MATH2301 Assignment 5

(1) A directed graph is strongly connected if for every i and j, there is a path from vertex i to vertex j. True or False:

(a) A graph with adjacency matrix A is strongly connected iff $\exists n|A^{*n}$ with all entries equal to $[1]$.

Counterexample: the graph of $A = \begin{pmatrix} [0] & [1] & [0] \\ [0] & [0] & [1] \\ [1] & [0] & [0] \end{pmatrix}$ is strongly connected but there is no $n|A^{*n} = \begin{pmatrix} [1] & [1] & [1] \\ [1] & [1] & [1] \\ [1] & [1] & [1] \end{pmatrix}$

$\therefore$ statement is false

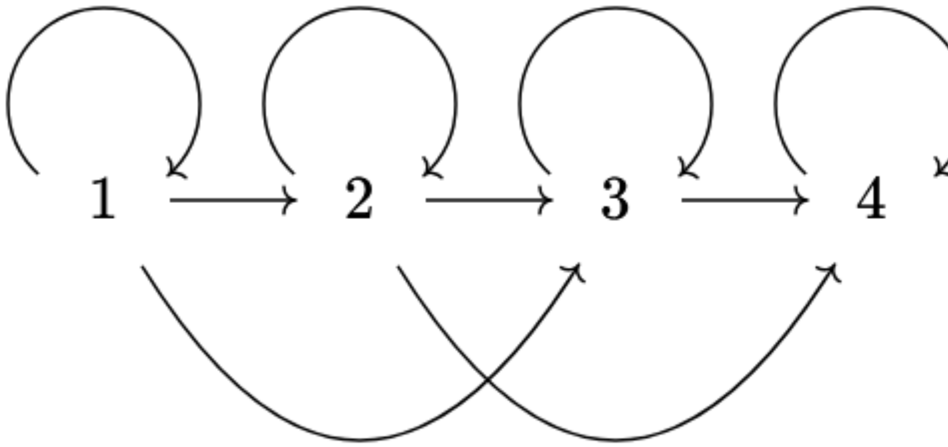
(b) A graph with adjacency matrix A is strongly connected iff $\exists n|\sum_{i=0}^{n} A^{*i}$ with all entries equal to [1].

if the graph of A is strongly connected then $\forall (i,j) \in A, \exists n|A^{*n}_{(i,j)} = [1]$ and since $[1] + [0] = [1]$ and $[1] + [1] = [1]$ then $\exists n|\sum_{i=0}^{n} A^{*i} = [1]$.

$\therefore$ statement is true.

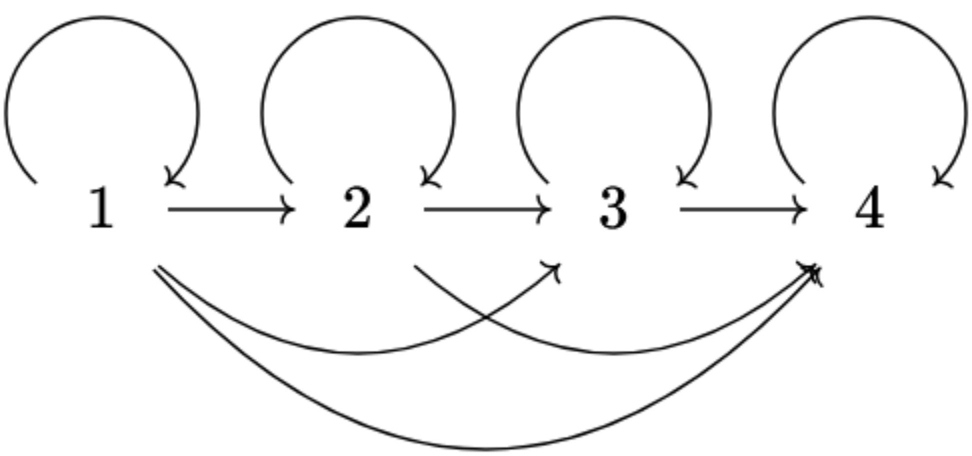
(2) Let G be the graph of the relation $R = \{(a,b) | 0 \leq b - a \leq 2\}$ on $S = \{1,2,3,4\}$. Draw G and G^+. Write down the adjacency matrix of both graphs.

