A Brief History of QED

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1 Introduction

Quantum electrodynamics is a branch of quantum field theory which is concerned with the interaction of electrons and photons. Richard Feynman, Julian Schwinger, and Sin-Itiro Tomonaga jointly won the 1965 Nobel Prize for their work on this theory. This paper will attempt to explain, at least qualitatively, their work and the history surrounding it. The first section will cover the development of quantum mechanics and QED until 1947 and will give brief biographies of Feynman and Schwinger. The following sections will discuss Feynman's and Schwinger's approaches to QED. The last section of the paper will discuss the impacts of QED in physics today.

1.1 The Beginnings of QED

Quantum mechanics was developed as a way to understand the behavior of particles (e.g. photons, electrons, and quarks) which seem to exhibit classical properties of both particles and waves. In quantum mechanics, a particle is described as a vector $|\psi(t)\rangle$, which satisfies the Schrodinger equation [**1**]

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1}$$

Here, the wave function ψ represents a probability amplitude. Rather than describing a particle's exact position in space, the wave function is related to the probability that a particle can be detected at a given location at a given time. [**1**]

1.2 Dirac, Heisenberg, and Pauli's Work in QED

In his 1927 paper "Quantum Theory of Emission and Absorption of Radiation,"[**2**] Paul Dirac both coined the term "quantum electrodynamics" and introduced quantum field theory as a way of understanding the quantum characteristics of photons. Quantum field theory, as opposed to traditional quantum mechanics, treats particles as excitations of an underlying field. This is necessary because photons are zero-mass particles, and therefore travel at the speed of light, so the non-relativistic equations of traditional quantum mechanics cannot provide an accurate description of them. Dirac considered the electromagnetic radiation field as a system of quantum harmonic oscillators using the Hamiltonian formalism. The Hamiltonian formalism is an alternative way of considering classical dynamical systems. It generally has the form $\mathcal{H} = T + V$, where T is the kinetic energy and V is the potential energy of the system. The Hamiltonian formalism is somewhat similar to the Lagrangian formalism, which will be discussed in more detail in section 2.1. Dirac was then able to write the Hamiltonian of the system: [**3**]

$$H = H_0 + H_{int} \tag{2}$$

Here, H_0 is the Hamiltonian of the atoms and the field, which contains the energy of an infinite set of harmonic oscillators describing the photons. H_{int} is an interaction term that describes the interactions between the field and the atoms and accounts for the creation and annihilation of photons. This formulation, and Dirac's subsequent work in relativistic quantum mechanics, were quite successful. They predicted numbers that agreed with experiments, and predicted the existence of the positron, which was discovered in 1932. [**3**]

1.2.1 Problems with pre-war QED

Despite their apparent success, pre-WWII theories of QED were plagued with difficulties, including infinite divergences, in which the mathematical theory predicted infinity for physical quantities that were obviously finite. One such problem is that of infinite charge density in a finite vacuum.This infinite charge density arises from the electron and positron "seas" proposed by Dirac. Pauli solved this problem rather elegantly, by saying that the charge of all the electrons in a finite vacuum must exactly cancel the charge of all the positrons in the same vacuum. While some problems were rather easily solved, other infinite divergences remained. Most troubling among these were the so-called ultraviolet divergences, which predicted that the self-energy of an electron diverges to infinity as its radius approaches zero and that the effective charge of a particle also approaches infinity as the distance to the particle approaches zero. [**3**]

Physicists now deal with such infinite divergences using a mathematical procedure known as renormalization. Essentially, renormalization involves subtracting an infinite term from another, larger infinite term in order to get a finite result. This idea was first introduced to QED in the late 1930s by Hendrik Kramers. While some of his ideas were correct, his work remained largely unknown. Additionally, Kramers based his work on an older, nonrelativistic model of the electron, and planned to develop the relativistic case after he had worked out the simpler, nonrelativistic model. Unfortunately, the nonrelativistic model failed to account for several important physical phenomena in the theory. Still, Kramers' idea of making a mathematical correction to existing theories of QED proved to be important. It was not until Feynman and Schwinger's contributions to QED that these difficulties could be resolved. [**3**]

1.3 Biographical Information on Feynman

Richard Feynman, born in 1918, grew up in Far Rockaway, Queens, New York. From before Richard was born, his father was determined that his son would be a scientist. To this end, Melville Feynman encouraged his son's curiosity, teaching him how to observe and intelligently question his surroundings. As a young child, Feynman was interested in electronics, and he built and repaired radios for fun. In high school, Feynman also developed an interest in mathematics. He maintained private math notebooks, in which he invented his own notation for trigonometric functions, wrote out tables of logarithms calculated using his own methods, and taught himself calculus. At fourteen, he discovered Euler's identity, and covered an entire page with "THE MOST REMARKABLE FORMULA IN MATH: $e^{i\pi}+1=0$." [**4**] Feynman graduated from Far Rockaway High School with excellent grades in math and science, and somewhat decent grades in other subjects. In his senior yearbook, his classmates voted him "Mad Genius."

