Parallel Sorting

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Logistics

Today

- Parallel Sorting: Quicksort
- Thursday: Mini-exam 2

Reading: Grama Ch 9

- Sorting
- Focus on 9.4: Quicksort

Exercise: Quick Review 1/2

1. What is Amdahl's law? What does it say about the speedup achievable by parallel programs?
2. How does one calculate the following for a parallel algorithm
   - S: Speedup
   - E: Efficiency
   - C: Cost
3. How does the Efficiency of a parallel usually change if the number of processors P increases but the problem size stays the same? If number of procs stays the same but problem size increases?

Exercise: Quick Review 2/2

4. What was the major benefit that Cannon's Algorithm provided over a naive implementation of parallel matrix multiply?
5. What type of MPI calls would be used in Cannon's Algorithm?
6. In broad strokes, how was the LU factorization parallelized?
7. What type of MPI calls would be used in the LU Factorization?

Sorting

- Much loved computation problem
- What is the best complexity of general purpose (comparison-based) sorting algorithms?
- What are some algorithms which have this complexity?
- What are some other sorting algorithms which aren't so hot?
- What issues need to be addressed to parallelize any sorting algorithm?

Parallel Sorting Base algorithm

Prospects of parallelizing standard $O(N \log N)$ sorting algorithms...

Heap Sort

- Manipulates a global array
- Very serial in nature: repeatedly percolate array elements up heap, swap to end of heap, repeat
- Random access to entire array is a must, not good for distributed memory

Merge Sort

- Has a nice recursive decomposition, but...
- Merging two sorted arrays on separate processors to produce a larger array would involve prohibitive communication
- Will look later at Odd-Even sort which has a similar flavor

This leaves the king of sorting for a parallel implementation...

Partition and Quicksort

- Quicksort has $O(N \log N)$ average complexity
- In-place, low overhead sorting, recursive

Partition

- Partition: select `pivot` value
- Rearrange elements so
  - Left array is $\leq$ `pivot`
  - Right array is $>$ `pivot`
  - `pivot` is in "middle"

```
// A is an array, lo/hi are
// inclusive boundaries
algorithm partition(A, lo, hi) is
  pivot := A[hi]
  boundary := lo
  for j := lo to hi do
    if A[j] <= pivot then
      swap A[boundary] with A[j]
      boundary++
  swap A[boundary] with A[hi]
  return boundary
```

Quicksort

- Partition into two parts
- Recurse on both halves
- Bail out when boundaries `lo/hi` cross

```
algorithm quicksort(A, lo, hi) is
 if lo < hi then
   p := partition(A, lo, hi)
   quicksort(A, lo, p - 1)
   quicksort(A, p + 1, hi)
```

Practical Parallel Sorting Setup

- Input array A of size N is already spread across P processors (no need to scatter)

```
P0: A[] = { 84 31 21 28 }
P1: A[] = { 17 20 24 84 }
P2: A[] = { 24 11 31 99 }
P3: A[] = { 13 32 26 75 }
```

- Goal: Numbers sorted across processors. Smallest on P0, next smallest on P1, etc.

```
P0: A[] = { 11 13 17 20 }
P1: A[] = { 21 24 24 26 }
P2: A[] = { 28 31 32 33 }
P3: A[] = { 75 84 84 99 }
```

- Want to use P processors as effectively as possible
- Favor bulk communication over many small messages

Exercise: Parallel Quicksort

- Find a way to parallelize quicksort
- **Hint:** The last step is each processor sorting its own data using a serial algorithm. Try to arrange data so this is possible.

```
START:
P0: A[] = { 84 32 21 28 }
P1: A[] = { 17 20 25 85 }
P2: A[] = { 24 11 31 99 }
P3: A[] = { 13 32 26 75 }

GOAL
P0: A[] = { 11 13 17 20 }
P1: A[] = { 21 24 25 26 }
P2: A[] = { 28 31 32 33 }
P3: A[] = { 75 84 85 99 }
```

```
SERIAL ALGORITHM
algorithm quicksort(A, lo, hi) is
 if lo < hi then
   p := partition(A, lo, hi)
   quicksort(A, lo, p - 1)
   quicksort(A, p + 1, hi)

algorithm partition(A, lo, hi) is
  pivot := A[hi]
  boundary := lo
  for j := lo to hi - 1 do
    if A[j] <= pivot then
      swap A[boundary] with A[j]
      boundary++
  swap A[i] with A[hi]
  return boundary
```

