Modelling and quantifying mortality and longevity risk

Module B1 : Multi-population Mortality Models

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Overview

In this module:

- Motivation for combining models for multiple populations
- Estimation of multi-population models

Motivation to combine populations

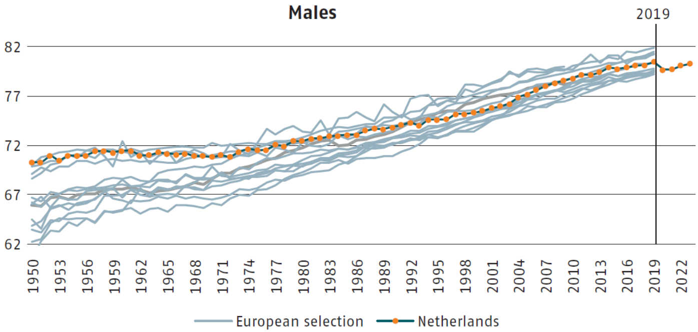
Micro vs Macro Risk

- In mortality models, we can distinguish (at least) two sources of uncertainty:
  - Macro Longevity Risk
    The changes over time in the probabilities of dying
    (due to random innovations in time effect κ_t), and
  - Micro Longevity Risk
    The Poisson-distributed number of deaths *given* these probabilities
    (i.e. the Grim Reaper throwing his coins for any individual).
- Second risk becomes relatively smaller if we increase the size of our portfolio.
- The first one does not. This is of fundamental importance for the risk management of life insurance products.

Correlation structure

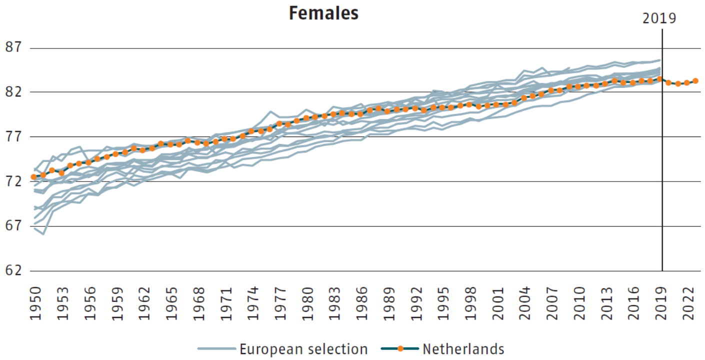
- The single factor Lee-Carter model creates perfect dependence between changes for different ages.
- Correlation between changes in $\ln \mu_{x_1,t}$ and $\ln \mu_{x_2,t}$ equals one for all ages x_1, x_2.
- In this module we discuss multi-population models such as the Dutch Actuarial Society AG2020/AG2022 based on (Li and Lee, 2005).
- Introducring multiple stochastic factors and data from different countries allows us to distinguish between country-specific and more general trends in mortality changes.

Period Life Expectancies at birth ♂



European peers and the Netherlands

Period Life Expectancies at birth ♀

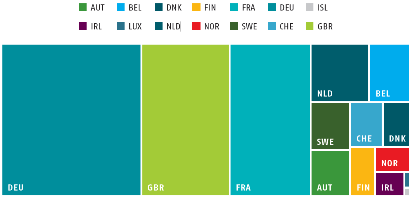


European Peers and the Netherlands

Looking beyond the border

- KAG model structure (pre-pandemic !) distinguishes
  - Long-term trend which is common among Western-European countries (medical improvements will not stop at a border)
  - Short term deviations from this trend for any single country (effects such as start/end of smoking epidemic may vary per population)
- We only take countries which are comparable in wealth: the top half of the European Union in terms of GDP (per capita). Using more data, we aim to enhance stability of forecasts.
- Main assumption that mortality in countries will not diverge from common European trend can be tested over time by studying the development of the country-specific time series.

