Modelling and quantifying mortality and longevity risk

Module A.2 on Single population stochastic mortality models

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Outline

Learning outcomes of this module

In this module you will learn

- how to reason along cohort (or: dynamic) instead of period thinking
- how actuarial lifetime data can be available at different levels of granularity
- how to build up the survival likelihood from individual lifetime data
- the Lee & Carter model: specification, intuition, calibration and time dynamics
- from Lee & Carter to a generic class of single population stochastic mortality models.

Motivation

In Module A.1 we gave a recap of the EPV expressions for basic life insurance and annuity products.

Clearly, the long-term horizon is distinctive for the valuation of life contingent risks.

Reflecting on possible evolutions in mortality rates is essential when valuing these types of risks.

Hereto, we will

- apply cohort thinking instead of the period thinking developed in Module A.1
- develop and use best estimates + scenarios for future mortality rates.

Motivation

Survival probabilities: period vs cohort

Consider (x), a policyholder aged x in year t, with future lifetime $T_{x,t}$.

Period thinking:

$$ {}_k p_{x,t} \quad = \quad Pr(T_{x,t} \geq k) = p_{x,t} \cdot p_{x+1,t} \cdot \ldots \cdot p_{x+k-1,t} $$

Cohort thinking:

$$ {}_k p_{x,t} \quad = \quad Pr(T_{x,t} \geq k) = p_{x,t} \cdot p_{x+1,t+1} \cdot \ldots \cdot p_{x+k-1,t+k-1} $$

Need for mortality projections and scenarios (beyond most recent t) in the cohort approach!

The period life expectancy for an x year old in year t is

$$
\begin{aligned}
E^{\text{per}}[T_{x,t}] &= \frac{1-\exp\left(-\mu_{x,t}\right)}{\mu_{x,t}} \\
&+ \sum_{k\geq 1}\left(\prod_{j=0}^{k-1}\exp\left(-\mu_{x+j,t}\right)\right)\frac{1-\exp\left(-\mu_{x+k,t}\right)}{\mu_{x+k,t}},
\end{aligned}
$$

cfr. the expression derived in Module A.1, under piecewise constant force of mortality, and $p_{x,t} = \exp\left(-\mu_{x,t}\right)$.

The cohort life expectancy for an x year old in year t is

$$
\begin{aligned}
E^{\text{coh}}[T_{x,t}] &= \frac{1-\exp\left(-\mu_{x,t}\right)}{\mu_{x,t}} \\
&+ \sum_{k\geq 1}\left(\prod_{j=0}^{k-1}\exp\left(-\mu_{x+j,t+j}\right)\right)\frac{1-\exp\left(-\mu_{x+k,t+k}\right)}{\mu_{x+k,t+k}}.
\end{aligned}
$$

Collecting and analyzing lifetime data

Throughout the modules and tutorials we will consider actuarial lifetime data being collected and processed in different ways:

- (M.A.1; T1) in (period) life tables, with entries (e.g.) λ_x, d_x, q_x
- (M.A.2-M.B.1; T1-2) number of deaths and corresponding exposure-to-risk, collected at population level per age x and year t combination $\Rightarrow$ to estimate $\mu_{x,t}$
- (M.B.3-M.C.2; T3) idem, but now collected at a finer level of granularity, e.g., weeks instead of years
- (M.C.1) as detailed event data on policyholders or pension plan members ($\sim$ seriatim data).

M is for Module and T for Tutorial.

The age-period survival likelihood

The age-period survival likelihood

- Given integer age x and calendar year or period t, we assume:

$$\mu_{x+\tau,t+\tau} \quad = \quad \mu_{x,t} \qquad\qquad 0 \leq \tau < 1,$$

  the piecewise constant force of mortality but now in the age-period setting.

- Using this assumption and $0 \leq \epsilon \leq 1$, we find:

$$_{\epsilon}p_{x,t} \quad = \quad \exp\left(-\int_0^{\epsilon} \mu_{x+\tau,t+\tau} d\tau\right) = \exp\left(-\epsilon \cdot \mu_{x,t}\right).$$

- Our goal is to put together a suitable likelihood function to estimate $\mu_{x,t}$, for every age x and period t.

