Modelling and quantifying mortality and longevity risk

Module A.1 on Basic concepts of lifetime distributions, mortality data and visualisations

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Outline

Learning outcomes of this module

In this module you will learn:

- how to define, denote and understand basic concepts related to lifetime data, relevant for the course
- how basic demographic markers evolve / have evolved as a function of age and time.

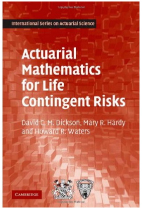
In this module we take a so-called period approach and consider the lifetime distribution or life table applicable to a specific period (e.g., a year).

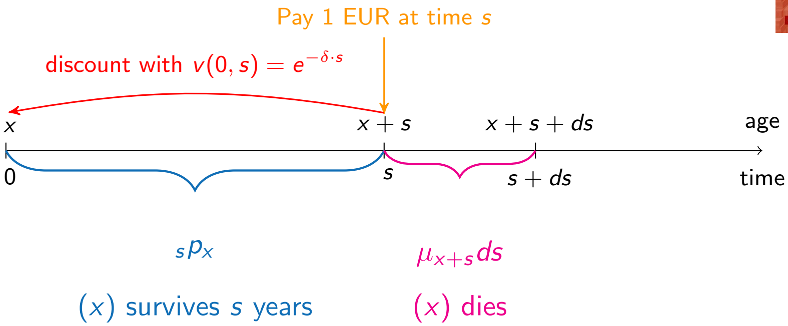
We'll switch to cohort (or: dynamic) thinking from Module A.2 on.

Motivation

Motivation

Life insurance mathematics 101 - whole life insurance (continuous)

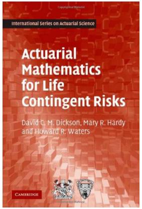


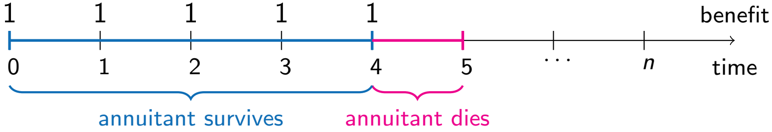


What is the Actuarial Present Value (or Expected Present Value) of this cashflow?

Motivation

Life insurance mathematics 101 - whole life annuity-due (discrete)





What is the Actuarial Present Value (or Expected Present Value) of this cashflow?

Motivation

Life insurance mathematics 101

A (detailed) recap of relevant actuarial value formulas for life insurance and life annuity products will follow later on.

These introductory examples motivate the need to master lifetime (or survival) distributions and related concepts when valuing life-contingent risks:

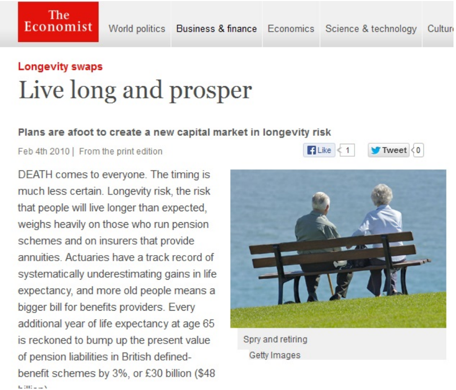
- T_x and $K_x = \lfloor T_x \rfloor$, and their cdf, pdf, survival function, expected value
- survival probabilities, e.g, ${}_sp_x$
- death probabilities, e.g., ${}_sq_x$
- ...

Motivation

Longevity risk

'Longevity risk, the risk that people will live longer than expected, weighs heavily on those who run pension schemes and on insurers that provide annuities. Actuaries have a track record of systematically underestimating gains in life expectancy, and more old people means a bigger bill for benefits providers.'

(The Economist, *Live long and prosper*, February 4, 2010.)



