Vector Calculus and the Topology of Domains in 3-Space

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Suppose you have a vector field defined on a bounded domain in 3-space. How can you tell whether your vector field is the gradient of some function? Or the curl of another vector field? Can you find a nonzero field on your domain that is divergence-free, curl-free, and tangent to the boundary? How about a nonzero field that is divergence-free, curl-free, and orthogonal to the boundary?

To answer these questions, you need to understand the relationship between the calculus of vector fields and the topology of their domains of definition. The Hodge Decomposition Theorem provides the key by decomposing the space of vector fields on the domain into five mutually orthogonal subspaces that are topologically and analytically meaningful. This decomposition is useful not only in mathematics, but also in fluid dynamics, electrodynamics, and plasma physics. Furthermore, carrying out the proof provides a pleasant introduction to homology and cohomology theory in a familiar setting, and a chance to see both the general Hodge theorem and the de Rham isomorphism theorem in action.

Our goal is to give an elementary exposition of these ideas. We provide three sections of background information early in the paper: one on solutions of the Laplace and Poisson equations with Dirichlet and Neumann boundary conditions, one on the Biot-Savart law from electrodynamics, and one on the topology of compact domains in 3-space. Near the end, we see how everything we have learned provides answers to the four questions that we have posed. We close with a brief survey of the historical threads that led to the Hodge Decomposition Theorem, and a guide to the literature.

1. STATEMENT OF THE THEOREM. Let Ω be a compact domain in 3-space with smooth boundary $\partial\Omega$. Let $\mathrm{VF}(\Omega)$ be the infinite-dimensional vector space of all smooth vector fields in Ω, on which we use the L^2 inner product

$$\langle V, W\rangle = \int_\Omega V\cdot W \, \mathrm{d(vol)}.$$

In this paper, "smooth" always means "all partial derivatives of all orders exist and are continuous".

Hodge Decomposition Theorem. *The space* $\mathrm{VF}(\Omega)$ *is the direct sum of five mutually orthogonal subspaces:*

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}$$

with

$$\begin{aligned}
\ker \mathit{curl} &= \phantom{\mathrm{FK} \oplus{}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}\\
\mathrm{im}\, \mathit{grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus{}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}\\
\mathrm{im}\, \mathit{curl} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}\\
\ker \mathit{div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG},
\end{aligned}$$

where

$$\begin{array}{lll}
\mathrm{FK} = \textit{fluxless knots} & = \{\nabla \cdot V = 0,\ V \cdot \boldsymbol{n} = 0,\ \text{all interior fluxes are } 0\} \\
\mathrm{HK} = \textit{harmonic knots} & = \{\nabla \cdot V = 0,\ \nabla \times V = \mathbf{0},\ V \cdot \boldsymbol{n} = 0\} \\
\mathrm{CG} = \textit{curly gradients} & = \{V = \nabla\varphi,\ \nabla \cdot V = 0,\ \text{all boundary fluxes are } 0\} \\
\mathrm{HG} = \textit{harmonic gradients} & = \{V = \nabla\varphi,\ \nabla \cdot V = 0,\ \varphi \text{ is locally constant on } \partial\Omega\} \\
\mathrm{GG} = \textit{grounded gradients} & = \{V = \nabla\varphi,\ \phi|_{\partial\Omega} = 0\}.
\end{array}$$

Furthermore,

$$\mathrm{HK} \cong H_1(\Omega; \mathbf{R}) \cong H_2(\Omega, \partial\Omega; \mathbf{R}) \cong \mathbf{R}^{\text{genus of } \partial\Omega}$$

$$\mathrm{HG} \cong H_2(\Omega; \mathbf{R}) \cong H_1(\Omega, \partial\Omega; \mathbf{R}) \cong \mathbf{R}^{(\#\text{ components of } \partial\Omega) - (\#\text{ components of } \Omega)}.$$

The meanings of the conditions in the theorem are explained in the following section.

2. ORGANIZATION OF THE PROOF. We now describe the various subspaces that appear in the Hodge Decomposition Theorem, explain the conditions that define them, and organize the proof of the theorem into four propositions.

Consider the subspace of VF(Ω) consisting of all divergence-free vector fields on Ω that are tangent to $\partial\Omega$. These vector fields are used to represent incompressible fluid flows within fixed boundaries, and magnetic fields inside plasma containment devices; see **[6]**, **[7]**, **[31]**, **[49]**, and **[55]**. When a problem in geometric knot theory is transformed to one in fluid dynamics, the knot is thickened to a tubular neighborhood of itself, and is then represented by an incompressible fluid flow (sometimes called a *fluid knot*) in this neighborhood, much like blood pumping miraculously through an arterial loop as in Figure 1; see **[4]**, **[1]**, **[35]**, **[36]**, **[39]**, and **[10]**.

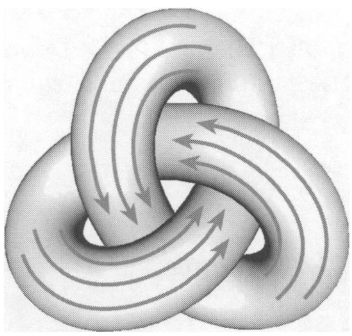

Figure 1. "Blood flow."

With this last application in mind, we define

$$\mathrm{K} = \textit{knots} = \{V \in \mathrm{VF}(\Omega) : \nabla \cdot V = 0,\ V \cdot \mathbf{n} = 0\},$$

where $\mathbf{n}$ is the unit outward normal to the boundary of Ω, and where the condition $\nabla \cdot V = 0$ holds throughout Ω, while $V \cdot \mathbf{n} = 0$ holds only on its boundary. In general, when we state conditions for our vector fields using $\mathbf{n}$, they are understood to apply only on the boundary of Ω.

At the same time, we define

$$\mathrm{G} = \textit{gradients} = \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi\}$$

for some smooth real-valued function φ defined on Ω.

Proposition 1. *The space* $\mathrm{VF}(\Omega)$ *is the direct sum of two orthogonal subspaces:*

$$\mathrm{VF}(\Omega) = \mathrm{K} \oplus \mathrm{G}.$$

The proof of this proposition in Section 7 is straightforward and brief.

As our development unfolds, we further decompose K into an orthogonal direct sum of two subspaces, and G into an orthogonal direct sum of three subspaces.

We start by decomposing the subspace K.

Let Σ denote any smooth orientable surface in Ω whose boundary $\partial\Sigma$ lies in the boundary $\partial\Omega$ of the domain Ω. We call Σ a *cross-sectional surface* and write $(\Sigma, \partial\Sigma) \subset (\Omega, \partial\Omega)$. Orient Σ by picking one of its two unit normal vector fields $\mathbf{n}$. Then, for any vector field V on Ω, we define the *flux of V through* Σ to be

$$\Phi = \int_\Sigma V \cdot \mathbf{n}\, \mathrm{d(area)}.$$

Assume that V is divergence-free and tangent to $\partial\Omega$. Then the value of Φ depends only on the homology class of Σ in the relative homology group $H_2(\Omega, \partial\Omega; \mathbf{R})$. For example, if Ω is an n-holed solid torus, then $H_2(\Omega, \partial\Omega; \mathbf{R})$ is generated by disjoint oriented cross-sectional disks $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, positioned so that cutting Ω along these disks produces a simply-connected region, as in Figure 2.The fluxes $\Phi_1, \Phi_2, \ldots, \Phi_n$ of V through these disks determine the flux of V through any other cross-sectional surface.

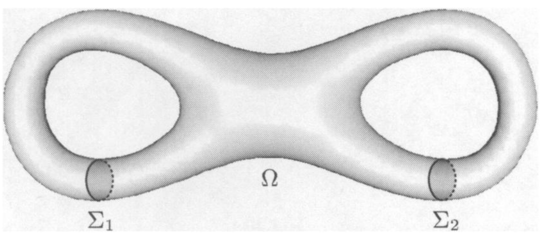

Figure 2. A domain with cross-sectional disks.

If the flux of V through every cross-sectional surface vanishes, then we say *all interior fluxes are* 0, and define

$$\mathrm{FK} = \textit{fluxless knots} = \{V \in \mathrm{VF}(\Omega) : \nabla \cdot V = 0,\ V \cdot \mathbf{n} = 0,\ \text{all interior fluxes are } 0\}.$$

Next we define

$$\mathrm{HK} = \textit{harmonic knots} = \{V \in \mathrm{VF}(\Omega) : \nabla \cdot V = 0,\ \nabla \times V = \mathbf{0},\ V \cdot \mathbf{n} = 0\}.$$

We show in Section 9 that the space of harmonic knots is isomorphic to the homology group $H_1(\Omega, \mathbf{R})$ and also, via Poincaré duality, to the relative homology group

$H_2(\Omega, \partial\Omega; \mathbf{R})$. This is a *finite-dimensional* vector space, with dimension equal to the (total) genus of $\partial\Omega$, which is obtained by adding the genera of the boundary components of the domain Ω.

Proposition 2. *The subspace* K *is the direct sum of two orthogonal subspaces:*

$$\mathrm{K} = \mathrm{FK} \oplus \mathrm{HK}.$$

This is the most challenging of the four propositions to prove, since its proof requires the fact (Lemma 2) that the harmonic knots HK on the domain Ω are rich enough to reflect a significant portion of its topology. The argument occupies Sections 8, 9, and 10.

Now we decompose the subspace G of gradient vector fields. Define

$$\mathrm{DFG} = \textit{divergence-free gradients} = \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \nabla \cdot V = 0\},$$

which is the subspace of gradients of harmonic functions, and

$$\mathrm{GG} = \textit{grounded gradients} = \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0\},$$

which is the subspace of gradients of functions that vanish on the boundary of Ω.

Proposition 3. *The subspace* G *is the direct sum of two orthogonal subspaces:*

$$\mathrm{G} = \mathrm{DFG} \oplus \mathrm{GG}.$$

The proof of this proposition in Section 11 is straightforward and brief.

Next we decompose the subspace DFG of divergence-free gradient vector fields.

If V is a vector field defined on Ω, we say that *all boundary fluxes of V are zero* when the flux of V through *each component* of $\partial\Omega$ is zero.

We define

$$\begin{aligned} \mathrm{CG} &= \textit{curly gradients} \\ &= \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \nabla \cdot V = 0,\ \text{all boundary fluxes are } 0\}. \end{aligned}$$

We call CG the subspace of *curly gradients* because, as we see in Section 14, these are the only gradient vector fields that lie in the image of *curl*.

Finally, we define the subspace

$$\begin{aligned} \mathrm{HG} &= \textit{harmonic gradients} \\ &= \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \nabla \cdot V = 0,\ \varphi \text{ is locally constant on } \partial\Omega\}, \end{aligned}$$

where the stated boundary condition means that φ is constant on each component of $\partial\Omega$.

We show in Section 12 that the space of harmonic gradients is isomorphic to the homology group $H_2(\Omega; \mathbf{R})$ and also, via Poincaré duality, to the relative homology group $H_1(\Omega, \partial\Omega; \mathbf{R})$. This is a finite-dimensional vector space, with one generator for each component of $\partial\Omega$, and one relation for each component of Ω. Hence, its dimension is equal to the number of components of $\partial\Omega$ minus the number of components of Ω.

Proposition 4. *The subspace* DFG *is the direct sum of two orthogonal subspaces:*

$$\mathrm{DFG} = \mathrm{CG} \oplus \mathrm{HG}.$$

The proof of this proposition requires the fact (Lemma 3) that the harmonic gradients HG on the domain Ω are rich enough to reflect a portion of its topology. The argument, which occupies Sections 12 and 13, is easier than that of Proposition 2, but is more substantial than those of Propositions 1 and 3.

With Propositions 1 through 4 in hand, the Hodge decomposition of $\mathrm{VF}(\Omega)$,

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

follows immediately, and we can then obtain the related expressions for the kernels of *curl* and *div*, and the images of *grad* and *curl*. Three of these four follow immediately, but we need a brief argument to confirm the image of *curl*.

