

Thesis Proposal

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Linearized Rotating Shallow-Water Equations with Damping

Tide Model

$$u_t + \frac{f}{\epsilon} u^\perp + \frac{\beta}{\epsilon^2} \nabla(\eta - \eta') + g(u) = F$$
$$\eta_t + \nabla \cdot (Hu) = 0,$$

- ▶ u : nondimensional two dimensional velocity field tangent to Ω
- ▶ u^\perp : $(-u_2, u_1)$ velocity rotated by $\pi/2$
- ▶ η : nondimensional free surface elevation above the height at a state of rest
- ▶ $\nabla\eta'$: spatially varying tidal forcing
- ▶ ϵ : Rossby number (small)
- ▶ f : spatially-dependent non-dimensional Coriolis parameter
- ▶ β : Burger number (small)
- ▶ H : spatially varying non-dimensional fluid depth at rest
- ▶ $g(u)$: monotonic damping function

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Acoustic Wave Equation

First-Order Equation

$$\begin{aligned}qu_t + \nabla p &= 0, \\k^{-1}p_t + \nabla \cdot u &= 0,\end{aligned}$$

on some domain $\Omega \times [0, T] \subset \mathbb{R}^d \times \mathbb{R}$ with $d = 2, 3$ and Ω assumed to be polyhedral

Acoustic Wave Equation

Boundary and Initial Conditions

We impose the initial conditions:

$$p(x, 0) = p_0(x)$$

$$u(x, 0) = u_0(x)$$

with the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$ where ν is the unit outward normal to Ω

Acoustic Wave Equation

Weak Form

Multiplying and integrating by parts gives

$$(u_t, v) + (\nabla p, v) = (f, v), \quad v \in H_0(\text{div})$$

$$(p_t, w) + (\nabla \cdot u, w) = (g, w), \quad w \in L_0^2$$

Acoustic Wave Equation

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\downarrow

$$(u_t, v) - (p, \nabla \cdot v) + \underbrace{\langle p, v \cdot \nu \rangle_{\partial\Omega}}_{=0} = (f, v),$$

$$(p_t, w) + (\nabla \cdot u, w) = (g, w),$$

Acoustic Wave Equation

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where $u : [0, T] \rightarrow V \equiv H_0(\text{div})$ and $p : [0, T] \rightarrow W \equiv L_0^2$

Acoustic Wave Equation

Semidiscrete Mixed Form

The semidiscrete mixed formulation is to find $u_h : [0, T] \rightarrow V_h$ and $p_h : [0, T] \rightarrow W_h$ such that

$$\begin{aligned}(u_{h,t}, v_h) - (p_h, \nabla \cdot v_h) &= (f, v_h), \\ (p_{h,t}, w_h) + (\nabla \cdot u_h, w_h) &= (g, w_h),\end{aligned}$$

$\forall v_h \in V_h$ and $w_h \in W_h$ where $V_h \subset V$ and $W_h \subset W$

Acoustic Wave Equation

Crank Nicolson

Partitioning $[0, T]$ into timesteps $0 \equiv t_0 < t_1 < \dots < t_N$ where $t_i = i\Delta t$ and applying Crank Nicolson leads to

$$\begin{aligned} \left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h \right) - \left(\frac{p_h^{n+1} + p_h^n}{2}, \nabla \cdot v_h \right) &= \left(f^{n+\frac{1}{2}}, v_h \right), \\ \left(\frac{p_h^{n+1} - p_h^n}{\Delta t}, w_h \right) + \left(\nabla \cdot \frac{u_h^{n+1} + u_h^n}{2}, w_h \right) &= \left(g^{n+\frac{1}{2}}, w_h \right), \end{aligned}$$

where $u_h(t_n) \approx u_h^n \in V_h$, $p_h(t_n) \approx p_h^n \in W_h$, $f^{n+\frac{1}{2}} = \frac{f(t_{n+1}) + f(t_n)}{2}$, and similarly for g .

Acoustic Wave Equation

Crank Nicolson

Reshuffling terms leads to

$$\begin{aligned}(u_h^{n+1}, v_h) - \frac{\Delta t}{2} (p_h^{n+1}, \nabla \cdot v_h) &= \tilde{F}, \\ (p_h^{n+1}, w_h) + \frac{\Delta t}{2} (\nabla \cdot u_h^{n+1}, w_h) &= \tilde{G},\end{aligned}$$

where

$$\begin{aligned}\tilde{F} &= (u_h^n, v_h) + \frac{\Delta t}{2} (p_h^n, \nabla \cdot v_h) + \left(f^{n+\frac{1}{2}}, v_h\right), \\ \tilde{G} &= (p_h^n, w_h) - \frac{\Delta t}{2} (\nabla \cdot u_h^n, w_h) + \left(g^{n+\frac{1}{2}}, w_h\right).\end{aligned}$$

Acoustic Wave Equation

Discretization

Let $\{\phi_i\}_{i=1}^{|W_h|}$ and $\{\psi_i\}_{i=1}^{|V_h|}$ be bases for W_h and V_h respectively. Then we can define mass matrices

$$M_{ij} = (\phi_j, \phi_i),$$

$$\tilde{M}_{ij} = (\psi_j, \psi_i).$$

We can formulate the semidiscrete mixed form as

$$\begin{bmatrix} \tilde{M} & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} u_t \\ p_t \end{bmatrix} + \begin{bmatrix} 0 & -D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

where $D_{ij} = (\operatorname{div} \psi_i, \phi_j)$, is the discrete div operator and F and G are the vectors (f, v_h) and (g, w_h) respectively.

Acoustic Wave Equation

Discretization

Then our final Crank Nicolson discretization is

$$\mathcal{A}_h \begin{bmatrix} u_h^{n+1} \\ p_h^{n+1} \end{bmatrix} = \begin{bmatrix} \tilde{F} \\ \tilde{G} \end{bmatrix},$$

where

$$\mathcal{A}_h = \begin{bmatrix} \tilde{M} & -\frac{\Delta t}{2} D^T \\ \frac{\Delta t}{2} D & M \end{bmatrix},$$