- Consider n individuals alive with age in $[x, x+1)$, followed throughout period $[t, t+1)$.
- Denote for every individual $i \in \{1, \ldots, n\}$:

$$\delta_i = \begin{cases} 1 & \text{if individual } i \text{ dies at age } x \\ 0 & \text{otherwise.} \end{cases}$$

- Let τ_i be the fraction of the period lived by individual i, then

$$\sum_{i=1}^{n} \delta_i = d_{x,t} \quad \text{and} \quad \sum_{i=1}^{n} \tau_i = E_{x,t},$$

with $d_{x,t}$ the number of deaths at age x in period $[t, t+1)$ and $E_{x,t}$ the corresponding (central) exposure to risk.

- The contribution of individual i to the survival likelihood becomes:
  - ${}_{\tau_i}p_{x,t} = \exp\left(-\tau_i \cdot \mu_{x,t}\right)$ if the individual survives
  - ${}_{\tau_i}p_{x,t} \cdot \mu_{x+\tau_i,t+\tau_i} = \exp\left(-\tau_i \cdot \mu_{x,t}\right) \cdot \mu_{x,t}$ if the individual dies.
- Combined, we get:

$$\exp\left(-\tau_i \cdot \mu_{x,t}\right) \cdot \left(\mu_{x,t}\right)^{\delta_i}.$$

- Assuming independence between individuals, $\mu_{x,t}$ can be estimated from

$$\mathcal{L}(\mu_{x,t}) = \prod_{i=1}^{n} \exp\left(-\tau_i \cdot \mu_{x,t}\right) \cdot \left(\mu_{x,t}\right)^{\delta_i} = \exp\left(-E_{x,t} \cdot \mu_{x,t}\right) \cdot \left(\mu_{x,t}\right)^{d_{x,t}}.$$

- This likelihood reminds us of a Poisson assumption for the number of deaths r.v. $D_{x,t}$ (see Brouhns, Denuit & Vermunt, 2002, IME)

$$D_{x,t} \sim \text{POI}(E_{x,t} \cdot \mu_{x,t}),$$

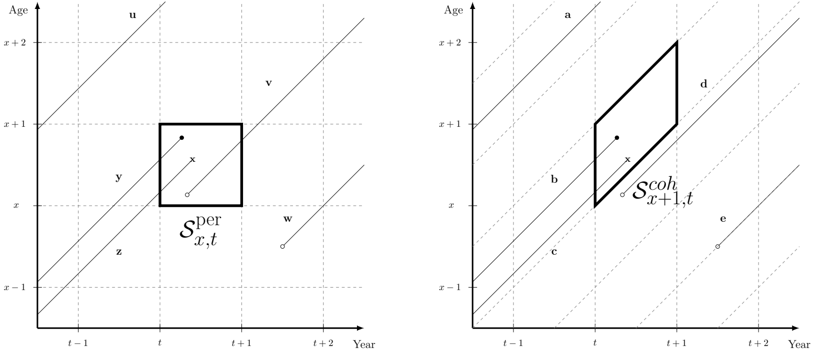
because

$$P(D_{x,t} = d_{x,t}) = \frac{(E_{x,t} \cdot \mu_{x,t})^{d_{x,t}}}{d_{x,t}!} \cdot \exp\left(-E_{x,t} \cdot \mu_{x,t}\right) \propto \exp\left(-E_{x,t} \cdot \mu_{x,t}\right) \cdot (\mu_{x,t})^{d_{x,t}}.$$

- The maximum likelihood estimate of $\hat{\mu}_{x,t}$ then becomes:

$$\hat{\mu}_{x,t} = \frac{d_{x,t}}{E_{x,t}} = m_{x,t},$$

the central death rate introduced in Module 1.



Be careful: different data sources may collect observations on $d_{x,t}$ and $E_{x,t}$ in different ways!

On the left: exact age or period age (as used by HMD). On the right: completed age or cohort age (as used, e.g., by CBS or StatBel). More details in Devriendt et al. (EAJ, 2017).

Single population stochastic mortality models: the Lee & Carter model

Our goal is to design a stochastic mortality projection model for a given population.

Examples of requirements imposed on such a model:

- capture trends over both ages as well as periods
- be able to generate future scenarios of mortality rates so that best estimates and intervals can be obtained
- be robust, time-consistent and biologically reasonable
- have good performance on in sample statistical measures as well as out-of-time back-tests
- (as industry standard or academic research) be reproducible and fully transparent.

More detailed discussion in Antonio, Devriendt et al. (2017, EAJ), KAG 2014, 2016, 2018, 2020 and 2022.