The following characterizations of the five orthogonal direct summands of $\mathrm{VF}(\Omega)$ attest to their naturality:

$$\begin{aligned}
\mathrm{FK} &= (\ker \mathit{curl})^{\perp} \\
\mathrm{HK} &= (\ker \mathit{curl}) \cap (\mathrm{im}\, \mathit{grad})^{\perp} \\
\mathrm{CG} &= (\mathrm{im}\, \mathit{grad}) \cap (\mathrm{im}\, \mathit{curl}) \\
\mathrm{HG} &= (\ker \mathit{div}) \cap (\mathrm{im}\, \mathit{curl})^{\perp} \\
\mathrm{GG} &= (\ker \mathit{div})^{\perp}.
\end{aligned}$$

We can also write

$$\mathrm{HK} = (\ker \mathit{curl})/(\mathrm{im}\, \mathit{grad}) \cong H_1(\Omega; \mathbf{R})$$

and

$$\mathrm{HG} = (\ker \mathit{div})/(\mathrm{im}\, \mathit{curl}) \cong H_2(\Omega; \mathbf{R}),$$

which gives us a chance to see two instances of the de Rham Isomorphism Theorem [**15**] within the Hodge Decomposition Theorem.

3. EXAMPLES OF VECTOR FIELDS FROM THE FIVE SUMMANDS OF VF(Ω). In this section we give examples and pictures of vector fields from the five subspaces that appear in the Hodge Decomposition Theorem.

Let Ω_0 be the round ball of radius 1, centered at the origin in 3-space. Since the genus of $\partial\Omega_0$ is zero, there are no harmonic knots, so $\mathrm{HK} = \mathbf{0}$. Since Ω_0 and $\partial\Omega_0$ are both connected, there are no harmonic gradients, so $\mathrm{HG} = \mathbf{0}$. Thus

$$\mathrm{VF}(\Omega_0) = \mathrm{FK} \oplus \mathrm{CG} \oplus \mathrm{GG}.$$

We give examples of vector fields from these three subspaces first, so that we can use the round ball Ω_0 as the common domain.

Consider the vector field

$$V = -y\,\mathbf{i} + x\,\mathbf{j}.$$

This is the velocity field for rotation of 3-space about the z-axis at constant angular speed, and is divergence-free and tangent to the boundary of the ball Ω_0; see Figure 3. Since $\mathrm{HK} = 0$, the vector field V belongs to the subspace FK of fluxless knots.

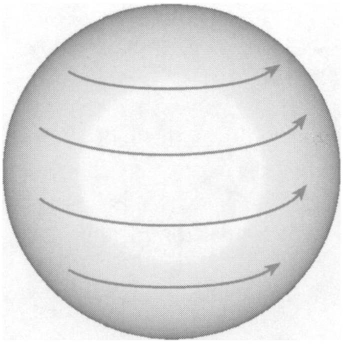

Figure 3. A vector field in the subspace FK of fluxless knots.

Now consider the constant vertical vector field $V = \mathbf{k}$; see Figure 4. This vector field is divergence-free and has zero flux through the one and only component of $\partial\Omega_0$, so it belongs to the subspace CG of curly gradients.

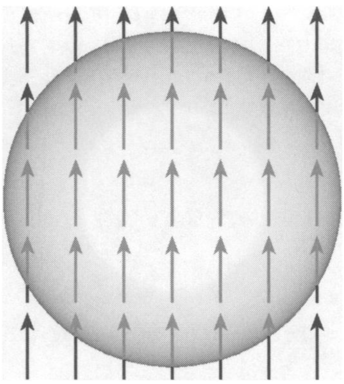

Figure 4. A vector field in the subspace CG of curly gradients.

The vector field V is the gradient of the harmonic function z. We can use this same construction with *any* harmonic function φ defined *on all of* R^3, and the resulting vector field $V = \nabla\varphi$ is a curly gradient on *any* compact domain Ω.

Now consider, on our round ball Ω_0 of radius 1, the function $r^2 - 1 = x^2 + y^2 + z^2 - 1$, which has constant value zero on the boundary of Ω_0. Then the vector field

$$V = \nabla(r^2 - 1) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

belongs to the subspace GG of grounded gradients; see Figure 5.

We must abandon the round ball to present examples of harmonic knots and harmonic gradients.

Let Ω_1 be a solid torus of revolution about the z-axis. Since the genus of $\partial\Omega_1$ is one, there are harmonic knots: $\mathrm{HK} \cong \mathbf{R}^1$. But since Ω_1 and $\partial\Omega_1$ are both connected, there are still no harmonic gradients. Thus

$$\mathrm{VF}(\Omega_1) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{GG}.$$

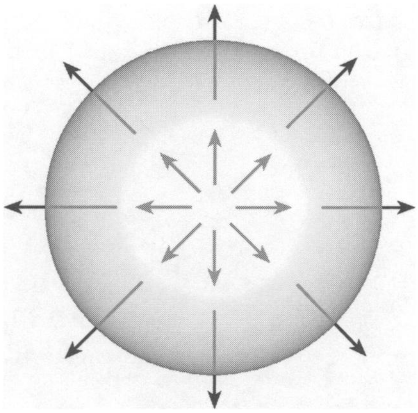

Figure 5. A vector field in the subspace GG of grounded gradients.

Using cylindrical coordinates (r, φ, z),

$$V = \frac{1}{r}\,\widehat{\varphi},$$

which is the magnetic field due to a steady current running up the z-axis; see Figure 6. It is divergence-free and curl-free and tangent to the boundary of the solid torus Ω_1, hence it belongs to the subspace HK of harmonic knots.

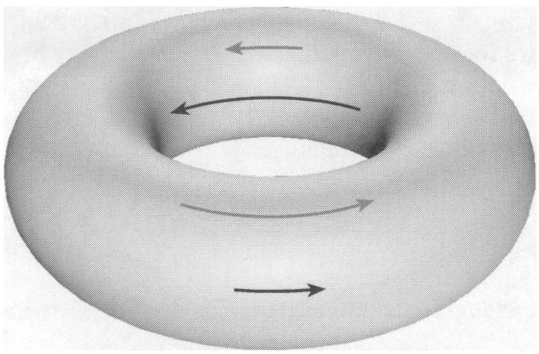

Figure 6. A vector field in the subspace HK of harmonic knots.

We switch domains again to get an example of a harmonic gradient.

Let Ω_2 be the region between two concentric round spheres, say of radii 1 and 2, centered at the origin. This domain has harmonic gradients, since $\partial\Omega_2$ has two components, while Ω_2 has only one. Thus $\mathrm{HG} \cong \mathbf{R}^1$. But Ω_2 has no harmonic knots, since each boundary component has genus zero. Thus

$$\mathrm{VF}(\Omega_2) = \mathrm{FK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}.$$

Using spherical coordinates (r, θ, φ), consider the harmonic function $-1/r$ and its gradient vector field

$$V = \nabla\left(-\frac{1}{r}\right) = \frac{1}{r^2}\,\widehat{\mathbf{r}},$$

which is just the inverse-square field; see Figure 7. Since the harmonic function $-1/r$ is constant on each component of $\partial\Omega$, the vector field V belongs to the subspace HG of harmonic gradients.

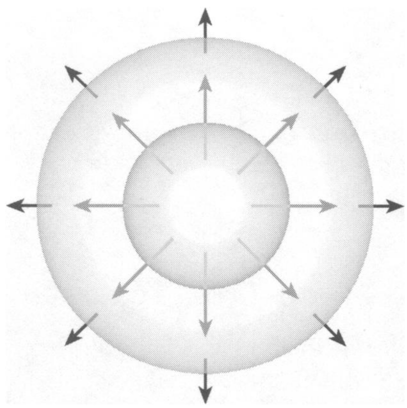

Figure 7. A vector field in the subspace HG of harmonic gradients.

4. SOLUTION OF THE LAPLACE AND POISSON EQUATIONS WITH DIRICHLET AND NEUMANN BOUNDARY CONDITIONS. We need to use some basic results from analysis about solvability of the Dirichlet and Neumann problems for the scalar Laplace and Poisson equations. We state them here, and point the reader to the literature for details.

We continue to work on a bounded domain Ω in $\mathbf{R}^3$ with connected components $\Omega_1, \Omega_2, \dots, \Omega_k$ and with smooth boundary $\partial\Omega$. The *Laplace operator* (or *Laplacian*) acts on scalar functions on Ω and is defined by

$$\Delta\varphi = \nabla \cdot \nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2},$$

where (x, y, z) are rectangular coordinates in R^3.

Given a function f on Ω, the *Poisson equation* is

$$\Delta\varphi = f \quad \text{on } \Omega.$$

If $f = 0$, then this is called the *Laplace equation*, and solutions of $\Delta\varphi = 0$ are called *harmonic functions*. This is a partial differential equation for the unknown function φ on Ω, and in order to have a problem that is reasonable (in either the mathematical or physical sense), one must impose additional conditions on φ. Typically, these restrict the values of φ or its derivatives on $\partial\Omega$. For example, if u is a function on $\partial\Omega$, we can require that

$$\varphi = u \quad \text{on } \partial\Omega.$$

This is called a *Dirichlet boundary condition*. Alternatively, we can specify that

$$\frac{\partial\varphi}{\partial\mathbf{n}} = v \quad \text{on } \partial\Omega$$

for some function v on $\partial\Omega$, where $\partial\varphi/\partial\mathbf{n}$ is the normal derivative $\nabla\varphi \cdot \mathbf{n}$ on $\partial\Omega$. This is called a *Neumann boundary condition*.

The Dirichlet Problem. *Given functions f on Ω and u on $\partial\Omega$, find a function φ on Ω satisfying*

$$\Delta\varphi = f \ \ \text{on } \Omega \ \ \text{and} \ \ \varphi = u \ \ \text{on } \partial\Omega.$$

The Neumann Problem. *Given functions f on Ω and v on $\partial\Omega$, find a function φ on Ω satisfying*

$$\Delta\varphi = f \ \ \text{on } \Omega \ \ \text{and} \ \ \frac{\partial\varphi}{\partial\boldsymbol{n}} = v \ \ \text{on } \partial\Omega.$$

We do not need the most general possible results: we assume that our data f, u, and v are smooth functions on their domains of definition. In particular, we mean that f is smooth on the closed domain Ω, not just on its interior.

Theorem 1.

(a) *The Dirichlet problem has a unique solution φ for every smooth function f on Ω and u on $\partial\Omega$.*

(b) *The Neumann problem for smooth functions f on Ω and v on $\partial\Omega$ has a solution φ if and only if*

$$\int_{\Omega_i} f \, \mathrm{d(vol)} = \int_{\partial\Omega_i} v \, \mathrm{d(area)}$$

for each $i = 1, 2, \ldots, k$. The difference between any two solutions is a function that is constant on each component Ω_i of Ω.

The solution functions φ for both the Dirichlet and Neumann problems are smooth on the closed domain Ω and satisfy $\Delta\varphi = f$ on the closed domain.

We refer the reader to [**20**] and [**23**] for a variety of approaches to the proof of this theorem.

5. THE BIOT-SAVART FORMULA FOR MAGNETIC FIELDS. The basic results from electrodynamics used throughout this paper concern the Biot-Savart formula for the magnetic field generated by a given current distribution, and the formulas for its divergence and curl. We state them here, and again point the reader to the literature for details.

Let Ω be a compact domain in 3-space with smooth boundary $\partial\Omega$, and let V be a smooth vector field on Ω. If we think of V as a distribution of current throughout Ω, then the Biot-Savart formula

$$\mathrm{BS}(V)(y) = \frac{1}{4\pi}\int_{\Omega} V(x) \times \frac{y - x}{|y - x|^3} \, \mathrm{d}(\mathrm{vol}_x) \tag{1}$$

gives the resulting magnetic field $\mathrm{BS}(V)$ throughout 3-space.

The magnetic field $\mathrm{BS}(V)$ is well-defined on all of 3-space, that is, the improper integral (1) converges for every $y \in \mathbf{R}^3$. Furthermore, $\mathrm{BS}(V)$ is continuous on all of $\mathbf{R}^3$,

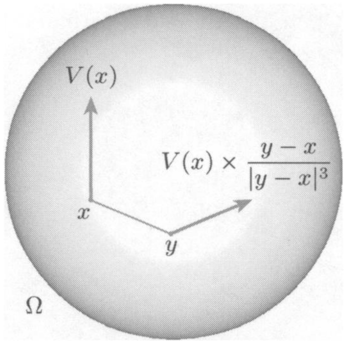

Figure 8. The Biot-Savart integrand.

though its derivatives typically experience a jump discontinuity as one crosses $\partial\Omega$. Nevertheless, $\mathrm{BS}(V)$ is of class C^∞ on Ω and on the closure Ω' of $\mathbf{R}^3 - \Omega$.

