Problem Set 01

[10] 1. Write down a differential equation for the time-dependent, continuous-energy, neutron diffusion equation in a multiplying media, without a source term. Describe what each term represents.

Solution:

$$\underbrace{\frac{1}{v}\frac{\partial \phi}{\partial t}}_{1} = -\underbrace{\nabla \cdot (-D\nabla\phi)}_{2} - \underbrace{\Sigma_t \phi}_{3} + \underbrace{\int_0^\infty \Sigma_s(E' \to E)\phi(E')dE'}_{4} + \underbrace{\chi(E)\int_0^\infty \nu(E')\Sigma_f(E')\phi(E')dE'}_{5}$$

1. $\frac{1}{v}\frac{\partial \phi}{\partial t}$ represents the time rate of change of the total number of neutrons.
2. $-\nabla \cdot (-D\nabla\phi)$ represents the divergence of the neutron current within a control volume.
3. $-\Sigma_t \phi$ represents the rate at neutrons are removed from one particular energy via all possible interactions with other particles within a control volume.
4. $\int_0^\infty \Sigma_s(E' \to E)\phi(E')dE'$ represents the rate at which neutrons are scattered into one particular energy from all other energies within a control volume.
5. $\chi(E)\int_0^\infty \nu(E')\Sigma_f(E')\phi(E')dE'$ represents the rate at which neutrons of a given energy are being produced through fission within the control volume.

[15] 2. Evaluate the following complex integral by integrating over the left half of the complex plane. γ is an arbitrary constant that ensures that all singularities are within the integration contour. a and b are positive, arbitrary, real constants.

$$I = \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{e^{s\,t}}{(s+a)(s+b)} ds$$

HINT: It will be easier to use a change of variable to recenter the complex plane around γ, such as $s = \gamma + \mathrm{i}\omega$.

Solution:

To evaluate the integral, recognize that it is the inverse of the Laplace Transform. While you can use a table and partial fraction expansions, you can instead evaluate this integral directly.

$$I = \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{e^{s\,t}}{(s+a)(s+b)} ds$$

This integral can be evaluated with an application of the Residue Theorem and Jordan's Lemma. To properly evaluate the integral, you must use a contour that encompasses all of the singularities on the left half of the complex plane. The contour should be a semicircle; this is where Jordan's Lemma comes into play. By using the change of variable hinted at in the problem, you get to the following:

$$\frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{e^{s\,t}}{(s+a)(s+b)} ds = \sum_{k=1}^{n} Res(f, z^k) + \frac{e^{\gamma}}{2\,\pi} \oint_{-\infty}^{\infty} \frac{e^{\mathrm{i}\,\omega\,t}}{(\omega - \mathrm{i}(a+\gamma))(\omega - \mathrm{i}(b+\gamma))}\, d\omega$$

The contour integral on the right is equal to zero by Jordan's Lemma. This means that the total integral evaluates to:

$$I = \frac{e^{-a\,t} - e^{-b\,t}}{b-a}$$

[15] 3. A one-dimensional slab reactor consists of two regions: a multiplying core of length a, and vacuum outside of the core. Using one-group diffusion theory, determine a such that the reactor will be critical. Express a in terms of relevant nuclear properties.

Solution:

For a critical, one-dimensional reactor, the steady-state neutron balance is give by:

$$D\nabla^2\phi - \Sigma_a\phi = -\nu\Sigma_f\phi$$

A critical reactor can be expressed in terms of the lowest eigenvalue representation of the Helmholtz Equation:

$$\nabla^2\phi + B_g^2\phi = 0$$

B_g^2 is the geometric buckling term, which, for a slab, is given as $\left(\frac{\pi}{\tilde{a}}\right)^2$.

Using some algebra,

$$\frac{\nabla^2\phi}{\phi} = \frac{\Sigma_a - \nu\Sigma_f}{D} = -B_g^2$$

$$\begin{aligned}
-B_g^2 &= \frac{\Sigma_a - \nu\Sigma_f}{D} \\
\left(\frac{\pi}{\tilde{a}}\right)^2 &= \frac{\nu\Sigma_f - \Sigma_a}{D} \\
\frac{\pi}{\tilde{a}} &= \sqrt{\frac{\nu\Sigma_f - \Sigma_a}{D}} \\
\tilde{a} &= \frac{\pi\sqrt{D}}{\sqrt{\nu\Sigma_f - \Sigma_a}} \\
a + 2z_o &= \frac{\pi\sqrt{D}}{\sqrt{\nu\Sigma_f - \Sigma_a}}
\end{aligned}$$

Here, z_o is given by:

$$z_o = 0.7123\lambda_{tr} \simeq 2.1\,D$$

Giving for a final solution:

$$a \simeq \frac{\pi\sqrt{D}}{\sqrt{\nu\Sigma_f - \Sigma_a}} - 4.2\,D$$

[15] 4. Use a computer to plot the solution to the following coupled ODEs. Plot $y(t)$ and $x(t)$ on the same graph for $t \in [0, 10]$.

$$\begin{aligned} \frac{dy}{dt} &= 0.4\,x(t) - y(t) \\ \frac{dx}{dt} &= -0.4\,y(t) \end{aligned} \quad , \qquad \begin{aligned} y(0) &= 1 \\ x(0) &= 0 \end{aligned}$$

Solution:

First, obtain an analytic solution to the system. Do not fret; this is not required for points.

Laplace Transform both equations.

$$\begin{aligned} \mathscr{L}\left[\frac{dy}{dt}\right] &= 0.4\,\mathscr{L}[x] - \mathscr{L}[y] \\ \mathscr{L}\left[\frac{dx}{dt}\right] &= -0.4\,\mathscr{L}[y] \\ s\,\tilde{y} - 1 &= 0.4\,\tilde{x} - \tilde{y} \\ s\,\tilde{x} &= -0.4\,\tilde{y} \end{aligned}$$

Solve for $\tilde{x}$ in terms of $\tilde{y}$.

$$\begin{aligned} s\,\tilde{x} &= -0.4\,\tilde{y} \\ \tilde{x} &= -0.4\frac{\tilde{f}}{s} \end{aligned}$$

Eliminate $\tilde{x}$ from the equation for $\tilde{y}$ and solve for $\tilde{x}$.

$$\begin{aligned} s\,\tilde{y} + \tilde{y} + 0.16\,\frac{\tilde{y}}{s} &= 1 \\ \tilde{y}\left(s + 1 + \frac{0.16}{s}\right) &= 1 \\ \tilde{y}\left(\frac{s^2 + s + 0.16}{s}\right) &= 1 \end{aligned}$$

$$\begin{aligned} \tilde{y} &= \frac{s}{s^2 + s + 0.16} \\ \tilde{y} &= \frac{s}{(s + 0.8)(s + 0.2)} \end{aligned}$$

Apply the Inverse Laplace Transform.

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left[\frac{s}{(s + 0.8)(s + 0.2)}\right] \\ y(t) &= \frac{4}{3}e^{-0.8\,t} - \frac{1}{3}e^{-0.2\,t} \end{aligned}$$

Integrating $y(t)$ with respect to t will yield $x(t)$.

$$x(t) = \frac{2}{3}\left(e^{-0.8\,t} - e^{-0.2\,t}\right)$$

Giving the final solutions of

$$\begin{aligned} y(t) &= \frac{4}{3}e^{-0.8\,t} - \frac{1}{3}e^{-0.2\,t} \\ x(t) &= \frac{2}{3}\left(e^{-0.8\,t} - e^{-0.2\,t}\right) \end{aligned}$$

See next page for graph.

Question 4: Plot of y(t) and x(t) vs. t.