *coequidistant* or *covertical* (their complex spines respectively intersect or are asymptotic) have connected intersection. (This is related to Lemma 3.3.1(2) of Mostow [128] in the coequidistant case.) Finally various exceptional cases of bisector intersections are discussed. A particularly interesting case arising from Dirichlet regions of groups in $\mathbf{PO}(2,1)$ is discussed at the end. For simplicity we only consider the case $n = 2$.

9.1 Pairs of spinal spheres

We adopt the following notation. For $i = 1, 2$, denote by $\mathfrak{E}_i$ the bisector with spine σ_i and complex spine Σ_i. Denote the orthogonal projection onto Σ_i by $\Pi_i : \overline{\mathbf{H}_{\mathbb{C}}^n} \longrightarrow \bar{\Sigma}_i$. Denote the corresponding spinal sphere by $\mathfrak{S}_i = \partial\mathfrak{E}_i$. Denote the vertices of $\mathfrak{E}_i$ by $q_i^{\pm} = \partial\sigma_i$. Recall that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ (respectively $\mathfrak{S}_1$ and $\mathfrak{S}_2$) are:

1. *coequidistant* if and only if $\Sigma_1 \cap \Sigma_2 \neq \emptyset$,
2. *covertical* if and only if $\Sigma_1 \parallel \Sigma_2$,
3. *comeridianal* if and only if they possess a common meridian,
4. *cospinal* if and only if $\Sigma_1 = \Sigma_2$.

9.1.1 *Coequidistant, covertical and cotranchal spinal spheres*

$\mathfrak{E}_1$ and $\mathfrak{E}_2$ (respectively $\mathfrak{S}_1$ and $\mathfrak{S}_2$) are *cotranchal* if and only if they possess a common slice. (A family of cotranchal spinal spheres is drawn in Fig. 9.1.)

Lemma 9.1.1 *A coequidistant (respectively covertical) pair of spinal spheres is never cotranchal.*

Proof The complex spines Σ_1, Σ_2 are orthogonal to a common slice Σ. The three points of intersection lie in a totally real geodesic subspace (compare Lemma 3.3.2), forming a triangle with two right angles, a contradiction. □

Two non-cotranchal spinal hypersurfaces always intersect transversely:

Theorem 9.1.2 *Let $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathbf{H}_{\mathbb{C}}^n$ be two bisectors containing x. Then either $\mathfrak{E}_1$ meets $\mathfrak{E}_2$ transversely at x (in which case the slice of $\mathfrak{E}_1$ containing x is transverse to the slice of $\mathfrak{E}_2$ containing x) or a common slice $S \subset \mathfrak{E}_1 \cap \mathfrak{E}_2$ contains x.*

Proof Since $T_x\mathfrak{E}_1$ and $T_x\mathfrak{E}_2$ are real hyperplanes in $T_x\mathbf{H}_{\mathbb{C}}^n$, either $\mathfrak{E}_1$ and $\mathfrak{E}_2$ intersect transversely at x or $T_x\mathfrak{E}_1 = T_x\mathfrak{E}_2$. Suppose that $\mathfrak{E}_1$ is not transverse to $\mathfrak{E}_2$ at x. Let S_i be the slice of $\mathfrak{E}_i$ containing x. Since T_xS_i is the maximal complex subspace of $T_x\mathfrak{E}_i$, S_1 and S_2 have the same tangent space at x and since they are totally geodesic, $S_1 = S_2$ as claimed. □

By Theorem 5.1.7, $\mathfrak{E}_1, \mathfrak{E}_2$ are coequidistant if and only if they are defined as equidistant hypersurfaces from a single point (from now on we allow the degenerate case where this point can lie on one of the spines). This case is

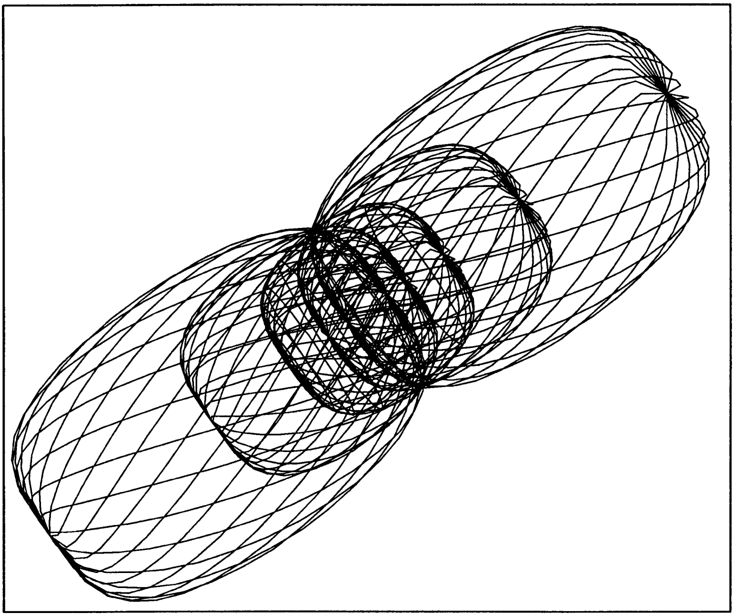

FIG. 9.1. Spinal spheres with a common slice

particularly important since the sides of Dirichlet polyhedra for discrete groups are all mutually coequidistant. Similarly the sides of Ford polyhedra for discrete groups are mutually covertical.

The following result (in the coequidistant case) is proved in Ehrlich–ImHof [45] in the more general context of a Hadamard manifold:

Corollary 9.1.3 *Two coequidistant or covertical bisectors intersect transversely.*

Proof By Theorem 9.1.2, two bisectors $\mathfrak{E}_1, \mathfrak{E}_2$ which do not intersect transversely have a common slice S. Since $S \perp \Sigma_i$ for $i = 1, 2$, it follows $\Sigma_1 \asymp \Sigma_2$, contradicting $\mathfrak{E}_1$ and $\mathfrak{E}_2$ being coequidistant or covertical. □

The following fact (due to Mostow [128]) will be useful in understanding the cotranchal case. It is "dual" to the fact that two distinct points determine a unique geodesic.

Theorem 9.1.4 *(Mostow) Two bisectors possessing two common slices are identical.*

Proof Suppose S_1, S_2 are distinct complex geodesics, both of which are slices of bisectors $\mathfrak{E}_1, \mathfrak{E}_2$. Since $S_1 \asymp S_2$, there exists a unique common orthogonal complex geodesic Σ, which must be the complex spine of each $\mathfrak{E}_i$. Furthermore the spine of $\mathfrak{E}_i$ is the unique geodesic spanned by the points $\Sigma \cap S_1$ and $\Sigma \cap S_2$. Thus $\mathfrak{E}_1$ and $\mathfrak{E}_2$ have the same spine, and are hence equal. □

9.1.2 *Examples of disconnected bisector intersections*

Here is one way to generate examples of pairs of bisectors with disconnected intersection. Begin with a pair $\mathfrak{E}_1, \mathfrak{E}_2$ of bisectors whose intersection is the union of a complex geodesic and an $\mathbb{R}^2$-plane along a geodesic, as in §5.3.5. The corresponding pair of spinal spheres $\mathfrak{S}_j = \partial\mathfrak{E}_j$ intersect in a chain C and an $\mathbb{R}$-circle R such that $C \cap R$ is a pair of distinct points. (Recall that such pairs arose in the discussion of Cartan's configuration of seven $\mathbb{R}$-circles and a chain.) A generic perturbation of such a pair of bisectors (respectively spinal spheres) has disconnected intersection.

Choose coordinates so that:

1. $\mathfrak{S}_1$ is the horizontal plane $v = 0$ (vertices at $(0,0)$ and ∞);
2. the common slice C is the unit circle $|\zeta| = 1$, $v = 0$;
3. the common meridian R is the real axis $\mathrm{Im}(\zeta) = v = 0$.

Since R is a meridian of $\mathfrak{S}_2$, the vertices of $\mathfrak{S}_2$ lie on R and must be of the form $(u_1, 0)$ and $(u_2, 0)$ where $u_1, u_2 \in \mathbb{R}$. Since C is a slice of $\mathfrak{S}_2$, inversion in C interchanges the vertices of $\mathfrak{S}_2$ (Theorem 5.2.1), so $u_1u_2 = 1$. For each $u_1 \neq 0, \pm 1, \infty$, take $u_2 = 1/u_1$ to obtain such a pair $(\mathfrak{S}_1, \mathfrak{S}_2)$. Let $a = (u_1+u_2)/2$. Then $\mathfrak{S}_2$ is given in Heisenberg coordinates $(x+iy, v)$ by

$$v = f(x,y)$$

where

$$f(x,y) = -\frac{y}{x-a}(x^2+y^2-1).$$

Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2$ equals

$$f^{-1}(0) \times \{0\} = C \cup R$$

as claimed. The level sets of the function f (with $a = 0$) are drawn in Fig. 9.2.

Let $v_0 \in \mathbb{R}$. Define $\mathfrak{S}_1(v_0)$ to be the image of $\mathfrak{S}_1$ under vertical translation by v_0 (so $\mathfrak{S}_1(v_0)$ is defined by the equation $v = v_0$). Then $\mathfrak{S}_1(v_0) \cap \mathfrak{S}_2$ corresponds to the level set $f^{-1}(v_0)$. For v_0 sufficiently small, this level set is disconnected. Fig. 9.3 and 9.4 depict these spinal spheres in Heisenberg space.

9.1.3 *Interpretation in terms of orthogonal projection*

Another way to understand disconnected bisector intersections involves the slice decomposition (§5.1.2). Recall that $\mathfrak{E}_2$ is the inverse image of the spine $\sigma_2 \subset \Sigma_2$ under the orthogonal projection $\Pi_{\Sigma_2} : \mathbf{H}^n_{\mathbb{C}} \longrightarrow \Sigma_2$ onto the complex spine Σ_2 of $\mathfrak{E}_2$. We interpret the intersection $\mathfrak{E}_1 \cap \mathfrak{E}_2$ in terms of orthogonal projection $\Pi_{\Sigma_2}(\mathfrak{E}_1)$ of $\mathfrak{E}_1$ on $\mathfrak{E}_2$. Components of $\mathfrak{E}_1 \cap \mathfrak{E}_2$ correspond to the intersections of the spine σ_2 with the projected image $\Pi_{\Sigma_2}(\mathfrak{E}_1) \subset \Sigma_2$. In particular, $\Pi_{\Sigma_2}(\mathfrak{E}_1)$ is geodesically convex if and only if $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is connected for every bisector $\mathfrak{E}_2$ with complex spine Σ_2. Compare the discussion in §7.3.10.

Figure 7.4 depicts a projection of a bisector onto a complex geodesic which is convex. Figures 7.5, 7.6, 7.7, 7.8, 7.9 depict projections which are not convex.