The magnetic field $\mathrm{BS}(V)$ is divergence-free in the sense that

$$\nabla \cdot \mathrm{BS}(V) = 0 \quad \text{on } \Omega \text{ and on } \Omega'.$$

The most important formula concerns the curl of the magnetic field:

$$\begin{aligned}\nabla_y \times \mathrm{BS}(V)(y) = \left\{ \begin{array}{ll} V(y) & \text{for } y \in \Omega \\ \mathbf{0} & \text{for } y \in \Omega' \end{array} \right\} &+ \frac{1}{4\pi}\nabla_y \int_\Omega \frac{\nabla_x \cdot V(x)}{|y-x|}\, \mathrm{d(vol}_x) \\ &- \frac{1}{4\pi}\nabla_y \int_{\partial\Omega} \frac{V(x)\cdot \mathbf{n}}{|y-x|}\, \mathrm{d(area}_x),\end{aligned} \tag{2}$$

where ∇_x differentiates with respect to x, while ∇_y differentiates with respect to y. This formula is simply Maxwell's law

$$\nabla \times B = J + \frac{\partial E}{\partial t},$$

where $B = \mathrm{BS}(V)$ represents the magnetic field, $J = V$ is the current distribution, and the time-dependent electric field E is due to a changing charge distribution within Ω if $\nabla \cdot V \neq 0$, and a changing charge distribution on $\partial\Omega$ if $V \cdot \mathbf{n} \neq 0$. The magnetic permeability μ_0 and the electric permittivity ϵ_0, which usually appear in Maxwell's equation, have been suppressed.

Formula (2) for the curl of the magnetic field contains a wealth of information. For example, if our vector field V is divergence-free, then the first integral on the right-hand side of (2) vanishes. If V is tangent to the boundary of Ω, then the second integral vanishes. If both conditions hold, that is, if $V \in \mathrm{K}$, which according to Proposition 2 equals $\mathrm{FK} \oplus \mathrm{HK}$, then we get the familiar statement that the curl of the magnetic field is the current flow:

$$\nabla_y \times \mathrm{BS}(V)(y) = \left\{ \begin{array}{ll} V(y) & \text{for } y \in \Omega, \\ \mathbf{0} & \text{for } y \in \Omega'. \end{array} \right.$$

In particular, this tells us that on the domain Ω the image of *curl* contains $\mathrm{FK} \oplus \mathrm{HK}$.

If the vector field V is divergence-free but not necessarily tangent to the boundary of Ω, then (2) tells us that

$$\nabla_y \times \mathrm{BS}(V)(y) = \left\{ \begin{array}{ll} V(y) & for\ y \in \Omega \\ \mathbf{0} & for\ y \in \Omega' \end{array} \right\} - \frac{1}{4\pi}\nabla_y \int_{\partial\Omega} \frac{V(x)\cdot\mathbf{n}}{|y-x|}\,\mathrm{d}(\mathrm{area}_x). \tag{3}$$

We use (3) at the end of the paper to show that on the domain Ω the image of *curl* is precisely FK $\oplus$ HK $\oplus$ CG.

We also want to consider the magnetic field B in the domain Ω that is caused by running a current I through a smooth loop C' in the complement of Ω; see Figure 9.

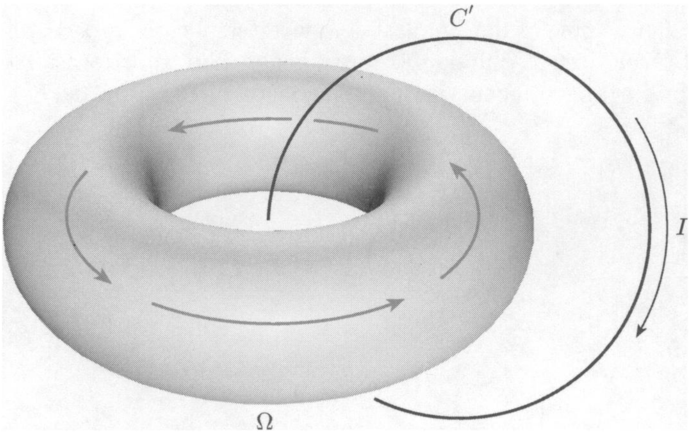

Figure 9. A linking current.

In this case, the Biot-Savart formula for the magnetic field takes the form

$$B(y) = \frac{1}{4\pi}\int_{C'} I(x) \times \frac{y-x}{|y-x|^3}\,dx, \tag{4}$$

where the vector field $I(x)$ along the oriented loop C' has constant length I and is tangent to C' in the direction of its orientation.

The integral (4) blows up along the loop C', but this is of no concern to us because we are interested only in the restriction of B to the domain Ω. In the domain Ω, the vector field B is smooth and has the following properties:

$$\begin{aligned} &\text{(a)} \quad \nabla \cdot B = 0 \\ &\text{(b)} \quad \nabla \times B = \mathbf{0} \\ &\text{(c)} \quad \int_C B \cdot ds = \mathrm{Link}(C, C')I, \end{aligned} \tag{5}$$

where C is a loop in Ω and Link(C, C') denotes its linking number with C'.

We can view magnetic fields caused by currents running in loops in the following way. Replace the loop C' by a small tubular neighborhood N', and let V' be a current distribution in N' that is divergence-free, tangent to the boundary, and has flux I through each cross-sectional disk of N'. Then the magnetic field B caused by the current I through the loop C' is the limit of the magnetic fields BS(V') as the tubular neighborhood N' shrinks down to C'.

With this point of view, the properties (5a), (5b), and (5c) are easy to derive; in particular, (5c) is Ampère's Law. See **[25]** and **[13]** for further details.

The history of the Biot-Savart Law is discussed in **[51]**. It contains extensive translations of the works of Oerstead, Biot, Savart, and Ampère, and a detailed analysis of this fascinating period of scientific discovery and of the interactions among its principals.

6. TOPOLOGY OF COMPACT DOMAINS IN 3-SPACE. The Hodge Decomposition Theorem shows how the structure of the space of vector fields defined on a compact domain in 3-space reflects the topology of the domain. We give here a brief overview of this topology.

Figure 10 shows some of the simplest compact domains in 3-space: a round ball, a solid torus of revolution, a solid double torus, and a solid triple torus. These domains are bounded by surfaces of genus zero, one, two, and three, respectively.

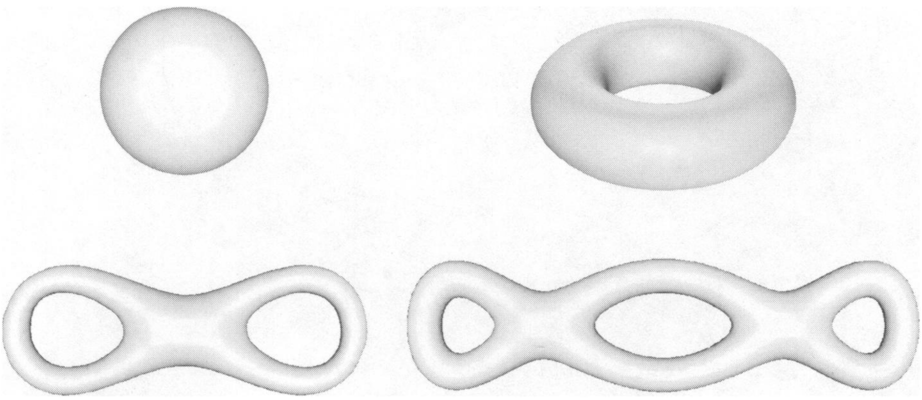

Figure 10. Domains of various topological types.

The tri-lobed domain in Figure 11 is geometrically more complicated than a round ball, but is *topologically equivalent* (or *homeomorphic*) to it: there is a one-to-one correspondence between the two domains that is continuous in each direction.

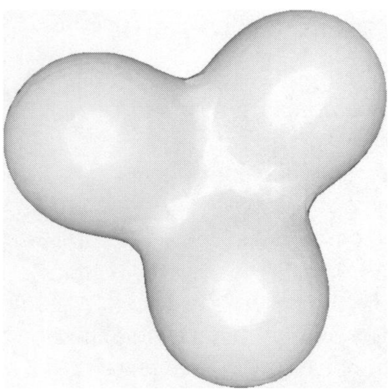

Figure 11. A domain homeomorphic to a ball.

Domains in 3-space can have several components, as shown by the pair of linked solid tori on the left in Figure 12. And the boundary of a connected domain can have several components, as shown by the region between two concentric spheres, on the right in Figure 12.

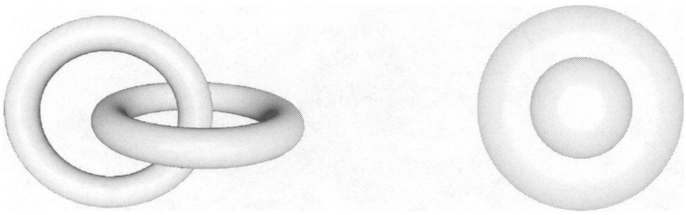

Figure 12. Domains with several boundary components.

Two topologically distinct domains can nevertheless have topologically equivalent boundaries. The domain pictured in Figure 13, which looks like a rolling pin with a knotted hole, is topologically distinct from a solid torus, yet the two domains have topologically equivalent boundaries. This is a case of "You can't tell a book by its cover."

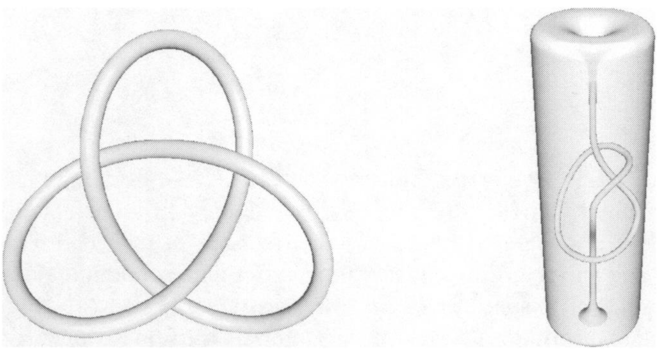

Figure 13. You can't tell a book by its cover.

In Figure 14, we see three domains: a solid torus linked with a solid double torus, and a potato-shaped domain (homeomorphic to a round ball) containing them both. The space outside the torus and double torus, but inside the potato, is a domain whose boundary consists of all three surfaces.

Now we get down to the business of describing the topological concepts needed for the Hodge Decomposition Theorem.

Let Ω be a compact domain in 3-space, with smooth boundary. The basic topological information about Ω that we need is given by its *homology with real coefficients*.

The *absolute homology* of Ω consists of the vector spaces $H_i(\Omega; \mathbf{R})$, for $i = 0, 1, 2, 3$, while the *relative homology* of Ω modulo its boundary $\partial\Omega$ consists of the vector spaces $H_i(\Omega, \partial\Omega; \mathbf{R})$ for the same values of i. We use only homology with real coefficients, so henceforth we suppress the symbol "$\mathbf{R}$".

The absolute homology vector space $H_0(\Omega)$ is generated by equivalence classes of points in Ω, with two points deemed equivalent if they can be connected by a path in Ω.

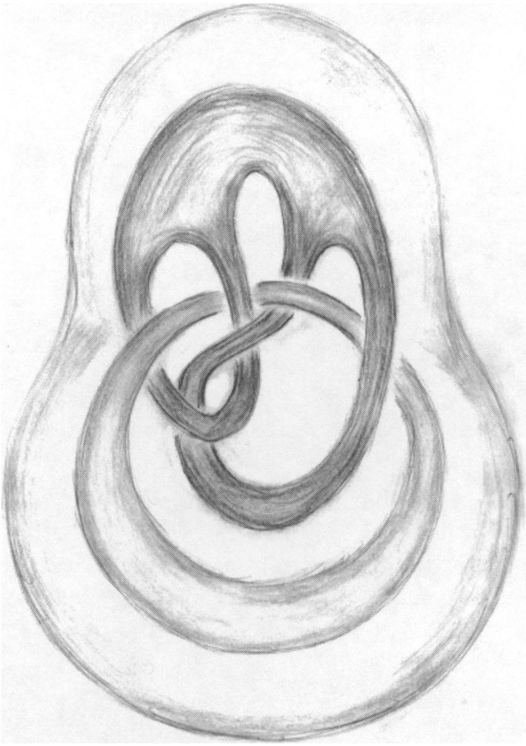

Figure 14. No limits to their beauty.