The Economist | World politics | Business & finance | Economics | Science & technology | Cultur

Longevity swaps

Live long and prosper

Plans are afoot to create a new capital market in longevity risk

Feb 4th 2010 | From the print edition

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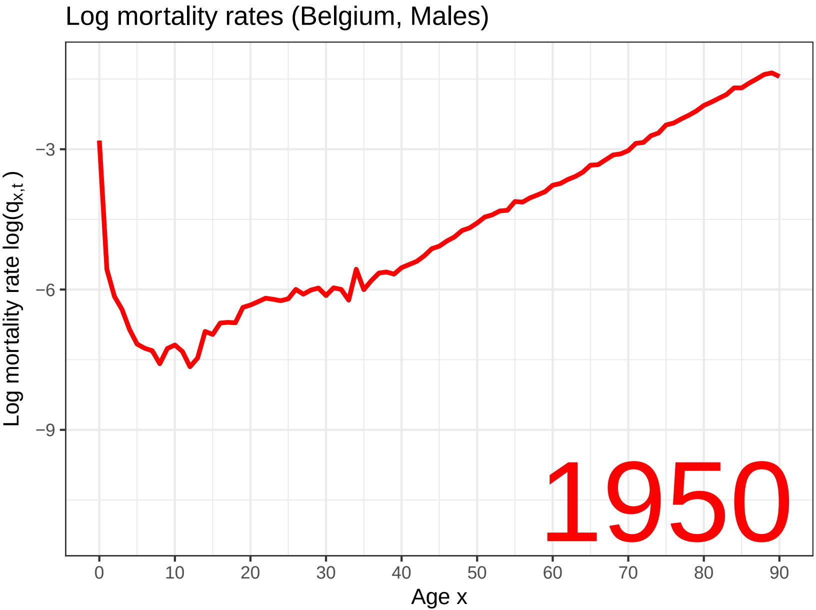
DEATH comes to everyone. The timing is much less certain. Longevity risk, the risk that people will live longer than expected, weighs heavily on those who run pension schemes and on insurers that provide annuities. Actuaries have a track record of systematically underestimating gains in life expectancy, and more old people means a bigger bill for benefits providers. Every additional year of life expectancy at age 65 is reckoned to bump up the present value of pension liabilities in British defined-benefit schemes by 3%, or £30 billion ($48

Spry and retiring

Getty Images


Log mortality rates (Belgium, Males)
Log mortality rate $\log(q_{x,t})$
−3
−6
−9
0
10
20
30
40
50
60
70
80
90
Age x
1950

Basic concepts of lifetime distributions and mortality data

The random future lifetime

- Let T_x be the remaining random lifetime for a person aged x.
- Connection between T_x and T_0:

$$P(T_x > t) \quad = \quad P(T_0 > x + t | T_0 > x) = \frac{P(T_0 > x + t)}{P(T_0 > x)}.$$

- A person alive at age x will die at the random age $x + T_x$.

- Survival and death probabilities:

$$ {}_tp_x = P(T_x > t) = P(T_0 > t + x | T_0 > x), $$

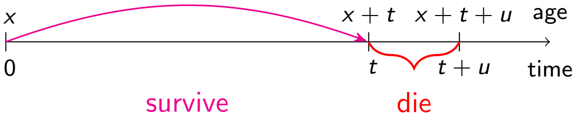
and

$$ {}_tq_x = 1 - {}_tp_x = P(T_x \leq t) = P(T_0 \leq x + t | T_0 > x). $$

${}_t\boldsymbol{p}_{\boldsymbol{x}}$ is the probability that an individual aged x will survive to age $x + t$.

${}_t\boldsymbol{q}_{\boldsymbol{x}}$ is the probability that an individual aged x will die before reaching age $x + t$.

${}_{t|u}q_x$ = probability that (x) will die between the ages of $x+t$ and $x+t+u$.



(x) survives for t years, but will die before age $x+t+u$:

$${}_{t|u}q_x = {}_tp_x \cdot {}_uq_{x+t}.$$

- Throughout our modules we will use the 'exact age' definition of q_x. (Some more details will follow in Module A.2.)

- Therefore:

  $q_x := {}_1q_x$ is the probability that an individual aged exactly x will die before reaching age $x+1$.