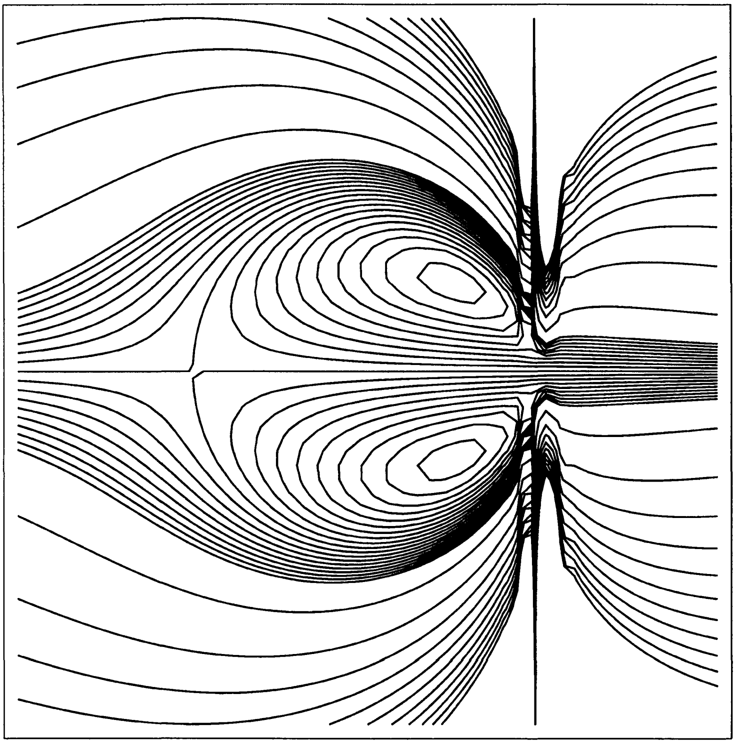

FIG. 9.2. Level sets of function defining an infinite spinal

9.1.4 *The hyperbola associated to a pair of spinal spheres*

Suppose that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are not cotranchal. Then Corollary 9.1.3 implies that each slice $q_2(t)^\perp$ of $\mathfrak{E}_2$ intersects $\mathfrak{E}_1$ transversely. In that case Theorem 7.3.9 (or Lemma 3.2.7(1) of Mostow [128]) implies that the intersection $q_2(t)^\perp \cap \bar{\mathfrak{E}}_1$ must be one of the following:

1. empty,
2. a point of $\partial\mathbf{H}^n_{\mathbb{C}}$,
3. a hypercycle (with two endpoints on $\partial\mathbf{H}^n_{\mathbb{C}}$).

The restriction of orthogonal projection

$$\Pi_2|_{\mathfrak{E}_1\cap\mathfrak{E}_2} : \mathfrak{E}_1 \cap \mathfrak{E}_2 \longrightarrow \Sigma_2$$

is a fibration onto an open submanifold of the line $\sigma_2 \subset \Sigma_2$ with these hypercycles as fibers. Therefore:

Lemma 9.1.5 *Suppose $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are not cotranchal. Then each component of $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is an open 2-disc, each component of $\overline{\mathfrak{E}_1} \cap \overline{\mathfrak{E}_2}$ is a point or a closed 2-disc, and each component of $\mathfrak{S}_1 \cap \mathfrak{S}_2$ is either a point or a circle.*

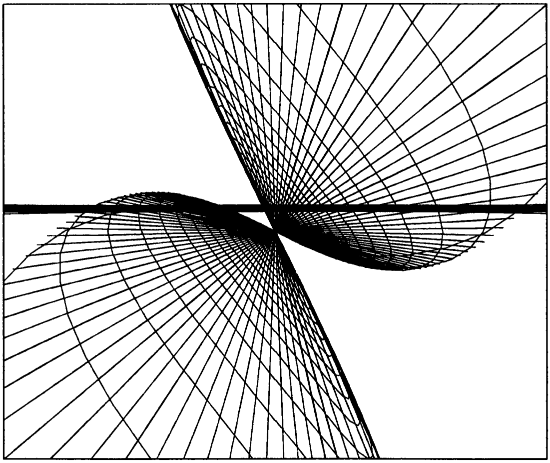

FIG. 9.3. A disconnected intersection of two spinal spheres

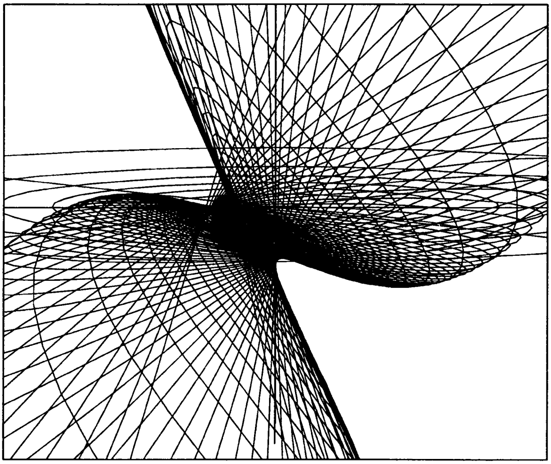

FIG. 9.4. Disconnected intersection of spinal spheres: another view

To analyze the relationship between the hypersurfaces $\mathfrak{C}_1$ and $\mathfrak{C}_2$, find a pair of null vectors $Q_2^{\pm}$ representing the vertices $q_2^{\pm}$ normalized as follows:

$$\langle Q_2^-, Q_2^+ \rangle = 2. \tag{9.1}$$

For $0 < t < \infty$, the vectors

$$Q_2(t) = \frac{1}{2}(tQ_2^+ + t^{-1}Q_2^-)$$

satisfy $\langle Q_2(t), Q_2(t)\rangle = 1$ and are polar to the slices of $\mathfrak{E}_2$. The interaction between $\mathfrak{E}_1$ and the slice $q_2(t)^\perp$ of $\mathfrak{E}_2$ polar to $Q_2(t)$ is described by the complex number

$$\eta(t) = \eta(q_1^-, q_1^+; q_2(t))$$

discussed in §7.3.1. We claim that $\eta(t)$ describes a component of a hyperbola as t ranges over $\mathbb{R}_+$. Specifically,

$$\eta(t) = At^{-2} + B + Ct^2$$

where

$$\begin{aligned} A = A(q_1^-, q_1^+; Q_2^-, Q_2^+) &= \frac{\langle Q_1^-, Q_2^-\rangle\langle Q_2^-, Q_1^+\rangle}{4\langle Q_1^-, Q_1^+\rangle} \\ B = B(q_1^-, q_1^+; Q_2^-, Q_2^+) &= \frac{\langle Q_1^-, Q_2^-\rangle\langle Q_2^+, Q_1^+\rangle + \langle Q_1^-, Q_2^+\rangle\langle Q_2^-, Q_1^+\rangle}{4\langle Q_1^-, Q_1^+\rangle} \\ C = C(q_1^-, q_1^+; Q_2^-, Q_2^+) &= \frac{\langle Q_1^-, Q_2^+\rangle\langle Q_2^+, Q_1^+\rangle}{4\langle Q_1^-, Q_1^+\rangle}. \end{aligned} \tag{9.2}$$

(We shall henceforth abuse language and simply refer to this curve as a "hyperbola.") By Theorem 7.3.1(5), this hyperbola lies completely within the half-plane $\mathrm{Re}(z) \leq \frac{1}{2}$. By Theorem 7.3.9, the slice $q_2(t)^\perp$ of $\mathfrak{E}_2$ meets $\mathfrak{E}_1$ if and only if $\eta(t) \in \mathfrak{P}$.

Thus the *connected components of* $\mathfrak{E}_1 \cap \mathfrak{E}_2$ (respectively the *connected components of* $\partial\mathfrak{E}_1 \cap \partial\mathfrak{E}_2$) *bijectively correspond* to the *connected components of* $\eta^{-1}(\mathfrak{P}) \subset \mathbb{R}_+$.

In the cotranchal case the situation is quite similar: by Theorem 7.3.1(7), $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are cotranchal if and only if the hyperbola $\eta(t)$ passes through the complex number $\frac{1}{2}$ (the corresponding slice of $\mathfrak{E}_2$ is the common slice). In any case, we have the following general result:

Theorem 9.1.6 *The intersection $\mathfrak{E}_1 \cap \mathfrak{E}_2$ of two bisectors has at most two connected components. Similarly the intersections $\bar{\mathfrak{E}}_1 \cap \bar{\mathfrak{E}}_2$ and $\mathfrak{S}_1 \cap \mathfrak{S}_2$ have at most two connected components.*

Proof The connected components of $\mathfrak{E}_1 \cap \mathfrak{E}_2$ correspond to the connected components of the set of all $t > 0$ such that $\eta(t) \in \mathfrak{P}$. Now $\eta(t) \in \mathfrak{P}$ if and only if $t^{-4}f(t^2) < 0$ where

$$t^{-4}f(t^2) = 2\mathrm{Re}(\eta(t)) + \mathrm{Im}(\eta(t))^2 - 1$$

and $f(s)$ is the polynomial

$$\begin{aligned} f(s) = {} & \mathrm{Im}(C)^2 s^4 + \big(2\mathrm{Im}(B)\mathrm{Im}(C) + 2\mathrm{Re}(C)\big)s^3 \\ & + (\mathrm{Im}(B)^2 + 2\mathrm{Re}(B) + 2\mathrm{Im}(A)\mathrm{Im}(C) - 1)s^2 \\ & + (2\mathrm{Im}(A)\mathrm{Im}(B) + 2\mathrm{Re}(A))s + \mathrm{Im}(A)^2. \end{aligned}$$

Thus

$$\eta^{-1}(\mathfrak{P}) = \{t > 0 \mid f(t^2) < 0\}$$

which has at most two connected components since $f(s)$ is a quartic polynomial with positive leading term. $\square$

9.1.5 *Example: cospinal pairs*

As an example, consider the case that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are cospinal (as discussed in §5.3.1); that is, all four vertices lie on a single chain. Choose coordinates so that their cross-ratio equals the real number r; in Heisenberg coordinates, they are represented by the points

$$\{(0,0),(0,1)\}, \qquad \{(0,r),\infty)\}$$

respectively. We then have

$$\eta(t) = i(r-1)t^{-2} + \frac{1}{2} + i\frac{r}{4}t^2$$

which lies on the vertical line $\mathrm{Re}(z) = \frac{1}{2}$. There are several cases. If $r > 1$, the spines are ultraparallel and the hyperbola degenerates into the vertical ray

$$\frac{1}{2} + i\left[\sqrt{r(r-1)}, \infty\right)$$

which completely misses $\mathfrak{P}$. If $r = 1$, the spines are asymptotic and the hyperbola degenerates into the open ray

$$\frac{1}{2} + i\mathbb{R}_+$$

which approaches but misses, $\mathfrak{P}$. If $0 < r < 1$ the spines intersect and the hyperbola describes all of the line $\frac{1}{2} + i\mathbb{R}$, which meets $\mathfrak{P}$ exactly at the complex number $\frac{1}{2}$. Similarly if $r = 0$, the spines are asymptotic, the hyperbola is now the open ray

$$\frac{1}{2} - i\mathbb{R}_+$$

missing $\mathfrak{P}$. If $r < 0$ the spines are ultraparallel and the degenerate hyperbola

$$\eta(\mathbb{R}_+) = \frac{1}{2} - i\left[\sqrt{r(r-1)}, \infty\right)$$

misses $\mathfrak{P}$.

