Exploration of the Spruce Budworm Model

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1 The Spruce Budworm Model

We explored the spruce budworm population model, a classical example of a nonlinear system. The model describes the competition between intrinsic population growth of the budworm and predation by birds.

Using a numerical implementation (RK45) and an interactive Streamlit application, we analyze the qualitative behavior of the system by studying its equilibria and time evolution under different parameter choices.

The population dynamics of the spruce budworm are governed by the ODE

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}, \tag{1}$$

where:

- $x(t)$ is the dimensionless budworm population,
- r is the intrinsic growth rate,
- k is the carrying capacity of the forest.

The first term represents logistic growth, while the second term models predation by birds. Due to the nonlinearity of the predation term, the system can have multiple equilibrium points.

2 Exploration Questions

2.1 Q1: Multiple Equilibria

For $r = 0.5$ and $k = 10$, equilibrium points satisfy

$$rx^*\left(1 - \frac{x^*}{k}\right) = \frac{(x^*)^2}{1 + (x^*)^2}. \tag{2}$$

Using the phase portrait generated by the application, we observe that the system has four equilibrium points (two stable and two unstable equilibria):

The stability is determined by the sign of $\frac{d}{dx}\left(\frac{dx}{dt}\right)$ at each equilibrium, which shows that equilibria alternate in stability.

The stable points correspond to population levels that over the time population approaches, while the unstable equilibria represent population levels where small deviations grow over time, driving the population away from the point.

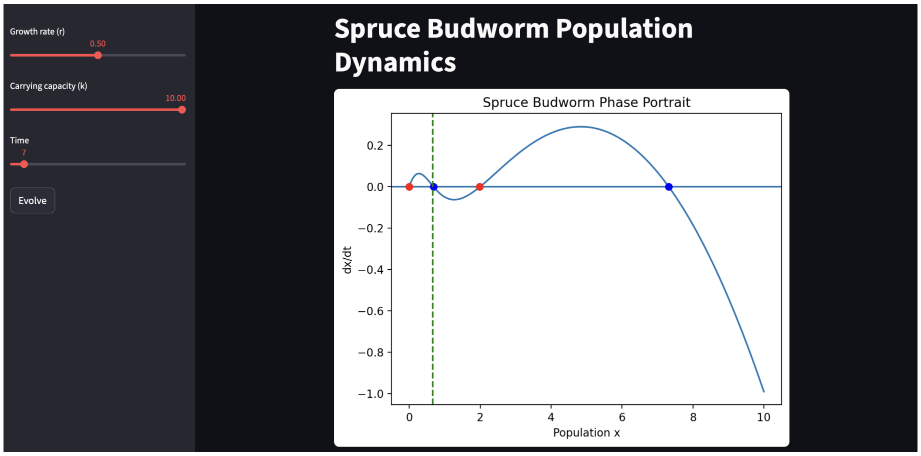
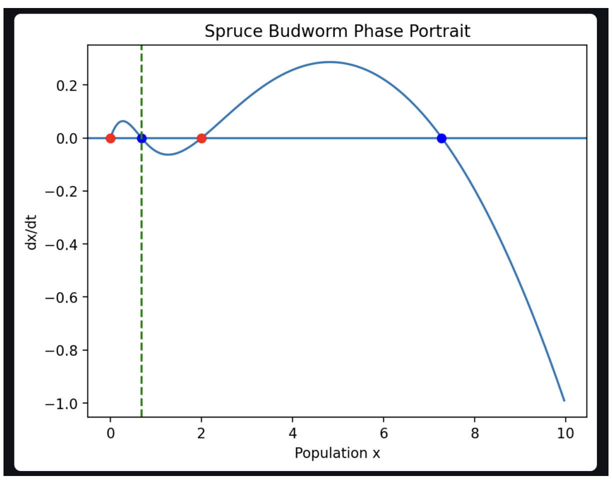


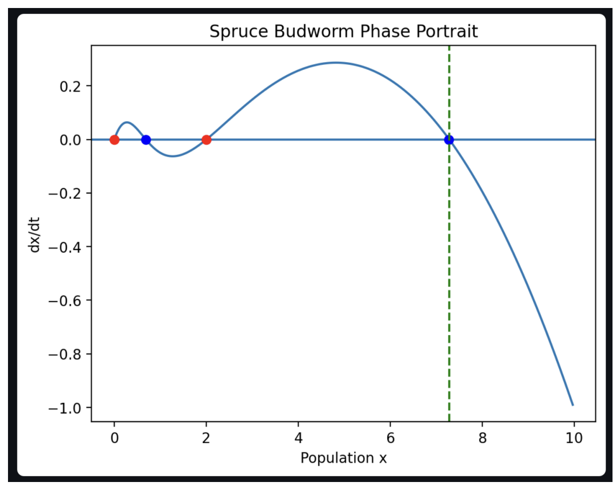
Figure 1: Phase portrait for $r = 0.5$, $k = 10$, showing two stable (blue) and two unstable equilibria (red).

2.2 Q2: Bistability

To investigate bistability, we simulate the system using two different initial conditions while keeping $r = 0.5$ and $k = 10$ fixed. As we do not control the initial points directly in the Streamlit application, we tried to estimate the two different initial conditions $x_0 \approx 1$ and $x_0 \approx 8$ (playing around with different values for r and k to change the initial condition from low to high)



(a) $x_0 \approx 1$



(b) $x_0 \approx 8$

Figure 2: Time evolution for two initial conditions showing bistability.

The simulations show that the population that started low remained low (it converged to a low population equilibrium), while the population that started large, remained in large and converged to a different equilibrium point. However, if we changed our r and k values, the two populations could converge to the same equilibrium if there was only one stable equilibrium point instead of two (which was the case for $r = 0.5$ and $k = 10$).

This demonstrates the concept of bistability, which is that the long-term behavior of the system depends on the initial condition.

2.3 Q3: Hysteresis

Hysteresis describes a history dependent behavior in a dynamical system. To study this phenomenon, we slowly increased the carrying capacity k from 5 to 15 and then decrease it back to 5, allowing the system to equilibrate at each step.

The results show that the population does not follow the same path when k is increased and decreased. Once the population transitions to the high equilibrium during the increase of k, it remains there even when k is reduced below the value at which the transition occurred.