CSCI 2021: Binary Floating Point Numbers

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Logistics

Reading Bryant/O'Hallaron

- Ch 2.4-5: Floats, Wed/Fri
- 2021 Quick Guide to GDB
- Next week: Ch 3.1-7: Assembly Intro

Goals this Week

- Discuss Bitwise ops (Integer Slides)
- `gdb` introduction
- Floating Point layout

Feedback Survey

- Open on Canvas
- Anonymous: be honest!
- Due Wed 2/17 for 1 EP

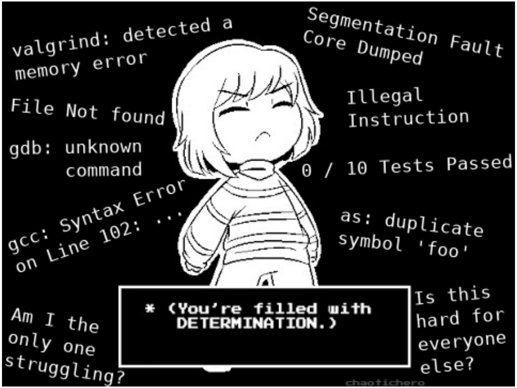
Labs/HW

- Lab05: Bit operations
- HW05: Bits, Floats, GDB

Project 2

Discuss later this week

Don't Give Up, Stay Determined!



- If Project 1 / Exam 1 went awesome, count yourself lucky
- If things did not go well, Don't Give Up
- Spend some time contemplating why things didn't go well, talk to course staff about it, learn from any mistakes
- There is a LOT of semester left and plenty of time to recover from a bad start

Parts of a Fractional Number

The meaning of the "decimal point" is as follows:

$$\begin{aligned} 123.406_{10} = & 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0 + \\ & 4 \times 10^{-1} + 0 \times 10^{-2} + 6 \times 10^{-3} \\ = & 123.406_{10} \end{aligned} \qquad \begin{aligned} 123 &= 100 + 20 + 3 \\ 0.406 &= \frac{4}{10} + \frac{6}{1000} \end{aligned}$$

Changing to base 2 induces a "binary point" with similar meaning:

$$\begin{aligned} 110.101_2 = & 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + \\ & 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\ = & 6.625_{10} \end{aligned} \qquad \begin{aligned} 6 &= 4 + 2 \\ 0.625 &= \frac{1}{2} + \frac{1}{8} \end{aligned}$$

One *could* represent fractional numbers with a **fixed point** e.g.

- 32 bit fractional number with
- 10 bits left of Binary Point (integer part)
- 22 bits right of Binary Point (fractional part)

BUT most applications require a more flexible scheme

Scientific Notation for Numbers

"Scientific" or "Engineering" notation for numbers with a fractional part is

Standard	Scientific	`printf("%.4e",x);`
123.456	1.23456×10^{2}	`1.2346e+02`
50.01	5.001×10^{1}	`5.0010e+01`
3.14159	3.14159×10^{0}	`3.1416e+00`
0.54321	5.4321×10^{-1}	`5.4321e-01`
0.00789	7.89×10^{-3}	`7.8900e-03`

- **Always** includes one **non-zero** digit left of decimal place
- Has some **significant** digits after the decimal place
- Multiplies by a **power of 10** to get actual number

Binary Floating Point Layout Uses Scientific Convention

- Some bits for integer/fractional part
- Some bits for exponent part
- All in base 2: 1's and 0's, powers of 2

Conversion Example

Below steps convert a decimal number to a fractional binary number equivalent then adjusts to scientific representation.

```
float fl = -248.75;

              7  6  5  4 3 2 1 0   -1  -2
-248.75 = -(128+64+32+16+8+0+0+0).(1/2+1/4)
        = -11111000.11 *2^0
           76543210 12
        = -1111100.011 *2^1
           6543210 123
        = -111110.0011 *2^2
           543210 1234
           ...
             MANTISSA    EXPONENT
        = -1.111100011 * 2^7
           0 123456789
```

Mantissa $\equiv$ Significand $\equiv$ Fractional Part

Principle and Practice of Binary Floating Point Numbers

- In early computing, computer manufacturers used similar principles for floating point numbers but varied specifics
- Example of Early float data/hardware
  - Univac: 36 bits, 1-bit sign, 8-bit exponent, 27-bit significand[1]
  - IBM: 32 bits, 1-bit sign, 7-bit exponent, 24-bit significand[2]
- Manufacturers implemented circuits with different rounding behavior, with/without infinity, and other inconsistencies
- Troublesome for reliability: code produced different results on different machines
- This was resolved with the adoption of the IEEE 754 Floating Point Standard which specifies
  - Bit layout of 32-bit `float` and 64-bit `double`
  - Rounding behavior, special values like Infinity
- Turing Award to William Kahan for his work on the standard