9.1.6 *Example: comeridianal pairs*

As another example, suppose that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are comeridianal; that is, $q_1^{\pm}, q_2^{\pm}$ lie on an $\mathbb{R}$-plane. (Comeridianal pairs of bisectors are discussed in §5.3.5.) Again let their cross-ratio be r. In Heisenberg coordinates they are represented by the points

$$\{(0,0),(1,0)\}, \qquad \{\infty,(r,0)\}$$

respectively. Then

$$\eta(t) = -\frac{(r-1)^2}{2}\, t^{-2} + \left(\frac{1}{2} + r(r-1)\right) - \frac{r^2}{2}\, t^2$$

describes the degenerate hyperbola

$$\left(-\infty, \frac{1}{2} + r(r-1) - |r(r-1)|\right]$$

which completely lies within $\mathfrak{P}$. Note that when $r(r-1) > 0$, then the hyperbola equals

$$\mathbb{R}_{\leq \frac{1}{2}} = \mathfrak{P} \cap \mathbb{R}$$

in which case $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are cotranchal (the common slice corresponds to $\eta = \frac{1}{2}$). This is the case related to Cartan's configuration (see Corollary 5.3.8).

We can deform this last example, as in §9.1.2, to produce examples of disconnected bisector intersections. Let $\mathfrak{C}_1(v_0)$ denote the bisector corresponding to $v = v_0$ and $\mathfrak{C}_2$ the bisector with vertices $(2.0, 0.0)$ and $(0.5, 0.0)$. For $v_0 = 0$, the pair $(\mathfrak{C}_1(v_0), \mathfrak{C}_2)$ is comeridianal and the hyperbola degenerates into the ray $\mathbb{R}_{\leq \frac{1}{2}}$, but for $v_0 \neq 0$ small, the corresponding hyperbola is nondegenerate and intersects $\mathfrak{P}$ in a pair of disjoint arcs. Figure 9.5 depicts the hyperbola associated to the pair with $v_0 = -0.3$.

9.1.7 *Angular invariant and complex cross-ratio*

The coefficients A, B, C of the hyperbola

$$\eta(t) = At^{-2} + B + Ct^2 \in \mathbb{C}$$

relate to the angular invariant and the complex cross-ratio of the four vertices $q_1^{\pm}, q_2^{\pm} \in \partial \mathbf{H}_{\mathbb{C}}^n$ of the spinal spheres $\mathfrak{S}_1, \mathfrak{S}_2$ as follows. Normalize the coefficients to obtain more intrinsic expressions. The coefficients A, B, C were defined in (9.2) and depend on the choice of null vectors $Q_2^{\pm}$ satisfying (9.1). Replacing

$$\begin{aligned} Q_2^+ &\mapsto \lambda Q_2^+ \\ Q_2^- &\mapsto \bar{\lambda}^{-1} Q_2^- \end{aligned}$$

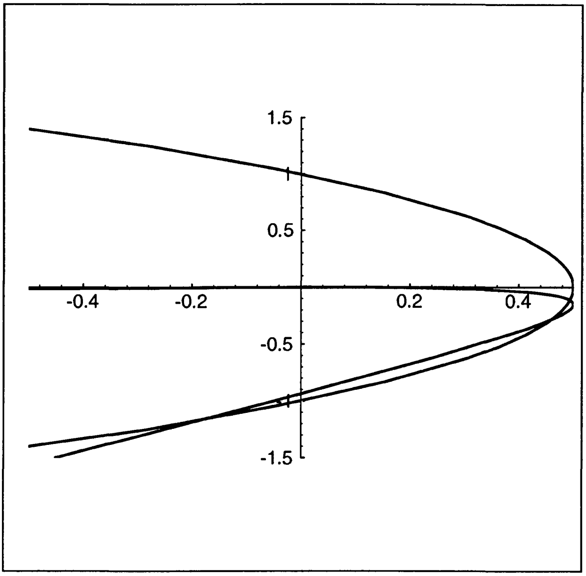

FIG. 9.5. η-hyperbola corresponding to disconnected bisector intersection

replaces

$$A \mapsto |\lambda|^{-2} A$$
$$B \mapsto B$$
$$C \mapsto |\lambda|^{2} C$$

and only affects the speed of the hyperbola. Thus normalize the coefficients (replacing t by a positive multiple) to assume that

$$|A| = |C|.$$

The common absolute value is then $\sqrt{|AC|}$.

The resulting normalization replaces A and C by

$$\frac{A}{|A|}\sqrt{|AC|} = A\sqrt{\left|\frac{C}{A}\right|}, \qquad \frac{C}{|A|}\sqrt{|AC|} = C\sqrt{\left|\frac{A}{C}\right|}$$

respectively. Then the directions of the complex numbers A and C are given in terms of Cartan invariants by

$$\frac{A}{|A|} = -\exp(i\mathbb{A}(q_1^-, q_2^-, q_1^+))$$
$$\frac{C}{|C|} = -\exp(i\mathbb{A}(q_1^-, q_2^+, q_1^+)).$$

The coefficients B and AC relate to Korányi–Reimann complex cross-ratios by

$$4AC = \mathbf{X}\{q_1^+, q_2^-, q_2^+, q_1^-\}\ \mathbf{X}\{q_1^+, q_2^+, q_2^-, q_1^-\},$$
$$2B = \mathbf{X}\{q_1^+, q_2^-, q_2^+, q_1^-\} + \mathbf{X}\{q_1^+, q_2^+, q_2^-, q_1^-\}.$$

9.2 Connected bisector intersections

9.2.1 *Tangencies of spinal spheres*

The proof that bisector intersections are connected involves a continuity argument. A key step is the direct analysis of pairs of spinal spheres which intersect tangentially. Figures 9.6 and 9.7 depicts some slices and meridians of a pair of tangentially intersecting spinal spheres. The spinal spheres themselves are depicted in Figs. 9.8 and 9.9.

Theorem 9.2.1 *Let $\mathfrak{S}_1.\mathfrak{S}_2 \subset \partial\mathbf{H}^n_{\mathbb{C}}$ be spinal spheres which intersect tangentially at a given point $p \in \partial\mathbf{H}^n_{\mathbb{C}}$. Suppose that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ do not have a common slice at p. Then exactly one of the following possibilities occurs:*

1. *$\mathfrak{S}_1 \cap \mathfrak{S}_2 = \{p\}$;*
2. *$\mathfrak{S}_1 \cap \mathfrak{S}_2 - \{p\}$ is a point;*
3. *$\mathfrak{S}_1 \cap \mathfrak{S}_2 - \{p\}$ is a simple closed curve;*
4. *$\mathfrak{S}_1 \cap \mathfrak{S}_2$ is a simple closed curve containing p;*
5. *$\mathfrak{S}_1 \cap \mathfrak{S}_2$ is a union of two simple closed curves each containing p.*

The proof of Theorem 9.2.1 involves an explicit normal form for such tangencies by computing the slices and meridians through p in Heisenberg coordinates. Let C_i (respectively R_i) denote the slice (respectively the meridian) of $\mathfrak{S}_i$ containing p. Our immediate goal is to determine the vertices of such a pair and then directly calculate the intersection of such a pair. We first determine the $\mathbb{R}$-circles R_1, R_2. The calculation will be in terms of the antipode q_i of p with respect to the spinal sphere $\mathfrak{S}_i$, that is the unique point q_i so that

$$C_i \cap R_i = \{p, q_i\}.$$

9.2.2 *$\mathbb{R}$-circles adapted to intersecting chains*

Consider two distinct chains C_1 and C_2 intersecting at a point p. We determine the points $q_i \in C_i$ which will be antipodal to p with respect to $\mathfrak{S}_i$. By Lemma 4.4.13, for any points $q_i \in C_i$ not equal to p, there exist $\mathbb{R}$-circles R_i containing p, q_i such that

$$T_pR_i = (T_pC_1 + T_pC_2) \cap E_p \tag{9.3}$$

for $i = 1, 2$.

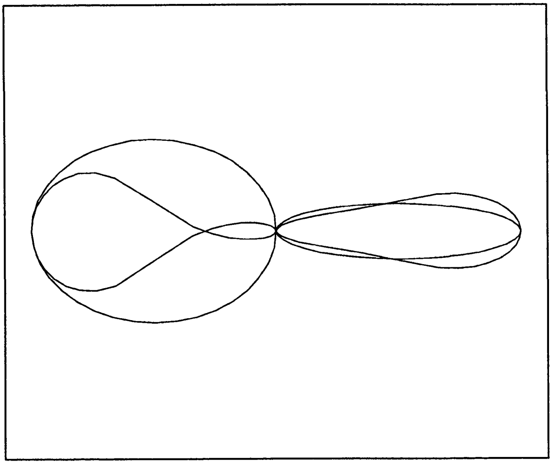

FIG. 9.6. Slices and meridians of tangent spinal spheres

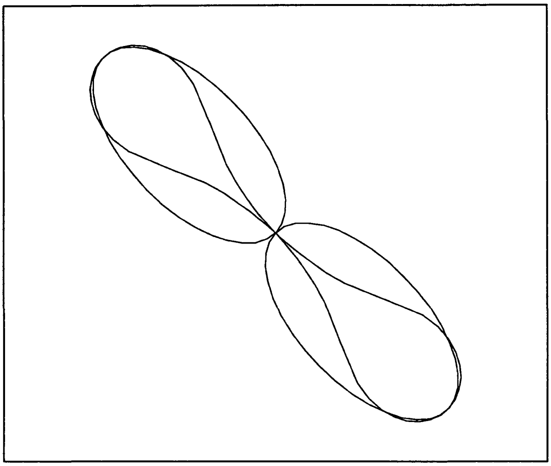

FIG. 9.7. Slices and meridians: another view

Lemma 9.2.2 *Suppose that $C_1, C_2 \subset \partial\mathbf{H}^n_{\mathbb{C}}$ are distinct chains intersecting at p and suppose $p \neq q_1 \in C_1$.*