G:



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

G^+ :



$$A^+ = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) Let A be the boolean adjacency matrix of G. Prove that G is transitive iff $A = A + A^{*2}$.

The adjacency matrix of any G^+ with n nodes is $\sum_{i=1}^{n-1} A^{*i}$

if G is transitive then $G = G^+$

$\implies A = A + \sum_{i=2}^{n-1} A^{*i} = A + A^{*2}$ (If G is transitive, the matrix A already includes paths of length 2. Thus, A^{*2} and beyond contribute nothing more to the sum.)

$\therefore G = G^+ \implies A = A + A^{*2}$

$A = A + A^{*2}$ implies every path of length 2 (or any further length by recursiveness) between (i, k) is already represented by a direct path. So all transitive connections are already in A which means G must be transitive.

$\therefore G = G^+ \iff A = A + A^{*2}$

(4) Using the criterion in the previous problem, determine if the following adjacency matricies define transitive graphs:

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$A^{*2} = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix}$$
$$A + A^{*2} = \begin{pmatrix} [1] & [1] \\ [1] & [1] \end{pmatrix}$$
$$A + A^{*2} \neq A \implies G \neq G^{+}$$

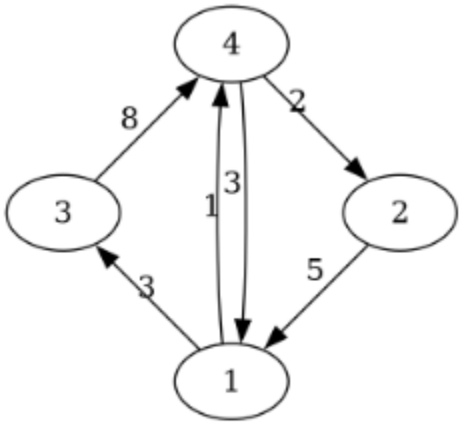
(b) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$A^{*2} = \begin{pmatrix} [1] & [1] & [1] \\ [0] & [1] & [1] \\ [0] & [0] & [1] \end{pmatrix}$$
$$A + A^{*2} = \begin{pmatrix} [1] & [1] & [1] \\ [0] & [1] & [1] \\ [0] & [0] & [1] \end{pmatrix}$$
$$A + A^{*2} \neq A \implies G \neq G^{+}$$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$A^{*2} = \begin{pmatrix} [0] & [0] & [1] \\ [0] & [0] & [0] \\ [0] & [0] & [0] \end{pmatrix}$$
$$A + A^{*2} = \begin{pmatrix} [0] & [1] & [1] \\ [0] & [0] & [1] \\ [0] & [0] & [0] \end{pmatrix}$$
$$A + A^{*2} = A \implies G = G^{+}$$

(5) Find the minimum cost of paths between any pair of verticies in the following graph. Assume every vertex has loops of length 0 (not shown).



$$W = \begin{pmatrix} 0 & \infty & 3 & 1 \\ 5 & 0 & \infty & \infty \\ \infty & \infty & 0 & 8 \\ 3 & 2 & \infty & 0 \end{pmatrix}$$

$$W^{\odot 2} = \begin{pmatrix} 0 & 3 & 3 & 1 \\ 5 & 0 & 8 & 6 \\ 11 & 10 & 0 & 8 \\ 3 & 2 & 6 & 0 \end{pmatrix}$$

$$W^{\odot 3} = \begin{pmatrix} 0 & 3 & 3 & 1 \\ 5 & 0 & 8 & 6 \\ 11 & 10 & 0 & 8 \\ 3 & 2 & 6 & 0 \end{pmatrix}$$

The matrix $W^{\odot 3}$ has entries with the lengths of the shortest paths between pairs of matricies.