- The survival function $S_0(t)$ of T_0 and $S_x(t)$ of T_x:

$$
\begin{aligned}
S_0(t) &= P(T_0 > t), \\
S_x(t) &= P(T_x > t) = {}_t p_x .
\end{aligned}
$$

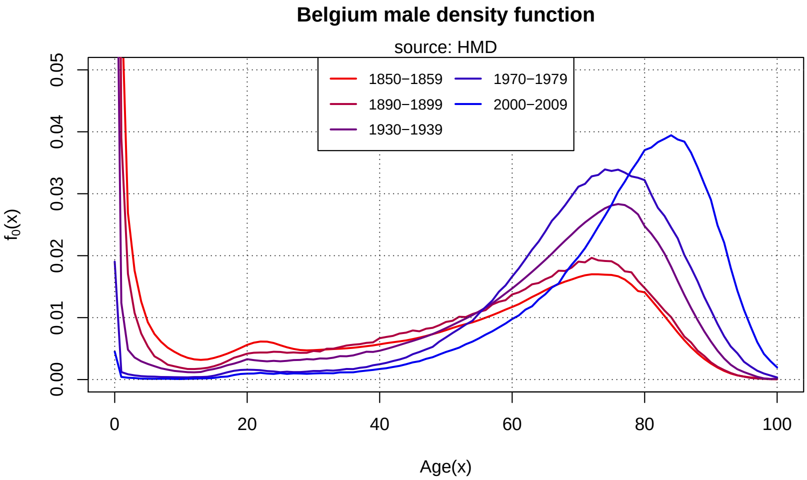
- And also:

  $f_0(t)$, $f_x(t)$, the probability density function of the future lifetime distribution of a 0 year old, respectively x year old, evaluated in t

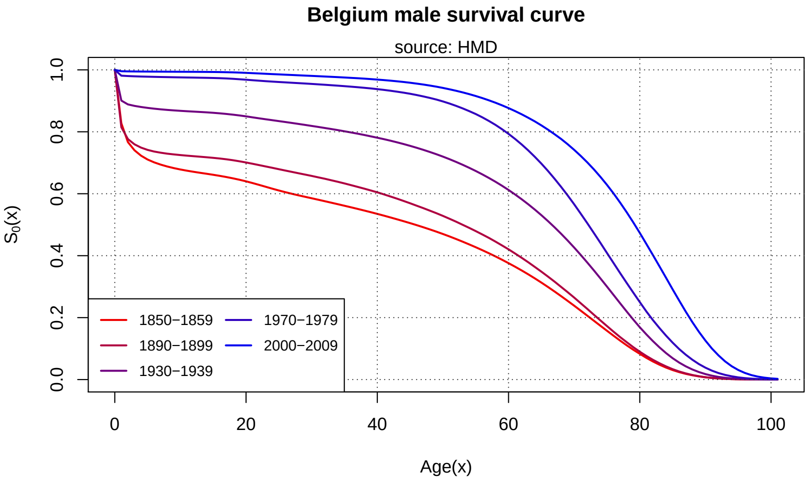
  $F_0(t) = Pr(T_0 \leq t)$ and $F_x(t) = Pr(T_x \leq t)$ for the cumulative distribution function of the future lifetime distribution of a 0 year old, respectively an x year old.

Mortality trends and visuals

Rectangularization and expansion



Rectangularization and expansion



We find

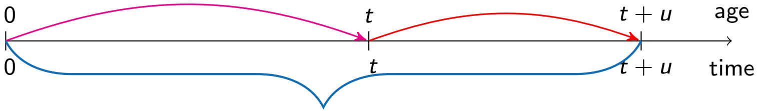
$$\begin{aligned} S_0(t+u) &= \mathrm{P}(T_0 > t+u) \\ &= \mathrm{P}(T_0 > t) \cdot \mathrm{P}(T_0 > t+u \,|\, T_0 > t) \\ &= \mathrm{P}(T_0 > t) \cdot \mathrm{P}(T_t > u) \\ &= S_0(t) \cdot S_t(u). \end{aligned}$$

Similarly,

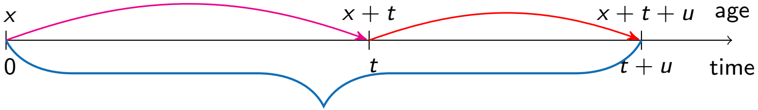
$$S_x(t+u) = S_x(t) \cdot S_{x+t}(u).$$

Multiplication rule for survival probabilities

Pictured



$$S_0(t+u) = S_0(t) \cdot S_t(u)$$



$$S_x(t+u) = S_x(t) \cdot S_{x+t}(u)$$

Your turn:

use the multiplication rule for survival probabilities to express ${}_kp_x$, the k–year survival probability (with $k \in \mathbb{N}$), as a function of 1–year survival probabilities $(p_x, \; p_{x+1}, \; \ldots)$

(recall: in this module we apply period thinking).

