Games, graphs, and machines

Partial orders 2

August 7, 2024

Product poset

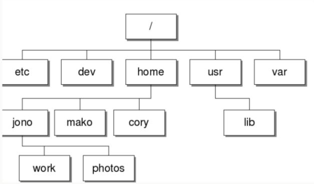
Let $\leq$ be the usual order on $\mathbb{R}$. Define $\preceq$ on $\mathbb{R} \times \mathbb{R}$ by

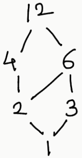
$$(a, b) \preceq (c, d) \text{ if } a \leq b \text{ and } c \leq d.$$

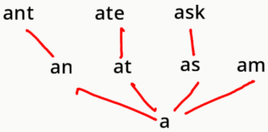
1. Give an example of two incomparable elements under $\preceq$.
2. Plot all elements that are $\preceq (2, 3)$.
3. Plot all elements (x, y) with $(1, 1) \preceq (x, y) \preceq (2, 3)$.

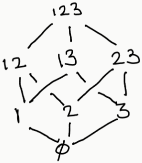
Max/min

In all the examples so far, identify the maximum (if it exists), the minimum (if it exists), all maximal elements, all minimal elements.









Product poset $\mathbb{R}^2$

Immediate successors

Let S be the divisor poset of 60. What are the immediate successors of 3?

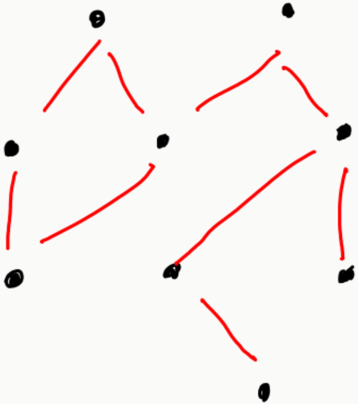
Immediate successors

Let S be the poset of words with $\preceq$ given by prefix.

- What are the immediate successors of "ant"?
- What is an element that succeeds "ant" but is not an immediate successor?

Rank function

Find a rank function on the following poset.



Rank functions

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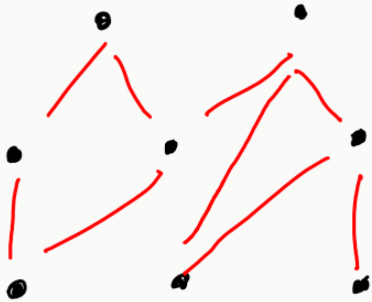

Chains

A *chain* in a poset is a sequence of elements $a_1, \ldots, a_n$ such that

$$a_1 \preceq a_2 \preceq \cdots \preceq a_n.$$

The number n is the *length* of the chain.

Find a chain of length 3 in the subset poset of $\{1, 2, 3, 4\}$.

Maximal chains

- What could be the meaning of a *maximal chain*?
- Find a maximal chain in the subset poset of $\{1, 2, 3, 4\}$.

Maximal chains

- Prove that any maximal chain in the subset poset of $\{1, \cdots, 100\}$ has length 100.

Maximal chains

- Prove that any maximal chain in the subset poset of $\{1, \cdots, 100\}$ has length 100.
- Is a similar statement true for the divisor poset of 100?

A theorem

A poset in which all maximal chains have the same (finite) length is called a *graded poset*.

Theorem
A graded poset of length n has a rank function.

Verify the theorem for the subset poset of $\{1, \ldots, n\}$.