Thesis Proposal

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- Robert Kirby
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Linearized Rotating Shallow-Water Equations with Damping

Tide Model

$$u_t + \frac{f}{\epsilon} u^{\perp} + \frac{\beta}{\epsilon^2} \nabla(\eta - \eta') + g(u) = F$$
$$\eta_t + \nabla \cdot (Hu) = 0,$$

Linearized Rotating Shallow-Water Equations with Damping

Tide Model

$$
\begin{gathered}
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\end{gathered}
$$

- u: nondimensional two dimensional velocity field tangent to Ω
- $u^{\perp}$: $(-u_2, u_1)$ velocity rotated by $\pi/2$
- η: nondimensional free surface elevation above the height at a state of rest
- $\nabla \eta'$: spatially varying tidal forcing

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- ϵ: Rossby number (small)
- f: spatially-dependent non-dimensional Coriolis parameter
- β: Burger number (small)
- H: spatially varying non-dimensional fluid depth at rest
- $g(u)$: monotonic damping function

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Acoustic Wave Equation

First-Order Equation

$$
\begin{aligned}
qu_t + \nabla p &= 0, \\
k^{-1}p_t + \nabla \cdot u &= 0,
\end{aligned}
$$

on some domain $\Omega \times [0, T] \subset \mathbb{R}^d \times \mathbb{R}$ with $d = 2, 3$ and Ω assumed to be polyhedral

Acoustic Wave Equation

Boundary and Initial Conditions

We impose the initial conditions:

$$p(x,0) = p_0(x)$$
$$u(x,0) = u_0(x)$$

with the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$ where ν is the unit outward normal to Ω

Acoustic Wave Equation

Weak Form

Multiplying and integrating by parts gives

$$(u_t, v) + (\nabla p, v) = (f, v), \quad v \in H_0(\mathrm{div})$$
$$(p_t, w) + (\nabla \cdot u, w) = (g, w), \quad w \in L_0^2$$

Acoustic Wave Equation

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$$\downarrow$$
$$(u_t, v) - (p, \nabla \cdot v) + \underbrace{\langle p, v \cdot \nu \rangle_{\partial\Omega}}_{=0} = (f, v),$$
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where $u : [0, T] \to V \equiv H_0(\mathrm{div})$ and $p : [0, T] \to W \equiv L_0^2$

Acoustic Wave Equation

Semidiscrete Mixed Form

The semidiscrete mixed formulation is to find $u_h : [0, T] \to V_h$ and $p_h : [0, T] \to W_h$ such that

$$(u_{h,t}, v_h) - (p_h, \nabla \cdot v_h) = (f, v_h),$$
$$(p_{h,t}, w_h) + (\nabla \cdot u_h, w_h) = (g, w_h),$$

$\forall\, v_h \in V_h$ and $w_h \in W_h$ where $V_h \subset V$ and $W_h \subset W$

Acoustic Wave Equation

Crank Nicolson

Partitioning $[0, T]$ into timesteps $0 \equiv t_0 < t_1 < \cdots < t_N$ where $t_i = i\Delta t$ and applying Crank Nicolson leads to

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h \right) - \left(\frac{p_h^{n+1} + p_h^n}{2}, \nabla \cdot v_h \right) = \left(f^{n+\frac{1}{2}}, v_h \right),$$

$$\left(\frac{p_h^{n+1} - p_h^n}{\Delta t}, w_h \right) + \left(\nabla \cdot \frac{u_h^{n+1} + u_h^n}{2}, w_h \right) = \left(g^{n+\frac{1}{2}}, w_h \right),$$

where $u_h(t_n) \approx u_h^n \in V_h$, $p_h(t_n) \approx p_h^n \in W_h$, $f^{n+\frac{1}{2}} = \frac{f(t_{n+1})+f(t_n)}{2}$, and similarly for g.