- We start from the seminal contribution by Lee & Carter (1992, JASA) who specify $m_{x,t}$, the central death rate, as:

$$\ln m_{x,t} \;=\; \beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t + \epsilon_{x,t}.$$

  The dependence on age is governed by the sequences of $\beta_x^{(1)}$'s and $\beta_x^{(2)}$'s, where $x \in \{1, \ldots, X\}$. The dependence on time by the κ_t's where $t \in \{1, \ldots, T\}$.

  The error terms $\epsilon_{x,t}$ (with mean 0 and variance σ_ϵ^2) reflect influences not captured by the model.

- Useful readings: Lee-Carter model and Girosi & King (2007).

- Identifiability issue? (See Nielsen & Nielsen, 2010.)
- Consider the set of parameters

$$\boldsymbol{\theta} \;=\; (\beta_1^{(1)}, \ldots, \beta_X^{(1)}, \beta_1^{(2)}, \ldots, \beta_X^{(2)}, \kappa_1, \ldots, \kappa_T).$$

- These are not identified without additional constraints, because for any scalar c and $d \neq 0$

$$\begin{aligned} E[\ln m_{x,t}] &= \beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t \\ &= (\beta_x^{(1)} - \beta_x^{(2)} c) + \frac{\beta_x^{(2)}}{d} \cdot d(\kappa_t + c) \\ &= \tilde{\beta}_x^{(1)} + \tilde{\beta}_x^{(2)} \cdot \tilde{\kappa}_t. \end{aligned}$$

Thus, the parametrizations via $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ are equivalent.

- Solution?

  We impose an identification scheme (or: constraints) when estimating the parameters.

  For example, with Lee & Carter:

$$
\begin{aligned}
\sum_x \beta_x^{(2)} &= 1 \\
\sum_t \kappa_t &= 0.
\end{aligned}
$$

The Lee & Carter stochastic mortality model

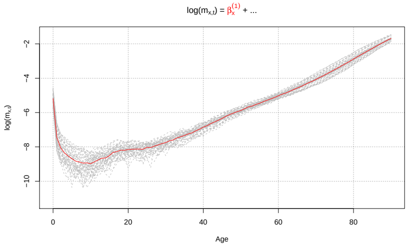
Interpretation - under the given constraints and ignoring the error terms

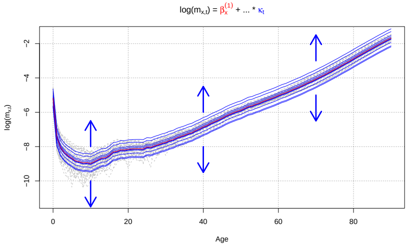
Let's try to give some intuition for the age and period-specific parameters:

$$\beta_x^{(1)} = \frac{1}{T}\sum_t \ln m_{x,t}.$$

$$\kappa_t = \sum_x \left(\ln m_{x,t} - \beta_x^{(1)}\right)$$

$$\frac{d}{dt}\ln m_{x,t} = \beta_x^{(2)}\frac{d}{dt}\kappa_t.$$





Interpretation - under the given constraints and ignoring the error terms

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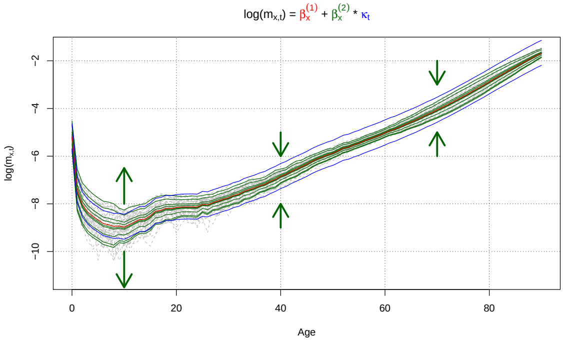
$$\kappa_t = \sum_x \left(\ln m_{x,t} - \beta_x^{(1)}\right).$$

$$\frac{d}{dt}\ln m_{x,t} = \beta_x^{(2)} \frac{d}{dt}\kappa_t.$$



- To calibrate the $\beta_x^{(1)}$, $\beta_x^{(2)}$ and κ_t parameters, recall our finding from the age–period survival likelihood derivation (see Brouhns, Denuit & Vermunt, 2002, IME):

$$D_{x,t} \sim \text{Poi}(E_{x,t} \cdot \mu_{x,t}),$$

  which holds for all ages x and periods t under consideration.