1. *There exists a unique $\mathbb{R}$-circle R' through p and q_1 which meets C_2.*
2. *There is a unique $\mathbb{R}$-circle R through p, q_1 such that*

$$T_pR = (T_pC_1 + T_pC_2) \cap E_p.$$

3. *R is the image of R' under complex reflection of order 4 in C_1.*

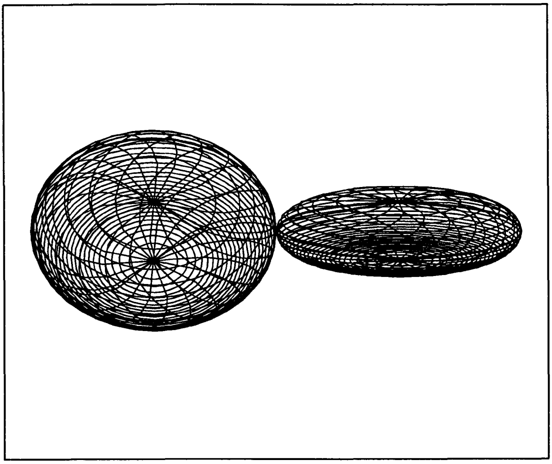

FIG. 9.8. Pair of tangent spinal spheres

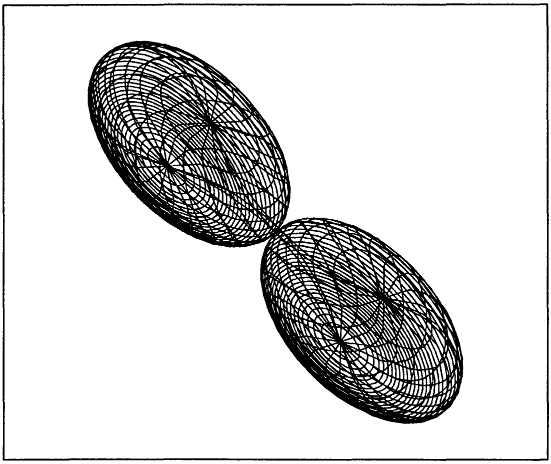

FIG. 9.9. Tangent spinal spheres: another view

Proof Choose Heisenberg coordinates so that p corresponds to ∞ and C_1 and C_2 are the vertical chains $\zeta = -1$ and $\zeta = +1$ respectively. The stabilizer of the pair (C_1, C_2) consists of vertical translations. We may fix coordinates so that

$$q_1 = (-1, -u), \qquad q_2 = (1, u)$$

for a unique $u \in \mathbb{R}$. The $\mathbb{R}$-circles through p, q_1 (respectively p, q_2) are Euclidean straight lines through $(-1, -u)$ (respectively $(1, u)$) which lie in the prolongation of the contact plane at $(-1, -u)$ (respectively $(1, u)$). Such curves are given parametrically in Heisenberg coordinates by

$$R_1 = \{(-1 + s\xi_1, -u + 2s\ \mathrm{Im}(\xi_1)) \mid s \in \mathbb{R}\}$$
$$R_2 = \{(1 + s\xi_2, u - 2s\ \mathrm{Im}(\xi_2)) \mid s \in \mathbb{R}\}$$

where ξ_1, ξ_2 are unit complex numbers. When ξ_1 is real, R_1 is the unique $\mathbb{R}$-circle with

$$R_1 \cap C_1 = \{p, q_1\}$$

and which intersects C_2. (R_1 bounds a totally real geodesic 2-plane orthogonal to the complex geodesics bounded by C_1 and C_2.)

Tangent vectors to C_1, C_2, R_1 and R_2 are computed by the applying inversion ι in the unit circle

$$|\zeta| = 1,\ v = 0$$

to obtain chains $\iota(C_i)$ and $\mathbb{R}$-circles $\iota(R_i)$ passing through the origin $(0,0)$. (The chain $\iota(C_1)$ is centered at $(-\frac{1}{2}, -u)$ with radius $\frac{1}{2}$ and $\iota(C_2)$ is centered at $(\frac{1}{2}, u)$ with radius $\frac{1}{2}$.) Writing $\zeta = x + iy$,

$$T_{(0,0)}\iota(C_1) = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial y}\right)$$
$$T_{(0,0)}\iota(C_2) = \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial y}\right)$$
$$T_{(0,0)}\iota(R_1) = \left(\mathrm{Re}(\xi_1)\frac{\partial}{\partial x} + \mathrm{Im}(\xi_1)\frac{\partial}{\partial y}\right)$$
$$T_{(0,0)}\iota(R_2) = \left(\mathrm{Re}(\xi_2)\frac{\partial}{\partial x} + \mathrm{Im}(\xi_1)\frac{\partial}{\partial y}\right)$$

so that

$$\xi_1 = \xi_2.$$

Therefore $T_{(0,0)}C_1 + T_{(0,0)}C_2K$ is spanned by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial v}$ and

$$(T_{(0,0)}C_1 + T_{(0,0)}C_2) \cap E_0 = \left(\frac{\partial}{\partial y}\right)$$

In particular $T_{(0,0)}R_i \subset T_{(0,0)}C_1 + T_{(0,0)}C_2$ if and only if ξ_i is purely imaginary. The proof of Lemma 9.2.2 is complete. □

9.2.3 *Spinal spheres with given slice and meridian*

Now we determine spinal spheres $\mathfrak{S}_i$ with C_i as a slice and R_i as a meridian. Such spinal spheres are necessarily tangent at p since we have already arranged that

$$T_p\mathfrak{S}_1 = T_pC_1 + T_pR_1 = T_pC_2 + T_pR_2 = T_p\mathfrak{S}_2.$$

The vertices of $\mathfrak{S}_i$ are then any pair of points on $R_i - \{p, q_i\}$ related by inversion in C_i. Thus there exist $s, t > 0$ so that the vertices of $\mathfrak{S}_1$ are

$$(-1 + is, -u + 2s), \qquad (-1 - is, -u - 2s)$$

and the vertices of $\mathfrak{S}_2$ are

$$(1 + it, u - 2t), \qquad (1 - it, u + 2t)$$

in Heisenberg coordinates. Applying the characterization for spinal spheres in terms of the Cartan invariant (Corollary 7.1.5), $\mathfrak{S}_1$ is described by the equation

$$v - 2y + u = \frac{x+1}{y}\left\{(x+1)^2 + y^2 + s^2\right\} \tag{9.4}$$

and $\mathfrak{S}_2$ is described by the equation

$$v + 2y - u = \frac{x-1}{y}\left\{(x-1)^2 + y^2 + t^2\right\}. \tag{9.5}$$

Subtracting (9.5) from (9.4) to eliminate v, we see that points of the intersection $\mathfrak{S}_1 \cap \mathfrak{S}_2$ correspond to solutions $(x, y) \in \mathbb{R}^2$ of the equation

$$6x^2 + 6y^2 + x(s^2 - t^2) - 2uy + (2 + s^2 + t^2) = 0,$$

or equivalently,

$$\left(x + \frac{s^2 - t^2}{12}\right)^2 + \left(y - \frac{u}{6}\right)^2 + \frac{F(s,t,u)}{144} = 0 \tag{9.6}$$

where

$$F(s,t,u) = 48 + 24(s^2 + t^2) - (s^2 - t^2)^2 - 4u^2. \tag{9.7}$$

If $F(s,t,u) > 0$, then there are no solutions to (9.6). Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2$ consists solely of the point p. This corresponds to case 1 of Theorem 9.2.1.

If $F(s,t,u) = 0$, then (9.6) has the unique solution

$$x = \frac{t^2 - s^2}{12}, \qquad y = \frac{u}{6} \tag{9.8}$$

However, this point may be the point at infinity, if the v-coordinate corresponding to (9.8) is infinite. By (9.4) and (9.5), the v-coordinate has a pole for finite values of x and y if and only if $y = 0$, which can only occur if $u = 0$. Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \{p\}$ if $u = F(s,t,u) = 0$. If $u \neq 0 = F(s,t,u)$, then (9.6) has a unique solution corresponding to a finite point and $\mathfrak{S}_1 \cap \mathfrak{S}_2$ consists of two points, which is case 2 of Theorem 9.2.1.

Finally suppose that $F(s,t,u) < 0$. In that case the solutions of (9.6) define a circle. Once again, there may be solutions of (9.6) which correspond to ∞, depending on whether the circle defined by (9.6) intersects the line $y = 0$. There are three cases, depending on whether this intersection has none, one, or two points. The circle and the line are disjoint if and only if

$$\frac{u^2}{36} + \frac{F(s,t,u)}{144} > 0$$

a condition we rewrite as

$$(s^2 - t^2)^2 < 24(2 + s^2 + t^2)$$

in which case $\mathfrak{S}_1 \cap \mathfrak{S}_2$ is the disjoint union of p with a circle (case 3 of Theorem 9.2.1). There is exactly one solution of (9.6) corresponding to ∞ if and only if

$$(s^2 - t^2)^2 = 24(2 + s^2 + t^2)$$

and in this case $\mathfrak{S}_1 \cap \mathfrak{S}_2$ is homeomorphic to a circle (case 4 of Theorem 9.2.1). Finally there are two solutions of (9.6) corresponding to ∞ if and only if

$$(s^2 - t^2)^2 > 24(2 + s^2 + t^2)$$

and in this case $\mathfrak{S}_1 \cap \mathfrak{S}_2$ is connected but p divides $\mathfrak{S}_1 \cap \mathfrak{S}_2$ into two components (case 5 of Theorem 9.2.1). The proof of Theorem 9.2.1 is concluded.

Lemma 9.2.3 *Suppose that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are coequidistant or covertical spinal spheres which are tangent at a point $p \in \partial\mathbf{H}^n_{\mathbb{C}}$. Then $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \{p\}$.*

Proof We continue the computation from the proof of Theorem 9.2.1. Since $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are coequidistant or covertical, they cannot be cotranchal (Lemma 9.1.1). Thus the slices of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ containing p are distinct. Choose Heisenberg coordinates as above; then the complex spines Σ_1, Σ_2 are given by polar vectors

$$\tilde{c}_1 = \begin{bmatrix} -2i \\ u + is^2 \\ -u + i(2 - s^2) \end{bmatrix}, \qquad \tilde{c}_2 = \begin{bmatrix} 2i \\ u - it^2 \\ u + i(2 - t^2) \end{bmatrix}$$

(in Heisenberg coordinates these are chains centered at $(-1, -u)$ and $(1, u)$ respectively with respective radii $\sqrt{3 - 2s^2}$ and $\sqrt{3 - 2t^2}$). The condition that Σ_1 and Σ_2 intersect or are asymptotic is that

$$\langle\tilde{c}_1, \tilde{c}_2\rangle\langle\tilde{c}_2, \tilde{c}_1\rangle - \langle\tilde{c}_1, \tilde{c}_1\rangle\langle\tilde{c}_2, \tilde{c}_2\rangle \le 0;$$

that is,

$$G(s,t,u) = (s^2 - t^2)^2 - 8(s^2 + t^2) + 4(4 + u^2) \le 0. \tag{9.9}$$

By (9.7) we have

$$F(s,t,u) = -G(s,t,u) + 16(s^2 + t^2) + 64 > 0.$$

Thus there are no solutions to (9.6) and $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \{p\}$ as desired. □

At a point of tangency between coequidistant or covertical spinal spheres $\mathfrak{S}_1$ and $\mathfrak{S}_2$, the spheres intersect with second-order contact:

Lemma 9.2.4 *Suppose that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are coequidistant or covertical spinal hypersurfaces which are tangent at a point p. Then in a local coordinate system at p which linearizes $\mathfrak{S}_1$, $\mathfrak{S}_2$ is expressed as the graph of a definite quadratic function locally on $\mathfrak{S}_1$.*