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[1] Floating Point Arithmetic

[2] IBM Hexadecimal Floats

IEEE 754 Format: *The* Standard for Floating Point

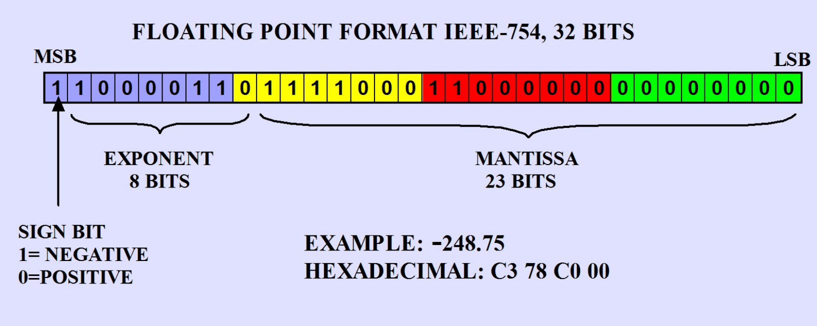
`float`	`double`	Property
32	64	Total bits
1	1	Bits for sign (1 neg / 0 pos)
8	11	Bits for Exponent multiplier (power of 2)
23	52	Bits for Fractional part or **mantissa**
7.22	15.95	Decimal digits of accuracy[3]

- Most commonly implemented format for floating point numbers in hardware to do arithmetic: processor has physical circuits to add/mult/etc. for this bit layout of floats
- Numbers appear in three categories

Category	**Description**	**Exponent**
Normalized	most common like `1.0` and `-9.56e37`	mixed 0/1
Denormalized	very close to zero and `0.0`	all 0's
Special	extreme/error values like `Inf` and `NaN`	all 1's

[3] Wikipedia: IEEE 754

Example float Layout of -248.75: float_examples.c



Source: IEEE-754 Tutorial, www.puntoflotante.net

Color: 8-bit blocks, **Negative**: highest bit, leading 1

Exponent: high 8 bits, 2^7 encoded with bias of -127

```
  1000_0110 - 0111_1111
= 128+4+2   - 127
= 134 - 127
= 7
```

Fractional/Mantissa portion is

```
1.111100011...
^ |||||||||
| explicit low 23 bits
|
implied leading 1
not in binary layout
```

Normalized Floating Point: General Case

- A "normalized" floating point number is in the standard range for `float/double`, bit layout follows previous slide
- Example: `-248.75 = -1.111100011 * 2^7`

Exponent is in **Bias Form** (not Two's Complement)

- Unsigned positive integer minus constant **bias number**
- **Consequence**: exponent of 0 is not bitstring of 0's
- **Consequence**: tiny exponents like -125 close to bitstring of 0's; this makes resulting number close to 0
- 8-bit exponent `1000 0110 = 128+4+2 = 134`
  so exponent value is `134 - 127 = 7`

Integer and Mantissa Parts

- The leading `1` before the binary point is **implied** so does not show up in the bit string
- Remaining fractional/mantissa portion shows up in the low-order bits

Fixed Bit Standards for Floating Point

IEEE Standard Layouts

Kind	Sign Bit	Exponent Bits	Bias	Exp Range	Mantissa Bits
`float`	31 (1)	30-23 (8 bits)	-127	-126 to +127	22-0 (23 bits)
`double`	63 (1)	62-52 (11 bits)	-1023	-1022 to +1023	51-0 (52 bits)

Standard allows hardware to be created that is as efficient as possible to do calculation on these numbers

Consequences of Fixed Bits

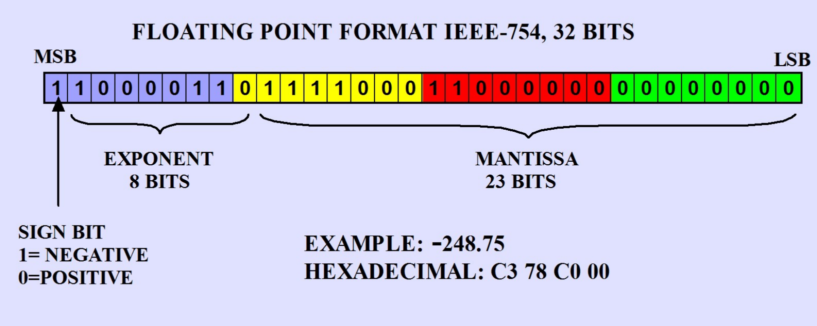
- Since a fixed # of bit is used, **some numbers cannot be exactly represented**, happens in any numbering system:
- Base 10 and Base 2 cannot represent $\frac{1}{3}$ in finite digits
- Base 2 cannot represent $\frac{1}{10}$ in finite digits

  ```
  float f = 0.1;
  printf("0.1 = %.20e\n",f);
  0.1 = 1.00000001490116119385e-01
  ```

  Try `show_float.c` to see this in action

Exercise: Quick Checks

1. Represent 7.125 in binary using "binary point" notation
2. What distinct parts are represented by bits in a floating point number (according to IEEE)
3. What is the "bias" of the exponent for 32-bit `floats`
4. What does the number 1.0 look like as a `float`?