Feynman did his undergraduate work at MIT, during which time he published two papers. The first was a joint paper with Manuel Vallarta on cosmic rays, and the second was his senior thesis, "The Forces in Molecules." [**5**] While this thesis was not extremely relevant to Feynman's later work, it presented a computational technique for finding forces on individual nuclei within solids that saved a great deal of time. The paper was well-received by MIT physics faculty and was published in the Physical Review. Feynman was accepted as a graduate student at Princeton, and began studying there in the fall of 1939.

At Princeton, Feynman's biggest influence was John Archibald Wheeler, who would later become his thesis adviser. Feynman and Wheeler worked jointly on an action-at-a-distance theory of electron interaction. This theory eliminated the field, assumed that charges did not act on themselves, and showed that interactions between electrons are symmetric with respect to time. [**4**] Feynman's doctoral thesis quantized the classical Wheeler-Feynman theory. In order to do so, Feynman invented the path integral formulation of quantum mechanics, which became enormously important to the development of QED.

After receiving his PhD from Princeton in 1942, Feynman moved to New Mexico, where he worked at Los Alamos National Laboratory until the fall of 1945. Los Alamos was one of the most unique scientific gatherings of the 20th century. Almost all of the leading scientists in the US were there, working frantically to develop the atomic bomb before any other country did. Feynman worked in the theoretical division, which was headed by Hans Bethe. Bethe and Feynman worked closely together, and Feynman's energy and eccentricity complemented Bethe's slow and methodical approach to the various problems associated with building a nuclear bomb. [**4**] Bethe made Feynman a group leader within the theoretical division. During their time at Los Alamos, Bethe became a close mentor to Feynman. After World War II, Bethe offered Feynman a teaching appointment at Cornell.

While at Cornell, Feynman published his most important works in QED, including "A Relativistic Cutoff for Classical Electrodynamics" [**6**] and "An Operator Calculus Having

Applications in Quantum Electrodynamics." [**7**] These papers laid out in detail the mathematics behind his space-time approach to QED, which will be discussed below. After leaving Cornell in 1951, Feynman spent a year on sabbatical in Brazil, during which time he taught at Centro Brasileiro de Pesguisas Fisicas and learned to play bongos. Feynman remained an enthusiastic amateur bongo drummer for the rest of his life. After his sabbatical, Feynman started teaching at Caltech, where he worked on theories of superfluidity and quantum gravity. Feynman was also a gifted teacher, and his transcribed lectures, both on introductory physics and on special topics such as QED, continue to sell copies today. Jointly with Schwinger and Tomonaga, Feynman received the 1965 Nobel Prize in physics for his contributions to the development of QED. Feynman died of cancer in 1988, and is widely remembered as one of the greatest scientists of the twentieth century. [**4**]

1.4 Biographical Information on Schwinger

Like Feynman, Julian Schwinger grew up in New York City. He attended an advanced high school, and as a result, entered the City College of New York as a sophomore in 1933, at the age of fifteen. While at City College, Schwinger began reading journals and following the progress of theoretical physics. He was particularly influenced by Dirac's papers on quantum field theory. In fact, Schwinger wrote a short paper on applying Dirac's unitary transformation of the Heisenberg-Pauli theory to the second-quantized electron field. He never published this paper, but it was important to his later work.[**3**]

During World War II, Schwinger did not work on the atomic bomb at Los Alamos. Instead, he worked as a theoretician at the Radiation Laboratory at MIT. After the war, Schwinger taught at Harvard and later at UCLA. While at Harvard, Schwinger published several influential papers on QED. Somewhat shy and introverted, Schwinger was known for his polished lectures. He made a point of rehearsing so that he could lecture without notes, and detested being interrupted for questions. Nevertheless, Schwinger was well-loved by all of his students, and he supervised a total of 73 doctoral dissertations. Schwinger had his office hours on Wednesday afternoons, and routinely had a line out the door to see him. Because he spent as much time as he felt was necessary with each student, he rarely saw all of the students in line. Still, one visit with Schwinger gave the average student enough to work on for several weeks, so this was not a huge problem. [**8**] While Schwinger worked on other theories later in his career, he is mostly remembered for his contributions to QED.