- We now impose the Lee & Carter specification on $\mu_{x,t}$, i.e.

$$\mu_{x,t} = \exp\left(\beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t\right),$$

  subject to a set of constraints, e.g., $\sum_x \beta_x^{(2)} = 1$ and $\sum_t \kappa_t = 0$.

All together, we estimate the $\beta_x^{(1)}$, $\beta_x^{(2)}$ (for $x \in \{1, \ldots, X\}$) and the κ_t (for $t \in \{1, \ldots, T\}$) parameters from the following Poisson likelihood:

$$\begin{aligned}
\mathcal{L}(\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\kappa} | \boldsymbol{d},\ \boldsymbol{E}) &= \prod_t \prod_x P(D_{xt} = d_{xt}) \\
&\Downarrow \\
L(\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\kappa} | \boldsymbol{d},\ \boldsymbol{E}) &= \sum_t \sum_x \left[d_{xt}(\beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t) - E_{xt} \exp\left(\beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t\right) \right] + c,
\end{aligned}$$

with c a constant that does not depend on the parameters in $\boldsymbol{\beta}^{(1)}$, $\boldsymbol{\beta}^{(2)}$, $\boldsymbol{\kappa}$.

Advantages:

- the integer character of $D_{x,t}$ is recognized.
- the Poisson law allows to get rid of the assumption of homoscedasticity.
- Maximum Likelihood Estimation can be used.
- the following balance property holds:

$$\sum_t d_{xt} \quad = \quad \sum_t E_{xt} \exp\left(\beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t\right),$$

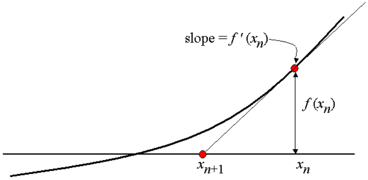
because $\frac{\partial}{\partial \beta_x^{(1)}} \log \mathcal{L}(\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\kappa}) = 0$.

- We take the derivative of the POI log-likelihood function wrt $\beta_x^{(1)}$, $\beta_x^{(2)}$ and κ_t $(\forall x, t)$ and put these equal to zero.

- We solve these equations using univariate Newton–Raphson steps:

  starting from x_0 the nth Newton–Raphson iteration step obtains x_{n+1} from x_n:

$$x_{n+1} \quad = \quad x_n - \frac{f(x_n)}{f'(x_n)}.$$



- We aim to estimate the $\beta_x^{(1)}$, $\beta_x^{(2)}$ and κ_t by maximizing the following log–likelihood:

$$L(\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\kappa}) \;=\; \sum_t \sum_x \left[d_{xt}(\beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t) - E_{xt} \exp\left(\beta_x^{(1)} + \beta_x^{(2)} \cdot \kappa_t\right) \right] + c.$$

- Hereto, we use the following Newton–Raphson steps:

$$\begin{aligned}
\hat{\beta}_x^{(1)(k+1)} &= \hat{\beta}_x^{(1)(k)} - \frac{\sum_t \left[d_{xt} - e_{xt} \exp\left(\hat{\beta}_x^{(1)(k)} + \hat{\beta}_x^{(2)(k)} \hat{\kappa}_t^{(k)}\right) \right]}{-\sum_t e_{xt} \exp\left(\hat{\beta}_x^{(1)(k)} + \hat{\beta}_x^{(2)(k)} \hat{\kappa}_t^{(k)}\right)} \\
\hat{\kappa}_t^{(k+1)} &= \hat{\kappa}_t^{(k)} - \frac{\sum_x \left[d_{xt} - e_{xt} \exp\left(\hat{\beta}_x^{(1)(k+1)} + \hat{\beta}_x^{(2)(k)} \hat{\kappa}_t^{(k)}\right) \right] \hat{\beta}_x^{(2)(k)}}{-\sum_x e_{xt} \exp\left(\hat{\beta}_x^{(1)(k+1)} + \hat{\beta}_x^{(2)(k)} \hat{\kappa}_t^{(k)}\right) \left(\hat{\beta}_x^{(2)(k)}\right)^2} \\
\hat{\beta}_x^{(2)(k+1)} &= \hat{\beta}_x^{(2)(k)} - \frac{\sum_t \left[d_{xt} - e_{xt} \exp\left(\hat{\beta}_x^{(1)(k+1)} + \hat{\beta}_x^{(2)(k)} \hat{\kappa}_t^{(k+1)}\right) \right] \hat{\kappa}_t^{(k+1)}}{-\sum_t e_{xt} \exp\left(\hat{\beta}_x^{(1)(k+1)} + \hat{\beta}_x^{(2)(k)} \hat{\kappa}_t^{(k+1)}\right) \left(\hat{\kappa}_t^{(k+1)}\right)^2}.
\end{aligned}$$