Proof Continuing the computation from the proof of Theorem 9.2.1, we compute the tangent planes of the images of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ under inversion ι in the unit circle. The equation of $\iota(\mathfrak{S}_1)$ is

$$\begin{aligned}(1+s^2)v^2 - (3+s^2)vy + 3y^2 - uy^3 + (1+s^2)y^4 \\ + \left\{1 - uv + (3+s^2)y^2\right\} x \\ + \left\{3 - uy + 2(1+s^2)y^2\right\} x^2 \\ + (3+s^2)x^3 \\ + (1+s^2)x^4 = 0\end{aligned} \tag{9.10}$$

and the equation of $\iota(\mathfrak{S}_2)$ is

$$\begin{aligned}(1+t^2)v^2 + (3+t^2)vy + 3y^2 - uy^3 + (1+t^2)y^4 \\ + \left\{-1 - uv - (3+t^2)y^2\right\} x \\ + \left\{3 - uy + 2(1+t^2)y^2\right\} x^2 \\ - (3+t^2)x^3 \\ + (1+t^2)x^4 = 0\end{aligned} \tag{9.11}$$

both of which have tangent plane $x = 0$ at the origin (as computed in Lemma 5.2.9 above). Expanding (9.10) about the origin,

$$x = -(1+s^2)v^2 + (3+s^2)vy - 3y^2 + \text{ (higher-order terms)}$$

and expanding (9.11) about the origin,

$$x = (1+t^2)v^2 + (3+t^2)vy + 3y^2 + \text{ (higher-order terms)}.$$

In a local coordinate system (ξ, η, ν) which agrees with the coordinate system (x, y, v) to first order and for which $\mathfrak{S}_1$ is locally given as $\xi = 0$, we see that $\mathfrak{S}_2$ is described by

$$\xi = (2 + s^2 + t^2)\nu^2 + (t^2 - s^2)\nu\eta + 6\eta^2 + \text{ (higher-order terms)}.$$

Its Hessian is described by the matrix

$$\begin{bmatrix} 6 & (t^2-s^2)/2 \\ (t^2-s^2)/2 & 2+s^2+t^2 \end{bmatrix}$$

which has determinant

$$12 + 6(s^2 + t^2) - \frac{(s^2 - t^2)^2}{4} = -\frac{G(s,t,u)}{4} + 16 + (u^2 + 4s^2 + 4t^2)$$
$$> -\frac{G(s,t,u)}{4} \geq 0$$

(where $G(s,t,u)$ is defined in (9.9)). By the Morse lemma, in these local coordinates $\iota(\mathfrak{S}_2)$ is the graph of a positive definite quadratic function near the origin. (This also follows since p is isolated in $\mathfrak{S}_1 \cap \mathfrak{S}_2$ and the Hessian is nondegenerate.) The proof of Lemma 9.2.4 is complete. □

9.2.4 *Bisectors with orthogonal complex spines*

The next ingredient in the proof of Theorem 9.2.6 is the special case of pairs of spinal spheres with orthogonal complex spines.

Theorem 9.2.5 *Suppose $\mathfrak{E}_1, \mathfrak{E}_2$ are bisectors with respective spines σ_1, σ_2 such that their respective complex spines Σ_1, Σ_2 are orthogonal complex geodesics. Let $o = \Sigma_1 \cap \Sigma_2$ and let $\rho_j = \rho(o, \Sigma_j)$. Then the intersection $\overline{\mathfrak{E}_1} \cap \overline{\mathfrak{E}_2}$ is empty, or a point, or a closed 2-disc, depending on whether the quantity*

$$\tanh^2\left(\frac{\rho_1}{2}\right) + \tanh^2\left(\frac{\rho_2}{2}\right)$$

is greater than, equal to or less than 1 respectively.

Proof Choose coordinates in the ball model so that o corresponds to the origin and

$$\Sigma_1 = \mathbb{B}^1 \times \{0\}$$
$$\Sigma_2 = \{0\} \times \mathbb{B}^1.$$

Let ι_j denote the (anti-holomorphic) reflection $\Sigma_j \to \Sigma_j$ then fixing σ_j; then $\mathfrak{E}_j$ equals the hypersurface equidistant from o and $\iota_j(o)$. Since the stabilizer of the triple (o, Σ_1, Σ_2) is the subgroup represented by diagonal matrices

$$\begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} \in \mathbf{U}(2)$$

where $|\zeta_1| = |\zeta_2| = 1$, the points $o, \iota_1(o), \iota_2(o)$ are represented by negative vectors

$$O = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad I_1 = \begin{bmatrix} \sinh(\rho_1) \\ 0 \\ \cosh(\rho_1) \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 \\ \sinh(\rho_2) \\ \cosh(\rho_2) \end{bmatrix}.$$

Elements $z \in \overline{\mathfrak{E}_1} \cap \overline{\mathfrak{E}_2}$ are represented by negative vectors

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \in \mathbb{C}^{2,1}$$

satisfying

$$\langle I_1, Z\rangle\langle Z, I_1\rangle = \langle I_2, Z\rangle\langle Z, I_2\rangle = \langle O, Z\rangle\langle Z, O\rangle = 1$$

so that the coordinates of z satisfy

$$|z_1 - \coth(\rho_1)| = \operatorname{csch}(\rho_1), \qquad |z_2 - \coth(\rho_2)| = \operatorname{csch}(\rho_2). \tag{9.12}$$

Now the solution to (9.12) in $\mathbb{C}^2$ closest to the origin is given by

$$z_j = \coth(\rho_j) - \operatorname{csch}(\rho_j) = \tanh\left(\frac{\rho_j}{2}\right)$$

for $j = 1, 2$. In particular this point lies in $\mathbf{H}^2_{\mathbb{C}}$ (respectively $\partial\mathbf{H}^2_{\mathbb{C}}$, outside $\mathbf{H}^2_{\mathbb{C}}$) if and only if

$$\tanh^2\left(\frac{\rho_1}{2}\right) + \tanh^2\left(\frac{\rho_2}{2}\right) < 1 \tag{9.13}$$

(respectively $= 1$, or > 1) as desired. □

If the intersection is nonempty, then it is a connected surface diffeomorphic to a 2-disc.

The η-hyperbola associated to a pair of bisectors with orthogonal $\mathbb{C}$-spines degenerates into a closed ray as follows. Choose vectors representing the two pairs of vertices

$$Q_1^- = \begin{bmatrix} i \\ 0 \\ -\operatorname{sech}(\rho_1) + i\tanh(\rho_1) \end{bmatrix}, \qquad Q_1^+ = \begin{bmatrix} \sinh(\rho_1) + i \\ 0 \\ \cosh(\rho_1) \end{bmatrix}$$

$$Q_2^- = \begin{bmatrix} 0 \\ i \\ -\operatorname{sech}(\rho_2) + i\tanh(\rho_2) \end{bmatrix}, \qquad Q_2^+ = \begin{bmatrix} 0 \\ \sinh(\rho_2) + i \\ \cosh(\rho_2) \end{bmatrix}$$

from which the Korányi–Reimann complex cross-ratios are

$$\mathbf{X}\{q_1^+, q_2^+, q_2^-, q_1^-\} = \frac{(1 - i\sinh(\rho_1))(1 - i\sinh(\rho_2))}{4}$$

$$\mathbf{X}\{q_1^+, q_2^-, q_2^+, q_1^-\} = \frac{(1 - i\sinh(\rho_1))(1 + i\sinh(\rho_2))}{4}$$

and the coefficients of the hyperbola (renormalized so that $|A| = |C|$) are

$$\frac{A}{|A|} = -\operatorname{sech}(\rho_1) + i\tanh(\rho_1) = \frac{C}{|C|}$$

(from which it follows that the hyperbola degenerates into a ray) and

$$\eta(t) = \frac{-1 + i\sinh(\rho_1)}{4}\left((\cosh(\rho_2) - 1) + \frac{1}{2}\cosh(\rho_2)(t - t^{-1})^2\right).$$

As t ranges over $\mathbb{R}^+$, this describes a closed ray in $\mathbb{C}$ which lies on a line through 0. In particular $\eta^{-1}(\mathfrak{P})$ is connected: it is a closed interval, or a point, or empty depending on whether the endpoint of the ray $\eta(1)$ lies in $\mathfrak{P}$, $\partial\mathfrak{P}$, or $\mathbb{C} - \bar{\mathfrak{P}}$ respectively. This condition is determined by the function expressing the geodesic curvature (see Theorem 7.3.7)

$$\frac{\operatorname{Im}(\eta(1))}{\sqrt{1 - 2\operatorname{Re}(\eta(1))}}$$

which compares to 1 by

$$\left(\frac{\operatorname{Im}(\eta(1))}{\sqrt{1 - 2\operatorname{Re}(\eta(1))}}\right)^2 - 1 = \left(\sinh^2\left(\frac{\rho_1}{2}\right)\sinh^2\left(\frac{\rho_2}{2}\right) - 1\right) \times \left(1 + \sinh^2\left(\frac{\rho_1}{2}\right)\tanh^2\left(\frac{\rho_2}{2}\right)\right). \tag{9.14}$$

Since (9.13) is equivalent to

$$\sinh\left(\frac{\rho_1}{2}\right)\sinh\left(\frac{\rho_2}{2}\right) < 1$$

we obtain an alternative proof of the trichotomy of Theorem 9.2.5.

Using the preceding results, we now prove that the intersection of two coequidistant or covertical spinal spheres is connected. An ordered pair $(\mathfrak{S}_1, \mathfrak{S}_2)$ of oriented spinal spheres determines a quadruple $(q_1^-, q_1^+, q_2^-, q_2^+)$ where $q_j^+ \neq q_j^-$ for $j = 1, 2$. Denote the space of pairs of spinal spheres by $\mathfrak{M}$. The map

$$\begin{aligned} \mathfrak{M} &\longrightarrow (\partial\mathbf{H}^n_{\mathbb{C}} \times \partial\mathbf{H}^n_{\mathbb{C}} - \Delta) \times (\partial\mathbf{H}^n_{\mathbb{C}} \times \partial\mathbf{H}^n_{\mathbb{C}} - \Delta) \\ (\mathfrak{S}_1, \mathfrak{S}_2) &\longmapsto (q_1^-, q_1^+, q_2^-, q_2^+). \end{aligned}$$

is a diffeomorphism. Let Σ_j denote the complex spine of $\mathfrak{S}_j$ and let B be the constant term in the expression $\eta(t)$ for the hyperbola. (2.21) and (3.3.2) imply

$$-1 + 4\operatorname{Re}\left(B(q_1^-, q_1^+; Q_2^-, Q_2^+)\right) = \begin{cases} \cosh^2(\rho(\Sigma_1, \Sigma_2)/2) & \text{if } \Sigma_1)(\Sigma_2 \\ 1 & \text{if } \Sigma_1 \parallel \Sigma_2 \\ \cos^2(\angle(\Sigma_1, \Sigma_2)) & \text{if } \Sigma_1 \between \Sigma_2. \end{cases}$$

Theorem 9.2.6 *The intersection of two coequidistant or covertical spinal hypersurfaces is connected.*

Proof Suppose that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are coequidistant or covertical spinal spheres whose intersection is disconnected. Let σ_j (respectively Σ_j) denote the spine (respectively $\mathbb{C}$-spine) of $\mathfrak{S}_j$. Then there exists a smooth path $\{g_t\}_{0\leq t\leq 1} \subset \mathbf{PU}(2,1)$ such that $g_0 = 1$ and $\Sigma(t) = g_t(\Sigma_2)$ intersects Σ_1 at an angle monotonically increasing to π_2.