1.5 The Shelter Island Conference

After World War II, many physicists were unsure of what to do with themselves. Having spent the last several years frantically working on either the Manhattan project or other technology for war, American physicists had not had an opportunity to discuss other research problems. So, in June of 1947, 23 prominent physicists met in Shelter Island, New

York for a conference on the Foundations of Quantum Mechanics. Most of the conference proceedings concerned experimental results such as the measurement of the Lamb shift and the electron magnetic moment. [**3**] Both of these quantities were slightly different from Dirac's predictions, which showed that further work was needed in QED. Both Feynman and Schwinger were present at this conference, and it greatly influenced their work in QED over the next few years.

2 Feynman's Approach to QED

Richard Feynman is most widely known for his unconventional space-time approach to quantum mechanics. Feynman's thesis, "The Principle of Least Action in Quantum Mechanics," [**9**] is based on a classical theory of action-at-a-distance that Feynman developed with John Wheeler at Princeton. The basic idea of this action-at-a-distance theory is to eliminate the electromagnetic field and consider objects acting directly on one another. Feynman's thesis quantizes this theory. In doing so, Feynman invented the path integral formulation of quantum mechanics, which became extremely important in his later work in QED.

2.1 Feynman's Thesis: The Path Integral Formulation of Quantum Mechanics

Feynman's theory makes use of the Lagrangian where traditional quantum mechanics uses the Hamiltonian. Because of the way these functions are defined, all systems which have Hamiltonians have Lagrangians, but some systems which have Lagrangians do not also have Hamiltonians. Feynman chose to use the Lagrangian because he wanted to represent one such system: two particles interacting through an oscillator, with the coordinate for the oscillator eliminated as if the two particles were interacting directly with some delay. The system with the oscillator has a Hamiltonian, but once the coordinate of the oscillator is eliminated, the system possesses only a Lagrangian.

In his thesis, Feynman first applies his approach to eliminate the oscillator for a classical system, then applies it to a quantum mechanical system. Similarly, we will briefly discuss the Lagrangian formalism in classical mechanics before moving on to its applications in quantum mechanics.

2.1.1 The Lagrangian Formalism

The Lagrangian formalism is a reformulation of Newtonian mechanics which makes it much easier to solve certain problems. It is especially useful for problems involving non-Cartesian coordinate systems, objects with constraints on their motion, and systems of several objects. The Lagrangian formulation uses a set of generalized coordinates for positions and velocities, rather than the usual x, y, and z Cartesian coordinates. We usually write these coordinates (q, q'). For example, we might say that a system of two particles, each having an x, y, and z coordinate, is described completely by $(x_1, y_1, z_1, x_2, y_2, z_2)$. In a generalized coordinate system, we would instead describe this system with $(q_1, q_2, q_3, q_4, q_5, q_6)$. [**10**] It is important to note that these generalized coordinates are not limited to positions; they can also represent angles and other quantities and can account for constraints on the system being studied.

In general, the Lagrangian of a system has the form $L = T - V$, where T is the kinetic energy and V is the potential energy of the system. This is similar, but not identical, to the Hamiltonian formalism described above. This definition is based on a set of partial differential equations known as Lagrange's equations, which are [**10**]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} = \frac{\partial L}{\partial q_k} \tag{3}$$

for conservative systems and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} = \frac{\partial L}{\partial q_k} + Q'_k \tag{4}$$

for nonconservative systems, where

$$Q'_k = \frac{\partial V}{\partial q_k} \tag{5}$$

A detailed discussion of these equations is outside the scope of this paper. Still, it is necessary to understand the general idea of the Lagrangian formalism in order to understand definition of action and the principle of least action.

2.1.2 The Principle of Least Action

The action, $\mathcal{S}$, of a system is defined to be [**10**]

$$\mathcal{S}[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt \tag{6}$$

The principle of least action states that a system will take whatever path minimizes the change in the action of the system. This principle is widely applicable in all areas of physics, and can be used to derive Newton's laws, the conservation of momentum and energy and Lagrange's equations of motion. Feynman thought the principle of least action was even more fundamental to nature than Newton's second law, and saw it as an ideal way to approach quantum mechanics and QED.