The Lee & Carter model

Calibration strategy - more details

- If the estimates for $\beta_x^{(1)}$, $\beta_x^{(2)}$ and κ_t do not satisfy the identifiability constraints, we transform the calibrated parameters as follows:

$$
\begin{aligned}
\widetilde{\beta_x^{(1)}} &\leftarrow \widehat{\beta_x^{(1)}} + \widehat{\beta_x^{(2)}}\bar{\kappa} \\
\widetilde{\kappa}_t &\leftarrow (\widehat{\kappa}_t - \bar{\kappa})\widehat{\beta_\bullet^{(2)}} \\
\widetilde{\beta_x^{(2)}} &\leftarrow \widehat{\beta_x^{(2)}} / \widehat{\beta_\bullet^{(2)}},
\end{aligned}
$$

with:

- $\widehat{\beta_\bullet^{(2)}}$ the sum (over x) of the $\widehat{\beta_x^{(2)}}$'s
- $\bar{\kappa}$ the average (over t) of the $\hat{\kappa}_t$'s.

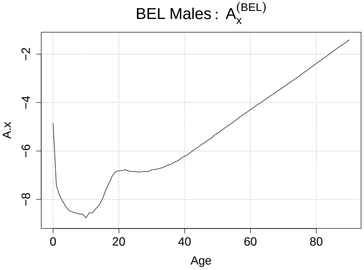
With the calibrated $\boldsymbol{\beta}^{(1)}$, $\boldsymbol{\beta}^{(2)}$ and $\boldsymbol{\kappa}$, we obtain

$$\hat{\mu}_{xt} \quad = \quad \exp\left(\hat{\beta}_x^{(1)} + \hat{\beta}_x^{(2)} \cdot \hat{\kappa}_t\right)$$

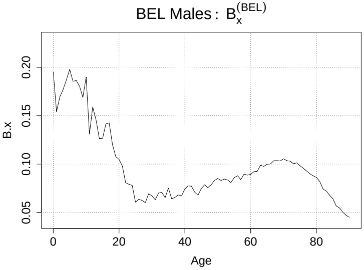
and also

$$\begin{aligned} \hat{q}_{xt} = 1 - \hat{p}_{xt} \quad &= \quad 1 - \exp\left(-\hat{\mu}_{xt}\right) \\ &= \quad 1 - \exp\left\{-\exp\left(\hat{\beta}_x^{(1)} + \hat{\beta}_x^{(2)} \cdot \hat{\kappa}_t\right)\right\}, \end{aligned}$$

under the assumption of piecewise constant force of mortality.



BEL Males : $A_x^{(BEL)}$
A.x
−2
−4
−6
−8
0
20
40
60
80
Age



BEL Males : $B_x^{(BEL)}$
B.x
0.20
0.15
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20
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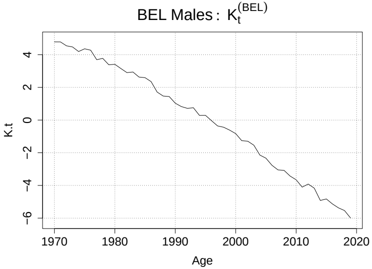
Age



BEL Males : $K_t^{(BEL)}$
K.t
4
2
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−2
−4
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1970
1980
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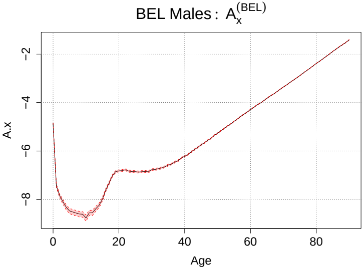
2000
2010
2020
Age



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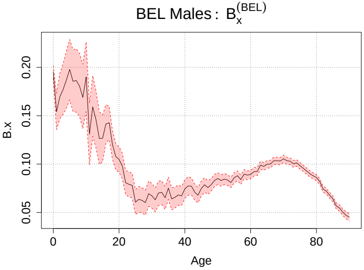

