# NE 155, Class 28, S15
# Taylor Series Methods and Runge-Kutta

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## 1 Introduction

This lecture will cover:

- A reminder of the PRKE we learned yesterday
- A derivation of Forward Euler for approximating solutions to initial value problems
- A way to step through time with Runge-Kutta methods
- And application of this to a PRKE simulation
- An exploration of the importance of delayed neutrons for reactor control

## 2 PRKE

Point Reactor Kinetics Equations (PRKE).

$$\frac{dn(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \tag{1}$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda} n(t) - \lambda_i C_i(t) \tag{2}$$

where

$$\begin{aligned}
i &\in [1, 6] \\
n &= \text{neutron population} && (3) \\
\beta &= \text{fraction of neutrons that are delayed} && (4) \\
\lambda_i &= \text{effective decay constant of the ith precursor}[\frac{1}{s}] && (5) \\
C_i(t) &= \text{delayed neutron concentration due to the ith precursor} && (6) \\
\Lambda &= \text{effective neutron lifetime} && (7) \\
&\equiv (v\nu\Sigma_F)^{-1} \\
\rho(t) &= \text{reactivity} && (8) \\
&\equiv \frac{k(t) - 1}{k(t)} \\
&\equiv \frac{\nu\Sigma_F - \Sigma_a(1 + L^2 B_g^2)}{\nu\Sigma_F}
\end{aligned}$$

and

$$k \equiv \frac{\nu\Sigma_F / \Sigma_a}{1 + L^2 B_g^2}.$$

The PRKEs allow a nuclear engineer to remove the spatial aspects of the reactor from consideration to simplify the analysis of this initial value problem. In particular, the assumptions that have gone into this set of equations include:

- there is no external neutron source
- One energy group ($\phi(\vec{r}, E, t) = \phi(\vec{r}, t)$)
- Separation of variables ($\phi(\vec{r}, t) = S(t)\psi(\vec{r})$)
- $\beta_i$ and $\lambda_i$ are constant over the transient

- There is no feedback (e.g. flux is low, feedbacks are slow/weak, $\rho(t)$ is known)

**Exercise:** *What initial conditions need to be defined in order to solve this initial value problem (eqn.* (2)*?)*

# 3 IVP

Recall that the initial value problem takes the form:

$$u'(t) = f(u(t), t), \text{ for } t > t_0 \tag{9}$$

where

$$u(t_0) = u_0 \tag{10}$$

In such a problem, we desire to compute $u(t_1)$, $u(t_2)$, $u(t_n)$ and so on.

There are many methods for solving initial value problems numerically. This lesson will introduce a simple one, Forward Euler, which is derived from a Taylor series expansion. We will then use Forward Euler to introduce a two-stage, explicit Runge-Kutta method as well, for higher accuracy.

# 4 Taylor Series Derivation of Forward Euler

The simplest method for finding $u(t_n)$ is Forward Euler, which approximates $u(t_n)$. Let's call this approximation, $U^n$. In this notation, Forward Euler is based on replacing $u'(t_n)$ with $(U^{n+1} - U^n)/\Delta t$, where $\Delta t$ is the width of the timestep. The Forward Euler method arises from a Taylor series expansion of $u(t_{n+1})$ about $u(t_n)$:

$$u(t_{n+1}) = u(t_n) + \Delta t u'(t_n) + \frac{1}{2}\Delta t^2 u''(t_n) + \cdots \tag{11}$$

With this, the $O(\Delta t^2)$ terms can be dropped to give:

$$u(t_{n+1}) \approx u(t_n) + \Delta t u'(t_n) \tag{12}$$

And, based on equation (9) we can replace $u'(t_n)$ with $f(u(t_n), t_n)$:

$$\frac{1}{2}\Delta t^2 u''(t_n) + \cdots \tag{13}$$

$$u(t_{n+1}) = u(t_n) + \Delta t f(u(t_n), t_n) \tag{14}$$

This expression gives a truncation error of order $O(\Delta t^2)$. More accurate schemes can be derived with a Taylor series expansion by retaining higher order terms in equation (11). Since we are only given $u'(t_n) = f(u(t_n), t_n)$, however, the computation of such schemes requires repeated recursive differentiation of this function, and can get quite messy.

# 5 Runge-Kutta Methods

Runge-Kutta is a method used in practice to get a higher order approximation *without* explicitly calculating higher order derivatives.

