How to Multiply Matrices

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CSCI 698 Teaching Demo

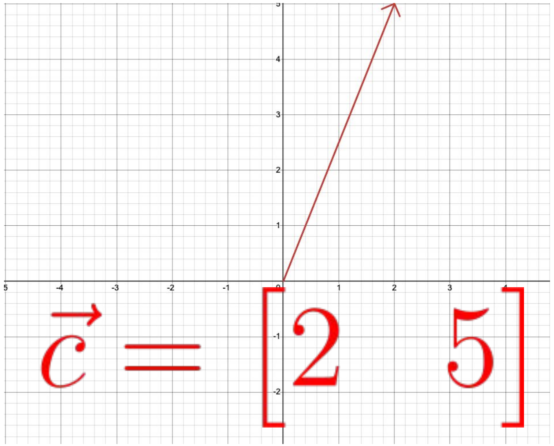
Matrix multiplication is the foundation of neural networks and modern AI tools like ChatGPT!

Defining Vectors

- A **vector** is a one-dimensional sequence of numbers.
  - Usually written as a lowercase letter with an arrow over it
  - An n-dimensional vector can represent coordinates in n-dimensional Cartesian space
  - Individual numbers in a vector are called **components** and written with subscripts
  - Examples:

$$\vec{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$



$$\vec{c} = \begin{bmatrix} 2 & 5 \end{bmatrix}$$

Vector Multiplication (Dot/Scalar Product)

- A **scalar** is a single number - e.g. 1 or 7.
- Let **x** and **y** be two vectors with the same length (**dimension**):

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

- The **dot product** or **scalar product** of two vectors is the sum of the element wise products:

$$\vec{x} \cdot \vec{y} = (x_1 y_1) + (x_2 y_2) + (x_3 y_3)$$

- Vectors MUST have the same length to take their dot product

Dot Product Example

$$\vec{u} = \begin{bmatrix} 3 & 4 \end{bmatrix} \;\; \vec{v} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = (u_1 v_1) + (u_2 v_2)$$

$$\vec{u} \cdot \vec{v} = (3 \times 1) + (4 \times 2)$$

$$\vec{u} \cdot \vec{v} = (3) + (8) = 11$$

Defining Matrices

- A **matrix** is a m-by-n grid of numbers.
  - Usually written as a capital letter
  - m rows by n columns
  - Can think of a matrix as a set of row vectors and/or a set of column vectors
  - Matrix **elements** are numbered with their row number, then their column number
  - Examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

2 x 3 (2 rows, 3 columns)

3 x 3 (3 rows, 3 columns)

Rules for Matrix Multiplication

- To multiply matrices A and B, **A must have the same # of columns as B has rows!**
- If A is m x n and B is n x p, the product C is m x p.
- Can multiply:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$$

(3 x 2) x (2 x 4) = (3 x 4)

- **Cannot** multiply:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \quad Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix}$$

(2 x 3) x (2 x 3)

How To Multiply Matrices

- When multiplying A and B, each element c_{ij} of the product matrix C is the dot product of the ith row of A and the jth column of B

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathrm{C} = \mathrm{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{11} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

Matrix Multiplication Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 1*1+2*2+3*3 & 1*4+2*5+3*6 \\ 4*1+5*2+6*3 & 4*4+5*5+6*6 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 4+10+18 \\ 4+10+18 & 16+25+36 \end{bmatrix} = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}$$