Likewise, $H_1(\Omega)$ is generated by equivalence classes of oriented loops in Ω, with two loops equivalent if their difference is the boundary of an oriented surface in Ω. The space $H_2(\Omega)$ is generated by equivalence classes of closed oriented surfaces in Ω, with two such surfaces equivalent if their difference is the boundary of some oriented subregion of Ω. The space $H_3(\Omega)$ is always zero.

The relative homology vector space $H_0(\Omega, \partial\Omega)$ is always zero. The space $H_1(\Omega, \partial\Omega)$ is generated by equivalence classes of oriented paths whose endpoints lie on $\partial\Omega$, with two such paths regarded as equivalent if their difference, augmented as necessary by paths on $\partial\Omega$, is the boundary of an oriented surface in Ω. The space $H_2(\Omega, \partial\Omega)$ is generated by equivalence classes of oriented surfaces whose boundaries lie on $\partial\Omega$, with two such surfaces equivalent if their difference, augmented as necessary by portions of $\partial\Omega$, is the boundary of some oriented subregion of Ω. And finally, $H_3(\Omega, \partial\Omega)$ has as a basis the oriented components of Ω, since these are the subregions of Ω whose boundaries lie on $\partial\Omega$.

The homology of the domain of definition plays an important role in vector calculus. Stokes' Theorem and the divergence theorem help us to understand it.

Suppose that V is a smooth vector field on Ω, and that C is a smooth oriented loop in Ω. Then $\int_C V \cdot \mathrm{d}s$ is called the *circulation of V around C*.

If V is curl-free, then the circulation of V around C depends only on the homology class of C in $H_1(\Omega)$. This is a consequence of Stokes' Theorem, for if the oriented loops C and C' together bound a surface S, meaning that $\partial S = C - C'$, then

$$\int_C V \cdot ds - \int_{C'} V \cdot ds = \int_{C-C'} V \cdot ds = \int_{\partial S} V \cdot ds = \int_S \nabla \times V \cdot \mathbf{n}\, \mathrm{d(area)} = 0.$$

We can also integrate V along an oriented path P. If the endpoints of P lie on $\partial\Omega$, and if V is curl-free and orthogonal to the boundary of Ω, then the value of the integral $\int_P V \cdot ds$ depends only on the relative homology class of P in $H_1(\Omega, \partial\Omega)$. This is again a consequence of Stokes' Theorem.

Now suppose that S is a smooth oriented surface without boundary in Ω, and consider the flux $\int_S V \cdot \mathbf{n}\, d(\text{area})$ of V through S. If V is divergence-free, then this flux depends only on the homology class of S in $H_2(\Omega)$. This is a consequence of the divergence theorem, for if the closed oriented surfaces S and S' together bound a subregion Ω' of Ω, then

$$\begin{aligned}\int_S V \cdot \mathbf{n}\, d(\text{area}) - \int_{S'} V \cdot \mathbf{n}\, d(\text{area}) &= \int_{S-S'} V \cdot \mathbf{n}\, d(\text{area}) \\ &= \int_{\partial\Omega'} V \cdot \mathbf{n}\, d(\text{area}) = \int_{\Omega'} \nabla \cdot V\, d(\text{vol}) = 0.\end{aligned}$$

We can also compute the flux of V through a smooth surface Σ in Ω when Σ has a non-empty boundary. If $\partial\Sigma \subset \partial\Omega$, and if V is divergence-free and tangent to the boundary of Ω, then the value of the flux integral $\int_\Sigma V \cdot \mathbf{n}\, d(\text{area})$ depends only on the relative homology class of Σ in $H_2(\Omega, \partial\Omega)$. This is also a consequence of the divergence theorem.

We continue now with our study of the homology vector spaces of a domain Ω, whose dimensions are determined by three integers: the number of components of Ω, the number of components of $\partial\Omega$, and the total genus of $\partial\Omega$; see Table 1.

TABLE 1.

Absolute Homology	Dimension	Relative Homology	Dimension
$H_0(\Omega)$	# comp Ω	$H_0(\Omega, \partial\Omega)$	0
$H_1(\Omega)$	total genus of $\partial\Omega$	$H_1(\Omega, \partial\Omega)$	# comp $\partial\Omega$ − # comp Ω
$H_2(\Omega)$	# comp $\partial\Omega$ − # comp Ω	$H_2(\Omega, \partial\Omega)$	total genus of $\partial\Omega$
$H_3(\Omega)$	0	$H_3(\Omega, \partial\Omega)$	# comp Ω

The confirmation of these dimensions, which we do not carry out here, relies on four tools:

- *Poincaré duality*, which compares the absolute and relative homology vector spaces of Ω;
- *Alexander duality*, which compares the homology vector spaces of Ω with those of the closure Ω' of its complementary domain $\mathbf{R}^3 - \Omega$;
- *The Mayer-Vietoris sequence*, which interweaves the homology of Ω and Ω' with that of their intersection $\Omega \cap \Omega' = \partial\Omega = \partial\Omega'$ and of their union $\Omega \cup \Omega' = \mathbf{R}^3$;
- *The long exact homology sequence* for the pair $(\Omega, \partial\Omega)$.

Poincaré duality provides the following isomorphisms:

$$\begin{aligned} H_0(\Omega) &\cong H_3(\Omega, \partial\Omega) \qquad & H_1(\Omega) &\cong H_2(\Omega, \partial\Omega) \\ H_2(\Omega) &\cong H_1(\Omega, \partial\Omega) \qquad & H_3(\Omega) &\cong H_0(\Omega, \partial\Omega).\end{aligned}$$

The dimensions of the homology vector spaces listed in Table 1 are clearly consistent with these isomorphisms. Poincaré duality actually has more to say, and we return to it later in this section.

Alexander duality provides the isomorphisms

$$H_0(\Omega) \cong H_2(\Omega'), \qquad H_1(\Omega) \cong H_1(\Omega'), \qquad H_2(\Omega) \cong \tilde{H}_0(\Omega'),$$

where the tilde over $H_0(\Omega')$ reduces the dimension of this vector space by one. These isomorphisms tell us that the homology of the closed complementary domain Ω' depends only on the homology of Ω, and not on how Ω is embedded in 3-space. This is in sharp contrast to the way fundamental groups behave: if Ω is homeomorphic to a solid torus, then the fundamental group $\pi_1(\Omega')$ of its closed complementary domain contains a wealth of information about the way Ω is embedded (i.e., knotted) in 3-space.

We also use the Mayer-Vietoris sequence:

$$\cdots \to H_{i+1}(\mathbf{R}^3) \to H_i(\partial\Omega) \to H_i(\Omega) \oplus H_i(\Omega') \to H_i(\mathbf{R}^3) \to \cdots,$$

from which we learn:

(1) Every two-dimensional homology class in Ω is represented by a linear combination of the components of $\partial\Omega$, and likewise every two-dimensional homology class in Ω' is represented by a linear combination of the components of $\partial\Omega' = \partial\Omega$.

(2) Every one-dimensional homology class in Ω is represented by a closed curve on $\partial\Omega$, and likewise for Ω'.

(3) Every one-dimensional homology class on $\partial\Omega$ can be expressed as the sum of a class that bounds in Ω (such a class is represented by a curve on $\partial\Omega$ that bounds a surface in Ω) and a class that bounds in Ω'.

We use items (2) and (3) in what follows.

We end this section by mentioning one of the finer aspects of Poincaré and Alexander duality, involving intersection and linking numbers.

Let k denote the genus of $\partial\Omega$, which is the common dimension of the three homology vector spaces

$$H_1(\Omega) \cong H_2(\Omega, \partial\Omega) \cong H_1(\Omega').$$

Poincaré and Alexander duality actually guarantee that we can choose loops and surfaces that represent bases for these three vector spaces so that they link and intersect one another in a very special way. We illustrate this in Figure 15, in which Ω is a solid double torus, and the common dimension k is 2.

Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ be a family of cross-sectional surfaces in Ω, whose boundaries lie on $\partial\Omega$, which represent a basis for $H_2(\Omega, \partial\Omega)$. In Figure 15, Σ_1 and Σ_2 appear as two disjoint disks.

Let $C_1, C_2, \ldots, C_k$ be a family of loops in the interior of Ω that represent a basis for $H_1(\Omega)$, chosen so that the intersection number of C_i with Σ_j is δ_{ij} (that is, it is 1 if $i = j$ and 0 if $i \neq j$). The specific details of Poincaré duality guarantee that this can be done. It is easy to make these loops disjoint. If we push the boundaries of the cross-sectional surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ slightly into the exterior region $\mathbf{R}^3 - \Omega$, we get a family $C_1', C_2', \ldots, C_k'$ of loops in $\mathbf{R}^3 - \Omega$ that represent a basis for $H_1(\Omega')$. It is

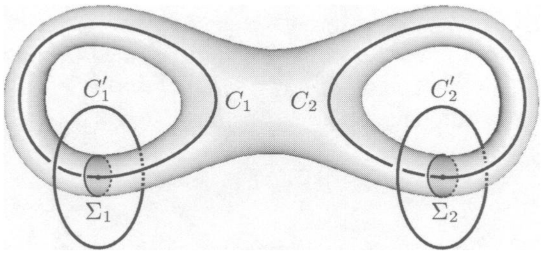

Figure 15. Related bases for $H_1(\Omega) \cong H_2(\Omega, \partial\Omega) \cong H_1(\Omega')$.

easy to make these loops disjoint. Furthermore, the linking number of C_i with C_j' is automatically δ_{ij}, as shown in Figure 15.

We caution the reader that Figure 15 depicts this situation in its simplest aspect. A cross-sectional surface Σ_j may have several boundary curves, in which case C_j' consists of several loops. And it may be impossible to make the cross-sectional surfaces disjoint from one another. Nevertheless, the bases may be chosen with the asserted intersection and linking numbers.

We refer back to this discussion, and choose such conveniently related bases for $H_1(\Omega)$, $H_2(\Omega, \partial\Omega)$, and $H_1(\Omega')$ several times in the sections ahead.

For further information about Poincaré and Alexander duality, see **[45]** and **[5]**.

7. PROOF OF PROPOSITION 1. Armed with the analytical and topological prerequisites from the preceding sections, we turn to the proofs of the propositions that lead to the Hodge Decomposition Theorem. First up is Proposition 1, which asserts that the space $\mathrm{VF}(\Omega)$ decomposes orthogonally as $\mathrm{K} \oplus \mathrm{G}$, which means splitting all smooth vector fields on Ω into those that are divergence-free and tangent to the boundary, and those that are gradients of smooth functions.

The argument is straightforward. Let V be an arbitrary smooth vector field on Ω, let f be the smooth function $f = \nabla \cdot V$ defined on Ω, and let g be the smooth function $g = V \cdot \mathbf{n}$ defined on $\partial\Omega$.

As a consequence of the divergence theorem, we have

$$\int_{\Omega_i} f \,\mathrm{d(vol)} = \int_{\Omega_i} \nabla \cdot V \,\mathrm{d(vol)} = \int_{\partial\Omega_i} V \cdot \mathbf{n} \,\mathrm{d(area)} = \int_{\partial\Omega_i} g \,\mathrm{d(area)}$$

for each component Ω_i of Ω. Thus, by Theorem 1, we have a solution φ of the Poisson equation $\Delta\varphi = f$ on Ω with Neumann boundary condition $\partial\varphi/\partial\mathbf{n} = g$ on $\partial\Omega$.