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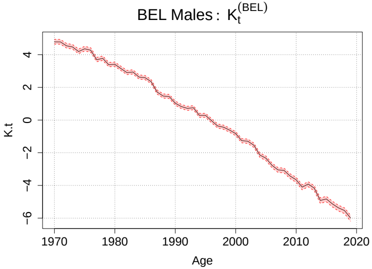

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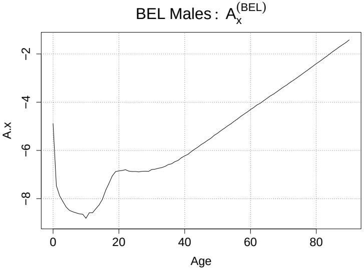
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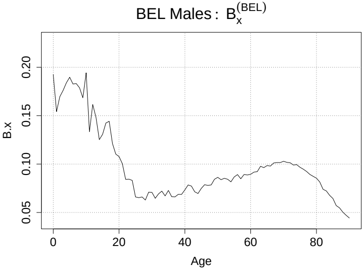
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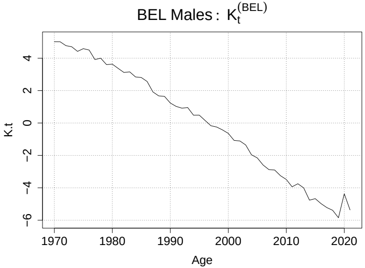
BEL Males : $K_t^{(BEL)}$
K.t
4
2
0
−2
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1970
1980
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Age

- Use ARIMA toolbox to find a suitable time series model for (the calibrated) κ_t.
- With the Lee-Carter model, typically the random walk with drift is used, i.e. ARIMA(0,1,0).
- Thus,

$$\begin{aligned}\hat{\kappa}_t &= \hat{\kappa}_{t-1} + \theta + \epsilon_t \\ \epsilon_t &\sim N(0, \sigma^2),\end{aligned}$$

where θ is the drift parameter, ϵ_t are the i.i.d. error terms and σ^2 their common variance.

- Consider the RWD for the calibrated κ_t

$$\begin{aligned} \kappa_t &= \kappa_{t-1} + \theta + \epsilon_t \\ \epsilon_t &\sim N(0, \sigma^2). \end{aligned}$$

- This implies that h time units into the future:

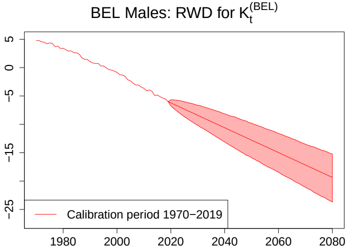
$$\kappa_{t+h} = \kappa_t + h \cdot \theta + \sum_{s=0}^{h-1} \epsilon_s$$

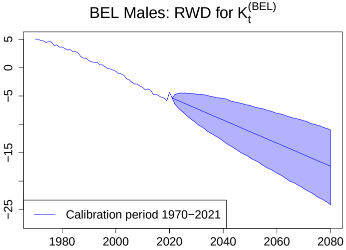
with

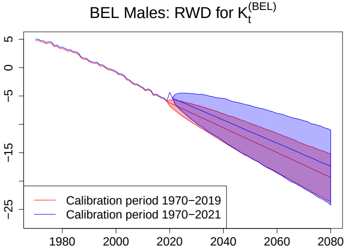
$$E(\kappa_{t+h}) = \kappa_t + h \cdot \theta \text{ and } \mathrm{Var}(\kappa_{t+h}) = h \cdot \sigma^2.$$