The force of mortality

- Force of mortality (or: hazard rate) at age x, μ_x:

$$\begin{aligned} \mu_x &= \mu_0(x) = \lim_{dx \to 0+} \frac{1}{dx} P[T_0 \le x + dx | T_0 > x] \\ &\Updownarrow \\ &= \lim_{dx \to 0+} \frac{1}{dx} P[T_x \le dx]. \end{aligned}$$

- Meaning of the highlighted components in this definition?
  - rate of event occurrence per unit of time
  - take limit $dx \to 0+ \Rightarrow$ instantaneous rate of occurrence.

- We obtain:

$$\begin{aligned}\mu_x &= \lim_{dx\to 0+} \frac{1}{dx} \frac{P[x < T_0 \leq x + dx]}{P(T_0 > x)} = \lim_{dx\to 0+} \frac{F_0(x+dx) - F_0(x)}{dx \cdot S_0(x)} \\ &= \frac{1}{S_0(x)} \frac{d}{dx} F_0(x) = \frac{1}{S_0(x)} \left\{ -\frac{d}{dx} S_0(x) \right\} = \frac{f_0(x)}{S_0(x)} = -\frac{d}{dx} \ln S_0(x).\end{aligned}$$

- Integrating over $(0, t)$ then yields:

$$\int_0^t \mu_x dx = -(\log S_0(t) - \log S_0(0)),$$

thus:

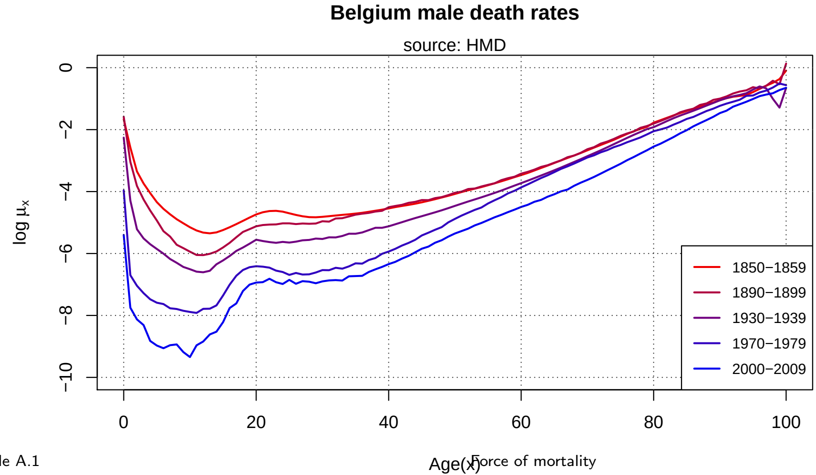
$${}_t p_0 = S_0(t) = \exp\left(-\int_0^t \mu_x dx \right).$$

This implies the following important expression:

$$
\begin{aligned}
{}_tp_x = S_x(t) &= \frac{S_0(x+t)}{S_0(x)} = \exp\left(-\int_x^{x+t} \mu_r dr\right) \\
&= \exp\left(-\int_0^t \mu_{x+s} ds\right).
\end{aligned}
$$

This is a general expression in statistics, connecting the survival function and the hazard rate!

Logarithm of the force of mortality, $\log(\mu_x)$, versus age (x) for Belgian males in four selected time periods.



- λ_x the expected number of survivors at age x (out of initial group λ_0), where

$$\lambda_{x+1} = \lambda_x \cdot (1 - q_x).$$

- d_x is the expected number of deaths at age x (out of initial group λ_0), where

$$d_x \;=\; \lambda_x - \lambda_{x+1} = \lambda_x \cdot q_x.$$

- E_x is the exposure to risk at age x, i.e. the expected total time lived by λ_x people between age x and $x+1$:

$$\lambda_x \int_0^1 {}_u p_x du \;=\; \int_0^1 \lambda_{x+u} du := E_x.$$

The central death rate at age x, denoted m_x, is then defined as a weighted average of the force of mortality over $[x, x+1)$:

$$
\begin{aligned}
m_x &= \frac{\int_0^1 S_0(x+u) \cdot \mu_{x+u} du}{\int_0^1 S_0(x+u) du} \\
&\vdots \\
&= \frac{q_x \cdot \lambda_x}{\lambda_x \int_0^1 \ {}_u p_x du} = \frac{d_x}{\lambda_x \int_0^1 \ {}_u p_x du} \\
&= \frac{d_x}{E_x}.
\end{aligned}
$$