Define the vector fields $V_2 = \nabla\varphi$ and $V_1 = V - V_2$. Then on $\partial\Omega$, we have

$$V_2 \cdot \mathbf{n} = \nabla\varphi \cdot \mathbf{n} = \frac{\partial\varphi}{\partial\mathbf{n}} = g = V \cdot \mathbf{n},$$

and hence $V_1 \cdot \mathbf{n} = 0$ on $\partial\Omega$. Also, on Ω we have

$$\nabla \cdot V_2 = \nabla \cdot \nabla\varphi = \Delta\varphi = f = \nabla \cdot V,$$

and hence $\nabla \cdot V_1 = 0$.

Thus the vector field V_1 is divergence-free on Ω and is tangent to $\partial\Omega$, while V_2 is of course a gradient vector field. This shows that $\mathrm{VF}(\Omega) = \mathrm{K} + \mathrm{G}$, so we must now check that this is an orthogonal direct sum.

To this end, let V_1 denote any smooth divergence-free vector field on Ω that is tangent to $\partial\Omega$, and let $V_2 = \nabla\varphi$ denote any smooth gradient vector field on Ω.

In the following, we leave out the expressions "d(vol)" and "d(area)" from our integrals where there is little chance of confusion. Using the product rule

$$\nabla \cdot (\varphi V_1) = (\nabla\varphi) \cdot V_1 + \varphi(\nabla \cdot V_1),$$

we have

$$\begin{aligned}
\langle V_1, V_2 \rangle &= \int_\Omega V_1 \cdot V_2 = \int_\Omega V_1 \cdot \nabla\varphi \\
&= \int_\Omega (\nabla \cdot (\varphi V_1) - \varphi(\nabla \cdot V_1)) = \int_\Omega \nabla \cdot (\varphi V_1) \\
&= \int_{\partial\Omega} (\varphi V_1) \cdot \mathbf{n} = 0,
\end{aligned}$$

where we have used both conditions on V_1, namely, that it is divergence-free and that it is tangent to $\partial\Omega$.

Hence our two summands K and G are orthogonal, and in particular their sum is direct. Thus we have shown that $\mathrm{VF}(\Omega) = \mathrm{K} \oplus \mathrm{G}$, which completes the proof of Proposition 1.

8. A LEMMA ABOUT FLUXLESS KNOTS. We begin preparing for the proof of Proposition 2. Recall the subspace of *fluxless knots*,

$$\mathrm{FK} = \{V \in \mathrm{VF}(\Omega) : \nabla \cdot V = 0,\ V \cdot \mathbf{n} = 0,\ \text{ all interior fluxes are } 0\}.$$

We use the following lemma in the proof of Proposition 2.

Lemma 1. *The fluxless knots are also given by*

$$\mathrm{FK} = \{\nabla \times U : \nabla \cdot U = 0,\ U \times \boldsymbol{n} = \mathbf{0}\}.$$

That is, we claim that every fluxless knot V is the curl of a divergence-free vector field U that is orthogonal to the boundary of Ω, and vice versa, that every such curl is a fluxless knot.

To begin, let V be a fluxless knot. Thinking of V as a current distribution, let $\mathrm{BS}(V)$ be the resulting magnetic field given by (1), and let B denote its restriction to the domain Ω.

Since B is a magnetic field, we know that it is divergence-free: $\nabla \cdot B = 0$. Since V is divergence-free and is tangent to $\partial\Omega$, we know from the curl formula (2) that $\nabla \times B = V$.

We want to adjust B so that it becomes orthogonal to the boundary of Ω, without altering the facts that it is divergence-free and that its curl is V. We do this by subtracting from B the gradient of an appropriate harmonic function.

Consider the vector field $B^{\parallel}|_{\partial\Omega}$ along $\partial\Omega$ obtained from the component of B that is parallel to $\partial\Omega$. We claim that this vector field on $\partial\Omega$ is the gradient of some smooth real-valued function f defined on $\partial\Omega$.

To verify this claim, consider the circulation of $B^{\parallel}|_{\partial\Omega}$ (equivalently, of B) around a closed curve C on $\partial\Omega$. Since $\nabla \times B = V$ is tangent to $\partial\Omega$, Stokes' Theorem tells us that the value of this integral depends only on the homology class of C in $H_1(\partial\Omega)$.

If C bounds a surface Σ inside Ω, then by Stokes' Theorem,

$$\begin{aligned}\int_C B \cdot ds = \int_\Sigma (\nabla \times B) \cdot \mathbf{n} = \int_\Sigma V \cdot \mathbf{n} \\ = \text{flux of } V \text{ through } \Sigma = 0,\end{aligned} \tag{6}$$

because V is fluxless by hypothesis.

If C bounds a surface Σ outside Ω, then again by Stokes' Theorem,

$$\int_C B \cdot ds = \int_C \mathrm{BS}(V) \cdot ds = \int_\Sigma (\nabla \times \mathrm{BS}(V)) \cdot \mathbf{n} = 0, \tag{7}$$

since outside of Ω, we have $\nabla \times \mathrm{BS}(V) = \mathbf{0}$ by the curl formula (2).

But we saw in item (3) of Section 6 that every closed curve C on $\partial\Omega$ is homologous to the sum of a closed curve that bounds inside Ω, and a closed curve that bounds outside Ω. Hence by (6) and (7), $\int_C B \cdot ds = 0$. Therefore $B^{\|}|_{\partial\Omega} = \nabla^{\|} f$ for some smooth function $f : \partial\Omega \to R$, where $\nabla^{\|} f$ denotes the gradient of f taken along the surface $\partial\Omega$.

Now let $\varphi : \Omega \to R$ be the solution of Laplace's equation $\Delta\varphi = 0$, with Dirichlet boundary condition $\varphi|_{\partial\Omega} = f$.

Note that $\nabla \cdot \nabla\varphi = \Delta\varphi = 0$ and that $\nabla \times \nabla\varphi = \mathbf{0}$. Hence, if $U = B - \nabla\varphi$, then $\nabla \cdot U = 0$ and $\nabla \times U = V$.

Furthermore, for any vector T tangent to $\partial\Omega$, we have

$$B \cdot T = (B^{\|}|_{\partial\Omega}) \cdot T = (\nabla^{\|} f) \cdot T = (\nabla\varphi) \cdot T,$$

and hence

$$U \cdot T = (B - \nabla\varphi) \cdot T = 0,$$

which tells us that $U \times \mathbf{n} = \mathbf{0}$. Thus we have shown that the fluxless field V is the curl of a divergence-free field U that is orthogonal to $\partial\Omega$, as desired.

To complete the proof of Lemma 1, we must start with a vector field U on Ω that satisfies $\nabla \cdot U = 0$ and $U \times \mathbf{n} = \mathbf{0}$, then define $V = \nabla \times U$, and show that V is a fluxless knot, that is, $\nabla \cdot V = 0$, $V \cdot \mathbf{n} = 0$, and all interior fluxes of V are zero.

Clearly $\nabla \cdot V = \nabla \cdot (\nabla \times U) = 0$. Also, all interior fluxes of V are zero because

$$\int_\Sigma V \cdot \mathbf{n} = \int_\Sigma (\nabla \times U) \cdot \mathbf{n} = \int_{\partial\Sigma} U \cdot ds = 0,$$

where the last integrand is identically zero because $\partial\Sigma \subset \partial\Omega$ and U is orthogonal to $\partial\Omega$.

It remains to show that V is tangent to the boundary of Ω. Since we have already proved Proposition 1, we can do this by showing that V is orthogonal, in the sense of the L^2 inner product on $\mathrm{VF}(\Omega)$, to every gradient vector field. This tells us that $V \in \mathrm{K}$, and so in particular V is tangent to $\partial\Omega$.

Let φ be any smooth function on Ω. We show that $\langle V, \nabla\varphi \rangle = 0$. We make use of the identity

$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B),$$

with U playing the role of A and $\nabla\varphi$ the role of B.

We compute:

$$\begin{aligned}
\langle V, \nabla\varphi\rangle &= \langle \nabla \times U, \nabla\varphi\rangle \\
&= \int_\Omega (\nabla \times U) \cdot \nabla\varphi \\
&= \int_\Omega (\nabla \cdot (U \times \nabla\varphi) + U \cdot (\nabla \times \nabla\varphi)) \\
&= \int_\Omega \nabla \cdot (U \times \nabla\varphi) \\
&= \int_{\partial\Omega} (U \times \nabla\varphi) \cdot \mathbf{n} = 0,
\end{aligned}$$

where the last integrand is identically zero because U is orthogonal to $\partial\Omega$.

Thus V is L^2-orthogonal to every gradient vector field, and so it lies in K, and hence is tangent to the boundary of Ω. This completes the proof of Lemma 1. Note that we did not use the fact that $\nabla \cdot U = 0$.

9. HARMONIC KNOTS. Still preparing for the proof of Proposition 2, we need to demonstrate the abundance of *harmonic knots*. Recall the definition:

$$\mathrm{HK} = \{V \in \mathrm{VF}(\Omega) : \nabla \cdot V = 0,\ \nabla \times V = \mathbf{0},\ V \cdot \mathbf{n} = 0\}.$$

Lemma 2. $\mathrm{HK} \cong H_1(\Omega) \cong H_2(\Omega, \partial\Omega) \cong \mathbf{R}^{\text{genus of } \partial\Omega}$.

We begin small and specific, and suppose that Ω is a solid torus of revolution, so that $H_1(\Omega)$ is one-dimensional. How can we get a vector field that generates HK?

Let C be the core circle inside Ω, and let C' be a circle in $\mathbf{R}^3 - \Omega$ that passes through the hole in Ω. Thus C and C' link once; see Figure 16.

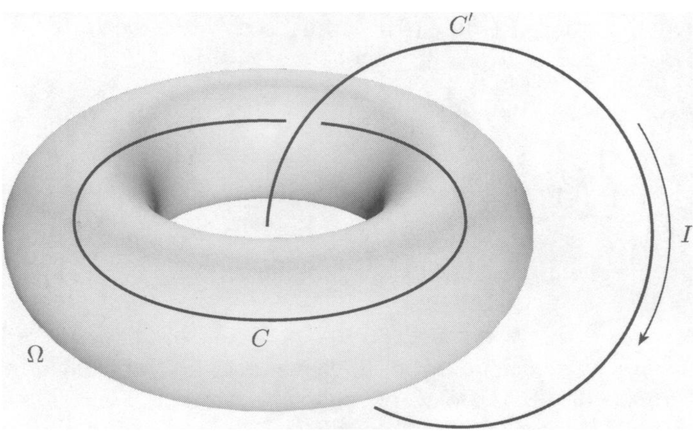

Figure 16. Finding a harmonic knot.

Run a current I through the loop C' and let B denote the restriction of its magnetic field to the domain Ω. Then, as indicated in Section 5, the field B has the following properties:

$$\nabla \cdot B = 0, \qquad \nabla \times B = \mathbf{0}, \qquad \int_C B \cdot ds = I. \tag{8}$$

In general, we cannot expect B to be tangent to the boundary of Ω. We plan to fix this, without disturbing the three properties (8) of B, by subtracting from B a gradient vector field that is divergence-free, (automatically) curl-free, and has the same normal component as B along $\partial\Omega$.

To this end, consider the smooth real-valued function $g = B \cdot \mathbf{n}$ on $\partial\Omega$. Since B is divergence-free, we have $\int_{\partial\Omega} g \, d(\text{area}) = 0$. Thus we can find a solution φ of Laplace's equation $\Delta\varphi = 0$ with Neumann boundary condition $\partial\varphi/\partial\mathbf{n} = g$ on $\partial\Omega$.

The gradient vector field $\nabla\varphi$ has the following properties:

$$\nabla \cdot (\nabla\varphi) = \Delta\varphi = 0, \qquad \nabla \times \nabla\varphi = \mathbf{0},$$
$$\nabla\varphi \cdot \mathbf{n} = \frac{\partial\varphi}{\partial\mathbf{n}} = g, \qquad \int_C \nabla\varphi \cdot ds = 0.$$

Now define $V = B - \nabla\varphi$. Then V satisfies:

$$\nabla \cdot V = 0, \qquad \nabla \times V = \mathbf{0},$$
$$V \cdot \mathbf{n} = 0, \qquad \int_C V \cdot ds = I.$$

Thus V is a nonzero harmonic knot in Ω.