Answers: Parallel Quicksort Ideas 1 / 3

- Select a global shared Pivot value and broadcast to all procs
- Select pivot so that half data elements got to lower processors, half got to higher processors
- Redistribute low data to low procs, high data to high procs
- Split procs into low / high group, and recurse
- When each proc is on its own, sort locally

Answers: Parallel Quicksort Ideas 2 / 3

```
A[] = { 84 32 21 11 | 17 20 25 85 | 24 28 31 99 | 13 33 26 75 }
        P0            P1            P2            P3

Partition(pivot=26) on each processor
A[] = { 21 11 84 32 | 17 20 25 85 | 24 28 31 99 | 13 26 33 75 }
Boundary:   ^                ^       ^                ^
Counts: P0: 2         P1: 3         P2: 1         P3: 2

Calculate which data goes where
A[] = { 21 11 84 32 | 17 20 25 85 | 24 28 31 99 | 13 26 33 75 }
        P0 P0 P2 P2   P0 P0 P1 P2   P1 P2 P3 P3   P1 P1 P3 P3

Re-arrange so values <= 26 on P0 and P1, > 26 on P2 and P3
A[] = { 21 11 17 20 | 25 24 13 26 | 84 32 85 28 | 31 99 33 75 }
        P0            P1            P2            P3

Split the world: 2 groups
A[] = { 21 11 17 20 | 25 24 13 26}|{84 32 85 28 | 31 99 33 75 }
        P0            P1            P2            P3
```

Answers: Parallel Quicksort Ideas 3 / 3

```
Each half partitions on different pivot value
        P0-P1: Partition(pivot=20)  P2-P3: Partition(pivot=33)
A[] = { 11 17 20 21 | 13 25 24 26}|{28 32 84 85 | 31 33 99 75 }
Boundary:      ^       ^                ^              ^
Counts: P0: 3          P1: 1            P2: 2          P3: 2

Calculate which data goes where
A[] = { 11 17 20 21 | 13 25 24 26}|{28 32 84 85 | 31 33 99 75 }
        P0 P0 P0 P1   P0 P1 P1 P1   P2 P2 P3 P3   P2 P2 P3 P3

Re-arrange values to proper processors
A[] = { 11 17 20 13 | 21 25 24 25}|{28 32 31 33 | 84 85 99 75 }
        P0             P1             P2             P3

Split the world: 4 groups
A[] = { 11 17 20 13}|{21 25 24 25}|{28 32 31 33}|{84 85 99 75 }
        P0             P1             P2             P3

4 groups == 4 processors, all processors sort locally
A[] = { 11 13 17 20}|{21 24 25 25}|{28 31 32 33}|{75 84 85 99 }
        P0             P1             P2             P3
```

Quicksort Difficulties

Communication

- Determine which data go to which processors, how many send/receives are required
- Opportunity for **all-to-all communications** in MPI

Recursing

- Recursive step of algorithm requires smaller "worlds"
- Use MPI's **communicator splitting** capability

Pivot Value Selection

- In example, pivot values were cherry-picked to get even distribution of data among processors
- A bad pivot splits data unevenly, is annoying for serial Quicksort, shaves off processors in parallel quicksort destroying efficiency