Crank Nicolson

Reshuffling terms leads to

$$
\begin{aligned}
\left(u_h^{n+1}, v_h\right) - \frac{\Delta t}{2}\left(p_h^{n+1}, \nabla \cdot v_h\right) &= \tilde{F}, \\
\left(p_h^{n+1}, w_h\right) + \frac{\Delta t}{2}\left(\nabla \cdot u_h^{n+1}, w_h\right) &= \tilde{G},
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\end{aligned}$$

where

$$\begin{aligned}
\tilde{F} &= \left(u_h^n, v_h\right) + \frac{\Delta t}{2}\left(p_h^n, \nabla \cdot v_h\right) + \left(f^{n+\frac{1}{2}}, v_h\right), \\
\tilde{G} &= \left(p_h^n, w_h\right) - \frac{\Delta t}{2}\left(\nabla \cdot u_h^n, w_h\right) + \left(g^{n+\frac{1}{2}}, w_h\right).
\end{aligned}$$

Acoustic Wave Equation

Discretization

Let $\{\phi_i\}_{i=1}^{|W_h|}$ and $\{\psi_i\}_{i=1}^{|V_h|}$ be bases for W_h and V_h respectively. Then we can define mass matrices

$$M_{ij} = (\phi_j, \phi_i),$$
$$\tilde{M}_{ij} = (\psi_j, \psi_i).$$

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$$M_{ij} = (\phi_j, \phi_i),$$
$$\tilde{M}_{ij} = (\psi_j, \psi_i).$$

We can formulate the semidiscrete mixed form as

$$\begin{bmatrix} \tilde{M} & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} u_t \\ p_t \end{bmatrix} + \begin{bmatrix} 0 & -D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

where $D_{ij} = (\mathrm{div}\psi_i, \phi_j)$, is the discrete div operator and F and G are the vectors (f, v_h) and (g, w_h) respectively.

Discretization

Then our final Crank Nicolson discretization is

$$\mathscr{A}_h \begin{bmatrix} u_h^{n+1} \\ p_h^{n+1} \end{bmatrix} = \begin{bmatrix} \tilde{F} \\ \tilde{G} \end{bmatrix},$$

where

$$\mathscr{A}_h = \begin{bmatrix} \tilde{M} & -\frac{\Delta t}{2} D^T \\ \frac{\Delta t}{2} D & M \end{bmatrix},$$

Crank Nicolson (again)

Applying Crank Nicolson again in the time derivative without discretizing in space becomes

$$u^{n+1} + \frac{\Delta t}{2} \nabla p^{n+1} = u^n - \frac{\Delta t}{2} \nabla p^n$$
$$p^{n+1} + \frac{\Delta t}{2} \nabla \cdot u^{n+1} = p^n - \frac{\Delta t}{2} \nabla \cdot u^n.$$

Preconditioning

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\end{aligned}
$$

Therefore, at each timestep we have a discretization of the coefficient operator $\mathscr{A}$, described as

$$
\mathscr{A} = \begin{pmatrix} I & \epsilon\,\mathrm{grad} \\ \epsilon\,\mathrm{div} & I \end{pmatrix}, \text{ where } \epsilon = \frac{\Delta t}{2}
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By discretizing in the finite element space $H_0(\mathrm{div}) \times L_0^2$, we recover our finite dimensional coefficient operator, $\mathscr{A}_h$.

Preconditioning

Mardal/Winther

We claim $\mathscr{A}$ is an isomorphism mapping $H(\mathrm{div}) \times L^2$ onto $H(\mathrm{div})^* \times L^2$, its dual space. Our goal is then to find a preconditioner $\mathscr{B}$ which maps $H(\mathrm{div})^* \times L^2$ onto $H(\mathrm{div}) \times L^2$.

Preconditioning

Mardal/Winther

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$$\mathscr{B} = \begin{bmatrix} \beta I - \alpha \operatorname{grad} \operatorname{div} & 0 \\ 0 & \gamma I \end{bmatrix}$$

Preconditioning

Discrete Form

Similarly, the discrete preconditioner is of the form

$$\mathscr{B}_h = \begin{bmatrix} \alpha(\nabla\cdot, \nabla\cdot) + \beta(\cdot,\cdot) & 0 \\ 0 & \gamma(\cdot,\cdot) \end{bmatrix}.$$

For this preconditioner, our goal is to look at the eigenvalues of $\mathscr{B}_h^{-1}\mathscr{A}_h$ as a function of α, β, and γ. We want to choose these parameters so that our preconditioner is robust in terms of Δt and h, thus eliminating the dependence on ϵ.

Conjectures

Conjecture 1

The operator $\mathscr{B}_h^{-1}\mathscr{A}_h$ has mesh-independent eigenvalue clustering

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The condition number κ of $\mathscr{B}_h^{-1}\mathscr{A}_h$ depends on Δt, α, β, and γ

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Conjecture 1

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Conjecture 2

The condition number κ of $\mathscr{B}_h^{-1}\mathscr{A}_h$ depends on Δt, α, β, and γ

Conjecture 3

The correct choice of α, β, and γ can force κ to be of order 1

Questions?

Thank you!