The vector field V has circulation $\int V \cdot ds = I$ around every loop that goes once the long way around Ω, since for curl-free vector fields the circulation depends only on the homology class of the loop.

The flux Φ of V through any cross-sectional disk Σ of Ω is independent of the cross-section because V is divergence-free and is tangent to $\partial\Omega$. If Φ were zero, then V would be a fluxless harmonic knot and hence (by the soon to be proved orthogonality of FK and HK) would be zero, contrary to fact. Thus its flux $\Phi \neq 0$.

We note that V and its real multiples aV are the *only* harmonic knots defined on Ω. For if V^* is a harmonic knot in Ω with flux Φ^* through a cross-sectional disk Σ, then $V^* - (\Phi^*/\Phi)V$ is a fluxless harmonic knot, and hence is zero.

It follows that, in this simple case, the subspace HK of harmonic knots on our torus of revolution satisfies

$$\text{HK} \cong \mathbf{R}^1 \cong H_1(\Omega) \cong H_2(\Omega, \partial\Omega) \cong \mathbf{R}^{\text{genus of } \partial\Omega},$$

as claimed in Lemma 2.

Now suppose that Ω is an arbitrary compact domain in 3-space, with smooth boundary $\partial\Omega$.

Recall from Section 6 that, by Poincaré duality, $H_1(\Omega) \cong H_2(\Omega, \partial\Omega)$, while by Alexander duality, $H_1(\Omega) \cong H_1(\mathbf{R}^3 - \Omega)$. Let k be the common dimension of these three vector spaces.

Let $C_1, C_2, \ldots, C_k$ be a family of disjoint loops Ω that represents a basis for $H_1(\Omega)$; let $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ be a family of cross-sectional surfaces in Ω that represents a basis for $H_2(\Omega, \partial\Omega)$; and let $C_1', C_2', \ldots, C_k'$ be a family of disjoint loops in $\mathbf{R}^3 - \Omega$ that represents a basis for $H_1(\mathbf{R}^3 - \Omega)$. These bases are to be chosen, as indicated at the end of Section 6, so that the intersection number of C_i with Σ_j is δ_{ij}, and so that the linking number of C_i with C_j' is also δ_{ij}.

Now run currents $I_1, I_2, \ldots, I_k$ through the loops $C'_1, C'_2, \ldots, C'_k$, and let B be the corresponding magnetic field in Ω. Then in the domain Ω we have

$$\nabla \cdot B = 0, \qquad \nabla \times B = \mathbf{0}, \qquad \int_{C_i} B \cdot ds = I_i. \tag{9}$$

Exactly as in the simple case of the torus of revolution, we subtract from B the gradient $\nabla\varphi$ of a harmonic function φ that satisfies the Neumann boundary condition $\partial\varphi/\partial\mathbf{n} = g = B \cdot \mathbf{n}$ on $\partial\Omega$.

Then the three conditions (9) for B are also satisfied by the vector field $V = B - \nabla\varphi$, and moreover $V \cdot \mathbf{n} = 0$. Thus $V = V(I_1, I_2, \ldots, I_k)$ is a nonzero harmonic knot, provided that at least one of the currents I_i is nonzero.

Let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be the fluxes of V through the cross-sectional surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$. If these were all zero, then V would be a fluxless harmonic knot, and again, by the soon to be proved orthogonality of FK and HK, would be zero. Therefore at least one of the Φ_i must be nonzero. And the linear transformation that takes the current data $(I_1, I_2, \ldots, I_k)$ to the flux data $(\Phi_1, \Phi_2, \ldots, \Phi_k)$ must be nonsingular, and therefore must be an isomorphism.

It follows that the fields $V(I_1, I_2, \ldots, I_k)$ are the only harmonic knots in Ω. For if V' is a harmonic knot in Ω with fluxes $\Phi_1, \Phi_2, \ldots, \Phi_k$ through the cross-sectional surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$, then one of our fields $V(I_1, I_2, \ldots, I_k)$ must have the same flux values, and hence $V' - V(I_1, I_2, \ldots, I_k)$ is a fluxless harmonic knot, and is therefore zero. So $V' = V(I_1, I_2, \ldots, I_k)$.

Thus $\mathrm{HK} \cong \mathbf{R}^k \cong H_1(\Omega) \cong H_2(\Omega, \partial\Omega) \cong \mathbf{R}^{\text{genus of } \partial\Omega}$, which completes the proof of Lemma 2.

10. PROOF OF PROPOSITION 2. With all the groundwork behind us, we now easily complete the proof of Proposition 2. We must show that the subspace of divergence-free vector fields that are tangent to the boundary of Ω is the orthogonal direct sum of the subspace of fluxless knots and the subspace of harmonic knots:

$$\mathrm{K} = \mathrm{FK} \oplus \mathrm{HK}.$$

Let V be a divergence-free vector field defined in Ω and tangent to its boundary. Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ be a family of cross-sectional surfaces in Ω that form a basis for the relative homology $H_2(\Omega, \partial\Omega)$. Let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be the fluxes of V through these surfaces.

According to Lemma 2 and its proof, there is a harmonic knot V_H in Ω with precisely these flux values. Let $V_F = V - V_H$. Then V_F is fluxless. Thus every divergence-free vector field V defined in Ω and tangent to its boundary can be written as the sum of a fluxless knot V_F and a harmonic knot V_H.

Now we show that the fluxless knots are orthogonal to the harmonic knots.

To this end, let $V \in \mathrm{FK}$ be a fluxless knot. That is, $\nabla \cdot V = 0$, $V \cdot \mathbf{n} = 0$, and all interior fluxes are zero. By Lemma 1, we can write $V = \nabla \times U$, where $\nabla \cdot U = 0$ and $U \times \mathbf{n} = \mathbf{0}$.

Let $W \in \mathrm{HK}$ be a harmonic knot. That is, $\nabla \cdot W = 0$, $\nabla \times W = \mathbf{0}$, and $W \cdot \mathbf{n} = 0$. Again using the formula

$$\nabla \cdot (U \times W) = (\nabla \times U) \cdot W - U \cdot (\nabla \times W),$$

we have

$$
\begin{aligned}
\langle V, W\rangle &= \langle \nabla \times U, W\rangle = \int_\Omega (\nabla \times U)\cdot W \\
&= \int_\Omega (\nabla\cdot(U\times W) + U\cdot(\nabla\times W)) = \int_\Omega \nabla\cdot(U\times W) \\
&= \int_{\partial\Omega} (U\times W)\cdot \mathbf{n} = 0,
\end{aligned}
$$

because U is orthogonal to $\partial\Omega$.

This completes the proof of Proposition 2.

11. PROOF OF PROPOSITION 3. We must show that the subspace of gradient vector fields on Ω is the orthogonal direct sum of the subspace of divergence-free gradient fields and the subspace of grounded gradient fields (that is, gradients of functions that vanish on $\partial\Omega$):

$$
\mathrm{G} = \mathrm{DFG} \oplus \mathrm{GG}.
$$

The argument is straightforward. We start with a gradient vector field $V = \nabla\varphi$, where φ is any smooth function on Ω. Let φ_1 be a solution of the Laplace equation $\Delta\varphi_1 = 0$ on Ω, with Dirichlet boundary condition $\varphi_1|_{\partial\Omega} = \varphi|_{\partial\Omega}$, and let $\varphi_2 = \varphi - \varphi_1$. Then $V_1 = \nabla\varphi_1$ and $V_2 = \nabla\varphi_2$ satisfy $V = V_1 + V_2$.

We note that $\nabla\cdot V_1 = \nabla\cdot\nabla\varphi_1 = \Delta\varphi_1 = 0$, so that $V_1 \in \mathrm{DFG}$. Similarly, $\varphi_2|_{\partial\Omega} = \varphi|_{\partial\Omega} - \varphi_1|_{\partial\Omega} = 0$, so that $V_2 \in \mathrm{GG}$. Thus the subspaces DFG and GG certainly span G.

Now we want to show that the divergence-free gradients are orthogonal to the grounded gradients. To this end, let $V \in \mathrm{DFG}$. Thus $V = \nabla\varphi$ with $\nabla\cdot V = \Delta\varphi = 0$. Likewise, let $W \in \mathrm{GG}$, and write $W = \nabla\psi$ with $\psi|_{\partial\Omega} = 0$.

We use the identity

$$
\nabla\cdot(\psi\nabla\varphi) = \nabla\psi\cdot\nabla\varphi + \psi\Delta\varphi,
$$

to get

$$
\begin{aligned}
\langle V, W\rangle &= \int_\Omega V\cdot W = \int_\Omega \nabla\varphi\cdot\nabla\psi \\
&= \int_\Omega \Big(\nabla\cdot(\psi\nabla\varphi) - \psi\Delta\varphi\Big) = \int_\Omega \nabla\cdot(\psi\nabla\varphi) \\
&= \int_{\partial\Omega} (\psi\nabla\varphi)\cdot\mathbf{n} = 0,
\end{aligned}
$$

where we have used the facts that $\Delta\varphi = 0$ and $\psi|_{\partial\Omega} = 0$.

This completes the proof of Proposition 3.

12. HARMONIC GRADIENTS. Before carrying out the proof of Proposition 4, we need to demonstrate the abundance of *harmonic gradients*. Recall the definition:

$$
\mathrm{HG} = \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \nabla\cdot V = 0,\ \varphi \text{ constant on each component of } \partial\Omega\}.
$$

Lemma 3. $\mathrm{HG} \cong H_2(\Omega) \cong H_1(\Omega, \partial\Omega) \cong \mathbf{R}^{(\#\text{ components of }\partial\Omega)-(\#\text{ components of }\Omega)}$.

Let Ω have k components $\Omega_1, \Omega_2, \dots, \Omega_k$, and for $1 \le i \le k$, let $\partial\Omega_i$ have r_i components $\partial\Omega_{i1}, \partial\Omega_{i2}, \dots, \partial\Omega_{ir_i}$. Thus $\partial\Omega$ has $r = r_1 + r_2 + \cdots + r_k$ components:

$$\partial\Omega = \bigcup\{\partial\Omega_{ij} : 1 \le i \le k,\ 1 \le j \le r_i\}.$$

Now let

$$\{c_{ij} : 1 \le i \le k,\ 1 \le j \le r_i\}$$

be a set of r constants, subject to the k relations

$$\sum_{j=1}^{r_i} c_{ij} = 0, \qquad \text{for } 1 \le i \le k.$$

Let φ be a solution of the Laplace equation $\Delta\varphi = 0$ on Ω with Dirichlet boundary conditions $\varphi|_{\partial\Omega_{ij}} = c_{ij}$. Then the divergence-free vector field $V = \nabla\varphi$ belongs to HG. Since we can alter φ on each component Ω_i of Ω by an additive constant (with different constants on different components) without changing $V = \nabla\varphi$, we lose no vector fields by imposing the k relations $\sum_j c_{ij} = 0$ for $1 \le i \le k$.

By uniqueness of solutions of the Dirichlet problem for the Laplace equation, there are no other elements of HG. This finishes the proof of Lemma 3.

Now let Φ_{ij} be the flux of V through the boundary component $\partial\Omega_{ij}$. Since V is divergence-free, we have

$$\sum_{j=1}^{r_i} \Phi_{ij} = 0, \qquad \text{for } 1 \le i \le k.$$

We use the following lemma in the proof of Proposition 4.

Lemma 4. *The linear map from boundary values of harmonic functions to the fluxes of their gradients through the components of $\partial\Omega$, which takes the $(r-k)$-dimensional vector space*

$$\left\{(c_{ij}) : 1 \le i \le k,\ 1 \le j \le r_i,\ \sum_{j=1}^{r_i} c_{ij} = 0 \, for\ 1 \le i \le k\right\}$$

to the $(r-k)$-dimensional vector space

$$\left\{(\Phi_{ij}) : 1 \le i \le k,\ 1 \le j \le r_i,\ \sum_{j=1}^{r_i} \Phi_{ij} = 0 \, for\ 1 \le i \le k\right\},$$

is an isomorphism.

Suppose not. Then there is some set of boundary values (c_{ij}), not identically zero, for which the corresponding set of fluxes (Φ_{ij}) is identically zero.