- The life expectancy (or expected lifetime) for a newborn is:

$$E[T_0] = \int_0^{\infty} t \cdot f_0(t)dt = \int_0^{\infty} S_0(t)dt.$$

- So, the expected lifetime for an (x)-year old is:

$$E[T_x] = \int_0^{\infty} t \cdot f_x(t)dt = \int_0^{+\infty} S_x(t)dt = \frac{1}{S_0(x)} \int_0^{+\infty} S_0(x+t)dt.$$

Piecewise constant force of mortality

To calculate the life expectancy from a given life table with p_x for integer ages x, assume:

$$\mu_{x+t} \quad = \quad \mu_x \qquad \text{with } t \in [0, 1).$$

That is: the force of mortality is piecewise constant between two integer ages.

This is equivalent to

$$\begin{aligned}
{}_tp_x &= 1 - {}_tq_x = \exp\left(-\int_0^t \mu_{x+\tau} d\tau\right) \\
&= \exp\left(-\int_0^t \mu_x d\tau\right) \\
&= \exp(-t \cdot \mu_x) = \{\exp(-\mu_x)\}^t = (1 - q_x)^t = p_x^t,
\end{aligned}$$

for $t \in [0, 1]$.

Under this piecewise constant assumption we get:

$$\begin{aligned} E[T_x] &= \int_0^{+\infty} S_x(t)dt = \int_0^{+\infty} {}_t p_x dt = \sum_{k\geq 0} {}_k p_x \int_0^1 {}_t p_{x+k} dt \\ &= \sum_{k\geq 0} {}_k p_x \frac{p_{x+k}-1}{\ln p_{x+k}} \\ &= \frac{p_x - 1}{\ln p_x} + \sum_{k\geq 1} \left(\prod_{j=0}^{k-1} p_{x+j} \right) \frac{p_{x+k}-1}{\ln p_{x+k}}, \end{aligned}$$

where ${}_k p_x = \prod_{j=0}^{k-1} p_{x+j}$ and $p_x = \exp(-\mu_x)$.

Life expectancy

UDD

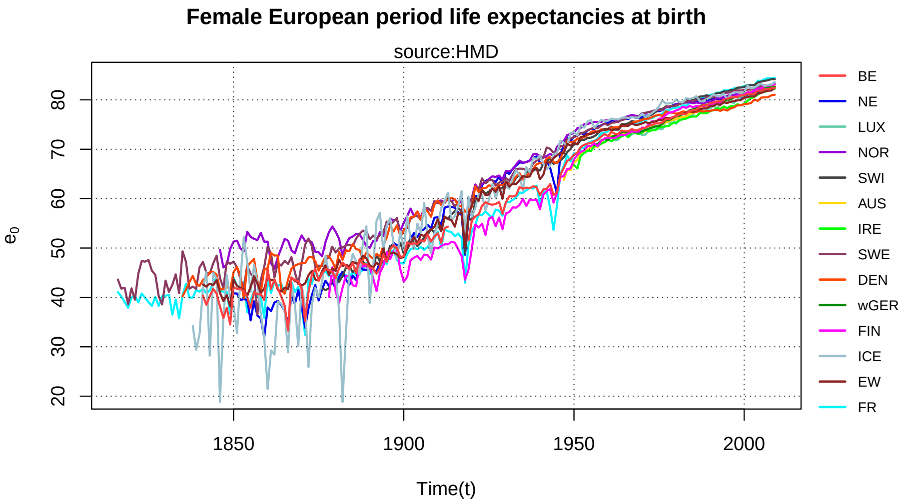
Alternatively, under the Uniform Distribution of Deaths (UDD):

$$\lambda_{x+t} \quad = \quad \lambda_x + t \cdot (\lambda_{x+1} - \lambda_x) \qquad t \in [0,1].$$

This assumption implies: ${}_tq_x = 1 - {}_tp_x = 1 - \frac{\lambda_{x+t}}{\lambda_x} = \frac{\lambda_x - \lambda_{x+t}}{\lambda_x} = t \cdot q_x$ for $t \in [0,1]$.