2.1.3 Applications to Quantum Mechanics

At this point, a perfectly valid question would be "why did Richard Feynman want to go to the trouble of completely reformulating quantum mechanics?" The answer has to do with quantum field theory, and the idea of representing a field as a harmonic oscillator, which is then eliminated and treated as if the particles are interacting directly. It also has to do with the philosophy behind Feynman's approach. The Hamiltonian (and therefore traditional quantum mechanics) describes a system at a specific time. By contrast, Feynman, in calculating the action of a system, integrates over all space and time. Consider an example presented by Feynman in section 3.5 of his thesis [**9**]:

$$\mathcal{A} = \int_{-\infty}^{\infty} \frac{m\dot{x}(t)^2}{2} - V(x(t)) + k^2(\dot{x}(t))(\dot{x}(t+T_0))dt \tag{7}$$

Here, the first term represents the kinetic energy, the second term the potential energy as a function of position, and the third term the interaction of classical particle with itself via a mirror. As we can see, Feynman is integrating over all time to obtain the action $\mathcal{A}$. In the quantum case for the same system [**9**]

$$\begin{aligned} \langle \chi | F | \psi \rangle = \int \chi * (q_{T_2}) exp\{\frac{i}{\hbar}\mathcal{A}(q_{T_2}...q2, q_1, q_0, q_{-1}...q_T)\} \\ \times F(...q_1, q_0...) \times \psi(q_{T_1}) \frac{\sqrt{g}dq_{T_2}...\sqrt{g}dq_{T_1}}{A(T_2 - t_m)...A(t_{-m} - T_1)} dt \end{aligned} \tag{8}$$

we can see that Feynman is taking a similar approach. He starts by assuming that the particle has wave functions χ at time T_1 and ψ at T_2 , where T_1 is assumed to be large and negative, and T_2 is assumed to be large and positive. This is analogous to integrating over all time in the classical case, because the interaction is assumed to go to zero before T_1 and after T_2 , so the particle is freely moving as described by the wave functions χ and ψ . To put it more simply, because we are considering the interaction of the particle with itself, we don't care about the times when it is not interacting with itself, and can discount these times in our analysis of all possible paths. From this analysis, we can say that every possible path of the particle through space and time contributes to the probability that a particle takes a specific path through the space and time. The contribution of each of these paths is weighted according to its action $\mathcal{A}$. This is the general idea of the path integral formulation, which is the basis for Feynman's space-time approach to QED.

2.2 Feynman's Formulation of QED

We recall that quantum electrodynamics aims to describe interactions between electrons and photons. According to Feynman and his path integral approach, all possible paths for all such interactions can be broken down into some combination of three basic events, each of which has an associated probability amplitude. We will discuss these events in detail, but first, it is necessary to understand what is meant by "probability amplitude."

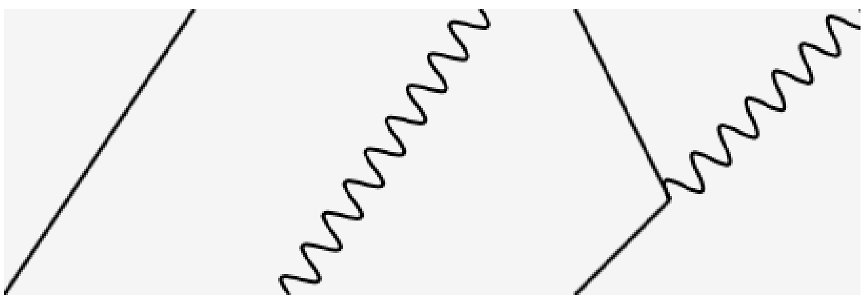
2.2.1 Probability Amplitudes

Feynman says that all we have to do to find the probability of any event is to draw a bunch of little arrows, add them up, and take the square of the length of the resulting arrow. [**11**]

More technically, we find the probability amplitude (a complex number) for each sub-event that makes up the event in which we are interested. We do this for every possible way the event can happen. We then add all of these amplitudes as complex numbers, Complex numbers add like vectors in the complex plane, so the sum of several complex numbers is the sum of the real parts plus the sum of the imaginary parts. The probability of the event of interest is the square of the magnitude of the sum of these complex numbers. This calculation seems simple enough, and it is — as long as we are willing to accept some error in our results. The error arises because it is very hard to account for every possible path an interaction can take. In most cases, we are willing to accept a reasonably good approximation.