All-to-All Personalized Communication

All-to-all personalized communication: like every processor scattering to every other processor.

```
                     BEFORE
P0: send[] ={A0, B0, C0, D0}  recv[] = { -,  -,  -,  -,}
P1: send[] ={A1, B1, C1, D1}  recv[] = { -,  -,  -,  -,}
P2: send[] ={A2, B2, C2, D2}  recv[] = { -,  -,  -,  -,}
P3: send[] ={A3, B3, C3, D3}  recv[] = { -,  -,  -,  -,}

                 MPI_Alltoall(...);

                     AFTER
P0: send[] ={A0, B0, C0, D0}  recv[] = {A0, A1, A2, A3}
P1: send[] ={A1, B1, C1, D1}  recv[] = {B0, B1, B2, B3}
P2: send[] ={A2, B2, C2, D2}  recv[] = {C0, C1, C2, C3}
P3: send[] ={A3, B3, C3, D3}  recv[] = {D0, D1, D2, D3}
```

MPI_Alltoall

- Standard version: every processor gets a slice of sendbuf, same sized data
- Vector version allows different sized slices (appropriate for quicksort)

```
int MPI_Alltoall(
 void *sendbuf, int sendcount, MPI_Datatype sendtype,
 void *recvbuf, int recvcount, MPI_Datatype recvtype,
 MPI_Comm comm);

int MPI_Alltoallv(
 void *sendbuf, int sendcounts[], int sdispls[], MPI_Datatype sendtype,
 void *recvbuf, int recvcounts[], int rdispls[], MPI_Datatype recvtype,
 MPI_Comm comm);
```

Exercise: Redistribution during Quicksort

- After `partition`, procs will redistribute data via all-to-all
- Perform All-Gather to get counts in table to the right

Element Count vs Pivot

Proc	<=	>
P0	2	2
P1	3	1
P2	1	3
P3	2	2

Each Proc must calculate its own Count/Displ arrays for all-to-all:

P#		P0	P1	P2	P3		P0	P1	P2	P3	P#
P0	RecvCount	2	2	0	0	SendCount	2	0	2	0	P0
P1		0	1	1	2		2	1	1	0	P1
P2		2	1	1	0		0	1	1	2	P2
P3		0	0	2	2		0	2	0	2	P3
P0	RecvDispl	0	2	4	4	SendDispl	0	0	2	0	P0
P1		0	0	1	2		0	2	3	4	P1
P2		0	2	3	4		0	0	1	2	P2
P3		0	0	0	2		0	0	2	2	P3

- Describe the process of calculating `RecvCount`
- Given `RecvCount`, how can one calculate `RecvDispl`

Answers: Redistribution during Quicksort

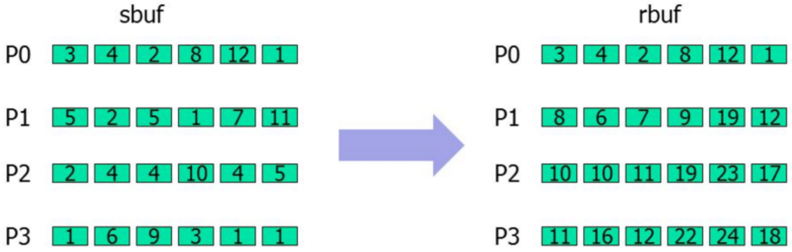
- `RecvCount` can be calculate through an iterative process
- Compute the **prefix sum** of for below/above pivot counts

```
El Count vs Pivot          PS[]: PREFIX SUMS
| Proc | <= | > |          | Proc | <= | > |
|------+----+---|          |------+----+---|
| P0   |  2 | 2 |   ===>   | P0   |  2 | 2 |
| P1   |  3 | 1 |          | P1   |  5 | 3 |
| P2   |  1 | 3 |          | P2   |  6 | 6 |
| P3   |  2 | 2 |          | P3   |  8 | 8 |
```

- Know each proc stores `N / P = 4` elements
- Procs receiving `<= pivot`, proc # `i`, scan column 0 for
  - First partner is proc `F` where `PS[F,0] <= 4*i`
  - Last partner is proc `L` where `PS[L,0] >= 4*(i+1)`
- Procs receiving `> pivot`, proc # `i`, scan column 1 for
  - First partner is proc `F` where `PS[F,1] <= 4*(i-2)`
  - Last partner is proc `L` where `PS[L,1] >= 4*(i-2+1)`
- Actual code will need to do additional arithmetic (e.g. P1 receives 1 element from itself)
- `RecvDispl` is the **prefix sum** of `RecvCount`