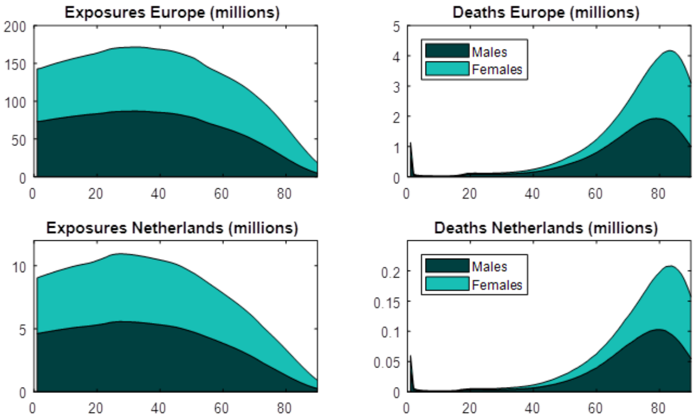
Datsets for Netherlands and European peer group



Left: Countries included in analysis
(European countries with GDP per person above the European average),

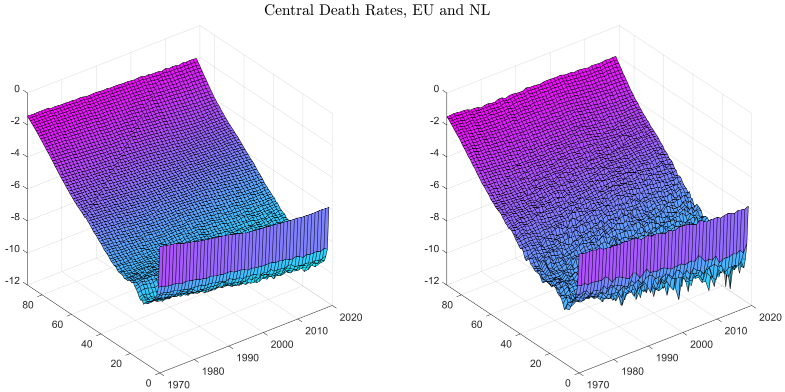
Right: Relative sizes for death counts (2019).
(Germany 787.320, Netherlands 126.995)

Datsets for Netherlands and European peer group



Dataset sizes for exposures and death counts.

Raw data



Log central death rates, $\ln(D_{xt}/E_{xt})$
for females in Europe (left) and the Netherlands (right).

Stochastic Model for the Dutch Population

- AG2020 (model of the Dutch Actuarial Society)

$$\begin{aligned} D_{xtg} &\sim \mathrm{Poisson}(E_{xtg}\,\mu_{xtg}) \\ \ln \mu_{xtg} &= \ln \mu_{xtg}^{\mathrm{EU}} + \ln \mu_{xtg}^{\mathrm{NL}} \end{aligned}$$

- with long term trend fitted using countries in Europe with comparable GDP

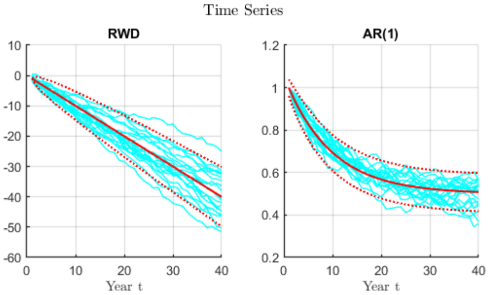
$$\begin{aligned} \ln \mu_{xtg}^{\mathrm{EU}} &= A_x^g + B_x^g\, K_t^g \\ K_{t+1}^g &= K_t^g + \theta^g + \epsilon_{t+1}^g \end{aligned}$$

- and short term dynamics which describe difference with this European trend

$$\begin{aligned} \ln \mu_{xtg}^{\mathrm{NL}} &= \alpha_x^g + \beta_x^g\, \kappa_t^g \\ \kappa_{t+1}^g &= a^g\, \kappa_t^g + \delta_{t+1}^g + c^g \end{aligned}$$

with $|a^g| < 1$, and $Z_t := (\epsilon_t^m, \delta_t^m, \epsilon_t^f, \delta_t^f)$ i.i.d. Gaussian with mean $(0,0,0,0)$ and a given 4×4 covariance matrix C which does not change over time.