In the space

$$\mathcal{E} = \operatorname{Emb}(S^2, S^3) \times \operatorname{Emb}(S^2, S^3)$$

of pairs of smooth embeddings $S^2 \hookrightarrow S^3$ pairs of transverse spheres form a dense open set $\mathcal{E}_{\text{transverse}}$. The function $N : \mathcal{E}_{\text{transverse}} \longrightarrow \mathbb{Z}$ which associates to a pair the number of connected components of the intersection is locally constant. Let $\mathcal{E}_{\text{tangent}}$ denote the subset of $\mathcal{E}$ consisting of pairs which intersect transversely except for exactly one point of tangency with second-order contact. Then $\mathcal{E}_{\text{transverse}} \cup \mathcal{E}_{\text{tangent}}$ is a connected dense open subset. By Corollary 9.1.3, Lemma 9.2.3, and Lemma 9.2.4, the path

$$\mathfrak{S}(t) = (\mathfrak{S}_1, g_t(\mathfrak{S}_2))$$

in $\mathcal{E}$ lies in $\mathcal{E}_{\text{transverse}} \cup \mathcal{E}_{\text{tangent}}$. By reparametrizing this path we may assume that $\mathfrak{S}(t)$ intersects $\mathcal{E}_{\text{tangent}}$ transversely. If $\mathfrak{S}(t)$ crosses $\mathcal{E}_{\text{tangent}}$ transversely at $t = t_0$ then the tangential intersection in $\mathfrak{S}_1 \cap g_{t_0}\mathfrak{S}_2$ can either bifurcate to a transverse intersection (which must be a circle) or disappear. By Theorem 9.2.5, $N(\mathfrak{S}(1)) \leq 1$ and the above remarks (together with Lemma 9.2.3) imply that $N(\mathfrak{S}(t)) \leq 1$ for all $t \leq 1$. □

9.2.5 *Cotranchal spinal spheres*

Theorem 9.2.7 *Let $\mathfrak{C}_1, \mathfrak{C}_2$ be distinct bisectors which possess a common slice C_0. Let σ_1, σ_2 be their respective spines. Then exactly one of the following possiblities occurs:*

1. *$\mathfrak{C}_1 \cap \mathfrak{C}_2 = C_0$ and the spines σ_1, σ_2 are intersecting geodesics inside a complex geodesic C which is orthogonal to C_0 at $\sigma_1 \cap \sigma_2$ (cospinal case).*
2. *There is an $\mathbb{R}$-plane R intersecting C_0 in a geodesic and $\mathfrak{C}_1 \cap \mathfrak{C}_2 = C_0 \cup R$ and the spines σ_1, σ_2 are ultraparallel geodesics in R—the common orthogonal geodesic to σ_1 and σ_2 inside R lies in C_0 (comeridianal case).*
3. *The spines σ_1, σ_2 do not lie in a common totally geodesic subspace and $\mathfrak{C}_1 \cap \mathfrak{C}_2$ is the union of C_0 and a surface R diffeomorphic to a 2-disc and $R \cap C_0$ is a hypercycle in C_0.*
4. *The spines σ_1, σ_2 do not lie in a common totally geodesic subspace and $\mathfrak{C}_1 \cap \mathfrak{C}_2$ equals C_0.*

Proof Choose coordinates so that the vertices of $\mathfrak{C}_1$ are represented by a null vector $q_1^- \in \mathbb{C}^{2,1}$ corresponding to the point at infinity in Heisenberg space and a null vector $q_1^+ \in \mathbb{C}^{2,1}$ corresponding to the origin in Heisenberg space. Furthermore C_0 is the complex geodesic $\mathbb{B}^1 \times \{0\} \subset \mathbf{H}^2_{\mathbb{C}}$, bounded by the chain in Heisenberg space with center the origin and unit radius (the unit circle) and we denote by $c_0 \in \mathbb{C}^{2,1}$ a vector polar to C_0. Let ι_0 denote inversion in C_0; then $q_1^{\pm}, c_0, \iota_0$ are represented by

$$q_1^{\pm} = \begin{bmatrix} 0 \\ 1 \\ \pm 1 \end{bmatrix}, \qquad c_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \iota_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now C_0 is a slice of a bisector with vertices u_-, u_+ if and only if ι_0 interchanges u_- and u_+ (Proposition 5.2.1). Furthermore (by applying a Heisenberg rotation about the vertical axis $\mathbb{V}$) we may assume that one of the vertices q_2^- corresponds to the point in $\mathcal{H}$ with Heisenberg coordinates (r, v) where $r \geq 0$ and the other vertex q_2^+ is represented by the unique complex vector linearly dependent with $\iota_0(q_2^-)$ such that $\langle q_2^-, q_2^+ \rangle = 2$. In coordinates we have

$$q_2^{\pm} = \left\{(r^2-1)^2 + v^2\right\}^{-1/2} \begin{bmatrix} \mp 2r \\ 1 - r^2 + iv \\ \mp(1 + r^2 - iv) \end{bmatrix}.$$

Now the bisector $\mathfrak{E}_1$ with vertices $q_1^{\pm}$ consists of all $z = (z_1, z_2) \in \mathbf{H}_{\mathbb{C}}^2$ satisfying $\mathrm{Im}(z_2) = 0$; that is, $z_2 \in \mathbb{R}$. Since $\langle q_2^-, q_2^+ \rangle = 2$, it follows that $z \in \mathfrak{E}_2$ if and only if $\mathbb{A}(z, q_2^-, q_2^+) = 0$; that is,

$$\begin{aligned} 0 &= \mathrm{Im}(\langle q_2^+, z\rangle \langle z, q_2^-\rangle \langle q_2^-, q_2^+\rangle) \\ &= \mathrm{Im}(\langle q_2^+, z\rangle \langle z, q_2^-\rangle) \\ &= 2\,\mathrm{Im}((1 + r^2 - iv - 2r\bar{z}_1)(1 - r^2 - iv)z_2). \end{aligned}$$

Thus the intersection $\mathfrak{E}_1 \cap \mathfrak{E}_2$ consists of all $z \in \mathbf{H}_{\mathbb{C}}^2$ satisfying the equations

$$\mathrm{Im}(z_2) = 0 \tag{9.15}$$

$$\mathrm{Im}((1 + r^2 - iv - 2r\bar{z}_1)(1 - r^2 - iv)z_2) = 0. \tag{9.16}$$

The solutions of these equations fall into two classes. If $z_2 = 0$, then (9.15) and (9.16) are automatically satisfied and define points on the complex geodesic C_0. If $z_2 \neq 0$, then (9.15) implies z_2 is a nonzero real number and (9.16) reduces to

$$\mathrm{Im}\left((1 + r^2 - iv - 2r\bar{z}_1)(1 - r^2 - iv)\right) = 0. \tag{9.17}$$

For fixed t, let C_t denote the complex geodesic $z_2 = t$; for $-1 < t < 1$ these are the slices of $\mathfrak{E}_1$. Now (9.17) defines the Euclidean straight line consisting of all (z_1, t), where z_1 satisfies

$$\mathrm{Im}\left(\bar{z}_1(1 - r^2 - iv)\right) = -\frac{v}{r}. \tag{9.18}$$

The point on the line defined by (9.18) of minimum norm equals (ζ_0, t) where

$$\zeta_0 = i\frac{1 - r^2 - iv}{(1 - r^2)^2 + v^2}\,\frac{v}{r}.$$

A solution of (9.18) on C_t exists if and only if

$$|\zeta_0|^2 + t^2 < 1$$

for some $-1 < t < 1$; that is, if $|\zeta_0| < 1$, or equivalently,

$$0 < (r^2 - 1)\left\{r^2(r^2 - 1) + v^2\right\}. \tag{9.19}$$

The set $\mathcal{S}$ of solutions $(r\xi, v) \in \mathcal{H}$ of (9.19) is the disjoint union of the set $r > 1$ (the outside of a right circular cylinder) and its image under ι_0. More intrinsically, if Π_1 is orthogonal projection onto the complex spine $\Sigma_1 \approx \mathfrak{H}^1$ of $\mathfrak{C}_1$, then

$$\mathcal{S} = \Pi_1^{-1}(H_1 \cup H_2)$$

where H_1 and H_2 are the horoballs in $\Sigma_1 \approx \mathfrak{H}^1$ defined by

$$H_1 = \{z \in \mathfrak{H}^1 \mid \mathrm{Re}(z) > 1\},$$
$$H_2 = \left\{z \in \mathfrak{H}^1 \;\middle|\; \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$$

Now $(r, v) \notin \mathcal{S}$ if and only if $\mathfrak{C}_1 \cap \mathfrak{C}_2 = C_0$.

Suppose that $(r, v) \in \mathcal{S}$; interchanging q_2^- and q_2^+ assume that $r > 1$. Then the solution of (9.18) defines a hypercycle in each C_t for

$$|t| < \sqrt{1 - |\zeta_0|^2}.$$

In particular as $t \to 0$, that is as the slices C_t approach C_0, these hypercycles approach a hypercycle h_0 in C_0 whose geodesic curvature equals

$$\left|\frac{v}{r}\right| \frac{1}{(r^2 - 1)^2 + v^2}.$$

The surface $S \subset \mathbb{C}^2$ defined by (9.15) and (9.18) is an $\mathbb{R}$-linear 2-plane such that $\mathfrak{C}_1 \cap \mathfrak{C}_2 = S \cap \mathbf{H}_{\mathbb{C}}^2$ and $S \cap C_0 = h_0$.

The coordinates involved in these calculations explicitly determine the hyperbola associated to this pair of spinal spheres. The relevant Korányi–Reimann complex cross-ratios are

$$\mathbf{X}\{q_1^-, q_2^-, q_2^+, q_1^+\} = \frac{r^4 + 4v^2}{(r^2 - 1)^2 + rv^2}$$
$$\mathbf{X}\{q_1^-, q_2^+, q_2^-, q_1^+\} = \frac{1}{(r^2 - 1)^2 + rv^2}$$

from which identities (7.23) and (7.24) follow. The hyperbola associated to the pair is then given by

$$\begin{aligned}
\eta(t) &= \eta(q(t); q_1^-, q_1^+) \\
&= \frac{1}{2} - \frac{1}{2}\frac{r^2}{(r^2 - 1)^2 + v^2}(t - t^{-1})^2 + \frac{i}{2}\frac{v}{(r^2 - 1)^2 + v^2}(t^2 - t^{-2}) \\
&= \frac{1}{2} - \frac{t - t^{-1}}{(r^2 - 1)^2 + v^2}\left\{\frac{r^2}{2}(t - t^{-1}) - i\frac{v}{2}(t + t^{-1})\right\}
\end{aligned}$$

which describes a component of a hyperbola in $\mathbb{C}$ for $t > 0$.

The slice $q(t)^\perp$ of $\mathfrak{C}_2$ intersects $\mathfrak{C}_1$ transversely if and only if the geodesic curvature $\Phi(\eta(t))$ of the hypercycle $q(t)^\perp \cap \mathfrak{C}_1$ satisfies $\Phi(\eta(t)) < 1$ where

$$\begin{aligned}\Phi(\eta(t)) &= \frac{|\mathrm{Im}(\eta(t))|}{\sqrt{1 - 2\,\mathrm{Re}(\eta(t))}} \\ &= \left|\frac{v}{2r}\right| \frac{t + t^{-1}}{\sqrt{(r^2-1)^2 + v^2}}.\end{aligned}$$

The exceptional case 1 occurs when $r = 0$, in which case $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are cospinal. This case was already treated in §5.3.1.