Runge-Kutta uses two stages. The first stage is an update using Euler's method, approximating $u(t_{n+1/2})$.

$$U^{n+1/2} = U^n + \frac{1}{2}\Delta t f(U^n) \tag{15}$$

$$\tag{16}$$

The second stage evaluates the function, $f$, at the midpoint to estimate the slope.

$$U^{n+1} = U^n + \Delta t f(U^{n+1/2}) \tag{17}$$

These equations can be combined into a single expression:

$$U^{n+1} = U^n + \Delta t f(U^n + \frac{1}{2}\Delta t f(U^n)) \tag{18}$$

This approximation, because it uses two points, like a centered approximation, is order $O(\Delta t)$ accurate.

A generic r-stage Runge-Kutta method can be expressed as:

$$Y_1 = U^n + \Delta t \sum_{j=1}^{r} a_{1j} f(Y_j, t_n + c_j \Delta t) \tag{19}$$

$$Y_2 = U^n + \Delta t \sum_{j=2}^{r} a_{2j} f(Y_j, t_n + c_j \Delta t) \tag{20}$$

$$\vdots$$

$$Y_r = U^n + \Delta t \sum_{j=r}^{r} a_{rj} f(Y_j, t_n + c_j \Delta t) \tag{21}$$

$$U^{n+1} = U^n + \Delta t \sum_{j=r}^{r} a_{rj} f(Y_j, t_n + c_j \Delta t) \tag{22}$$

# 6 Application to PRKE

Each of these can be applied to the PRKE. In particular, let's consider the application of a Forward Euler.

To avoid confusion with the multiplication factor, the width of our timestep will be called $\Delta t$.

$$n(t_{n+1}) = n(t) + \Delta t \left[ \frac{\rho(t_n) - \beta}{\Lambda} n(t_n) + \sum_{j=1}^{j=J} \lambda_j \zeta_j \right] \tag{23}$$

$$\tag{24}$$

# 7 Delayed Neutrons and Reactor Control

These delayed neutrons are critical to controlling the reactor. To capture the reasons why, we will need the following definitions.

$$\rho = \text{reactivity} \tag{25}$$
$$= \frac{k-1}{k} \tag{26}$$
$$k = \text{multiplication factor} \tag{27}$$
$$(k < 1) \rightarrow \text{negative reactivity} \tag{28}$$
$$(k > 1) \rightarrow \text{positive reactivity} \tag{29}$$
$$(k = 1) \rightarrow \text{critical} \tag{30}$$
$$\beta = \text{delayed neutron fraction} \tag{31}$$
$$(\rho < \beta) \text{ delayed supercriticality} \tag{32}$$
$$(\rho > \beta) \text{ prompt supercriticality} \tag{33}$$
$$l = \text{mean neutron lifetime} \tag{34}$$

## 7.1 Units of Reactivity

Note that the units of $\rho$ can be confusing.

| Unit | Definition | Example |
|---|---|---|
| $\Delta k$ | actual PRKE units | 0.0005 |
| $\%\Delta k$ | percent notation of $\Delta k$ | 0.05% |
| pcm | per cent mille | 50pcm |
| Dollars | $\frac{\Delta k}{\beta}$ | \$1 |
| Cents | 100 cents per dollar | 100 cents |
| Milli-beta | 1000 milli-beta per dollar | 1000 milli-beta |

Table 1: Common units of reactivity.

## 7.2 Thought Experiment

If there were no delayed neutrons, then the time constant for power increase would be approximately $l_p$, the prompt neutron lifetime. That isn't the case, but if it were, the reactor power would

proceed thus:

$$l = \text{mean generation time} \tag{35}$$

$$n(t+l) = n(t) + l\frac{dn}{dt} = k_\infty n(t) \tag{36}$$

such that

$$\frac{dn}{dt} = \left(\frac{k-1}{l}\right) n(t) \tag{37}$$

which gives

$$n(t) = n_0 e^{\frac{(k-1)t}{l}} \tag{38}$$

characterized by the time constant

$$T = \text{reactor period} \tag{39}$$

$$= \frac{l}{k-1} \tag{40}$$

In a universe without delayed neutrons, the mean neutron lifetime ($l$) would be the prompt neutron lifetime ($l_p$). Noting that the prompt neutron lifetime is about $2 \times 10^{-5} s$, take a moment to think about the implications of this.

**Exercise** *If a control rod were moved to introduce an excess reactivity of* $0.0005\Delta k$*, what would the power be one second later?*