Prefix Sums / Scan

Prefix Sums or Prefix Scans are supported in parallel via MPI




```
int MPI_Scan(const void *sendbuf, void *recvbuf, int count,
             MPI_Datatype datatype, MPI_Op op, MPI_Comm comm)
```

- Similar to reduction but only add on values from procs <= `proc_id`
- op can be sum/max/min/etc.
- In simple Quicksort implementations, **don't use parallel prefix scan** as this does not yield enough info to calculate send/receive partners

Overall Flow

1. Pivot selection (open question how to do this right)
2. Broadcast of pivot value
3. Each processor `partition`'s its data
4. All-gather to get element/pivot counts
5. Calculate send/receives
6. Redistribute data via `MPI_Alltoallv()`
7. And then...

Splitting the World

```
int MPI_Comm_split(MPI_Comm comm, int color, int key,
                   MPI_Comm *newcomm);
```

- `comm` is the old communicator (start with `MPI_COMM_WORLD`)
- `color` is which sub-comm to go into
  - Colors `0,1` splits into 2 communicators
  - Colors `0,1,2,3` splits into 4 communicators
  - Etc.
- `key` establishes rank in new sub-comm, usually `proc_id`
- `newcomm` is filled in with a new communicator
- Examine `04-mpi-code/comm_split.c`
- In Quicksort, new comm is different for lower/upper procs

Exercise: Pivot Selection

- So far have assumed a "good" pivot can be found
- Pivot evenly splits N/2 data, half to lower # processors, half to upper

Discuss the following questions with a neighbor

1. What if the pivot is poorly selected? E.g. 1/4 below pivot, 3/4 above? Could the algorithm adapt?
2. How could one avoid a bad pivot? Discuss a some strategies
3. Is there a way to avoid recusing entirely?

Answers: Pivot Selection 1/2

Discuss the following questions with a neighbor

1. What if the pivot is poorly selected? E.g. 1/4 below pivot, 3/4 above? Could the algorithm adapt?

   With some additional computation, can split the world unevenly: 1/4 procs assigned to "low" numbers, 3/4 to "high" numbers. Still broken if a tiny fraction of the array is lower/higher than the pivot: should just try another pivot at that point or use a scheme that prevents poor pivot selection.

2. How could one avoid a bad pivot? Discuss a some strategies

   Lots of these exist, some mentioned in the textbook such as having a randomly selected processor compute its median and broadcast it as the pivot (main text of Grama) or have processors sample random elements, perform All-Gather, then compute a median as the common pivot (Grama Exercise 9.21).

Answers: Pivot Selection 1/2

3. Is there a way to avoid recusing entirely, e.g. single multiway pivot?

   Grama Exercise 9.20 explores this:

   - *Each proc samples elements, often around* $\log(N)$ *elements, and procs perform an All-Gather*
   - *All procs use common sample to select* $P-1$ *common pivots.*
   - *Elements between pivots are sent directly to final destination procs in an All-to-All communication.*
   - *Local sorting commences.*

   In short: With 4 procs, estimate quartile boundaries based on sampling, give bottom 25% of elements to Proc 0, etc. and sort locally.