Dynamics in the long run



Common European trend for all in peer group (Left, RWD)

$$K_{t+1}^{g} = K_{t}^{g} + \theta^{g} + \epsilon_{t+1}^{g},$$

combined with transient effect (Right, AR(1)) that describes individual country's deviation from the common trend:

$$\kappa_{t+1}^{g} = a^{g}\, \kappa_{t}^{g} + \delta_{t+1}^{g} + c^{g}.$$

Estimation of multi-population models

Maximum Likelihood estimation

- First we fit European data using $\ln \mu_{xtg}^{\mathrm{EU}} = A_x^g + B_x^g K_t^g$

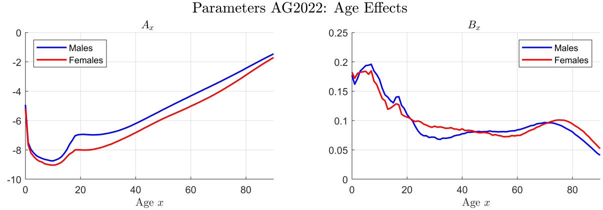
$$\max_{(A_x^g, B_x^g, K_t^g)} \prod_{x \in \mathcal{X}^{\mathrm{EU}}} \prod_{t \in \mathcal{T}^{\mathrm{EU}}} \frac{(E_{xtg}^{\mathrm{EU}} e^{A_x^g + B_x^g K_t^g})^{D_{xtg}^{\mathrm{EU}}} \exp(-E_{xtg}^{\mathrm{EU}} e^{A_x^g + B_x^g K_t^g})}{D_{xtg}^{\mathrm{EU}}!},$$

- then the Dutch deviation $\ln \mu_{xtg}^{\mathrm{NL}} = \ln \mu_{xtg}^{\mathrm{EU}} + \alpha_x^g + \beta_x^g \kappa_t^g$

$$\max_{(\alpha_x^g, \beta_x^g, \kappa_t^g)} \prod_{x \in \mathcal{X}^{\mathrm{NL}}} \prod_{t \in \mathcal{T}^{\mathrm{NL}}} \frac{(E_{xtg}^{\mathrm{NL}} e^{\hat{A}_x^g + \hat{B}_x^g \hat{K}_t^g + \alpha_x^g + \beta_x^g \kappa_t^g})^{D_{xtg}^{\mathrm{NL}}} \exp(-E_{xtg}^{\mathrm{NL}} e^{\hat{A}_x^g + \hat{B}_x^g \hat{K}_t^g + \alpha_x^g + \beta_x^g \kappa_t^g})}{D_{xtg}^{\mathrm{NL}}!}$$

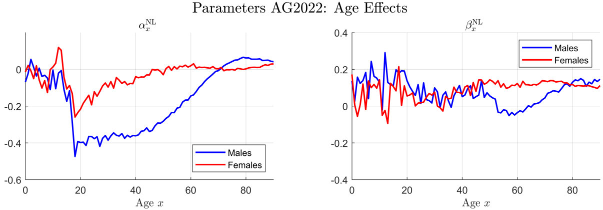
  using estimates of first step.
- Note that we treat time series K and κ as parameters here.
- In numerical procedures we maximize the logarithm of the likelihood.

Age Parameters for the Dutch Population, AG2022

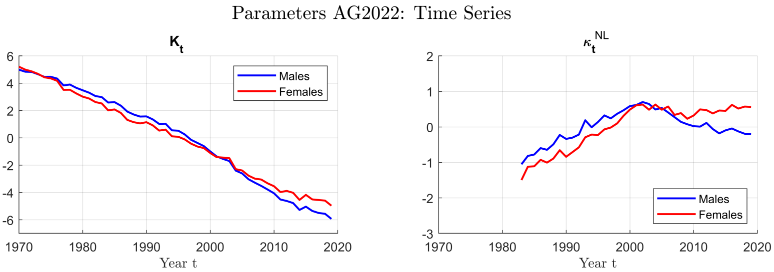


$$\ln \mu_{xtg}^{\mathrm{EU}} = \boxed{A_x^g} + \boxed{B_x^g} K_t^g$$

$$\ln \mu_{xtg}^{\mathrm{NL}} = \boxed{\alpha_x^g} + \boxed{\beta_x^g} \kappa_t^g$$