Let V be the corresponding vector field, constructed as in the proof of Lemma 3. Then V lies in both CG and HG, and so (by their soon to be proved orthogonality) must be zero. Thus φ must be constant on each component Ω_j of Ω, and since $\sum_j c_{ij} = 0$, each $c_{ij} = 0$. Thus the linear map in question must be an isomorphism. This proves Lemma 4.

We pause to contrast harmonic knots with harmonic gradients.

Harmonic knots are smooth vector fields in Ω that are divergence-free, curl-free, and tangent to the boundary of Ω.

Harmonic gradients, we claim, are smooth vector fields in Ω that are divergence-free, curl-free, and orthogonal to the boundary of Ω. Every harmonic gradient certainly has these three features, so suppose we are given a vector field V with these three properties. We must show that it is a harmonic gradient.

To see this, first recall that a curl-free vector field on Ω is a gradient vector field if and only if it has zero integral around a family of loops that generates $H_1(\Omega)$. Since the map $H_1(\partial\Omega) \to H_1(\Omega)$ is onto, as noted in Section 6, these loops may be chosen on the boundary of Ω, where the integral is sure to be zero because V is orthogonal to the boundary. This shows that V must be a gradient vector field: $V = \nabla\varphi$, for some smooth function φ on Ω. Since V is divergence-free, the function φ is harmonic. Since V is orthogonal to the boundary of Ω, φ must be constant on each component of $\partial\Omega$. Thus V is a harmonic gradient.

13. PROOF OF PROPOSITION 4. With the groundwork done, we now complete the proof of Proposition 4. We must show that the subspace of divergence-free gradient vector fields on Ω is the orthogonal direct sum of the subspace of curly gradients and the subspace of harmonic gradients:

$$\mathrm{DFG} = \mathrm{CG} \oplus \mathrm{HG}.$$

Recall the definitions:

$$\mathrm{CG} = \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \nabla\cdot V = 0,\ \text{all boundary fluxes are } 0\}$$

$$\mathrm{HG} = \{V \in \mathrm{VF}(\Omega) : V = \nabla\varphi,\ \nabla\cdot V = 0,\ \varphi \text{ is constant on each component of } \partial\Omega\}.$$

Start with a vector field $V \in \mathrm{DFG}$. Thus $V = \nabla\varphi$ and $\nabla\cdot V = \Delta\varphi = 0$.

Let (Φ_{ij}) be the fluxes of V through the components $\partial\Omega_{ij}$ of $\partial\Omega$, where $\partial\Omega_{ij}$ is the j-th connected component of the boundary of the i-th connected component of Ω. Since V is divergence-free, we know that $\sum_j \Phi_{ij} = 0$ for $1 \le i \le k$.

Using the isomorphism established in Lemma 4, we choose constants (c_{ij}) with $\sum_j c_{ij} = 0$ so that the harmonic function ψ with $\psi|_{\partial\Omega_{ij}} = c_{ij}$ provides us with a vector field $V_2 = \nabla\psi$ in HG with fluxes Φ_{ij} through $\partial\Omega_{ij}$.

Then $V_1 = V - V_2 = \nabla(\varphi - \psi)$ has zero flux through each component $\partial\Omega_{ij}$ of $\partial\Omega$, and hence $V_1 \in \mathrm{CG}$. Since $V_2 \in \mathrm{HG}$, these two subspaces certainly span DFG.

It remains to check that these subspaces are orthogonal.

Let $V \in \mathrm{CG}$. Thus $V = \nabla\varphi$, $\nabla\cdot V = \Delta\varphi = 0$, and all boundary fluxes of V are zero.

Let $W \in \mathrm{HG}$. Thus $W = \nabla\psi$, $\nabla\cdot W = \Delta\psi = 0$, and ψ is constant on each component $\partial\Omega_{ij}$ of $\partial\Omega$.

Then, once again using the formula

$$\nabla\cdot(\psi\nabla\varphi) = \nabla\psi\cdot\nabla\varphi + \psi\Delta\varphi,$$

we have

$$\langle V, W\rangle = \int_\Omega \nabla\varphi\cdot\nabla\psi = \int_\Omega (\nabla\cdot(\psi\nabla\varphi) - \psi\Delta\varphi)$$

$$
\begin{aligned}
&= \int_\Omega \nabla \cdot (\psi \nabla \phi) = \int_{\partial\Omega} \psi \nabla \varphi \cdot \mathbf{n} \\
&= \sum_{i,j} \int_{\partial\Omega_{ij}} \psi \nabla \varphi \cdot \mathbf{n} = \sum_{i,j} c_{ij} \int_{\partial\Omega_{ij}} \nabla \varphi \cdot \mathbf{n} \\
&= 0,
\end{aligned}
$$

where we have used the facts that $\Delta\varphi = 0$, that ψ has constant value c_{ij} on the component $\partial\Omega_{ij}$ of $\partial\Omega$, and that the flux of $V = \nabla\varphi$ through each boundary component $\partial\Omega_{ij}$ is zero.

This completes the proof of Proposition 4.

14. PROOF OF THE HODGE DECOMPOSITION THEOREM. Let Ω be a compact domain in 3-space with smooth boundary. The desired orthogonal direct sum decomposition,

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

is an immediate consquence of Propositions 1 through 4, as are the three partial sums

$$
\begin{aligned}
\ker \mathit{curl} &= \phantom{\mathrm{FK} \oplus{}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\
\operatorname{im} \mathit{grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus{}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\
\ker \mathit{div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG}.
\end{aligned}
$$

It remains to prove

Lemma 5.

$$\operatorname{im} \mathit{curl} = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}.$$

It is easy to see that

$$\operatorname{im} \mathit{curl} \subset \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}, \tag{10}$$

as follows. Suppose that $V = \nabla \times U$. Then we know that $\nabla \cdot V = 0$ and that the flux of V through every closed surface in Ω is zero. Let φ be a solution of the Laplace equation $\Delta\varphi = 0$ with Neumann boundary data $\partial\varphi/\partial\mathbf{n} = V \cdot \mathbf{n}$ along $\partial\Omega$. Then $V_2 = \nabla\varphi$ lies in CG. The vector field $V_1 = V - V_2$ is divergence-free and tangent to $\partial\Omega$, and hence lies in FK $\oplus$ HK. Writing $V = V_1 + V_2$ establishes the inclusion (10).

Suppose now that $V \in \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}$. This tells us that $\nabla \cdot V = 0$ and that the flux of V through each component of $\partial\Omega$ is zero. We want to write $V = \nabla \times U$ for some U. The key to this is formula (3) for divergence-free fields:

$$\nabla_y \times \mathrm{BS}(V)(y) = \left\{ \begin{matrix} V(y) \text{ for } y \in \Omega \\ \mathbf{0} \text{ for } y \in \Omega' \end{matrix} \right\} - \frac{1}{4\pi} \nabla_y \int_{\partial\Omega} \frac{V(x) \cdot \mathbf{n}}{|y - x|} \, \mathrm{d}(\mathrm{area}_x). \tag{3}$$

To show that V is a curl, it is sufficient to show that the last term in (3) is a curl, and we do this as follows.

Construct another domain Ω^* by taking a large ball containing Ω in its interior, and then removing the interior of Ω, as shown in Figure 17. The boundary components of Ω^* consist of the boundary components of Ω plus the boundary of the ball.

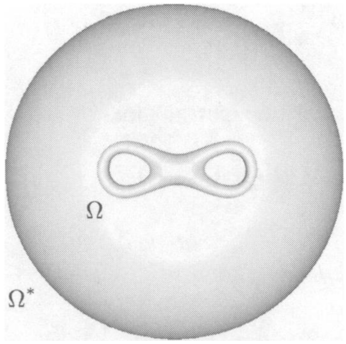

Figure 17. Verifying the image of curl.

We now solve a Neumann problem for the Laplacian on Ω^*. That is, we find a harmonic function φ^* on Ω^* with $\partial\varphi^*/\partial\mathbf{n}^* = -V \cdot \mathbf{n}$ on each of the boundary components that Ω^* shares with Ω, and $\partial\varphi^*/\partial\mathbf{n}^* = 0$ on the boundary of the ball. Then we let $V^* = \nabla\varphi^*$. Equation (3) for V^* is

$$\nabla_y \times \mathrm{BS}(V^*)(y) = \left\{ \begin{matrix} V^*(y) \text{ for } y \in \Omega^* \\ \mathbf{0} \text{ for } y \in \Omega^{*\prime} \end{matrix} \right\} - \frac{1}{4\pi}\nabla_y \int_{\partial\Omega^*} \frac{V^*(x) \cdot \mathbf{n}^*}{|y-x|}\, \mathrm{d}(\text{area}_x).$$

In the complementary domain $\Omega^{*\prime}$, this equation is

$$\begin{aligned} \nabla_y \times \mathrm{BS}(V^*)(y) &= -\frac{1}{4\pi}\nabla_y \int_{\partial\Omega^*} \frac{V^* \cdot \mathbf{n}^*}{|y-x|}\, \mathrm{d}(\text{area}_x) \\ &= \frac{1}{4\pi}\nabla_y \int_{\partial\Omega} \frac{V(x) \cdot \mathbf{n}}{|y-x|}\, \mathrm{d}(\text{area}_x). \end{aligned}$$

Since $\Omega \subset \Omega^{*\prime}$, this equation holds in Ω.

Thus in Ω we have

$$\nabla \times \big(\mathrm{BS}(V) + \mathrm{BS}(V^*)\big) = V,$$

which shows that V is in the image of curl, and hence that

$$\operatorname{im} \mathit{curl} = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG},$$

as claimed.

Wrapping up our business, note that we have already established the isomorphisms

$$\mathrm{HK} \cong H_1(\Omega; \mathbf{R}) \cong H_2(\Omega, \partial\Omega; \mathbf{R}) \cong \mathbf{R}^{\text{genus of } \partial\Omega}$$
$$\mathrm{HG} \cong H_2(\Omega; \mathbf{R}) \cong H_1(\Omega, \partial\Omega; \mathbf{R}) \cong \mathbf{R}^{(\#\text{ components of } \partial\Omega)-(\#\text{ components of } \Omega)}$$

in Lemmas 2 and 3. Finally, the characterizations of the five orthogonal direct summands of $\mathrm{VF}(\Omega)$, namely,

$$\begin{aligned} \mathrm{FK} &= (\ker \mathit{curl})^{\perp} \\ \mathrm{HK} &= (\ker \mathit{curl}) \cap (\operatorname{im} \mathit{grad})^{\perp} \\ \mathrm{CG} &= (\operatorname{im} \mathit{grad}) \cap (\operatorname{im} \mathit{curl}) \end{aligned}$$

$$\begin{aligned} \mathrm{HG} &= (\ker div) \cap (\operatorname{im} curl)^{\perp} \\ \mathrm{GG} &= (\ker div)^{\perp} \end{aligned}$$

follow immediately from the various partial sums.

15. ANSWERS TO THE FOUR QUESTIONS.

Question 1. *Given a vector field defined on a compact domain in 3-space, how do you know whether it is the gradient of a function?*

Let V be our vector field, defined on the compact domain Ω with smooth boundary; we want to know if there is a smooth real-valued function φ on Ω with $V = \nabla\varphi$.

If V is a gradient vector field, then its circulation around every loop is zero. Conversely, if the circulation of V around every loop is zero, then $V = \nabla\varphi$ with $\varphi(x)$ defined by integrating V along any path from a base point x_i in each connected component of Ω to the point x. But this is not a very practical test.

Answer to Question 1. *The vector field V is a gradient if and only if $\nabla \times V = 0$ and the integrals of V around the loops $C_1, C_2, \ldots, C_k$ are zero, where these loops represent a basis for the one-dimensional homology $H_1(\Omega)$.*

We now prove that our answer is correct. We can see that the conditions are necessary, so suppose that V satisfies them. We saw in Section 6 that if V is curl-free, then its circulation around any loop C depends only on the homology class of C in $H_1(\Omega)$. But any C is homologous to a linear combination of the basis loops $C_1, \ldots, C_k$, around each of which V has circulation zero, and so V must also have circulation zero around C. Then V is certainly a gradient field.

