Double Pendulum Workings

katsca

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1 Finding the Lagrangian

Our pendulum is made of two uniform rods of length l_1 and l_2 and mass m_1 and m_2.

We can define the coordinates of the centre of masses of the rods assuming the pivot is at (0,0) at:

$$x_1 = \frac{1}{2} l_1 sin(\theta_1) \tag{1}$$

$$y_1 = -\frac{1}{2} l_1 cos(\theta_1) \tag{2}$$

$$x_2 = \frac{1}{2} l_2 sin(\theta_2) + l_1 sin(\theta_1) \tag{3}$$

$$y_1 = -\frac{1}{2} l_2 cos(\theta_2) - l_1 cos(\theta_1) \tag{4}$$

We can find the velocities as follows:

$$\dot{x_1} = \frac{1}{2} l_1 \dot{\theta}_1 cos(\theta_1) \tag{5}$$

$$\dot{y_1} = \frac{1}{2} l_1 \dot{\theta}_1 sin(\theta_1) \tag{6}$$

$$\dot{x_2} = \frac{1}{2} l_2 \dot{\theta}_2 cos(\theta_2) + l_1 \dot{\theta}_1 cos(\theta_1) \tag{7}$$

$$\dot{y_1} = \frac{1}{2} l_2 \dot{\theta}_2 sin(\theta_2) + l_1 \dot{\theta}_1 cos(\theta_1) \tag{8}$$

The kinetic energy is defined for a uniform rod as the following:

$$T = \frac{1}{2} m_1 * (\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2} m_2 * (\dot{x_2}^2 + \dot{y_2}^2) + \frac{1}{2} I_1 \dot{\theta_1}^2 + \frac{1}{2} I_2 \dot{\theta_2}^2 \tag{9}$$

Where for a uniform rod the moments of inertia are:

$$I_1 = \frac{1}{12} m_1 l_1^2 \tag{10}$$

$$I_2 = \frac{1}{12} m_2 l_2^2 \tag{11}$$

We can also define the potential energy as the following:

$$V = m_1 g y_1 + m_2 g y_2 \tag{12}$$

By subbing in equations (1-8) into equations (9) and (12) we will get:

$$\begin{aligned} T = &\frac{1}{8} m_1 (l_1 \dot{\theta_1})^2 + \frac{1}{2} m_2 (l_1 \dot{\theta_1})^2 + \frac{1}{8} m_2 (l_2 \dot{\theta_2})^2 + \frac{1}{2} m_2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} cos(\theta_1 - \theta_2) \\ &+ \frac{1}{2} I_1 \dot{\theta_1}^2 + \frac{1}{2} I_2 \dot{\theta_2}^2 \end{aligned} \tag{13}$$

$$V = -\frac{1}{2} m_1 g l_1 cos(\theta_1) - m_2 g (l_1 cos(\theta_1) + \frac{1}{2} l_2 cos(\theta_2)) \tag{14}$$

We know then that the Lagrangian is just simply L = T - V

$$\begin{aligned} L = &\frac{1}{8} m_1 (l_1 \dot{\theta_1})^2 + \frac{1}{2} m_2 (l_1 \dot{\theta_1})^2 + \frac{1}{8} m_2 (l_2 \dot{\theta_2})^2 + \frac{1}{2} m_2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} cos(\theta_1 - \theta_2) \\ &+ \frac{1}{2} I_1 \dot{\theta_1}^2 + \frac{1}{2} I_2 \dot{\theta_2}^2 + \frac{1}{2} m_1 g l_1 cos(\theta_1) + m_2 g (l_1 cos(\theta_1) + \frac{1}{2} l_2 cos(\theta_2)) \end{aligned} \tag{15}$$

2 Finding the Euler-Lagrange Equations

The ELE are generally defined by:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_i}}) - \frac{\partial L}{\partial \theta_i} = 0, \quad i = 1, 2 \tag{16}$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} m_1 g l_1 sin(\theta_1) - m_2 g l_2 sin(\theta_2) - \frac{1}{2} m_2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} sin(\theta_1 - \theta_2) \tag{17}$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} m_2 g l_2 sin(\theta_2) + \frac{1}{2} m_2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} sin(\theta_1 - \theta_2) \tag{18}$$

$$\frac{\partial L}{\partial \dot{\theta_1}} = \frac{1}{4} m_1 l_1^2 \dot{\theta_1} + m_2 l_1^2 \dot{\theta_1} + I_1 \dot{\theta_1} + \frac{1}{2} m_2 l_1 l_2 \dot{\theta_2} cos(\theta_1 - \theta_2) \tag{19}$$

$$\begin{aligned} \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_1}}) = &\frac{1}{4} m_1 l_1^2 \ddot{\theta_1} + m_2 l_1^2 \ddot{\theta_1} + I_1 \ddot{\theta_1} + \frac{1}{2} m_2 l_1 l_2 \ddot{\theta_2} cos(\theta_1 - \theta_2) \\ &- \frac{1}{2} m_2 l_1 l_2 \dot{\theta_2} (\dot{\theta_1} - \dot{\theta_2}) sin(\theta_1 - \theta_2) \end{aligned} \tag{20}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{4} m_2 l_2^2 \dot{\theta}_2 + I_2 \dot{\theta}_2 + \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 cos(\theta_1 - \theta_2) \tag{21}$$

$$\begin{aligned} \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_2}) =& \frac{1}{4} m_2 l_2^2 \ddot{\theta}_2 + I_2 \ddot{\theta}_2 + \frac{1}{2} m_2 l_1 l_2 \ddot{\theta}_1 cos(\theta_1 - \theta_2) \\ & - \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) sin(\theta_1 - \theta_2) \end{aligned} \tag{22}$$

This means our equation for motion for θ_1 is:

$$\begin{aligned} 0 =& (\frac{m_1 l_1}{4} + m_2 l_1 + \frac{m_1 l_1}{12}) \ddot{\theta}_1 + (\frac{m_2 l_2}{2} cos(\theta_1 - \theta_2)) \ddot{\theta}_2 \\ & + (\frac{m_2 l_2}{2} sin(\theta_1 - \theta_2)) \dot{\theta}_2 + \frac{m_1 g}{2} sin(\theta_1) + m_2 g sin(\theta_1) \end{aligned} \tag{23}$$

Our equation for motion for θ_2 is:

$$\begin{aligned} 0 =& (\frac{m_2 l_2}{4} + \frac{m_2 l_2}{12}) \ddot{\theta}_2 + (\frac{m_2 l_1}{2} cos(\theta_1 - \theta_2)) \ddot{\theta}_1 \\ & - (\frac{m_2 l_1}{2} sin(\theta_1 - \theta_2)) \dot{\theta}_1 + \frac{m_2 g}{2} sin(\theta_2) \end{aligned} \tag{24}$$

We then get the two equations we want to solve by eliminating $\ddot{\theta}_1$ and then $\ddot{\theta}_2$. We solve the two equations left using scipy_ivp with the Explicit Runge-Kutta method of order 5(4).