Modelling and quantifying mortality and longevity risk

Module D2 : Pricing Longevity Risk

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“Pricing” of longevity risk (and “derivatives”)

Pricing Longevity Risk

- Most insurers / pension funds are short macro longevity.
- Macro Longevity risk cannot be diversified away in similar products if it is consistent across ages. It is therefore a risk which should be compensated in pricing.
- If no (liquid) market for longevity products exist, it is impossible to use riskneutral pricing methodology.
- Since macro longevity risk cannot be neutralized, universal fair price cannot be obtained so it must be based on risk appetite:
  - Use explicit model for risk aversion (eg. utility indifference pricing)
  - Use actuarial premium principle (eg. standard deviation)
  - Use asset performance criterion (eg. Sharpe ratio).

Pricing Longevity Risk: Insurance Company

From point of view of protection providing insurance company:

- Longevity swap risk is based on shock to mortality rates
- This translates into higher capital requirements (SCR, EC) which are based on mortality shocks, other shocks and their correlations
- If more capital needs to be set aside this translates, using a cost-of- capital method, into higher risk margins
- Risk premium charged should compensate total costs. Price depends on all elements in red above.

Pricing Longevity Risk: Investors

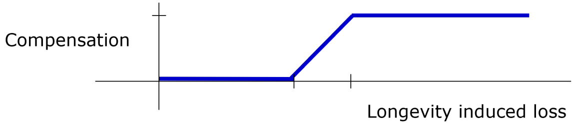
From point of view of protection provider (if not short longevity himself):

- May be interested in earning risk premium on an asset class which is usually uncorrelated with other asset classes
- Pricing principles 'supply side':
  - Specify Sharpe ratio:
    excess rate of return per standard deviation, often quoted: 25% (Sharpe ratio equity)
  - Specify transformed probabilities
    (from best estimates of probabilities), to include risk premium
    - By defining a Wang transform
      may introduce arbitrage, strongly discouraged
    - By defining a market price of diffusion risk
      can be done in consistent, arbitrage-free way

Pricing Longevity Risk: Insurance Company

- Two parties may not agree on uncertainty in, or even best estimates of, mortality rates. If they don't, longevity swap basically becomes a 'model bet'
- Diversification effects mean that pension fund will not have the full exposure to mortality shock, which reduces critical risk premium
- If pension fund just wants protection against extreme longevity scenario's, it may be more natural to look at a longevity collar, paying

$$\min(P_{\max},\ \max(0, {}_kp_{65} - {}_k\hat{p}_{65} - P_{\min})$$



Future liquidity of Longevity Swaps

- Most successful derivative structures in the market place
  - are not ‘model bets’ or ‘parameter bets’ (short term profit),
  - but are repeatable under similar circumstances (long term reduction of uncertainty).
- Example: commodity futures.
- Important in pension context:
  - How well can off-setting effects of mortality and its hedge be communicated to participants in adverse scenarios?
  - This is extra difficult due to structure of swap product.

Future liquidity of Longevity Swaps

- Is it really a derivative, or a form of (re-)insurance ?
- Compare (classification of) credit default swaps.
- No tradeable underlying so (almost) no asset pricing theory ...

Utility Indifference Pricing

- Economic agent with wealth W needs to consider whether he/she would like to avoid an immediate stochastic payoff $-X$ with $\mathbb{E}X = 0$ for a certain price P. Since X adds risk without return, a risk averse agent is willing to pay to avoid this risk. The maximal price P the agent would be willing to pay should satisfy

$$U(W - P) = \mathbb{E}U(W - X).$$

for a utility function U.

Utility Indifference Pricing

- Taylor Expansion of $U(W-P) = \mathbb{E}U(W-X)$ gives

$$U(W) - PU'(W) + o(P) = \mathbb{E}[U(W) - XU'(W) + \tfrac{1}{2}X^2U''(W) + o(X^2)]$$

  so

$$P \approx -\tfrac{1}{2}\frac{U''(W)}{U'(W)}\mathbb{E}X^2 \;=\; \tfrac{1}{2}r_U(W)\mathrm{Var}X$$

  with $r_U(x) = -U''(x)/U'(x)$ the Arrow-Pratt measure of absolute risk aversion.
- For HARA class it has the form

$$r_U(x) = \frac{1}{a+bx}$$

  for certain a and b.
- Not utility function but risk aversion function is natural representation of risk attitude of an agent.

Utility Indifference Pricing

- For exponential utility function $U(x) = -e^{-\gamma x}/\gamma$ (with $\gamma > 0$), we find for current wealth W, risky payoff X and indifference premium P

$$U(W-P) = \mathbb{E}U(W-X) \qquad \Rightarrow \qquad P = \frac{\ln \mathbb{E}[e^{\gamma X}]}{\gamma},$$

  so premium (and risk aversion) do not depend on wealth W.