Time Series for the Dutch Population, AG2022

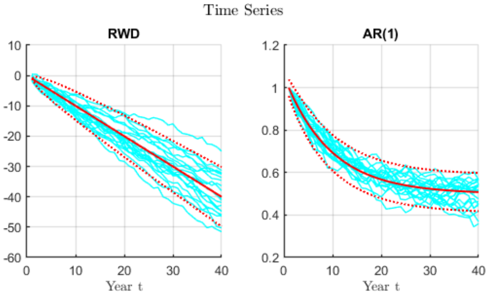


$$\ln \mu_{xtg}^{\mathrm{EU}} = A_x^g + B_x^g \, K_t^g, \qquad K_{t+1}^g = K_t^g + \theta^g + \epsilon_{t+1}^g$$

$$\ln \mu_{xtg}^{\mathrm{NL}} = \alpha_x^g + \beta_x^g \, \kappa_t^g, \qquad \kappa_{t+1}^g = a^g \, \kappa_t^g + \delta_{t+1}^g + c^g$$

Notice that we (only) use recent data for the Dutch deviation series κ_t^g.

Dynamics in the long run



Common European trend for all in peer group (Left, RWD)

$$K_{t+1}^{g} = K_{t}^{g} + \theta^{g} + \epsilon_{t+1}^{g},$$

combined with transient effect (Right, AR(1)) that describes individual country's deviation from the common trend:

$$\kappa_{t+1}^{g} = a^{g}\,\kappa_{t}^{g} + \delta_{t+1}^{g} + c^{g}.$$

Random Walk with Drift

- Dynamics for K are assumed to follow (we suppress superscripts g):

$$K_t = K_{t-1} + \theta + \epsilon_t$$

  with i.i.d. Gaussian $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ and $\theta < 0$.
- This implies that

$$K_t = K_0 + t\theta + \sum_{s=0}^{t-1} \epsilon_s$$

  so

$$\mathbb{E}K_t = K_0 + t\theta$$

  and

$$\mathrm{Var}(K_t) = t\sigma_\epsilon^2.$$

First Order Autoregressive Model

- For both genders, dynamics for κ are assumed to follow

$$\kappa_t = a\kappa_{t-1} + \delta_t + c$$

with i.i.d. Gaussian sequence $\delta_t \sim N(0, \sigma_\delta^2)$.

- This implies that

$$\kappa_t = \tfrac{c}{1-a} + (\kappa_0 - \tfrac{c}{1-a})a^t + \sum_{s=0}^{t-1} a^s \delta_{t-s}$$

- We thus find that mean, variance and autocorrelation converge to fixed values

$$\begin{aligned}
\mathbb{E}\kappa_t &= \tfrac{c}{1-a} + (\kappa_0 - \tfrac{c}{1-a})a^t \quad \overset{t\uparrow\infty}{\rightarrow} \quad \tfrac{c}{1-a} \\
\mathrm{Var}\,\kappa_t &= \sigma_\delta^2 \sum_{s=0}^{t-1} a^{2s} = \sigma_\delta^2 \frac{1-a^{2t}}{1-a^2} \quad \overset{t\uparrow\infty}{\rightarrow} \quad \frac{\sigma_\delta^2}{1-a^2} \\
\rho_{t,t+s} &= a^s \sqrt{\frac{1-a^{2t}}{1-a^{2(t+s)}}} \quad \overset{t\uparrow\infty}{\rightarrow} \quad a^s
\end{aligned}$$

Prediction of future values: RWD

- If today is time t, value at a future time $t+h$ can be written as

$$K_{t+h} = K_t + h\theta + \sum_{s=0}^{h-1} \epsilon_{t+s}$$

  which has the Gaussian distribution $N(K_t + h\theta, \sigma_\epsilon^2 h)$.
- So our best point estimate for the future value at time $t+h$ is

$$K_t + h\theta$$

  and the 95% confidence bounds are

$$\pm 1.96\sigma_\epsilon\sqrt{h}$$

  on top of that value.