Source: IEEE-754 Tutorial, www.puntoflotante.net

The diagram above may help in recalling IEEE 754 layout

Answers: Quick Checks

1. Represent 7.125 in binary using a "binary point"
   - $7_{10} = 111_2$
   - $0.125_{10} = \frac{1}{8} = 2^{-3} = 0.001_2$
   - $7.125_{10} = 111.001_2$
2. What distinct parts are represented by bits in a floating point number (according to IEEE 754)
   - Sign, Exponent, and Mantissa/Fractional Portion
3. What is the "bias" of the exponent for 32-bit `floats` (according to IEEE 754)
   - Bias is -127 which is subtracted from the unsigned value of the 8 exponent bits to get the actual exponent
4. What does the number 1.0 look like as a `float`?
   - Positive: sign bit of `0`
   - Exponent is 0, so sign bits total 127:

     ```
     0111 1111
     8    4
     ```
   - Mantissa has implied leading 1 and all 0's so:

     ```
     000 0000 0000 0000 0000 0000
     23  20   16   12   8    4
     ```

Special Cases: See `float_examples.c`

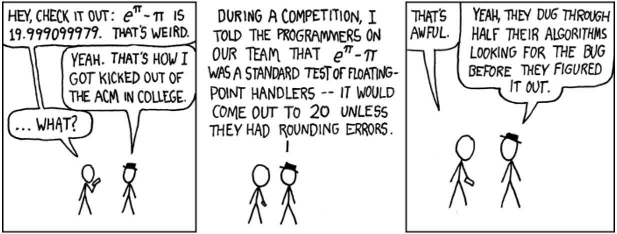
Denormalized values: Exponent bits all 0

- Fractional/Mantissa portion evaluates *without* implied leading one, still an unsigned integer though
- Exponent is $Bias + 1$: 2^{-126} for `float`
- Result: very small numbers close to zero, smaller than any other representation, degrade uniformly to 0
- Zero: bit string of all 0s, optional leading 1 (*negative zero*);

Special Values

- **Infinity**: exponent bits all `1`, fraction all `0`, sign bit indicates $+\infty$ or $-\infty$
- Infinity results from overflow/underflow or certain ops like `float x = 1.0 / 0.0;`
- `#include <math.h>` gets macro `INFINITY` and `-INFINITY`
- **NaN**: not a number, exponent bits all `1`, fraction has some 1s
- Errors in floating point like `0.0 / 0.0`

Other Float Notes



Source: XKCD #217

Approximations and Roundings

- Approximate $\frac{2}{3}$ with 4 digits, usually `0.6667` with standard rounding in base 10
- Similarly, some numbers cannot be exactly represented with fixed number of bits: $\frac{1}{10}$ approximated
- IEEE 754 specifies various rounding modes to approximate numbers

Clever Engineering

- IEEE 754 allows floating point numbers to sort using signed integer routines
- Bit patterns for `float` follows are ordered the same as bit patterns for signed `int`
- Integer comparisons are usually fewer clock cycles than floating comparisons

Sidebar: The Weird and Wonderful Union

- Bitwise operations like & are not valid for `float`/`double`
- Can use pointers/casting to get around this OR...
- Use a **union**: somewhat unique construct to C
- Defined like a `struct` with several fields
- BUT fields occupy the same memory location (!?!)
- Allows one to treat a byte position as multiple different types, ex: `int / float / char[]`
- Memory size of the union is the **max** of its fields

```
// union.c
typedef union { // shared memory
  float fl;      // an int
  int in;        // a float
  char ch[4];    // char array
} flint_t;       // 4 bytes total

int main(){
  flint_t flint;
  flint.in = 0xC378C000;
  printf("%.4f\n",   flint.fl);
  printf("%08x %d\n",flint.in,flint.in);
  for(int i=0; i<4; i++){
    unsigned char c = flint.ch[i];
    printf("%d: %02x '%c'\n",i,c,c);
  }
}
```

```
| Symbol           | Mem   |  Val |
|------------------+-------+------|
| flint.ch[3]      | #1027 | 0xC3 |
| flint.ch[2]      | #1026 | 0x78 |
| flint.ch[1]      | #1025 | 0xC0 |
| flint.in/fl/ch[0] | #1024 | 0x00 |
| i                | #1020 | ?    |
```