- For γ very close to zero we can use a second order moment approximation for exponential utility:

$$P \;\approx\; \mathbb{E}[X] + \tfrac{1}{2}\gamma \mathrm{Var}[X].$$

- For general utility functions we find:

$$P \;=\; W - U^{-1}(\mathbb{E}U(W-X)).$$

A Portfolio of Annuities

- As an example, we use the exponential utility second-order moment approximation to analyze longevity risk in a portfolio.
- Take M policyholders indicated by $i \in \{1, 2, \dots M\}$ with ages $x(i)$ who are alive today and die at the stochastic time T_i. Then discounted value of all annual annuity payments (of 1 euro) to policyholder i equals

$$A(i) \; := \; \sum_{k=1}^{\omega} d_{0,k} \mathbf{1}_{T_i > k}$$

  where $\omega = 120$ is enough to make sure no one survives after ω yearly periods.
- Value of whole annuity portfolio is thus

$$A \; := \; \sum_{i=1}^{M} A(i) \; = \; \sum_{i=1}^{M} \sum_{k=1}^{\omega} d_{0,k} \mathbf{1}_{T_i > k}.$$

  We assume that discount rates $d_{0,k}$ are deterministic here.

Systematic and Non-Systematic Longevity Risk

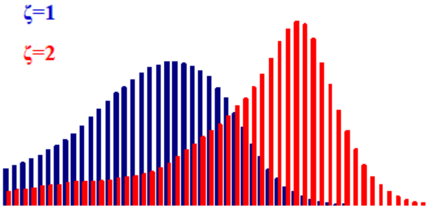
- Now assume that survival probabilities are stochastic themselves: they depend on a stochastic scenario ζ, so

$$\mathbb{P}(T_i > k|\zeta) \ := \ {}_k p_{x(i)}(\zeta).$$

- We do not yet know which mortality scenario will materialize, only that there is a distribution

$$\mathbb{P}(\zeta = j) = \tilde{p}_j$$

for possible scenarios $j \in \{1, 2, ..., J\}$ with probabilities $\tilde{p}_j$ that sum to one.



Implied distribution of age at death, under two scenarios.

Systematic and Non-Systematic Longevity Risk

- If we use our second order moment approximation for exponential indifference premiums, we should charge policyholder i a premium $P(i)$ such that

$$
\begin{aligned}
P = \sum_{i=1}^{M} P(i) &= \mathbb{E}[\sum_{i=1}^{M} A(i)] + \tfrac{1}{2}\gamma \,\mathrm{Var}[\sum_{i=1}^{M} A(i)] \\
&= \mathbb{E}[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d_{0,k}\mathbf{1}_{T_i>k}] + \tfrac{1}{2}\gamma \,\mathrm{Var}[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d_{0,k}\mathbf{1}_{T_i>k}] \\
&= \sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} \mathbb{E}[{}_k p_{x(i)}(\zeta)] + \tfrac{1}{2}\gamma \,\mathrm{Var}[\sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} \mathbf{1}_{T_i>k}]
\end{aligned}
$$

- First term is simply expectation over all longevity scenarios.
- In second term we have interaction of systematic longevity risk (variance due to unknown scenario ζ, 'macro') and idiosyncratic longevity risk (variance due to individual deaths given the scenario, 'micro').

Variance Decomposition Rule

- We can decompose the risk premium component in the variance using the following general rule for any stochastic variables X and Y:

$$\mathrm{Var}[X] \;=\; \mathbb{E}[\mathrm{Var}[X|Y]] + \mathrm{Var}[\mathbb{E}[X|Y]]$$

- This rule follows easily when we write

$$\begin{aligned}\mathbb{E}[\mathrm{Var}[X|Y]] &= \mathbb{E}[\ \mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2\ ] \\ \mathrm{Var}[\mathbb{E}[X|Y]] &= \mathbb{E}[\ \mathbb{E}[X|Y]^2\ ] - (\mathbb{E}[\mathbb{E}[X|Y])^2\end{aligned}$$

since summation then gives

$$\mathbb{E}(\ \mathbb{E}[X^2|Y]\ ) - (\mathbb{E}[X])^2 \;=\; \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \;=\; \mathrm{Var}[X].$$

Decomposition of Longevity Risk Premia

- We decompose the variance term for the premium, using this rule for $Y = \zeta$:

$$\begin{aligned}\tfrac{1}{2}\gamma \mathrm{Var}[\sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} \mathbf{1}_{T_i > k}] &= \tfrac{1}{2}\gamma \mathbb{E}[\mathrm{Var}[\sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} \mathbf{1}_{T_i > k} \,|\zeta]\,] \\ &\quad + \tfrac{1}{2}\gamma \mathrm{Var}[\,\mathbb{E}[\sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} \mathbf{1}_{T_i > k} \,|\zeta]\,]\end{aligned}$$

- If policyholder deaths are independent for a given survival table ζ, we find that this equals

$$\sum_{i=1}^{M} \tfrac{1}{2}\gamma \mathbb{E}[\mathrm{Var}[\sum_{k=1}^{\omega} d_{0,k} \mathbf{1}_{T_i > k} \,|\zeta]\,] + \tfrac{1}{2}\gamma \mathrm{Var}[\,\sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} {}_k p_{x(i)}(\zeta)\,].$$

Decomposition of Longevity Risk Premia

$$\sum_{i=1}^{M} \tfrac{1}{2}\gamma \mathbb{E}[\mathrm{Var}[\sum_{k=1}^{\omega} d_{0,k} \mathbf{1}_{T_i>k} \;|\zeta]\;] + \tfrac{1}{2}\gamma \mathrm{Var}[\;\sum_{k=1}^{\omega} d_{0,k} \sum_{i=1}^{M} {}_k p_{x(i)}(\zeta)\;]$$

Note:

- First part of premium (average over all scenarios for variance in payoff of policyholder i) can be attributed to individual policyholder.
- Second part (extra variance due to joint dependence on longevity trends) does not diversify and must be spread over individuals in the collective!

Key References

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