- Under the UDD assumption we find: (using $\sum_{k\geq 0} {}_kp_x \cdot q_{x+k} = 1$)

$$
\begin{aligned}
E[T_x] &= \int_0^{+\infty} {}_tp_x dt = \sum_{k\geq 0} {}_kp_x \int_{t=0}^{1} {}_tp_{x+k} dt \\
&= \sum_{k\geq 0} \int_{t=0}^{1} {}_kp_x \cdot (1 - t \cdot q_{x+k}) dt \\
&= \sum_{k\geq 0} {}_kp_x \left(1 - \frac{q_{x+k}}{2}\right) = -\frac{1}{2} + \sum_{k\geq 0} {}_kp_x .
\end{aligned}
$$


Female European period life expectancies at birth
source:HMD
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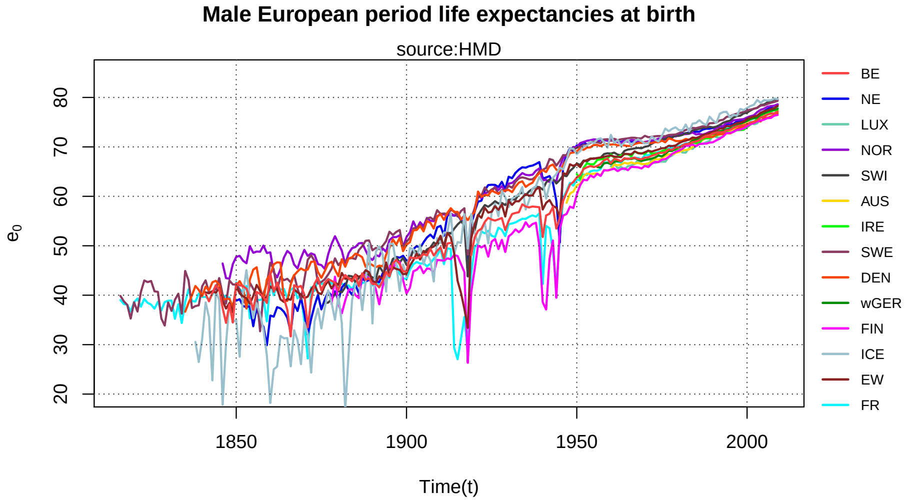
Male European period life expectancies at birth
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With constant interest rate, the Expected Present Value (EPV) of Actuarial Present Value (APV) of:

- a 1 EUR benefit whole life insurance product in continuous time

$$\bar{A}_x \quad = \quad E[e^{-\delta \cdot T_x}] = \int_0^{\infty} e^{-\delta \cdot t} \cdot \ {}_t p_x \cdot \mu_{x+t} dt$$

- a 1 EUR benefit whole life insurance product in discrete time

$$A_x \quad = \quad E[v^{K_x+1}] = \sum_{k=0}^{+\infty} v^{k+1} \ {}_{k|}q_x = v \cdot q_x + v^2 \cdot \ {}_{1|}q_x + v^3 \cdot \ {}_{2|}q_x + \ldots,$$

where ${}_{k|}q_x = \ {}_k p_x \cdot \ q_{x+k}$.

With constant interest rate, the Expected Present Value (EPV) of Actuarial Present Value (APV) of:

- a 1 EUR benefit whole life annuity in continuous time

$$\bar{a}_x = \int_0^{\infty} e^{-\delta t} \cdot \ {}_t p_x dt.$$

- a 1 EUR benefit whole life annuity in discrete time

$$\begin{aligned} \ddot{a}_x &= 1 + v \cdot p_x + v^2 \cdot \ {}_2 p_x + v^3 \cdot \ {}_3 p_x + \ldots \\ &= \sum_{k=0}^{\infty} v^k \cdot \ {}_k p_x. \end{aligned}$$

Wrap-up

Via this module you have acquired insights in:

- essential characteristics of the distribution of the random variable T_x, the future lifetime r.v. of an (x)–year old: q_x, p_x, μ_x, $E[T_x]$
- how to extract or calculate these characteristics from (the entries in) a given period life table
- qualitative and visual insights in the evolution of mortality over ages and periods.

So far, we took a so-called period approach and considered the lifetime distribution or life table applicable to a specific period (e.g., a year).