MATH2301 Assignment 4

(1) Consider a graph whose adjacency matrix is $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find the number of paths of length 4 from 1 to 3

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $A^4_{1,3} = 10$ there are 10 paths of length 4 from 1 to 3.

(2)

(a) Find (without explicit calculation) an example of a 4×4 nonzero adjacency matrix such that all powers beyond the 10th power are 0. Justify briefly.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is connected only one way so there are no paths with length more than 1.

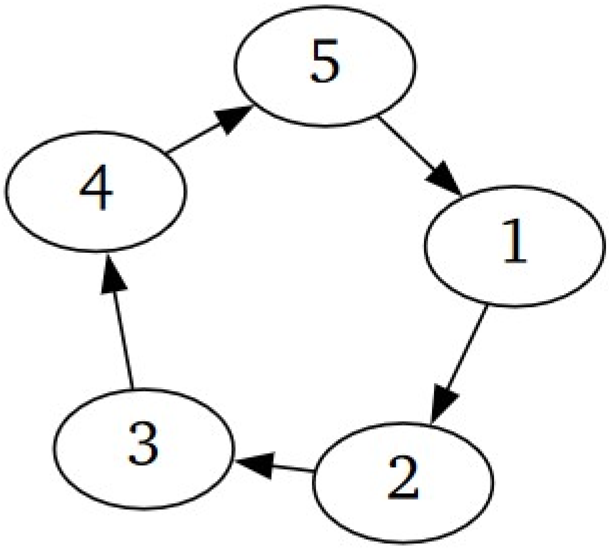
(b) Is it true that the 8th power of any such matrix (not just your example!) must also be zero?

A graph with 4 nodes will either loop indefinitely or have a max path of 3. So a 4×4 adjacency matrix where all powers beyond the 10th power are 0 must have all powers beyond the 4th power as 0.

(c) Is it true that the cube of any such matrix (not just your example) must also be zero?

No.

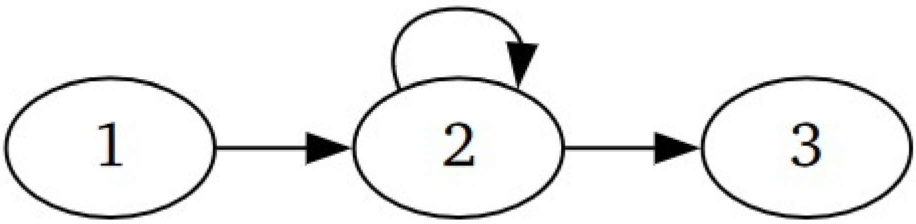
(3) Let G be the directed pentagon below, Let A be the adjacency matrix of G. Describe all positive powers of A.



$$A^k = \begin{cases} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & k = 5n+1, n \in \mathbb{N} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} & k = 5n+2, n \in \mathbb{N} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} & k = 5n+3, n \in \mathbb{N} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} & k = 5n+4, n \in \mathbb{N} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & k = 5n, n \in \mathbb{N} \end{cases}$$

*in this case $\mathbb{N}$ includes 0.

(4) Let G be the directed snail below, Let A be the adjacency matrix of G. Describe all positive powers of A.



$$A^k = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} & k = 1 \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} & k \in \mathbb{Z}, k \geq 2 \end{cases}$$

(5) Let $(S, \preceq)$ be a finite poset. Let G be the directed graph of the relation $\preceq$ and let A be the adjacency matrix of G. Let I be the identity matrix of the same size as A. True or false: some positive power of $A - I$ must be zero. If true, justify it. Otherwise give an example where no positive power of $A - I$ is zero.

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subtracting any adjacency matrix A by I is simply the act of removing the reflexive edges from G, which can be treated as a conversion of $(S, \preceq)$ to $(S, \prec)$. Since the graph of $(S, \prec)$ is finite and asymetric, there is a maximum length path which exists. Therefore after a certain power $A - I$ is zero.