Exercise: Odd-Even Sort

- Variant of bubble sort which splits bubbling into odd/even phases
- $O(N^2)$ complexity of serial algorithm
- There is potential for parallelism here: what is it?
  - Consider simple case where each $P = N$: each proc hold a single number
  - What can be parallelized and how?

```
ODD_EVEN_SORT(A[]) {
  N = length(A[])
  for(r=0 to N-1){
    if(r is even){
      for(i=0; i<N-1; i+=2){
        compare_exchange(A, i, i+1);
      }
    }
    if(r is odd){
      for(i=1; i<N-1; i+=2){
        compare_exchange(A, i, i+1);
      }
    }
  }
}

COMPARE_EXCHANGE(A[], i, j){
  if(A[i] > A[j]){
    temp = A[i]
    A[i] = A[j]
    A[j] = temp
  }
}
```

Answers: Odd-Even Sort

- There is potential for parallelism here: what is it?
- Consider simple case where each $P = N$: each proc hold a single number
- What can be parallelized and how?
  - *The inner loops of `compare_exchange()` can be executed in parallel as it involves communication between 2 procs to potentially exchange elements but only with a single partner.*
  - *Even iterations, lower evens exchange with higher odds*
  - *Odd iterations lower odds exchange with higher evens*
  - *Exchange can be done via a Send/Receive of elements and then "keeping" the appropriate element, min on lower proc, max on higher proc*

Odd-Even Sort in Practice

- As before, unrealistic to have $P = N$, rather each proc holds N/P elements of the array `A[]`
- `COMPARE_EXCHANGE()` becomes `COMPARE_SPLIT()`

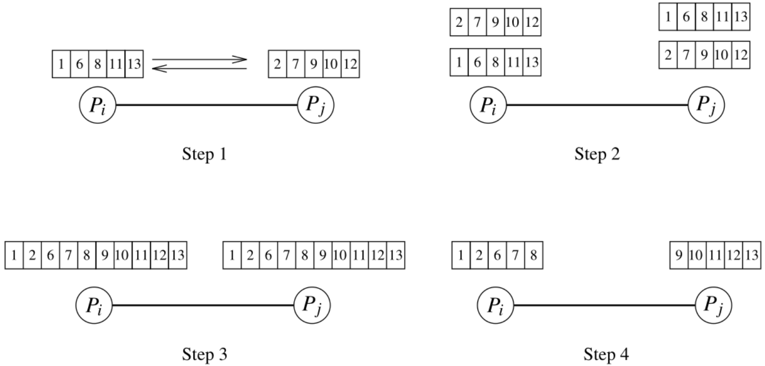


Figure 9.2 A compare-split operation. Each process sends its block of size n/p to the other process. Each process merges the received block with its own block and retains only the appropriate half of the merged block. In this example, process P_i retains the smaller elements and process P_j retains the larger elements.

Analysis of Odd-Even Sort

- Initially all procs sort their local array: $O(\frac{N}{P}\log\frac{N}{P}$
- Conduct P Outer iterations of `ODD_EVEN_SORT()`
- Each odd/even inner loop is done in parallel by all procs communicating with a neighbor
- Neighbor procs exchange ararys: $O(t_s + t_w\frac{N}{P})$
- Each proc then performs a compare/split: $O(\frac{2N}{P}$
- Overall complexity of parallel algorithm:

$$\begin{aligned} T_{par} &= O\left(\frac{N}{P}\log\frac{N}{P}\right) + P \times \left(O\left(t_s + t_w\frac{N}{P}\right) + O\left(\frac{2N}{P}\right)\right) \\ &= O\left(\frac{N}{P}\log\frac{N}{P}\right) + O(N) + O(N) \end{aligned}$$

Isoefficiency? : Reported in textbook as $O(P2^P)$, linear increase in P requires exponential increase in problem size to maintain efficiency. Verifying this is a good exercise.

Sorting Extras

Odd-Even Sort to Shell Sort

- Allowing bigger "moves" in odd-even sort can improve practical efficiency of algorithm
- Shell Sort provides a mechanism for this: neighbors selected according to a "gap" scheme, less known sort with yet mysterious complexity analysis

Sorting Hardware

- Grama Ch 9.1 discusses Sorting networks, specialized hardware which can implement sorting
- With N processors, can implement Bitonic Sort in a sorting network and achieve $T_{par} = O(\log^2 N)$
- Hardware that implements sorting networks is not common but...
- Can do this in GPUs and may revisit this algorithm when we do CUDA programming