Although we didn't use the Hodge Decomposition Theorem to reach this conclusion, we can use it to reflect on what we have just said. The condition $\nabla \times V = 0$ tells us that V lies in the kernel of *curl*, namely, $\mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}$. The condition that V has zero circulation around the basis loops for $H_1(\Omega)$ tells us that the HK-component of V is zero, and hence that V lies in $\mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}$, which is just the subspace of gradient fields.

Question 2. *Given a vector field defined on a compact domain in 3-space, how do you know whether it is the curl of another vector field?*

Let V be our vector field defined on the compact domain Ω with smooth boundary; we want to know if there is a vector field U on Ω with $\nabla \times U = V$.

If V is a curl, then its flux through every closed surface is zero, as an immediate consequence of Stokes' Theorem. Conversely, if the flux of V through every closed surface S in Ω is zero, then V is a curl. Again, this is not a very practical test.

Answer to Question 2. *The vector field V is the curl of another vector field if and only if $\nabla \cdot V = 0$ and the flux of V through each component of $\partial\Omega$ is zero.*

Confirmation that we have given the correct answer comes from the Hodge Decomposition Theorem. The condition $\nabla \cdot V = 0$ tells us that V lies in the kernel of *div*, namely, $\mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG}$. The condition that the flux of V through each component of $\partial\Omega$ is zero tells us that the HG-component of V is zero, and hence that V lies in $\mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG}$, which is just the image of *curl*.

Question 3. *Can you find a nonzero vector field on a given compact domain in 3-space that is divergence-free, curl-free, and tangent to the boundary?*

Answer to Question 3. *Such a vector field exists if and only if at least one boundary component of the domain has nonzero genus.*

This is simply a part of the Hodge Decomposition Theorem. The vector fields we are looking for are the harmonic knots; they form the subspace HK of VF(Ω), and this is isomorphic to $H_1(\Omega)$. To see whether $H_1(\Omega)$ is nonzero, we simply look at the boundary components of Ω and make sure that they are not all of genus zero. If this is true, we can refer to the proof of the Hodge Decomposition Theorem (see Section 9) to see how the harmonic knots were constructed from magnetic fields.

Question 4. *Can you find a nonzero vector field on a given compact domain in 3-space that is divergence-free, curl-free, and orthogonal to the boundary?*

Answer to Question 4. *Such a vector field exists if and only if at least one component of the domain has more than one boundary component.*

This is also just a part of the Hodge Decomposition Theorem. The vector fields we are looking for are the harmonic gradients; they form the subspace HG of VF(Ω), and this is isomorphic to $H_2(\Omega)$. To see whether $H_2(\Omega)$ is nonzero, we simply look for a connected component of Ω that has more than one boundary component. If such a component is present, we can again refer to the proof of the Hodge Decomposition Theorem (see Section 12) to see how the harmonic gradients were constructed from solutions to the Dirichlet problem for the Laplace equation.

16. HISTORICAL THREADS. In this section, we trace the threads that led from potential theory, functional analysis, fluid dynamics, electrodynamics, and topology to the Hodge Decomposition Theorem. We begin with existence and uniqueness of solutions to the Dirichlet problem for the Laplace equation, which appears in the Hodge Decomposition Theorem in the characterization of the subspace HG of harmonic gradients.

By the early part of the nineteenth century, mathematicians appreciated the importance of the potential equation $\Delta\varphi = 0$ in problems concerning gravitation and heat conduction. In 1813, Poisson [**38**] corrected an earlier misconception of Laplace, who had assumed that the gravitational potential φ satisfies $\Delta\varphi = 0$ everywhere (inside or outside the mass), to the equation $\Delta\varphi = -4\pi\rho$ (where ρ is the mass density) that now bears his name. In the same paper, Poisson noted the usefulness of the potential function and equation in the study of electricity.

Then George Green published his 1828 treatise [**24**] on the application of mathematical analysis to electricity and magnetism, which was neglected until its importance was recognized in the 1850s by William Thomson (Lord Kelvin). In it, Green derived his famous identities and developed the notion now known as "Green's function", whose existence is equivalent to the solvability of the Dirichlet problem for the Laplace equation. Although Green did not give a complete proof, his paper was the first to give the variational characterization of the Dirichlet problem, now known as "Dirichlet's principle": in order to solve $\Delta\varphi = 0$ on a domain Ω subject to the boundary condition $\varphi = f$ on $\partial\Omega$, one should seek to minimize the integral $\int_\Omega |\nabla\varphi|^2 \mathrm{d}(vol)$ over the set of functions φ that satisfy the boundary condition. Thomson called further attention to this idea, and Riemann used it in two dimensions to solve problems in complex variable theory.

In the 1870s, the proof of the existence of solutions to the Dirichlet problem via Dirichlet's principle was called into question by Weierstrass. Subsequently, Weierstrass's student, H.A. Schwarz, and Carl Neumann and others developed proofs of existence that were not based on Dirichlet's principle. It remained for Hilbert [**27**] in 1900 to establish the validity of Dirichlet's principle as a method for proving existence.

The Fourier analysis approach to partial differential equations and Fredholm's work on integral equations motivated the development of functional analysis at the beginning of the twentieth century. Hilbert and his successors, especially Friedrich Riesz [**41**]–[**43**] and Maurice Frechet [**21**], introduced the notion of functions as points of a space on which the geometry of the L^2 inner product could be exploited.

The notion of orthogonally complementary subspaces of functions grew out of the study of self-adjoint integral operators to which Fredholm's alternative applies, as well as out of the work of Erhard Schmidt, Hermann Weyl, and others on eigenvalue problems for self-adjoint differential operators, where eigenspaces corresponding to different eigenvalues are orthogonal.

We turn now to the relationship between the topology of a domain in 3-space and the nature and variety of fluid flows that can be defined there. This appears in the Hodge Decomposition Theorem in the characterization of the subspace HK of harmonic knots.

An "ideal fluid" is incompressible and has no internal friction (viscosity). In 1858, Hermann von Helmholtz [**26**] studied the motion of such fluids without assuming that the velocity is the gradient of a potential function, as had usually been done in earlier treatments by Euler, Lagrange, and others. Helmoltz's paper contained a wealth of new ideas, which had a powerful effect on Tait, Thomson, and Maxwell.

Helmholtz introduced the curl of a velocity field V to measure the local rotation of the elements of the fluid, and then faced the problem of reconstructing V from knowledge of its curl. He showed how to get one solution, and stated that all others are obtained from this one by adding the gradient of a "multi-valued potential function" that could be chosen to satisfy the boundary conditions.

He also introduced the *simple-connectivity* of a three-dimensional domain, extending the sense in which Riemann [**40**] had used it the year before for surfaces, and pointed out that in a simply-connected domain bounded by closed surfaces, irrotational (i.e., curl-free) fluid motion is uniquely determined by boundary conditions. In particular, if the normal component of the fluid velocity along the boundary vanishes, then the fluid must be at rest.

Helmoltz defined *multiply-connected* three-dimensional domains Ω in terms of the maximum number of cross-sectional surfaces $(\Sigma, \partial\Sigma) \subset (\Omega, \partial\Omega)$ that can be placed in the domain without disconnecting it, again extending the sense in which Riemann used this idea for surfaces.

In July 1858, the Scottish physicist Peter Guthrie Tait read Helmholtz's article and immediately made an English translation [**46**] for his personal use, which was published nine years later, after Helmholtz had a chance to look it over and revise it. Tait saw how to use Hamilton's quaternions to express Helmholtz's decomposition of infinitesimal fluid motions into translations, deformations, and rotations, and later saw how to express Euler's equations of fluid motion in a similar fashion [**47**], [**48**]. The compactness of Tait's quaternionic expressions foreshadowed those of contemporary vector calculus.

Since the mid 1850s, William Thomson had been thinking of the space between the smallest parts of matter as filled with a continuous and material medium undergoing rotary motions around material atoms and molecules. Thomson and Tait were friends and collaborators, and Thomson was impressed with Tait's smoke ring experiments that illustrated Helmholtz's mathematical predictions of vortex interaction, vibration, and stability. He felt that closed vortices might provide stable dynamical configurations in a universal medium that were consistent with his general beliefs about the nature of atoms, and that their vibrations might give a possible explanation of atomic spectra. Spurred by these beliefs (which he maintained for almost twenty years before gradually abandoning them), Thomson, in his 1869 paper [**50**] on vortex motion, continued the mathematical investigations begun by Helmholtz.

Thomson introduced an embryonic version of one-dimensional homology $H_1(\Omega)$ in which one counted the number of "irreconcilable" closed paths inside the domain Ω. This was subject to the standard confusion of the time between homology and homotopy of paths: homology was the appropriate notion in this setting, but the definitions were those of homotopy.

He also introduced a primitive version of two-dimensional relative homology $H_2(\Omega, \partial\Omega)$ in which one counted the maximum number of "barriers", meaning cross-sectional surfaces $(\Sigma, \partial\Sigma) \subset (\Omega, \partial\Omega)$, that one could erect without disconnecting the domain Ω. Thomson pointed out, just as we did in Section 6, that while these barriers might be disjoint in simple cases, in general one must expect them to intersect one another.

He recognized that one was counting the "same thing" by two different means, and, translating to fluid flows, also saw that this was the same as counting the maximum number of linearly independent vector fields in Ω that were divergence-free, curl-free, and tangent to the boundary (*harmonic knots* in our terminology), since these would be in one-one correspondence with the values of their integrals along a maximal family of irreconcilable closed paths inside Ω.

Then James Clerk Maxwell, a former schoolmate of Tait, started to think seriously about Thomson's topological ideas, and discussed them repeatedly with him while Thomson was completing his paper on vortex motions. Maxwell came to understand that, independent of complicated appearances, the connectivity of a domain in 3-space is determined by and can be counted in terms of the connectivities of the surfaces on its boundary. Furthermore, he saw that the connectivity of a domain is the same as that of its complement in 3-space, the key statement of Alexander duality in this setting. Maxwell included an exposition of all this in his *Treatise on Electricity and Magnetism* [**34**]; see the preliminary chapter *On the Measurement of Quantities*, and Chapter IV, *General Theorems*.

17. A GUIDE TO THE LITERATURE. To the best of our knowledge, the Hodge Decomposition Theorem for vector fields on bounded domains in 3-space first appeared in Hermann Weyl's 1940 paper [**54**]. Hodge's book [**28**], which appeared in 1941 with a second edition in 1952, is the standard reference for the Hodge Decomposition Theorem for differential forms on manifolds without boundary, and provides a good guide to the early literature. The corresponding results for differential forms on manifolds with boundary, and the related boundary value problems, have their early treatments in the 1950s in [**17**], [**14**], and [**22**].

The Hodge Decomposition Theorem is closely related to the de Rham Isomorphism Theorem for differential forms. De Rham's 1960 book [**15**] is the standard reference for this on closed manifolds; it provides a good guide to the early literature. Duff's 1952 paper [**16**] develops de Rham theory for differential forms on manifolds with boundary, and solves the basic boundary value problems in the subject. Duff credits Tucker [**52**] with conjecturing many of these results in 1941.

The Hodge Decomposition Theorem provides a framework for vector analysis in 3-space that is used in many studies in fluid dynamics and plasma physics: mathematical theory of viscous incompressible flow [**29**]; stationary solutions of the Navier-Stokes equation [**19**]; self-adjoint realizations of the curl operator [**37**], [**57**]; Beltrami fields and force-free magnetic fields [**30**], [**32**], [**33**]; and discrete eigenstates of plasmas [**55**], [**56**].

Three works have been especially helpful to us in the preparation of this paper: the 1957 notes of Blank, Friedrichs, and Grad [**6**], which seem to us closest in spirit to our own views; the beautiful treatment of Hodge theory in Frank Warner's 1971 book [**53**]; and the excellent 1995 book of Gunter Schwarz [**44**]. In addition we recommend the papers of Giles Auchmuty [**2**], [**3**], which, in particular, deal with the case where the boundary of the domain is not of class C^{∞}.

Finally, the Hodge Decomposition Theorem plays an important role in our own works on writhing of knots, helicity of vector fields, the spectral theory of the Biot-Savart and curl operators, and applications to plasma physics [**8**]–[**13**].

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