The Lee & Carter model

Projecting the time series of calibrated κ_t, Belgian males

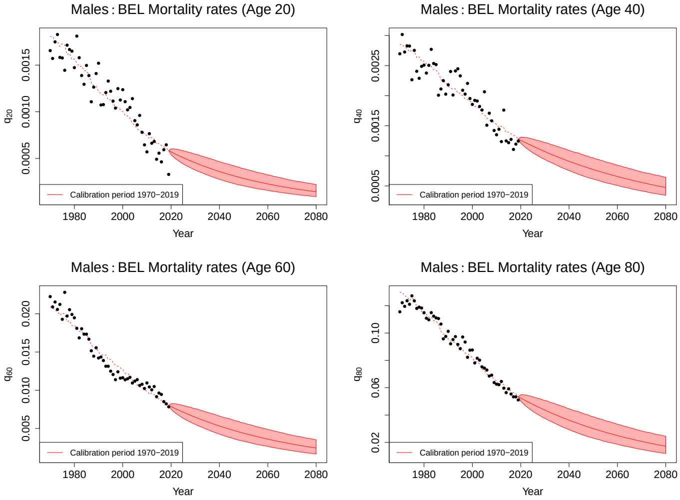






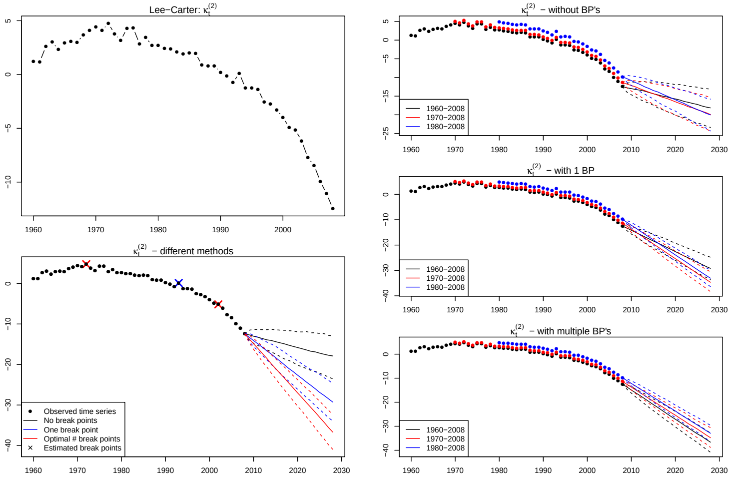
The Lee & Carter model

Projections of $q_{x,t}$ for $x \in \{20, 40, 60, 80\}$, Belgian males, 1970 - 2019



The Lee & Carter model

Sensitivity wrt calibration period - Van Berkum, Antonio & Vellekoop (2016, SAJ)



Single population stochastic mortality models: a generic class of models

Single population mortality models

Generic framework

A generic class of models ('LifeMetrics models', cfr. work of prof. Andrew Cairns):

$$\log \mu_{x,t} \;=\; \beta_x^{(1)}\kappa_t^{(1)}\gamma_{t-x}^{(1)} + \ldots + \beta_x^{(N)}\kappa_t^{(N)}\gamma_{t-x}^{(N)}$$

or

$$\text{logit } q_{x,t} \;=\; \beta_x^{(1)}\kappa_t^{(1)}\gamma_{t-x}^{(1)} + \ldots + \beta_x^{(N)}\kappa_t^{(N)}\gamma_{t-x}^{(N)}$$

where

- $\beta_x^{(k)}$ = age effect for component k
- $\kappa_t^{(k)}$ = period effect for component k
- $\gamma_{t-x}^{(k)}$ = cohort effect for component k.

M1	Lee & Carter (1992)	$\log \mu_{x,t} = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)}$
M2	Renshaw & Haberman (2006)	$\log \mu_{x,t} = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)}$
M3	Currie (2006)	$\log \mu_{x,t} = \beta_x^{(1)} + n_a^{-1}\kappa_t^{(2)} + n_a^{-1}\gamma_{t-x}^{(3)}$
M5	Cairns-Blake-Dowd (CBD, 2006)	$\text{logit } q_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$
M6	CBD(2), with cohort effect	$\text{logit } q_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}$
M7	CBD(3), quadratic age	$\text{logit } q_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \kappa_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}_x^2)$ $+\gamma_{t-x}$
M8	CBD(4), cohort effect $\searrow$ over time	$\text{logit } q_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}(x_c - x)$

With n_a number of ages considered, x_c and $\hat{\sigma}_x^2$ the mean and variance of the ages in the considered age range.

Be careful: CBD models (M5-M8) specifically designed for pensioner ages.