Floating Point Operation Efficiencies

- Floating Point Operations per Second, **FLOPS** is a major measure for numerical code/hardware efficiency
- Often used to benchmark and evaluate scientific computer resources, (e.g. top super computers in the world)
- Tricky to evaluate because of
  - A single FLOP (add/sub/mul/div) may take 3 clock cycles to finish: **latency 3**
  - Another FLOP **can start** before the first one finishes: **pipelined**
  - Enough FLOPs lined up can get **average 1 FLOP per cycle**
  - FP Instructions may automatically operate on multiple FPs stored in memory to feed pipeline: **vectorized ops**
  - Generally referred to as **superscalar**
  - Processors schedule things **out of order** too
- All of this makes micro-evaluation error-prone and pointless
- Run a real application like an N-body simulation and compute

$$\text{FLOPS} = \frac{\text{number of floating ops done}}{\text{time taken in seconds}}$$

Top 5 Super Computers Worldwide, Nov 2017

Rank	System	#Cores	Rmax (TFlop/s)	Rpeak (TFlop/s)	Power (kW)
1	Sunway TaihuLight *China* Sunway MPP	10,649,600	93,014.6	125,435.9	15,371
2	Tianhe-2 (MilkyWay-2) *China* TH-IVB-FEP Cluster	3,120,000	33,862.7	54,902.4	17,808
3	Piz Daint *Switzerland* Cray XC50	361,760	19,590.0	25,326.3	2,272
4	Gyoukou *Japan* ZettaScaler-2.2 HPC system	19,860,000	19,135.8	28,192.0	1,350
5	Titan *USA* Cray XK7	560,640	17,590.0	27,112.5	8,209

https://www.top500.org/lists/2017/11/

Top 5 Super Computers Worldwide, Nov 2018

Rank	System	#Cores	Rmax (TFlop/s)	Rpeak (TFlop/s)	Power (kW)
1	Summit *United States* IBM POWER9 22C 3.07GHz	2,397,824	143,500.0	200,794.9	9,783
2	Sierra *United States* IBM POWER9 22C 3.1GHz,	1,572,480	94,640.0	125,712.0	7,438
3	Sunway TaihuLight *China* Sunway MPP	10,649,600	93,014.6	125,435.9	15,371
4	Tianhe-2A *China* TH-IVB-FEP Cluster	4,981,760	61,444.5	100,678.7	18,482
5	Piz Daint *Switzerland* Cray XC50, Xeon E5-2690v3	387,872	21,230.0	27,154.3	2,384

https://www.top500.org/list/2018/11/

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3	Sunway TaihuLight *China* Sunway MPP	10,649,600	93,014.6	125,435.9	15,371
4	Tianhe-2A *China* Xeon 2.2GHz	4,981,760	61,444.5	100,678.7	18,482
5	Frontera, *United States* Dell 6420, Xeons 2.7GHz	448,448	23,516.4	38,745.9	??

https://www.top500.org/list/2019/11/

Top 5 Super Computers Worldwide, June 2020

Rank	System	#Cores	Rmax (TFlop/s)	Rpeak (TFlop/s)	Power (kW)
1	Fugaku, *Japan / Fujitsu* Fujitsu A64FX 2.2GhZ (Arm)	7,299,072	415,530.0	513,854.7	28,335
2	Summit *United States* IBM POWER9 22C 3.07GHz (Power)	2,397,824	143,500.0	200,794.9	10,096
3	Sierra *United States* IBM POWER9 22C 3.1GHz (Power)	1,572,480	94,640.0	125,712.0	7,438
4	Sunway TaihuLight *China* Sunway SW26010 (custom RISC)	10,649,600	93,014.6	125,435.9	15,371
5	Tianhe-2A *China* Intel Xeon 2.2GHz (x86-64)	4,981,760	61,444.5	100,678.7	18,482

https://www.top500.org/lists/top500/2020/06/

Top 5 Super Computers Worldwide, November 2020

Rank	System	#Cores	Rmax (TFlop/s)	Rpeak (TFlop/s)	Power (kW)
1	Fugaku, *Japan / Fujitsu* Fujitsu A64FX 2.2GhZ (Arm)	7,299,072	415,530.0	513,854.7	28,335
2	Summit *United States* IBM POWER9 22C 3.07GHz (Power)	2,397,824	143,500.0	200,794.9	10,096
3	Sierra *United States* IBM POWER9 22C 3.1GHz (Power)	1,572,480	94,640.0	125,712.0	7,438
4	Sunway TaihuLight *China* Sunway SW26010 (custom RISC)	10,649,600	93,014.6	125,435.9	15,371
5	Selene *USA, NVIDIA/AMD* AMD EPYC 7742 64C 2.25GHz (x86-64)	555,520	63,460.0	79,215.0	2,646

https://www.top500.org/lists/top500/2020/06/