Another exceptional case occurs when $v = 0$. In that case all four vertices $q_1^\pm, q_2^\pm$ lie on the $\mathbb{R}$-plane corresponding to the real axis. This is case 2. Here the spinal spheres are comeridianal. This especially curious case was treated in Corollary 5.3.8. Case 3 (respectively 4) occurs when $r, v \neq 0$ and (9.19) holds (respectively does not hold). The proof of Theorem 9.2.7 is complete. □

9.3 Dirichlet and Ford polyhedra

The theory of bisectors and extors was motivated by the study of fundamental polyhedra for discrete groups, in particular Dirichlet and Ford fundamental polyhedra. Suppose Γ is a discrete group of isometries of $X = \mathbf{H}^n_{\mathbb{C}}$ and choose a point $x \in X$, the *base point* of our polyhedron. Then the *Dirichlet region for* Γ *centered at* x is defined as

$$\Delta_x(\Gamma) = \{y \in X \mid \rho(y,x) \leq \rho(y,\gamma x) \quad \forall \gamma \in \Gamma\}.$$

Clearly $\Delta_x(\Gamma)$ is the intersection of equidistant half-spaces

$$\Delta_x(\Gamma) = \bigcap_{1 \neq \gamma \imath \Gamma} \mathfrak{H}_{x,\gamma x}$$

where the *equidistant half-space* is defined by

$$\mathfrak{H}(x,y) = \{u \in X \mid \rho(u,x) < \rho(u,y)\}.$$

When the base point x tends to an ideal point, one can define the *Ford* fundamental domains using the *Busemann functions* defined in §4.1.2. The Dirichlet fundamental domains converge to *Ford fundamental domains.* Rewrite the defining half-spaces of the Dirichlet region $\Delta_x(\Gamma)$ as

$$\mathfrak{H}(x,\gamma x) = \{u \in X \mid \rho_x(u) < \rho_x(\gamma^{-1}u)\}$$

where $\rho_x(u) = \rho(x,u)$. Now move x_t along a geodesic to an ideal point q not fixed by γ. The distance functions ρ_{x_t} normalized by

$$\tilde{\rho}_{x_t}(u) = \rho_{x_t}(u) - t$$

converge to Busemann functions h_Q as in §4.1.2 and clearly

$$\mathfrak{H}(x_t, \gamma x_t) = \{u \in X \mid \tilde{\rho}_{x_t}(u) < \tilde{\rho}_{x_t}(\gamma^{-1}u)\}.$$

Thus the half-spaces $\mathfrak{H}(x_t, \gamma x_t)$ converge to half-spaces

$$\mathfrak{H}(Q, \gamma Q) = \{u \in X \mid h_Q(u) < h_Q(\gamma^{-1}(u))\},$$

obtaining the *Ford fundamental domain*

$$\Delta_Q(\Gamma) = \{u \in \mathbf{H}^n_{\mathbb{C}} \mid h_Q(u) < h_Q(\gamma^{-1}(u))\}.$$

Exercise 9.3.1 *Relate this definition of Ford fundamental domains to the classical definition (see for example Ford [53]) of Ford fundamental domains for Kleinian groups using isometric circles.*

The general picture of fundamental polyhedra and Dirichlet (and Ford) fundamental polyhedra has been extensively described in the literature, see for example Beardon [9], Maskit [118], Epstein–Petronio [49] and de Rham [40]. The analogous theory for discrete groups acting on simply connected complete Riemannian manifolds of nonpositive curvature is discussed in detail in Ehrlich–ImHof [45] to which we refer.

In particular, Δ is a *fundamental domain* and its orbit $\gamma(\mathrm{int}(\Delta)$, for $\gamma \in \Gamma$, *tiles* X: all the $\gamma(\mathrm{int}(\Delta))$ are disjoint and

$$\bigcup_{\gamma\in\Gamma} \gamma\Delta = X.$$

Furthermore Δ is a topological manifold with boundary, whose boundary contains open smooth submanifolds, called *sides,* which intersect transversely along codimension 2 smooth submanifolds, called *edges,* etc. This defines a smooth stratification of Δ. Furthermore for each side $S \subset \partial\Delta$, there exists $g \in \Gamma$ such that $g(S)$ is another side of Δ. This collection of *side-pairing transformations* generates Γ and defines a collection of identifications of Δ such that the corresponding identification space of Δ is homeomorphic to the orbit space X/Γ.

Suppose that $e \subset \partial\Delta$ is an edge. Then e is a submanifold of codimension 2 which arises as the intersection of two sides S_0 and S_1 of Δ. Let g_1 be the side-pairing transformation corresponding to S_1. Then $g_1(S_1)$ is another side of Δ whose closure contains the edge $g_1(e)$. This edge $g_1(e)$ is the intersection of two sides, one of which is g_1S_1. Denote the other side containing $g_1(e)$ by S_2 and the side-pairing transformation corresponding to S_2 by g_2. Inductively, suppose that sides $S_1, S_2, \ldots, S_j$ of Δ have been defined with corresponding side-pairing transformations $g_1, g_2, \ldots, g_j$ such that $g_jg_{j-1}\cdots g_2g_1(e)$ is an edge contained in the closure of the side $g_j(S_j)$; define S_{j+1} as the other sides whose closure contains $g_j \dots g_2 g_1(e)$. In particular

$$g_j g_{j-1} \cdots g_2 g_1(e) = g_j(S_j) \cap S_{j+1}. \tag{9.20}$$

If Δ has finitely many faces and edges, eventually $S_k = S_0$. The sequence

$$(S_1, g_1), \dots, (S_k, g_k)$$

is called the *side-pairing cycle* associated with the edge e.

When a fundamental domain Δ arises by a Dirichlet–Ford construction, one can say somewhat more. If S is a side of $\partial\Delta$ lying in the bisector $\mathfrak{E}(x, \gamma x)$, then $\gamma^{-1} S$ is a side of $\partial\Delta$ which lies in the bisector

$$\gamma^{-1}\mathfrak{E}(x, \gamma x) = \mathfrak{E}(x, \gamma^{-1} x).$$

Then γ^{-1} is the side-pairing transformation for S and γ is the side-pairing transformation for $\gamma^{-1}(S)$. In particular for the side-pairing cycle above,

$$S_j \subset \mathfrak{E}(x, g_j^{-1} x).$$

(9.20) implies that

$$\begin{aligned} g_j \cdots g_1(e) &= g_j(S_j) \cap S_{j+1} \\ &\subset g_j \mathfrak{E}(x, g_j^{-1} x) \cap \mathfrak{E}(x, g_{j+1}^{-1} x) \end{aligned}$$

and

$$\begin{aligned} e &= g_1^{-1} \cdots g_{j-1}^{-1} \mathfrak{E}(x, g_j^{-1} x) \ \cap \ g_1^{-1} \cdots g_{j-1}^{-1} g_j^{-1} \mathfrak{E}(x, g_{j+1}^{-1} x) \\ &= \mathfrak{E}(g_1^{-1} \cdots g_{j-1}^{-1} x, g_1^{-1} \cdots g_{j-1}^{-1} g_j^{-1} x) \\ &\qquad \cap \ \mathfrak{E}(g_1^{-1} \cdots g_{j-1}^{-1} g_j^{-1} x, g_1^{-1} \cdots g_{j-1}^{-1} g_j^{-1} g_{j+1} x) \end{aligned}$$

In particular e is equidistant from all points

$$g_1^{-1}(x),\ g_2^{-1} g_1^{-1}(x),\ \dots,\ g_{k-1}^{-1} \cdots g_2^{-1} g_1^{-1}(x),\ g_k^{-1} g_{k-1}^{-1} \cdots g_2^{-1} g_1^{-1}(x) = x.$$

Giraud's theorem implies that unless e is a flat edge (in which case e lies in a complex hyperplane H and the g_j are complex reflections in H), then at most three bisectors contain e. Thus the cycle has length $k = 3$.

We discuss this construction briefly, when Γ is a discrete group which preserves a totally geodesic subspace S containing the base point x. Two quite different types of behavior arise, depending on whether S is complex or totally real.

We begin with the case when S is complex. (See §4.3.7.) If $x, y \in S$, then

$$\mathfrak{H}(x,y) = \Pi^{-1}(\mathfrak{H}(x,y) \cap S)$$

where $\Pi : X \longrightarrow S$ is orthogonal projection. In particular if $x \in S$ and S is Γ-invariant, then

$$\Delta_x(\Gamma) = \Pi^{-1}(\Delta_x(\Gamma) \cap S)$$

where $\Delta_x(\Gamma) \cap S$ is the Dirichlet fundamental region for Γ acting on S. In particular, since Π is a fibration, the combinatorial type of the Dirichlet region for Γ acting on X corresponding to that of the Dirichlet region for Γ acting on S.

When S is totally real, the situation is more complicated. As in §4.4.13, a fundamental domain can be defined completely analogously using the orthogonal projection onto S, but unlike orthogonal projections onto complex totally geodesic subspaces, the inverse image of a half-space in a totally real subspace S under orthogonal projection will not be a metric half-space. In fact, Giraud's theorem implies that the combinatorial type will generally not be preserved. However, some of the basic combinatorics will be preserved, owing to the following fact:

Theorem 9.3.2 *Consider three points $x_0, x_1, x_2 \in \mathbf{H}^2_{\mathbb{R}}$ and let $\mathfrak{H}_i = \mathfrak{H}(x_i, x_0)$ for $i = 1, 2$. The half-spaces $\mathfrak{H}_i$ are disjoint if and only if the intersections $\mathfrak{H}_i \cap \mathbf{H}^2_{\mathbb{R}}$ are disjoint.*

Proof Clearly if $\mathfrak{H}_i \cap \mathbf{H}^2_{\mathbb{R}}$ intersect, then the half-spaces $\mathfrak{H}_i$ intersect. Conversely, suppose that $\mathfrak{H}_i \cap \mathbf{H}^2_{\mathbb{R}}$ are disjoint. The half-plane $\mathfrak{H}_i \cap \mathbf{H}^2_{\mathbb{R}}$ is bounded by the geodesic line bisecting the segment from x_0 to x_i. The disjointness of $\mathfrak{H}_1 \cap \mathbf{H}^2_{\mathbb{R}}$ and $\mathfrak{H}_2 \cap \mathbf{H}^2_{\mathbb{R}}$ implies that the three points x_1, x_0, x_2 lie on either a hypercycle or horocycle in $\mathbf{H}^2_{\mathbb{R}}$ (in that order). In particular there exists a hyperbolic or parabolic 1-parameter subgroup

$$\phi : \mathbb{R} \longrightarrow \mathbf{SO}(2,1)$$

stabilizing $\mathbf{H}^2_{\mathbb{R}}$ and $t_1 < 0 < t_2$ such that

$$\begin{aligned} x_1 &= \phi(t_1)(x_0) \\ x_2 &= \phi(t_2)(x_0). \end{aligned}$$