2.2.2 Feynman Diagrams

The three basic events, which make up almost every other event that can happen in the universe, are: A) a photon moves through time and space, B) an electron moves through time and space, and C) an electron emits or absorbs a photon at a specific point in time and space. [**12**]

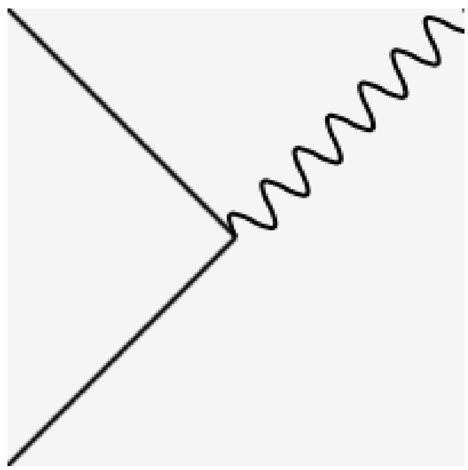


The Feynman diagrams for these three events are shown above. In these diagrams, the horizontal axis represents space, and the vertical axis represents time. The straight lines represent electrons, and the wavy lines represent photons. Each of these events has a probability amplitude associated with it, and we use these amplitudes to calculate the probabilities of things happening.

First, consider event A, the photon moving through time and space. Call the photon's initial position A and its final position B. It is important to remember that A and B are not just locations in space — they are also specific points in time. Thus, the probability amplitude is not just related to the probability that the photon will travel from place to place, but to the probability that the photon will do so at a specific time. The probability amplitude of the photon moving from A to B is given by the function P(A to B), which is a somewhat complicated function of a quantity I, which is, for a photon moving in one dimension: $I = (X_2 - X_1)^2 - (T_2 - T_1)^2$. [**11**] For particles moving in more that one dimension, terms of the form $(Y_2 - Y_1)^2$ can be added as needed, according to the Pythagorean theorem.

So, we can see that the probability amplitude is a function only of the distance traveled and the time taken. Next, consider event B, the electron moving through time and space. Its probability amplitude, E(A to B), is closely related to P(A to B). This formula is also quite complicated, but the general idea is that it depends on all of the possible paths the electron can take. The probability amplitude of the third basic event, C, which is the emission or absorption of a photon or an electron, is much simpler than the other two. It is a simple coupling number, $j \approx -.1$. [**11**]

We can combine these basic rules to compute probability amplitudes for any number of events. For example, say we want to calculate the probability amplitude that an electron will travel from A to B, emitting a photon at C along the way, which then ends up at D. The Feynman diagram for the situation looks like this: [**12**]



In order to calculate the probability amplitude, we first need to identify the sub-events (in terms of the basic three) that make up the event of interest. In this case, the electron travels from A to C, then it emits a photon, then it travels from C to B, and the photon travels from C to D. So, we can say that the probability amplitude for this whole event is

$$E(AtoC) \times j \times E(CtoB) \times P(CtoD) \tag{9}$$

Now, suppose we want to calculate the probability amplitude that the electron will travel from A to B and emit a photon anywhere along its path. In this case, we have to account not only for every possible path of the electron, but for the possibility that the photon is emitted at any point along any of those paths and this is only in the case where we are limiting ourselves to one photon emission - what if the photon were emitted, absorbed, and emitted again? This is a valid "path" for the same event, because, just as in the previous case, we start with an electron at A and end with an electron at B and a photon at D. Similarly, a path in which the photon is emitted and absorbed twice before the final emission is equally valid. It is easy to see that, while the ideas behind the diagrams and probability amplitudes are quite simple, the calculations get complicated rather quickly. As Feynman points out, an important part of any such calculation is deciding how accurate your answer needs to be — generally, as accurate as current experiments are.