Prediction of future values: AR(1)

- If today is time t, value at a future time $t+h$ can be written as

$$\kappa_{t+h} = \frac{c}{1-a} + (\kappa_t - \frac{c}{1-a})a^h + \sum_{s=0}^{h-1} a^s \delta_{t+h-s}$$

  with distribution $N(\frac{c}{1-a} + (\kappa_t - \frac{c}{1-a})a^h, \ \frac{\sigma_\delta^2(1-a^{2h})}{1-a^2})$.
- So our best point estimate for the future value h years from now is

$$\frac{c}{1-a} + (\kappa_t - \frac{c}{1-a})a^h$$

  i.e. the current value κ_t times an exponentially decreasing weight plus a limit value to which it converges.
- The 95% confidence bounds are

$$\pm 1.96\sigma_\delta \frac{\sqrt{1-a^{2h}}}{\sqrt{1-a^2}}$$

  on top of that value. This equals $1.96\sigma_\delta$ for $h=1$ but converges to a fixed strictly positive value $1.96\sigma_\delta/\sqrt{1-a^2}$ for $h \to \infty$.

Time Series for the Dutch Population, AG2022

- Calibration of time series

$$\begin{aligned} K_{t+1}^{g} &= K_t^{g} + \theta^{g} + \epsilon_{t+1}^{g}, \\ \kappa_{t+1}^{g} &= a^{g}\,\kappa_t^{g} + c^{g} + \delta_{t+1}^{g}, \end{aligned}$$

uses $Y_t = X_t\Theta + Z_t$, $Z_t \sim N(0_4, C)$ iid, for

$$Y_t = \begin{bmatrix} K_{t+1}^m - K_t^m \\ K_{t+1}^v - K_t^v \\ \kappa_{t+1}^m \\ \kappa_{t+1}^v \end{bmatrix}, \quad X_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_t^m & 0 & 1 & 0 \\ 0 & 0 & 0 & \kappa_t^v & 0 & 1 \end{bmatrix}, \quad \Theta = \begin{bmatrix} \theta^m \\ \theta^v \\ a^m \\ a^v \\ c^m \\ c^v \end{bmatrix}, \quad Z_t = \begin{bmatrix} \epsilon_{t+1}^m \\ \epsilon_{t+1}^v \\ \delta_{t+1}^m \\ \delta_{t+1}^v \end{bmatrix}.$$

- Maximal likelihood estimate for parameters $(\hat{\Theta}, \ \hat{C})$ based on observations $\{(Y_t, X_t) : \ 1 \leq t \leq T-1\}$ is

$$(\hat{\Theta}, \hat{C}) = \underset{(\Theta, C)}{\arg\max} \ \ln \prod_{t=1}^{T-1} \left((2\pi|C|)^{-1/2} \exp[-\tfrac{1}{2}(Y_t - X_t\Theta)' C^{-1} (Y_t - X_t\Theta)] \right).$$

Key References

- Barrieu et al. (2010), Understanding, modelling and managing longevity risk: key issues and main challenges, *Scandinavian Actuarial Journal* 3, 203-231.
- Brouhns, N., Denuit, M., and Vermunt, J.K. (2002) A Poisson log-bilinear regression approach to the construction of projected lifetables. *Insurance: Mathematics and Economics* 31(3), 373-393.
- Koninklijk Actuarieel Genootschap (2022) Projection Table AG2022.
- Lee, R.D. and Carter, L.R. (1992). Modeling and forecasting U.S. mortality. *Journal of the American Statistical Association* 87(419), 659-675.
- Li, N. and Lee, R. (2005) Coherent mortality forecasts for a group of populations: An extension of the Lee-Carter method. *Demography* 42(3), 575-594.
- Pitacco et al. (2009) Modelling Longevity Dynamics for Pensions and Annuity Business, Oxford University Press.