For $z \in \mathbf{H}^2_{\mathbb{C}}$ the function

$$f(t) = \cosh^2\left(\frac{\rho(z, \phi(t)x_0)}{2}\right)$$

is a convex function of $t \in \mathbb{R}$. (Compare Phillips [140].) In particular

$$(t_2 - t_1)f(0) < t_2 f(t_1) + (-t_1)f(t_2)$$

or, equivalently,

$$0 < t_2(f(t_1) - f(0,z)) + (-t_1)(f(t_2) - f(0)). \tag{9.21}$$

Now

$$\begin{aligned} z \in \mathfrak{H}(x_1, x_0) &\iff f(t_1) < f(0) \\ &\iff t_2(f(t_1) - f(0)) < 0 \\ &\implies (-t_1)(f(t_2) - f(0)) > 0 \text{ (by (9.21))} \\ &\iff f(0) < f(t_2) \\ &\iff z \in \mathfrak{H}(x_0, x_2) \end{aligned}$$

proving that $\mathfrak{H}(x_1, x_0) \subset \mathfrak{H}(x_0, x_2)$; that is, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are disjoint. □

Exercise 9.3.3 *Let Γ be a cyclic subgroup of $\mathbf{PO}(2,1)$ of order $n < \infty$ fixing a point $x_0 \in \mathbf{H}^2_{\mathbb{R}}$ and let $x \in \mathbf{H}^2_{\mathbb{R}}$ satisfy $x \neq x_0$. Then the Dirichlet tesselation of $\mathbf{H}^2_{\mathbb{R}}$ based at x consists of n sectors meeting at x_0. Determine the Dirichlet tesselation of $\mathbf{H}^2_{\mathbb{C}}$ for Γ based at x.*

APPENDIX A

COMMENTS ON GIRAUD'S PAPER

In the summer of 1993 Ossip Shvartsman visited College Park from Moscow. Through him I learned of the work of M. Georges Giraud on automorphic functions of several variables, which developed the ideas of Picard [141, 142], which were so influential in the more recent work of Deligne and Mostow [38, 39, 129] and others (for example, Holzapfel [88] and Yoshida [172]). The paper, "Sur certaines fonctions automorphes de deux variables" [65], contains many interesting results which seem to have gone unnoticed in more recent literature on complex hyperbolic geometry. (Several *Comptes Rendus* notes [58, 59, 62, 60, 66, 67] summarize these results.)

1. Classification des substitutions linéaires (p.45)
2. Étude du polyèdre fondamental rayonné (p.55)
3. Les sommets paraboliques (p.77)
4. Relations algébriques entre fonctions de M.Picard a groups linéaires. Systèmes d'équations linéaires aux derivées partielles (p.99)
5. Classification. Application de la méthode de rayonnement a certains groups spéciaux (p.122)
6. Sommets et arêtes paraboliques (p.136)
7. Relations algebriques entre fonctions hyperabéliennes de M.Picard. Systemes d'equations lineaires aux derivees partielles (p.159)

Giraud's paper divides in two parts, the first of which ("Groupes linéaires de M. Picard") deals with complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$. It comprises the first four chapters. The second part, "Fonctions hyperabéliennes," deals with $\mathbf{H}^2_{\mathbb{R}} \times \mathbf{H}^2_{\mathbb{R}}$ and occupies the last three chapters, beginning with p.122. (Functions automorphic with respect to subgroups of the orthogonal group $\mathbf{O}(n, 2)$ are discussed in [62].) We only discuss the first part on complex hyperbolic geometry, which at that time was called "fonctions hyper-Fuchsiennes."

Chapter one gives a classification of the elements of $\mathbf{PU}(2,1)$ and a description of the conjugacy classes. Transformations in $\mathbf{PU}(2,1)$ are called "substitutions de M. Picard" and a subgroup of $\mathbf{PU}(2,1)$ is called a "groupe linéaire." Suppose that $A \in \mathbf{SU}(2,1)$. Giraud observes that $|\det(A)| = 1$ and that if λ is an eigenvalue of A, then so is $\bar{\lambda}^{-1}$ (§1.2, p.46). The fixed points ("points doubles") of the associated collineation correspond to eigenvectors of A. §3 discusses the case when the absolute values of the eigenvalues are distinct. The hyperbolic elements are put in the canonical form

$$\begin{bmatrix} e^{i\theta}\cosh(u) & 0 & e^{i\theta}\sinh(u) \\ 0 & e^{-2i\theta} & 0 \\ e^{i\theta}\sinh(u) & 0 & e^{i\theta}\cosh(u) \end{bmatrix}.$$

Regular elliptic elements are discussed in §4. Singular elliptic elements are discussed in §5; these elements are "complex reflections" about points or complex geodesics. Giraud calls these elements "substitutions elliptiques à plan double" and distinguishes between point reflections and line reflections by whether the fixed line is exterior or interior to the ball. What I have called "ellipto-parabolic" elements (two coincident eigenvalues), Giraud calls "paraboliques à deux points doubles" and finds the normal form:

$$\begin{bmatrix} \frac{2+\epsilon i}{2}e^{i\theta} & 0 & \frac{\epsilon i}{2}e^{i\theta} \\ 0 & e^{-2i\theta} & 0 \\ \frac{-\epsilon i}{2} & 0 & \frac{2-\epsilon i}{2}e^{i\theta} \end{bmatrix}.$$

Next (§7, p.52) he discusses the unipotent elements, beginning with the identity. The special unipotent elements he calls "paraboliques à plan double." §8 discusses the regular unipotents ("paraboliques à point double unique"). §9 discusses the convergence of orbits of cyclic parabolic groups to the fixed points.

The second chapter discusses the structure of the Dirichlet fundamental region ("polyèdre fondamental rayonné"), its faces of codimension 1 ("faces"), its faces of codimension 2 ("arêtes", which I will translate as "ridges"), and its vertices (sommets). In §II.1, "Nature des faces," (p.55), Giraud computes the bisector equidistant from a pair of points q, q' in $\mathbf{H}^2_{\mathbb{C}}$ and identifies it with the projective image of an indefinite Hermitian form in $\mathbb{C}^{2,1}$. He considers their full analytic continuation to *extors,* hypersurfaces E in $\mathbb{P}^2_{\mathbb{C}}$ which extend metric bisectors. Giraud calls extended bisectors "surfaces d'équidistance," "multiplicités d'équidistance." (In [58, 59], extors are called "faux-plan.")

§2, "Propriétés des surfaces d'équidistance" (p.56), develops the basic geometry of bisectors. To describe the results, consider an extended bisector $E \subset \mathbb{P}^2_{\mathbb{C}}$. §2.I, "La surface d'équidistancea un point double unique," shows that E has a unique singular point p. The complex geodesic ("plan") C polar to p contains the points q, q' used to define E as a bisector (§2.II). He calls C the "polaire" of E and shows that p lies outside the ball (§2.III) assuming that q, q' lie inside the ball. Indeed, any point on $C - E$ can serve as a point q to define E (§2.IV). The bisector E is a union of complex geodesics (the "slices" of E, following Mostow) polar to p (orthogonal to C) (§2.V). Lemme 3 (p.59) is our Lemma 8.3.6, used later in the discussion of "arêtes gauches." From this lemma, Giraud deduces Theorem 8.3.3 (Theorème 4, p.62).

Theorème 5 discusses the intersection of cospinal bisectors (when $\Sigma_1 = \Sigma_2$, the $\mathbb{C}$-spines coincide). Their spines σ_1, σ_2 are geodesics in the common $\mathbb{C}$-spine Σ which intersect. The intersection $\mathfrak{E}_1 \cap \mathfrak{E}_2$ consists of two complex lines, one of which lies inside complex hyperbolic space (and contains the intersection of the spines). The other line is the slice $\Pi_\Sigma^{-1}(\sigma_1 \cap \sigma_2)$. (The case of this intersection is discussed also by Mostow.)

Next (Theorème 6), Giraud proves that given a complex geodesic C, there exist infinitely many bisectors containing C as a slice, all of which have the same $\mathbb{C}$-spine. Indeed, if Σ is any complex geodesic orthogonal to C, bisectors $\mathfrak{E}$ having C as a slice and Σ as $\mathbb{C}$-spine correspond to geodesics $\sigma \subset \Sigma$ passing through the point $\Pi_\Sigma(C) = C \cap \Sigma$. Evidently σ is the spine of $\mathfrak{E}$: $\mathfrak{E} = \Pi_\Sigma^{-1}(\sigma)$ and $\sigma = \mathfrak{E} \cap \Sigma$.

The codimension 1 strata, the *faces* of a polyhedron, are open subsets of bisectors. Faces meet along the codimension 2 strata, the *ridges*. There are two types of ridges: "flat edges" and "rough edges." A *flat edge* is a complex geodesic ("arête plane") and a *rough edge* ("arête gauche") is a trisector belonging to only three bisectors as above (as in Theorem 8.3.3).

§7 discusses "invisible ridges" ("arêtes non apparentes") on the Dirichlet fundamental region $\Delta = \Delta_\Gamma(x_0)$ of a discrete group Γ with "centre" x_0. An *invisible ridge* is a ridge R belonging to only two faces. Giraud shows that an invisible ridge must be a complex geodesic. The bisector containing such a ridge is transformed into itself by an element $\gamma \in \Gamma$; necessarily $\gamma^2 = 1$ and $R \subset \Delta \cap \gamma(\Delta)$. The involutions in $\mathbf{PU}(2,1)$ are complex reflections about either complex geodesics or points in $\mathbf{H}^2_\mathbb{C}$. Giraud shows by direct calculation that in the former case R must be the fixed-point set of γ.

§8–§9 introduce terminology: Giraud calls a complex projective line a "plan," and remarks that through every pair of points in projective space is a unique line. The absolute of a hyperbolic anti-polarity is "l'hypersphère," and Giraud calls a real geodesic in $\mathbf{H}^2_\mathbb{C}$ a "pseudo-droite." He remarks that every pair of points in $\mathbf{H}^2_\mathbb{C}$ lies on a unique geodesic. Using the cross-ratio formula, Giraud defines "la pseudo-distance" as the Bergman distance between two points, and §10–§16 develop trigonometric formulas such as the laws of cosines. In §17, Giraud uses these formulas to prove that these geodesic curves are indeed length minimizing. §18–§20 associate a triangle to edges arising from bisector intersections.

The third chapter, "Sommets paraboliques," relates the geometry at an ideal vertex of a fundamental polyhedron to the function theory. The first theorem states that if a theta-automorphic function for a discrete group Γ is singular on $\partial\mathbf{H}^n_\mathbb{C}$, then no face of a fundamental polyhedron meets $\partial\mathbf{H}^n_\mathbb{C}$. The possible discrete groups arising as stabilizers of ideal vertices are listed.

The final chapter of the first part describes the space of automorphic forms for a group Γ. Starting with a rational function $R(\xi, \eta)$ on $\mathbb{B}^2$, Giraud constructs an automorphic form $\Theta(x, y)$ for Γ by the usual process. The main result is that if the fundamental polyhedron meets $\partial\mathbf{H}^2_\mathbb{C}$ only at ideal vertices, then the field of such functions has transcendence degree 2.

Giraud's monograph [64] summarizes this theory of automorphic functions, before Giraud's research shifted into partial differential equations. In his obituary [23] to Giraud, Cartan writes of Giraud's early work in discrete groups and automorphic forms in complex hyperbolic geometry: "Georges Giraud s'éleve à de plus vastes généralisations, envisageant le problème par son côte géométrique, dont in poursuit très loin."

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