3 Schwinger's Approach to QED

In contrast to Feynman's work, which was based on the path integral formulation of quantum mechanics, Schwinger took a conservative approach based on Dirac, Pauli, and Heisenberg's relativistic quantum mechanics. He started from the Schrodigner equation for two systems, with interaction Hamiltonian H_{12}: [**3**]

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = (H_1 + H_2 + H_{12})\Psi(t) \tag{10}$$

By making a contact transformation and following the methods used by Dirac and Tomonaga, he arrived at

$$i\hbar c\frac{\delta\Psi}{\delta\sigma} = \mathcal{H}'\Psi(t) \tag{11}$$

where

$$\mathcal{H}' = \frac{i}{2}[\mathcal{S},\mathcal{H}] = \frac{i}{2\hbar c^3}\int_{\pm\infty}^{\sigma}[j_\mu(x')A_\mu(x'), j_\nu(x)A_\nu(x)]d\omega' \tag{12}$$

Schwinger said that this treatment could be extended to a problem involving any number of electrons. In his lecture at the 1948 Pocono Conference (discussed in detail below), he also discussed the self-energy of the photon and of the electron. Using his method, Schwinger showed that the self-energy of the photon is finite, and that the self energy of the electron is only logarithmically divergent.[**3**] Feynman and Schwinger reached similar conclusions in many cases. The important distinction between their work is in the philosophy. Feynman formulated the path integral approach to quantum mechanics and built his version of QED on that, while Schwinger used existing mathematical formalisms to arrive at his conclusions.

4 QED after Feynman and Schwinger

4.1 The Pocono Conference

In 1948, American physicists gathered once again for a conference on important developments in physics. Similar to the Shelter Island Conference, the Pocono Conference was held the Pocono Manor Inn in eastern Pennsylvania. At this conference, both Schwinger and Feynman presented their work in QED on the same day. Schwinger's talk met a great deal of acclaim from the other physicists at the conference. Because he used mathematical ideas with which all of the attendees were familiar, they were able to follow his talk. Feynman's talk, on the other hand, was not nearly as well-received. Feynman's thinking was primarily physical, and his mathematics were based on formulas and tricks that he had invented based on his physical intuition. Another problem arose when Feynman tried to talk about positrons being electrons moving backward in time. For the most part, the theoretical physicists present found Feynman's work to be almost gimmicky, while they found Schwinger's work to be mathematically elegant. [**3**]

4.2 Freeman Dyson's Work

Freeman Dyson, born in 1923, grew up in Britain, attending first Winchester College and later Trinity College at Cambridge. He had a strong background in both mathematics and physics. Despite being a staunch pacifist, Dyson spent World War II working as a civilian scientist at the Royal Air Force Bomber Command Headquarters. After the war, Dyson applied for fellowship in Trinity College. Once accepted, he worked primarily in physics. He applied to study with Hans Bethe at Cornell, and was accepted. In the 1947–48 academic year, Dyson frequently interacted with Feynman, listening attentively as Feynman developed his formulation of QED during the year. Dyson thought that Feynman had valuable ideas, but had trouble communicating them effectively. After learning from Feynman at Cornell, Dyson attended the 1948 Michigan Summer School in Ann Arbor, where Schwinger lectured on QED. Dyson quickly saw that Feynman's and Schwinger's methods arrived at the same answers. He also saw that Schwinger's method was very similar to that of Tomonaga, although Tomonaga's was much less mathematically elaborate. So Dyson set out to "translate Feynman into the language that other people could use." In 1949, he published "The Radiation Theories of Tomonaga, Schwinger, and Feynman," [**13**] which was notable for two reasons. First, it showed that Schwinger's and Feynman's theories were mathematically equivalent, and second, it was the first paper not written by Feynman to use Feynman diagrams. [**3**]

4.3 QED Today

Quantum electrodynamics is the basis for most of modern particle physics. It is the basis for quantum chromodynamics (the idea that quarks have a property called color that is analogous to charge and is carried by different colored gluons) and the standard model. It also has applications outside of theoretical physics. Because it describes the most fundamental interactions among electrons and photons, QED can describe almost all chemical and biological phenomena, and is important in electronics and nanotechnology. Unlike most theories in particle physics, QED has been widely used for over fifty years and has yet to see breakdowns, even at extremely high energies. Because of its broad applications and enduring success, Dr. Kimball Milton, who worked with Schwinger for many years, calls QED the "greatest success of mankind." [**8**]

4.4 Conclusion

This paper summarized the development of QED from the early twentieth century the present. It especially focused on Richard Feynman's space-time approach to quantum mechanics and QED, and compared this approach with Julian Schwinger's more traditional mathematical formalism. Finally, the paper briefly discussed the impacts of QED on physics today.

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