M9	Plat (2009)	$\text{logit } q_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + (x-\bar{x})\kappa_t^{(3)} + (\bar{x}-x)_+\kappa_t^{(4)} + \gamma_{t-x}^{(3)}$
M10	Haberman & Renshaw (2011)	$\text{logit } q_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + (\bar{x}-x)\kappa_t^{(3)} + (\bar{x}-x)_+\kappa_t^{(4)}$
M11	Haberman & Renshaw (2011)	$\text{logit } q_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + (\bar{x}-x)\kappa_t^{(3)} + (\bar{x}-x)_+\kappa_t^{(4)} + b(x)\kappa_t^{(5)} + \gamma_{t-x}^{(3)}$
M12	Haberman & Renshaw (2011)	$\text{logit } q_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + (\bar{x}-x)\kappa_t^{(3)} + (\bar{x}-x)_+\kappa_t^{(4)} + (x_c-x)\gamma_{t-x}^{(3)}$
M13	O'Hare & Li (2011)	$\text{logit } q_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + (\bar{x}-x)\kappa_t^{(3)} + c(x)\kappa_t^{(4)} + \gamma_{t-x}^{(3)}$
M14	Borger et al. (2014)	$\text{logit } q_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + (x-\bar{x})\kappa_t^{(3)} + (x_{\text{young}}-x)_+\kappa_t^{(4)} + (x-x_{\text{old}})_+\kappa_t^{(5)} + \gamma_{t-x}^{(3)}$

Here: $b(x) = (x-\bar{x})^2 - \hat{\sigma}_x^2$ and $c(x) = (\bar{x}-x)_+ + [(\bar{x}-x)_+]^2$.

Identifiability issues with models as complex as M9.

Project multiple time dependent effects in the models M9-M14: not easy to find suitable time series specification!

Ad hoc strategy to avoid jumps or irregularities in cohort effects: only estimate cohort effects when ‘enough’ observations are available $\Rightarrow$ use weights in Poisson likelihood (Plat, 2009)

Projecting cohort effects is challenging!

Some overview papers, with model comparisons:

- Haberman & Renshaw (2011, IME)
- Lovasz (2011, EAJ)
- Cairns et al. (2009, NAAJ; 2011, IME)
- van Berkum, Antonio & Vellekoop (SAJ, 2016)
- ...

CBDX - a workhorse mortality model from the Cairns-Blake-Dowd family by Dowd, Cairns & Blake (2020, Annals of Actuarial Science):

- 'workhorse mortality model for the adult age range', hence: excluding accident hump and younger ages
- CBDX, is for Cairns-Blake-Dowd with additional non-parametric age effect:

$$\log \mu_{x,t} \quad = \quad \alpha_x + \sum_{i=1}^{K} \beta_x^{(i)} \kappa_t^{(i)} + \gamma_{t-x},$$

where $\beta_x^{(1)} = 1$, $\beta_x^{(2)} = (x - \bar{x})$ and $\beta_x^{(3)} = (x - \bar{x})^2 - \sigma_x^2$.

StMoMo - an R package for stochastic mortality modeling by Villegas, Kaishev & Millossovich (2018, JSS):

- random component:

$$D_{xt} \sim \text{POI}(E^c_{xt} \cdot \mu_{xt}) \quad \text{or} \quad D_{xt} \sim \text{BIN}(E^0_{xt}, q_{xt})$$

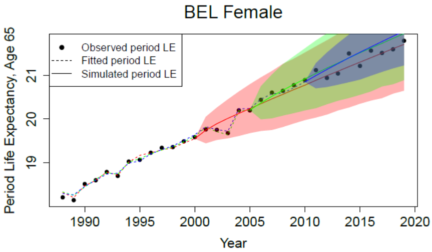
- systematic component:

$$\eta_{xt} = \alpha_x + \sum_{i=1}^{N} \beta^{(i)}_x \kappa^{(i)}_t + \beta^{(0)}_x \gamma_{t-x}$$

- link function $g(.)$ is logarithmic (with POI) or logit (with BIN).

Criteria to take into account (see Modules B):

- requirements regarding age range and calibration period to be considered
- good in-sample fits that capture trends over ages and time periods
- robustness and consistency, biologically reasonable scenarios
- reproducible and fully transparent



- good performance on in-sample statistical measures (e.g., AIC, BIC, residual plots) and out-of-time back-tests.

Wrap-up

That's a wrap!

After this module you are able to:

- write down the Lee & Carter model specification, give some intuition to the model's parameters, explain the identifiability issues and propose constraints to tackle these
- reflect on mortality projections with the Lee & Carter model, including a discussion of attention points such as the (long term) time series dynamics and the choice of the calibration period
- transform the scenarios for future κ_t and $\mu_{x,t}$'s into scenarios for quantities of interest to the actuary, e.g. $E[T_{x,t}]$, the EPV of a life annuity issued to x in year t, $q_{x,t}$'s.
- reflect on alternative stochastic mortality models from the LifeMetrics toolbox, for a single population.